WELL POSEDNESS FOR PRESSURELESS FLOW

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Abstract. We study the uniqueness problem for pressureless gases. Previous results on this topic are only known for the case when the initial data is assumed to be a bounded measurable function. This assumption is unnatural because the solution is in general a Radon measure. In this paper, the uniqueness of weak solution is proved for the case when the initial data is a Radon measure. We show that, besides Oleinik entropy condition, it is also important to require the energy to be weakly continuous initially. Our uniqueness result is obtained in *the same functional space* as the existence theorem.

Key Words: Pressureless flow, generalized potential, Oleinik entropy condition

1. Introduction

The one-dimension pressureless gases read

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x = 0, \end{cases} \tag{1.1}$$

where $\rho \geq 0$ denotes the density of mass and u the velocity.

Recently, the problem of pressureless gas dynamics has attracted a great deal of attention (see [1,3-5,10,12,16,19,21,22]). Physically this problem is studied as the sticky particle model in many different areas of physics (see [23]). Mathematically the solutions to these equations are naturally measures and this poses new challenges to the analysis of the solutions. The existence of global weak solution of (1.1) is first obtained independently by Brenier, Grenier [5] and E, Rykov and Sinai [10]. An explicit formula of weak solution was obtained in [10] by Generalized Variational Principles (GVP). Wang, Huang and Ding [21] extended their results to the general case when u_0 is not continuous. Similar results were also obtained by Chen, Li, Zhang [16]. Boudin [4] showed that the weak solution could also be obtained as limits of solutions to a viscous pressureless model.

After the existence of weak solution is obtained, people naturally want to know whether the weak solution is unique or not. It is well known that in general, the weak solution of conservation laws of hyperbolic system is not unique without entropy condition. The question is which entropy condition is suitable for pressureless gases. The authors of [1,5,10] found that the Lax entropy condition was insufficient to guarantee the uniqueness for (1.1). In [10], E, Rykov and Sinai pointed out further that Oleinik entropy condition might be necessary. Along this direction, Wang and Ding [22] proved the uniqueness

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of weak solution for the case when the initial data ρ_0 is of bounded measurable function. Similar results were also obtained by Bouchut and James [3]. Their results showed that the conjecture of [10] was true, i.e. the Oleinik entropy condition was suitable for pressureless gases. However, the uniqueness results of [3,22] are far from satisfactory because the solution of (1.1) is in general a Radon measure.

In the present paper, we establish the uniqueness of weak solution to (1.1) when the initial data is a Radon measure. We show that, besides Oleinik entropy condition, it is also important to require the energy to be weakly continuous initially, i.e. even though the system (1.1) loses energy as time goes on, it does not do so right away. This condition is called energy condition. A weak solution of (1.1) is called entropy solution if it satisfies Oleinik entropy condition and energy condition (see definition 2 below for detail). Without the energy condition, the weak solution satisfying Oleinik entropy condition is not unique (see section 4). This indicates that the energy condition is necessary and sufficient for the uniqueness.

Our proofs are based on the idea that any entropy solution should coincide with an entropy solution generated by the generalized potential,

$$F(y;x,t) = \int_{0+0}^{y-0} tu_0(\eta) + \eta - xdm_0(\eta), \tag{1.2}$$

with $m_0(x) = \rho_0([0,x[)$. This naturally leads to the uniqueness of entropy solution. It is noted that $F(y;x,t)$ only depends on the initial data. Obviously, our idea includes two parts:

Part I) constructing an entropy solution (ρ_s, u_s) by $F(y;x,t)$,

Part II) showing that any entropy solution (ρ, u) coincides with (ρ_s, u_s).

For part I, we give an explicit formula of entropy solution. We note that our constructive procedure is essentially equivalent to that of [10] by GVP. We thank anonymous referee for reminding us the relation between the two kinds of constructive procedure. For part II, we first establish a uniqueness theorem for linear equation with discontinuous coefficients (see section 3). By the new uniqueness theorem for linear equation, we prove the uniqueness of entropy solution of (1.1) in the region $t \geq t_1 > 0$. Namely, any entropy solution in the region $t \geq t_1$ (denoted by (ρ_{t_1}, u_{t_1})) can be expressed explicitly by its value at $t = t_1$. This is possible because (1.1) has only one family of characteristics. Then, we study the convergence of the sequence (ρ_{t_1}, u_{t_1}) as $t_1 \to 0$. We show that the limit of (ρ_{t_1}, u_{t_1}) coincides with (ρ_s, u_s). This leads to the uniqueness of entropy solution of (1.1) for general initial data. It is worthwhile to point out that *our uniqueness theorem is obtained in the same functional space as the existence theorem.* We note here that the usual initial condition (i.e. the weak continuity of the mass ρ, the momentum ρu at the initial time), together with the Oleinik entropy condition, is not sufficient to guarantee the uniqueness of the limits as $t_1 \to 0$.

Before recalling the definition of weak solution introduced by Wang, Huang and Ding (see [21]), we introduce some notations first.

If ρ, u are bounded measurable functions, then $m(x,t) = \oint_{(0,0)}^{(x,t)} \rho dx - \rho u dt$ is independent of integral path and satisfies $m_x = \rho$ due to the fact that the first equation of (1.1) is conserved. Therefore the system (1.1) becomes, by the new variable $m(x,t)$,

$$\begin{cases} m_t + um_x = 0, \\ (m_x u)_t + (m_x u^2)_x = 0. \end{cases} \tag{1.3}$$

We focus our attention on the system (1.3) instead of (1.1).

Definition 1 Let $m(x,t)$ be of bounded variation locally in x and $u(x,t)$ be bounded and measurable to m_x. Assume that the measures m_x and um_x are weakly continuous in t. $(\rho, u) = (m_x, u)$ is called a weak solution of (1.1) or (m, u) is called a weak solution of (1.3), if

$$\begin{cases} \displaystyle\iint \varphi_t m \; dx \; dt - \iint \varphi u \; dm \; dt = 0, \\ \displaystyle\iint \psi_t u + \psi_x u^2 \; dm \; dt = 0, \end{cases} \tag{1.4}$$

holds for all $\varphi,\ \psi \in C_0^\infty(R_+^2)$. Here $\int\int \cdots dmdt$ denotes Lebesgue-Stieltjes integral.

The initial value is understood in the following sense: as $t \to +0$, the measures ρ and ρu weakly converge to ρ_0, $\rho_0 u_0$ respectively, where u_0 is bounded and measurable to ρ_0.

Definition 2 Let (ρ, u) be a weak solution of (1.1). (ρ, u) is called an entropy solution of (1.1) if

$$\frac{u(x_2,t) - u(x_1,t)}{x_2 - x_1} \le \frac{1}{t}, \tag{1.5}$$

holds for any $x_1 < x_2$, almost everywhere $t > 0$ and the measure ρu^2 weakly converges to $\rho_0 u_0^2$ as $t \to 0$.

Our main results are

Theorem 1 (Existence theorem) Let $\rho_0 \ge 0 \in M_{loc}(R)$ (or $m_0(x) = \rho_0([0,x[)$ be increasing) and u_0 be bounded and measurable to ρ_0, then the system (1.1) admits at least one entropy solution.

Theorem 2 (Uniqueness theorem) Let $\rho_0 \ge 0 \in M_{loc}(R)$ (or $m_0(x) = \rho_0([0,x[)$ be increasing) and u_0 be bounded and measurable to ρ_0. Let (m_1, u_1) and (m_2, u_2) be the two entropy solutions of (1.1) with the same initial data ρ_0, u_0 in the above sense, then

$$m_1 = m_2 \quad \text{a.e.} \tag{1.6}$$

and

$$u_1 = u_2 \quad \text{a.e} \tag{1.7}$$

with respect to the measure $m_{1x} = m_{2x}$.

An outline of this paper is as follows. In section 2, we construct the entropy solution and study its structure in detail. In section 3, we prove a new uniqueness theorem for linear equation with discontinuous coefficient. In section 4, we study the general uniqueness for pressureless gases.

2. Existence of entropy solution

In this section, we use the generalized potential introduced in [21] to construct an entropy solution. As in [21], let

$$F(y;x,t)=\int_{0+0}^{y-0} tu_0(\eta)+\eta-xdm_0(\eta). \tag{2.1}$$

where $m_0(x)=\rho_0([0,x[)$ is an increasing function. Since $\rho_0=0$ is trivial, we assume that, there always exist $a<b$ such that $\rho_0([a,b[)>0$, i.e. the support of ρ_0 is not empty set.

Before constructing the entropy solution, we give some lemmas first.

Lemma 2.1 For any point (x,t), as a function of y, $F(y;x,t)$ has a finite low bound.

Proof: For any $y_1<y_2\le x-Mt$, or $y_1>y_2\ge x+Mt$, we have

$$F(y_1;x,t)-F(y_2;x,t)=t\int_{y_2-0}^{y_1-0} u_0(\eta)-\frac{x-\eta}{t}\,dm_0(\eta)\ge 0,$$

where $M=\|u_0\|_{L^\infty}$. This implies the Lemma.

Let

$$\begin{aligned}&\nu(x,t)=\min\nolimits_y F(y;x,t),\\ &S(x,t)=\{y;\exists y_n\to y,\ \text{s.t. } F(y_n;x,t)\to\nu(x,t)\}.\end{aligned} \tag{2.2}$$

Since $F(y;x,t)$ is left continuous (not continuous) with respect to y, for any $y_0\in S(x,t)$, the following holds

$$\nu(x,t)=\begin{cases} F(y_0;x,t), & \text{if } F(y;x,t) \text{ achieves its minimum at } y_0,\\ F(y_0+0;x,t), & \text{otherwise.}\end{cases} \tag{2.3}$$

Thus, we have

Lemma 2.2 Assume that $y_0\in S(x,t)$ and $[m_0(y_0)]=m_0(y_0+0)-m_0(y_0-0)>0$, then

$$\begin{aligned}&\nu(x,t)=\min\nolimits_y F(y;x,t)=F(y_0;x,t), && \text{if}\quad x\le y_0+tu_0(y_0),\\ &\nu(x,t)=\min\nolimits_y F(y;x,t)=F(y_0+0;x,t), && \text{if}\quad x>y_0+tu_0(y_0).\end{aligned} \tag{2.4}$$

Proof: Since u_0 is measurable to ρ_0, u_0 must be well defined at the point y_0 where ρ_0 has a positive mass. If $x\le y_0+tu_0(y_0)$, we compute

$$F(y_0+0;x,t)-F(y_0-0;x,t)=(tu_0(y_0)+y_0-x)[m_0(y_0)]\ge 0.$$

This indicates that $F(y;x,t)$ achieves its minimum at y_0 due to the left continuity of $F(y;x,t)$. In the same fashion, the second term of (2.4) also holds.

Lemma 2.3 Let (x_n,t_n) and $y_n\in S(x_n,t_n)$ converge to (x,t) and y_0, respectively, then $y_0\in S(x,t)$.

Proof. First we show that $\nu(x,t)$ is a continuous function of x and t. Note that $F(y;x,t)$ is continuous in x,t and $\nu(x_n,t_n)=F(y_n;x_n,t_n)$ or $F(y_n+0,x_n,t_n)$, it is easy to see $\nu(x,t)\le\liminf_{n\to\infty}\nu(x_n,t_n)$.

On the other hand, we compute

$$F(y;x,t)\ge\nu(x_n,t_n)+F(y;x,t)-F(y;x_n,t_n).$$

Letting $n\to\infty$ gives $\nu(x,t)\ge\limsup_{n\to\infty}\nu(x_n,t_n)$. This implies

$$\nu(x,t)=\lim_{n\to\infty}\nu(x_n,t_n).$$

Now we choose a subsequence of y_n (still denoted by y_n) such that $\nu(x_n,t_n) = F(y_n;x_n,t_n)$ (or $\nu(x_n,t_n) = F(y_n+0,x_n,t_n)$) always holds. Without loss of generality, we assume that $\nu(x_n,t_n) = F(y_n;x_n,t_n)$. Due to $|F(y_n;x,t) - F(y_n;x_n,t_n)| = o(1)$, we have $F(y_n;x,t) \to \nu(x,t)$, as $n \to \infty$. Thus, the definition of $S(x,t)$ gives $y_0 \in S(x,t)$.

Let

$$y_m(x,t) = \inf_{y\in S(x,t)} y,\ y^m(x,t) = \sup_{y\in S(x,t)} y.$$

From Lemma 2.1, $\inf\{y; y \in spt\{\rho_0\}\}$ is finite if $y_m(x,t) = -\infty$. In the same way, $\sup\{y; y \in spt\{\rho_0\}\}$ is also finite if $y^m(x,t) = +\infty$.

