Error Bounds for the Large Time Step Glimm Scheme Applied to Scalar Conservation Laws[1]

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Abstract

In this paper we derive an L^1 error bound for the large time step, i.e. large Courant number, version of the Glimm scheme when used for the approximation of solutions to a genuinely nonlinear, i.e. convex, scalar conservation law for a generic class of piecewise constant data. We show that the error is bounded by $O(\Delta x^{1/2}|\log \Delta x|)$ for Courant numbers up to 1. The order of the error is the same as that given by Hoff and Smoller [5] in 1985 for the Glimm scheme under the restriction of Courant numbers up to 1/2.

1. Introduction

The large time step (LTS) Glimm scheme was introduced by LeVeque [9] in 1985. It is a generalization of the Glimm scheme allowing larger time steps. The usual geometric Courant-Friedrichs-Lewy (CFL) stability restriction is satisfied by an automatic increase of the spatial stencil when interpreting these explicit schemes as difference schemes. Numerical experiments illustrate that these schemes can give better numerical results than the Glimm scheme for calculating discontinuous solutions. By involving fewer projections, the projection error is reduced when taking larger time steps. For details, refer to Section 2.

In this paper we derive bounds for the error in the LTS Glimm difference approximation to the solution of a single conservation law

$$v_t + f(v)_x = 0 \quad \text{for} \quad -\infty < x < +\infty, t > 0 \tag{1}$$

with initial condition

$$v(x,0) = v_0(x) \quad \text{for} \quad -\infty < x < +\infty \tag{2}$$

where $v_0(x)$ is a function of bounded variation and f is a C^2 function satisfying $f'' > 0$ on the closed convex hull of the range of the data v_0. By v we denote the unique exact weak solution of bounded variation satisfying, as usual, an additional entropy condition. We let u be the approximate solution of (1) generated from the LTS Glimm scheme, Δx the spatial mesh length and Δt the time step. The approximations considered in this paper are based upon an equidistributed sequence $\{\theta_n\}$, as will be explained below. For the classical Glimm scheme the following was proved by Hoff and Smoller [5]. Given any interval $[a,b] \subseteq \mathbb{R}$ and any real numbers $\epsilon, t > 0$ there exits a constant $C > 0$, independently of the mesh parameters, such that the following error bound holds

$$\|u(\cdot,t) - v(\cdot,t)\|_{L^1[a,b]} \le C \left(\|u(\cdot,0) - v(\cdot,0)\|_{L^1[\tilde{a},\tilde{b}]} + \Delta x^{1/6} |\log \Delta x| \right)$$

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where $\tilde{a} = a - Lt - \epsilon$, $\tilde{b} = b + Lt + \epsilon$, and $L = \sup_{|v| \leq \|v_0\|_\infty} |f'(v)|$. In addition, they showed that for a generic class of piecewise constant data v_0, the error bound can be improved to $O(\Delta x^{1/2}|\log \Delta x|)$ when $u(\cdot, 0)$ is suitably chosen, namely piecewise constant.

The equidistributed sequence $\{\theta_n\}_{n\in\mathbb{N}}$, that we will use, is required to satisfy the following condition: *There exists a constant $c > 0$ such that, given any $\theta \in [0,1]$ and natural numbers $K, N \in \mathbb{N}$, the estimate*

$$\left| \frac{\#\{n \in [K+1, K+N] : \theta_n \leq \theta\}}{N} - \theta \right| \leq \frac{c \log N}{N} \tag{3}$$

holds. See Hua and Wang [6] or Kuipers and Niederreiter [7] for a description of such sequences.

In Section 2 we give a description of the LTS Glimm scheme. The Section 3 is then devoted to the derivation of the error estimate for the scheme when it is applied to solve (1) and (2) for a generic class of piecewise constant initial data. We obtain the following theorem.

Theorem: *Let u denote the approximate solution of (1) for the initial data u_0 generated from the LTS Glimm scheme. A particular equidistributed sequence $\{\theta_n\}$ as described above is used for the approximation. We assume that the Courant number $\mathcal{C}$ satisfies the restriction $\mathcal{C} \leq 1$, see Section 2 for the definition of the Courant number. Further, take v to be the exact admissible weak solution for the initial data v_0. We additionally assume that*

(a) the initial data v_0 are piecewise constant,

(b) the exact solution v contains no interactions of more than two waves at a time,

(c) the numerical initial data are given by averaging over mesh cells $[x_i, x_{i+1}[$, i.e. $u_0(x) = \frac{1}{\Delta x}\int_{x_i}^{x_{i+1}} v_0(x)dx \quad$ for $x \in [x_i, x_{i+1}[, \quad i \in \mathbb{Z}$, see Section 2.

Then for any $T, X, \epsilon > 0$ there exists a constant $C > 0$ independently of Δx and Δt such that for $0 \leq t \leq T$ and $\Delta x < \epsilon$

$$\|u(\cdot,t) - v(\cdot,t)\|_{L^1[-X,X]} \leq C\left(\|u(\cdot,0) - v(\cdot,0)\|_{L^1[-\widetilde{X},\widetilde{X}]} + \Delta x^{1/2}|\log \Delta x|\right). \tag{4}$$

Here $\widetilde{X} = X + Lt + \epsilon$ and L as defined above. In addition, if the exact weak solution v contains no interactions of shocks with rarefaction waves, then we obtain the stronger estimate

$$\|u(\cdot,t) - v(\cdot,t)\|_{L^1[-X,X]} \leq C\left(\|u(\cdot,0) - v(\cdot,0)\|_{L^1[-\widetilde{X},\widetilde{X}]} + \Delta x|\log \Delta x|\right). \tag{5}$$

$\square$

For systems of conservation laws, under the restriction that the total variation of the initial data is small, error bounds similar to the form (4) were given by Bressan and Marson [2].

The difficulty in obtaining the theorem comes from the fact that, for the LTS scheme, the neighboring waves in the approximate solution may interact within a cell when the Courant number $\mathcal{C} \geq 1/2$. This leads to problems in estimating the error, especially when a rarefaction wave interacts with a shock within a cell. In the latter case new rarefaction waves may be generated in the neighboring cells. We deal such difficulties in Section 3 when proving the above theorem. The analysis presented in this paper is only a first step towards the ultimate goal, namely obtaining error estimates for much larger Courant numbers. Our result is analogous to that given by Hoff and Smoller [5] in 1985 for the classical Glimm scheme, i.e. $\mathcal{C} \leq \frac{1}{2}$. In addition, we give a modified proof of their Theorem 5.1 in [5].

It would be of great interest to extend our estimates to arbitrarily large Courant numbers. But, in order to get these error bounds we have to prove the entropy consistency of the LTS Glimm scheme for arbitrary Courant numbers. This is unfortunately still a difficult and open problem.

One would also like to have bounds for the LTS Gimm scheme in the case of general intial data. We see two ways to try to deal with this problem. One way could be that we approximate the initial data, say of bounded variation, by a piecewise constant function. Take it to be constant on each cell by taking the avarage of these initial data on each cell. It may be set equal to the limits of the initial data at plus and minus infinity for x larger than a positive number and less than a negative number respectively. It is then easy to deduce that the L^1 difference between the solution with the general initial data and the solution with the approximate initial data is of order Δx. It would seem that we could then apply the triangle inequality to obtain the error bound for the LTS Glimm scheme for general initial data. But, the problem is that we cannot generally obtain a subdivision consisting of R_i, which satisfy the conditions given in the proof of the theorem at the end of the paper, even when the approximate initial data have a fixed number of discontinuities.

