Generalized Characteristics and the Uniqueness of Entropy Solutions to Zero-pressure Gas Dynamics

Jiequan Li, Gerald Warnecke
Institut für Analysis und Numerik
Otto-von-Guericke-Universität Magdeburg
PSF 4120
39016 Magdeburg, Germany

Dedicated to C. M. Dafermos on the occasion of his 60th birthday

July 10, 2001

Abstract

The system of zero-pressure gas dynamics conservation laws describes the dynamics of free particles sticking under collision while mass and momentum are conserved both at the discrete and continuous levels. The existence of such solutions was established in [CLZ, *Science in China* **40**, *1997*; ERS, *Comm. Math. Phys.* **177**, *1996*]. In this paper we are concerned with the uniqueness of entropy solutions. We first introduce additionally to the Oleinik entropy condition a cohesion condition. Both conditions together form our extended concept of an admissibility condition for solutions to the system. The cohesion condition is automatically satisfied by the solutions obtained in the existence results mentioned above. Further, we regularize such a given admissible solution so that generalized characteristics are well-defined. Through limiting procedures the concept of generalized characteristics is then extended to a very large class of admissible solutions containing vacuum states and singular measures. Next we use the generalized characteristics and the dynamics of the center of mass in order to prove that all entropy solutions are equal in the sense of distributions.

Key words: zero-pressure gas dynamics, uniqueness, entropy condition, cohesion condition, generalized characteristics

1. Introduction

We are concerned with the uniqueness problem for solutions to the zero-pressure gas dynamics system of conservation laws

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + (\rho u^2)_x &= 0, \end{aligned} \tag{1}$$

with the initial data

$$(\rho, u)(t=0, x) = (\rho_0, u_0)(x), \tag{2}$$

where the dependent variables ρ and u are the particle mass density and the particle velocity respectively. The independent variables t and x are the time and space coordinates. This system is connected to the sticky particle model in e.g. Shandarin and Zeldovich [SZe], which describes the motion of free particles sticking under collision while the conservation laws of mass and momentum hold. This model can be described both at the discrete and continuous levels, see E et al. [ERS]. Formally, we can obtain this model from the compressible Euler equations by letting the pressure drop to zero or from the Boltzmann equation by letting the temperature go to zero, see Bouchut [B]. A rigorous proof of the reduction from the Euler equations to the system (1) taking account of the structure of solutions can be found in Li [L]. Recently (1) was related to scalar conservation laws and even Hamilton-Jacobi equations in Brenier and Grenier [BG].

The system (1) is non-strictly hyperbolic. It has the velocity u as double eigenvalue. For smooth solutions it is easily seen that the system decouples in the following manner, see E et al. [ERS]. Using the first equation, the second becomes the Burgers equation

$$u_t \quad + \quad \left(\frac{u^2}{2}\right)_x \quad = \quad 0$$

which may be solved in the usual manner. Using the known velocity u the first equation for the non-negative mass density ρ may then also be solved as a scalar equation.

Since the density may develop into a singular measure, even when the initial data are regular, the solution of (1) and (2) must be considered in the sense of signed measures, i.e. in the sense of distributions which here are signed measures. These will be a measure m for the density ρ and a signed measure I for the momentum ρu, which is the other conserved dependent variable.

Definition 1.1: *We denote by $\mathcal{B}$ the σ-algebra of Borel measurable subsets of $\mathbb{R}$ and by $\mathcal{M}(\mathbb{R})$ the space of signed Borel measures on $\mathbb{R}$. Take $m, I \in L^\infty(\mathbb{R}^+, \mathcal{M}(\mathbb{R}))$, i.e. we assume that $(m, I)(t, \cdot)$ are signed Borel measures on $\mathbb{R}$ for any $t \in [0, \infty[$. By $(m, I)(t, \Delta)$ we denote the signed measure of the Borel measurable set $\Delta \in \mathcal{B}$ at time $t \in \mathbb{R}^+ = [0, \infty[$.*

Further, let the measure $I(t, \cdot)$ for the momentum be absolutely continuous with respect to the measure for the particle mass density $m(t, \cdot)$. Then a velocity $u(t, x)$ can be defined using the Radon-Nikodym Theorem as the density function given by

$$u(t,x) = \frac{dI}{dm}, \qquad \text{i.e.} \quad \int_\Delta dI = \int_\Delta u \, dm. \tag{3}$$

For clarity we also use the notation $m(t, dx) := dm$ and $I(t, dx) := dI$. We say that the pair of signed measures (m, I) is a **measure solution** *of the system (1) for the initial data (2) if and only if the equations*

$$\begin{aligned} \int\!\!\int_{\mathbb{R}^+ \times \mathbb{R}} [\phi_t m(t, dx) + \phi_x I(t, dx)] dt + \int_{-\infty}^{\infty} \phi(0, x) m(0, dx) &= 0, \\ \int\!\!\int_{\mathbb{R}^+ \times \mathbb{R}} [(\phi_t + \phi_x u(t, x)) I(t, dx)] dt + \int_{-\infty}^{\infty} \phi(0, x) I(0, dx) &= 0, \end{aligned} \tag{4}$$

are satisfied for all $\phi \in C^\infty(\mathbb{R}^+ \times \mathbb{R})$. □

Based on this definition, the Cauchy problem was solved in Cheng et al. [CLZ]. An alternative proof may be found in E et al. [ERS]. Explicit solutions to Riemann problems were given by Sheng and Zhang [SZh] who also proved stability properties. For completeness, we cite the existence theorem in [CLZ].

Theorem 1.2 (Existence): *Let the initial data* $(m_0, I_0) = (m(0,\cdot), I(0,\cdot)) \in [\mathcal{M}(\mathbb{R})]^2$ *with* $m_0 \geq 0$ *be such that the measure* $I(0,\cdot)$ *is absolutely continuous with respect to* $m(0,\cdot)$*, the total mass* $m_0(\mathbb{R})$ *is finite, and the condition*

$$\int_0^x \eta \, m(0, d\eta) \to +\infty, \quad as \ |x| \to \infty \tag{5}$$

is assumed to hold if $\operatorname{supp} m_0$ *is unbounded. Then a measure solution of the system (1) for these initial data exists.* □

Remark: The assumption that the initial mass satisfies $m_0(\mathbb{R}) < \infty$ can be replaced by the local finiteness of m_0 because the test functions $\phi(x,t)$ in Definition 1.1 can be taken in $C_0^\infty(\mathbb{R}^+ \times \mathbb{R})$. Our assumption is made just for simplicity of the presentation.

It is well-known that some entropy condition should be supplemented in order to obtain the uniqueness of solutions and to select the physically relevant solution. The obvious entropy inequality

$$\partial_t[\rho S(u)] + \partial_x[\rho u S(u)] \leq 0, \tag{6}$$

for all smooth convex function S is not sufficient to guarantee this, see [B,LZY]. A careful reading of the existence proof in [CLZ, ERS] reveals that the solution constructed there satisfies the classical Oleinik condition. This was also pointed out by [ERS, WD].

Oleinik Condition: *There exists a constant* $E > 0$ *such that*

$$\frac{u(t, x+a) - u(t,x)}{a} \leq \frac{E}{t}, \tag{7}$$

for all $x \in \mathbb{R}$*,* $a > 0$ *and* $t \in]0, \infty[$*.* □

Under this condition, the uniqueness of solutions to (1) was proved in [WD, BJ] when the initial data (2) are boundedly measurable functions. This is obviously far from being satisfactory since the solutions and the initial data to (1) should naturally be in the same space of measures.

However, a trivial counterexample illustrates that some additional condition should be imposed. Let (2) be the simple initial state

$$(m_0, I_0)(x) = (\delta_0, \overline{u}\delta_0), \tag{8}$$

where δ_0 is the standard Dirac measure with unit mass at zero and $\overline{u}$ is a constant. Basically, the initial data for the velocity are $\overline{u}$ at $x = 0$ and completely arbitrary elsewhere because the mass density vanishes. We can construct the infinite family of solutions

$$m(t,\cdot) := \rho_1 \delta_{(x - a_1 t)} + \rho_2 \delta_{(x - a_2 t)}, \quad u(t,x) := \begin{cases} a_1, & x = a_1 t, \\ a_2, & x = a_2 t, \\ 0, & \text{everywhere else.} \end{cases} \tag{9}$$

We impose conservation of mass by taking $\rho_1 + \rho_2 = 1$, with $\rho_1, \rho_2 \geq 0$. Choosing $a_1 < a_2$ we have conservation of momentum by setting $\rho_1 a_1 + \rho_2 a_2 = \overline{u}$. These solutions all satisfy Definition 1.1 and the Oleinik condition. The Oleinik condition is satisfied in the sense that the function u in (9) may be replaced equivalently almost everywhere with respect to $m(t,\cdot)$ for example by a linear interpolant between the values $0, a_1, a_2, 0$ that has a maximal slope E/t for an appropriate constant

E, since the values of u on the set where the mass density vanishes may be chosen arbitrarily. Note that this interpretation of the Oleinik condition prevents any further such splitting of the mass from occuring at times $t > 0$. The slope of any interpolant must become infinite in that case. The factor $1/t$ in the Oleinik condition allows this to occur only at $t = 0$.

But, the Oleinik condition does not reflect the fact that the free particles should not even separate initially. This example motivates the following condition suggested by the first author.

Cohesion Condition: *For $x_0 \in \mathbb{R}$, if $m_0(\{x_0\}) > 0$, then writing $t \to 0^+$ for the limit from above, i.e. using $t > 0$, we require that*

$$\lim_{t\to 0^+} m\left(t, \left\{x \in \mathbb{R}; \left|\frac{x - x_0}{t} - u_0(x_0)\right| \le \epsilon\right\}\right) = m_0(\{x_0\}) \tag{10}$$

for all $\epsilon > 0$. □

The Oleinik and the cohesion conditions together form an **entropy condition** for measure solutions of the zero-pressure gas dynamics initial value problem (1) and (2). We point out that the solution constructed in Theorem 1.2 satisfies this condition. Note further that the cohesion condition prevents the splitting up of singular masses at time $t = 0$. But both the cohesion and Oleinik's condition together do not prevent singular masses from separating from pieces of regular mass at time $t = 0$. Similarily one may have also $u_0(x^-) < u_0(x^+)$ for some $x \in \mathbb{R}$. Here $u_0(x^\pm)$ denotes the limits obtained by approaching x from above and below respectively. Sometimes we will also use the notation $x \pm 0$ instead. In all cases of initial separation of velocity a vacuum is created due to conservation of mass. This must be taken into account in the analysis in this paper. At later times any separation of mass is prevented by the Oleinik condition.

In this paper we will establish as the main result the uniqueness of these solutions to the system (1) with the initial condition (2) under the restriction of the above entropy condition. The uniqueness theorem is stated as follows.

Theorem 1.3 (Uniqueness): *Measure solutions to the system (1) under the initial condition (2) satisfying the entropy condition are unique in the following sense. Assume that (m_i, I_i) for $i = 1, 2$ are two measure solutions of (1) and (2) satisfying Definition 1.1 and the Oleinik condition (7) as well as the cohesion condition (10). Then they are equal, i.e. the equation*

$$\begin{aligned}\int\int_{\mathbb{R}^+\times\mathbb{R}} \phi m_1(t, dx)dt &= \int\int_{\mathbb{R}^+\times\mathbb{R}} \phi m_2(t, dx)dt, \\ \int\int_{\mathbb{R}^+\times\mathbb{R}} \phi I_1(t, dx)dt &= \int\int_{\mathbb{R}^+\times\mathbb{R}} \phi I_2(t, dx)dt,\end{aligned} \tag{11}$$

hold for all bounded test functions $\phi \in C^\infty(\mathbb{R}^+ \times \mathbb{R})$.

The method of proof adopted here is basically a method of characteristics, namely generalized characteristics, see Dafermos [D]. We have to overcome two main difficulties in extending the concept of generalized characteristics in this paper: One is the possible presence of vacuum states in the solutions, where no velocity is specified. The other is the irregularity of the solution, i.e. the solutions do not have to be functions of bounded variation but may contain singular measures. For this purpose, we first regularize the entropy solutions so that the generalized characteristics, see Dafermos [D], for the regularized solutions are well-defined. Then we mainly concentrate on the limit properties of the regularized solutions. An important tool in the ensuing analysis are the characteristic maps which are flow maps of the flow generated by the generalized characteristics. It is shown that they naturally satisfy the properties of conservation of mass and momentum, which is to be expected. Further, the dynamic behaviour of the center of mass of sets, subjected to the characteristic maps in time, is studied. The results thus obtained finally lead to our uniqueness proof. As a by–product, we also verify the Generalized Variational Principle used in [ERS].

After this paper was finished, we were informed that Huang and Wang in [HW] reproved the existence result and gave a uniqueness result for solutions to (1) independently. They utilized the Lebesgue–Stieltjes integral to equivalently define weak solutions in terms of the distribution function for (1). Then they proved uniqueness of solutions. The uniqueness result there was obtained via the adjoint problem. It needed a new existence and uniqueness theorem for a linear equation with discontinuous coefficients. Their entropy condition consists of the Oleinik condition and an energy condition that states that the energy should be weakly continuous initially. It must be pointed out that the sticky particles in this model loose energy immediately under collision as time evolves, see [B], and so one cannot guarantee that the energy is continuous in the solution. Therefore, their condition does not reflect a natural property of the model. Here we instead use the initial cohesion condition which reflects the physical fact that the sticky particles never split, even initially.

The rest of this paper is organized into four parts. In Section 2, we regularize a given entropy solution and prove that the regularized solution is equivalent to the original one. The generalized characteristics are defined in Section 3 and their properties are investigated carefully in Section 4 to facilitate the proof of uniqueness. Finally the proof of uniqueness, i.e. Theorem 1.3, is given in Section 5.

This paper completes the program of giving an existence and uniqueness theory for the initial value problem (1) and (2).

2 Entropy solutions as limits of regularized measures

The measure solutions of the system (1) under the initial conditions (2) are not sufficiently regular to apply the theory of generalized characteristics directly. In order to prove our uniqueness theorem, we show that they may be obtained as limits of regularized mesures. These are regularizations of the entropy solutions constructed in [CLZ, ERS].