Let

$$y_M(x,t) = \begin{cases} y_\sharp, & if\ (x,t) \in A(x,t)\ and\ y_\sharp \in spt\{\rho_0\}, \\ y_m, & otherwise, \end{cases} \tag{2.5}$$

$$y^M(x,t) = \begin{cases} y^\sharp, & if\ (x,t) \in B(x,t)\ and\ y^\sharp \in spt\{\rho_0\}, \\ y^m, & otherwise, \end{cases} \tag{2.6}$$

where

$$A(x,t) = \{(x,t); \exists y_\sharp > y_m, s.t. \rho_0 = 0\ on\ (y_m,y_\sharp)\ and\ \forall y \in (y_m,y_\sharp), y \in S(x,t)\},$$

$$B(x,t) = \{(x,t); \exists y^\sharp < y_m, s.t. \rho_0 = 0\ on\ (y^\sharp,y^m)\ and\ \forall y \in (y_\sharp,y^m), y \in S(x,t)\}.$$

Let

$$y_*(x,t) = \min\{y_M(x,t), y^M(x,t)\}, \qquad y^*(x,t) = y^M(x,t). \tag{2.7}$$

It is easy to see $y_*(x,t), y^*(x,t) \in S(x,t) \cap spt\{\rho_0\}$.

Analogous to [21], for each point (x_0,t_0), we introduce the left and right backward generalized characteristics L_1, L_2 as follows

$$L_1:\ x = x_0 + \frac{x_0 - y_*(x_0,t_0)}{t_0}(t-t_0) \quad \text{for} \quad t < t_0 \tag{2.8}$$

$$L_2:\ x = x_0 + \frac{x_0 - y^*(x_0,t_0)}{t_0}(t-t_0) \quad \text{for} \quad t < t_0 \tag{2.9}$$

We claim that there is only one minimum point of $F(y;x,t)$ for each (x,t) along the backward lines L_1, L_2.

Lemma 2.4 For any $y_0 \in S(x_0,t_0)$, $y_*(x,t) = y^*(x,t)$ holds along the lines

$$L: x = x_0 + \frac{x_0 - y_0}{t_0}(t-t_0), \quad t < t_0.$$

Furthermore, $y_*(x,t) = y^*(x,t) \le y_*(x_0,t_0)$ along the line L_1 and $y_*(x,t) = y^*(x,t) = y^*(x_0,t_0)$ along the line L_2.

Proof: Let $y_0 \in S(x_0,t_0)$. Without loss of generality, we assume that $\nu(x_0,t_0) = F(y_0;x_0,t_0)$. On the line L, for any $y \ne y_0$, we compute

$$\begin{aligned} F(y;x,t) - F(y_0;x,t) &= \int_{y_0-0}^{y-0} tu_0(\eta) + \eta - x dm_0(\eta) \\ = \quad & \frac{t}{t_0}\int_{y_0-0}^{y-0} t_0u_0(\eta) + \eta - x_0\ dm_0(\eta) + \frac{t_0-t}{t_0}\int_{y_0-0}^{y-0} \eta - y_0\ dm_0(\eta) \\ = \quad & \frac{t}{t_0}(F(y;x_0,t_0) - \nu(x_0,t_0)) + \frac{t_0-t}{t_0}\int_{y_0-0}^{y-0} \eta - y_0\ dm_0(\eta) \ge 0, \end{aligned} \tag{2.10}$$

which infers $y_0 \in S(x,t)$ and $y_*(x,t) = y^*(x,t)$ along the line L. Now we prove Lemma 2.4 for L_1, i.e. $y_0 = y_*(x_0,t_0)$. There are four subcases.

(1) $\exists y_1 < y_0 < y_2, s.t.\ \rho_0 = 0$ on $(y_1, y_0) \cup (y_0, y_2)$;
(2) $\exists y_1 < y_0, s.t.\ \rho_0 = 0$ on (y_1, y_0) and for any $y_2 > y_0, \int_{y_0+0}^{y_2-0} dm_0 > 0$;
(3) $\exists y_2 > y_0, s.t.\ \rho_0 = 0$ on (y_0, y_2) and for any $y_1 < y_0, \int_{y_1+0}^{y_0-0} dm_0 > 0$;
(4) For any $y_1 < y_0 < y_2$, the integrals $\int_{y_1+0}^{y_0-0} dm_0$ and $\int_{y_0+0}^{y_2-0} dm_0$ are positive.

It is easy to see $y_*(x,t) = y^*(x,t) = y_0$ for the subcases (1),(3) and (4) and $y_*(x,t) = y^*(x,t) \le y_0$ for (2). In the same way, we can also prove $y_*(x,t) = y^*(x,t) = y_0$ along the line L_2.

In terms of Lemma 2.4, for each point (x,t), the backward generalized characteristics L_1, L_2 and X-axis form a characteristic triangle. We denote it $\Delta(x,t)$. Fix the time t, as x changes, all these characteristic triangles $\Delta(x,t)$ never interact with each other except X-axis. Thus, we have the following lemma.

Lemma 2.5 y_*, y^* are increasingly monotonic in x. In particular, $y^*(x_1,t) \le y_*(x_2,t)$ holds for any $x_1 < x_2$.

Proof: If there exist two points (x_1,t) and (x_2,t) such that $x_1 < x_2$, $y^*(x_1,t) > y_*(x_2,t)$, then the line $L_2(x_1)$ (defined in (2.9)) must interact with $L_1(x_2)$ at a point (x_s,t_s), with $t_s > 0$. From Lemma 2.4, we have $y_*(x_s,t_s) = y^*(x_s,t_s) = y^*(x_1,t) \le y_*(x_2,t)$. This contradicts the assumption $y^*(x_1,t) > y_*(x_2,t)$.

Lemma 2.6 Each point (x_1,t_1) at $t_1 > 0$ uniquely determines a Lipschitz continuous curve $x = x(t)$, $x_1 = x(t_1)$. In particular, at every $t \in \{\tau; \tau \ge t_1\}$,

$$x'(t) = \begin{cases} \dfrac{x - y_*}{t}, & if\ t \in V_1 \cap (V_2 \cup V_3), \\ \dfrac{x(t_a) - x(t)}{t_a - t}, & if\ t \in V_1 \cap V_4, \\ \lim\limits_{x_2, x_1 \to x \pm 0} \dfrac{\int_{\tilde{y}_*(x_1,t)+0}^{\tilde{y}^*(x_2,t)-0} u_0(\eta) dm_0(\eta)}{\int_{\tilde{y}_*(x_1,t)+0}^{\tilde{y}^*(x_2,t)-0} dm_0(\eta)}, & if\ t \in V_5 \cup (V_1/(V_2 \cup V_3 \cup V_4)), \end{cases} \tag{2.11}$$

where

$$\begin{aligned} V_1 = \ & \{t; y_*(x,t) = y^*(x,t)\} \\ V_2 = \ & \{(x,t); \exists t_a > t,\ s.t.\ y_*(x(t_a),t_a) = y^*(x(t_a),t_a),\ and\ (x(t),t) \in \Delta(x(t_a),t_a)\}, \\ V_3 = \ & \{t; \forall t_a > t, y_*(x(t_a),t_a) < y^*(x(t_a),t_a),\ and\ y_*(x(t_a),t_a) \to y_*(x,t), \\ & y^*(x(t_a),t_a) \to y^*(x,t),\ as\ t_a \to t\}, \\ V_4 = \ & \{(x,t); \forall t_a > t,\ s.t.\ y_*(x(t_a),t_a) = y^*(x(t_a),t_a),\ and\ (x(t),t) \in R_+^2/\Delta(x(t_a),t_a)\}, \\ V_5 = \ & \{t; y_*(x,t) < y^*(x,t)\}, \end{aligned}$$

and

$$\begin{aligned} & x'(t) = \lim_{t'',t' \to t+0} \tfrac{x(t'') - x(t')}{t'' - t'}, \\ & \tilde{y}^*(x_2,t) = y^*(x_2,t) + H(x_2 - y_*(x+0,t) - u_0(y_*(x+0,t))t)(x_2 - x), \\ & \tilde{y}_*(x_1,t) = y_*(x_1,t) + H(u_0(y_*(x-0,t))t + y_*(x-0,t) - x_1)(x_1 - x), \\ & H(x) = \begin{cases} 1, & x > 0, \\ 0, & x \le 0. \end{cases} \end{aligned}$$

Proof. Consider any line $t = t_0 > t_1$. Let

$$a(t_0) = \sup\{x; x - \tfrac{x - y_*(x,t_0)}{t_0}(t_0 - t_1) < x_1\},$$
$$b(t_0) = \inf\{x; x - \tfrac{x - y_*(x,t_0)}{t_0}(t_0 - t_1) > x_1\}.$$

Obviously, $a(t_0) \le b(t_0)$ holds. If $a(t_0) < b(t_0)$, then for any $c \in (a(t_0), b(t_0))$, we have $c - \frac{c - y_*(c,t_0)}{t_0}(t_0 - t_1) = x_1$ which contradicts the fact that all characteristic triangles $\Delta(x, t_0)$, $(-\infty < x < \infty)$ never interact with each other except X-axis. Thus $a(t_0) = b(t_0)$ always holds. Therefore, for any $t = t_0$, there always exists a unique point $x = x(t_0) = a(t_0)$ on the line $t = t_0$. When time t_0 changes, these points $x = x(t_0)$ form a curve $x = x(t)$ with $x_1 = x(t_1)$ in $t > t_1$. We denote it $L(x_1, t_1)$. In particular, for any $t_2 > t > t_1$, the characteristic $\Delta(x, t_2)$ does not contain the points $(x(t_1), t_1)$ and $(x(t), t)$ if $x \neq x(t_2)$. This means that the curve $L(x_1, t_1)$ coincides with the curve $L(x_2, t_2)$ originated from (x_2, t_2) in $t \ge t_2$, if $x_2 = x(t_2), x_1 = x(t_1), t_2 \ge t_1$.

We first consider the case $y_*(x,t) < y^*(x,t)$. Let $y_* = y_*(x,t), y^* = y^*(x,t)$. By the definition of y_*, y^*, it is easy to check

$$\int_{y_*+0}^{y^*-0} dm_0(\eta) > 0, \tag{2.12}$$

or

$$\rho_0 = 0 \ on \ (y_*, y^*), (y_*, y^*) \cap S(x,t) = \emptyset, \tag{2.13}$$

which indicates

$$\Delta(x(t), t) \subset \Delta(x(t'), t'), \forall t' > t. \tag{2.14}$$

In fact, if $\Delta(x(t'), t') \cap \Delta(x(t), t) \cap R_+^2 = \emptyset$, we have $y^*(x(t'), t') \le y_*$ or $y_*(x(t'), t') \ge y^*$. Without loss of generality, we assume that $y^*(x(t'), t') \le y_*$. Now we choose a sequence of points (x_n, t') with $x_n > x(t') = a(t')$, $x_n \to x(t')$, as $n \to \infty$. By the definition of $a(t')$, we have $\varsigma_n = x_n - \frac{x_n - y_*(x_n,t')}{t'}(t' - t_1) > x(t_1)$. This implies $y_*(x_n, t') \ge y^*$. Let $y_0 = \liminf_{n\to\infty} y_*(x_n, t')$, then

$$y_0 \ge y^* > y_* > y^*(x(t'), t'). \tag{2.15}$$

On the other hand, from Lemma 2.3, we have $y_0 \in S(x(t'), t')$. This contradicts the definition of y^* due to (2.12) and (2.13).

