Existence and Uniqueness of Solutions for a Non-uniformly Parabolic Equation[1]

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Abstract. In this paper we study a generalized Burgers equation $u_t + (u^2)_x = tu_{xx}$, which is a non-uniformly parabolic equation for $t > 0$. We show the existence and uniqueness of classical solutions to the initial value problem of the generalized Burgers equation with rough initial data belonging to $L^\infty(\mathbb{R})$.

Key Words: Non-uniformly parabolic equation, existence, uniqueness, classical solutions

AMS subject classification: 35K55, 35B65

1. Introduction

In this paper we will consider the following initial value problem for a non-uniformly parabolic equation

$$u_t + (u^2/2)_x = tu_{xx}, \quad x \in \mathbb{R}, \ t > 0 \tag{1}$$

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with the initial data

$$u(x,0) = u_0(x), \quad u_0 \in L^\infty(\mathbb{R}). \tag{2}$$

The equation (1) can be derived from the system of compressible Navier-Stokes equations with cylindrical symmetry, the ideal gas law as well as the polytropic equation of state when the method of multiple scales is used, see Sachdev [10], Leibovich and Seebass [8]. The initial value problem (1) and (2) also occurs in the study of the propagation of finite-amplitude sound waves in variable-area ducts, see Crighton [1]. The equation (1) and the corresponding systems have attracted attention in the mathematical community due to the fact that the corresponding Riemann problems admit self-similar solutions, see e.g. Dafermos [3], Slemrod [12] or Tzavaras [14]. The long time behavior of solutions to the initial value problem has been studied, e.g. by Crighton and Scott [2] as well as Scott [11] under the assumption of the well-posedness of the initial value problem (1), (2) without verification.

In this paper we show the existence and uniqueness of classical solutions to the initial value problem (1), (2) when the initial data only belong to $L^\infty(\mathbb{R})$. It is straightforward to extend the results obtained in this paper to the type of the equation (1) with a general convex flux function in its convection term instead of the quadratic function considered here.

This paper is organized as follows: In the Section 2, we show that in the definition of weak solutions to the initial value problem (1), (2) we may use more general test functions that do not have compact support. This allows us to use solutions to the adjoint problem as test functions. Section 3 is devoted to the uniqueness of weak solution. It is shown by a nonlinear version of the Holmgren method, which was used by Oleinik [9] and Hoff [5] for convex conservation laws. We estimate the decay rates of solutions, as well as their derivatives, to the adjoint parabolic equation for the difference of two solutions to the initial value problem (1), (2). In Section 4 we show that the weak solutions of the initial value problem (1), (2) are classical solutions in the sense that they have all of the continuous derivatives occurring in equation (1). The slightly stronger version of a one-sided Lipschitz condition that was given by Tadmor [13] is used in the process of the proof.

Finally in Section 5 we indicate how the existence of the weak solutions to the initial value problem (1), (2) may also be obtained via a finite difference scheme with variable time steps. As matter of fact the scheme can be used as a numerical method for the computation of approximate solutions to this problem. It is interesting to note that the first n_0 steps of the scheme proposed here use a constant time step as used when the Lax-Friedrichs scheme is taken to approximate conservation laws. This number n_0 depends only on the supremum norm of the initial data. The first n_0 steps of the scheme deal with the non-uniform parabolicity as t tends to 0 and ensure that the scheme satisfies the CFL stability condition. Thereafter variable time steps are used in order to be consistent with the nonuniform parabolic equation (1) and to ensure global stability of the scheme.

2. Existence of weak solutions

In this section we discuss the existence of weak solutions to the initial value problem (1) and (2). First we give the definition of the weak solutions to this problem.

Definition 2.1 *A bounded measurable function u is called a* **weak solution** *of the initial value problem (1) and (2) if it satisfies the following conditions:*

$$\int_{t_1}^{t_2}\int_{-\infty}^{\infty}[u\phi_t + \frac{u^2}{2}\phi_x + tu\phi_{xx}]\, dxdt - \int_{-\infty}^{\infty}(u\phi)(x,\cdot)|_{t_1}^{t_2}\, dx = 0 \tag{3}$$

for any $0 < t_1 < t_2$ and all $\phi \in C_0^2(\mathbb{R}\times\mathbb{R}_+)$ where $\mathbb{R}_+ = [0,\infty[$. Additionally we assume that the solution u satisfies for any $t > 0$ the one-sided Lipschitz condition

$$L^+[u(\cdot,t)] := ess\sup_{x\neq y}\left(\frac{u(x,t)-u(y,t)}{x-y}\right)^+ \leq \frac{1}{t} \tag{4}$$

and for the initial data we require that $u(\cdot,t)$ tends to u_0 in $L^1_{loc}(\mathbb{R})$ as $t \to 0$. □

We easily obtain the following basic existence result.