Let (m, I) be an entropy solution of the system (1) satisfying the initial conditions (2). The distribution function of the particle mass distribution $m(t, \cdot)$ is defined as

$$F(t,x) := m(t,]-\infty, x]), \qquad \text{for all } x \in \mathbb{R}, \quad t \in \mathbb{R}^+. \tag{12}$$

It is non-decreasing, has jumps at all singular points of m and is continuous from the right. Obviously, in the limit one obtains $F(t, -\infty) = 0$ and $F(t, +\infty) = m(t, \mathbb{R})$. Also we set

$$U := \max_{(t,x)\in\mathbb{R}^+\times\mathbb{R}} |u(t,x)|. \tag{13}$$

We further define a continuous regularization by one-sided averaging given as

$$F^\delta(t,x) := \frac{1}{\delta}\int_0^\delta F(t+s, x+Us)ds, \quad \text{for all } \ x \in \mathbb{R}, \ \delta > 0, t \geq 0. \tag{14}$$

Now we show

Lemma 2.1: *The limit of the regularizations $F^\delta(t,x)$ exists, i.e. we may set*

$$\overline{F}(t,x) := \lim_{\delta\to 0} F^\delta(t,x), \qquad \text{for all } x\in\mathbb{R},\ t\geq 0. \tag{15}$$

Proof: Take $0<\tau<T<+\infty$, $x_0\in\mathbb{R}$ and let $\delta,\ \epsilon>0$ be sufficiently small. We define two test functions $\eta\in C_0^0(]0,\infty[)$ and $\phi\in C^\infty(\mathbb{R})$ taking the following values

$$\eta(t)=\begin{cases} 1 & \text{for} \quad t\in[\tau+\delta,T],\\ 0 & \text{for} \quad t\notin[\tau,T+\delta],\\ \in[0,1] & \quad\text{elsewhere;}\end{cases} \qquad \phi(x)=\begin{cases} 1 & \text{for} \quad x\leq x_0,\\ 0 & \text{for} \quad x\geq x_0+\epsilon,\\ \in[0,1] & \quad\text{elsewhere.}\end{cases}$$

We additionally require that the function η is linear on the interval $[\tau,\tau+\delta]$, i.e. increasing, as well as on $[T,T+\delta]$, i.e. decreasing there. Further, we want to take ϕ as a non-increasing function on the interval $[x_0,x_0+\epsilon]$, i.e. $\phi'(x)\leq 0$. Using these test functions and U given by (13) we introduce the test function

$$\Phi(t,x) := \phi(x-Ut)\eta(t), \quad \text{for } x\in\mathbb{R}, \quad t\in\mathbb{R}^+.$$

Then $\Phi\in C^0([\tau,T+\delta]\times\mathbb{R})$ is bounded and piecewise differentiable.

The test function Φ is not smooth enough in order to be used in (4). This defect can easily be mended by mollifying Φ. Since (4) must hold for the mollified functions, it must also hold for Φ itself by taking the limit of the mollification parameter and using the Lebesgue Dominated Convergence Theorem.

We now divide the integral of the first equation of (4) into three parts I_1, I_2 and I_3. Note that

$$\Phi_t+\Phi_x u = \phi'(x-Ut)(u-U)\eta(t)+\phi(x-Ut)\eta'(t).$$

Using the fact that $\phi'=0$ for $x\leq x_0$ and then the analogous property of η, we obtain

$$\begin{aligned} I_1 &:= \int_\tau^{T+\delta}\int_{]-\infty,x_0+Ut]}(\Phi_t+\Phi_x u)m(t,dx)dt\\ &= \int_\tau^{T+\delta}\int_{]-\infty,x_0+Ut]}\eta'(t)m(t,dx)dt\\ &= \int_\tau^{T+\delta}\eta'(t)F(t,x_0+Ut)dt\\ &= \int_T^{T+\delta}\eta'(t)F(t,x_0+Ut)dt+\int_\tau^{\tau+\delta}\eta'(t)F(t,x_0+Ut)dt.\end{aligned}$$

The remaining two parts can be expressed as

$$I_2 := \int_\tau^{T+\delta}\int_{]x_0+Ut,x_0+\epsilon+Ut]}\phi'(x-Ut)(u-U)\eta(t)m(t,dx)dt,$$

and

$$I_3 := \int_\tau^{T+\delta}\int_{]x_0+Ut,x_0+\epsilon+Ut]}\phi(x-Ut)\eta'(t)m(t,dx)dt.$$

Because $\phi'(x)\leq 0$ and $u-U\leq 0$, we conclude that

$$I_2\geq 0.$$

Furthermore, we take the distribution function and obtain the estimate

$$|I_3| \leq \int_\tau^{T+\delta} |\eta'(t)| \left[F(t, x_0 + \epsilon + Ut) - F(t, x_0 + Ut)\right] dt.$$

In view of the boundedness of $\eta'(t)$ and the continuity from the right hand side of $F(t, \cdot)$, we arrive at the conclusion that

$$\lim_{\epsilon \to 0} I_3 = 0$$

by the Lebesgue Dominated Convergence Theorem.

Our assumptions on η above imply that, taking appropriate one-sided derivatives, we have

$$\eta'(t) = \begin{cases} \frac{1}{\delta} & \text{for} \quad t \in [\tau, \tau + \delta], \\ -\frac{1}{\delta} & \text{for} \quad t \in [T, T + \delta], \\ 0 & \text{elsewhere.} \end{cases}$$

Now we take the first equation of (4) with Φ inserted and consider the limit $\epsilon \to 0$. This gives us for all $\tau < T$ by the results obtained above

$$\begin{aligned} 0 = & \lim_{\epsilon \to 0} \int_\tau^{T+\delta} \int_{-\infty}^{+\infty} (\Phi_t + \Phi_x u) m(t, dx) dt \\ \geq & \frac{1}{\delta} \int_\tau^{\tau+\delta} F(t, x_0 + Ut) dt - \frac{1}{\delta} \int_T^{T+\delta} F(t, x_0 + Ut) dt \\ = & F^\delta(\tau, x_0 + U\tau) - F^\delta(T, x_0 + UT). \end{aligned}$$

Since this holds for any $\tau < T$ we have shown the monotonicity of F^δ with respect to t. Thus we can conclude that

$$\begin{aligned} F^{2\delta}(t, x_0 + Ut) = & \frac{1}{2\delta} \int_0^{2\delta} F(t + s, x_0 + U(t + s)) ds \\ = & \frac{1}{2} \left[\frac{1}{\delta} \int_0^\delta F(t + s, x_0 + U(t + s)) ds + \frac{1}{\delta} \int_\delta^{2\delta} F(t + s, x_0 + U(t + s)) ds \right] \\ = & \frac{1}{2} \left[F^\delta(t, x_0 + Ut) + F^\delta(t + \delta, x_0 + Ut + U\delta) \right] \\ \geq & F^\delta(t, x_0 + Ut). \end{aligned}$$

Since the distribution function is also non-negative, the inferior limit of $F^\delta(t, x_0 + Ut)$ exists as δ tends to zero. Due to the monotonicity in δ it is the limit, i.e. (15) holds. $\square$

It is easily seen that $F^\delta(t, \cdot)$ and $\overline{F}(t, \cdot + 0)$ are also distribution functions, which respectively determine their corresponding measures $m^\delta(t,]-\infty, x])$ and $\overline{m}(t,]-\infty, x])$, i.e.

$$m^\delta(t,]-\infty, x]) = F^\delta(t, x), \quad \overline{m}(t,]-\infty, x]) = \overline{F}(t, x + 0), \quad \text{for all } \ x \in \mathbb{R}. \tag{16}$$

From the lemma we conclude that the family of measures $m^\delta(t, \cdot)$ converges to the limit measure $\overline{m}(t, \cdot)$ in the sense of measures as δ tends to zero.

Analogously as in (15), we can define the distribution function

$$\underline{F}(t, x) := \lim_{\delta \to 0} \frac{1}{\delta} \int_0^\delta F(t + s, x - Us) ds, \quad \text{for all } x \in \mathbb{R}, \quad t \geq 0.$$

It again determines a measure to be denoted by $\underline{m}(t,x)$. We claim that $\overline{m}(t,\cdot)=\underline{m}(t,\cdot)$. In fact, since the distribution function F is non-decreasing in x, we have

$$\underline{F}(t,x) \le \frac{1}{\delta}\int_0^\delta F(t+s,x-Us))ds \le \frac{1}{\delta}\int_0^\delta F(t+s,x+Us))ds.$$

Taking the limit gives $\underline{F}(t,x) \le \overline{F}(t,x)$. Similarly, for all $x_1 < x_2$ we find for $\delta > 0$ sufficiently small that

$$\frac{1}{\delta}\int_0^\delta F(t+s,x_1+Us))ds \le \frac{1}{\delta}\int_0^\delta F(t+s,x_2-Us))ds$$

holds. Again taking the limit $\delta \to 0$, we obtain

$$\overline{F}(t,x_1) \le \underline{F}(t,x_2).$$

So, $\overline{F}(t,x)=\underline{F}(t,x)$ at their points of continuity. This proves the above claim.

Having determined this we can also define another variant to be used later

$$\tilde{F}(t,x) := \lim_{\delta\to 0}\frac{1}{\delta}\int_0^\delta F(t+s,x)ds,$$

which determines a measure, which we continue to denote by $\overline{m}(t,\cdot)$.

Lemma 2.2: *The measures $m(t,\cdot)$ and $\overline{m}(t,\cdot)$ are equal in the sense of* (11).

Proof: We define for $t \ge 0$

$$A(t) := \int_{-\infty}^{\infty}\phi(t,x)dF(t,x).$$

Take arbitrary $T > \tau \ge 0$ and a bounded test function $\phi \in C^\infty(\mathbb{R}^+ \times \mathbb{R})$. Using Fubini's theorem to change the order of integration for the regularized distribution function $\overline{F}$ gives

$$\begin{aligned}
\int_\tau^T\int_{-\infty}^{\infty}\phi(t,x)d\overline{F}(t,x)dsdt &= \int_\tau^T\int_{-\infty}^{\infty}\phi(t,x)\left(\lim_{\delta\to 0}\frac{1}{\delta}\int_t^{t+\delta}dF(s,x)\right)dt\\
&= \int_\tau^T\lim_{\delta\to 0}\frac{1}{\delta}\int_t^{t+\delta}\left[\int_{-\infty}^{\infty}\phi(t,x)dF(s,x)\right]dsdt\\
&= \int_\tau^T\lim_{\delta\to 0}\frac{1}{\delta}\int_t^{t+\delta}A(s)dsdt\\
&\quad+\int_\tau^T\lim_{\delta\to 0}\frac{1}{\delta}\int_t^{t+\delta}\int_{-\infty}^{\infty}[\phi(t,x)-\phi(s,x)]dF(s,x)dsdt\\
&=: I_1+I_2.
\end{aligned}$$

Making use of the boundedness of ϕ, we have the obvious estimate

$$|I_2| \le (T-\tau)\,m_0(\mathbb{R})\sup_{x\in\mathbb{R},|s-t|\le\delta}|\phi(t,x)-\phi(s,x)|.$$

Since ϕ is continuous, the right hand side vanishes in the limit $\delta \to 0$. On the other hand we split I_1 into three parts

$$\begin{aligned}
I_1 &= \lim_{\delta\to 0}\left[\frac{1}{\delta}\int_\tau^T ds\int_{s-\delta}^{s}A(s)dt - \frac{1}{\delta}\int_\tau^{\tau+\delta}ds\int_{s-\delta}^{\tau}A(s)dt + \frac{1}{\delta}\int_T^{T+\delta}ds\int_{s-\delta}^{s}A(s)dt\right]\\
&= \lim_{\delta\to 0}[I_{11}+I_{12}+I_{13}].
\end{aligned}$$

Then we observe that

$$I_{11} = \int_\tau^T A(s)ds,$$
$$|I_{12}| \leq \|\phi\| m_0(\mathbb{R}) \frac{1}{\delta} \delta^2 \to 0, \quad \text{as } \delta \to 0,$$

$$|I_{12}| \leq \|\phi\| m_0(\mathbb{R}) \frac{1}{\delta} \delta^2 \to 0, \quad \text{as } \delta \to 0.$$

This gives in the limit

$$\int_\tau^T \int_{-\infty}^\infty \phi(t,x) d\overline{F}(t,x) ds dt = \int_\tau^T A(s) ds = \int_\tau^T \int_{-\infty}^\infty \phi(t,x) dF(t,x) ds,$$

i.e. the desired equality of the measures. □

So we will not have to destinguish between the measures m and $\overline{m}$ anymore. We will be using the distribution functions $\overline{F}$ and $\underline{F}$ though.

Lemma 2.3: *For all $x < y$, $t \in \mathbb{R}^+$, U given by (13) and $\delta > 0$ the measure m satisfies the estimate*

$$m(t, [x,y]) \leq m(t+\delta, [x - \delta U, y + \delta U]). \tag{17}$$

Proof: Letting $\delta \to 0$ then by Lemma 2.1 we have for $\tau < T$

$$\overline{F}(\tau, x_0 + U\tau + 0) \leq \overline{F}(T, x_0 + UT + 0).$$

This gives

$$\overline{F}(t, y+0) \leq \overline{F}(t+\delta, y + U\delta + 0) \tag{18}$$

by setting $x_0 + U\tau = y$, $\tau = t$, $T = t + \delta$. On the other hand we have

$$\underline{F}(t, x+0) \geq \underline{F}(t+\delta, x - U\delta + 0). \tag{19}$$

Therefore, by Lemma 2.1, (18) and (19) together show that (17) holds. □

We observe that since $\delta > 0$ we have

$$\overline{F}(t_0, x_0 - \delta U) \leq \overline{F}(t_0 + \delta, x_0) \leq \overline{F}(t_0, x_0 + \delta U)$$

at all points of continuity of $\overline{F}(t_0, \cdot)$. By letting $\delta \to 0$ we get

$$\overline{F}(t_0, x_0) \leq \lim_{\delta \to 0} \overline{F}(t_0 + \delta, x_0) \leq \overline{F}(t_0, x_0).$$

Thus by (16) we have for $t > t_0$ at all points of continuity of $m(t_0, \cdot)$ the continuity in time from above

$$\lim_{t \to t_0^+} m(t, \cdot) = m(t_0, \cdot). \tag{20}$$

in the sense of measures.

Now we turn to the discussion of the conservation law of momentum. First, we note the Gallilean invariance of our system of equations, i.e. we may consider solutions in a moving frame of reference.

For bounded velocities this can be used to make all velocities positive and the momentum a positive measure.

Lemma 2.4 *If (m, I) is a measure solution of (1) satisfying our entropy condition, then*

$$\overline{m}(t,\cdot) := m(t,\cdot - tv),$$
$$\overline{I}(t,\cdot) := I(t,\cdot - tv) + vm(t,\cdot - tv)$$

for all $v \in \mathbb{R}$ is also a measure solution of (1) satisfying our entropy condition. □

This lemma can be found in Bouchut [B]. By choosing v large enough, we may regard $I(t,\cdot)$ as a non-negative Radon measure. Therefore, we will assume without loss of generality that all velocities u are positive throughout the paper. Also all of the above proofs of the properties of $m(t,\cdot)$ can be used analogously to treat $I(t,\cdot)$. Thus we can obtain the following results for $I(t,\cdot)$ without giving repetitive proofs.

Lemma 2.5: *There exists a limit of regularized measures $\overline{I}(t,\cdot)$ equal to $I(t,\cdot)$ in the sense of the second equality of (11). Furthermore, it has the following properties.*

(i) *For $x < y$, $\delta > 0$, $t \geq 0$, as well as $U > 0$ defined in (13)*

$$I(t,[x,y]) \leq I(t+\delta,]x-\delta U, y+\delta U[),$$

and at all points of continuity of $I(t_0,\cdot)$ we have the continuity from above in time

$$\lim_{t\to t_0^+} I(t,\cdot) = I(t_0,\cdot), \qquad t_0 > 0,$$

in the sense of measures.