Let $x' = x(t'),\ x'' = x(t''),\ t'' > t' > t$, and

$$y_, = y_*(x', t'), \quad y' = y^*(x', t')$$
$$y_{,,} = y_*(x'', t''), \quad y'' = y^*(x'', t'').$$

From (2.14), we have

$$\begin{aligned} &\Delta(x,t) \subset \Delta(x', t') \subset \Delta(x'', t''), \\ &y_{,,} \le y_, \le y_* \le y^* \le y' \le y''. \end{aligned} \tag{2.16}$$

It is easy to check that $y_*(x+0, t) = \lim_{t''\to t+0} y''$ and $y_*(x-0, t) = \lim_{t''\to t+0} y_{,,}$. Without loss of generality, we assume that $y_2 < y', y''$, $y_1 > y_,, y_{,,}$ and

$$\nu(x', t') = F(y'; x', t') = F(y_,; x', t'),$$
$$\nu(x'', t'') = F(y''; x'', t'') = F(y_{,,}; x'', t'').$$

By the definition of $\nu(x,t)$, we have

$$F(y''; x'', t'') - F(y_,; x'', t'') \le F(y''; x', t') - F(y_,; x', t'), \tag{2.17}$$

i.e.,

$$\int_{y_{,}-0}^{y''-0} t''u_0+\eta-x''\,dm_0(\eta)\le\int_{y_{,}-0}^{y''-0} t'u_0+\eta-x'\,dm_0(\eta), \tag{2.18}$$

which gives

$$\frac{x''-x'}{t''-t'}\int_{y_{,}-0}^{y''-0} dm_0(\eta)-\int_{y_{,}-0}^{y''-0} u_0dm_0(\eta)\ge 0. \tag{2.19}$$

In the same way, we also have

$$F(y_{,,};x'',t'')-F(y';x'',t'')\le F(y_{,,};x',t')-F(y';x',t'), \tag{2.20}$$

which gives

$$\frac{x''-x'}{t''-t'}\int_{y_{,,}-0}^{y'-0} dm_0(\eta)-\int_{y_{,,}-0}^{y'-0} u_0dm_0(\eta)\le 0. \tag{2.21}$$

Combining (2.19) and (2.21), we have

$$\frac{\int_{y_{,}-0}^{y''-0} u_0dm_0(\eta)}{\int_{y_{,}-0}^{y''-0} dm_0(\eta)}\le\frac{x''-x'}{t''-t'}\le\frac{\int_{y_{,,}-0}^{y'-0} u_0dm_0(\eta)}{\int_{y_{,,}-0}^{y'-0} dm_0(\eta)}.$$

Letting $t''\to t+0$ yields

$$x'(t)=\frac{\int_{y_*(x-0,t)-0}^{y_*(x+0,t)+0} u_0dm_0}{\int_{y_*(x-0,t)-0}^{y_*(x+0,t)+0} dm_0}=\lim_{x_2,x_1\to x\pm0}\frac{\int_{\tilde y_*(x_1,t)+0}^{\tilde y^*(x_2,t)-0} u_0(\eta)dm_0(\eta)}{\int_{\tilde y_*(x_1,t)+0}^{\tilde y^*(x_2,t)-0} dm_0(\eta)}, \tag{2.22}$$

where we have used the fact that $\tilde y^*(x_2,t)\to y_*(x+0,t)+0$ and $\tilde y_*(x_1,t)\to y_*(x-0,t)-0$, as $x_2,x_1\to x\pm 0$.

Next we turn to the case $y_*=y^*$. There are three subcases.

(1) $t\in V_1\cap(V_2\cup V_3)$; (2) $t\in V_1\cap V_4$; (3)$t\in V_1/(V_2\cup V_2\cup V_3)$.

Proof of (1): If $t\in V_1\cap V_3$, then the curve $x=x(t), t<t''$ lies in $\Delta(x(t''),t'')$. Thus, we have

$$\frac{x''-y''}{t''}\le\frac{x''-x'}{t''-t'}\le\frac{x''-y_{,,}}{t''}, \tag{2.23}$$

which indicates $\lim_{t'',t'\to t+0}\frac{x(t'')-x(t')}{t''-t'}=\frac{x-y_*}{t}$ by letting $t'',t'\to t+0$. If $t\in V_1\cap V_2$, then $(x(t),t)$ is located on the line L_{t_a}: $\varsigma(\tau)=x(t_a)+\frac{x(t_a)-y_*}{t_a}(\tau-t_a)$. It is easy to check that $(x(\tau),\tau), t<\tau<t_a$ is also located on the line L_{t_a}. Thus, we have

$$\frac{x''-x'}{t''-t'}=\frac{x-y_*}{t}, t''>t'>t.$$

Proof of (2): There exists $t_a>t$ such that for any $t<t'\le t_a$, $y_*(x(t'),t')=y_*(x(t),t)=y_*$ and $\Delta(x(t'),t')$ does not contain the point $(x(t),t)$. Obviously, we have

$$x(t_a)-\frac{x(t_a)-y_*}{t_a}(t_a-t)<x(t), \tag{2.24}$$

or,

$$x(t_a)-\frac{x(t_a)-y_*}{t_a}(t_a-t)>x(t). \tag{2.25}$$

Without loss of generality, we assume that

$$x(t_a)-\frac{x(t_a)-y_*}{t_a}(t_a-t)<x(t).$$

Now we choose a sequence of (x_n, t_a) with $x_n > x(t_a)$, $x_n \to x(t_a), n \to \infty$. It is observed that for any x, $\Delta(x, t'), t < t' \le t_a$, does not contain the point $(x(t), t)$, we have, from (2.24),

$$y_{t_a} = \liminf y_*(x_n, t_a) \ge x(t_a) - \frac{x(t_a) - x(t)}{t_a - t} t_a > y_*. \tag{2.26}$$

Furthermore, $y_{t_a} = y^m(x(t_a), t_a) \in S(x(t_a), t_a) \cap spt\{\rho_0\}$ due to Lemma 2.3.

Define

$$\overline{L}_{t_a} : \xi = x(t_a) + \frac{x(t_a) - y_{t_a}}{t_a}(\tau - t_a), \tau < t_a. \tag{2.27}$$

From Lemma 2.4, we have

$$y_*(\xi, \tau) = y^*(\xi, \tau) = y_*. \tag{2.28}$$

Let (x_s, t_s) be the interaction point of $\overline{L}_{t_a}$ with the line

$$L(t, t_a) : \xi = x(t) + \frac{x(t) - y_*}{t}(\tau - t), \tau > 0.$$

In view of (2.26), we have $t_s \ge t$. If $t_s > t$, then (2.28) gives $\Delta(x(t), t) \subset \Delta(x_s, t_s)$. This contradicts the fact that for any x, $\Delta(x, t_s)$ does not contain the point $(x(t), t)$. Therefore $t_s = t$ and then $(x(t), t) \in \overline{L}_{t_a}$. In the same way, we also have $(x(t), t) \in L_{t'}$, if $t < t' < t_a$, where $L_{t'}$ is defined as in (2.27). It is easy to see $y_{t_a} = y_{t'}$. This implies that any line $L_{t'}$ coincides with L_{t_a}. Thus, we have

$$\frac{x'' - x'}{t'' - t'} = \frac{x(t_a) - x(t)}{t_a - t}, t'' > t' > t.$$

Proof of (3) is similar to the case $y_* < y^*$, we omit it here.

On the other hand, it is easy to see $x(t)$ is Lipschitz continuous in t by the argument above. Therefore the lemma 2.6 is proved.

We call $x = x(t)$ in Lemma 2.6 the forward generalized characteristic generated by (x_1, t_1). Obviously, the curve $x = x(t)$ is Lipschitz continuous. We define

$$u(x,t) = \begin{cases} \lim\limits_{t'', t' \to t+0} \dfrac{\overline{x}(t'') - \overline{x}(t')}{t'' - t'}, & if\ |x - y_*(x,t)| \le Mt, \\ M, & if\ x - y_*(x,t) > Mt, \\ -M, & if\ x - y_*(x,t) < -Mt \end{cases} \tag{2.29}$$

where the curve $\xi = \overline{x}(\tau)$ is originated from the point (x, t).

Obviously, $u(x, t)$ is well defined in R^2_+. Some properties of u follow from Lemma 2.6.

Lemma 2.7

(i) $u(x-0,t) = \frac{x - y_*(x-0,t)}{t}$, $\quad u(x+0,t) = \frac{x - y_*(x+0,t)}{t}$, if $|x - y_*(x \pm 0, t)| \le Mt$.

(ii) For almost everywhere $t > 0$, $u(x, t)$ satisfies

$$u(x+0,t) < u(x,t) < u(x-0,t). \tag{2.30}$$

(iii) For any $x_1 < x_2$, almost everywhere $t > 0$, $u(x, t)$ satisfies Oleinik entropy condition. i.e.

$$\frac{u(x_2,t) - u(x_1,t)}{x_2 - x_1} \le \frac{1}{t}.$$

Proof. (i), (ii) are easy due to Lemmas 2.1, 2.5-2.6. We only prove (iii). For any $x_1 < x_2$, without loss of generality, we assume that, $|x_2 - y_*(x_2+0,t)| \le Mt$ and $|x_1 - y_*(x_1-0,t)| \le$

Mt. We compute

$$\begin{aligned} &u(x_2,t)-u(x_1,t)\le u(x_2-0,t)-u(x_1+0,t)\\ &=\frac{x_2-y_*(x_2-0,t)}{t}-\frac{x_1-y_*(x_1+0,t)}{t}\le\frac{x_2-x_1}{t}. \end{aligned}$$

Therefore (iii) is proved.

Now we construct the solution $m(x,t)$. First we introduce the forward generalized characteristics originated from X-axis. For each point $\eta\in spt\{\rho_0\}$, let

$$a(\eta,t)=\begin{cases}\eta+u_0(\eta)t, & if\ N_1\cup N_2=\emptyset,\\ \inf\{x;x\in N_2\}, & if\ N_1=\emptyset, N_2\neq\emptyset,\\ \sup\{x;x\in N_1\}, & if\ N_1\neq\emptyset,\end{cases}\tag{2.31}$$

$$b(\eta,t)=\begin{cases}\eta+u_0(\eta)t, & if\ N_1\cup N_2=\emptyset,\\ \sup\{x;x\in N_1\}, & if\ N_1\neq\emptyset, N_2=\emptyset,\\ \inf\{x;x\in N_2\}, & if\ N_2\neq\emptyset,\end{cases}\tag{2.32}$$

where $N_1=\{x;y_*(x,t)<\eta\}$, $N_2=\{x;y_*(x,t)>\eta\}$.

Note that $(\eta,0)$ is located on the X-axis, $a(\eta,t)$ may not be equal to $b(\eta,t)$. Namely, each point $(\eta,0)$ may generate at least one Lipschitz continuous curve $x(t), x(0)=\eta$. If there is only one forward generalized characteristic $x(t)$ from $(\eta,0)$, we denote $x(\eta,t)=x(t)$. If there are at least two curves from $(\eta,0)$ where ρ_0 has a positive mass, let

$$x(\eta,t)=\begin{cases}a(\eta,t), & \text{if } \eta+u_0(\eta)t\le a(\eta,t),\\ \eta+u_0(\eta)t, & \text{if } a(\eta,t)\le\eta+u_0(\eta)t\le b(\eta,t),\\ b(\eta,t), & \text{if } b(\eta,t)\le\eta+u_0(\eta)t.\end{cases}\tag{2.33}$$

For each point $\eta\in R/spt\{\rho_0\}$, let $c(\eta,t)=\eta+Mt$, then there exists $t_\eta>0$ such that $c(\eta,t)$ does not interact with other curves $x(\tilde\eta,t),\tilde\eta\in spt\{\rho_0\}$, as $0<t<t_\eta$. Let $\tilde t_\eta$ be the first time when $c(\eta,t)$ interacts with a curve $x(\eta_0,t)$, $\eta_0\in spt\{\rho_0\}$, let

$$x(\eta,t)=\begin{cases}\eta+Mt, & 0<t\le\tilde t_\eta,\\ x(\eta_0,t), & t>\tilde t_\eta.\end{cases}$$

It is easy to see $x(\eta,t)$ is increasing in η and Lipschitz continuous in $t>0$. Thus $x=x(\eta,t)$ is well defined with respect to the measure ρ_0 due to Wang and Ding [22]. By Lemma 2.6, we have,

$$\frac{\partial x(\eta,t)}{\partial t}=u(x(\eta,t),t),\ if\ |x-y_*(x(\eta,t),t)|\le Mt\ and\ \eta\in spt\{\rho_0\}.\tag{2.34}$$

In view of the definition of y_* and the forward generalized characteristic, we define

$$m(x,t)=\begin{cases}\displaystyle\int_{0+0}^{y_*-0}dm_0, & F \text{ achieves its minimum at } y_*,\\ \displaystyle\int_{0+0}^{y_*+0}dm_0, & \text{otherwise},\end{cases}\tag{2.35}$$

$$q(x,t)=\begin{cases}\displaystyle\int_{0+0}^{y_*-0} u_0\,dm_0, & F \text{ achieves its minimum at } y_*,\\ \displaystyle\int_{0+0}^{y_*+0} u_0 dm_0, & \text{otherwise},\end{cases}\tag{2.36}$$

$$E(x,t)=\begin{cases}\displaystyle\frac{1}{2}\int_{0+0}^{y_*-0} u_0 u(x(\eta,t),t)\,dm_0, & F \text{ achieves its minimum at } y_*,\\ \displaystyle\frac{1}{2}\int_{0+0}^{y_*+0} u_0 u(x(\eta,t),t) dm_0, & \text{otherwise}.\end{cases}\tag{2.37}$$

It is noted that $u(x(\eta,t),t)$ is of bounded variation locally in x and satisfies

$$\frac{u(x(\eta_2,t),t)-u(x(\eta_1,t),t)}{x(\eta_2,t)-x(\eta_1,t)}\le\frac{1}{t}.$$

Thus $u(x(\eta,t),t)$ is also measurable to the measure ρ_0 due to Wang and Ding [22]. Obviously, $m(x,t),q(x,t)$ and $E(x,t)$ are of bounded variation locally in x. For these functions, we have the following properties.

