POINTWISE GREEN'S FUNCTION BOUNDS AND STABILITY OF RELAXATION SHOCKS

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Abstract. We establish sharp pointwise Green's function bounds and consequent linearized stability for smooth traveling front solutions, or relaxation shocks, of general hyperbolic relaxation systems of dissipative type, under the necessary assumptions ([G,Z.1,Z.4]) of spectral stability, i.e., stable point spectrum of the linearized operator about the wave, and hyperbolic stability of the corresponding ideal shock of the associated equilibrium system, with no additional assumptions on the structure or strength of the shock. Restricting to Lax type shocks, we establish the further result of nonlinear stability with respect to small $L^1 \cap H^2$ perturbations, with sharp rates of decay in L^p, $2 \le p \le \infty$, for weak shocks of general simultaneously symmetrizable systems; for discrete kinetic models, and initial perturbation small in $W^{3,1} \cap W^{3,\infty}$, we obtain sharp rates of decay in L^p, $1 \le p \le \infty$, for (Lax type) shocks of arbitrary strength. This yields, in particular, nonlinear stability of weak relaxation shocks of the discrete kinetic Jin–Xin and Broadwell models, for which spectral stability has been established in [HL,JH] and [KM], respectively.

Our analysis follows the basic pointwise semigroup approach introduced by Zumbrun and Howard [ZH] for the study of traveling waves of parabolic systems; however, significant extensions are required to deal with the nonsectorial generator and more singular short-time behavior of the associated (hyperbolic) linearized equations. Our main technical innovation is a systematic method for refining large-frequency (short-time) estimates on the resolvent kernel, suitable in the absence of parabolic smoothing. This seems particularly interesting from the viewpoint of general linear theory, replacing the zero-order estimates of existing theory with a series expansion to arbitrary order. The techniques of this paper should have further application in the closely related case of traveling waves of systems with partial viscosity, for example in compressible gas dynamics or MHD.

Section 1. Introduction

A variety of nonequilibrium processes in continuum mechanics can be modeled by *hyperbolic relaxation systems* of general form

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix}_x = \begin{pmatrix} 0 \\ \tau^{-1} q(u,v) \end{pmatrix}, \tag{1.1}$$

$u,\ f \in \mathbb{R}^n$, $v,\ g,\ q \in \mathbb{R}^r$, where

$$\text{Re}\ \sigma\big(q_v(u, v^*(u))\big) < 0 \tag{1.2}$$

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along a smooth equilibrium manifold defined by

$$q(u, v^*(u)) \equiv 0, \tag{1.3}$$

and τ is a (usually small) parameter determining relaxation time: for example, non-thermal equilibrium gas dynamics [W,MR,Lev], traffic dynamics [LW,AR,Li], and multiphase flow [BGDV,BV,NT,MaP]; for a general discussion of relaxation models, we refer the reader to [L.2,CLL] or to the surveys [N,Yo.4].

A particularly natural class of relaxation systems are those obtained by discretizing continuous kinetic models such as Boltzmann or Vlasov–Poisson equations, for which the lefthand side of (1.1) reduces to a simple linear transport operator: for example, Broadwell and other lattice gas models [PI]. Here, loosely following Natalini [N], we define the class of *discrete kinetic models* as systems of form (1.1) for which f and g are linear, constant coefficient. These include in particular the subclass of semilinear relaxation schemes, such as the Jin–Xin [JX] and Natalini [N] models, that have enjoyed considerable recent popularity as a method for numerical approximation of hyperbolic systems.

The relaxation system (1.1) can be viewed "to zeroth order" as a regularization of the associated *equilibrium*, or "relaxed" system of conservation laws

$$u_t + f^*(u)_x = 0, \tag{1.4}$$

$f^*(u) := f(u, v^*(u))$, which system may or may not be hyperbolic (see [JX,BGDV,BV] for examples of nonhyperbolicity). To "first order," it may be approximated, at least formally, by an associated parabolic system of conservation laws

$$u_t + f^*(u)_x = \tau(B^*(u)u_x)_x, \tag{1.5}$$

where

$$\begin{aligned} B^*(u) &:= -f_v q_v^{-1}(g_u^* - v_u^* f_u^*) \\ &= -f_v q_v^{-1}(g_u - g_v q_v^{-1} q_u + q_v^{-1} q_u (f_u - f_v q_v^{-1} q_u), \end{aligned} \tag{1.6}$$

$g^* := g(u, v^*(u))$, is determined by Chapman–Enskog expansion as described in, e.g., [Wh,L.2,Z.1]. Here, the order referred to is with respect to relaxation time τ; however, as pointed out by Liu in his seminal work [L.2], this formal approximation has a more readily justifiable interpretation in terms of *large time*, rather than small τ behavior. It is this point of view that is relevant for the present work, in which we investigate large time behavior for a *fixed* relaxation time. From now on, we take without loss of generality $\tau = 1$, and suppress the parameter τ.

Note that, in the frequently occurring case that $r < n$, the $n \times n$ matrix B^* is necessarily singular, since it factors through the $r \times r$ matrix q_v^{-1}. Thus, the proper analogy is to "real viscosity," or partially parabolic systems of conservation laws. Indeed, this analogy holds even in the case that B^* is nonsingular, since the high frequency behavior of the original system (1.1) is by necessity hyperbolic, irregardless of formal expansion; see [Ro,Sl.1–2,Ma] for interesting discussions on the validity/meaning of the Chapman–Enskog expansion in the high-frequency regime. The analogy is in fact much deeper than this. As pointed out by Zeng [Ze.2], the two problems (relaxation/real viscosity) are essentially dual in the linear constant–coefficient case. A similar duality, at the structural level, may be seen in the variable-coefficient case [Z.4]

A particularly interesting phenomenon, suggested by (1.4), is the existence of smooth *traveling front solutions*

$$
\begin{aligned}
&(u,v)(x,t) = (\bar u, \bar v)(x-st)\\
&\lim_{z\to\pm\infty} (\bar u, \bar v) = (u_\pm, v_\pm),
\end{aligned}
\tag{1.7}
$$

of (1.1), where by necessity $v_\pm = v^*(u_\pm)$ and $u_\pm$ corresponds to a shock solution of (1.4). See [L.2] for a treatment of existence in the general case $n = r = 1$, and [YoZ,FZe] for generalizations in case n or $r > 1$; further results/discussion are given in Section 1.1, below. Such traveling waves are known as *relaxation shocks* or *relaxation profiles.* The question of their stability was investigated in [L.2] in the general case $n = r = 1$, for which the equilibrium system is scalar, under the assumption of "weak," or small-amplitude shocks. This analysis has been generalized to strong shocks in the special case of the 2×2 Jin–Xin model [MaN]. For the 3×3 Broadwell model, a standard discrete kinetic model for which $n = 2$, $r = 1$, Szepessy and Xin [S,SX] have announced the result of linear and nonlinear stability of weak shocks, extending partial, results of [KM,CaL]: to our knowledge, the only such result for the system case $n > 1$ occurring in most physical applications. However, to date there is no complete analysis of stability for any other case, in particular for the case $n > 1$, $r >> 1$ arising through approximation of Boltzmann or Vlasov–Poisson equations by moment closure or discretization. For strong shocks, there is no complete analysis for *any* system such that n or $r > 1$.