Theorem 2.2. *Let $u_0 \in L^\infty(\mathbb{R})$. Then there exists a weak solution u of the initial value problem (1), (2) having the properties that $u(\cdot,t)$ converges in $L^1_{loc}(\mathbb{R})$ to u_0 for $t \to 0$, it satisfies a one-sided Lipschitz condition (4) and the bound*

$$|u(x,t)| \leq \|u_0\|_{L^\infty} = M.$$

Proof. We use the vanishing viscosity method. Consider for $\varepsilon > 0$ the nonsingular parabolic equation

$$u_t + (u^2/2)_x = (t+\varepsilon)u_{xx}, \quad 0 < \varepsilon \leq 1.$$

The existence of weak solutions for the uniformly parabolic case $\varepsilon > 0$ and the properties of the singular limit $\varepsilon \to 0$ follow by standard theory along the lines of Oleinik [9] and

Kružkov [7] analogously as in the case of conservation laws with convex flux functions. The proof of the one-sided Lipschitz condition can be given along the line of the argument by Tadmor [13] with slight modifications. $\square$

Corollary 2.3. *If equality (3) holds for $\phi \in C_0^2(\mathbb{R} \times \mathbb{R}_+)$ this implies that the equality (3) holds for all*

$$\begin{aligned} &\phi \in C^2(\mathbb{R} \times \mathbb{R}_+), \\ &\phi, \phi_x, \phi_{xx} \textit{ and } \phi_t \textit{ as functions of } x \textit{ belonging to } L^1(\mathbb{R}) \textit{ for any } t > 0. \end{aligned} \tag{5}$$

Proof. We introduce for $N > 0$ a cut-off function

$$\xi_N(x) = \int_{-\infty}^{\infty} \chi(x-y)\eta_N(y)\, dy$$

by taking a standard non-negative mollifying function $\chi \in C_0^\infty(\mathbb{R})$ with unit mass supported in the interval $[-1, 1]$. The function η_N is the characteristic function of the interval $[-N-1, N+1]$, i.e.

$$\eta_N(x) = \begin{cases} 1, & \text{for } |x| \le N+1 \\ 0, & \text{for } |x| > N+1. \end{cases}$$

It is easy to see that

$$\begin{aligned} &\xi_N \in C_0^\infty(\mathbb{R}), \\ &\xi_N(x) = \begin{cases} 1, & \text{for } |x| \le N \\ 0, & \text{for } |x| \ge N+2. \end{cases} \\ &|\xi_N(x)| \le 1, \quad |\xi_N'(x)| \le C-1, \quad |\xi_N''(x)| \le C_1, \end{aligned} \tag{6}$$

where

$$C := \int_{-\infty}^{\infty} |\chi'(s)|\, ds + 1, \quad C_1 := \int_{-\infty}^{\infty} |\chi''(s)|\, ds. \tag{7}$$

Now consider the quantity $Q(u, \phi)$ obtained by substituting ϕ with the properties (5), instead of a $\phi \in C_0^2(\mathbb{R} \times \mathbb{R}_+)$, into the left hand side of (3), i.e.

$$\begin{aligned} Q(u,\phi) &= \int_{t_1}^{t_2}\int_{-\infty}^{\infty} (u\phi_t + \frac{u^2}{2}\phi_x + tu\phi_{xx})\, dxdt - \int_{-\infty}^{\infty} (u\phi)(x,\cdot)|_{t_1}^{t_2}\, dx \\ &= Q(u, \phi\xi_N) + Q(u, \phi(1-\xi_N)). \end{aligned} \tag{8}$$

It is easy to see from (5), (6) that $\phi\xi_N \in C_0^2(\mathbb{R})$. Therefore it follows by the assumption of the corollary that it is an admissible test function for which

$$\begin{aligned} Q(u,\phi\xi_N) &= \int_{t_1}^{t_2}\int_{-\infty}^{\infty} (u(\phi\xi_N)_t + \frac{u^2}{2}(\phi\xi_N)_x + tu(\phi\xi_N)_{xx})\, dxdt - \int_{-\infty}^{\infty} (u\phi\xi_N)(x,\cdot)|_{t_1}^{t_2}\, dx \\ &= 0. \end{aligned} \tag{9}$$