Another approach could be the use of the wave tracing analysis of as Hoff and Smoller have done in the first part of [5], or see Bressan and Marson [2]. The wave tracing analysis works only for the original Glimm scheme because of the fact that the approximate solution constructed by the Glimm scheme is the exact solution of the conservation law between the time steps t_n and t_{n+1}. But, this fact does not hold for the LTS Glimm scheme because of the linearization of interactions of waves.

We consider the error bounds for the LTS Glimm scheme since the large time step method is an approach to increase the efficiency of this numerical method for conservation laws. To our knowledge this paper is first one to deal with the issue of error bounds for the LTS Glimm scheme. To get the error bounds careful estimates and new analysis were required.

2. A description of the LTS scheme

The well known Glimm [3] and Godunov [4] schemes are semianalytical techniques for calculating discontinuous solutions to the initial value problem for conservation laws (1) and (2), see also Smoller [11]. In their basic versions they take piecewise constant approximations of the solution on each time level by projection. For the time evolution they then use exact weak solutions to Riemann problems, see below. They differ only in the manner the projection at each time step is carried out.

The Courant number of a numerical calculation is defined as $\mathcal{C} = \max(|f'(u)|) \cdot \Delta t/\Delta x$. It relates the speed of numerical information, given by the ratio of the spatial mesh size Δx to the time step Δt, to the speed of the characteristics of the conservation law. In order to make numerical computations more efficient, it is reasonable to try larger Courant numbers $\mathcal{C}$, i.e. to use fewer time steps. But, if $\mathcal{C} \geq 1/2$, neighboring waves may interact. It is difficult, even impossible to deal with the interactions exactly. Based on the idea of linearizing the interactions of these nonlinear waves, LeVeque [8] in 1984 introduced a large time step version of the Godunov scheme. He showed that it gives a conservative and consistent difference scheme for abitrarily large Courant numbers while being very convenient to compute at the same time. He illustrated that one could expect improved numerical accuracy with suitable larger Courant numbers, usually between 1 and 3. This observation, that fewer projections due to larger time steps lead to a higher accuracy, is also backed up by the error estimates of Lucier [10]. A parallel large time step version of the Glimm scheme and similar results were subsequently given by LeVeque [9], Brenier [1] and Wang [12,13]. In addition, Wang [12] obtained the consistency of the LTS Glimm scheme for systems of conservation laws in one space dimension for any choice of time step and convergence of the scheme for the case that $\mathcal{C} \leq 1$, if the total variation of v_0 is sufficiently small and each eigenvalue of the system does not change its' sign. Using the Roe approximate Riemann solver as building block, similar results were obtained by Brenier [1]. The entropy consistency of these schemes was studied by Wang and Warnecke [14,15].

We give a more precise description of the LTS Glimm scheme. We discretize $\mathbb{R} \times [0, \infty[$ by taking the spatial mesh length $h = \Delta x$ and the time step $\tau = \Delta t$. We let $x_i = ih$ for $i \in \mathbb{Z}$ denote the mesh points of our spatial grid and $t_j = j\tau$ for $j \in \mathbb{N}_0$ the time levels. Suppose that the approximate solution $u_h(x, t)$ to the initial value problem (1) and (2) has been constructed for $t < t_j$ for some given $j \in \mathbb{N}_0$. In the LTS Glimm scheme, one obtains a piecewise constant approximation at this time level by setting

$$u_h(x, t_j) = u_{ij} = u_h(x_i + \theta_j h, t_j - 0) \quad \text{for } x \in [x_i, x_{i+1}[$$

where $\{\theta_j | j = 1, 2, \cdots\}$ is an equidistributed sequence in]0,1]. This means that the projection to piecewise constant functions at each time level t_j is achieved by a *random evaluation* of the solution obtained on the time interval $[t_{j-1}, t_j]$.

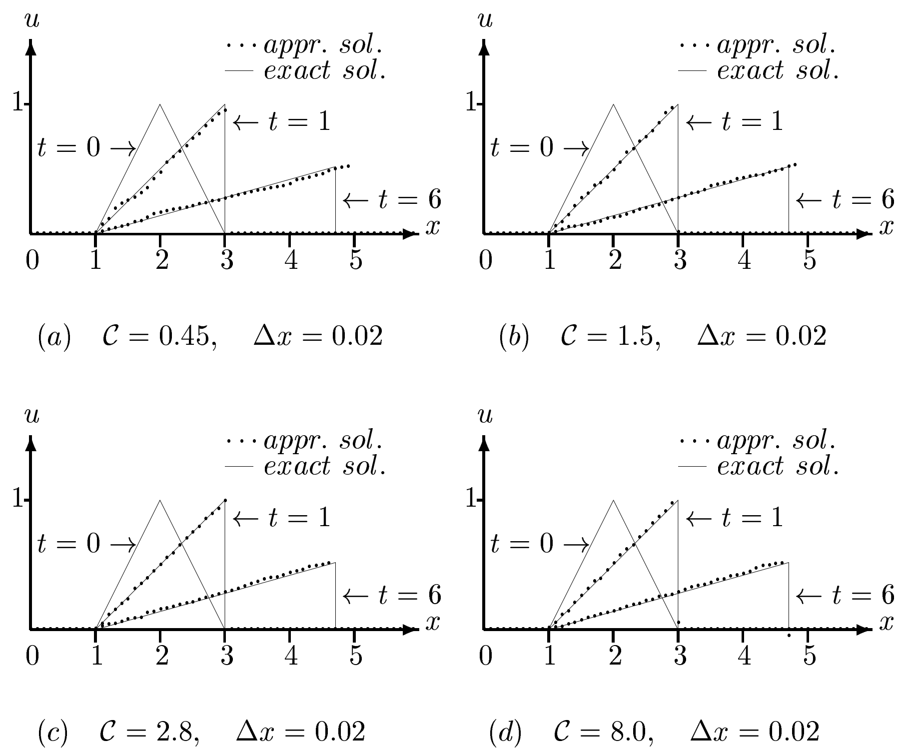
Thereby, we have at each time level t_j a piecewise constant step function as approximate solution. Now we must describe the time evolution on the interval $[t_j, t_{j+1}]$ for any $j \in \mathbb{N}_0$. For each $i \in \mathbb{Z}$, we solve locally the *Riemann problem* for only two constant states

$$\begin{aligned} u_t + f(u)_x &= 0 \\ u|_{t=t_j} &= \begin{cases} u_{i-1,j} & x < x_i \\ u_{i,j} & x > x_i. \end{cases} \end{aligned}$$

For each $i \in \mathbb{Z}$ we denote this solution by $u_{ij}(x, t)$ for $t > t_j$. Taking the linear superposition of these solutions over $i \in \mathbb{Z}$ we get

$$u_h(x, t) := u_h(x, t_j) + \sum_{i \in \mathbb{Z}} [u_{ij}(x, t) - u_{ij}(x, t_j)]$$

as approximate solution to the conservation law (1) on the space-time strip $S_j = \{(x, t) | -\infty < x < +\infty, t_j \leq t < t_{j+1}\}$. If the equation (1) were linear, this solution would be exact for arbitrarily large times.