Useful necessary conditions for stability have been obtained for shock profiles of the Jin–Xin relaxation model in [Z.1,G] and for shock profiles of general relaxation and real viscosity models in [Z.4], generalizing corresponding results of [GZ,BSZ,ZS] in the strictly parabolic case. Strengthened versions of the one-dimensional hyperbolic stability criterion of Erpenbeck–Majda [Er,M.1–3], these conditions yield interesting results of *spectral instability* (hence linearized and nonlinear instability), similarly as in the strictly parabolic case (see e.g. [GZ,FreZ,ZS,Z.4]). Notably, the above-mentioned analyses apply to shocks of arbitrary strength and type (i.e., Lax, under-, or overcompressive).

However, to our knowledge, for $n > 1$, the only positive stability result for relaxation shocks other than the ones of [KM,CaL,S] is a partial result recently established by H. Liu [Liu] of linear and nonlinear *zero-mass stability* of weak shocks of the Jin–Xin system, i.e., stability with respect to perturbations whose integral in the u variable is zero; see also a subsequent and closely related proof by Humpherys

[Hu] of *spectral stability* of small-amplitude Jin–Xin shocks, defined as stability of the point spectrum of the linearized operator about the wave. (Note: spectral stability is implied by zero-mass stability, see discussion, [ZH] or [Hu]). Being based on energy methods, none of these previous analyses for systems ([KM,S,Liu,Hu]) generalize to the case of strong (i.e., large-amplitude) shocks. Moreover, they each depend to some degree on special structure of the respective systems treated; hence, the extent to which they generalize to other systems is not clear.

In the present paper, following the general philosophy set out in [ZH] for the study of viscous shock waves, we investigate the linearized and nonlinear stability of relaxation shocks of general systems (1.1), with arbitrary n, r, but *assuming spectral stability.* As in the parabolic case, what makes this enterprise nontrivial is that the essential spectrum of the linear generator accumulates at the imaginary axis, so that linearized decay cannot immediately be concluded from spectral stability, and moreover can be at most at time-algebraic and not exponential rate. Indeed, much of the technical difficulty of the problem lies in this usually trivial step. Spectral stability can then be investigated separately as an ODE problem, or checked numerically as in [Br.1–2]. An important advantage of this approach is that it is completely general, applying in principle to shocks *of all types* (i.e., Lax, under-, or overcompressive) *and strengths.*[1]

More precisely, we study linearized stability for systems (1.1) satisfying appropriate dissipation conditions as in [Yo.4] or [Ze.2], and nonlinear stability for weak Lax shocks of (general) symmetrizable systems (defined Section 7.2), or strong Lax shocks of discrete kinetic models. Our main example will be the Jin–Xin model, for which calculations are particularly transparent. In this case, combining our results with the existing spectral stability results of [Liu,Hu], we obtain the result that Lax-type Jin–Xin shocks of sufficiently weak strength are nonlinearly stable under small $L^1 \cap L^\infty$ perturbations, with no additional hypotheses: the first such result for a relaxation shock in the case n, $r > 1$. Likewise, using the zero-mass stability result of [KM] (recall: zero-mass implies spectral stability), we recover a more detailed version of the result of [S,SX] that weak Broadwell shocks are nonlinearly stable, with the additional information of sharp rates of decay in all L^p. (Note: No rates of decay are given in [S,SX].) In the course of our analysis, we develop extremely detailed bounds on the Green's function of the linearized evolution equations about the traveling wave, of interest in their own right. These generalize bounds obtained in [ZH,Z.2–3] for shock profiles of strictly parabolic systems, and in [Ze.1,LZe], [Ze.2] for constant solutions of partially parabolic and relaxation systems, respectively.

Our results are of interest from the physical point of view as some of the first complete shock stability results for systems with realistic dissipative mechanism such as relaxation or partial viscosity: in particular the first positive stability results for strong relaxation shocks (namely, the reduction to the numerically verifiable spectral stability condition $(\mathcal{D})$), the first for quasilinear systems, and the first results for relaxation shocks of any strength to give rates of decay. From the technical side,

[1] However, see Remark 2 just below, regarding subshocks.

they are of interest as a blueprint for the application of the pointwise semigroup methods of [ZH] to the case that the associated semigroup is C^0 and not analytic: in particular, when the Green's function $G(x,t;y)$ is a distribution, lying outside the class of smooth functions for $t>0$.

With regard to the latter, technical issue, we mention related analyses carried out by Dodd [Do] and Howard–Zumbrun [HZ.2] for shock profiles of dissipative–dispersive conservation laws, for which the semigroups associated with the linearized equations are likewise merely C^0. The analysis of [HZ.2] in particular proceeds from a somewhat similar point of view to that followed here, and could be considered as an intermediate point between the analysis of [ZH] and that carried out here. The main technical innovation distinguishing the present treatment is a systematic method for refining high-frequency bounds on the resolvent kernel to any desired accuracy; for comparison, the zeroth order step in this algorithm gives bounds at the level of [ZH], and the first order approximately at the level of [HZ.2][2], whereas here we require bounds to the second order. In effect, we generate an arbitrary order asymptotic expansion, limited only by smoothness of the coefficients of the linearized operator, of the resolvent about an essential singularity at infinity, with strong error bounds valid almost up to the boundary of analyticity/essential spectrum. This, and other techniques of the paper should apply also to systems with incomplete parabolicity, such as the equations of compressible gas dynamics or MHD.

We now describe our results in somewhat greater detail. Consider a smooth traveling wave solution (1.7) of relaxation system (1.1). Besides (1.2)–(1.3), we make the following, rather standard assumptions (see, e.g., [Yo.1–3,YoZ,Z.4]):

(H0) $f,\, g,\, q \in C^3$ in the general case, $f,\, g,\, q \in C^2$ for discrete kinetic models.

(H1) $\sigma\left((df,dg)^t(u,v)\right)$ real, semi-simple with constant multiplicity, and different from s.

(H2) $\sigma\left(df^*(u_\pm)\right)$ real, distinct, and different from s.

(H3) $\mathrm{Re}\ \sigma\left(i\xi\begin{pmatrix} df\\ dg\end{pmatrix}(u_\pm,v_\pm)+\begin{pmatrix} 0\\ dq\end{pmatrix}(u_\pm,v_\pm)\right)\le \dfrac{-\theta|\xi|^2}{1+|\xi|^2},\quad \theta>0$, for all $\xi\in\mathbb{R}$.

(H4) The set of solutions of (1.7) forms a smooth manifold $(\bar u^\delta,\bar v^\delta)$, $\delta\in\mathcal{U}\in\mathbb{R}^\ell$.