Now let us estimate $Q(u,\phi(1-\xi_N))$ as follows

$$
\begin{aligned}
|Q(u,\phi(1-\xi_N))| &= \Big| \int_{t_1}^{t_2}\int_{-\infty}^{\infty} [u(\phi(1-\xi_N))_t + \frac{u^2}{2}(\phi(1-\xi_N))_x + tu(\phi(1-\xi_N))_{xx}]\, dxdt \\
&\quad - \int_{-\infty}^{\infty} (u\phi(1-\xi_N))(x,\cdot)|_{t_1}^{t_2}\, dx\Big| \\
&\le \int_{t_1}^{t_2}\int_{|x|\ge N} |[u(\phi(1-\xi_N))_t| + |\frac{u^2}{2}(\phi(1-\xi_N))_x| \\
&\quad + |tu(\phi(1-\xi_N))_{xx}]|\, dxdt + \int_{|x|\ge N} |(u\phi(x,\cdot)|_{t_1}^{t_2}(1-\xi_N))\, dx
\end{aligned}
$$

where (6) was used. We consider the properties of ϕ given by (5), the boundedness of u, and the estimates on ξ_N in (6). Then we have for any given $\varepsilon > 0$, taking $N = N(\varepsilon)$ large enough,

$$|Q(u,\phi(1-\xi_N))| \le \varepsilon. \tag{10}$$

It follows from (8), (9) and the arbitrariness of ε in (10) that $Q(u,\phi) = 0$. □

3. Uniqueness of the weak solution

We will show that weak solutions of the initial value problem (1), (2) which satisfy Definition 2.1 are unique. The method of proof we give is a nonlinear version of the method of Holmgren that Hoff [5] and Oleinik [9] used for convex scalar conservation laws.

We take two solutions u and v and mollify them using a standard nonnegative mollifying function $\chi \in C_0^\infty(\mathbb{R})$ with unit mass that is supported in $[-1,1]$ to obtain for any $\delta \in]0,t_1[$

$$
\begin{aligned}
u_\delta(x,t) &= \frac{1}{\delta^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \chi\left(\frac{x-y}{\delta}\right)\chi\left(\frac{t-\tau}{\delta}\right) u(y,\tau)\, dyd\tau, \\
v_\delta(x,t) &= \frac{1}{\delta^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \chi\left(\frac{x-y}{\delta}\right)\chi\left(\frac{t-\tau}{\delta}\right) v(y,\tau)\, dyd\tau.
\end{aligned}
$$

Then we consider the *adjoint problem*, namely the backward initial value problem for any given $0 < t_1 < t_2$

$$\phi_t + \frac{1}{2}(u_\delta + v_\delta)\phi_x + t\phi_{xx} = 0, \qquad t \in]t_1,t_2[, \tag{11}$$

$$\phi(x,t_2) = \psi(x), \tag{12}$$

where $\psi \in C_0^\infty(\mathbb{R})$ is any given function. The equation (11) is a linear uniformly parabolic equation with smooth coefficients. It has smooth solutions and satisfies the maximum principle, see Friedman [4]. We take

$$M_0 := \|\psi(\cdot)\|_{L^\infty}, \quad M_1 := \|\psi'(\cdot)\|_{L^\infty}, \quad M_2 := \|\psi''(\cdot)\|_{L^\infty}, \quad \text{and} \quad \mathrm{supp}\psi \subset [-K, K].$$

First we prove the following properties of the solution to the adjoint problem.

Lemma 3.1. *The solution of the adjoint problem (11) and (12) satisfies the following estimates for $t \in [t_1, t_2]$*

$$\begin{aligned}
|\phi(x,t)| &\le M_0 \min\left\{1, \exp\left(C(t_2-t) + \frac{(M+1)(t_2-t)+K+2-|x|}{t_2}\right)\right\}, && (13)\\
|\phi_x(x,t)| &\le \frac{t_2}{t_1} M_1 \min\left\{1, \exp\left(C(t_2-t) + \frac{(M+1)(t_2-t)+K+2-|x|}{t_2}\right)\right\}, && (14)\\
|\phi_{xx}(x,t)| &\le \left(\frac{t_2^2}{t_1^2} M_2 + \frac{C_1 t_2^3 M M_1}{t_1^3 \delta^2}\right) \\
&\qquad \cdot \exp\left(C(t_2-t) + \frac{(M+1)(t_2-t)+K+2-|x|}{t_2}\right) && (15)
\end{aligned}$$

and

$$|\phi_t(x,t)| \le t_2 |\phi_{xx}(x,t)| + M|\phi_x(x,t)|, \tag{16}$$

where the constants C as well as C_1 were specified in (7) and M in Theorem 2.2. Then by Corollary 2.3 the function ϕ is admissible as a test function in (3).