(a) $\mathcal{C} = 0.45$, $\Delta x = 0.02$ (b) $\mathcal{C} = 1.5$, $\Delta x = 0.02$

(c) $\mathcal{C} = 2.8$, $\Delta x = 0.02$ (d) $\mathcal{C} = 8.0$, $\Delta x = 0.02$

Figure 1. Comparison between the approximate solution and exact solution

To initialize the scheme at time $t = 0$, i.e. for $j = 0$, one takes the usual L^2 projection of the initial data onto piecewise constant functions given by integral averages over the mesh cells $[x_i, x_{i+1}[$

$$u_h(x,0) = u_{i0} = \frac{1}{h}\int_{x_i}^{x_{i+1}} u_0(x)dx \quad \text{for } x \in [x_i, x_{i+1}[, \quad i \in \mathbb{Z}.$$

When $\mathcal{C} \leq 1/2$, the LTS Glimm scheme reduces to the classical Glimm scheme. Therefore, it is a proper generalization of this scheme.

The following is an example to illustrate the robustness of the LTS Glimm scheme. Under different Courant numbers, the approximate solutions calculated using the above scheme for the well known Burgers equation

$$u_t + (\frac{1}{2}u^2)_x = 0 \quad \text{for } -\infty < x < +\infty, t > 0$$

with initial condition

$$u|_{t=0} = \begin{cases} 1 - |x-2| & |x-2| \leq 1 \\ 0 & |x-2| > 1 \end{cases}$$

are compared with the exact solution in Figure 1.

3. Error estimate

To prove our main theorem stated above, we will first need a few technical lemmas. Some of these lemmas, namely the Lemmas 2,3,4 and 8, can be proven with a procedure analogous that used in proving the corresponding lemmas for staggered meshes in the paper of Hoff and Smoller [5]. Although we will use a fixed mesh instead of a staggered mesh for notational convenience, this in no way affects the error estimates discussed in this paper. We will suppose that the reader has access to the paper [5] and do not repeat for the Lemmas 2,3,4 and 8 the proofs essentially given there. We want to focus attention on the specific difficulties caused by the interaction of waves due to the large time steps.

Take $M = \|u\|_\infty$ to be the upper bound of absolute value of the approximate solution. According to the above assumption that f is a C^2 function satisfying $f'' > 0$, there exist constants $c_0, c_1 > 0$ such that

$$\max_{|u|\leq M} f''(u) \leq c_1, \quad \min_{|u|\leq M} f''(u) \geq c_0,$$

i.e. in the following we assume that $c_0 \leq f''(u) \leq c_1$. For convenience, we adopt the notation

$$r^p := \Delta x^p |\log \Delta x| = O(\Delta t^p |\log \Delta x|).$$

The following lemma is a slight generalization of Lemma 5.2 in [5]. We give a proof of this lemma to make our exposition more self-contained since this lemma plays a key role in our analysis.

Lemma 1: *If there exist bounded functions $\alpha(x,t), \beta(t)$ and $\omega(x,t)$ such that, for $t \in [0,T]$, $x \in [-X,X]$,*

$$u(x,t) = v(x + \alpha(x,t), t + \beta(t)) + \omega(x,t), \tag{6}$$

then, for $0 \leq t \leq T$, we obtain the estimate

$$\| u(\cdot,t) - v(\cdot,t) \|_{L^1[-X,X]} \leq C\Big(\| \alpha \|_\infty + \| \beta \|_\infty \Big) + \| \omega(\cdot,t) \|_{L^1[-X,X]} . \tag{7}$$

Proof: We have

$$\begin{aligned}\int_{-X}^{X}|u(x,t)-v(x,t)|dx &\le \int_{-X}^{X}|v(x+\alpha,t+\beta)-v(x,t+\beta)|dx\\ &\quad+\int_{-X}^{X}|v(x,t+\beta)-v(x,t)|dx+\int_{-X}^{X}|\omega(x,t)|dx.\end{aligned}$$

Consider the derivative v_x of the $BV-$solution v as a measure, then we get

$$\begin{aligned}\int_{-X}^{X}|v(x+\alpha,t+\beta)-v(x,t+\beta)|dx &\le \int_{-X}^{X}\int_{x-\|\alpha\|_\infty}^{x+\|\alpha\|_\infty}|v_x(s,t+\beta)|dsdx\\ &\le 2\parallel\alpha\parallel_\infty TV(v(\cdot,t+\beta)).\end{aligned}$$

By a similar estimate for the second integral, we obtain the conclusion immediately. □

Lemma 2: *For any equidistributed sequence $\{\theta_n\}$ satisfying (3), there exists a constant C depending only on the sequence such that for any number s satisfying $\left|s\frac{\Delta t}{\Delta x}\right|\le 1$ the bound*

$$\left|\sum_{m=1}^{n}\frac{\operatorname{sgn}(s\Delta t-\theta_m\Delta x)+\operatorname{sgn}(s\Delta t+(1-\theta_m)\Delta x)}{2}\Delta x-st_n\right|\le C\Delta x\log n$$

holds. Here $t_n=n\Delta t$.

Proof: The proof is essentially the same as the proof of Lemma 5.3 in [5]. □

For $s>0$ the term multiplying Δx in the sum in the inequality of this lemma is simply equal to 0 when $s\Delta t<\theta_m\Delta x$ and equal to 1 otherwise. This means that the shock moves forward by one grid cell in the latter case. For $s<0$ the term multiplying Δx in the sum is 0 when $s\Delta t>(1-\theta_m)\Delta x$ and -1 otherwise. This implies that the shock moves backward by one cell in the latter case. So this inequality is measuring the error in shock location for a shock with the speed s.

Now we analyse in detail the possible behavior of the approximate solution u and the exact solution v under the conditions given in our theorem. For this purpose, we express the numerical approximation u as a perturbation of the exact solution v in the form of (6). We have to discuss five special cases in order to prove the theorem.

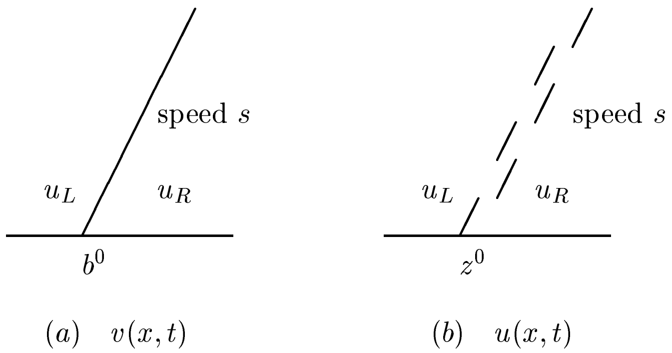


Figure 2

Case 1: Assume that the exact solution v consists of two constant states separated by a single shock emanating from the point $(b^0, 0)$, see Figure 2a, and that the numerical approximation u consists of the same two states with a discontinuity starting from the point $(z^0, 0)$, see Figure 2b.

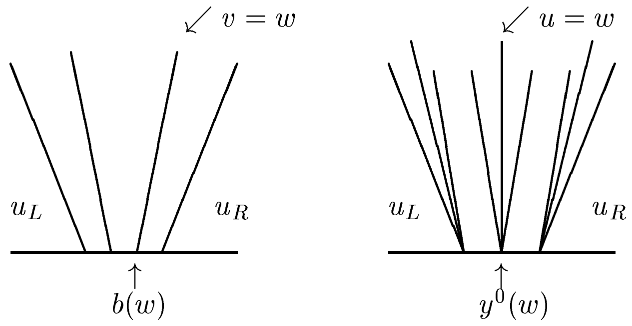
Lemma 3: *For Case 1, we can choose $\alpha = \alpha(x)$, $\beta = 0$, $\omega(x,t) = 0$, and obtain the estimate*

$$\| \alpha \|_\infty \leq \left| z^0 - b^0 \right| + Cr.$$

Proof: The proof of this lemma is essentially the same as one of the first case of Problem 1 in [5]. □

Case 2: The exact solution v is the rarefaction wave shown in Figure 3a. We may assume that the characteristic speeds in the interior of this wave are strictly increasing, so that each state $w \in (u_L, u_R)$ is attained at a unique point $b(w)$ by the initial data $v_0(x)$, namely we have $v_0(b(w)) = w$.