(ii) *For a Borel measurable set $\Delta \in \mathcal{B}$,*

$$|I(t,\Delta)| \leq U \cdot m(t,\Delta), \quad \text{for all } t \geq 0. \tag{21}$$

□

The inequality (21) is due to the fact that $I(t,\cdot)$ is absolutely continuous with respect to $m(t,\cdot)$. So by the Radon-Nikodym theorem, there exists the density function u, which is bounded and measurable with respect to $m(t,\cdot)$, such that

$$u(t,x) = \frac{dI}{dm}. \tag{22}$$

Note that by our assumptions the velocity is defined almost everywhere with respect to the measure m. Also due to the Oleinik condition (7) we may express u for any given $\tau > 0$ as

$$u(t,x) = cx - v(t,x), \quad x \in \mathbb{R}, t \geq \tau > 0,$$

where $c \geq E/t$ is a constant depending on τ and the function v is non-decreasing in x. Hence, since u is bounded, it has a bounded variation in x on any bounded domain for almost every $t \in \mathbb{R}^+$ in the sense of Lebesgue measure. We may appropriately define $u(t,\cdot)$ everywhere on $\mathbb{R}^+$ so that none of the relevant properties of u are lost.

We now proceed to show that a version of the Gauss theorem holds for the weak solutions we have defined. By choosing $\Phi \equiv 1$ this will show that the conservation of mass and momentum hold for all times $T > 0$. Also it shows that the solutions consider here are also solutions in the sense of [ERS].

Theorem 2.6: *For any measure solution (m,u) of the initial value problem (4), we have*

$$\begin{aligned} \int_0^T \int_{-\infty}^{\infty} (\Phi_t + \Phi_x u) m(t,dx)dt &= \int_{-\infty}^{\infty} \Phi(T,x) m(T,dx) - \int_{-\infty}^{\infty} \Phi(0,x) m_0(dx), \\ \int_0^T \int_{-\infty}^{\infty} (\Phi_t + \Phi_x u) I(t,dx)dt &= \int_{-\infty}^{\infty} \Phi(T,x) I(T,dx) - \int_{-\infty}^{\infty} \Phi(0,x) I_0(dx), \end{aligned} \tag{23}$$

for all $\Phi \in C^\infty(\mathbb{R}^+ \times \mathbb{R})$ and $T > 0$.

Proof: Let us define for $\delta > 0$

$$\eta(t) := \begin{cases} 1, & t \leq T, \\ 0, & t \geq T + \delta, \\ \frac{1}{\delta}(T + \delta - t), & t \in [T, T+\delta]. \end{cases}$$

Then η is piecewise linear, non-increasing and satisfies $\eta'(t) = 1/\delta$ for $t \in]T, T+\delta[$. By mollification we could aprroximate η in $C^\infty(\mathbb{R}^+)$ to obtain a proper test function. The proof below then follows by dominated convergence. For simplification we proceed with η as it is. Taking $\Phi(t,x)\eta(t)$ with any $\Phi \in C^\infty(\mathbb{R}^+ \times \mathbb{R})$ as our test function and substituting it into the first identity of (4), we obtain

$$\int_0^T \int_{-\infty}^{\infty} (\Phi_t + \Phi_x u) m(t,dx)dt + \int_T^{T+\delta} \int_{-\infty}^{\infty} [(\Phi\eta)_t + (\Phi\eta)_x u] m(t,dx)dt + \int_{-\infty}^{\infty} \Phi(0,x) m_0(dx) = 0.$$

So, in order to obtain (23), it suffices to prove that

$$\lim_{\delta \to 0} \int_T^{T+\delta} \int_{-\infty}^{\infty} [(\Phi\eta)_t + (\Phi\eta)_x u] m(t,dx)dt = - \int_{-\infty}^{\infty} \Phi(T,x) m(T,dx).$$

Note that

$$(\Phi\eta)_t + (\Phi\eta)_x u = (\Phi_t + \Phi_x u)\eta(t) + \Phi\eta'(t).$$

We get

$$\begin{aligned} \left| \int_T^{T+\delta} \int_{-\infty}^{\infty} (\Phi_t + \Phi_x u)\eta(t) m(t,dx)dt \right| &\leq \Big(\|\Phi_t\| + \|\Phi_x\| U \Big) \int_T^{T+\delta} m(t,\mathbb{R})dt \\ &\leq \Big(\|\Phi_t\| + \|\Phi_x\| U \Big) m_0(\mathbb{R})\delta \to 0, \qquad \text{as } \delta \to 0; \end{aligned}$$

and

$$\begin{aligned} \int_T^{T+\delta} \int_{-\infty}^{\infty} \Phi(t,x)\eta'(t) m(t,dx)dt &= \int_T^{T+\delta} \int_{-\infty}^{\infty} \Phi(T,x)\eta'(t) m(t,dx)dt \\ &\quad + \int_T^{T+\delta} \int_{-\infty}^{\infty} [\Phi(t,x) - \Phi(T,x)]\eta'(t) m(t,dx)dt = I_1 + I_2. \end{aligned}$$

Due to the continuity of Φ we get

$$\begin{aligned} |I_2| \leq & \sup_{(t,x)\in]T,T+\delta[\times\mathbb{R}} |\Phi(t,x)-\Phi(T,x)|\frac{1}{\delta}\int_T^{T+\delta} m(t,\mathbb{R})dt \\ \leq & \; m_0(\mathbb{R}) \sup_{(t,x)\in]T,T+\delta[\times\mathbb{R}} |\Phi(t,x)-\Phi(T,x)| \to 0, \quad \text{as } \delta \to 0. \end{aligned}$$

Next we have by the continuity from above in time of the measure m

$$\begin{aligned} I_1 = & -\frac{1}{\delta}\int_T^{T+\delta}\int_{-\infty}^{\infty} \Phi(T,x)m(t,dx)dt \\ = & -\int_{-\infty}^{\infty} \Phi(T,x)\left(\frac{1}{\delta}\int_T^{T+\delta} m(t,dx)dt\right) \\ = & \int_{-\infty}^{\infty} \Phi(T,x)d\underline{F}^{\delta}(T,x) \end{aligned}$$

and further

$$\underline{F}^{\delta}(T,x) = \frac{1}{\delta}\int_T^{T+\delta} m(t,]-\infty,x])dt \to \underline{F}(T,x), \quad \text{as } \delta \to 0.$$

Since the distribution function $\underline{F}(T,x)$ determines the measure $m(T,\cdot)$, we have

$$I_1 = \int_{-\infty}^{\infty} \Phi(T,x)d\underline{F}^{\delta}(T,x) \to \int_{-\infty}^{\infty} \Phi(T,x)m(T,dx)$$

as $\delta \to 0$.

With the same method, we can prove the second indentity of (23). □

3 Generalized characteristics associated with a regularized solution

This section is devoted to discussing the existence of generalized characteristics

$$\frac{dx}{dt} = u(t,x) \tag{24}$$

for the double eigenvalue u in the presence of vacuum states and singular measures. If u is a smooth function, we denote the forward in time solution curve of (24) through the point (t_0, x_0) by

$$x = x^0(s; t_0, x_0), \quad s \geq t_0. \tag{25}$$

We are aiming to extend these notions not only to the well known case where u has jumps, see Dafermos [D], but also the case of characteristics near a vacuum state or a singular measure. For this purpose the velocity function u has to be extended into the vacuum and then regularized. The generalized characteristics and the corresponding differential equation will be attained as appropriate limits.

We describe the trajectory of x_0 by the family of **characteristic maps**

$$x = h_{t_0,s}(x_0) = x^0(s; t_0, x_0), \quad s \geq t_0. \tag{26}$$

We will be also considering the inverses $h_{t_0,s}^{-1}$ which will not always be proper inverses as point mappings but may be set valued mappings, i.e. the image $h_{t_0,s}^{-1}(x)$ may be a set. Due to this property one may prefer to think of characteristic maps as maps between subsets of $\mathbb{R}$ from the outset. If the solution contains vacuum states the set $h_{t_0,s}^{-1}(x)$ may have gaps there. In this case we add these gaps to $h_{t_0,s}^{-1}(x)$ so that the set will always be an interval.

Consider a Borel measurable set $A \in \mathcal{B}$. Our goal is to prove that

$$m(s,A) = m(t_0, h_{t_0,s}^{-1}(A)), \quad I(s,A) = I(t_0, h_{t_0,s}^{-1}(A)), \quad \text{for all } A \in \mathcal{B}. \tag{27}$$

Throughout this section, the time t is restricted to lie in an interval $[\tau, T]$ with $0 < \tau < T < \infty$.

Definition 3.1: *A bounded function w will be called a* **quasi-Oleinik function** *if there are a positive constant $c > 0$ and a function v, non-decreasing in x, such that*

$$w(t,x) = cx - v(t,x), \quad \text{for all } (t,x) \in \mathbb{R}^+ \times \mathbb{R}.$$

□

Obviously, any bounded function satisfying the Oleinik condition (8) is also a quasi-Oleinik function. On the other hand, if w is a quasi-Oleinik function, then obviously

(a) the function w is of bounded variation in x for all times $t \in [\tau, T]$;

(b) for all $x_1 < x_2$ it satisfies the one-sided Lipschitz condition

$$w(t,x_2) - w(t,x_1) \le c(x_2 - x_1).$$

Since the vacuum state $\rho(t,x) = 0$ may appear in the solution (m, I) and the velocity $u(t,x)$ is not specified there, we will make some definitions for this case. First we introduce the definition of the vacuum set.

Definition 3.2 *The* **vacuum set** *at time $t \in \mathbb{R}^+$ is the set*

$$V(t) = \cup\{]a,b[; m(t,]a,b[) = 0\} \subset \mathbb{R}. \tag{28}$$

□

The set $V(t)$ is open, therefore it may be expressed as the union of at most countably many disjoint open intervals, two of which may be semi-infinite,

$$V(t) = \cup_{i=1}^{\infty}]a_i, b_i[. \tag{29}$$

We define the complementary set $\Omega(t) := \mathbb{R} \setminus V(t)$. For any $\delta > 0$ we set

$$u^\delta(t,x) := cx - \frac{\int_{]x-\delta,x+\delta[} v(t,\xi) m(t,d\xi)}{\int_{]x-\delta,x+\delta[} m(t,d\xi)} \tag{30}$$

where c and $v(t,x)$ are given as in Definition 3.1. This definition of values $u^\delta(t,x)$ makes sense on the set

$$\Omega^\delta(t) := \{x \in \mathbb{R}; m(t,]x-\delta, x+\delta[) > 0\} \supset \Omega(t). \tag{31}$$

But, we consider this definition only on $\Omega(t)$ and define the extension of u^δ to the complement $V(t)$ as follows. For any $x \in V(t)$, there exists an open interval $]a_i, b_i[$ as given above containing x. Suppose this is a finite interval. We take some δ as in (30) and choose $h \in]0, \delta[$, e.g. $h = \frac{\delta}{2}$, to define the linear interpolant

$$u^\delta(t,x) = \lim_{h \to 0} \left[\frac{b_i - x + h}{b_i - a_i + 2h} u^\delta(t, a_i - h) + \frac{x - a_i + h}{b_i - a_i + 2h} u^\delta(t, b_i + h) \right] \tag{32}$$

for $x \in V(t)$, where $u^\delta(t, a_i - h)$ and $u^\delta(t, b_i + h)$ are defined in (30). In case $V(t)$ contains semi-infinite intervals we choose a piecewise linear extension on the semi-infinite intervals that takes the velocity continuously to zero and does not increase the maximal slope. Then $u^\delta(t, x)$ is well-defined everywhere.

Lemma 3.3 *The extended velocity function u^δ has the following properties:*

(i) *it satisfies the quasi-Oleinik condition;*

(ii) *it satisfies in the limit*

$$\lim_{\delta\to 0} u^\delta(t, x) = u(t, x), \quad \textit{a.e. with respect to the measure } m \tag{33}$$

for all $t \in [\tau, T]$;

(iii) *for $T > t_0$ it is continuous from above in time*

$$\lim_{t\to t_0^+} u^\delta(t, x) = u^\delta(t_0, x), \tag{34}$$

for a fixed $\delta > 0$ and all $x \in \Omega(t)$, if x is a point of continuity for $u(t_0, \cdot)$.

Proof: It is evident that (i) holds. In fact, the function v^δ given by

$$v^\delta(t, x) := \frac{\int_{]x-\delta, x+\delta[} v(t, \xi) m(t, d\xi)}{\int_{]x-\delta, x+\delta[} m(t, d\xi)}, \quad x \in \Omega^\delta(t), \tag{35}$$

is non-decreasing in x. So u^δ given by (30) satisfies the quasi-Oleinik condition for $x \in \Omega^\delta(t)$. We can extend this conclusion onto the whole of $\mathbb{R}$ by using the definition of the extension u^δ given by (32).

Next, we proceed to prove (ii). Consider a point of continuity for u, say x_0. Since $u(t, x)$ can be expressed in the form

$$u(t, x) = cx - v(t, x),$$

we have that $v(t, x)$ is also continuous at x_0. From (35),

$$\left| v^\delta(t, x_0) - v(t, x_0) \right| \le \sup_{\xi \in]x_0-\delta, x_0+\delta[} |v(t, \xi) - v(t, x_0)| \to 0, \quad \text{as } \delta \to 0,$$

i.e.

$$\lim_{\delta\to 0} v^\delta(t, x_0) = v(t, x_0). \tag{36}$$

Thus (34) holds.

Now we deal with the points of discontinuity of u. As pointed out before, u satisfies the quasi-Oleinik condition, so it is of bounded variation, which implies that all points of discontinuity are countable. Therefore, it suffices to consider one such point, say x_0, satisfying $m(t, \{x_0\}) > 0$ and to show that (36) holds there. Since $\text{meas}(]x_0 - \delta, x_0 + \delta[\setminus \{x_0\}) \to 0$ as $\delta \to 0$,

$$m(t,]x_0 - \delta, x_0 + \delta[\setminus \{x_0\}) \to 0, \quad \text{as } \delta \to 0.$$

It follows that

$$\begin{aligned}\left|v^\delta(t,x_0)-v(t,x_0)\right| &= \left|\frac{\int_{]x_0-\delta,x_0+\delta[\setminus\{x_0\}} v(t,\xi)m(t,d\xi)}{m(t,]x_0-\delta,x_0+\delta[)}\right|\\ &\le \frac{1}{m(t,\{x_0\})}\cdot(U+c)\cdot m(t,]x_0-\delta,x_0+\delta[\setminus\{x_0\})\to 0, \quad \text{as } \delta\to 0.\end{aligned}$$

This shows that (36) holds.