Lemma 2.8 In the sense of Radon-Nikodym derivatives, for almost everywhere $t>0$, $m(x,t)$,$q(x,t)$ and $E(x,t)$ satisfy

(1) $q_x=um_x$, (2) $E_x=\frac{1}{2}u^2m_x$.

Proof. First we consider the simplest case that there is a neighborhood $U_{(x,t)}$ of (x,t), such that y_* keeps constant in $U_{(x,t)}$. Namely, for any point $(\xi,\tau)\in U_{(x,t)}$, $y_*(\xi,\tau)=y_*(x,t)$. In terms of Lemma 2.3, we also have $y^*(\xi,\tau)=y_*(x,t)$. There are two cases.

Case 1. $F(y;\xi,\tau)$ always achieves or never achieves its minimum at y_* in $U_{(x,t)}$. This case is trivial because m,q and E keep constant in $U_{(x,t)}$. We note that the case $|x-y_*(x,t)|>Mt$ belongs to Case 1.

Case 2. $F(y;\xi,\tau)$ achieves its minimum at y_* only for some points of $U_{(x,t)}$, not for the whole set $U_{(x,t)}$. Obviously, m,q,E take two values in $U_{(x,t)}$. When $m(x-0,t)=m(x+0,t)$, it is the same as subcase 1 by choosing a smaller neighborhood of (x,t). When $m(x-0,t)\neq m(x+0,t)$, $F(y;\xi,\tau)$ achieves its minimum at y_* only for $\xi<x$. Let $y_0=y_*(x,t)$, it is easy to see ρ_0 has a positive mass at the point y_0. Thus Lemma 2.2 and 2.6 give

$$u(x,t)=u_0(y_0)=\frac{x-y_0}{t}.\tag{2.38}$$

We compute

$$\lim_{x_2,x_1\to x\pm0}\frac{q(x_2,t)-q(x_1,t)}{m(x_2,t)-m(x_1,t)}=\frac{\int_{y_0-0}^{y_0+0}u_0dm_0(\eta)}{\int_{y_0-0}^{y_0+0}dm_0(\eta)}=u_0(y_0)=u(x,t),\tag{2.39}$$

$$\lim_{x_2,x_1\to x\pm0}\frac{E(x_2,t)-E(x_1,t)}{m(x_2,t)-m(x_1,t)}=\frac{\frac{1}{2}\int_{y_0-0}^{y_0+0}u_0u(x,t)dm_0(\eta)}{\int_{y_0-0}^{y_0+0}dm_0(\eta)}=\frac{1}{2}u^2(x,t).\tag{2.40}$$

Next we turn to the case that y_* is not a constant for any neighborhood of (x,t). Let $x_1<x<x_2$, $y_1=y_*(x_1,t)$, $y_2=y_*(x_2,t)$, then $y_1<y_2$. Analogous to Lemma 2.6, we first consider the case $y_*<y^*$. It is easy to check that

$$\lim_{x_2,x_1\to\pm0}\frac{q(x_2,t)-q(x_1,t)}{m(x_2,t)-m(x_1,t)}=\lim_{x_2,x_1\to x\pm0}\frac{\int_{\tilde{y}_*(x_1,t)+0}^{\tilde{y}^*(x_2,t)-0}u_0(\eta)dm_0(\eta)}{\int_{\tilde{y}_*(x_1,t)+0}^{\tilde{y}^*(x_2,t)-0}dm_0(\eta)}=u(x,t).\tag{2.41}$$

When $y_* = y^*$, there are three subcases. The proof of (3) is similar to the case $y_* < y^*$, we omit it here. For (1), if $t \in V_1 \cap V_3$, then we have $y_1, y_2 \to y_*$. Without loss of generality, we assume that

$$\nu(x_1,t) = F(y_1;x_1,t), \quad \nu(x_2,t) = F(y_2;x_2,t).$$

By the definition of $\nu(x,t)$, we have

$$\nu(x_2,t) \le F(y_1;x_2,t),$$
$$F(y_2;x_1,t) \ge \nu(x_1,t),$$

which indicates

$$\int_{y_1-0}^{y_2-0} u_0 dm_0(\eta) \le \int_{y_1-0}^{y_2-0} \frac{x_2-\eta}{t} dm_0(\eta), \tag{2.42}$$

$$\int_{y_1-0}^{y_2-0} u_0 dm_0(\eta) \ge \int_{y_1-0}^{y_2-0} \frac{x_1-\eta}{t} dm_0(\eta). \tag{2.43}$$

Therefore, we have

$$\begin{aligned} \frac{x_1-y_2}{t} &\le \frac{\int_{y_1-0}^{y_2-0} \frac{x_1-\eta}{t} dm_0(\eta)}{\int_{y_1-0}^{y_2-0} dm_0(\eta)} \le \frac{q(x_2,t)-q(x_1,t)}{m(x_2,t)-m(x_1,t)} \\ &\le \frac{\int_{y_1-0}^{y_2-0} \frac{x_2-\eta}{t} dm_0(\eta)}{\int_{y_1-0}^{y_2-0} dm_0(\eta)} \le \frac{x_2-y_1}{t}. \end{aligned} \tag{2.44}$$

Let $x_1 \to x-0$, $x_2 \to x+0$ in (2.44), we get $\lim_{x_2,x_1\to 0} \frac{q(x_2,t)-q(x_1,t)}{m(x_2,t)-m(x_1,t)} = \frac{x-y_*}{t}$.

In the same fashion, we can also prove (2.44) for the case $t \in V_1 \cap V_2$.

Now we study the case (2). Without loss of generality, we assume that

$$x(t_a) - \frac{x(t_a)-y_*}{t_a}(t_a-t) < x(t). \tag{2.45}$$

It is observed that $\rho_0 = 0$ on the interval (y_*, y_{t_a}). This infers $y_* \in S(x_1,t)$ when x_1 is close to x. Furthermore, $F(y;x_1,t)$ does not achieve its minimum at y_* if ρ_0 has a positive mass at y_*. By (2.44), we have

$$\begin{aligned} \frac{x_1-y_2}{t} &\le \frac{\int_{y_{t_a}-0}^{y_2-0} \frac{x_1-\eta}{t} dm_0(\eta)}{\int_{y_{t_a}-0}^{y_2-0} dm_0(\eta)} \le \frac{q(x_2,t)-q(x_1,t)}{m(x_2,t)-m(x_1,t)} \\ &\le \frac{\int_{y_{t_a}-0}^{y_2-0} \frac{x_2-\eta}{t} dm_0(\eta)}{\int_{y_{t_a}-0}^{y_2-0} dm_0(\eta)} \le \frac{x_2-y_{t_a}}{t}. \end{aligned}$$

Letting $x_2, x_1 \to x \pm 0$, we have

$$\lim_{x_2,x_1\to x\pm 0} = \frac{x(t_a)-t}{t_a-t},$$

where we have used the fact that $y_2 \to y_{t_a}$, as $x_2 \to x+0$. Thus the first part of Lemma 2.8 is proved.

On the other hand, it is observed that

$$x(\eta,t) = x, \text{ as } y_*(x-0,t) < \eta \in spt\{\rho_0\} < y_*(x+0,t), \tag{2.46}$$

$$x_1 \le x(\eta,t) \le x_2, \text{ as } y_1 < \eta \in spt\{\rho_0\} < y_2. \tag{2.47}$$

Thus, we have

$$u(x(\eta,t),t) = u(x,t), \text{ as } y_*(x-0,t) < \eta \in spt\{\rho_0\} < y_*(x+0,t),$$

$$\frac{x_1-y_2}{t}\le u(x(\eta,t),t)\le\frac{x_2-y_1}{t},\ \text{as } y_1<\eta\in spt\{\rho_0\}<y_2.$$

If ρ_0 has a positive mass on $y_*(x-0,t)$ and $y_*(x+0,t)$, then we have

$$\begin{cases} x(y_*(x-0,t),t)<x, & if\ \exists x_1<x, s.t.\ y_1=y_*(x-0,t) \\ & and\ \frac{x-y_*(x-0,t)}{t}>u_0(y_*(x-0,t)), \\ x(y_*(x-0,t),t)=x, & otherwise, \end{cases} \tag{2.48}$$

and

$$\begin{cases} x(y_*(x+0,t),t)>x, & if\ \exists x_2>x, s.t.\ y_2=y_*(x+0,t) \\ & and\ \frac{x-y_*(x+0,t)}{t}>u_0(y_*(x+0,t)), \\ x(y_*(x+0,t),t)=x, & otherwise. \end{cases} \tag{2.49}$$

It is noted that in the first relation of (2.48) and (2.49), $\tilde{y}^*(x_2,t)=y_*(x+0,t)$ and $\tilde{y}_*(x_1,t)=y_*(x-0,t)$ hold.

Applying Lemma 2.6, (2.46-2.49) and the fact that $q_x=um_x$, we have $E_x=\frac{1}{2}u^2m_x$.

Lemma 2.9

$$\int_{x_1}^{x_2} m(x,t)dx=\nu(x_1,t)-\nu(x_2,t), \tag{2.50}$$

$$\int_{t_1}^{t_2} q(x,t)dt=\nu(x,t_2)-\nu(x,t_1). \tag{2.51}$$

Proof. Let $y_*=y_*(x,t),\quad y_*'=y_*(x',t)$. Without loss of generality, we assume $F(y;x,t)$ achieves its minimum at y_* and $F(y;x',t)$ does not achieve its minimum at y_*', Then we have $\nu(x,t)=F(y_*;x,t)$ and $\nu(x',t)=F(y_*'+0;x',t)$. We compute

$$\begin{aligned} &\nu(x,t)-\nu(x',t) \\ &=[\nu(x,t)-F(y_*'+0;x,t)]+[F(y_*'+0;x,t)-\nu(x',t)] \\ &=[F(y_*;x,t)-F(y_*;x',t)]+[F(y_*;x',t)-\nu(x',t)]. \end{aligned}$$

In the first relation the first term on the right is ≤ 0, in the second relation the second term is ≥ 0. This means that $\nu(x,t)-\nu(x',t)$ lies between the limits

$$\begin{aligned} F(y_*'+0;x,t)-\nu(x',t)=(x'-x)m(x',t), \\ F(y_*;x,t)-F(y_*;x',t)=(x'-x)m(x,t). \end{aligned} \tag{2.52}$$

Let x',x be a subdivision of the interval (x_1,x_2), we get (2.50) by integrating (2.52) over (x_1,x_2).

Similar to (2.50), we also have (2.51). Lemma 2.9 is proved.

Proof of theorem 1: We now prove $(u(x,t),m(x,t))$ defined in (2.29), (2.35) is an entropy solution of (1.1). Form Lemma 2.9, we have $\nu_x=-m(x,t)$ and $\nu_t=q(x,t)$ in the sense of distribution. This infers $m_t+q_x=m_t+um_x=0$. In fact, for any $\varphi\in C_0^\infty(R_+^2)$, we have

$$\begin{aligned} &\iint m\varphi_t dxdt-u\varphi dmdt=\iint(m\varphi_t+q\varphi_x)dxdt \\ =&\iint(\nu_t\varphi_x-\nu_x\varphi_t)dxdt=0. \end{aligned} \tag{2.53}$$

Therefore the first relation of (1.4) is proved.