As for the numerical approximation u, we may assume that in the strip $\mathbb{R} \times [0, \Delta t]$ it consists of sub-rarefaction waves whose centers are a finite number of adjacent mesh points respectively, and that the sum of the wave strengths of these subwaves is the wave strength in v, see Figure 3b. If w is a state value taken by u in the interior of one of these subwaves, we denote by $y^0(w)$ the x-coordinate of the mesh point at which that particular subwave takes this value.



(a) $v(x,t)$ (b) $u(x,t)$

Figure 3

Lemma 4: *For Case 2, we can choose $\alpha = \alpha(x)$, $\beta = 0$, $\omega(x,t) = 0$, and obtain the estimate*

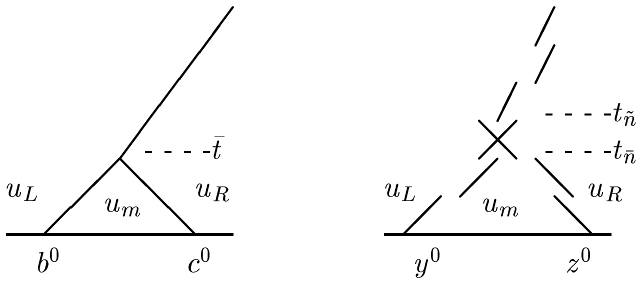
$$\| \alpha \|_\infty \leq C \left(\sup_w \left| b(w) - y^0(w) \right| + r \right),$$

where the supremum is taken over all w for which $y^0(w)$ is defined. In addition, we also have the bound

$$\left| u^n_{k+1} - u^n_k \right| \leq Cr/t_n \tag{8}$$

where $u^n_i = u(x, t_n)$ for $x \in [x_i, x_{i+1}[$.

Proof: The proof of Lemma 4 is the same as the one of Problem 3 in [5]. □



$(a)\quad v(x,t)$ $\qquad$ $(b)\quad u(x,t)$

Figure 4

Case 3: Both v and u consist of the interaction of two shock waves. The two shock speeds need not be of opposite sign. Although this is the situation we have chosen to illustrate in Figure 4.

Lemma 5: *For Case 3 we can choose α, β and ω, such that*

$$\begin{aligned} \| \alpha \|_\infty + \| \beta \|_\infty &\leq C\left(\left|y^0 - b^0\right| + \left|z^0 - c^0\right| + r\right) \\ \| \omega(\cdot,t) \|_{L^1[-X,X]} &\leq Cr. \end{aligned}$$

Proof: Let y^n, z^n denote the x-coordinates of two shocks in u at time t_n. Then

$$\begin{aligned} y^n &= y^0 + \sum_{m=1}^{n} \frac{\operatorname{sgn}(s_y \Delta t - \theta_m \Delta x) + \operatorname{sgn}(s_y \Delta t + (1-\theta_m)\Delta x)}{2} \Delta x \\ z^n &= z^0 + \sum_{m=1}^{n} \frac{\operatorname{sgn}(s_z \Delta t - \theta_m \Delta x) + \operatorname{sgn}(s_z \Delta t + (1-\theta_m)\Delta x)}{2} \Delta x, \end{aligned} \tag{9}$$

where s_y and s_z are the corresponding shock speeds and $s_y > s_z$. Now suppose that the two shocks begin their interaction in the cell $Y_{k\bar{n}} = [x_k, x_{k+1}] \times [t_{\bar{n}}, t_{\bar{n}+1}]$, then $y^{\bar{n}} + \Delta x = z^{\bar{n}}$. By taking the difference in (9) and applying Lemma 2, we obtain

$$z^0 - y^0 = (s_y - s_z)t_{\bar{n}} + O(\Delta x \log \bar{n}). \tag{10}$$

We claim that there exists an upper bound for $t_{\bar{n}}$. Because $f''(u) \geq c_0 > 0$, we have

$$\begin{aligned} |s_y - s_z| &= \left| \int_0^1 \left[f'(u_m + \theta(u_L - u_m)) - f'(u_m + \theta(u_R - u_m)) \right] d\theta \right| \\ &\geq \frac{1}{2} c_0 \left| u_R - u_L \right|. \end{aligned}$$

From (10) we find that

$$t_{\bar{n}} = \frac{z^0 - y^0}{s_y - s_z} + O(\Delta x \log t_{\bar{n}} - \Delta x \log \Delta t).$$

Therefore, we obtain

$$t_{\bar{n}} - C_2 \Delta x \log t_{\bar{n}} \leq C_3 \left| z^0 - y^0 \right| + C_4 r,$$

where C_2, C_3, C_4 are constants and C_3, $C_4 > 0$. This shows that $t_{\bar{n}}$ is bounded.

Let $(\bar{x}, \bar{t})$ be the point at which the two shocks in v interact, as shown in Figure 4a, where

$$\begin{aligned}\bar{x} &= b^0 + s_y\bar{t} = c^0 + s_z\bar{t} \\ \bar{t} &= \frac{c^0 - b^0}{s_y - s_z}.\end{aligned}$$

We have

$$\begin{aligned}\bar{t} - t_{\bar{n}} &= \frac{c^0 - b^0 - (z^0 - y^0)}{s_y - s_z} + O(-\Delta x \log t_{\bar{n}} + \Delta x \log \Delta t) \\ &= O\left(\left|y^0 - b^0\right| + \left|z^0 - c^0\right| + r\right). \end{aligned} \tag{11}$$

From (9), we get $y^{\bar{n}} = y^0 + s_y t_{\bar{n}} + O(r)$. Therefore, we obtain

$$\begin{aligned}\left|y^{\bar{n}} - \bar{x}\right| &= \left|y^0 - b^0\right| + |s_y|\,|t_{\bar{n}} - \bar{t}| + Cr \\ &\leq C\left(\left|y^0 - b^0\right| + \left|z^0 - c^0\right| + r\right)\end{aligned} \tag{12}$$

and similarly

$$\left|z^{\bar{n}} - \bar{x}\right| \leq C\left(\left|y^0 - b^0\right| + \left|z^0 - c^0\right| + r\right). \tag{13}$$

Now let $(x_{\tilde{k}}, t_{\tilde{n}})$ be the point at which the two shocks in u merge into one shock. Then we have $y^{\tilde{n}} = z^{\tilde{n}} = x_{\tilde{k}}$. If the two shocks in u merge into one shock in a finite number of time steps independently of Δt, then we obviously obtain

$$\begin{aligned}\bar{t} - t_{\tilde{n}} &= O\left(\left|y^0 - b^0\right| + \left|z^0 - c^0\right| + r\right) \\ \left|x_{\tilde{k}} - \bar{x}\right| &\leq C\left(\left|y^0 - b^0\right| + \left|z^0 - c^0\right| + r\right)\end{aligned} \tag{14}$$

holds.