To prove (iii), note first that by (20) and Lemma 2.5 we have that $(m,I)(t,\cdot)$ converges to $(m,I)(t_0,\cdot)$ as $t>t_0$ tends to t_0. Then we have

$$\begin{aligned}\lim_{t\to t_0^+}(m,I)(t,]x_0-\delta,x_0+\delta[) &= (m,I)(t_0,]x_0-\delta,x_0+\delta[),\\ \lim_{t\to t_0^+}\int_{]x_0-\delta,x_0+\delta[}\xi m(t,d\xi) &= \int_{]x_0-\delta,x_0+\delta[}\xi m(t_0,d\xi).\end{aligned}$$

for $x_0\in\Omega(t_0)$. Also, we note that on $\Omega(t_0)$ by (30), $v(t,x)=cx-u(t,x)$, and the absolute continuity of I with respect to m

$$u^\delta(t,x_0)=cx_0-\frac{c\int_{]x_0-\delta,x_0+\delta[}\xi m(t,d\xi)}{m(t,]x_0-\delta,x_0+\delta[)}+\frac{I(t,]x_0-\delta,x_0+\delta[)}{m(t,]x_0-\delta,x_0+\delta[)}.$$

So we obtain

$$\lim_{t\to t_0^+}u^\delta(t,x_0)=u^\delta(t_0,x_0).$$

Thus, as long as $x_0\in\Omega(t_0)$ and x_0 is a point of continuity for $u^\delta(t,\cdot)$ the limit (34) holds. □

We now regularize our extensions of the velocity u^δ in the following manner. We introduce a subset of $\Omega^\delta(t)$ for some $\eta\in]0,\delta[$ as

$$\Omega^{\delta+\eta}(t)=\{x\in\Omega^\delta(t);]x-\eta,x+\eta[\subset\Omega^\delta(t)\}. \tag{37}$$

On this set, we define two regularized continuous functions

$$\begin{aligned}u_-^{\delta\eta}(t,x) &= \frac{1}{\eta}\int_x^{x+\eta}u^\delta(t,\xi)d\xi-\frac{c\eta}{2},\\ u_+^{\delta\eta}(t,x) &= \frac{1}{\eta}\int_{x-\eta}^{x}u^\delta(t,\xi)d\xi+\frac{c\eta}{2}.\end{aligned} \tag{38}$$

Let $]a_i,b_i[$ be an open interval in (29) and set

$$\alpha_i:=a_i+\delta-\eta\le x\le b_i-\delta+\eta:=\beta_i.$$

We take again the linear interpolant

$$u_\pm^{\delta\eta}(t,x)=\frac{\beta_i-x}{\beta_i-\alpha_i}u_\pm^{\delta\eta}(t,\alpha_i)+\frac{x-\alpha_i}{\beta_i-\alpha_i}u_\pm^{\delta\eta}(t,\beta_i),\quad x\in]\alpha_i,\beta_i[. \tag{39}$$

Then these two functions have the following properties.

Lemma 3.4: *The functions $u_\pm^{\delta\eta}(t,x)$ are Lipschitz continuous in x, continuous from above in t, and they satisfy the quasi-Oleinik condition.*

Proof: By the definition of $u_\pm^{\delta\eta}(t,x)$, we have

$$\begin{aligned} u_-^{\delta\eta}(t,x) &= cx - \frac{1}{\eta}\int_x^{x+\eta} v^\delta(t,\xi)d\xi =: cx - v_-^{\delta\eta}(t,x), \\ u_+^{\delta\eta}(t,x) &= cx - \frac{1}{\eta}\int_{x-\eta}^{x} v^\delta(t,\xi)d\xi =: cx - v_+^{\delta\eta}(t,x), \end{aligned} \tag{40}$$

for $x \in \Omega^{\delta\eta}(t)$. Since the function v^δ is non-decreasing in x, the functions $v_\pm^{\delta\eta}$ are also non-decreasing in x, i.e. the functions $u_\pm^{\delta\eta}$ satisfy the quasi-Oleinik condition on $\Omega^{\delta\eta}(t)$. Note that the definition in (32) prevents slopes in u^δ from becoming positively infinite. Using (39), we conclude that the functions $u_\pm^{\delta\eta}$ satisfy the quasi-Oleinik condition on the whole of $\mathbb{R}^+ \times \mathbb{R}$.

Note that as in the proof of Lemma 3.3 we may rewrite u^δ to give the estimate

$$|u^\delta(t,x)| \le c\frac{\left|\int_{]x-\delta,x+\delta[}(x-\xi)m(t,d\xi)\right|}{m(t,]x-\delta,x+\delta[)} + \frac{\left|\int_{]x-\delta,x+\delta[}u(t,\xi)m(t,d\xi)\right|}{m(t,]x-\delta,x+\delta[)} \le c\delta + U.$$

This we may use to estimate

$$|u_-^{\delta\eta}(t,x) - u_-^{\delta\eta}(t,y)| = \frac{1}{\eta}\left|\int_x^y u^\delta(t,\xi)d\xi - \int_{x+\eta}^{y+\eta} u^\delta(t,\xi)d\xi\right| \le \frac{2(c\delta+U)}{\eta}|x-y|.$$

A similar argument can be given for $u_+^{\delta\eta}$. So the functions $u_\pm^{\delta\eta}$ are Lipschitz continuous in x.

It is left is to prove that the functions $u_\pm^{\delta\eta}$ are continuous from above in t. For $x \in \Omega^{\delta\eta}(t)$ we have $]x-\eta, x+\eta[\subset \Omega^\delta(t)$. By Lemma 3.3, the set of points at which u^δ is discontinuous in x for a fixed $t > \tau$ is at most countable and so has Lebesgue measure zero. Therefore in the interval $]x-\eta, x+\eta[$

$$\begin{aligned} \lim_{t\to t_0^+} u_-^{\delta\eta}(t,x) = & \lim_{t\to t_0^+}\frac{1}{\eta}\int_x^{x+\eta} u^\delta(t,\xi)d\eta - \frac{c\eta}{2} \\ = & \frac{1}{\eta}\int_x^{x+\eta} u^\delta(t_0,\xi)d\eta - \frac{c\eta}{2}. \end{aligned}$$

This can be extended to $x \in \mathbb{R}$ by (39). The continuity from above of $u_+^{\delta\eta}$ in t is true with an analogous proof. □

By this lemma, it is easy to see that the extended and regularized velocities $u_\pm^{\delta\eta}$ also have the following properties:

Corollary 3.5 **(a)** *We have the bounds*

$$u_-^{\delta\eta}(t,x) \le u^\delta(t,x) \le u_+^{\delta\eta}(t,x); \tag{41}$$

(b) *The values $u_-^{\delta\eta}(t,x)$ are increasing when $\eta \to 0^+$ while for $u_+^{\delta\eta}(t,x)$ they are decreasing, and*

$$\lim_{\eta\to 0^+} u_\pm^{\delta\eta}(t,x) = u^\delta(t,x) \tag{42}$$

at the points of continuity of u^δ;

(c) *Due to the quasi-Oleinik condition the functions $u_\pm^\delta(t,\cdot)$ are either continuous or they jump down with respect to x. From this it follows that for all sequences $(\eta_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ such that*

$\eta_n \to 0^+$ and $x_n \to x_0$ as $n \to \infty$, we have

$$\begin{aligned} \liminf_{n\to\infty} u_-^{\delta\eta_n}(t, x_n) &\geq u^\delta(t, x_0+0), \\ \limsup_{n\to\infty} u_+^{\delta\eta_n}(t, x_n) &\leq u^\delta(t, x_0-0). \end{aligned} \tag{43}$$

□

Lemma 3.6: *For all $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}$, there exists a unique solution*

$$x = x_\pm^{\delta\eta}(t) = x_\pm^{\delta\eta}(t; t_0, x_0) \tag{44}$$

to the equation

$$\frac{dx}{dt} = u_\pm^{\delta\eta}(t, x). \tag{45}$$

This solution has a limit as $\eta \to 0$

$$x^\delta(t) = x^\delta(t; t_0, x_0) = \lim_{\eta\to 0} x_\pm^{\delta\eta}(t; t_0, x_0), \tag{46}$$

which satisfies

$$\begin{aligned} x^\delta(t) &\geq x^\delta(s) + \int_s^t u^\delta(\alpha, x^\delta(\alpha)+0)d\alpha, \\ x^\delta(t) &\leq x^\delta(s) + \int_s^t u^\delta(\alpha, x^\delta(\alpha)-0)d\alpha, \\ |x^\delta(t) &- x^\delta(s)| \leq \|u^\delta\| \cdot |t-s|. \end{aligned} \tag{47}$$

Proof: The existence and uniqueness follows from Lemma 3.4 and Corollary 3.5. Since $u^{\delta\eta}$ is continuous from above in time, the solution (44) is Lipschitz continuous, i.e.

$$|x_\pm^{\delta\eta}(t_1) - x_\pm^{\delta\eta}(t_2)| \leq \|u_\pm^{\delta\eta}\| \cdot |t_1 - t_2|. \tag{48}$$

Making use of (41-43), we conclude that $x_-^{\delta\eta}(t)$ is increasing and $x_+^{\delta\eta}(t)$ is decreasing for $\eta \to 0^+$ and $x_-^{\delta\eta}(t) \leq x_+^{\delta\eta}(t)$. So their limits exist,

$$x_-^\delta(t) := \lim_{\eta\to 0^+} x_-^{\delta\eta}(t) \leq \lim_{\eta\to 0^+} x_+^{\delta\eta}(t) := x_+^\delta(t), \quad \text{for all } t \in [t_0, T].$$

With the definition of solutions, we have

$$x_\pm^{\delta\eta}(t) = x_\pm^{\delta\eta}(s) + \int_s^t u_\pm^{\delta\eta}(\alpha, x_\pm^{\delta\eta}(\alpha))d\alpha. \tag{49}$$

By the Fatou Lemma and (43), we get by letting $\eta \to 0^+$ that

$$\begin{aligned} x_-^\delta(t) &\geq x_-^\delta(s) + \int_s^t u^\delta(\alpha, x_-^\delta(\alpha)+0)d\alpha, \\ x_+^\delta(t) &\leq x_+^\delta(s) + \int_s^t u^\delta(\alpha, x_+^\delta(\alpha)-0)d\alpha, \end{aligned} \tag{50}$$

We use a contradiction argument to prove that $x_+^\delta(t) = x_-^\delta(t)$. Assume that there is a $t_1 \in [t_0, T]$ such that $x_+^\delta(t_1) > x_-^\delta(t_1)$. Set

$$t_2 := \sup\{t \in [t_0, t_1]; x_+^\delta(t) = x_-^\delta(t)\}.$$

Since the functions $x_\pm^\delta(t)$ are continuous in t, $x_+^\delta(t_2) = x_-^\delta(t_2)$. Taking $t = t_1$ and $s = t_2$ in (50) and using Lemma 3.3 giving the quasi-Oleinik condition which implies the one-sided Lipschitz condition for u^δ, we get

$$\begin{aligned} 0 < x_+^\delta(t_1) - x_-^\delta(t_1) \leq \;& x_+^\delta(t_2) - x_-^\delta(t_2) + \int_{t_2}^{t_1} [u^\delta(\alpha, x_+^\delta(\alpha) - 0) - u^\delta(\alpha, x_-^\delta(\alpha) + 0)] d\alpha \\ \leq \;& 0 + \int_{t_2}^{t_1} c \left[x_+^\delta(\alpha) - x_-^\delta(\alpha)\right] d\alpha, \end{aligned}$$

where $c > 0$ is the Lipschitz constant. We now have an integral Gronwall inequality with 0 initial data having only the trivial solution, i.e. $x_+^\delta(t_1) - x_-^\delta(t_1) = 0$. Thereby we obtain the contradiction we sought. □

Theorem 3.7: *For all $(t_0, x_0) \in [\tau, T] \times \mathbb{R}$ we have for $t \in [t_0, T]$ the uniform limit to a Lipschitz continuous function x^0, i.e.*

$$x^0(t) := x^0(t; t_0, x_0) = \lim_{\delta \to 0} x^\delta(t; t_0, x_0). \tag{51}$$

Proof: Choose $\delta_0 > 0$ such that the set of functions x^δ is equi-Lipschitz continuous on $[t_0, T]$ for $\delta \in [0, \delta_0]$, i.e.

$$|x^\delta(t_2) - x^\delta(t_1)| \leq (U + c\delta_0)|t_2 - t_1|, \quad \text{for all } \delta \in [0, \delta_0], \quad t_1, t_2 \in [t_0, T].$$

The compactness by the Arzela-Ascoli theorem of $x^\delta(\cdot)$ leads to the fact that there is a sequence $(\delta_n)_{n \in \mathbb{N}}$ such that $x^{\delta_n}(\cdot)$ has a uniform limit, denoted by $x^0(\cdot)$.

It remains to prove the uniqueness. Assume that there are two sequences $(\delta'_n)_{n' \in \mathbb{N}}$ and $(\delta''_n)_{n'' \in \mathbb{N}}$ converging to 0 such that

$$x_1^0(t) = \lim_{n \to \infty} x^{\delta'_n}(t), \quad x_2^0(t) = \lim_{n \to \infty} x^{\delta''_n}(t).$$

Since the velocity function u satisfies the quasi-Oleinik condition, we have analogously as in Corollary 3.5

$$\liminf_{n \to \infty} u^{\delta_n}(t, x_n) \geq u(t, x_0 + 0), \quad \limsup_{n \to \infty} u^{\delta_n}(t, x_n) \leq u(t, x_0 - 0).$$

for $\delta_n \to 0$ and $x_n \to x_0$. This gives in analogy to (50)

$$\begin{aligned} x_i^0(t) &\geq x_i^0(s) + \int_s^t u(\alpha, s_i^0(\alpha) + 0) d\alpha, \\ x_i^0(t) &\leq x_i^0(s) + \int_s^t u(\alpha, s_i^0(\alpha) - 0) d\alpha. \end{aligned} \tag{52}$$

With a similar contradiction argument as used in the proof of Lemma 3.6, we can prove $x_1^0(t) = x_2^0(t)$ for all $t \in [t_0, T]$. □

Note that the limit function x^0 also satisfies (52).

As pointed out at the beginning of this section, we call $x = x^0(t) = x^0(t; t_0, x_0)$ the **generalized characteristic curve** for all $(t_0, x_0) \in [\tau, T] \times \mathbb{R}$, $t \geq t_0$. Also we continue to use the notation $x = h_{t_0,t}(x_0) := x^0(t; t_0, x_0)$ for the family characteristic of maps. Then we have

Lemma 3.8: *The maps $h_{t_0,t} : \mathbb{R} \to \mathbb{R}$ are non-decreasing and continuous. Furthermore, they satisfy*

$$h_{t_1,t_3}(x) = h_{t_2,t_3}(h_{t_1,t_2}(x)) \tag{53}$$

for all $\tau \leq t_1 \leq t_2 \leq t_3 \leq T$.

Proof: Since the functions $u^{\delta\eta}$ are continuous, the solutions curves to (45) cannot intersect. This leads to the fact that

$$x_{\pm}^{\delta\eta}(t; x_1, t_0) \leq x_{\pm}^{\delta\eta}(t; x_2, t_0), \quad \text{ for all } x_1 \leq x_2, \ \ t \geq t_0 \geq \tau,$$

we obtain that $h_{t_0,t}$ is non-decreasing by letting $\eta \to 0$ and $\delta \to 0$. We can use the contradiction argument as in the proof of Lemma 3.6 to prove the continuity of $h_{t_0,t}$.

We proceed to prove (53). Denote

$$x^0(t) = x^0(t; x_0, t_1), \quad x_1^0(t) = x^0(t; x^0(t_2), t_2), \text{ for all } t \geq t_1, t_2.$$

Then we need to prove that $x^0(t_3) = x_1^0(t_3)$. Assume to the contrary that $x^0(t_3) < x_1^0(t_3)$, we get, as in (52),

$$\begin{aligned} x^0(t) &\geq x^0(s) + \int_s^t u(\alpha, x^0(\alpha) + 0)d\alpha, \\ x_1^0(t) &\leq x_1^0(s) + \int_s^t u(\alpha, x_1^0(\alpha) - 0)d\alpha. \end{aligned}$$

As in the last lemma, we get $x^0(t_3) = x_1^0(t_3)$. $\square$

Theorem 3.9: *For all bounded measurable functions $\phi : \mathbb{R} \to \mathbb{R}$ we have the conservation of mass*

$$\int_{-\infty}^{\infty} \phi(x)m(T, dx) = \int_{-\infty}^{\infty} \phi(h_{\tau,T}(x))m(\tau, dx) \tag{54}$$

for all $0 \leq \tau < T < \infty$.