Proof. The estimate (13): First let us transform the backward initial value problem (11), (12) into the following forward initial value problem

$$\begin{aligned}
L(\phi) &= \phi_\tau - \frac{u_\delta + v_\delta}{2}\phi_x - (t_2-\tau)\phi_{xx} = 0, \quad \tau \in]0, t_2 - t_1[, && (17)\\
\phi(x,0) &= \psi(x) && (18)
\end{aligned}$$

with $\tau = t_2 - t$. Let us introduce an auxilliary function

$$\Phi(x,\tau) = \exp\left(C\tau + \frac{(M+1)\tau + G(x) + 1}{t_2}\right) \tag{19}$$

where

$$G(x) = \int_{-\infty}^{\infty} \chi(x-y)(K-|y|)\, dy.$$

Here χ is the mollifying function introduced above. Using (7) the function G is easily seen to satisfy the estimates

$$|G'(x)| \le 1, \qquad |G''(x)| \le \int_{-\infty}^{\infty} |\chi'(s)|\, ds = C - 1.$$

A similar type of function Φ as in (19) was considered by Kružkov [7], see also Hörmander [6]. One may also show that

$$K - |x| \le G(x) + 1 \le K + 2 - |x|.$$

Since $\tau = t_2 - t$ we have

$$\exp\left(C(t_2 - t) + \frac{(M+1)(t_2 - t) + K + 2 - |x|}{t_2}\right) \ge \Phi(x,\tau). \tag{20}$$

Therefore in order to get the estimate (13) it is enough to prove that

$$|\phi(x,\tau)| \le M_0 \min(1, \Phi(x,\tau)). \tag{21}$$

By straightforward calculation we have

$$\begin{aligned} L(M_0\Phi \pm \phi) &= M_0\Phi\Big(C + \frac{M+1}{t_2} - \frac{u_\delta + v_\delta}{2}\frac{G'(x)}{t_2} - (t_2 - \tau)\frac{G'(x)^2}{t_2^2} \\ &\quad -(t_2 - \tau)\frac{G''(x)}{t_2}\Big) > M_0\Phi > 0 \end{aligned} \tag{22}$$

for $\tau \in]0, t_2 - t_1]$. It follows by the maximum principle for (22) that the minimum of $M_0\Phi \pm \phi$ in the strip $\mathbb{R} \times [0, t_2 - t_1]$ is taken when $\tau = 0$, so by $\|\psi(\cdot)\|_{L^\infty} \le M_0$ we obtain

$$(M_0\Phi \pm \phi)(x,\tau) \ge \inf_{x\in\mathbb{R}} (M_0\Phi \pm \phi)|_{\tau=0} \ge M_0 \pm \psi(x) \ge 0. \tag{23}$$

Therefore (21) is proved by (23) and the fact that $|\phi(x,\tau)| \le M_0$, which is obtained by directly applying the maximum principle to the initial value problem (17), (18).

The estimate (14): We have to prove that

$$|\phi_x(x,\tau)| \le \frac{t_2}{t_1} M_1 \min(1, \Phi(x,\tau)).$$

Differentiating $L(\phi)$ in (17) with respect to x and setting $\tilde{\phi} = \phi_x$ gives

$$\begin{aligned} &\tilde{\phi}_\tau - \frac{1}{2}(u_\delta + v_\delta)\tilde{\phi}_x - \frac{(u_\delta + v_\delta)_x}{2}\tilde{\phi} - (t_2 - \tau)\tilde{\phi}_{xx} = 0, \\ &\tilde{\phi}(x,0) = \psi'(x). \end{aligned}$$

Then using the transformation $\tilde{\phi} = \frac{t_2}{t_2 - \tau}\overline{\phi}$ we have

$$L_1(\overline{\phi}) = \overline{\phi}_\tau - \frac{u_\delta + v_\delta}{2}\overline{\phi}_x - \left[\frac{(u_\delta + v_\delta)_x}{2} - \frac{1}{t_2 - \tau}\right]\overline{\phi} - (t_2 - \tau)\overline{\phi}_{xx} = 0, \tag{24}$$

$$\overline{\phi}(x,0) = \psi'(x). \tag{25}$$

The one-sided Lipschitz condition (4) in Definition 2.1 implies that

$$\frac{(u_\delta + v_\delta)_x}{2} \le \frac{1}{t} = \frac{1}{t_2 - \tau}.$$

This allows us to apply the maximum principle to (24), see Friedman [4, Chapter 2]. Therefore we obtain

$$|\overline{\phi}(x,\tau)| \le \|\psi'(\cdot)\|_{L^\infty} = M_1 \tag{26}$$

and analogously to (22)

$$L_1(M_1\Phi \pm \overline{\phi}) > M_1\Phi > 0, \quad \tau \in]0, t_2 - t_1].$$

By the maximum principle we know that a non-positive minimum of $M_1\Phi \pm \overline{\phi}$ cannot be taken for $\tau \in]0, t_2 - t_1]$. But we have

$$\inf_{x\in\mathbb{R}} (M_1\Phi \pm \overline{\phi})|_{\tau=0} \ge M_1 \pm \psi'(x) \ge 0.$$

Therefore, we get

$$|\overline{\phi}| \le M_1\Phi. \tag{27}$$

The inequalities (26), (27) and the transformation give

$$|\phi_x(x,\tau)| = |\tilde{\phi}(x,\tau)| \le \frac{t_2}{t_1} M_1 \min(1, \Phi), \tag{28}$$

which implies the estimate (14) by (20).