Next we prove that it is true that the shocks merge in a finite number of time steps. First we discuss the case that the two shocks do not intersect in the rectangle $Y_{k\bar{n}}$, as is shown in Figure 5a. Due to $s_y > s_z$ we have $0 < (s_y - s_z)\Delta t \leq \Delta x$. For this case, let us consider the possibilities that $\theta_j \in]s_z\frac{\Delta t}{\Delta x}, s_y\frac{\Delta t}{\Delta x}[$ if $s_y > 0, s_z \geq 0$, or that $\theta_j \in]1 + s_z\frac{\Delta t}{\Delta x}, 1[\cup]0, s_y\frac{\Delta t}{\Delta x}[$ if $s_y > 0, s_z < 0$, or that $\theta_j \in]1 + s_z\frac{\Delta t}{\Delta x}, 1 + s_y\frac{\Delta t}{\Delta x}[$ if $s_y \leq 0, s_z < 0$ for some $j > \bar{n}$. Then the two shocks will merge into one shock from t_j. In the following we prove that these possibilities actually occur in a finite number of time steps. Let us suppose that $s_y > 0, s_z \geq 0$ and set $a = s_z\frac{\Delta t}{\Delta x}, b = s_y\frac{\Delta t}{\Delta x}$, then $0 \leq a < b \leq 1$. From (3) we obtain

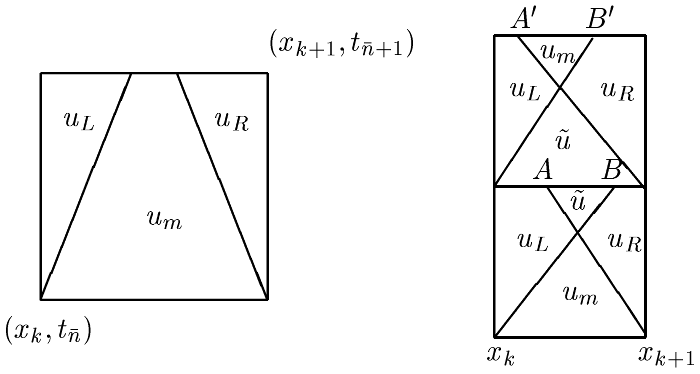
$$\left|\frac{\#\left\{m \in [\bar{n}+1, \bar{n}+N] : \theta_m \in]a, b[\right\}}{N} - (b-a)\right| \leq \frac{c \log N}{N},$$

and thus

$$\#\left\{m \in [\bar{n}+1, \bar{n}+N] : \theta_m \in]a, b[\right\} \geq N(b-a) - c\log N.$$

Obviously, there exists a natural number $\overline{N} \in \mathbb{N}$ such that $\overline{N}(b-a) - c\log\overline{N} \geq 1$. Therefore, we get

$$\#\left\{m \in [\bar{n}+1, \bar{n}+\overline{N}] : \theta_m \in]a, b[\right\} \geq 1.$$



(a) (b)

Figure 5

Namely, there exists an index $j \in [\bar{n}+1, \bar{n}+\overline{N}]$ such that $\theta_j \in]a, b[$. So the two shocks will merge into one shock in a finite number of time steps. Analogously, the same conclusion can be obtained for the two other possibilities.

In the second case the two shocks intersect in the rectangle $Y_{k\bar{n}}$, as shown in Figure 5b. We set $\tilde{u} = u_L + u_R - u_m$. The situation in Figure 5(b) is assuming that the sample point falls between the two shocks, i.e., $A \le x_k + \theta_{\bar{n}+1}\Delta x \le B$. If the two shocks do not intersect in the next time step, this case can be included in the first case above. Otherwise, the two shocks again intersect in the rectangle $Y_{k(\bar{n}+1)} = [x_k, x_{k+1}] \times [t_{\bar{n}+1}, t_{\bar{n}+2}]$, as shown in Figure 5b. In this case, the speed of the shock starting from $(x_k, t_{\bar{n}+1})$ is $s'_y = (f(u_L) - f(u_{\tilde{u}}))/(u_L - \tilde{u})$. That of the other shock starting from $(x_{k+1}, t_{\bar{n}+1})$ is $s'_z = (f(u_R) - f(u_{\tilde{u}}))/(u_R - \tilde{u})$. If $x_k + \theta_{\bar{n}+2}\Delta x$ falls into the interval $[x_k, x_{k+1} + s'_z\Delta t[$ or $[x_k + s'_y\Delta t, x_{k+1}[$, then $t_{\tilde{n}} = t_{\bar{n}+2}$ and $x_{\tilde{k}} = x_{k+1}$ or x_k. In this case (14) remains true.

Otherwise, the two shocks still intersect in $Y_{k(\bar{n}+2)}$. In the following we prove that the two shocks will stop intersecting in a cell after a finite number of time steps. We set

$$A = \min\{x_{k+1} + s_z\Delta t, x_{k+1} + s'_z\Delta t\}, \qquad B = \max\{x_k + s_y\Delta t, x_k + s'_y\Delta t\}.$$

As long as $x_k + \theta_j\Delta x$ can fall into the set $[x_k, A[\bigcup]B, x_{k+1}]$ for some $j \ge \bar{n}+1$, the two shocks will merge into one shock at time t_{j+1}. Therefore, to prove that (14) holds, we only need to prove that $x_k + \theta_j\Delta x$ falls into the set $[x_k, A[\bigcup]B, x_{k+1}]$ for some $j \ge \bar{n}+1$ in a finite number of time steps. First we prove that $A > x_k$. We have

$$\begin{aligned}
s_z &= \int_0^1 f'(u_m + \theta(u_R - u_m))d\theta \\
&= \int_0^1 \left[f'(u_m + \theta(u_R - u_m)) - f'(u_R)\right]d\theta + f'(u_R) \\
&= \int_0^1 f''(\xi)(1-\theta)(u_m - u_R)d\theta + f'(u_R) \\
&\ge \frac{1}{2}c_0(u_m - u_R) + f'(u_R).
\end{aligned}$$

Our restriction on the Courant number $\mathcal{C} = \max|f'(u)|\,\frac{\Delta t}{\Delta x} \le 1$ gives $f'(u)\Delta t \ge -\Delta x$. We thereby obtain the estimates

$$s_z\Delta t \ge \left[\frac{1}{2}c_0(u_m - u_R) + f'(u_R)\right]\Delta t > -\Delta x.$$

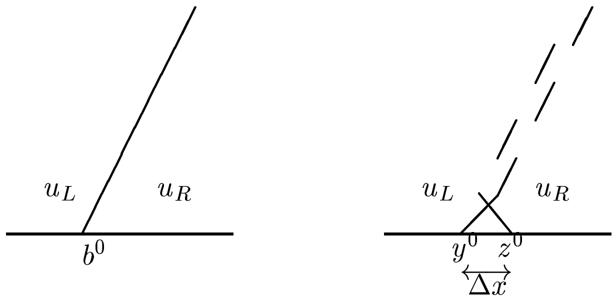
Similarly, $s'_z \Delta t > -\Delta x$. So $A > x_k$. Analogously, we can show $B < x_{k+1}$. As in the first case using (3), we can obtain the existence of an index $j \in [\bar{n}+1, \bar{n}+\overline{N}]$ such that $x_k + \theta_j \Delta x \in [x_k, A[$. So (14) also holds in this case.

Summing up (11), (12) and (14), it can be concluded that, for Case 3, we can choose α, β and ω such that

$$\begin{aligned} \| \alpha \|_\infty + \| \beta \|_\infty &\le C\left(\left|y^0 - b^0\right| + \left|z^0 - c^0\right| + r\right) \\ \| \omega(\cdot, t) \|_{L^1[-X,X]} &\le C\Delta x. \end{aligned}$$

The proof is complete. □

Case 4: The solution v contains a single shock. Due to initial averaging, the approximate solution u consists of the interaction of two shocks whose product is the shock in v, as shown in Figure 6. Here $z^0 = y^0 + \Delta x$.