Proof: First we assume ϕ is non-negative and non-decreasing with continuous and bounded derivatives. By using the solutions given in Lemma 3.6, we construct a test function by pulling ϕ back from time T to τ along the generalized characteristics, i.e.

$$\Phi_{\pm}^{\delta\eta}(\tau, x) := \phi(x_{\pm}^{\delta\eta}(T; x, \tau)). \tag{55}$$

Since Φ is constant along any generalized characteristic, we have

$$0 = \frac{d}{d\tau}\Phi_{\pm}^{\delta\eta} = \frac{\partial}{\partial\tau}\Phi_{\pm}^{\delta\eta} + \frac{\partial}{\partial x}\Phi_{\pm}^{\delta\eta}u_{\pm}^{\delta\eta}. \tag{56}$$

By the quasi-Oleinik property of $u_{\pm}^{\delta\eta}$ and Gronwall's inequality, we get

$$0 \leq \frac{\partial x_{\pm}^{\delta\eta}(T; x, \tau)}{\partial x} \leq e^{c(T-\tau)} \leq +\infty. \tag{57}$$

Substituting $\Phi_\pm^{\delta\eta}$ into (23) and using (56) gives

$$\begin{aligned}\int_{-\infty}^{\infty}\phi(x)m(T,dx)-\int_{-\infty}^{\infty}\phi(x_\pm^{\delta\eta}(T;x,\tau))m(\tau,dx) &= \int_\tau^T\int_{-\infty}^{\infty}\left(\frac{\partial}{\partial t}\Phi_\pm^{\delta\eta}+\frac{\partial}{\partial x}\Phi_\pm^{\delta\eta}u\right)m(t,dx)\\ &= \int_\tau^T\int_{-\infty}^{\infty}\frac{\partial}{\partial x}\Phi_\pm^{\delta\eta}(u-u_\pm^{\delta\eta})m(t,dx)dt.\end{aligned}$$

Remember that after Lemma 2.4 we made the assumption that all velocities are positive. Due to

$$0<u_-^{\delta\eta}\le u^\delta\le u_+^{\delta\eta}$$

by the definition of u^δ and $u_\pm^{\delta\eta}$ as well as the monotonicity of $\Phi_\pm^{\delta\eta}$ it follows that

$$\begin{aligned}\int_{-\infty}^{\infty}\phi(x)m(T,dx)-\int_{-\infty}^{\infty}\phi(x_-^{\delta\eta}(T;x,\tau))m(\tau,dx) &\ge \int_\tau^T\int_{-\infty}^{\infty}\frac{\partial}{\partial x}\Phi_-^{\delta\eta}(u-u^\delta)m(t,dx)dt,\\ \int_{-\infty}^{\infty}\phi(x)m(T,dx)-\int_{-\infty}^{\infty}\phi(x_+^{\delta\eta}(T;x,\tau))m(\tau,dx) &\le \int_\tau^T\int_{-\infty}^{\infty}\frac{\partial}{\partial x}\Phi_+^{\delta\eta}(u-u^\delta)m(t,dx)dt\end{aligned} \tag{58}$$

We introduce the notation

$$A_-^\delta:=\liminf_{\eta\to0}\frac{\partial}{\partial x}\Phi_-^{\delta\eta},\qquad A_+^\delta:=\limsup_{\eta\to0}\frac{\partial}{\partial x}\Phi_+^{\delta\eta}.$$

Then by the Lebesgue Dominated Convergence Theorem, we get from (58)

$$\begin{aligned}\int_{-\infty}^{\infty}\phi(x)m(T,dx)-\int_{-\infty}^{\infty}\phi(x_-^{\delta}(T;x,\tau))m(\tau,dx) &\ge \int_\tau^T\int_{-\infty}^{\infty}A_-^\delta(u-u^\delta)m(t,dx)dt,\\ \int_{-\infty}^{\infty}\phi(x)m(T,dx)-\int_{-\infty}^{\infty}\phi(x_+^{\delta}(T;x,\tau))m(\tau,dx) &\le \int_\tau^T\int_{-\infty}^{\infty}A_+^\delta(u-u^\delta)m(t,dx)dt.\end{aligned} \tag{59}$$

Since $|A_\pm^\delta|\le\|\phi'\|_\infty e^{c(T-\tau)}$ from (57), this shows that

$$\int_{-\infty}^{\infty}\phi(x)m(T,dx)-\int_{-\infty}^{\infty}\phi(x^\delta(T;x,\tau))m(\tau,dx)\le\int_\tau^T\int_{-\infty}^{\infty}\|\phi'\|e^{c(T-\tau)}|u-u^\delta|m(t,dx)dt. \tag{60}$$

We get by the Lebesgue Dominated Convergence Theorem taking the limit $\delta\to0$ that

$$\lim_{\delta\to0}\int_\tau^T\int_{-\infty}^{\infty}|u-u^\delta|m(t,dx)dt=0. \tag{61}$$

Therefore (54) holds for non-negative and non-decreasing functions with continuous derivatives. Since linear combinations of such functions can approximate all bounded and measurable functions, we complete the proof of lemma by such approximation. □

This lemma directly leads to the following conservation of mass and momentum property.

Corollary 3.10: *For all Borel sets $A\in\mathcal{B}$ and $T>\tau$*

$$m(T,A)=m(\tau,h_{\tau,T}^{-1}(A)),\qquad I(T,A)=I(\tau,h_{\tau,T}^{-1}(A)). \tag{62}$$

□

4 The geometric properties of generalized characteristics

In this section, we investigate the geometric properties of generalized characteristics that have been introduced in the last section. There the generalized characteristics were introduced as uniform limits of regularized characteristics in Theorem 3.7. We now show that they are solutions to the characteristic differential equation in a generalized sense. These properties will be needed to show the uniqueness of solutions we are seeking.

Theorem 4.1: *For all $\tau > 0$ and $x_0 \in \mathbb{R}$, the generalized characteristic curve $x = x^0(t) = x^0(t;\tau,x_0)$ satisfies*

$$\frac{dx^0(t)}{dt^+} = u(t, x^0(t)) \tag{63}$$

almost everywhere with respect to $m(t,\cdot)$. The derivative $\frac{d}{dt^+}$ denotes the one-sided derivative from above in time.

Proof: We only have to consider the support of the measure $m(t,\cdot)$, i.e. points $x = x^0(t)$ where $m(t,\cdot)$ does not vanish in an open neighborhood. Otherwise, x would be a point in an open subset of the vacuum set $V(t)$. We will consider the possible cases.

(a) For the motion of a single particle x_0 described by a point mass the conclusion is obvious. The same holds for a finite or countable number of non-interacting point masses.

(b) *Consider $t_1 > \tau$ such that $h^{-1}_{\tau,t_1}(x^0(t_1)) = \{x_0\}$ and $m(\tau, x_0) = 0$.*

Under such circumstances, at least on one side of $x = x^0(t)$ is not in the vacuum state. We only give the proof for the case that on both sides there is not almost everywhere in the sense of Lebesgue measure a vacuum state.

First let $(y_n)_{n\in\mathbb{N}}$ is an increasing sequence of real numbers which converges to $y_0 := x^0(t_1)$ as n tends to infinity. We set

$$x_n := \inf h^{-1}_{\tau,t}(\{y_n\}), \quad x^0_n(t) := x^0(t; x_n, \tau), \quad n = 1, 2, \cdots.$$

By the monotonicity of $h_{\tau,t}$, there is a limit $\overline{x}$ of x_n with $\overline{x} \le x_0$ and

$$x^0(t_1; \overline{x}, \tau) = \lim_{n\to\infty} x^0(t_1; x_n, \tau) = \lim_{y\to n} y_n = y_0 = x^0(t_1; x_0, \tau)$$

due to the continuity of $h_{\tau,t}$. So we have $\overline{x} \in h^{-1}_{\tau,t}(\{x^0(t_1)\}) = \{x_0\}$, i.e. $\overline{x} = x_0$.

Let us take $s \in [\tau, t_1]$, then $h^{-1}_{\tau,s}([x^0_n(s), x^0(s)]) = [x_n, x_0]$. Using (62) this gives

$$m(s, [x^0_n(s), x^0(s)]) = m(\tau, [x_n, x_0]) > 0, \quad I(s, [x^0_n(s), x^0(s)]) = I(\tau, [x_n, x_0]), \quad n = 1, 2, \cdots.$$

It follows that

$$u(s, x^0(s) - 0) = \lim_{n\to\infty} \frac{I(s, [x^0_n(s), x^0(s)])}{m(s, [x^0_n(s), x_0(s)])} = \lim_{n\to\infty} \frac{I(s, [x_n, x_0])}{m(s, [x_n, x_0])} = u(\tau, x_0 - 0). \tag{64}$$

Similar arguments lead to

$$u(s, x^0(s) + 0) = u(\tau, x_0 + 0), \quad \text{for all } s \in [\tau, t_1]. \tag{65}$$

On the other hand, (50) shows that

$$x^0(t) - x^0(s) \le \int_s^t u(\alpha, x^0(\alpha) - 0)d\alpha, \quad x^0_n(t) - x^0_n(s) \ge \int_s^t u(\alpha, x^0_n(\alpha) + 0)d\alpha$$

for all $\tau \le s \le t \le t_1$. Then taking $n \to \infty$, noting that $x_n^0(\alpha)$ is a monotone increasing sequence, and using the Fatou Lemma gives

$$x^0(t) - x^0(s) \ge \int_s^t \liminf_{n\to\infty} u(\alpha, x_n^0(\alpha) + 0) d\alpha \ge \int_s^t u(\alpha, x^0(\alpha) - 0) d\alpha. \tag{66}$$

With similar arguments, we can prove

$$x^0(t) - x^0(s) \le \int_s^t u(\alpha, x^0(\alpha) + 0) d\alpha. \tag{67}$$

Therefore we obtain

$$\int_s^t u(\alpha, x^0(\alpha) - 0) d\alpha \le \int_s^t u(\alpha, x^0(\alpha) + 0) d\alpha. \tag{68}$$

Since $u(t, \cdot)$ satisfies the Oleinik condition, i.e. $u(t, x+0) \le u(t, x-0)$, we get by comparing with (68) that

$$u(s, x^0(s) + 0) = u(s, x^0(s) - 0) = u(s, x^0(s))$$

holds almost everywhere for $s \in [\tau, t_1]$. Therefore using (65) and (66), we obtain (63) for $t \in [\tau, t_1]$. We point out in passing that due to (65) the characteristic curve passing through (τ, x_0) is a straight line.

(c) *Consider the case that $m(\tau, \{x_0\}) > 0$, but we do not have an isolated point mass surrounded by vacuum.*

Since $x^0(t)$ is Lipschitz continuous, $\dfrac{dx^0(t)}{dt}$ exists almost everywhere. Denote by $t_0 > \tau$ a point where the derivative $\dfrac{dx^0(t)}{dt}$ exists. We use a contradiction argument to prove that (63) is true there, i.e.

$$u_1 := \frac{dx^0(t_0)}{dt} = u(t_0, x^0(t_0)) =: u_0. \tag{69}$$

Assume on the contrary that without loss of generality $u_1 < u_0$. Take $u_2 \in]u_1, u_0[$, $t_1 > t_0$ and set $y_0 := x^0(t_0)$, $y_1 := x^0(t_1)$ as well as $y_2 := y_0 + (t_1 - t_0)u_2$. Furthermore take the continuous function ϕ given by

$$\phi(x) = \begin{cases} 0, & x \le y_1, \\ 1, & x \ge y_2 \\ \text{linear} & x \in [y_1, y_2], \end{cases}$$

i.e. $\phi'(x) = 1/(y_2 - y_1)$ for $x \in]y_1, y_2[$. Now we consider the test function

$$\Phi(t, x) = \phi(x + (t_1 - t)u_2), \qquad \text{for all } (t, x) \in [t_0, t_1] \times \mathbb{R}.$$

Substituting this into (23) and using (62) gives

$$\begin{aligned}
\int_{t_0}^{t_1} \int_{-\infty}^{\infty} \phi'(x + (t_1 - t)u_2)[u(t, x) - u_2] m(t, dx) dt &= \int_{-\infty}^{\infty} \phi(x) m(t_1, dx) \\
&\quad - \int_{-\infty}^{\infty} \phi(x + (t_1 - t_0)u_2) m(t_0, dx) \\
&\le m(t_1,]y_1, \infty[) - m(t_0, [y_0, \infty[) \qquad (70) \\
&\le m(t_0,]y_0, \infty[) - m(t_0, [y_0, \infty[) \\
&= -m(t_0, \{y_0\}) = -m(\tau, \{x_0\}) < 0.
\end{aligned}$$

Since $y_0 = x^0(t_0; x_0, \tau)$, it is easily checked that

$$y_0 = \bigcap_{t>t_0} h_{t_0,t}^{-1}(\{x^0(t)\}) = \lim_{t\to t_0^+} h_{t_0,t}^{-1}(\{x^0(t)\}).$$

It follows that

$$m(t_0, y_0) = \lim_{t\to t_0^+} m(t_0, h_{t_0,t}^{-1}(x^0(t))) = \lim_{t\to t_0^+} m(t, x^0(t)). \tag{71}$$

Similarly, we get

$$I(t_0, y_0) = \lim_{t\to t_0^+} I(t, x^0(t)). \tag{72}$$

So we have

$$u(t_0, x_0) = \frac{I(t_0, y_0)}{m(t_0, y_0)} = \lim_{t\to t_0^+} \frac{I(t, x^0(t))}{m(t, x^0(t))} = \lim_{t\to t_0^+} u(t, x^0(t)) = u_0. \tag{73}$$

The limits are taken from the above. When t_1 is sufficiently close to t_0 we have

$$y_1 \le x^0(t) + (t_1 - t)u_2 \le y_2$$

for all $t \in [t_0, t_1]$. Then

$$\phi'(x^0(t) + (t_1 - t)u_2) = \frac{1}{y_2 - y_1} = \frac{1}{t_1 - t_0}\left(u_2 - \frac{y_1 - y_0}{t_1 - t_0}\right)^{-1}. \tag{74}$$

Now we turn to the left-hand side of (70), which is divided into two parts, namely the singular and the regular part. The first is the singular part

$$\begin{aligned} I_1 &= \int_{t_0}^{t_1} \int_{x=x^0(t)} \phi'(x + (t_1 - t)u_2)[u(t,x) - u_2]m(t,dx)dt \\ &= \int_{t_0}^{t_1} \phi'(x^0(t) + (t_1 - t)u_2)[u(t, x^0(t)) - u_2]m(t, x^0(t))dt \\ &= \frac{1}{t_1 - t_0}\left(u_2 - \frac{y_1 - y_0}{t_1 - t_0}\right)^{-1} \int_{t_0}^{t_1} [u(t, x^0(t)) - u_2]m(t, x^0(t))dt \\ &\to (u_2 - u_1)^{-1}(u_0 - u_2)m(t_0, y_0) > 0 \end{aligned} \tag{75}$$

as $t_1 \to t_0^+$.