The proof of the second equation of (1.4) is slightly complicated. We need to introduce another generalized potential to get some information on q, E like Lemma 2.9. Similar to $F(y;x,t)$, we define

$$G(y;x,t)=\int_{0+0}^{y-0}(u_0(\eta)+k)(x(\eta,t)-x)dm_0(\eta), \tag{2.54}$$

where k is any constant satisfying $k>M$ and $x(\eta,t)$ is defined in (2.33). Since $x(\eta,t)$ is increasing in η and $x(y_*-0,t)\le x$, $x(y^*+0,t)\ge x$, it is easy to see $G(y;x,t)$ has a finite low bound for any (x,t) and $y_*(x,t)=G_*(x,t)\in S_G(x,t)$, where $G_*(x,t)$ and $S_G(x,t)$ are defined like (2.2) and (2.7). Let $\mu(x,t)=\min_y G(y;x,t)$. In the same way, Lemma 2.9 still holds for $G(y;x,t)$. Therefore, we have

$$\mu_x=-(\overline{q}+k\overline{m}), \tag{2.55}$$

$$\mu_t=2\overline{E}+kv(x,t), \tag{2.56}$$

where

$$\overline{m}(x,t)=\begin{cases}\displaystyle\int_{0+0}^{y_*-0}dm_0, & \text{If } G \text{ achieves its minimum at } y_*,\\ \displaystyle\int_{0+0}^{y_*+0}dm_0, & \text{otherwise,}\end{cases}$$

$$\overline{q}(x,t)=\begin{cases}\displaystyle\int_{0+0}^{y_*-0}u_0dm_0, & \text{If } G \text{ achieves its minimum at } y_*,\\ \displaystyle\int_{0+0}^{y_*+0}u_0dm_0, & \text{otherwise,}\end{cases}$$

$$\overline{E}(x,t)=\begin{cases}\displaystyle\frac{1}{2}\int_{0+0}^{y_*-0}u_0u(x(\eta,t),t)dm_0, & \text{If } G \text{ achieves its minimum at } y_*,\\ \displaystyle\frac{1}{2}\int_{0+0}^{y_*+0}u_0u(x(\eta,t),t)dm_0, & \text{otherwise,}\end{cases}$$

$$v(x,t)=\begin{cases}\displaystyle\int_{0+0}^{y_*-0}u(x(\eta,t),t)dm_0, & \text{If } G \text{ achieves its minimum at } y_*,\\ \displaystyle\int_{0+0}^{y_*+0}u(x(\eta,t),t)dm_0, & \text{otherwise.}\end{cases}$$

We claim that for any $t>0$,

$$m=\overline{m},\quad q=\overline{q},\quad E=\overline{E}\quad \text{a.e.}$$

In fact, due to $y_*(x,t)=G_*(x,t)$, only the case that $[m_0(y_*(x,t))]>0$ may cause $m(x,t)\neq\overline{m}(x,t)$. It is observed that such points $(y_*,0)$ are at most denumerable. Since the map from (x,t) to $(y_*,0)$ is not one to one, we have to consider the case that an interval maps to one point, i.e. $(y_*,0)$ generates at least two forward generalized characteristics. Let $x\in(a(\eta,t),b(\eta,t))$. There are two subcases:

(1) $y_*+u_0(y_*)t<x$, (2) $x<y_*+u_0(y_*)t$.

For (1), $F(y;x,t)$ and $G(y;x,t)$ do not arrive in their minimum at y_* in terms of Lemma 2.2. Thus we have $m=\overline{m}$. For (2), both $F(y;x,t)$ and $G(y;x,t)$ arrive in their minimum at y_*. So F and G always achieve or do not achieve their minimum at y_* at the same time. This infers $m(x,t)=\overline{m}(x,t)$. In the same fashion, we also have $q=\overline{q}$, $E=\overline{E}$, a.e.

Since $k>M$ is arbitrary, we have, from (2.55),(2.56),

$$\theta_x=-q,\quad \theta_t=2E \tag{2.57}$$

where

$$\theta(x,t)=\begin{cases}\displaystyle\int_{0+0}^{y_*-0}u_0(\eta)(x(\eta,t)-x)dm_0, & \text{If } G \text{ achieves its minimum at } y_*,\\ \displaystyle\int_{0+0}^{y_*+0}u_0(\eta)(x(\eta,t)-x)dm_0, & \text{otherwise.}\end{cases}\tag{2.58}$$

For any $\phi\in C_0^\infty(R_+^2)$, (2.57) and Lemma 2.8 give

$$\begin{aligned}&\iint\phi_t u dm dt+\iint\phi_x u^2 dm dt\\ &=\iint\theta_x\phi_{xt}dxdt-\iint\theta_t\phi_{xx}dxdt=0.\end{aligned}\tag{2.59}$$

Therefore the second relation of (1.4) is proved.

To prove (m,u) is an entropy solution of (1.1), it is sufficient to prove that m_x, um_x, u^2m_x are weakly continuous at $t=0$. We first consider the weak continuities of mass at the initial time. It is observed that $m_0(x)$ is continuous except at most denumerable points. We only study the point x where $m_0(x)$ is continuous. If there exists t_0 such that $|x-y_*(x,t_0)|>Mt_0$, then $\rho_0=0$ on $(x-Mt,x+Mt)$. This implies that for any $t<t_0$, $y_*(x,t)=y_*(x,t_0)$ and $m(x,t)\to m_0(x)$, as $t\to+0$. If there exists t_0 such that $\forall t<t_0$, $|x-y_*(x,t)|\le Mt$, then $y_*(x,t)\to x$ as $t\to 0$. From (2.35), we have $m(x,t)\to m_0(x)$. Thus, $m(x,t)\to m_0(x)$, a.e. This indicates the weak continuity of m_x at the initial time. In the same way, q_x is also weakly continuous initially.

Now we study the initial condition for the energy $\frac{1}{2}\rho u^2$. By (2.37), we only need to prove $u(x(\eta,t),t)\to u_0(\eta)$, a.e. with respect to the measure ρ_0, as $t\to 0+$.

Let η be the point satisfying

$$u_0(\eta_0)=\lim_{\eta_2,\eta_1\to\eta_0\pm 0}\frac{\int_{\eta_1-0}^{\eta_2+0}u_0(\eta)dm_0(\eta)}{\int_{\eta_1-0}^{\eta_2+0}dm_0(\eta)}.\tag{2.60}$$

We shall show $u(x(\eta_0,t),t)\to u_0(\eta_0)$, as $t\to 0+$. In view of the definition of $x(\eta,t)$, we have $y_*(x(\eta_0,t)-0,t)\le\eta_0$. Let $\overline{y}_*=\lim_{t\to 0+}y_*(x(\eta_0,t)-0,t)$. There are two subcases.

(a) $\overline{y}_*<\eta_0$; (b) $\overline{y}_*=\eta_0$.

Proof of (a): It is easy to see $\rho_0=0$ on $(\overline{y}_*,\eta_0)$ and $\overline{y}_*\in S(x(\eta_0,t),t)$ when t is sufficiently small. By Lemma 2.2, $F(y;x(\eta_0,t),t)$ does not achieve its minimum at $\overline{y}_*$ if ρ_0 has a positive mass at $\overline{y}_*$. This implies $\eta_0=y_*(x(\eta_0,t),t)$.

Let $\overline{y}=\eta_0+ht$, we have

$$\begin{aligned}&\int_{\eta_0-0}^{\overline{y}-0}u_0(\eta)dm_0(\eta)\ge\int_{\eta_0-0}^{\overline{y}-0}\frac{x(\eta_0,t)-\eta}{t}dm_0(\eta)\\ &\ge\int_{\eta_0-0}^{\overline{y}-0}\frac{x(\eta_0,t)-\eta_0}{t}dm_0(\eta)-h\int_{\eta_0-0}^{\overline{y}-0}dm_0(\eta).\end{aligned}\tag{2.61}$$

From Lemma 2.6, we have

$$\frac{\int_{\eta_0-0}^{\overline{y}-0}u_0(\eta)dm_0(\eta)}{\int_{\eta_0-0}^{\overline{y}-0}dm_0(\eta)}\ge u(x(\eta_0,t),t)-h.\tag{2.62}$$

Letting $t\to 0+$, we have

$$u_0(\eta_0)\ge\limsup_{t\to 0}u(x(\eta_0,t),t)-h.\tag{2.63}$$

Again letting $h \to 0$, we have

$$u_0(\eta_0) \geq \limsup_{t\to 0} u(x(\eta_0,t),t). \tag{2.64}$$

The case (b) can be treated similarly. Thus, (2.64) always holds. In the same fashion, we also have

$$u_0(\eta_0) \leq \liminf_{t\to 0} u(x(\eta_0,t),t), \tag{2.65}$$

which, together with (2.64), indicates

$$u_0(\eta_0) = \lim_{t\to 0} u(x(\eta_0,t),t). \tag{2.66}$$

Thus, $u(x(\eta,t),t) \to u_0(\eta)$, a.e. with respect to the measure ρ_0, as $t \to 0+$. Then we have $E(x,t) \to E_0(x)$ a.e. as $t \to 0+$ due to (2.37). Therefore Theorem 1 is proved.

Finally, it is worthwhile to point out that $m(x,t), q(x,t)$ and $E(x,t)$ can also be expressed by

$$m(x,t) = \begin{cases} \displaystyle\int_{0+0}^{y_0-0} dm_0, & F \text{ achieves its minimum at } y_0, \\ \displaystyle\int_{0+0}^{y_0+0} dm_0, & \text{otherwise}, \end{cases} \tag{2.67}$$

$$q(x,t) = \begin{cases} \displaystyle\int_{0+0}^{y_0-0} u_0\, dm_0, & F \text{ achieves its minimum at } y_0, \\ \displaystyle\int_{0+0}^{y_0+0} u_0 dm_0, & \text{otherwise}, \end{cases} \tag{2.68}$$

$$E(x,t) = \begin{cases} \displaystyle\frac{1}{2}\int_{0+0}^{y_0-0} u_0 u(x(\eta,t),t)\, dm_0, & F \text{ achieves its minimum at } y_0, \\ \displaystyle\frac{1}{2}\int_{0+0}^{y_0+0} u_0 u(x(\eta,t),t) dm_0, & \text{otherwise}, \end{cases} \tag{2.69}$$

with any $y_0 \in S(x,t)$. These are due to the fact that, for any $t > 0$, the points (x,t) satisfying $y_* < y^*$ are at most denumerable and $\rho_0 = 0$ on (y_0, y_*) for the point where $y_* = y^*$, if $y_0 < y_*$ (or $\rho_0 = 0$ on (y_*, y_0) if $y_0 > y_*$).

In addition, for any $(x_0,t_0) \in R^2_+$, from Lemma 2.4, it is easy to see

$$u(x,t) = \frac{x - y_0}{t}, \tag{2.70}$$

holds along the line

$$L : x = x_0 + \frac{x-y_0}{t_0}(t - t_0), t < t_0,$$

with any $y_0 \in S(x_0,t_0) \cap spt\{\rho_0\}$ and $|\frac{x-y_0}{t_0}| \leq M$.

3. Linear equation with discontinuous coefficient

This section is devoted to the existence and uniqueness of linear equation with discontinuous coefficient in the region $t \geq t_1 > 0$. In fact, we get an explicit formula of the weak solution if its value at $t = t_1$ is given.

We consider the following linear equation

$$\omega_t + u(x,t)\omega_x = 0, \tag{3.1}$$

where $u(x,t)$ is a given bounded measurable function satisfying

$$\frac{u(x_2,t) - u(x_1,t)}{x_2 - x_1} \leq \frac{1}{t}, \tag{3.2}$$

for any $x_1 < x_2$ and almost all $t > 0$.

Similar to definition 1, our definition of weak solution of (3.1) is

Definition 3.1 Suppose that $\omega(x,t)$ is of bounded variation locally in x and the measure ω_x is weakly continuous with respect to t, $\omega(x,t)$ is called a weak solution of (3.1) if

$$\iint \varphi_t \omega dxdt - \iint \varphi u d\omega dt = 0, \tag{3.3}$$

holds for all $\varphi \in C_0^\infty(R_+^2)$.

In the sense of definition 3.1, we have the following existence and uniqueness results.

Theorem 3.1 Suppose that $\omega(x,t_1) \in BV_{loc}(R)$, $t_1 > 0$ and $u(x,t)$ satisfies (3.2), then (3.1) admits only one weak solution in $t \geq t_1$.

Proof: Let $M = \|u_0\|_{L^\infty}$ and $u_\varepsilon = u * j_\varepsilon$, where j_ε is a standard mollifier, then $|u^\varepsilon| \leq M$ and $u_{\varepsilon x} \leq \frac{1}{t}$.

We denote $x = X_{t_1}^\varepsilon(\xi,t)$ the characteristic curve as follows,

$$\begin{cases} \frac{dx}{dt} = u_\varepsilon, \\ x(t_1) = \xi. \end{cases} \tag{3.4}$$

Some properties of these characteristics have been studied in [13,22]. We state them in Lemma 3.2.