With the possible exception of the "noncharacteristic condition," $s\notin\sigma(df,dg)^t$, these are precisely analogous to the hypotheses given in [ZH,Z.4] for the viscous case. Conditions (H1)–(H3), with θ set to zero in (H3), and relaxing the requirement of distinct eigenvalues in (H1), are essentially the standard set of hypothesees proposed by W.-A. Yong for general relaxation systems [Yo.1–3]; as discussed in [Y.4], these are satisfied for most known relaxation sytems, and indeed may be considered as criteria for a "reasonable" model. A slight difference here is that we have assumed strict hyperbolicity/nonsonicity of the equilibrium system, (H2), and stability, (H3),

[2] In fact, we make also key improvements generalizing the approach of [HZ.2] to the system case, and to operators with nonconstant principal part, see Remarks 8.16 and 4.10 below.

only at the endpoints $(u_\pm, v_\pm)$ of the shock, and not everywhere along the profile; this allows applications to interesting nonhyperbolic situations such as are discussed in [JX,BGDV,BV]: in particular, shocks of nonclassical, under- and overcompressive type are admitted in our treatment, along with the classical, Lax type. Existence of weak (necessarily Lax-type) relaxation profiles has been shown under the hypotheses of [Y.1–3] by Yong and Zumbrun [YoZ]; see [FreZ,BR] for related studies.

The slightly strengthened condition (H3) ($\theta > 0$) may be recognized as the *dissipativity* condition of Zeng–Kawashima [Ze.2,Kaw], and is likewise satisfied for a variety of physical models, though not all. The degenerate case $\theta = 0$ is quite delicate, and requires additional hypotheses on the nonlinear structure of (1.1); for a discussion of this interesting situation and related phenomena, see [Ze.2].

Remarks. 1. The noncharacteristic condition $s \notin \sigma(df, dg)^t$ is necessary in order that the traveling wave ODE be of nondegenerate type, a natural assumption in the context of stability of relaxation profiles. (See Section 1.1, below, for a statement of the traveling wave ODE).

2. In (H0)–(H4), we have made no assumptions on the structure or amplitude of the profile $(\bar u, \bar v)$. However, note that the assumption of a smooth profile is in effect a limitation on the strength of the shock, since, as is well known, profiles of sufficiently large amplitude will develop "subshocks," or jump discontinuities [Whi,L.2]. More precisely, this is a restriction on the amplitude with respect to the number of resolving modes in the system [BR,MR]: For example, the allowable amplitude for a moment closure system is a strictly increasing function of the number of moments, apparently approaching infinity in the limiting (infinite-dimensional) case of the Boltzmann equations. The stability of profiles containing subshocks is a very interesting problem for future investigation, and likewise the stability (or instability) of strong Boltzmann profiles. Of course, stability of even weak Boltzmann profiles is a fundamental open problem.

Example 1.1. The Jin–Xin model,

$$\begin{aligned} u_t + u_x &= 0, \\ v_t + a^2 v_x &= h(u) - v, \end{aligned} \tag{1.8}$$

u, $v \in \mathbb{R}^n$, was introduced in [JX] as a numerical scheme approximating solutions of the equilibrium system $u_t + h(u)_x = 0$. System (1.8) is in general nonstrictly hyperbolic, with $\sigma(df, dg)^t$ consisting of the two eigenvalues $\pm a$, each with constant multiplicity n. It is readily checked that (H0)–(H3) hold if and only if: (i) $h \in C^3$; (ii) $a \neq s$; (iii) the equilibrium system $u_t + h(u)_x = 0$ is (nonstrictly) hyperbolic at the endstates $u_\pm$, with $s \notin \sigma\big(dh(u_\pm)\big)$; and, (iv) there holds *at the endpoints of the shock* the subcharacteristic condition $a^2 > \sigma\big(dh(u_\pm)^2\big)$.

Likewise, one finds that the traveling wave ODE for a relaxation shock of speed s reduces to that for an "effective" viscous system,

$$u_t + h(u)_x = (a^2 - s^2) u_{xx}, \tag{1.9}$$

with constant, scalar viscosity $a^2 - s^2$, so that (H4) reduces to the corresponding requirement in the viscous case. Note that (iv) above implies that the effective viscosity $a^2 - s^2$ is positive whenever a profile exists, since in that case s must lie between the extreme values of the spectra of $dh(u_\pm)$. This furnishes a rich set of examples of relaxation shocks falling under our framework, through the existence theory for viscous traveling waves. Note that (1.8) is a discrete kinetic model, since its lefthand side is linear. ∎

1.1. Structure/Classification of Profiles. We begin by presenting some basic, but apparently new results relating the structure of relaxation profiles to the characteristic structure of the corresponding equilibrium shock in the relaxed system (1.4). These generalize observations of Majda and Pego in the viscous, strictly parabolic case [MP]; analogous results hold for viscous profiles with real (i.e., partially parabolic) viscosity, as can be seen by essentially the same arguments used here (see, for example, the unified framework for relaxation/real viscosity models given in [Z.4], Appendices A1/A2).

Without loss of generality taking $s = 0$, by a linear change in the flux functions if necessary, we may write the traveling wave ODE in the form:

$$\begin{aligned} f(u,v)' &= 0, \\ g(u,v)' &= q(u,v), \end{aligned} \tag{1.10}$$

where $'$ denotes d/dx. From (1.10), we immediately see that $q(u_\pm, v_\pm) = 0$ and $f(u) \equiv constant$, so that endstates $(u_\pm, v_\pm)$ lie on the equilibrium manifold, and satisfy the Rankine–Hugoniot jump condition

$$f^*(u_+) - f^*(u_-) = f(u_+, v_+) - f(u_-, v_-) = 0 \tag{RH}$$

for the equilibrium system (1.4): that is, (u_-, u_+) corresponds to a shock solution of (1.4). Let us now consider the behavior of solutions of (1.10) near the rest points $(u_\pm, v_\pm)$, for definiteness (u_+, v_+).

By assumption (H1) together with the Inverse Function Theorem, $(u,v) \to (f,g)$ may be inverted in a neighborhood of (u_+, v_+), reducing (1.10) to the r-dimensional ODE

$$g' = q\big(u(f_+, g), v(f_+, g)\big), \tag{1.11}$$

where f is held constant at value f_+. Linearizing about the rest point g_+, we obtain after a brief calculation the linearized equation

$$g' = (q_u, q_v) \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} g, \tag{1.12}$$

where the righthand side is understood to be evaluated at (u_+, v_+).

Lemma 1.2. *Under assumptions (H0)–(H3), the matrix*

$$(q_u, q_v) \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\pm} \tag{1.13}$$

has no center subspace, i.e., $(u_\pm, v_\pm)$ *are hyperbolic rest points of the reduced traveling wave ODE (1.11). In particular, traveling wave solutions (1.7) satisfy*

$$|(d/dx)^k \big((\bar u(x), \bar v(x)) - (u_\pm, v_\pm)\big)| \le Ce^{-\theta|x|}, \quad k = 0, \dots, 4, \tag{1.14}$$

as $x \to \pm\infty$.

Lemma 1.2 verifies the hypotheses of the Gap Lemma, [GZ,ZH][3], on which we shall rely to obtain the basic ODE estimates underlying our analysis in the low-frequency/large-time regime. A proof of Lemma 1.2 is given in Appendix A1, a description and proof of the Gap Lemma in Appendix A3.