We turn to the regular part and set $A_t := [y_1 - (t_1 - t)u_2, y_2 - (t_1 - t)u_2]$. Then

$$m(t, A_t - \{x^0(t)\}) \le m(t_0, [y_0 - 2(t_1 - t_0)U, y_0 + 2(t_1 - t_0)U] - \{y_0\}) \to m(t_0, \emptyset) = 0$$

as $t_1 \to t_0^+$. Thus

$$\begin{aligned} I_2 &= \int_{t_0}^{t_1} \int_{A_t - \{x^0(t)\}} \phi'(x + (t_1 - t)u_2)[u(t,x) - u_2]m(t,dx)dt \\ &\le \frac{1}{t_1 - t_0}\left(u_2 - \frac{y_1 - y_0}{t_1 - t_0}\right)^{-1} \int_{t_0}^{t_1} 2Um(t, A_t - \{x^0(t)\})dt \to 0 \end{aligned} \tag{76}$$

as $t_1 \to t_0^+$, where the Lebesgue Dominated Convergence Theorem is used. The summation of (75) and (76) converges to $(u_2 - u_1)^{-1}(u_0 - u_2)m(t, y_0) > 0$, which contradicts (70).

(d) *Consider the case that $m(\tau,\{x_0\}) = 0$ while x_0 is a point of continuity of $u(\tau,\cdot)$ and $h^{-1}_{\tau,t}(\{x^0(t)\})$ contains an open interval.*

We set

$$\Xi_t := h^{-1}_{\tau,t}(\{x^0(t)\}),$$

which is a closed set and by assumption $m(\tau,\Xi_t) > 0$. Therefore $m(t,x(t)) > 0$ by (62). It follows that

$$\frac{dx^0(t)}{dt} = u(t,x^0(t)), \qquad \text{for all } t > \tau.$$

Since x_0 is a point of continuity for $u(\tau,\cdot)$, $\cap_{t>\tau}\Xi_t = \{x_0\}$. Then we have

$$\begin{aligned} u(t,x^0(t)) = \;& \frac{I(t,x^0(t))}{m(t,x^0(t))} = \frac{I(t,\Xi_t))}{m(t,\Xi_t)} \\ = \;& \frac{\int_{\Xi_t} u(t,\xi)m(t,d\xi)}{\int_{\Xi_t} m(t,d\xi)} \to u(\tau,x_0), \qquad \text{as } t \to \tau. \end{aligned}$$

Therefore we conclude

$$\frac{dx^0(t)}{dt^+}|_{t=\tau} = \lim_{t\to\tau}\frac{x^0(t)-x_0}{t-\tau} = \lim_{t\to\tau}\frac{1}{t-\tau}\int_\tau^t u(s,x^0(s))ds = u(\tau,x_0).$$

(e) *Consider the case that $m(\tau,\{x_0\}) = 0$ while x_0 is a point of discontinuity of $u(\tau,\cdot)$.*

Since the set of these points is at most countable, it has Lebesgue measure zero. Therefore the proof of this theorem is complete. □

Next we will discuss the center of mass. We denote the first **moment of mass** by

$$K(t,A) := \int_A \xi\, m(t,d\xi), \qquad t \ge 0, \quad A \in \mathcal{B}. \tag{77}$$

Then the **center of mass** of a set $A \in \mathcal{B}$ with $m(t,A) > 0$ is defined as

$$M(t,A) := \frac{K(t,A)}{m(t,A)}. \tag{78}$$

Consider a point x with the properties that $m(t,\{x\}) = 0$ but the measure does not vanish in any open neigborhood of x, i.e. x does not lie in the the vacuum set $V(t)$. Then we can still define the center of mass for the one point set $\{x\} = A$. Note that by the Lebesgue Theorem one has for a Lebesgue point x that $M(t,A) = x$ by taking the limit of the center of mass on open neighborhoods of x. This holds almost everywhere in the sense of Lebesgue measure. We may assume that this holds everywhere in the complement of the vacuum set.

Lemma 4.2: *For all $A \in \mathcal{B}$, if $m(t,A) > 0$ and $t \ge \tau$, then*

$$M(t,A) = \frac{\int_{h^{-1}_{\tau,t}(A)}(\xi + (t-\tau)u(\tau,\xi))m(\tau,d\xi)}{\int_{h^{-1}_{\tau,t}(A)} m(\tau,d\xi)}. \tag{79}$$

Proof: First consider the case that $A = [y_1,y_2]$ is a closed interval. Let us set

$$[a(s),b(s)] := h^{-1}_{s,t}([y_1,y_2]), \quad k(s) := \int_{[a(s),b(s)]} \xi m(s,d\xi), \quad \text{for all } s \in [\tau,t].$$

Then

$$\begin{aligned}
\frac{k(s+\Delta s)-k(s)}{\Delta s} &= \frac{1}{\Delta s}\left\{\int_{[a(s+\Delta s),b(s+\Delta s)]}\xi m(s+\Delta s,d\xi)-\int_{[a(s),b(s)]}\xi m(s,d\xi)\right\}\\
&= \frac{1}{\Delta s}\left\{\int_{[a(s),b(s)]}x^0(s+\Delta s;\xi,s)m(s,d\xi)-\int_{[a(s),b(s)]}\xi m(s,d\xi)\right\}\\
&= \int_{[a(s),b(s)]}\frac{x^0(s+\Delta s;\xi,s)-x^0(s;\xi,s)}{\Delta s}m(s,d\xi).
\end{aligned}$$

Note the Lipschitz continuity of $x^0(t;\xi,s)$. Using Theorem 4.1, the Lebesgue Dominated Convergence Theorem as well as Corollary 3.10, we get

$$\frac{dk(s)}{ds^+}=\int_{[a(s),b(s)]}u(s,x^0(s;\xi,s))m(s,d\xi)=I(s,[a(s),b(s)])=I(\tau,[a(\tau),b(\tau)]).$$

It follows that

$$k(s)=k(\tau)+(s-\tau)I(\tau,[a(\tau),b(\tau)]),$$

i.e. for $s=t$

$$K(t,[y_1,y_2])=K(\tau,[a(\tau),b(\tau)])+(t-\tau)I(\tau,[a(\tau),b(\tau)])=\int_{[a(\tau),b(\tau)]}(\xi+(t-\tau)u(\tau,\xi))m(\tau,d\xi).$$

Together with (78) this gives (79). Then it is easy to extend the result to any $A\in\mathcal{B}$. $\square$

The previous lemma gives the formula for the shift of the center of mass of a set under the flow map due to the characteristic velocity. We introduce for the **shifted center of mass** of any Borel set $B\in\mathcal{B}$ the notation

$$C(B;\tau,t):=\frac{\int_B[\xi+(t-\tau)u(\tau,\xi)]m(\tau,d\xi)}{\int_B m(\tau,d\xi)}. \tag{80}$$

This quantity tells us where the center of mass of the set B considered at time τ would be at the later time t, if the generalized characteristics originating in B do not interact with others that start outside B at time τ. Note that the set B here is not necessarily an image under the map $h_{\tau,t}^{-1}$ as required for the validity of formula (79). In view of Lemma 4.2, we have

$$M(t,A)=C(h_{\tau,t}^{-1}(A);\tau,t). \tag{81}$$

We also introduce the **average velocity** on a set $A\in\mathcal{B}$ with $m(t,A)>0$ as

$$\bar{u}(t,A):=\frac{\int_A u(t,\xi)m(t,d\xi)}{m(t,A)}=\frac{d}{dt}C(A;\tau,t) \tag{82}$$

and again extend the definition to point sets $A=\{x\}$ when x is not in the vacuum set $V(t)$ by taking the appropriate limit. This leads to the formulae

$$C(A;\tau,t)=M(\tau,A)+(t-\tau)\bar{u}(\tau,A)$$

and for $t_0>t$

$$C(A;\tau,t_0)=M(\tau,A)+(t_0-\tau)\bar{u}(\tau,A)=C(A;,\tau,t)+(t_0-t)\bar{u}(\tau,A). \tag{83}$$

Let $0 < \tau < T < \infty$ and $y \in \mathbb{R}$, then we set $[a, b] := h_{\tau,T}^{-1}(y)$. We will be mostly interested in the case $a < b$, which means that generalized characteristics are merging and mass is being accumulated in y. Taking $x_0 \in [a, b]$, we consider the generalized characteristic $x^0(t) = x^0(t; x_0, \tau)$. All generalized characteristics originating in the interval $[a, b]$ must pass through y at time T, i.e. $x^0(T) = y$. We set

$$[c(s,t), d(s,t))] := h_{s,t}^{-1}(x^0(t)).$$

For the case $s = \tau$ we define

$$[c(t), d(t)] := [c(\tau, t), d(\tau, t)] = h_{\tau,t}^{-1}(x^0(t))$$

for $\tau \le s < t \le T$. By Lemma 3.8 we get that c is a non-increasing function of t, because increasing t does not make the interval $[c(t), d(t)]$ smaller. We have $\lim_{t\to s^+} c(t) = c(s)$ and set $c(t^-) := \lim\limits_{s\to t^-} c(s)$. Then it is obvious that $c(t^-) \ge c(t)$.

We regard c as a mapping $c : [\tau, T] \to [a, x_0]$. Then we can define a measure $m_t(\cdot)$, depending on x_0, such that $m_t([\tau, t]) = m(\tau, [c(t), x_0])$ for all $t \in [\tau, T]$. Also we construct a velocity function

$$v(t) := \begin{cases} u(\tau, c(t)), & \text{if} \quad c(t) = c(t^-), \\ \dfrac{I(\tau, [c(t), c(t^-)[)}{m(\tau, [c(t), c(t^-)[)}, & \text{if} \quad c(t) < c(t^-) \quad \text{and} \quad m(\tau, [c(t), c(t^-)[) > 0. \end{cases} \tag{84}$$

We remark that $v(t)$ makes no sense when $m(\tau, [c(t), c(t^-)[) = 0$. However, all the points at which $c(t) < c(t^-)$ are at most countable and have measure zero. So $v(t)$ is well-defined almost everywhere with respect to $m_t(\cdot)$.

Lemma 4.3: *Let $0 < \tau < T < \infty$, $y \in \mathbb{R}$, $[a, b] = h_{\tau,T}^{-1}(y)$. Then for any $x_0 \in [a, b]$ the measure $m_t(\cdot)$ in time is equivalent to the measure $m(t, \cdot)$ in space in the following sense*

$$\int_{[a,x_0]} [\xi + (T - \tau)u(\tau, \xi)]m(\tau, d\xi) = \int_{[\tau,T]} [x^0(t) + (T - t)v(t)]m_t(dt). \tag{85}$$

Proof: In order to prove the identity (85), we divide the right-hand side into two parts.

We have the case $c(t) < c(t^-)$, which implies that

$$\begin{aligned} C([c(t), c(t^-)]; \tau, t) \cdot m(\tau, [c(t), c(t^-)]) &= \int_{[c(t),c(t^-)]} [\xi + (t - \tau)u(\tau, \xi)]m(\tau, d\xi) \\ &= x^0(t)m(\tau, [c(t), c(t^-)]). \end{aligned} \tag{86}$$

Using (84), we get

$$I(\tau, [c(t), c(t^-)]) = \int_{[c(t),c(t^-)]} u(\tau, \xi)m(\tau, d\xi) = v(t)m(\tau, [c(t), c(t^-)]). \tag{87}$$

Adding (86) to (87) multiplied by $T - t$ gives for all points of discontinuity of the map c

$$\int_{[c(t),c(t^-)]} [\xi + (T - \tau)u(\tau, \xi)]m(\tau, d\xi) = \int_{\{t\}} [x^0(t) + (T - t)v(t))m_t(dt). \tag{88}$$

The measure m_t is singular at these points.

Since c is a non-increasing function of t, all the singular points at which $c(t) < c(t^-)$ are at most countable. We denote them by $t_1, t_2, \cdots$ and set

$$J = [\tau, T] \backslash \{t_1, t_2, \cdots\}, \qquad J' = [a, x_0] \backslash \bigcup_{i \in \mathbb{N}} [c(t_i), c(t_i^-)].$$

We now turn to the regular part on J. The restricted mapping $c : J \to J'$ is one-to-one. See part (b) in the proof of Theorem 4.1 where we showed that the characteristics are straight lines in this case. Therefore, from the definition of the map c and (84), we have

$$\begin{aligned} x^0(t) + (T-t)v(t) &= c(t) + (t-\tau)u(\tau, c(t)) + (T-t)u(\tau, c(t)) \\ &= c(t) + (T-\tau)u(\tau, c(t)). \end{aligned}$$

This means that for $\xi = c(t)$ we obtain

$$\int_{J'} [\xi + (T-\tau)u(\tau, \xi)]\, m(\tau, d\xi) = \int_J \left[x^0(t) + (T-t)v(t)\right] m_t(dt). \tag{89}$$

This, together with (88), gives (85). □

Lemma 4.4: *Let $0 < \tau < T < \infty$, $y \in \mathbb{R}$, $[a, b] = h_{\tau,T}^{-1}(y)$. Then*

$$C([a, x_0]; \tau, T) \ \geq y \ \geq \ C([x_0, b]; \tau, T) \tag{90}$$

for all $x_0 \in [a, b]$. The possibility that the strict inequality holds comes from the fact that the sets $[a, x_0]$ and $[x_0, b]$ are not images under the map $h_{\tau,t}^{-1}$ unless $a = x_0 = b$.

Proof: The case $a = x_0 = b$ is trivial. We must consider in detail the case $a < b$, where mass has been accumulated in y. Without loss of generality we assume that $a < x_0 < b$, i.e. mass is accumulated from both sides of the generalized characteristic originating at x_0. We continue to use the notations introduced before Lemma 4.3. We give the proof in detail for the interval $[c(t), x_0]$. We do not explicitly carry out all steps for $[x_0, d(t)]$ that follow analogously. First we claim that

$$C([c(t), c(t^-)]; \tau, t) = x^0(t). \tag{91}$$

Indeed, by the definition of $c(t)$ and (81), we have for $s \in [\tau, t]$ that

$$C([c(t), c(s)]; \tau, s) = M(s, [c(s,t), c(s,s)]).$$

Letting $s \to t^-$ from the below and using the continuity of $c(s,t)$ in s, we arrive at

$$C([c(t), c(t^-)]; \tau, t) \in [c(t,t), x^0(t)] = [x^0(t), x^0(t)].$$

This is just (91).

We turn to the proof of (90). First we prove it in three steps under the following two assumptions:

(a) $u(\tau, x_0^-) > u(\tau, x_0) > u(\tau, x_0^+)$;
(b) $m(\tau, [c(t), x_0[) > 0$ and $m(\tau,]x_0, d(t)]) > 0$ for all $t \in]\tau, T[$.

Afterwards we will consider the general case.

1^{st} **step**. We claim that there exists a $\delta > 0$ such that for all $t \in [\tau, \tau + \delta]$,

$$C([c(t), x_0]; \tau, t) - x^0(t) > 0. \tag{92}$$

Due to (85), the left-hand side of (92) can be expressed as

$$\frac{\int_{[\tau,t]} \left[x^0(s) + (t-s)v(s)\right] m_t(ds)}{\int_{[\tau,t]} m_t(ds)} - x^0(t) = \int_{[\tau,t]} (t-s) \left[v(s) - \frac{x^0(t) - x^0(s)}{t-s}\right] m_t(ds) \cdot m_t([\tau,t])^{-1}. \tag{93}$$

The definition (84) of $v(t)$ gives

$$\lim_{t \to \tau^+} v(t) = u(\tau, x_0 - 0).$$

Also, from Theorem 4.1, we have

$$\lim_{t \to \tau^+} \frac{x^0(t) - x^0(s)}{t-s} = \left.\frac{dx^0(t)}{dt^+}\right|_{t=\tau} = u(\tau, x_0), \quad s \in [\tau, t].$$

Based on the first assumption under consideration, (93) is positive for t sufficiently small. So the same holds for (92). An analogous inequality holds for the set $[x_0, d(t)]$.