The estimate (15): Differentiating $L(\phi)$, satisfying (17), twice with respect to x and setting $\hat{\phi} = \phi_{xx}$ gives

$$\begin{aligned}
&\hat{\phi}_\tau - \frac{1}{2}(u_\delta + v_\delta)\hat{\phi}_x - (u_\delta + v_\delta)_x\hat{\phi} - (t_2 - \tau)\hat{\phi}_{xx} = \frac{1}{2}(u_\delta + v_\delta)_{xx}\phi_x, \\
&\hat{\phi}(x,0) = \psi''(x).
\end{aligned}$$

Then using transformation $\hat{\phi} = \frac{t_2^2}{(t_2-\tau)^2}\underline{\phi}$ we have

$$\begin{aligned}
L_2(\underline{\phi}) &= \underline{\phi}_\tau - \frac{u_\delta + v_\delta}{2}\underline{\phi}_x - \left[(u_\delta + v_\delta)_x - \frac{2}{t_2 - \tau}\right]\underline{\phi} - (t_2 - \tau)\underline{\phi}_{xx} \\
&= \frac{1}{2}(u_\delta + v_\delta)_{xx}\phi_x \frac{(t_2 - \tau)^2}{t_2^2}, \\
\underline{\phi}(x,0) &= \psi''(x).
\end{aligned}$$

First we consider the case of a homogeneous right hand side

$$\begin{aligned}
&L_2(\underline{\phi}_1) = 0, \\
&\underline{\phi}(x,0) = \psi''(x).
\end{aligned}$$

The one-sided Lipschitz condition (4) in Definition 2.1 implies that

$$-\left[(u_\delta+v_\delta)_x-\frac{2}{t_2-\tau}\right]\geq 0. \tag{29}$$

Similar to the arguments to obtain (26) and (27) we get the estimate

$$\|\underline{\phi}_1(x,\tau)\|_{L^\infty}\leq M_2\min(1,\Phi(x,\tau)). \tag{30}$$

Now we consider the case of homogeneous initial data

$$\begin{aligned} L_2(\underline{\phi}_2) &= \frac{1}{2}(u_\delta+v_\delta)_{xx}\phi_x\frac{(t_2-\tau)^2}{t_2^2}, \\ \underline{\phi}_2(x,0) &= 0. \end{aligned} \tag{31}$$

It follows from the inequality (28) that

$$\left|\frac{1}{2}(u_\delta+v_\delta)_{xx}\phi_x\frac{(t_2-\tau)^2}{t_2^2}\right|\leq\frac{C_1t_2MM_1}{t_1\delta^2}\Phi \tag{32}$$

where C_1 was defined in (7). Straightforward calculation and the inequalities (29), (31) give

$$L_2\left(\frac{C_1t_2^2MM_1}{t_1\delta^2}\Phi(x,\tau)\pm\underline{\phi}_2\right)>\frac{C_1t_2MM_1}{t_1\delta^2}\Phi(x,\tau)\pm\frac{1}{2}(u_\delta+v_\delta)_{xx}\phi_x\frac{(t_2-\tau)^2}{t_2}>0,$$

for $\tau\in]0,t_2-t_1]$. This implies by the maximum principle that a non-positive minimum of the quantity $\frac{C_1t_2MM_1}{t_1\delta^2}\Phi(x,\tau)\pm\underline{\phi}_2$ cannot exist for $\tau\in]0,t_2-t_1]$. But we have

$$\inf_{x\in\mathbb{R}}\frac{C_1t_2MM_1}{t_1\delta^2}\Phi(x,0)\pm\underline{\phi}_2(x,0)\geq 0.$$

Therefore we obtain

$$|\underline{\phi}_2(x,\tau)|\leq\frac{C_1t_2MM_1}{t_1\delta^2}\Phi(x,\tau).$$

It follows from (30) and (32) that

$$\begin{aligned} |\phi_{xx}(x,\tau)| &= \left|\frac{t_2^2}{(t_2-\tau)^2}\underline{\phi}(x,\tau)\right|\leq\frac{t_2^2}{t_1^2}\left(|\underline{\phi}_1(x,\tau)|+|\underline{\phi}_2(x,\tau)|\right) \\ &\leq \frac{t_2^2}{t_1^2}\left(M_2+\frac{C_1t_2MM_1}{t_1\delta^2}\Phi(x,\tau)\right). \end{aligned}$$

Then the estimate (15) follows from (20).