We shall classify relaxation shocks mainly according to their *hyperbolic type*, i.e., the type of the corresponding hyperbolic shock for the equilibrium system (1.4). Specifically, let $i+$ denote the dimension of the stable subspace of $df^*(u_+)$, i_- denote the dimension of the unstable subspace of $df^*(u_-)$, and $i := i_+ + i_-$. Indices $i_\pm$ count the number of incoming characteristics from the right/left of the shock, while i counts the total number of incoming characteristics toward the shock. Then, the relaxed (hyperbolic) shock (u_-, u_+) is called:

$$\begin{cases} \text{Lax type} & \text{if} \quad i = n+1, \\ \text{Undercompressive (u.c.)} & \text{if} \quad i \le n, \\ \text{Overcompressive (o.c.)} & \text{if} \quad i \ge n+2. \end{cases}$$

In case all characteristics are incoming on one side, i.e., $i_+ = n$ or $i_- = n$, a shock is called *extreme*.

Similarly as in the parabolic case, [MP,ZH,ZS,Z.4], there is a close connection between the hyperbolic type of a shock profile, and the nature of the corresponding connection in the phase portrait of the r-dimensional reduced traveling wave ODE (1.11) on the n-dimensional submanifold

$$f(u, v) - su \equiv f(u_-, v_-) - su_-$$

of $\mathbb{R}^{n+r}$, obtained by integration of the u-equation $\big((f(u,v) - su\big) \equiv 0$ in the full traveling wave ODE. Define d_+ to be the dimension within this submanifold of the stable manifold at $(u+, v_+)$, and d_- the dimension of the unstable manifold at (u_-, v_-), and $d := d_- + d_+$. Then, we have:

[3] See also the version established in [KS], independently of and simultaneously to that of [GZ].

Lemma 1.3. *Under assumptions (H0)–(H3), there holds*

$$d - r = i - n. \tag{1.15}$$

Lemma 1.3 generalizes an analogous relation pointed out in the viscous case by Majda and Pego [MP] (see, e.g., Lemma 1.1 [Z.4]). However, the proof is different from the essentially algebraic one given in [MP], following instead an alternative argument suggested in Remark 2.3, [ZH], that is more closely linked to the dynamics of the traveling wave; for an exposition of this proof, see Appendix A1.

A complete description of the connection, of course, requires the further index ℓ defined in (H4) as the dimension ℓ of the connecting manifold between $(u_\pm, v_\pm)$ in the traveling wave ODE. Generically, one expects that ℓ should be equal to the surplus $d-r=i-n$. In case the connection is "maximally transverse" in this sense, i.e.:

$$\ell = \begin{cases} 1 & \text{undercompressive or Lax case,} \\ i-n & \text{overcompressive case,} \end{cases} \tag{1.16}$$

we call the shock "pure" type, and classify it according to its hyperbolic type; otherwise, we call it "mixed" under/overcompressive type. Throughout the paper, we shall assume that (1.16) holds, so that all relaxation profiles are of *pure, hyperbolic type*. This holds in particular for the weak profiles whose existence was established in [YoZ], which are always of pure Lax type.

Remark. In the case of the Jin–Xin model (1.8), relation (1.15) follows directly from the corresponding parabolic result of [MP], see discussion, Example 1.1.

1.2. Spectral vs. Linearized Stability. We next discuss spectral and linearized stability of profiles, and the relation between them: specifically, we show that spectral stability, appropriately defined, is necessary and sufficient for linearized stability. Linearizing (1.1) about the stationary solution $(\bar u, \bar v)$, we obtain the linearized equations

$$U_t = LU := -(AU)_x + QU, \tag{1.17}$$

where

$$A := \begin{pmatrix} df \\ dg \end{pmatrix}(\bar u, \bar v), \quad Q := \begin{pmatrix} 0 \\ dq(\bar u, \bar v) \end{pmatrix}, \tag{1.18}$$

and

$$U = (u, v)^t, \ u \in \mathbb{R}^n, \ v \in \mathbb{R}^r. \tag{1.19}$$

The appropriate notion of stability of shock profiles is *orbital stability*, or convergence to the manifold of stationary solutions $\{(\bar u^\delta, \bar v^\delta)\}$ defined in (H4). Likewise, the relevant notion of linearized stability is that of *linearized orbital stability*, defined as convergence to the *tangent stationary manifold*, consisting of stationary solutions $\{(\partial/\partial\delta_j)(\bar u^\delta, \bar v^\delta)\}$ of the linearized equations (1.17) [ZH]. An evident necessary condition for linearized orbital stability, (and thus for nonlinear orbital stability, appropriately defined) is

$$\text{(D1)} \qquad \sigma(L) \cap \{\text{Re } \lambda \geq 0\} = \{0\},$$

where σ refers to the spectrum of L with respect to some L^p, say L^2 for simplicity. We shall refer to condition (D1) as *strong spectral stability.*

A standard result of Henry [He] gives that the rightmost boundary of the essential spectrum $\sigma_{ess}(L)$ is given by the right envelope of the union of the (purely essential) spectra of the limiting, constant-coefficient operators $L_\pm$ as $x \to \pm\infty$, which by (H3) intersect the non-strictly-stable half-plane Re $\lambda \geq 0$ precisely at the origin $\lambda = 0$. Thus, (D1) is equivalent (as often the case for stability of traveling waves) to nonnegativity of the point spectrum:

$$\text{Re } \sigma_p(L) \cap \{\text{Re } \lambda \geq 0\} = \{0\}. \tag{1.20}$$

One might conjecture that sufficient conditions for linearized orbital stability would be (1.20) augmented with the requirement that $\lambda = 0$ be an ℓ-fold, semi-simple eigenvalue of L, where ℓ is the dimension of the tangent stationary manifold. However, the lack of spectral separation associated with accumulation of the essential spectrum of L at the stationary eigenvalue $\lambda = 0$ prevents us from drawing immediate conclusions in this direction. Indeed, it is not a priori clear how one should properly define eigenspace at a point of accumulation of essential spectrum.

As described in [ZH], such technical difficulties may be conveniently resolved using an extended, or "effective" spectrum based on the *Evans function* of [E.1–4,AGJ,GZ], defined precisely in Section 3, below. An analytic function playing a role for differential operators analogous to that played for finite-dimensional operators by the characteristic polynomial, the Evans function $D(\lambda)$ has the property that, on the resolvent set of L, its zeroes agree (in both location and multiplicity) with the eigenvalues of L [AGJ,GJ.1–2]. Taking this correspondence as the definition of effective point spectrum, we can thus extend the notion of eigenvalue past the essential spectrum boundary of L, via analytic extension of the Evans function [GZ,KS]. The associated effective eigenprojection may then be defined in terms of the induced meromorphic extension of the resolvent kernel, via calculus of residues [ZH]. For details, see Section 3, below.