Lemma 3.2:

(1) There exist a subsequence $X_{t_1}^{\varepsilon_i}(\xi,t)$ such that

$$\lim_{\varepsilon_i \to 0} X_{t_1}^{\varepsilon_i}(\xi,t) = X_{t_1}(\xi,t), \tag{3.5}$$

for all ξ and t. Furthermore, $X_{t_1}(\xi,t)$ is Lipschitz continuous with respect to t.

(2) If $X_{t_1}(\xi_1,t_0) = X_{t_1}(\xi_2,t_0)$ holds for some $\xi_1 < \xi_2$ and $t_0 \geq t_1 > 0$, then $X_{t_1}(\xi_1,t) = X_{t_1}(\xi_2,t)$ for all $t \geq t_0$. Furthermore, $X_{t_1}(\xi,t), \xi \in R$, cover the strip $\{t \geq t_1\}$.

(3) If $t_2 < t_1$, then $X_{t_2}(\xi,t) = X_{t_1}(\xi',t)$ holds for any $t \geq t_1$, where $\xi' = X_{t_2}(\xi,t_1)$.

(4) Let $U = \{\xi : \exists t > 0, s.t. X(\xi - 0,t) \neq X(\xi + 0,t)\}$, where $X(\xi,t)$ is defined by letting $t_1 = 0$ in (3.4), then for any $\xi \in R/U$ and almost $t > 0$,

$$\frac{\partial X(\xi,t)}{\partial t} = u(X(\xi,t),t).$$

(5) Let $\xi_{t_1}(x,t) = \sup\{\xi : X_{t_1}(\xi,t) < x\}$, $\eta_{t_1}(x,t) = \inf\{\xi : X_{t_1}(\xi,t) > x\}$, then the set $\Gamma_{t_1} = \{(x,t) : \xi_{t_1}(x,t) \neq \eta_{t_1}(x,t), t \geq t_1\}$ consists of at most countable Lipschitz continuous curves. We call these curves shock waves of u. Furthermore, $x - M(t-t_1) \leq \xi_{t_1}(x,t) \leq \eta_{t_1}(x,t) \leq x + M(t-t_1)$ and $\xi_{t_1}(x_1,t) \to \xi_{t_1}(x,t)$, as $x_1 \to x - 0$, $\eta_{t_1}(x_2,t) \to \eta_{t_1}(x,t)$, as $x_2 \to x + 0$.

Similar to $F(y;x,t)$, we define

$$H_{t_1}(y;x,t) = \int_{0+0}^{y-0} X_{t_1}(\eta,t) - x d\omega(\eta,t_1), \tag{3.6}$$

Since BV functions can always be expressed by the difference of two increasing functions, we assume that $\omega_0(x)$ is increasing. Thus the function $H_{t_1}(y;x,t)$ has a finite low bound for each fixed point (x,t). It is easy to see

$$H_{t_1}(\xi_{t_1}(x,t);x,t)=\min_y H_{t_1}(y;x,t), \tag{3.7}$$

always holds due to fact that $x=X_{t_1}(\xi_{t_1}(x,t),t)$ (see (2),(5) of Lemma 3.2). In another words, $H_{t_1}(y;x,t)$ always achieves its minimum at $\xi_{t_1}(x,t)$. We note here that (3.7) does not always hold when $t_1=0$ because the curves $X(\xi,t),\xi\in R$, do not cover the half space R^2_+.

Define

$$\omega(x,t)=\int_{0+0}^{\xi_{t_1}-0} d\omega(\eta,t_1), \tag{3.8}$$

$$q_2(x,t)=\int_{0+0}^{\xi_{t_1}-0} u(X_{t_1}(\eta,t),t)d\omega(\eta,t_1), \tag{3.9}$$

In terms of the existence result of Wang and Ding [22], we have $q_{2x}=u\omega_x$ in the sense of Radon-Nikodym derivative. On the other hand, it is easy to check that Lemma 2.9 still holds here. Therefore we have

$$\frac{\partial}{\partial x}\min_y H_{t_1}(y;x,t)=-\omega(x,t),\quad \frac{\partial}{\partial t}\min_y H_{t_1}(y;x,t)=q_2(x,t). \tag{3.10}$$

This implies that $\omega(x,t)$ is a weak solution of (3.1).

Now we study the uniqueness of weak solution of (3.1) in the region $t\geq t_1$.

Consider two weak solutions $\omega_1(x,t),\omega_2(x,t)$ satisfying $\omega_1(x,t_1)=\omega_2(x,t_1)$ a.e. at the line $t=t_1>0$. Let $z=\omega_1-\omega_2$, then (u,z) satisfies (3.3) with $z(x,t_1)=0$ a.e. in R^1. For any $N>0$, let

$$\Omega=\{(x,t);0<t<N/M,|x|<N-Mt\}.$$

To prove the uniqueness, it is sufficient to prove $z(x_0,t_0)=0$, for any Lebesgue point of $\Omega\cap\{t>t_1\}$.

We first choose a sequence of smooth functions u_h such that u_h converge pointwisely to u and $u_{hx}\leq\frac{1}{t}$. We note that choosing this approximating functions is possible because we can modify u by a special smooth kernel introduced by Wang and Ding in [22].

Let $C_h=\int\int_\Omega|u_h-u||dz|dt$, then $C_h\to 0$ as $h\to +0$.
From (3.3), we have

$$|\int\int(\psi_t+(\psi u_h)_x)zdxdt|\leq C_h\|\psi\|_{C_0},\quad \forall\psi\in C_0^\infty(\Omega). \tag{3.11}$$

For any $-N+t_0M<\xi<N-t_0M$, there exists a smooth curve $x=\tilde{X}(\xi,t)$ satisfying

$$\begin{cases}\dfrac{dx}{dt}=u_h,\\ x(t_0)=\xi.\end{cases} \tag{3.12}$$

Then

$$\tilde{X}(\xi,t)=\xi+\int_{t_0}^t u_h(\tilde{X}(\xi,s),s)ds. \tag{3.13}$$

We compute

$$\frac{\partial\tilde{X}(\xi,t)}{\partial\xi}=e^{\int_{t_0}^t u_{hx}ds}. \tag{3.14}$$

We now make a coordinate transformation

$$A:\quad (x,t)\to(\xi,t) \tag{3.15}$$

such that $A^{-1}(\xi,t)=(\tilde{X}(\xi,t),t)$.

From (3.13) and (3.14), (3.11) becomes

$$\begin{aligned}
&\iint(\psi_t+(\psi u_h)_x)zdxdt=\iint(\psi_t+u_h\psi_x+u_{hx}\psi)zdxdt\\
=&\iint(\frac{\partial}{\partial t}\psi(\tilde{X}(\xi,t),t)+u_{hx}\psi)e^{\int_{t_0}^t u_{hx}ds}zd\xi dt\\
=&\iint(\psi e^{\int_{t_0}^t u_{hx}ds})_t zd\xi dt\le C_h\|\psi\|_{C_0}.
\end{aligned}\tag{3.16}$$

For any $\varphi(\xi)\in C_0^\infty(-N+t_0M,N-t_0M)$ and $t_2'\in(t_1,t_0)$, let

$$\begin{aligned}
\phi(t)&=\int_{-\infty}^t(\alpha_\varepsilon(s-t_2')-\alpha_\varepsilon(s-t_1))ds,\\
\psi(\tilde{X}(\xi,t),t)e^{\int_{t_0}^t u_{hx}ds}&=\varphi(\xi)\phi(t),
\end{aligned}\tag{3.17}$$

where α_ε is a standard modifier.

Substituting (3.17) into (3.16) gives

$$\begin{aligned}
|\iint\varphi(\xi)\phi'(t)zd\xi\,dt| &\le C_h\|\varphi\phi e^{-\int_{t_0}^t u_{hx}ds}\|_{C_0}\\
&\le C_h\|\varphi\phi e^{\int_{\frac{t_1}{2}}^{t_0}\frac{ds}{s}}\|_{C_0}\\
&\le\frac{2t_0}{t_1}C_h\|\varphi\|_{C_0}.
\end{aligned}\tag{3.18}$$

or,

$$|\iint\varphi(\alpha_\varepsilon(t-t_2')-\alpha_\varepsilon(t-t_1))zd\xi dt|\le\frac{2t_0}{t_1}C_h\|\varphi\|_{C_0}.\tag{3.19}$$

Let $\varepsilon\to0$, we get

$$|\int\varphi(\xi)z(\tilde{X}(\xi,t_2'),t_2')d\xi|\le\frac{2t_0}{t_1}C_h\|\varphi\|_{C_0},\tag{3.20}$$

for almost all $t_1<t_2'<t_0$.

By approximating smooth functions, (3.20) gives

$$|\frac{1}{\sqrt{C_h}}\iint_{t_0-\sqrt{C_h}<t<t_0}\varphi(\xi)z(\tilde{X}(\xi,t),t)d\xi dt|\le\frac{2t_0}{t_1}C_h\|\varphi\|_{L^\infty},\tag{3.21}$$

for all $\varphi\in L^\infty(-N+t_0M,N-t_0M)$.

Choosing $\varphi(\xi)=\frac{1}{2\sqrt{C_h}}(H(\xi-x_0+\sqrt{C_h})-H(\xi-x_0-\sqrt{C_h}))$ with $H(s)$ the Heaviside function, we get

$$|\frac{1}{2C_h}\iint_D z(\tilde{X}(\xi,t),t)d\xi dt|\le\frac{2t_0}{t_1}\sqrt{C_h},\tag{3.22}$$

where $D=[x_0-\sqrt{C_h},x_0+\sqrt{C_h}]\times[t_0-\sqrt{C_h},t_0]$.

Since $A^{-1}(D)\subset\{(x,t);t_0-\sqrt{C_h}<t<t_0,|x-x_0|<\sqrt{C_h}+M|t-t_0|\}$, we have $\mathrm{mes}(A^{-1}(D))\le2(M+1)C_h$.

We compute

$$
\begin{aligned}
|z(x_0,t_0)| &\le \frac{1}{2C_h}\iint_D |z-z(x_0,t_0)|d\xi dt + |\frac{1}{2C_h}\iint_D z(\tilde{X}(\xi,t),t)d\xi dt| \\
&\le \frac{1}{2C_h}\iint_D |z-z(x_0,t_0)|d\xi dt + \frac{2t_0}{t_1}\sqrt{C_h} \\
&= \frac{1}{2C_h}\iint_{A^{-1}(D)} |z(x,t)-z(x_0,t_0)|e^{-\int_{t_0}^t u_{hx}ds}dxdt + \frac{2t_0}{t_1}\sqrt{C_h} \\
&\le \frac{2(M+1)}{\mathrm{mes}(A^{-1}(D))}\iint_{A^{-1}(D)} |z(x,t)-z(x_0,t_0)|dxdt + \frac{2t_0}{t_1}\sqrt{C_h}.
\end{aligned}
\tag{3.23}
$$

Letting $C_h \to +0$ in (3.23), we have $z(x_0,t_0)=0$. This proves Theorem 3.1.

By Theorem 3.1, any weak solution $\omega(x,t)$ of (3.1) must have the following formula if its value at $t=t_1$ is given,i.e.

$$
\omega(x,t) = \int_{0+0}^{\xi_{t_1}-0} d\omega(x,t_1). \tag{3.24}
$$

4. Uniqueness of entropy solution

This section is devoted to the uniqueness of entropy solution. First, we prove the uniqueness of entropy solution in the region $t \ge t_1 > 0$ with arbitrary t_1, by the uniqueness results for linear equation with discontinuous coefficient. In another word, any entropy solution can explicitly expressed by its value at the time $t = t_1 > 0$. Then, by letting $t_1 \to 0$, we show any entropy solution (ρ,u) of (1.1) coincides with the entropy solution (ρ_s,u_s) constructed in section 2 by the generalized potential $F(y;x,t)$ with the same initial data. This leads to our uniqueness theorem.

Proof of theorem 2: Suppose that $(\rho,u)=(m_x,u)$ is any entropy solution of (1.1) with initial data ρ_0,u_0, where ρ_0 is a non-negative Radon measure and u_0 is bounded and measurable to ρ_0. It is easy to see $m(x,t)$ is a weak solution of

$$
m_t + um_x = 0, \tag{4.1}
$$

in the sense of definition 3.1.

Next, we shall construct a function $q(x,t)$ which is a weak solution of a scalar transport equation

$$
q_t + uq_x = 0 \tag{4.2}
$$

and satisfies $q_x = um_x$.