The estimate (16): This follows from the equation (11) and the fact that $\|u_\delta\|_{L^\infty}\leq M$ and $\|v_\delta\|_{L^\infty}\leq M$. $\square$

Theorem 3.2. *Let u and v be two solutions of the initial value problem (1), (2) as obtained in Theorem 2.2. Then $u = v$ almost everywhere for $t > 0$.*

Proof. Since u and v are weak solutions of the initial value problem (1), (2) we have by (3)

$$\begin{aligned}\int_{-\infty}^{\infty} (u\phi)(x,\cdot)|_{t_1}^{t_2}\, dx &= \int_{t_1}^{t_2}\int_{-\infty}^{\infty} (u\phi_t + \frac{u^2}{2}\phi_x + ut\phi_{xx})\, dxdt,\\ \int_{-\infty}^{\infty} (v\phi)(x,\cdot)|_{t_1}^{t_2}\, dx &= \int_{t_1}^{t_2}\int_{-\infty}^{\infty} (v\phi_t + \frac{v^2}{2}\phi_x + vt\phi_{xx})\, dxdt.\end{aligned}$$

We subtract these two equations and set $d = u - v$. Then we get

$$\int_{-\infty}^{\infty} (d\phi)(x,\cdot)|_{t_1}^{t_2}\, dx = \int_{t_1}^{t_2}\int_{-\infty}^{\infty} d(\phi_t + \frac{u+v}{2}\phi_x + t\phi_{xx})\, dxdt$$

and therefore using the adjoint equation (11)

$$\int_{-\infty}^{\infty} (d\phi)(x,\cdot)|_{t_1}^{t_2}\, dx = \int_{t_1}^{t_2}\int_{-\infty}^{\infty} d\frac{u+v-(u_\delta+v_\delta)}{2}\phi_x\, dxdt. \tag{33}$$

Since $|u|, |v|, |u_\delta|$ and $|v_\delta|$ are bounded by the constant M considered in Theorem 2.2 and ϕ_x tends to 0 exponentially in x uniformly for $t \in [t_1, t_2]$ by Lemma 3.1, we have for any given $\varepsilon > 0$

$$\left|\int_{t_1}^{t_2}\int_{|x|\geq N} d\frac{u+v-(u_\delta+v_\delta)}{2}\phi_x\right|\, dxdt \leq \varepsilon \tag{34}$$

provided that N is large enough. From the estimate (14) for ϕ_x we also obtain on the complementary set $S_N^{t_1,t_2} = \{(x,t) \in \mathbb{R}\times\mathbb{R}_+ |\ |x| \leq N, t_1 \leq t \leq t_2\}$ the estimate

$$\begin{aligned}&\int_{t_1}^{t_2}\int_{|x|\leq N} d\frac{u+v-(u_\delta+v_\delta)}{2}\phi_x\, dxdt\\ &\qquad \leq M\|\phi_x\|_{L^\infty(S_N^{t_1,t_2})}\left(\|u-u_\delta\|_{L^1(S_N^{t_1,t_2})} + \|v-v_\delta\|_{L^1(S_N^{t_1,t_2})}\right)\\ &\qquad \leq \frac{t_2}{t_1}MM_1\left(\|u-u_\delta\|_{L^1(S_N^{t_1,t_2})} + \|v-v_\delta\|_{L^1(S_N^{t_1,t_2})}\right).\end{aligned} \tag{35}$$

Applying (34) and (35) to (33) and letting $\delta \to 0$ gives

$$\int_{-\infty}^{\infty} (d\phi)(x,\cdot)|_{t_1}^{t_2}\, dx \leq \varepsilon.$$

From the arbitrariness of ε we infer that

$$\int_{-\infty}^{\infty} (d\phi)(x,\cdot)|_{t_1}^{t_2}\, dx \leq 0$$

or

$$\int_{-\infty}^{\infty} (d\phi)(x,t_2)\ dx \le \int_{-\infty}^{\infty} (d\phi)(x,t_1)\ dx. \tag{36}$$

We observe that u and $v \to u_0$ in $L^1_{loc}(\mathbb{R})$ for $t \to 0$ by Theorem 2.2 and deduce from (13) the estimate

$$\phi(x,0) \le M_0 \exp\left(\frac{Ct_2 + M + 1 + (K + 2 - |x|)}{t_2}\right).$$

Then we obtain that the right hand side of (36) tends to zero as $t_1 \to 0$ and therefore

$$\int_{-\infty}^{\infty} d(x,t_2)\psi(x)\ dx \le 0.$$

By the arbitrariness of t_2 and the fact that the inequality must be satisfied by any test function ψ as well as by $-\psi$ this implies that $u(x,t) = v(x,t)$ almost everywhere on $\mathbb{R} \times \mathbb{R}_+$. □

4. The regularity of the weak solution

In this section we will show that the weak solutions of the initial value problem (1), (2) have all continuous derivatives occurring in equation (1) even for initial data $u_0 \in L^\infty(\mathbb{R})$.