The first main result of this paper, generalizing the corresponding result established for viscous, strictly parabolic, shocks in [ZH], is:

Theorem 1.4. *Under assumptions (H0)–(H3), shocks of "pure" hyperbolic type are $L^1 \cap L^p \to L^p$ linearly orbitally stable for $p > 1$ if and only if L has precisely ℓ effective eigenvalues in $\{\text{Re } \lambda \geq 0\}$ (necessarily at $\lambda = 0$), or, equivalently, there holds the Evans function condition:*

$(\mathcal{D})$ $\quad D(\cdot)$ *has precisely ℓ zeroes in $\{\text{Re } \lambda \geq 0\}$ (necessarily at $\lambda = 0$).*

As pointed out in [ZH], the apparently simple criterion ($\mathcal{D}$) in fact hides considerable subtlety in behavior, in particular in the critical case that (D1) holds, but ($\mathcal{D}$) does not: see [ZH], section 12 for examples of possible behaviors in this case.

Evidently, ($\mathcal{D}$) is equivalent to the pair of conditions:

($\mathcal{D}$1) $D(\cdot)$ has no zeroes in $\{\text{Re } \lambda \geq 0\} \setminus \{0\}$.

($\mathcal{D}$2) (ii) $(d/d\lambda)^\ell D(0) \neq 0$.

The first condition may be recognized as (D1), above. The second has been shown in [Z.4], Appendix A1 to be equivalent to

(D2) $\bar{\Delta} := \gamma\Delta \neq 0$,

where γ is a coefficient measuring transversality of the connecting manifold between $(u_\pm, v_\pm)$ (i.e., $\gamma \neq 0$ equivalent to transversality of the intersection of unstable/stable manifolds at $(u_-, v_-)/(u_+, v_+)$), and Δ is the one-dimensional "hyperbolic stability coefficient" defined in [ZS,Z.4], appropriate to the type (Lax, undercompressive, or overcompressive) of the wave. For example, in the Lax case, Δ is just the Liu-Majda determinant

$$\Delta := \det\, (r_1^{*-}, \dots, r_{n-i_-}^{*-}, r_{i_++1}^{*+}, \dots, r_n^{*+}, [u]), \tag{1.21}$$

for which $\Delta \neq 0$ is equivalent to hyperbolic stability of the corresponding ideal shock of the equilibrium system (1.4) [M.1–3]; here $r_j^{*\pm}$ are the eigenvectors of $df^*(u_\pm)$, in order of increasing eigenvalue, and $i_\pm$ are as defined above (Section 1.1). For the generalizations of (1.21) in the undercompressive and overcompressive cases, and their interpretations with respect to hyperbolic theory, see [ZS] or [Z.4]. (Note: the overcompressive version is cited also in (6.48)–(6.49) below). In the physical literature, transversality, $\gamma \neq 0$, is referred to as *structural* stability (i.e., stability of inner shock structure), and evolutionary (hyperbolic) stability $\Delta \neq 0$ as *dynamical* stability [BE].

Corollary 1.5. *Under assumptions (H0)–(H3), linear $L^1 \cap L^p \to L^p$ orbital stability of pure shocks, for $p > 1$, is equivalent to (D1)–(D2), or equivalently: strong spectral stability plus classical structural and dynamical stability.*

Remark 1.6. The necessary conditions obtained in [Go,Z.1,Z.4] were *signed conditions*

$$\gamma\Delta \geq 0, \tag{1.22}$$

with appropriate normalization of γ, Δ. For $\gamma\Delta \neq 0$, the sign of $\gamma\Delta$ is associated with a *stability index* determining the parity of the number of unstable eigenvalues Re $\lambda > 0$ of L, hence yields information intermediate to that given in (D1) and (D2). The advantage of (1.22) is that in the extreme, Lax shock case, the sign of the quantity on the lefthand side is explicitly computable as the product of (1.21) and a Wronskian determinant evaluated entirely at one infinity: two linear-algebraic quantities. For history/applications/further discussion of the stability index, see e.g. [GZ,BSZ,ZS,Z.4].

Note, with the result of Theorem 1.4, that (1.22) is strengthened to

$$\gamma\Delta > 0. \tag{1.23}$$

Remark 1.7. The twin conditions (D1)–(D2) reflect multiple scales present in the problem, (D1) corresponding to high-frequency and (D2) to low-frequency scales. As discussed in [Z.4], introduction, this dichotomy is natural from the point of view of formal, matched asymptotic expansion. For a generalization of (D1)–(D2) to multi-dimensions, see [ZS,Z.4].

Remark 1.8. In the case discussed in [YoZ], of weak shock profiles arising in regions where the equilibrium system is strictly hyperbolic, profiles are necessarily of Lax type, and transversal, $\gamma \neq 0$; moreover, the Liu-Majda condition $\Delta \neq 0$ is automatically satisfied, from standard hyperbolic considerations [M.1–3]. Thus, *linearized stability in this case is equivalent to strong spectral stability, (D1).* As remarked earlier, strong spectral stability is equivalent to zero mass stability, so this is indeed a substantial reduction of the problem. For example, zero-mass stability has been shown in the case of weak Jin–Xin profiles [Liu,Hu], whereas linearized stability has not. The verification of (D1) for weak relaxation profiles in this general setting we regard as an extremely interesting open problem in the one-dimensional theory.

1.3. Pointwise Green's Function Bounds. Theorem 1.4 is obtained as a consequence of detailed, pointwise bounds on the Green's function $G(x,t;y)$ of the linearized evolution equations (1.17) (more properly speaking, a distribution), which we now describe. For simplicity, we first restrict to the case that $(df,dg)^t$ is strictly hyperbolic, indicating afterward the extension to the general case.

Let $a_j(x)$, $j=1,\dots(n+r)$ denote the eigenvalues of $A(x)=(df,dg)^t(\bar u(x))$, and $l_j(x)$ and $r_j(x)$ be smooth families of associated left and right eigenvectors, respectively, normalized so that $l_j^t r_k = \delta_k^j$. These are well-defined, by strict hyperbolicity. Eigenvectors $a_j(x)$, and eigenvectors l_j, r_j correspond to hyperbolic characteristic speeds, and modes of propagation of the relaxation model (1.1). Likewise, let $a_j^{*\pm}$, $j=1,\dots,n$ denote the eigenvalues of $df^*(u_\pm)$, and $l_j^{+\pm}$, $r_j^{*\pm}$ associated left and right eigenvector, corresponding to hyperbolic characteristic speeds and modes of propagation of equilibrium system (1.4) at the endpoints $u_\pm$ of the relaxation profile.

Define local, *scalar dissipation rates*

$$\eta_j(x) := -l_j^t Q r_j(x), \quad j=1,\dots,n+r, \tag{1.24}$$

and asymptotic, *scalar diffusion rates*

$$\beta_j^{*\pm} := \left(l_j^{*t} B^* r_j^*\right)_\pm, \quad j=1,\dots,n, \tag{1.25}$$

where $B^*_\pm := B^*(u_\pm)$ is the effective diffusion defined in (1.6), predicted by formal, Chapman–Enskog expansion. As described in Appendix A2, these quantities arise in a natural way, through Taylor expansion of the (frozen-coefficient) Fourier symbol

$$-i\xi A(x) + Q(x) \tag{1.26}$$

of the linearized operator L about $\xi = \infty$ and $\xi = 0$, respectively. As a consequence of dissipativity, (H3), we have that

$$\eta_j^\pm := \eta_j(\pm\infty) > 0, \quad j = 1, \dots, n+r, \tag{1.27}$$

and

$$\beta_j^{*\pm} > 0, \quad j = 1, \dots, n. \tag{1.28}$$

Let us define also $a_j\pm := a_j(\pm\infty)$.