2^{nd} **step**. We set

$$P := \{t \in [\tau, T] : C([c(t), x_0]; \tau, t) \;\geq\; x^0(t) \;=\; C(\{x^0(t)\}; \tau, t) \;\geq\; C([x_0, d(t)]; \tau, t)\}.$$

In view of the first step, P is non-empty. We prove

$$t_0 := \sup P \in P$$

by contradiction. Assume that $t_0 \notin P$. Then there is an increasing sequence $(s_n)_{n \in \mathbb{N}} \subset P$ such that $s_n \to t_0$ as $n \to \infty$. Using the second assumption we have

$$x^0(s_n) \leq C([c(s_n), x_0]; \tau, s_n) = \frac{\int_{[c(s_n), x_0]} [\xi + (s_n - \tau) u(\tau, \xi)]\, m(\tau, d\xi)}{m(\tau, [c(s_n), x_0])}.$$

Letting $n \to \infty$, we get

$$x^0(t_0) \leq \frac{\int_{[c(t_0^-), x_0]} [\xi + (t_0 - \tau) u(\tau, \xi)]\, m(\tau, d\xi)}{m(\tau, [c(t_0^-), x_0])} = C([c(t_0^-), x_0]; \tau, t_0) =: C_1. \tag{94}$$

Note that by (91)

$$x^0(t_0) = C([c(t_0), c(t_0^-)], \tau, t_0) = C([c(t_0), x_0] \backslash [c(t_0^-), x_0]; \tau, t_0) =: C_2. \tag{95}$$

Therefore, we have using the mass ratios

$$\lambda_1 = \frac{m(\tau, [c(t_0^-), x_0])}{m(\tau, [c(t_0), x_0])} \quad \text{and} \quad \lambda_2 = \frac{m(\tau, [c(t_0), x_0] \backslash [c(t_0^-), x_0])}{m(\tau, [c(t_0), x_0])}$$

that

$$C([c(t_0), x_0]; \tau, t_0) = \lambda_1 C_1 + \lambda_2 C_2 \geq x^0(t_0).$$

Note that $\lambda_1, \lambda_2 \geq 0$ and $\lambda_1 + \lambda_2 = 1$. This contradicts the part of the assumption concerning $c(t)$. The proof for the part concerning $d(t)$ is analogous and this shows that $t_0 \in P$.

3^{rd} **step**. We want to prove that $T = t_0 \in P$, i.e. (90) holds. We again use a contradiction argument. Assume that $T > t_0$, then for $t \in [\tau, t_0]$ we have by (83)

$$C([c(t), c(t_0)[; \tau, t_0) = M(\tau, [c(t), c(t_0)[) + (t_0 - \tau)\bar{u}(\tau, [c(t), c(t_0)[), \tag{96}$$

$$C([c(t_0), x_0]; \tau, t_0) = M(\tau, [c(t_0), x_0]) + (t_0 - \tau)\bar{u}(\tau, [c(t_0), x_0]), \tag{97}$$

and

$$C(]x_0, d(t_0)]; \tau, t_0) = M(\tau,]x_0, d(t_0)]) + (t_0 - \tau)\bar{u}(\tau,]x_0, d(t_0)]). \tag{98}$$

By assumption (b) it is obvious that the center of mass satisfies

$$M(\tau, [c(t_0), x_0]) < x_0 < M(\tau, [x_0, d(t_0)])$$

Since $t_0 \in P$, we get by the definition of P and (97-98) that

$$\begin{aligned} \bar{u}(\tau, [c(t_0), x_0]) &= \frac{C([c(t_0), x_0]; \tau, t_0) - M(\tau, [c(t_0), x_0])}{t_0 - \tau} \\ &> \frac{C([x_0, d(t_0)]; \tau, t_0) - M(\tau, [x_0, d(t_0)])}{t_0 - \tau} = \bar{u}(\tau, [x_0, d(t_0)]). \end{aligned}$$

Since we are assuming mass accumulation, we have

$$\bar{u}(\tau, [c(t_0), x_0]) > u(t_0, x^0(t_0)) > \bar{u}(\tau, [x_0, d(t_0)]).$$

We must discuss two cases concerning $[c(t), x_0]$. The conclusion for $[x_0, d(t)]$ is obtained analogously.

(i) The left-hand side of $c(t_0)$ is in the vacuum state. For this case, as $t > t_0$ is very close to t_0,

$$x^0(t) = x^0(t_0) + (t - t_0)u(t_0, x^0(t_0)).$$

With this, we have using (97) and $t_0 \in P$

$$\begin{aligned} C([c(t), x_0]; \tau, t) &= C([c(t_0), x_0]; \tau, t) = C([c(t_0), x_0]; \tau, t_0) + (t - t_0)\bar{u}(\tau, [c(t_0), x_0]) \\ &\geq x^0(t_0) + (t - t_0)u(t_0, x^0(t_0)) = x^0(t). \end{aligned}$$

This would lead to $t \in P$ and a contradiction to $t > \sup P = t_0$.

(ii) The left-hand side of $c(t_0)$ is not in the vacuum state. Now we have by Lemma 4.3

$$\begin{aligned} C([c(t), c(t_0)[; \tau, t) - x^0(t) &= \int_{[c(t), c(t_0)[} [\xi + (t - \tau)u(\tau, \xi)]m(\tau, d\xi) \cdot m(\tau, [c(\tau), c(t_0)[)^{-1} - x^0(t) \\ &= \int_{]t_0, t]} [x^0(s) + (t - s)v(s)]m_t(ds) \cdot m_t(]t_0, t])^{-1} - x^0(t) \\ &= \int_{]t_0, t]} (t - s)\left[v(s) - \frac{x^0(t) - x^0(s)}{t - s}\right] m_t(ds) \cdot m_t(]t_0, t])^{-1}. \end{aligned}$$

Since

$$\lim_{t \to t_0} \left[v(s) - \frac{x^0(t) - x^0(s)}{t - s}\right] = u(t_0, x^0(t_0) - 0) - u(t_0, x^0(t_0)) > 0, \qquad s \in]t_0, t],$$

we get

$$C([c(t),c(t_0)[;\tau,t)-x^0(t)>0 \tag{99}$$

when $t>t_0$ is very close to t_0. On the other hand, by (96) we have

$$\begin{aligned} C([c(t_0),x_0];\tau,t)-x^0(t) = &\ C([c(t_0),x_0];\tau,t_0)+(t-t_0)\bar{u}(\tau,[c(t_0),x_0])-x^0(t) \\ \geq &\ (t-t_0)\left[\bar{u}(\tau,[c(t_0),x_0])-\frac{x^0(t)-x^0(t_0)}{t-t_0}\right]. \end{aligned}$$

Because of

$$\lim_{t\to t_0}\left[\bar{u}(\tau,[c(t_0),x_0])-\frac{x^0(t)-x^0(t_0)}{t-t_0}\right]=\bar{u}(\tau,[c(t_0),x_0])-u(t_0,x^0(t_0))>0,$$

we conclude that when $t>t_0$ is very close to t_0,

$$C([c(t_0),x_0];\tau,t)-x^0(t)>0.$$

This together with (99) again gives the $c(t)$ part of showing $t\in P$, which leads to a contradiction. Thus we have completed the proof of (90) under the restrictive assumptions.

We turn to the proof of the general case. We observe that there must exist a time $t_1\in[\tau,T]$ such that the above restrictive assumptions hold substituting $\tau=t_1$ and $x_0=x^0(t_1)$. Otherwise, there would never be accumulation of mass into the characteristic starting at x_0 at time τ from either side. Then we would be in the trivial case $a=x_0=b$. Therefore, we have

$$C([c(t_1,T),x^0(t_1)];t_1,T)\geq y, \qquad C([x^0(t_1),d(t_1,T)];t_1,T)\leq y. \tag{100}$$

By Lemma 3.8, we have $[a,x_0]=h^{-1}_{\tau,t_1}([c(t_1,T),x^0(t_1)])$. Using Corollary 3.10, the property of conservation of mass and momentum, and Lemma 4.2, we get

$$\begin{aligned} C([c(t_1,T),x^0(t_1)];t_1,T) &= \frac{\int_{[c(t_1,T),x^0(t_1)]}(\xi+(T-t_1)u(t_1,\xi))m(t_1,d\xi)}{\int_{[c(t_1,T),x^0(t_1)]}m(t_1,d\xi)} \\ &= \frac{\int_{[c(t_1,T),x^0(t_1)]}\xi m(t_1,d\xi)+(T-t_1)\int_{[c(t_1,T),x^0(t_1)]}I(t_1,d\xi)}{\int_{[a,x_0]}m(\tau,d\xi)} \\ &= \frac{\int_{[a,x_0]}(\xi+(t_1-\tau)u(\tau,\xi))m(\tau,d\xi)+(T-t_1)\int_{[a,x_0]}I(\tau,d\xi)}{\int_{[a,x_0]}m(\tau,d\xi)} \\ &= \frac{\int_{[a,x_0]}(\xi+(T-\tau)u(\tau,\xi))m(\tau,d\xi)}{\int_{[a,x_0]}m(\tau,d\xi)} \\ &= C([a,x_0];\tau,T). \end{aligned}$$

This, together with (100), gives (90).

Note that if $t_1=T$, then the equality signs may be taken in (100) because all mass accumulates in y. If $t_1=\tau$, this is exactly what we had proved above. Thus we have completed the proof of (90). □

Lemma 4.5: *For all $T>\tau>0$, $y\in\mathbb{R}$, let*

$$[a(t),b(t)]=h^{-1}_{t,T}(\{y\}), \qquad \textit{for all } t\in[\tau,T].$$

Then $a(t)$ and $b(t)$ have a convexity property, namely $\dfrac{da(t)}{dt^+}$ is decreasing and $\dfrac{db(t)}{dt^+}$ is increasing as t increases.

Proof: Take $t_0 \in [\tau, T]$, set $u_0 := u(t_0, a(t_0))$ and define

$$y = y(t) := a(t_0) + (t - t_0)u_0, \qquad \text{for all } t \in [\tau, T],$$

which is the tangent line to the curve $a(t)$ at $t = t_0$. We need to prove that $a(t) \leq y(t)$, for all $t \in [\tau, T]$. Without loss of generality, it suffices to prove $a(\tau) \leq y(\tau)$ and $a(T) \leq y(T)$.

By (81) we have for $[a(\tau), b] = h^{-1}_{\tau,t_0}(\{a(t_0)\})$ that

$$a(t_0) = M(t_0, \{a(t_0)\}) = C([a(\tau), b]; \tau, t_0).$$

We set

$$x_0 := \frac{\int_{[a(\tau),b]} \xi m(\tau, d\xi)}{m(\tau, [a(\tau), b])}$$

for the center of mass. Hence we have by using Corollary 3.10 for the momentum

$$a(t_0) = \frac{\int_{[a(\tau),b]} [\xi + (t_0 - \tau)u(\tau, \xi)] m(\tau, d\xi)}{m(\tau, [a(\tau), b])} = x_0 + (t_0 - \tau)u_0.$$

Then

$$y(\tau) = x_0 + (t_0 - \tau)u_0 + (\tau - t_0)u_0 = x_0 \in [a(\tau), b].$$

Therefore $a(\tau) \leq y(\tau)$.

On the other hand, we get from Lemma 4.4 that

$$y \leq C([a(t_0), a(t_0)]; t_0, T) = a(t_0) + (T - t_0)u(t_0, a(t_0)) = y(T).$$

Then $a(T) = y \leq y(T)$. □

5 The proof of uniqueness theorem

We finish the proof of the uniqueness theorem by following a series of lemmas. We set

$$[a(t), b(t)] := h^{-1}_{t,T}(\{y\}), \qquad \text{for all } t \in]0, T] \tag{101}$$

for all $T > 0$ and all $y \in \mathbb{R}$ not in the vacuum set, i.e. $y \notin V(T)$.

Lemma 5.1: *Let $x = a(t)$ and $x = b(t)$ for $t \in]0, T]$ be the two generalized characteristics given by (101). Then the following limits exist*

$$\begin{aligned} &\lim_{t\to 0} a(t) =: a(0) =: a, \qquad &&\lim_{t\to 0} b(t) =: b(0) =: b; \\ &\lim_{t\to 0} \frac{da(t)}{dt^+} =: a'(0) =: a', \quad &&\lim_{t\to 0} \frac{db(t)}{dt^+} =: b'(0) =: b'. \end{aligned} \tag{102}$$

Proof: By the Lipschitz continuity of $a(t)$ and $b(t)$ we know that the first two limits exist. By Lemma 4.5, we conclude that the last two limits exist. □

Lemma 5.2: *Using the notation in Lemma 5.1 let $a < x_0 < b$, $m_0(\{a\}) = 0$ and $m_0(\{x_0\}) = 0$. Then we have*

$$\lim_{t\to 0^+} m(t,[a(t),x^0(t;x_0)]) = m_0([a,x_0]). \tag{103}$$

Proof: This lemma is obtained by noting that $m(t,\cdot)$ converges to the initial measure $m_0(\cdot)$ in the sense of measures. □

It is in the following lemma that we will use the cohesion condition (10).

Lemma 5.3: *Using the notation in Lemma 5.1 let $a < x_0$, $m_0(\{a\}) > 0$ and $m_0(\{x_0\}) = 0$. Then*

$$\lim_{t\to 0^+} m(t,[a(t),x^0(t;x_0)]) = \begin{cases} m_0([a,x_0]), & a' \le u_0(a), \\ m_0(]a,x_0]) & a' > u_0(a). \end{cases} \tag{104}$$

Proof: First we note that

$$m_0(]a,x_0]) \le \liminf_{t\to 0^+} m(t,[a(t),x^0(t,x_0)]) \le \limsup_{t\to 0^+} m(t,[a(t),x^0(t,x_0)]) \le m_0([a,x_0]). \tag{105}$$

Then we prove this lemma by considering two cases.

(i) $a' > u_0(a)$. Choose u_1 such that $u_0 := u_0(a) < u_1 < a'$. As $t > 0$ is small enough, we have

$$a(t) = a + \int_0^t \frac{da(t)}{dt^+} dt. \tag{106}$$

Using (102) and the choice of u_1 above, we get

$$a + u_1 t < a(t).$$

Hence

$$m(t,[a(t),x^0(t,x_0)]) \le m(t,[a+u_1t, a+Ut]) + m(t,]a+Ut,x_0]). \tag{107}$$

We have a point mass particle initially sitting at a. The cohesion condition (10) prevents the particle from splitting into parts with velocities differing from $u_0(a)$. So no contribution from this particle may enter the set $[a+u_1t, a+Ut]$. This gives

$$\lim_{t\to 0^+} m(t,[a+u_1t,a+Ut]) = 0. \tag{108}$$

While

$$m(t,]a+Ut,x_0]) \to m_0(]a,x_0]), \quad \text{as } t \to 0. \tag{109}$$

The statements (107), (108) and (109) together give

$$\limsup_{t\to 0} m(t,[a(t),x^0(t,x_0)]) \le m_0(]a,x_0]). \tag{110}$$

So we arrive at (104) by comparing (110) with (105).