First we state a slightly stronger version of the one-sided Lipschitz condition (4) as follows.

Lemma 4.1. *For any weak solution u of the initial value problem (1), (2) the following estimate holds*

$$L^+(u(\cdot,t)) \le \frac{1}{\frac{1}{L^+(u(\cdot,0))} + t}.$$

Proof. This lemma can be proved along the line of argument given by Tadmor [13, Theorem 3.1] for a parabolic regularization of convex scalar conservation laws. We omit the detailed proof of this lemma here. □

Theorem 4.2. *Let $u_0 \in L^\infty(\mathbb{R})$. Then the weak solution u to the parabolic initial value problem (1), (2) has all continuous derivatives occurring in the equation (1).*

Proof. For any given $t_0 > 0$ consider the function

$$\tilde{u}(x,t) = \begin{cases} u(x,t), & 0 < t \le t_0, \\ \overline{u}(x,t), & t_0 < t. \end{cases}$$

Here $\overline{u}(x,t)$ is the solution of the following initial value problem

$$\overline{u}_t + (\overline{u}^2/2)_x = t\overline{u}_{xx}, \quad x \in \mathbb{R},\ t > t_0, \tag{37}$$

$$\overline{u}(x,t_0) = u(x,t_0). \tag{38}$$

The function $\overline{u}(x,t)$, as the solution of a uniformly parabolic equation (37), has all continuous derivatives occurring in (37), i.e. $\overline{u}_x, \overline{u}_{xx}$ and $\overline{u}_t$ are continuous.

Now we shall prove that $\tilde{u}$ is a weak solution of the initial value problem (1), (2). According to Definition 2.1 we only have to prove that $\overline{u}$ satisfies the one-sided Lipchitz condition (4). Applying Lemma 4.1 to the initial value problem (37), (38) implies that

$$L^+(\overline{u}(\cdot,t)) \leq \frac{1}{\frac{1}{L^+(u(\cdot,t_0))} + (t-t_0)}, \quad \text{for } t > t_0. \tag{39}$$

Since u is a weak solution of the initial value problem (1), (2) we have

$$L^+(u(\cdot,t_0)) \leq \frac{1}{t_0}. \tag{40}$$

Substituting (40) into (39) gives

$$L^+(\overline{u}(\cdot,t)) \leq \frac{1}{t} \quad \text{for } t > t_0.$$

By the uniqueness shown in Theorem 3.2 we deduce that

$$u(x,t) = \tilde{u}(x,t),\ t > 0.$$

Therefore, the derivatives u_x, u_{xx}, u_t are continuous for $t > t_0$. The arbitrariness of $t_0 > 0$ implies that u_x, u_{xx}, u_t are continuous for $t > 0$. $\square$

5. A finite difference scheme and concluding Remarks

In this section we shall indicate an alternative proof of the existence theorem for weak solutions to the initial value problem (1), (2) by a finite difference method. This also gives a means to compute approximate solutions numerically.

Consider a fixed mesh size l in space. Using variable time steps h_n, $n \in \mathbb{N}$, let the upper-half plane $t \geq 0$ be discretized using the grid points (x_j, t_n) with $x_j = jl$, $t_n = \sum_{i=1}^n h_i$, for $j \in \mathbb{Z}$, $n \in \mathbb{N}$. Consider the *Lax-Friedrichs scheme*

$$\frac{u_j^{n+1} - \frac{u_{j+1}^n + u_{j-1}^n}{2}}{h_n} + \frac{(u_{j+1}^n)^2/2 - (u_{j-1}^n)^2/2}{2l} = 0, \tag{41}$$

with discrete initial data obtained as the point values given by averaging over the *spatial cells* $[(j-\frac{1}{2})l, (j+\frac{1}{2})l]$ centered around x_j, i.e.

$$u_j^0 = \frac{1}{l}\int_{(j-\frac{1}{2})l}^{(j+\frac{1}{2})l} u_0(x)\, dx. \tag{42}$$