The important quantities η_j, β_j^* were identified by Zeng [Z2] in her study of decay to *constant* (necessarily equilibrium) *solutions* $(\bar u, \bar v) \equiv (u_\pm, v_\pm)$ of relaxation systems, corresponding at the linearized level to the study of the limiting equations

$$U_t = L_\pm U := -A_\pm U_x + Q_\pm U \tag{1.29}$$

as $x \to \pm\infty$ of the linearized evolution equations (1.17). Using Fourier Transform techniques, Zeng established the following sharp, pointwise bounds on the Green's function of (1.29), stated for reference in our own somewhat different notation:

Proposition 1.9(Ze.2). *Assuming (H0)–(H3), plus strict hyperbolicity of $(df, dg)^t$, the Green's function $G(x,t;y)$ associated with the linearized, constant-coefficient evolution equations (1.29) may be decomposed as*

$$G(x,t;y) = H + S + R, \tag{1.30}$$

where

$$H(x,t;y) := \sum_{j=1}^{n+r} e^{-\eta_j^\pm t} \delta(x - y - a_j^\pm t) r_j^\pm l_j^{\pm t}$$

and

$$S(x,t;y) := \sum_{j=1}^{n} \left(4\pi\beta_j^{*\pm}(1+t)\right)^{-1} e^{|x-y-a_j^{*\pm}t|^2/4\beta_j^{*\pm}(1+t)} R_j^{*\pm} L_j^{*\pm t},$$

denote hyperbolic and scattering terms, respectively, and R denotes a faster decaying residual. Here, $a_j^\pm$, $a_j^{*\pm}$, $l_j^\pm$, $l_j^{*\pm}$, $r_j^\pm$, $r_j^{*\pm}$, $\eta_j^\pm$ and $\beta_j^{*\pm}$ are as defined just above, and $R_j^{*\pm}$, $L_j^{*\pm}$ by

$$R_j^{*\pm} := \begin{pmatrix} r_j^{*\pm} \\ -q_v^{-1} q_u r_j^{*\pm}) \end{pmatrix}, \qquad L_j^{*\pm} := \begin{pmatrix} l_j^{*\pm t} \\ 0 \end{pmatrix}. \tag{1.31}$$

Note, as suggested by the derivation of constants $\eta_j^\pm$, $\beta_j^\pm$ in Appendix A2, that short-time ($\sim$ large-frequency) behavior is dominated by hyperbolic term H, while large-time ($\sim$ small-frequency) behavior is dominated by scattering term S. Moreover, the latter term is precisely that identified in [LZe] as describing large-time behavior for the Chapman–Enskog approximation (1.5), in agreement with the general philosophy of [L.2].

Our second main result, and the principal result of this paper, is the following, nonconstant-coefficient analog of Proposition 1.9:

Theorem 1.10. *Assuming (H0)–(H4), plus strict hyperbolicity of $(df, dg)^t$, and the spectral stability criterion $(\mathcal{D})$ (equivalently, (D1)–(D2)), the Green's function $G(x,t;y)$ associated with the linearized evolution equations (1.17) may in the Lax or overcompressive case be decomposed as*

$$G(x,t;y) = H + E + S + R, \tag{1.32}$$

where, for $y \le 0$:

$$H(x,t;y) := \sum_{j=1}^{n+r} e^{-\bar\eta_j t}\delta_{x-\bar a_j t}(-y) r_j(x) l_j^t(y), \tag{1.33}$$

(1.34)

$$E(x,t;y) := \sum_{a_k^- > 0} [c_{k,-}^{j,0}] \frac{\partial}{\partial \delta_j} \begin{pmatrix} \bar u^\delta(x) \\ \bar v^\delta(x) \end{pmatrix} L_k^{*-t} \left(\text{errfn}\left(\frac{y + a_k^{*-} t}{\sqrt{4\beta_k^{*-} t}} \right) - \text{errfn}\left(\frac{y - a_k^{*-} t}{\sqrt{4\beta_k^{*-} t}} \right) \right),$$

and

(1.35)

$$\begin{aligned} S(x,t;y) :=\ & \chi_{\{t\ge1\}} \sum_{a_k^{*-}<0} R_k^{*-} L_k^{*-t} (4\pi\beta_k^- t)^{-1/2} e^{-(x-y-a_k^{*-}t)^2/4\beta_k^{*-}t} \\ & + \chi_{\{t\ge1\}} \sum_{a_k^{*-}>0} R_k^{*-} L_k^{*-t} (4\pi\beta_k^{*-} t)^{-1/2} e^{-(x-y-a_k^{*-}t)^2/4\beta_k^{*-}t} \left(\frac{e^x}{e^x + e^{-x}} \right) \\ & + \chi_{\{t\ge1\}} \sum_{a_k^{*-}>0,\, a_j^{*-}<0} [c_{k,-}^{j,-}] R_j^{*-} L_k^{*-t} (4\pi\bar\beta_{jk}^{*-} t)^{-1/2} e^{-(x-z_{jk}^{*-})^2/4\bar\beta_{jk}^{*-}t} \left(\frac{e^{-x}}{e^x + e^{-x}} \right), \\ & + \chi_{\{t\ge1\}} \sum_{a_k^{*-}>0,\, a_j^{*+}>0} [c_{k,-}^{j,+}] R_j^{*+} L_k^{*-t} (4\pi\bar\beta_{jk}^{*+} t)^{-1/2} e^{-(x-z_{jk}^{*+})^2/4\bar\beta_{jk}^{*+}t} \left(\frac{e^x}{e^x + e^{-x}} \right) \end{aligned}$$

denote hyperbolic, excited, and scattering terms, respectively, and R denotes a faster decaying residual (rates given in Proposition 6.1, below). Symmetric relations hold for $y \ge 0$.

Here, the averaged dissipation and convection rates $\bar\eta_j(y,t)$ and $\bar a_j(y,t)$ in (1.33) denote the time-averages over $[0,t]$ of $\eta_j(x)$ and $a_j(x)$, respectively, along characteristic paths $z_j = z_j(y,t)$ defined by

$$dz_j/dt = a_j(z_j), \quad z_j(0) = y, \tag{1.36}$$

while, in (1.35),

$$z_{jk}^{*\pm(y,t)} := a_j^{*\pm}\left(t - \frac{|y|}{|a_k^{*-}|}\right) \tag{1.37}$$

and

$$\bar\beta_{jk}^{*\pm}(x,t;y) := \frac{|x^\pm|}{|a_j^{*\pm}t|}\beta_j^{*\pm} + \frac{|y|}{|a_k^{*-}t|}\left(\frac{a_j^\pm}{a_k^{*-}}\right)^2\beta_k^{*-}, \tag{1.38}$$

represent, respectively, approximate scattered characteristic paths and the time-averaged diffusion rates along those paths. In all equations, a_j, $a_j^{*\pm}$, l_j, $L_j^{*\pm}$, r_j, $R_j^{*\pm}$, η_j and $\beta_j^{*\pm}$ are as defined just above, and scattering coefficients $[c_{k,-}^{j,i}]$, $i=-,0,+$, are constants, uniquely determined by