We choose a smooth function $\phi(x) \in C_0^\infty(R)$, $\int_{-\infty}^{+\infty}\phi(x)dx = 1$. Let $q(x,t)=p(x,t)-f(t)$, where

$$
\begin{aligned}
p(x,t) &= \int_{0+0}^{x-0} udm, \\
f(t) &= \int_{-\infty}^{+\infty}\phi(x)(p(x,t)-p(x,t_1))dx + \int_{t_1}^t h(\tau)d\tau, \\
h(t) &= \int_{-\infty}^{+\infty}\phi(x)udp.
\end{aligned}
$$

Obviously, $q_x = p_x = um_x$. For any $\varphi \in C_0^\infty(R\times R_{t_1})$ with $R_{t_1} = \{t \in R : t \ge t_1\}$, let

$$
g(t) = \int_{-\infty}^{+\infty}\varphi(y,t)dy, \quad \psi(x,t) = \int_{-\infty}^{x}(\varphi(y,t)-\phi(x)g(t))dy,
$$

then $g(t) \in C_0^\infty(R_{t_1})$, $\psi \in C_0^\infty(R\times R_{t_1})$ and $\psi_x(x,t) = \varphi(x,t)-\phi(x)g(t)$.

Substituting ψ in the second relation of (1.4), we have

$$\iint \psi_t u dm dt + \iint \psi_x u^2 dm dt = 0,$$

which indicates

$$-\iint \psi_{xt} q dx dt + \iint \psi_x u dq dt = 0,$$

i.e.

$$\iint (\varphi(x,t) - \phi(x)g(t))_t q dx dt - \iint (\varphi(x,t) - \phi(x)g(t)) u dq dt = 0. \tag{4.3}$$

From (4.3), we have

$$\begin{aligned}
&\iint \varphi_t q dx dt - \iint \varphi u dq dt \\
&= \iint \phi(x) g_t(t) q dx dt - \iint \phi(x) g(t) u dq dt \\
&= \iint \phi(x) g_t(t)(p(x,t) - f(t)) dx dt - \int g(t) h(t) dt \\
&= \int g_t(t) \Big(\int \phi(x) p\, dx - f(t) + \int_{t_1}^{t} h(\tau) d\tau \Big) dt \\
&= \int_{-\infty}^{+\infty} \phi(x) p(x, t_1) dx \int g_t dt = 0,
\end{aligned} \tag{4.4}$$

where we have used the definition of $f(t), h(t)$. Therefore $q(x,t)$ is a weak solution of (4.2) in the sense of definition 3.1. By the uniqueness of theorem 3.1 and (3.24), we have the following expressions of $m(x,t)$ and $q(x,t)$ in the region $t \geq t_1$.

$$m(x,t) = \int_{0+0}^{\xi_{t_1}(x,t)-0} dm(\eta, t_1), \tag{4.5}$$

$$q(x,t) = \int_{0+0}^{\xi_{t_1}(x,t)-0} dq(\eta, t_1), \tag{4.6}$$

where $\xi_{t_1}(x,t) = \sup\{\xi : X_{t_1}(\xi, t) < x\}$ and $X_{t_1}(\xi,t)$ is defined in (3.5).

We note that (4.5) and (4.6) are not our desired results because $\xi_{t_1}(x,t)$ depends on $u(x,t)$. However, they provide some information on the structure of the entropy solution. Next, we shall use (4.5),(4.6) to show

$$m(x,t) = -\frac{\partial}{\partial x} \min_y F_{t_1}(y; x, t), \qquad q(x,t) = \frac{\partial}{\partial t} \min_y F_{t_1}(y; x, t), \tag{4.7}$$

in the sense of distributions, where

$$F_{t_1}(y; x, t) = \int_{0+0}^{y-0} (t - t_1) u(\eta, t_1) + \eta - x dm(\eta, t_1), \tag{4.8}$$

is the generalized potential which only depends on the value of (m, q) at $t = t_1$. This leads to the uniqueness of entropy solution in the region $t \geq t_1$.

Since

$$m_t + q_x = m_t + u m_x = 0, \tag{4.9}$$

is conserved, we introduce the following generalized potential $\Phi_{t_1}(x,t)$

$$\Phi_{t_1}(x,t) = \oint_{(0,t_1)}^{(x,t)} m(x-0,t) dx - q(x-0,t) dt. \tag{4.10}$$

Obviously, $\Phi_{t_1}(x,t)$ is independent of the integral path and $(\Phi_{t_1})_x = m, (\Phi_{t_1})_t = -q$ hold in the sense of distributions.

Now we prove

$$\Phi_{t_1}(x,t) = -\min_y F_{t_1}(y;x,t), \tag{4.11}$$

by choosing two integral paths.

For any point $(x_0,t_0) \in R_+^2$, the first integral path is from $(0,t_1)$ to $(\xi_{t_1}(x_0,t_0),t_1)$ along the line $t=t_1$ and from $(\xi_{t_1}(x_0,t_0),t_1)$ to (x_0,t_0) along the curve $L' : x = X_{t_1}(\xi_{t_1}(x_0,t_0),t)$. Let $\xi_{t_1} = \xi_{t_1}(x_0,t_0)$ and $\tau_0 = t_0 - t_1$. We compute

$$\begin{aligned}
\Phi_{t_1}(x_0,t_0) &= \int_0^{\xi_{t_1}} m(\eta-0,t_1)d\eta + \int_{L'} m(x-0,t)dx - q(x-0,t)dt \\
&= \int_{0+0}^{\xi_{t_1}-0} \xi_{t_1} - \eta \; dm(\eta,t_1) - \tau_0 \int_{0+0}^{\xi_{t_1}-0} u(\eta,t_1)dm(\eta,t_1) \\
&\quad +(x_0-\xi_{t_1}) \int_{0+0}^{\xi_{t_1}-0} dm(\eta,t_1) \\
&= \int_{0+0}^{\xi_{t_1}-0} x_0 - \eta - \tau_0 u(\eta,t_1)dm(\eta,t_1),
\end{aligned} \tag{4.12}$$

where we have used (4.5) and (4.6) and the fact that $\xi_{t_1}(x,t) = \xi_{t_1}$ along the curve $X_{t_1}(\xi_{t_1},t)$.

On the other hand, we choose another integral path that is from $(0,t_1)$ to $(x_0 - v\tau_0, t_1)$ along the line $t=t_1$ and from $(x_0-v\tau_0,t_1)$ to (x_0,t_0) along the line $L : x = x_0 + v(t-t_0), t_1 \le t \le t_0$, with arbitrary v. We compute

$$\begin{aligned}
\Phi_{t_1}(x_0,t_0) &= \int_0^{x_0-v\tau_0} m(\eta-0,t_1)d\eta + \int_L mdx - qdt \\
&= \int_0^{x_0-v\tau_0} m(\eta-0,t_1)d\eta + \int_{t_1}^{t_0} (vm(x-0,t) - q(x-0,t))dt,
\end{aligned} \tag{4.13}$$

where $x = x_0 + v(t-t_0), t_1 \le t \le t_0$. Since v is a constant, from (4.1) and (4.2), we have,

$$\begin{aligned}
&\iint (\varphi_t + v\varphi_x)(q - vm)dxdt \\
&= \iint \varphi(u^2 - vu)dmdt - v \iint \varphi(u-v)dmdt \\
&= \iint \varphi(u-v)^2 dmdt \ge 0,
\end{aligned} \tag{4.14}$$

for any $\varphi \ge 0,\ \varphi \in C_0^\infty(R_+^2)$.

Let $y = x - vt$, $\hat{\varphi}(y,t) = \varphi(x,t)$, then (4.14) becomes

$$\iint \frac{\partial}{\partial t}\hat{\varphi}(\hat{q} - v\hat{m})dydt \ge 0. \tag{4.15}$$

Let $\tau = t - t_1, t \ge t_1$, then (4.15) implies

$$q(x,t) - vm(x,t) \le q(x-v\tau,t_1) - vm(x-v\tau,t_1) \text{ a.e.} \tag{4.16}$$

which indicates

$$q(x-0,t) - vm(x-0,t) \le q(x_0 - v\tau_0 - 0, t_1) - vm(x_0 - v\tau_0 - 0, t_1) \tag{4.17}$$

along the line $L : x = x_0 + v(t-t_0)$, $t_1 \le t \le t_0$.

From (4.13) and (4.17), we have

$$\begin{aligned}
\Phi_{t_1}(x_0,t_0) &\geq \int_0^{x_0-v\tau_0} m(\eta-0,t_1)d\eta + v\tau_0 m(x_0-v\tau_0-0,t_1) \\
&-\tau_0 q(x_0-v\tau_0-0,t_1) \\
&= \int_{0+0}^{x_0-v\tau_0-0}(x_0-v\tau_0-\eta)dm(\eta,t_1) + v\tau_0\int_{0+0}^{x_0-v\tau_0-0} dm(\eta,t_1) \\
&-\tau_0\int_{0+0}^{x_0-v\tau_0-0} u(\eta,t_1)dm(\eta,t_1) \\
&= \int_{0+0}^{x_0-v\tau_0-0} x_0-\eta-\tau_0 u(\eta,t_1)dm(\eta,t_1).
\end{aligned} \tag{4.18}$$

Since v is arbitrary, we have

$$\Phi_{t_1}(x_0,t_0) \geq \max_y \int_{0+0}^{y-0} x_0-\eta-\tau_0 u(\eta,t_1)dm(\eta,t_1). \tag{4.19}$$

Combining (4.12) and (4.19), we have

$$\begin{aligned}
\Phi_{t_1}(x_0,t_0) &= \max_y \textstyle\int_{0+0}^{y-0} x_0-\eta-(t_0-t_1)u(\eta,t_1)dm(\eta,t_1), \\
&= -\min_y F_{t_1}(y;x_0,t_0).
\end{aligned} \tag{4.20}$$

On the other hand, by existence theorem 1, we can construct a standard entropy solution $m_{t_1,s}, u_{t_1,s}$ satisfying $m_{t_1,s} = (\Phi_{t_1})_x$, $q_{t_1,s} = -(\Phi_{t_1})_t$, where $(q_{t_1,s})_x = u_{t_1,s}(m_{t_1,s})_x$. We note that the constructive procedure above starts from the line $t = t_1 > 0$. Since $m = (\Phi_{t_1})_x$, $q = -(\Phi_{t_1})_t$, we have $m = m_{t_1,s}, q = q_{t_1,s}$ a.e. in $t \geq t_1 > 0$. Therefore the uniqueness of entropy solution of (1.1) in the region $t \geq t_1 > 0$ is proved. Next, we shall show $m(x,t), q(x,t)$ only depends on the initial value. This leads to the uniqueness of entropy solution for pressureless gases.

For any (x_0,t_0), from (3.8), (4.5) and (2.67-2.69), we know that, $\xi_{t_1}(x_0,t_0)$ belongs to the set of minimum points of

$$F_{t_1}(y;x_0,t_0) = \int_{0+0}^{y-0} t_0 u(\eta,t_1)+\eta-x_0\ dm(\eta,t_1)$$

except denumerable points. i.e. $\xi_{t_1} \in S_{t_1}(x_0,t_0)$, where

$$S_{t_1}(x_0,t_0) = \{y; \exists y_n \to y, s.t. F_{t_1}(y_n;x_0,t_0) \to \min_y F_{t_1}(y;x_0,t_0)\}. \tag{4.21}$$

Now, we choose a sequence of $t_i > 0, i = 1,2,\cdots$ such that $t_i \to 0$, as $i \to \infty$. By Lemma 2.4 and (2.67-2.69), m and E keep constant along the straight lines $L_{t_i} : x = x_0 + \frac{x_0-\xi_{t_i}}{t_0-t_i}(t-t_0), t_i \leq t \leq t_0$. Furthermore,

$$u(x,t) = \frac{x_0-\xi_{t_i}}{t_0-t_i}, \tag{4.22}$$

holds along L_{t_i} due to (2.70) and (5) of Lemma 3.2 if $\xi_{t_i}(x_0,t_0) \in spt\{\rho_0\}$. If $\xi_{t_i}(x_0,t_0) \in R/spt\{\rho_0\}$, then m, q must be constant in some neighborhood V_s of (x,t), i.e. $\rho = m_x = 0$ in V_s. We redefine

$$u(x,t) = \frac{x_0-\xi_{t_i}}{t_0-t_i}.$$

Since $\xi_{t_i}(x,t)$ is increasing in x, the solution we redefine still satisfies Oleinik entropy condition. In view of Lemma 3.2, we have $|\xi_{t_i}-\xi_{t_j}| \leq M|t_i-t_j|$. Therefore, as $t_i \to 0$, ξ_{t_i}

converge toward $H_*(x_0,t_0)$ and L_{t_i} converge to a straight line $L_0: x=x_0+\frac{x_0-H_*(x_0,t_0)}{t_0}(t-t_0), 0\le t\le t_0$. Therefore $m(x,t)$ and $q(x,t)$ are constant along the line L_0 and $u(x,t)=\frac{x_0-H_*(x_0,t_0)}{t_0}$. Namely, $m(x_0,t_0), q(x_0,t_0)$ are the functions of $H_*(x_0,t_0)$.