We take the bound M for the data, as considered in Theorem 2.2. It is well known and easily seen that the scheme (41) is monotone and stable if the CFL-condition

$$\frac{Mh_n}{l} \leq 1 \qquad \text{for } n = 1, 2, \cdots \tag{43}$$

holds. Now we fix the time step $n_0 := [\frac{M^2}{2}] + 1 \in \mathbb{N}$. The time steps h_n will be defined as follows

$$h_n = l/M \quad \text{ for } \quad n \leq n_0 \tag{44}$$

$$h_n = \frac{l^2}{2\sum_{i=1}^{n} h_i} = \frac{l^2}{2t_n} \quad \text{ for } \quad n > n_0. \tag{45}$$

In (45) we have h_n given implicitly by a quadratic equation. Of this we take the positive solution given by

$$h_n = \frac{1}{2}\left(\sqrt{t_{n-1}^2 + 2l^2} - t_{n-1}\right), \quad n > n_0. \tag{46}$$

For any $n > n_0$ we have by (44) and (45) that

$$h_n = \frac{l^2}{2\sum_{i=1}^{n} h_i} < \frac{l^2}{2\sum_{i=1}^{n_0} h_i} = \frac{l^2}{2([\frac{M^2}{2}]+1)\frac{l}{M}} \leq \frac{l^2}{2\frac{M^2}{2}\frac{l}{M}} = \frac{l}{M}. \tag{47}$$

It follows from (44) and (47) that the CFL stability condition (43) is satisfied. The difference scheme (41) can be rewritten in the following equivalent form using (44) and (45) respectively

$$\frac{u_j^{n+1} - u_j^n}{h_n} + \frac{(u_{j+1}^n)^2/2 - (u_{j-1}^n)^2/2}{2l} = Ml\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{2l^2} \quad \text{for } n \leq n_0 \tag{48}$$

and

$$\begin{aligned} \frac{u_j^{n+1} - u_j^n}{h_n} + \frac{(u_{j+1}^n)^2/2 - (u_{j-1}^n)^2/2}{2l} &= \frac{l^2}{h_n}\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{2l^2} \\ &= t_n\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{l^2} \quad \text{for } n_0 \leq n. \end{aligned} \tag{49}$$

The scheme (49) is obviously consistent with the equation (1). Following the same line of arguments given by Oleinik [9, Sections 3 and 5] we can obtain the weak solution u of the initial value problem (1), (2) in the sense of Definition 1.2 as the limit of a subsequence of approximate solutions U_l constructed by the scheme (41) and (42) as step functions by setting

$$U_l(x,t) = u_j^n \quad \text{for} \quad (j-\frac{1}{2})l \le x < (j+\frac{1}{2})l, \quad t_n \le t < t_{n+1},$$

except for the fact that the weak solutions obtained here satisfy the slightly different one-sided Lipschitz condition

$$\operatorname*{ess\,sup}_{x\neq y}\left(\frac{u(x,t)-u(y,t)}{x-y}\right)^+ \le \frac{2}{t} \tag{50}$$

instead of (4). It is easy to see that the weak solutions of the initial value problem (1), (2) satisfying (50) are also unique and that (4) implies (50). So the weak solutions of (1), (2) obtained by the viscosity method in Section 2 and the difference scheme in this section are identical and are the unique classical solutions of (1), (2), due to the results of Sections 3 and 4.

We would like to remark that for any $l > 0$ the solutions of the difference scheme (41) and (42) with time steps given by (44), (45) are well defined for all $t > 0$. Indeed suppose there exists a finite number $T > 0$ such that $t_{n+n_0} \to T$ as $n \to \infty$. From (45) we have

$$t_{n_0+n} = t_{n_0+n-1} + \frac{l^2}{2t_{n_0+n}}. \tag{51}$$

Letting $n \to \infty$ in (51) gives

$$T = T + \frac{l^2}{2T}$$

which contradicts the assumption that T is finite.

Futher, all previous arguments remain true with the time steps taken to be (44) and

$$h_n = \frac{l^2}{2\sum_{i=1}^{n-1} h_i} = \frac{l^2}{2t_{n-1}} \text{ for } n > n_0. \tag{52}$$

The time step h_n given by (52) is slightly simpler than (46).

It is interesting that the solution of the non-uniformly parabolic equation (1) with the initial data (2) can be approximated by the Lax-Friedrichs scheme in the form (41), (42) if the first finite number of constant finite difference steps are given by (48). These steps deal with the non-uniform parabolicity of the equation (1). Then afterwards the use of specifically chosen variable time steps makes the scheme consistent with equation (1). By (45) the time steps are decreased with the order $\frac{1}{t}$ for fixed mesh size l in order to compensate for the growing diffusion coefficient. This is needed to keep the scheme stable. As is usual for second order parabolic equations, the time step is also of the order l^2 for $l \to 0$ to remain stable.

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