$$\sum_{a_j^{*-}<0}[c_{k,-}^{j,-}]r_j^{*-} + \sum_{a_j^{*+}>0}[c_{k,-}^{j,+}]r_j^{*+} + \sum_{j=1}^{\ell}[c_{k,-}^{j,0}]\int_{-\infty}^{+\infty}(\partial/\partial\delta_j)\bar u^\delta(s)ds = r_k^{*-} \tag{1.39}$$

for each $k=1,\dots n$, and satisfying

$$\sum_{a_k^->0}[c_{k,-}^{j,0}]L_k^{*-} = \sum_{a_k^+<0}[c_{k,+}^{j,0}]L_k^{*+} = \pi_j \tag{1.40}$$

for each $j=1,\dots,\ell$, where the constant vector π_j is the left zero effective eigenfunction associated with the right eigenfunction $(\partial/\partial\delta_j)(\bar u^\delta, \bar v^\delta)$. Similar, but more complicated formulae hold in the undercompressive case (see Remark 6.2, below).

In addition, G vanishes identically outside the hyperbolic domain of influence

$$z_1(y,t) \le x \le z_{n+r}(y,t), \tag{1.41}$$

z_j defined as above.

Theorem 1.10 will be established by Laplace Transform (i.e., semigroup) techniques generalizing the Fourier Transform approach of [Ze.1–2,LZe]. The extension of Fourier Transform techniques to the variable-coefficient case is of course a central issue in mathematical physics and analysis, associated with such topics as semigroup, Sturm–Liouville, and scattering theory [Ti,RS,LP]; however, many unresolved issues remain, particularly in connection with higher order or nonnormal differential operators. From this, general theoretical point of view, the results of Theorem 1.10, and the methods used to obtain them, seem of wider interest, independent of their application to stability of shock profiles.

Remarks. Just as in the constant–coefficient case, short-time/large-frequency behavior is dominated by hyperbolic term H, while large-time/small-frequency behavior is dominated by terms $E+S$ agreeing with those predicted for the partially parabolic Chapmann–Enskog approximation (1.5), or, equivalently, the "effective," strictly parabolic, system defined by Kawashima [Kaw].

Terms E and S are essentially those derived for strictly parabolic systems in [ZH,Z.2–3], combined with projection onto the equilibrium manifold. As discussed in those references, they may be understood by the caricature of Gaussian signals propagating for $x \gtrless 0$ according to the limiting, constant-coefficient equations at $\pm\infty$ (of the associated effective, strictly parabolic system), interacting with an "ideal" shock layer at $x=0$ according to the rules encoded by scattering coefficients $[c^{j,i}_{k,\pm}]$, $i=-,0,+$: That is, a signal initiating as a delta-function at $y<0$ first decomposes into n Gaussian signals traveling in each of the equilibrium characteristic modes r_k^{*-} with approximately equilibrium characteristic speed a_k^{*-}; each of these signals then travels until it reaches the shock layer $x=0$, whereupon it is reflected/transmitted in each of the outgoing characteristic modes $r_j^{*\pm}$, thereafter moving with approximate equilibrium characteristic speed $a_j^{*\pm} \gtrless 0$ along path $z_{jk}^{*\pm}$; at the same time, the incoming signal excites stationary modes, or "bound states," $(\partial/\partial\delta_j)(\bar u^\delta, \bar v^\delta)$. Incoming and outgoing Gaussian signals comprise the scattering term S, excited stationary modes the excited term E.

The "parabolic" terms $E+S$ are considerably more complicated in the variable-coefficient than in the constant–coefficient case; by contrast, the hyperbolic term H, since localized, has an appealingly simple generalization to the variable-coefficient case, consisting in the "frozen-coefficient" evolution described by (1.36).

The nonstrictly hyperbolic case. The result of Theorem 1.10 holds also in the case that $(df,dg)^t)$ has multiple characteristics, with the following modification. If a_j has multiplicity m_j, $j=1,\dots,J$, then $l_j(x)$, $r_j(x)$ are defined instead as $(n+r)\times m_j$ matrices of left and right eigenvectors of $(df,dg)^t(\bar u,\bar v)(x)$, normalized so that $l_j^t r_j \equiv I_{m_j}$. In addition to this "static" normalization, we make the "dynamic" normalization

$$l_j^t(r_j)'(x) \equiv 0. \tag{1.42}$$

This is always possible, as shown in Lemma 4.3 below; for discrete kinetic models, it may be accomplished simply by taking l_j, r_j constant. Then, defining the $m_j\times m_j$ matrix

$$\eta_j(x) := -l_j^t Q r_j(x), \tag{1.43}$$

and augmenting characteristic equation (1.36) with the *dissipative flow*

$$d\zeta_j/dt = -\eta_j(z_j)\zeta_j, \quad \zeta_j(y) = I_{m_j}, \tag{1.44}$$

governing the evolution of dissipation matrix $\zeta_j = \zeta_j(y,t) \in \mathbb{R}^{m_j \times m_j}$, we have:

Theorem 1.11. *Assuming (H0)–(H4) and the spectral stability criterion ($\mathcal{D}$) (equivalently, (D1)–(D2)), decomposition (1.32) holds also in the general (nonstrictly hyperbolic) case, with (1.33) replaced by*

$$
\begin{aligned}
H(x,t;y) &:= \sum_{j=1}^{J} r_j(x)\zeta_j(y,t)\delta_{x-\bar a_j t}(-y) l_j^t(y) \\
&= \sum_{j=1}^{J} r_j(x)\mathcal{O}(e^{-\eta_0 t})\delta_{x-\bar a_j t}(-y) l_j^t(y),
\end{aligned}
\tag{1.45}
$$

for some uniform $\eta_0 > 0$, with l_j, r_j, and ζ_j as defined just above.

That is, the trivial, *scalar flow* $\zeta(t) = e^{\int_0^t -\eta_j(z(s))ds} = e^{\bar{-}\eta_j t}$ must be replaced in the case of corresponding characteristics by genuine, system dynamics (1.44). In the strictly hyperbolic case, (1.45) reduces to (1.33), as it must. Likewise, condition (1.27) must be replaced by

$$
\mathrm{Re}\ \sigma(\eta_j^\pm) > 0, \quad j = 1, \dots, J. \tag{1.46}
$$

Remark 1.12. Similarly, we may relax the requirement of strict hyperbolicity of $df^*(u_\pm)$, under a mild additional assumption. If $a_j^{*\pm}$ has multiplicity $m_j^{*\pm}$, then, likewise, we define $l_j^{*\pm}$ and $r_j^{*\pm}$ as $n \times m_j^{*\pm}$ blocks, and

$$
\beta_j^{*\pm} := l_j^{*\pm t} B^{*\pm} r_j^{*\pm}. \tag{1.47}
$$

Make the additional assumption that each β_j be *diagonalizable*, without loss of generality diagonal. Then, (1.32) holds with (1.34)–(1.35) replaced by matrix versions as in (1.45) above. Diagonalizability of $\beta_j^{*\pm}$ follows, for example, from simultaneous symmetrizability of A, Q, which holds for most physical systems of interest; this is satisfied, in particular, for our main example, the Jin–Xin system [Yo.4].