It is easy to see the map from (x,t) to $H_*(x,t)$ is not one to one. If there exists $x_1<x_2$ such that $H_*(x_1,t)=H_*(x_2,t)$, then for any $x\in(x_1,x_2)$, $H_*(x,t)=H_*(x_1,t)$.

On the other hand, each point $(\eta,0)$ of X-axis generates at least one curve if we choose $t_1=0$ in (3.4) and let $\varepsilon_i\to 0$. Define

$$\Omega_s=\{\eta:\forall t>0,\exists x, s.t., \eta(x,t)-\xi(x,t)>0 \text{ and } \xi(x,t)\le\eta\le\eta(x,t)\},$$

where

$$\xi(x,t)=\sup\{\xi; X(\xi,t)<x\},\quad \eta(x,t)=\inf\{\xi; X(\xi,t)>x\}.$$

Since the weak solution of (3.1) admits at most countable shock waves due to Lemma 3.2, Ω_s is a zero measure set. It is easy to see, for any $\eta\in R/\Omega_s$, there is $t_0>0$ such that the curve $X(\eta,t), X(\eta,0)=\eta, 0<t\le t_0$ lies in R/Γ, with $\Gamma=\{(x,t);\xi(x,t)\ne\eta(x,t), t>0\}$. Therefore $H_*(X(\eta,t),t)=\eta$ holds for any $0<t\le t_0$ from Lemma 3.2.

Let $w(\eta)=\int_{0+0}^{\eta-0}dm_0(x)-m(x(\eta),t(\eta))$, where $(x(\eta),t(\eta))$ is any point located on the curve $X(\eta,t)$ and satisfies $H_*(x(\eta),t(\eta))=\eta$. It is easy to see $w(\eta)$ is independent of the choice of $(x(\eta),t(\eta))$, if $\eta\in R/(U\cup\Omega_s)$ with U introduced in (4) of Lemma 3.2. Thus $w(\eta)$ is well defined almost everywhere in R^1 due to the zero measure set U.

$\forall\phi(\eta)\in C_0^\infty(R^1)$, we compute

$$\begin{aligned}|\textstyle\int\phi(\eta)w(\eta)d\eta|\le\ & |\textstyle\int\phi(\eta)(\int_{0+0}^{\eta-0}dm(x,t(\eta))-m(x(\eta),t(\eta)))d\eta|\\ &+|\textstyle\int\phi(\eta)(\int_{0+0}^{\eta-0}dm(x,t(\eta))-\int_{0+0}^{\eta-0}dm_0(x))d\eta|.\end{aligned}\tag{4.23}$$

Take any $t_0>0$, let

$$Y_{t_0}=\{\eta\in spt\{\phi\}:\exists t(\eta)=t_0,\ s.t.\ H_*(x(\eta),t(\eta))=\eta\}.$$

Obviously, Y_{t_0} converges to $spt\{\phi\}/(\Omega_s\cap U)$ as $t_0\to 0$, i.e. $\lim_{t_0\to 0} meas\{spt\{\phi\}/Y_{t_0}\}=0$. Thus (4.23) becomes

$$\begin{aligned}|\textstyle\int\phi(\eta)w(\eta)d\eta|\le\ & |\textstyle\int\phi(\eta)(\int_{0+0}^{\eta-0}dm(x,t_0)-m(x(\eta),t_0))d\eta|\\ &+|\textstyle\int\phi(\eta)(\int_{0+0}^{\eta-0}dm(x,t_0)-\int_{0+0}^{\eta-0}dm_0(x))d\eta|\\ &+C meas\{spt\{\phi\}/Y_{t_0}\}.\end{aligned}\tag{4.24}$$

Since the mass is weakly continuous at $t=0$ and $|x(\eta)-\eta|\le Mt_0$, $\eta\in Y_{t_0}$, letting $t_0\to 0$ gives $w(\eta)=0$ a.e. in R^1. Similarly, let $z(\eta)=\int_{0+0}^{\eta-0}u_0dm_0-q(x(\eta),t(\eta))$, then we get $q(\eta)=0$ a.e. in R^1. It is noted that, for any $(x,t)\in R_+^2$, there exists a point η on X-axis such that $H_*(x,t)=\eta$. Thus, we have

$$m(x,t)=\int_{0+0}^{H_*(x,t)-0}dm_0(\eta),\ if\ \forall\tilde{x}\ne x, H_*(\tilde{x},t)\ne H_*(x,t),\tag{4.25}$$

$$q(x,t)=\int_{0+0}^{H_*(x,t)-0}u_0(\eta)dm_0(\eta),\ if\ \forall\tilde{x}\ne x, H_*(\tilde{x},t)\ne H_*(x,t),\tag{4.26}$$

Next we turn to the case that a whole interval maps into one point $H_*(x,t)$, i.e. $H_*(x,t)\in U$. Without loss of generality, we assume that $H_*(x,t)=0$ and denote $x_1(t)=X(0-0,t), x_2(t)=X(0+0,t)$. Obviously, $m(x,t), q(x,t)$ keep constant along the line $x = ct, x_1(t)/t < c < x_2(t)/t$. Therefore, the region V surrendered by $x_1(t), x_2(t)$ is convex and m, q are functions of $\xi = \frac{x}{t}$. Of course, $u(x,t) = \frac{x}{t}$ is also a function of ξ. Since the velocity u is always finite and $x_1'(t), x_2'(t)$ are monotone functions, $x_1'(t), x_2'(t)$ have finite limits as $t \to 0$. Let $\xi_1 = \lim_{t\to 0} x_1'(t)$, $\xi_2 = \lim_{t\to 0} x_2'(t)$. We define

$$m(\xi) = \begin{cases} \lim_{t\to 0} m(x_1(t) - 0, t), & \xi < \xi_1, \\ m(x,t), & \xi_1 < \xi < \xi_2, \\ \lim_{t\to 0} m(x_2(t) + 0, t), & \xi > \xi_2. \end{cases} \tag{4.27}$$

Choose a smooth function φ_ε satisfying $\varphi_\varepsilon = 1$ in $(x_1(t) - \varepsilon, x_2(t) + \varepsilon)$ and $\varphi_\varepsilon = 0$ in $(-\infty, x_1(t) - 2\varepsilon)$ and $(x_2(t) + 2\varepsilon, +\infty)$, we compute that

$$\begin{aligned} \int \varphi_\varepsilon dm(x,t) = & \int_{x_1(t)-2\varepsilon}^{x_1(t)-0} \varphi_\varepsilon dm(x,t) + [m(x_1(t),t)] \\ & + \int_{\frac{x_1(t)}{t}+0}^{\frac{x_2(t)}{t}-0} dm(\xi) + [m(x_2(t),t)] \\ & + \int_{x_2(t)+0}^{x_2(t)+2\varepsilon} \varphi_\varepsilon dm(x,t). \end{aligned} \tag{4.28}$$

where $[m(x,t)] = m(x+0,t) - m(x-0,t)$.

Since the first and fifth terms of right hand side of (4.28) tend to zero as $\varepsilon \to 0$ and m_x is weakly continuous at $t = 0$, let $t \to 0$, we have

$$\begin{aligned} [m_0(0)] &= \lim_{t\to 0}[m(x_1(t),t)] + \textstyle\int_{\xi_1+0}^{\xi_2-0} dm(\xi) + \lim_{t\to 0}[m(x_2(t),t)] \\ &= \textstyle\int_{\xi_1-0}^{\xi_2+0} dm(\xi). \end{aligned} \tag{4.29}$$

Similarly, we also have

$$[m_0(0)]u_0(0) = \int_{\xi_1-0}^{\xi_2+0} \xi dm(\xi), \tag{4.30}$$

$$\frac{1}{2}[m_0(0)]u_0^2(0) = \int_{\xi_1-0}^{\xi_2+0} \frac{1}{2}\xi^2 dm(\xi), \tag{4.31}$$

where we have used the energy condition in (4.31), i.e. $u^2 m_x$ is weakly continuous at $t = 0$.

Combining (4.29)-(4.30) gives

$$\int_{\xi_1-0}^{\xi_2+0} (\xi - u_0(0))^2 dm(\xi) = 0. \tag{4.32}$$

We claim that we can obtain a formula of m, q by initial value from (4.32). If $[m_0(0)] = 0$, then $m(\xi)$ is constant in V which infers that

$$m(x,t) = m(x_1(t) - 0, t) = m(x_2(t) + 0, t) = \int_{0+0}^{H_*(x,t)-0} dm_0(\eta). \tag{4.33}$$

When $[m_0(0)] > 0$, $m(\xi)$ must be discontinuous at $\xi = u_0(0)$ because $m(\xi)$ is increasing in ξ,. From (4.24), we have $\xi_1 \le u_0(0) \le \xi_2$. This means $m(\xi)$ takes two values determined by $\xi = u_0(0)$. Namely,

$$m(x,t) = \begin{cases} \displaystyle\int_{0+0}^{H_*(x,t)-0} dm_0(\eta), & if\ x < u_0(H_*(x,t))t, \\ \displaystyle\int_{0+0}^{H_*(x,t)+0} dm_0(\eta), & if\ x > u_0(H_*(x,t))t. \end{cases} \tag{4.34}$$

By (4.34), it is easy to check that

$$q(x,t) = \begin{cases} \displaystyle\int_{0+0}^{H_*(x,t)-0} u_0(\eta)dm_0(\eta), & if\ x < u_0(H_*(x,t))t, \\ \displaystyle\int_{0+0}^{H_*(x,t)+0} u_0(\eta)dm_0(\eta), & if\ x > u_0(H_*(x,t))t. \end{cases} \tag{4.35}$$

Therefore any entropy solution $(m(x,t), q(x,t))$ can be expressed by its initial value. Furthermore, $m(x,t)$ and $q(x,t))$ keep constant along the straight line which connects (x,t) and $(H_*(x,t),0)$. It is observed that the previous argument for $t_1 > 0$ still holds here if (m,q) can be expressed by its initial data. Therefore we have $\Phi_x = m(x,t)$, $\Phi_t = -q(x,t)$, where

$$\Phi(x,t) = \max_y \int_{0+0}^{y-0} x - \eta - tu_0(\eta)dm_0(\eta) = -\min_y F(y;x,t). \tag{4.36}$$

Assume that (m_s, u_s) be the standard entropy solution constructed by (4.36), then $m = \Phi_x = m_s$ and $q = -\Phi_t = q_s$ hold almost everywhere in R^2_+. Therefore the uniqueness Theorem 2 is proved.

Finally, we show that the energy condition is necessary for the uniqueness of entropy solution. Without the energy condition, we are able to construct at least two weak solutions satisfying Oleinik entropy condition with the same initial data. For example, we consider the initial data

$$(\rho_0, u_0) = \begin{cases} (1,0), & x < 0, \\ (\delta(x), 1), & x = 0, \\ (1,2), & x > 0. \end{cases} \tag{4.37}$$

We construct

$$(\rho_1, u_1) = \begin{cases} (1,0), & x < 0, \\ (0, x/t), & 0 \le x < t, \\ (\delta(x-t), 1), & x = t, \\ (0, x/t), & t < x \le 2t, \\ (1,2), & x > 0, \end{cases} \tag{4.38}$$

$$(\rho_2, u_2) = \begin{cases} (1,0), & x < 0, \\ (1/2t, x/t), & 0 \le x < 2t, \\ (1,2), & x > 0. \end{cases} \tag{4.39}$$

Obviously, both (ρ_1, u_1) and (ρ_2, u_2) are the weak solutions of (1.1) with initial data (4.37). Moreover, u_1, u_2 satisfy Oleinik entropy condition.

By direct computation, $u_1^2\rho_1$ weakly converges to $u_0^2\rho_0$. However, it is not right for ρ_2, u_2. Therefore ρ_1, u_1 is the entropy solution of (1.1), (4.37) in the sense of definition 1.2. This shows that the energy initial condition is necessary for the uniqueness of weak solution to pressureless gases (1.1).

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