(ii) $a' \le u_0(a)$. If the left-hand side is all in the vacuum state, take $u_1 < a'$. In this case (106) and (107) still hold. Note that we have $]a+u_1t, a(t)[\subset V(t)$. Then for t sufficiently small

$$m(t,[a(t),x^0(t,x_0)]) = m(t,]a+u_1t,a+2Ut[) + m(t,[a+2Ut,x^0(t,x_0)]),$$

and

$$\lim_{t\to 0} m(t,]a+u_1t, a+2Ut[) = m_0(\{a\}),$$

since $u_1 < u_0(a) < 2U$. In addition, we have

$$m(t, [a+2Ut, x^0(t,x_0)]) \geq m_0([a+3Ut, x_0-Ut]) \to m_0(]a, x_0[) = m_0(]a, x_0]), \quad \text{as } t \to 0. \tag{111}$$

Therefore, we obtain

$$\liminf_{t\to 0} m(t, [a(t), x^0(t,x_0)]) \geq m_0(\{a\}) + m_0(]a, x_0]) = m_0([a, x_0]). \tag{112}$$

This leads to (104) by taking into account (105).

If the left-hand side of $x = a(t)$ is not in the vacuum state, then for any $\epsilon > 0$ there is a $y_1 \notin V(T)$ with $y_1 < y$ such that $m(T,]y_1, y[) < \epsilon$. We set $[a_1(t), b_1(t)] := h_{t,T}^{-1}(\{y_1\})$. Then we get by Lemma 3.8 that $b_1(t) \leq a(t)$ for all $t \in]0, T[$. This implies

$$\lim_{t\to 0} b_1 \leq a. \tag{113}$$

Take a line segment $x = a + tu_1$ through the points (T, y_1) and $(0, a)$ with slope $u_1 = \frac{y_1 - a}{T}$. Then using the convexity of $a(t)$, see Lemma 4.5, gives

$$a + Tu_1 = y_1 < y \leq a + a'T \leq a + u_0(a)T.$$

This shows that $u_1 < u_0$. We use the convexity of $b(t)$ to obtain $b_1(t) \leq a + u_1t < a(t)$ for $t \in]0, T]$. Thus we have

$$m(t,]a+u_1t, a(t)[) \leq m(t,]b_1(t), a(t)[) = m(T,]y_1, y[) < \epsilon$$

and for t sufficiently small

$$\begin{aligned} m(t, [a(t), x^0(t,x_0)]) = &\quad m(t,]a+u_1t, x^0(t,x_0)]) - m(t,]a+u_1t, a(t)[) \\ \geq &\quad m(t,]a+u_1t, a+2Ut]) + m(t,]a+2Ut, x^0(t,x_0)[) - \epsilon. \end{aligned}$$

Note that $u_1 < u_0 < 2U$ holds. Then by the cohesion condition (10)

$$m(t,]a+u_1t, a+2Ut]) \to m_0(\{a\}), \quad \text{as } t \to 0.$$

Note that (111) is still true. Due to the arbitrariness of ϵ, we can prove that (112) also holds. Comparing this to (105) shows (104) is true for this case. We have completed the proof of this lemma. □

We unite cases like (103) and (104) by introducing new notations $\lfloor$ and $\rfloor$for the boundary points of intervals. Take any $x_0 < y_0$. We set

$$\lfloor x_0, y_0] := \begin{cases}]x_0, y_0] & \text{if } m_0(\{x_0\}) > 0 \text{ and } u_0(x_0) < u_0(x_0^+) \\ [x_0, y_0] & \text{otherwise} \end{cases}$$

or

$$[x_0, y_0\rfloor := \begin{cases} [x_0, y_0\rfloor & \text{if } m_0(\{y_0\}) > 0 \text{ and } u_0(y_0) > u_0(y_0^-) \\ [x_0, y_0] & \text{otherwise.} \end{cases}$$

Also we may combine these to the case $\lfloor x_0, y_0 \rfloor$. Further, we introduce the notation $[x_0, y_0\lfloor := [x_0, \infty[\setminus \lfloor y_0, \infty[$.

Then we have

$$\lim_{t\to 0^+} m(t, [a(t), x^0(t, x_0)]) = m_0(\lfloor a, x_0]). \tag{114}$$

Analogous to Lemmas 5.2 and 5.3, we get the lemma below.

Lemma 5.4: *Using the notation in Lemma 5.1 let $a < x_0$ and $m_0(\{x_0\}) = 0$, we have*

$$\lim_{t\to 0^+} (I, K)(t, [a(t), x^0(t, x_0)]) = (I_0, K_0)(\lfloor a, x_0]). \tag{115}$$

Proof: First we note that the cohesion condition (10) gives a similar inequality for I because the measure $I(t, \cdot)$ is absolutely continuous with respect to the measure $mI(t, \cdot)$. So we have

$$\lim_{t\to 0^+} I\left(t, \{x \in \mathbb{R}; \left|\frac{x - x_0}{t} - u_0(x_0)\right| \le \epsilon\}\right) = I(\{x_0\}),$$

for all $\epsilon > 0$, $x_0 \in \mathbb{R}$ and $m_0(\{x_0\}) > 0$. Similar arguments as in the proof of Lemmas 5.2 and 5.3 give the result for I. As for K, we can prove it by using (114). $\square$

For the other cases, we can proceed along similar lines.

Lemma 5.5: *Using the notation in Lemma 5.1 let $a < x_0 < b$ and $m_0(\lfloor a, x_0]) > 0$ and $m_0([x_0, b\rfloor) > 0$. Then we have*

$$C(\lfloor a, x_0]; 0, T) \ge y, \quad C([x_0, b\rfloor; 0, T) \le y. \tag{116}$$

Proof: By Lemma 4.4 we have

$$C([a(t), x^0(t, x_0)]; t, T) \ge y, \qquad \text{for all } t \in]0, T].$$

Lemma 4.2 shows that

$$C([a(t), x^0(t, x_0)]; t, T) = \frac{K(t, [a(t), x^0(t, x_0)]) + (T - t)I(t, [a(t), x^0(t, x_0)])}{m(t, [a(t), x^0(t, x_0)])} \ge y.$$

Taking the limit as $t \to 0$ and making use of (114) and (115), we get

$$\frac{K_0(\lfloor a, x_0]) + TI_0(\lfloor a, x_0])}{m_0(\lfloor a, x_0])} \ge y.$$

This is just (116). The result for b is proved analogously. $\square$

Lemma 5.6: *Using the notation in Lemma 5.1 take $x_0 < a$, i.e. x_0 is outside the interval $[a, b]$. If $m_0(\lfloor x_0, a\lfloor) > 0$ then we have*

$$C(\lfloor x_0, a\lfloor; 0, T) < y. \tag{117}$$

Proof: Without loss of generality, we assume $x_0 \notin V(0)$. Then there is a generalized characteristic curve denoted by $x = x^0(t, x_0)$ through x_0. As above, we can prove $\lim_{t\to 0^+} x^0(t, x_0) = x_0$. Since $x_0 < a$ we have $y_1 := x^0(T, x_0) < y$. We set $[a_1(t), b_1(t)] := h_{t,T}^{-1}(\{y_1\})$ for $t \in]0, T]$. By Lemma 5.1,

$$\lim_{t\to 0} b_1(t) := b_1, \qquad \lim_{t\to 0} \frac{db_1(t)}{dt^+} = b_1'.$$

Taking $[z_1, z_2] \supset\supset [b_1(t), a(t)]$ for any $t \in]0, T[$ such that $m_0(\{z_1\}) = 0$ and $m_0(\{z_2\}) = 0$, we can calculate by using Corollary 3.10 that

$$\begin{aligned} m(T,]y_1, y[) &= \lim_{t \to 0^+} m(t,]b_1(t), a(t)[) \\ &= \lim_{t \to 0^+} [m(t, [z_1, z_2]) - m(t, [z_1, b_1(t)]) - m(t, [a(t), z_2])] \\ &= m_0([z_1, z_2]) - m_0([z_1, b_1\rfloor) - m_0(\lfloor a, z_2]) \\ &=: m_0(\rfloor b_1, a\lfloor). \end{aligned}$$

With similar arguments to those in the process of proof of the above lemmas, we can get

$$\lim_{t \to 0^+} (I, K)(t,]b_1(t), a(t)[) = (I_0, K_0)(\rfloor b_1, a\lfloor).$$

Then by (81) we have

$$\begin{aligned} y > \; & C(]y_1, y[; T, T) = M(]y_1, y[, T) = C(]b_1(t), a(t)[; t, T) \\ = \; & \frac{K(t,]b_1(t), a(t)[) + (T - t) I(t,]b_1(t), a(t)[)}{m(t,]b_1(t), a(t)[)}. \end{aligned}$$

Letting $t \to 0^+$, we get

$$y > \frac{K_0(\rfloor b_1, a\lfloor) + T I_0(\rfloor b_1, a\lfloor)}{m_0(\rfloor b_1, a\lfloor)} = C(\rfloor b_1, a\lfloor; 0, T).$$

Lemma 5.5 applied to $a_1 < x_0 < b_1$ gives

$$y > y_1 \geq C([x_0, b_1\rfloor; 0, T), \qquad \text{if } m_0([x_0, b_1\rfloor) > 0.$$

By the definition of the center of mass, there exist specific mass ratios $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$ such that

$$C([x_0, a\lfloor; 0, T) = \lambda_1 C([x_0, b_1\lfloor; 0, T) + \lambda_2 C(\rfloor b_1, a\lfloor; 0, T) < y.$$

□

We remark that Lemmas 5.5 and 5.6 are actually the generalized variational principle in E et al. [ERS].

Proof of Theorem 1.3: Let (m, I) and $(\overline{m}, \overline{I})$ be two pairs of solutions satisfying the conditions of Definition 1.1 and the entropy condition. Correspondingly they determine two characteristic maps, denoted by $h_{t,T}$ and $\overline{h}_{t,T}$, respectively. We set

$$a := \liminf_{t \to 0^+} h_{t,T}^{-1}(\{y\}), \quad \overline{a} := \liminf_{t \to 0^+} \overline{h}_{t,T}^{-1}(\{y\}), \tag{118}$$

for $T > 0$ and $y \in \mathbb{R}$. Without loss of generality let us assume that $\lfloor a, \infty[\supset \lfloor \overline{a}, \infty[$, i.e. $a < \bar{a}$. We claim that $m_0(\lfloor a, \overline{a}\lfloor) = 0$. Otherwise, suppose that $m_0(\lfloor a, \overline{a}\lfloor) \neq 0$. By Lemma 5.5 we have, taking $x_0 := \bar{a}$,

$$C(\lfloor a, \overline{a}\lfloor; 0, T) \geq y.$$

But we also have from Lemma 5.6, taking $a := \bar{a}$, $x_0 := a$, that

$$C(\lfloor a, \overline{a}\lfloor; 0, T) < y.$$

These contradict each other.

Letting $x_0 \to \infty$ in Lemma 5.3, we easily get

$$\lim_{t\to 0^+} m(t,]a(t), \infty[) = m_0(\lfloor a, \infty[).$$

Then we have

$$\begin{aligned} m(T, [y, \infty[) &= m(t, [a(t), \infty[) = m_0(\lfloor a, \infty[) = m_0(\lfloor \overline{a}, \infty[) \\ &= \overline{m}(t, [a(t), \infty[) = \overline{m}(T, [y, \infty[), \end{aligned}$$

i.e.

$$m(T, \cdot) = \overline{m}(T, \cdot).$$

With analogous arguments, we can prove that

$$I(T, \cdot) = \overline{I}(T, \cdot).$$

Thus

$$u(T, \cdot) = \frac{dI(T, \cdot)}{dm(T, \cdot)} = \frac{d\overline{I}(T, \cdot)}{d\overline{m}(T, \cdot)} = \overline{u}(T, \cdot).$$

This completes the proof of Theorem 1.3. □

Acknowledgement Parts of this work were accomplished during visits of both authors at the Institute of Mathematics of Hebrew University in Jerusalem, Israel and they are very grateful for the hospitality extended to them. This work has benefitted from helpful discussions with Matania Ben-Artzi, Tong Zhang and Hailiang Liu. The work of both authors was partially supported by funding through the German-Israeli Science Foundation (GIF) grant I-318-195.06/93 and the DFG-funded 'Graduiertenkolleg Modellierung, Berechnung und Identifikation mechanischer Systeme' in Magdeburg, Germany.

References

[AH] R.K. Agarwal and D.W. Halt, A modified CUSP scheme in wave/particle split form for unstructed grid Euler flow. In: *Frontiers of Computational Fluid Dynamics*, D.A. Caughey and M.M. Hafez (Eds.), (1994) 155-163.

[B] F. Bouchut, On zero pressure gas dynamics. *Advances in kinetic theory and computing, Series on Advances in Mathematics for Applied Sciences* **22**, World Scientific, (1994) 171-190.

[BJ] F. Bouchut and F. James, Duality solutions for pressureless gases, monotone scalar conservation laws and uniqueness, *Comm. Part. Diff. Equat.*, **24**, (1999) 2173–2190.

[BG] Y. Brenier and E. Grenier, Sticky particles and scalar conservation laws, *SIAM J. Numer. Anal.* **35**, (1998) 2317-2328.

[CLZ] S. Cheng, J. Li and T. Zhang, Explicit construction of measure solutions of the Cauchy problem for the transportation equations. *Science in China, Series A* **40**, (1997) 1287-1299.

[D] C.M. Dafermos, Generalized characteristics in hyperbolic systems of conservation laws. *Arch. Rat. Mech. Anal.* **107**, (1989) 127-155.

[ERS] W. E, Yu.G. Rykov and Ya.G. Sinai, Generalized variational principles, global weak solutions and behavior with random initial data for systems of conservation laws arising in adhesion particle dynamics. *Comm. Math. Phys.* **177**, (1996) 349-380.

[HW] F. Huang and Z. Wang, Well posedness for pressureless flow, Preprint, *Institute of Applied Mathematics, the Chinese Academy of Sciences*, 2001.

[KPS] L. Kofman, D. Pogosyan and S. Shandarin, Structure of the universe in the two-dimensional model of adhesion. *Mon. Nat. R. Astr. Soc.* **242**, (1990) 200-208.

[L] J. Li, Note on the compressible Euler equations with zero temperature. *Appl. Math. Lett.* **14**, (2001) 519-523.

[LC] Y. Li and Y. Cao, Second order "large particle" difference method. *Sciences in China,* in Chinese, **A.** 8, (1985).

[LZY] J. Li, T. Zhang and S. Yang, *The two-dimensional Riemann problem in gas dynamics.* Pitman Monographs and Surveys in Pure and Applied Mathematics **98**, Longman, Harlow (1998).

[O] O. Oleinik, Uniqueness and a stability of the generalized solutions of the Cauchy problem for a quasilinear equations. *USP. Mat. Nauk.* **14**, (1959) 165-170; English transl. in *Amer. Math. Soc. Transl., Ser. 2* **33**, (1964) 285-290.

[SZe] S.F. Shandarin and Ya.B. Zeldovich, The large-scale structure of the universe: Turbulence, intermittency, structures in a self-gravitating medium. *Rev. Mod. Phys.* **61**, (1989) 185-220.

[SZh] W. Sheng and T. Zhang, The Riemann problem for the transportation equations in gas dynamics. *Mem. Amer. Math. Soc.* **137**, (1999).

[WD] Z. Wang and X. Ding, Uniqueness of generalized solution for the Cauchy problem of transportation equations, *Acta Math. Scientia*, **17**, (1997) 341–352.