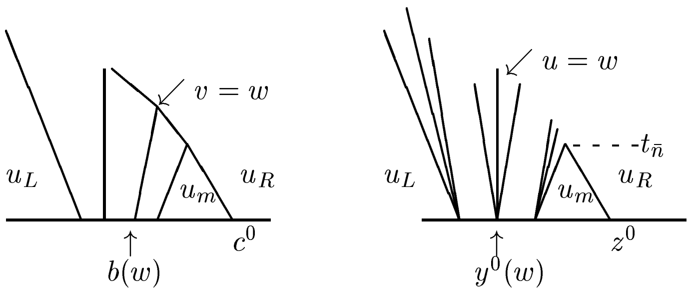
$(a) \quad v(x,t)$ $\qquad$ $(b) \quad u(x,t)$

Figure 6

Lemma 6: *For Case 4 we can choose α, β and ω, such that*

$$\begin{aligned} \| \alpha \|_\infty + \| \beta \|_\infty &\le C\left(\left|y^0 - b^0\right| + \left|z^0 - b^0\right| + r\right) \\ \| \omega(\cdot, t) \|_{L^1[-X,X]} &\le C\Delta x. \end{aligned}$$

Proof: From the proof of Lemma 5 we know that the two shocks in u will merge into one shock in a finite number of time steps independently of Δt. So thereby the conclusion follows immediately. □



$(a) \quad v(x,t)$ $\qquad$ $(b) \quad u(x,t)$

Figure 7

Case 5: The exact solution v consists of a shock wave and a rarefaction wave and their interaction, as shown in Figure 7. For the rarefaction wave we make the same assumptions and use the notations as in Case 2.

Lemma 7: *For Case 5 we can choose α, β and ω, such that*

$$\begin{aligned} \| \alpha \|_\infty + \| \beta \|_\infty &\leq C\left(\sup_w \left|b(w) - y^0(w)\right| + \left|z^0 - c^0\right| + r^{1/2}\right) \\ \| \omega(\cdot, t) \|_{L^1[-X,X]} &\leq C\Delta x. \end{aligned}$$

□

In order to prove Lemma 7, we will need the following lemma of Hoff and Smoller [5].

Lemma 8: *Assume that $\{g^m\}_{m=1}^\infty$ is a sequence of real numbers satisfying $|g^m| \leq 1$ and $\left|g^{m+1} - g^m\right| \leq cr$. If K is a positive integer satisfying $K = O(\Delta t^{-1/2})$, then*

$$\frac{1}{2n}\sum_{m=1}^{n}[sgn(g^m - \theta_m) + sgn(g^m + (1-\theta_m))] = \frac{1}{t_n}\sum_{j=1}^{\left\lfloor \frac{t_n}{K\Delta t}\right\rfloor} g^{jK}K\Delta t + O(r^{1/2}).$$

Here $\lfloor x \rfloor$ denotes the largest integer less than or equal to $x \in \mathbb{R}$.

Proof: The proof of Lemma 8 is essentially the same as the one of Lemma 5.4 in [5]. □

Proof of Lemma 7: Let s_u^{n+1} be the speed of the shock in u at $t \in]t_n, t_{n+1}[$, and suppose that the shock wave and the rarefaction waves in u begin to interact in the rectangle $Y_{k\bar{n}} = [x_k, x_{k+1}] \times [t_{\bar{n}}, t_{\bar{n}+1}]$. Further suppose that the waves in v begin to intersect in the point $(\bar{x}, \bar{t})$. Similarly to the proof of the first half of Lemma 5, we easily obtain that

$$\begin{aligned} \bar{t} - t_{\bar{n}} &= O\left(\left|y^0(u_m) - b(u_m)\right| + \left|z^0 - c^0\right| + r\right) \\ \bar{x} - x_k &= O\left(\left|y^0(u_m) - b(u_m)\right| + \left|z^0 - c^0\right| + r\right). \end{aligned} \tag{15}$$

Because $f''(u) \geq c_0 > 0$, we know that there exists a constant $c > 0$ such that $\bar{t} \geq c$. Hence, from (15) we can conclude that, when Δt is sufficiently small, there exists a constant $\tilde{c} > 0$ such that $t_{\bar{n}} \geq \tilde{c}$.

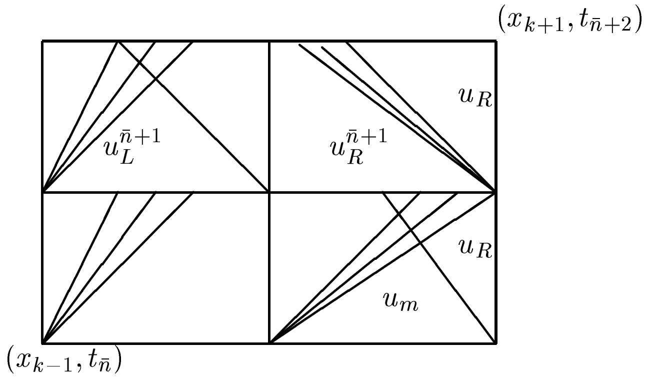


Figure 8

When $n > \bar{n}$, the shock wave and the rarefaction waves may intersect in some cells. In this case, some new rarefaction waves with the same direction as the shock wave may appear to the right of the shock, as shown in Figure 8. In the following we first prove that, when Δt is small enough, at most one such rarefaction wave appears to the right of the shock. In this case the new rarefaction wave is contained in the cell adjacent to the cell in which the shock lies. For this purpose we prove that $s_u^{n+1} > f'(u_R)$ for $n \geq \bar{n}$ when Δt is sufficiently small.

Now we suppose the shock wave and the rarefaction waves begin to intersect between $t_{\bar{n}}$ and $t_{\bar{n}+1}$ and a new rarefaction wave with the same direction as the shock wave appears to the right of the shock between $t_{\bar{n}+1}$ and $t_{\bar{n}+2}$. For $n \geq \bar{n}$ let the left state of the shock be u_L^{n+1} and the right state u_R^{n+1} between t_{n+1} and t_{n+2}. Without loss of generality we assume $u_L^{\bar{n}} > u_0$, where u_0 satisfies $f'(u_0) = 0$. Otherwise there is no interaction between the shock and the rarefaction wave with $u_L^{\bar{n}}$ as its left state. Then we obtain

$$\begin{aligned} s_u^{\bar{n}+1} - f'(u_R) &= \int_0^1 [f'(u_L^{\bar{n}} + \theta(u_R - u_L^{\bar{n}})) - f'(u_R)]d\theta \\ &= \int_0^1 f''(\xi)(1-\theta)(u_L^{\bar{n}} - u_R)d\theta \\ &\geq \frac{c_0}{2}(u_L^{\bar{n}} - u_R) \\ &> \frac{c_0}{2}(u_0 - u_R) \end{aligned}$$

giving $s_u^{\bar{n}+1} > f'(u_R)$.

In the following we will prove that

$$s_u^{n+1} - s_u^n = O(r). \tag{16}$$

When $n \leq \bar{n}$, then (16) is true due to Lemma 3. For $n \geq \bar{n}$, we note first that

$$\begin{aligned} \left|s_u^{n+2} - s_u^{n+1}\right| &= \left|\int_0^1 \left[f'(u_L^{n+1} + \theta(u_R^{n+1} - u_L^{n+1})) - f'(u_L^n + \theta(u_R^n - u_L^n))\right] d\theta\right| \\ &= \left|\int_0^1 f''(\xi)\left[u_L^{n+1} - u_L^n + \theta(u_R^{n+1} - u_R^n + u_L^n - u_L^{n+1})\right] d\theta\right| \\ &\leq \frac{1}{2}c_1\left(\left|u_L^{n+1} - u_L^n\right| + \left|u_R^{n+1} - u_R^n\right|\right). \end{aligned}$$

From (8) in Lemma 4 we have

$$\left|u_L^{n+1} - u_L^n\right| \leq 2Cr/t_n \leq 2Cr/t_{\bar{n}}, \quad \left|u_R^{n+1} - u_R^n\right| \leq Cr/t_n \leq 2Cr/t_{\bar{n}}.$$

With a constant $c^* = 2Cc_1/t_{\bar{n}}$ that is independent of n this gives $\left|s_u^{n+1} - s_u^n\right| \leq c^* r$ and consequently (16).

The fact that $s_u^{\bar{n}+1} > f'(u_R)$ together with (16) proves that $s_u^{n+1} > f'(u_R)$ for $n \geq \bar{n}$ when Δt is sufficiently small. Specifically, we can easily infer that

$$s^{\bar{n}+k+1} - f'(u_R) \geq \frac{c_0}{2}(u_0 - u_R) - kc^* r.$$

With this result we can conclude that the distance between the approximate shock and the state u_R does not exceed Δx, even if there has been a rarefaction wave on the right hand side of the shock. This means that no new rarefaction wave can be produced to the right of the shock in the next steps.

Again from (3) we deduce that

$$\#\left\{m \in [J+1, J+N] : 1 + f'(u_R)\tfrac{\Delta t}{\Delta x} < \theta_m \leq 1 + [f'(u_R) + \tfrac{c_0}{4}(u_0 - u_R)]\tfrac{\Delta t}{\Delta x}\right\}$$
$$\geq \tfrac{c_0}{4}(u_0 - u_R)\tfrac{\Delta t}{\Delta x} \cdot N - c\log N.$$

Consequently, there exists an $\overline{N} \in \mathbb{N}$ such that for all $J \in \mathbb{N}$

$$\#\left\{m \in [J+1, J+\overline{N}] : 1 + f'(u_R)\frac{\Delta t}{\Delta x} < \theta_m \leq 1 + [f'(u_R) + \frac{c_0}{4}(u_0 - u_R)]\frac{\Delta t}{\Delta x}\right\} \geq 1. \qquad (17)$$

Now we choose Δx so small that

$$\overline{N}c^* r < \frac{c_0}{4}(u_0 - u_R)$$

and obtain

$$s^{\bar{n}+k} > f'(u_R) + \frac{c_0}{4}(u_0 - u_R) \quad \text{for } 1 \leq k \leq \overline{N}.$$

From (17) there exists a natural number $m_1 \in [\bar{n}+1, \bar{n}+\overline{N}]$, such that

$$\Delta x + f'(u_R)\Delta t < \theta_{m_1}\Delta x < \Delta x + \left[f'(u_R) + \frac{c_0}{4}(u_0 - u_R)\right]\Delta t < \Delta x + s^{\bar{n}+k}\Delta t.$$

At this time step the rarefaction wave to the right of the shock will disappear, i.e. $u_R^{m_1} = u_R$, because the neighboring random evaluations take place to the left of the shock wave and to the right of the rarefaction wave.

Suppose that the shock wave and the rarefaction waves begin again to intersect between say times t_{m^*} and t_{m^*+1} and a new rarefaction wave with the same direction as the shock wave appears to the right of the shock between times t_{m^*+1} and t_{m^*+2}. Then we may take m^* for $\bar{n}$ in the above argument and obtain the same result.

Now let z^n be the x-coordinate of the shock at $t = t_n$, then

$$z^n = z^0 + \sum_{m=1}^{n}\left[\operatorname{sgn}(s_u^m\Delta t - \theta_m\Delta x) + \operatorname{sgn}(s_u^m\Delta t + (1-\theta_m)\Delta x)\right]\frac{\Delta x}{2}. \qquad (18)$$

Taking $K = O(\Delta t^{-1/2})$ and using Lemma 8 for $g^m = s_u^m\frac{\Delta t}{\Delta x}$, we have

$$\begin{aligned}
z^n &= z^0 + n\Delta x\left[\frac{1}{2n}\sum_{m=1}^{n}\left(\operatorname{sgn}(s_u^m\frac{\Delta t}{\Delta x} - \theta_m) + \operatorname{sgn}(s_u^m\frac{\Delta t}{\Delta x} + 1 - \theta_m)\right)\right] \\
&= z^0 + n\Delta x\left[\frac{1}{t_n}\sum_{j=1}^{\lfloor \frac{t_n}{K\Delta t}\rfloor} s_u^{jK}\cdot\frac{\Delta t}{\Delta x}\cdot K\Delta t\right] + O(r^{1/2}) \qquad (19)\\
&= z^0 + \sum_{j=1}^{\lfloor \frac{t_n}{K\Delta t}\rfloor} s_u^{jK}K\Delta t + O(r^{1/2}).
\end{aligned}$$

Now we extend the sequences z^n and s_u^n by piecewise linear interpolation to obtain functions $z(t)$ and $s_u(t)$ defined for all $t > 0$. From (16) and (18), we have

$$\begin{aligned} \left|s_u(t) - s_u(t')\right| &\leq c\left|t - t'\right| \cdot |\log \Delta t| \\ \left|z(t) - z(t')\right| &\leq c\left|t - t'\right|. \end{aligned} \tag{20}$$

Thus we can rewrite the sum in (19) as

$$\begin{aligned} \sum_{j=1}^{\lfloor \frac{t_n}{K\Delta t} \rfloor} s_u^{jK} K\Delta t &= \sum_{j=1}^{\lfloor \frac{t_n}{K\Delta t} \rfloor} \left[\int_{(j-1)K\Delta t}^{jK\Delta t} s_u(t)dt + \int_{(j-1)K\Delta t}^{jK\Delta t} [s_u^{jK} - s_u(t)]dt \right] \\ &= \int_0^{t_n} s_u(t)dt + O(K^2 \Delta t^2) \left|\log \Delta t\right| M \\ &= \int_0^{t_n} s_u(t)dt + O(r^{1/2}). \end{aligned}$$

Hence, from (19), we have that

$$z(t) = z(0) + \int_0^t s_u(t)dt + O(r^{1/2}) \tag{21}$$

holds for $t = t_n$. Using (20), we deduce that (21) holds for all $t > 0$.

Let $c(t)$ and $s_v(t)$ be the x-coordinate and speed of the shock in v at time t respectively, then

$$c(t) = c(0) + \int_0^t s_v(t)dt.$$

Subtracting from (21), we have

$$z(t) - c(t) = z(0) - c(0) + \int_0^t [s_u(t) - s_v(t)]dt + O(r^{1/2}). \tag{22}$$

We show below that

$$|s_u(t) - s_v(t)| \leq c\left(\sup_w \left|b(w) - y^0(w)\right| + r + |z(t) - c(t)|\right). \tag{23}$$

To prove this, we let $\tilde{v}$ be the solution of (1) with initial data

$$\tilde{v}(x,0) = \begin{cases} v(x,0) & x < c^0 \\ u_m & x \geq c^0 \end{cases}$$

and $\tilde{u}$ the approximate solution with initial data

$$\tilde{u}(x,0) = \begin{cases} u(x,0) & x < z^0 \\ u_m & x \geq z^0 \end{cases}.$$

Then $\tilde{v} = v(x,t)$ for $x < c(t)$ and $\tilde{u}_k^n = u_k^n$ for $x_k < z^n$. Now fix a time t_n and let $\bar{v} = v(c(t_n)-, t_n) = \tilde{v}(c(t_n), t_n), \bar{u} = u(z(t_n)-, t_n)$. Then the Rankine-Hugoniot condition gives

$$\begin{aligned} |s_u(t_n) - s_v(t_n)| &= \left| \int_0^1 \left[f'(\bar{u} + \theta(u_R^n - \bar{u})) - f'(\bar{v} + \theta(u_R - \bar{v})) \right] d\theta \right| \\ &= \left| \int_0^1 f''(\xi) \left[\theta(u_R^n - u_R) + (1-\theta)(\bar{u} - \bar{v}) \right] d\theta \right| \\ &\leq c\left|\bar{u} - \bar{v}\right| + cr. \end{aligned}$$

Here $u_R^n - u_R = O(r)$ is used. However $\tilde{v}(x,t_n)$ achieves the value $\bar{u}$ at some point $\tilde{x}$, namely $\bar{u} = \tilde{v}(\tilde{x},t_n)$. When $t_n \le \min\{t_{\bar{n}},\bar{t}\}$, because $s_u(t_n) = s_v(t_n) = s$, obviously (23) holds. When $t_n > \min\{t_{\bar{n}},\bar{t}\}$, from (15), obviously $|\tilde{v}(\tilde{x},t_n) - \tilde{v}(x,t_n)| \le c\,|\tilde{x}-x|$. In this case

$$\begin{aligned} |s_u(t_n) - s_v(t_n)| &\le c\left(|\tilde{v}(\tilde{x},t_n) - \tilde{v}(c(t_n),t_n)| + r\right) \\ &\le c'\left(|\tilde{x} - z(t_n)| + |z(t_n) - c(t_n)| + r\right). \end{aligned}$$

Noting that $\tilde{u}(z(t_n),t_n) = \bar{u} = \tilde{v}(\tilde{x},t_n)$, from Lemma 4, we have

$$|\tilde{x} - z(t_n)| \le \| \alpha \|_\infty \le C\left(\sup_w \left|b(w) - y^0(w)\right| + r\right).$$

Thus we obtain

$$|s_u(t) - s_v(t)| \le c\left(\sup_w \left|b(w) - y^0(w)\right| + r + |z(t) - c(t)|\right).$$

That is, (23) holds for $t = t_n$. So it also holds for all $t > 0$.

By substituting (23) into (22) and applying Gronwall's inequality, we can get

$$|z(t) - c(t)| \le c\left(\left|z^0 - c^0\right| + \sup_w \left|b(w) - y^0(w)\right| + O(r^{1/2})\right).$$

Summing up the above results, we conclude that we can choose α, β and ω, such that

$$\begin{aligned} \| \alpha \|_\infty + \| \beta \|_\infty &\le C\left(\sup_w \left|b(w) - y^0(w)\right| + \left|z^0 - c^0\right| + r^{1/2}\right) \\ \| \omega(\cdot,t) \|_{L^1[-X,X]} &\le C\Delta x. \end{aligned}$$

The proof is complete. □

Having discussed the above cases, now we prove our theorem.

Proof of the theorem: We subdivide the upper half-plane into nonoverlapping trapezia, i.e. quadrilateral domains R_i whose bottom and top boundaries are line segments parallel to the x-axis on the lines $t = a_i$ and $t = b_i$ respectively with $a_i < b_i$. They are to be chosen in such a way that in each trapezium R_i the exact solution v consists of either of a single wave entering R_i at the bottom and leaving R_i at the top, or of two waves entering R_i at the bottom with the product of their interaction leaving R_i at the top. The left and right boundaries of R_i can be chosen as suitable lines, see Figure 9. By the "bottom" of R_i we mean the interior of $\partial R_i \cap \{t = a_i\}$; similarly for the "top".

Under the conditions assumed in our theorem, the above subdivision is feasible. We can first use some trapezia R_i of the above kind to contain all the domains in which there are two waves and their interaction and then subdivide other parts. We can choose a proper subdivision such that any bounded rectangle $[-X,X] \times [0,T]$ is covered by a finite number of R_i. Now suppose that Δt is sufficiently small and $\bigcup_{i \in I} R_i \supset [-X,X] \times [0,T]$, where I is a finite index set. If $I_1 = \{j | j \in I, a_j = 0\}$, then we apply for each such initial trapezium R_j with $j \in I_1$ the Lemmas 1-7. We can conclude using (7) that the error bound (4) holds in $\bigcup_{i \in I_1} R_i$. Assume that w^* is a wave which enters $R_{j'}$ from one of the initial R_j, then the existence of α and β for $(x,t) \in \partial R_j \bigcap \{t = b_j\}$ with $j \in I_1$ gives that the difference of the "initial" position of w^* in $R_{j'}$, namely as $t = t_{j'}$, and the corresponding position of the exact solution is bounded by

$\| \alpha \|_\infty + \| \beta \|_\infty = O(r^{1/2})$. Therefore, applying the Lemmas 1-7 for $R_{j'}$, we obtain that (4) also holds in $R_{j'}$. Proceeding by induction on j towards increasing time, we conclude that (4) holds in $\bigcup_{i \in I} R_i$. So (4) holds in $[-X, X] \times [0, T]$.

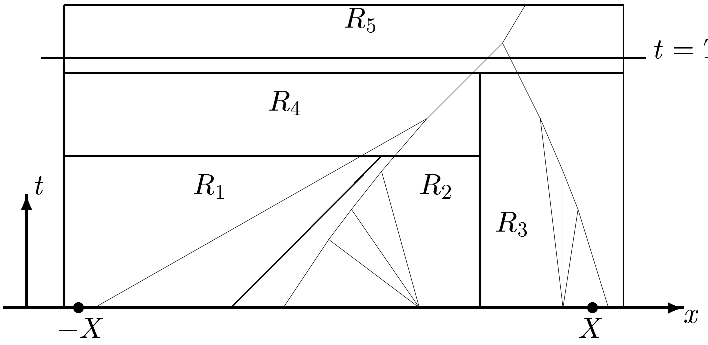


Figure 9

When there are no interactions between shock waves and rarefaction waves, we do not need Lemma 7 and can similarly conclude that (5) holds. So the theorem has been proved. $\square$

In the proof of Theorem 5.1 in [5], each R_i was taken as a rectangle in the subdivision of above properties. The following example shows that such a subdivision may be impossible to achieve in general. Let v consist of the waves as shown in Figure 10.

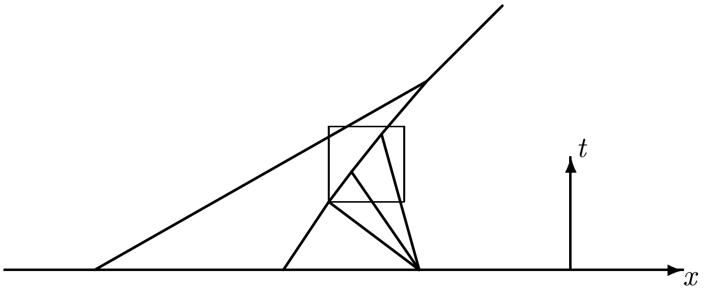


Figure 10

Obviously any rectangle which contains the interaction part of the shock wave and rarefaction wave in the figure must contain a part of the other shock. As long as the restrictions on the left and right boundaries of R_i are relaxed, the conclusion of the Theorem 5.1 can be obtained with no change in the proof.

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