1.4. Nonlinear Stability. Finally, we discuss our results on nonlinear stability. In this part of the analysis, we restrict our attention to weak Lax shocks of *simultaneously symmetrizable models* (defined precisely in Section 7.2, below), or strong Lax shocks of *discrete kinetic models* (defined as above as the subclass of models for which f, g are linear, constant-coefficient). Simultaneously symmetrizable models of the dissipative type considered here admit favorable energy estimates [Kaw,Ze.2] allowing us to bound nonlinear derivative terms and close our nonlinear iteration; however, these energy estimates appear to require the weak shock assumption. Discrete kinetic models, on the other hand, do not have any such derivative terms to begin with, since nonlinearities enter only through the collision term q.

With these restrictions, we obtain our third main result:

Theorem 1.13. *Let $(\bar u(x), \bar v(x))$ be a Lax type relaxation profile of a general relaxation model (1.1), satisfying (H0)–(H3) and the spectral stability criterion ($\mathcal{D}$) (equivalently, (D1)–(D2)). If, also, $(\bar u, \bar v)$ is sufficiently weak, in the sense that $|(\bar u', \bar v')|_{L^\infty}$ is sufficiently small relative to the parameters of (1.1), and (1.1) is simultaneously symmetrizable (with coefficients $q \in C^3$ now also in the case of discrete kinetic models), then, provided that initial perturbation $(u_0, v_0) - (\bar u, \bar v)$ is bounded by ζ_0 in L_1 and H^2, for ζ_0 sufficiently small, the solution $(u,v)(x,t)$ of (1.1) with initial data (u_0, v_0) satisfies*

$$|(u,v)(x,t) - (\bar u, \bar v)(x - \delta(t))|_{L^p} \le C\zeta_0 (1+t)^{-\frac{1}{2}(1-1/p)}, \tag{1.48}$$

for all $2 \le p \le \infty$, for some $\delta(t)$ satisfying

$$|\dot\delta(t)| \le C\zeta_0 (1+t)^{-\frac{1}{2}} \tag{1.49}$$

and

$$|\delta(t)| \le C\zeta_0. \tag{1.50}$$

In particular, $(\bar u, \bar v)$ is nonlinearly orbitally stable from $L^1 \cap H^2$ to L^p, for all $p \ge 2$. For weak shocks of simultaneously symmetrizable discrete kinetic models ($q \in C^2$) and data merely small in $L^1 \cap H^1$, we obtain (1.48) for $p = 2$, but only boundedness $|(u,v)(x,t) - (\bar u, \bar v)(x - \delta(t))|_{L^\infty} \le C\zeta_0$, for $p = \infty$, and interpolated rates for p between 2 and ∞.

For discrete kinetic models, not necessarily simultaneously symmetrizable, with $q \in C^3$, and data small in $W^{3,1} \cap W^{3,\infty}$, we obtain the rates (1.48) for all $1 \le p \le \infty$, for shocks of arbitrary strength.

Theorem 1.13 is obtained using a modified version of the argument used in [Z.2] to treat the strictly parabolic case, new complications arising from the need to control contributions of the singular component H of the Green's function. Similar results should be possible for nonclassical, over- and under-compressive shocks by more detailed, pointwise analyses in the spirit of [L.1,ZH] and [Z.5], respectively.

Remark 1.14. As pointed out by W.-A. Yong [Yo.4], the condition of simultaneously symmetrizability is satisfied for most systems of interest in applications: for example, the discrete kinetic models of Platkowski–Illner [PI]; the numerical approximation schemes of Jin–Xin [JX] and their generalizations by Natilini [N]; the BGK models of Bouchut [B]; and, most notably, perhaps, for *all* of the extended thermodynamic models in the moment closure heirarchies of Levermore or Dreyer [Lev,Dre]; further examples may be found among the general relaxation models described in [CLL,Yo.1–3].

Remark 1.15. It is an interesting question whether the limitation to weak shocks for general relaxation models in Theorem 1.13 is only a technical artifact of our analysis, or a genuine difference in nonlinear behavior between quasilinear and semilinear models. We point out that this limitation arises very late in our argument, and at the (nonlinear) level of energy estimates only. On the other hand, discrete kinetic models are in some sense closer to original kinetic models such as Boltzmann equations than are quasilinear models based on moment closure assumptions, hence it is conceivable that they truly give a more faithful approximation of behavior in the large-amplitude regime.

Remark 1.16. Theorem 1.13 in particular shows that *strong spectral stability implies nonlinear stability*, for weak shocks of general simultaneously symmetrizable models, or for strong shocks of discrete kinetic models. As noted above, strong spectral stability follows for weak relaxation profiles of the Jin–Xin and Broadwell models from the results of [KM] and [Liu,Hu], respectively, thus yielding complete nonlinear stability results for these two cases. A very interesting open problem is spectral stability of weak shocks of general dissipative relaxation systems, as identified by Zeng/Kawashima [Kaw,Ze.2], or, likewise, spectral stability of strong shocks of special discrete kinetic models.

Remark 1.17. By a more refined, pointwise analysis, it should be possible to show that $\delta(t)$ approaches a limit as $t \to \infty$, which may then be explicitly determined using conservation of mass; however, it is clear that there can be no uniform rate of convergence for general $L^1 \cap L^\infty$ perturbations, since interaction with the shock layer (and thus formation of the final asymptotic state) may be delayed for arbitrarily long time by moving the perturbation to spatial infinity. Likewise, it should be possible to show that the perturbed shock converges in L^1 to a translated profile superimposed with diffusion waves in the far field: see, for example, related results obtained in [L.1,ZH] for localized data in the strictly parabolic case.

Plan of the paper. In Section 2, we establish the basic spectral representation of G via inverse Laplace Transform of the resolvent kernel. In Section 3, we make a careful construction of the resolvent kernel; using this representation, we make careful resolvent kernel estimates in Sections 4 and 5, in high- and low-frequency regimes, respectively. Combining these results, we derive in Section 6 the Green's function bounds that comprise the main results of the paper, establishing Theorems 1.10–1.11 and a partial generalization yielding necessity of $(\mathcal{D})$ for linearized stability. Finally, in Section 7, we establish sufficiency of $(\mathcal{D})$ for linearized stability, completing the proof of Theorem 1.4, and carry out the nonlinear iteration establishing nonlinear stability, Theorem 1.13. For completeness, we carry out in Appendices A1–A4 various auxiliary calculations/lemmas used throughout the paper.

Note: Subsequent to the completion of our analysis, F. Rousset [R] has announced the partial result of zero-mass stability of Jin–Xin relaxation shocks satisfying the spectral stability condition of [G,Z.1,Z.4] (the same one assumed here): that is, the extension to strong shocks of the zero-mass result of H. Liu [Liu]. However, his analysis appears to be incomplete in the high-frequency regime; specifically, the Gap Lemma of [GZ,ZH] is incorrectly cited at a key step in the proof of his high-frequency resolvent estimates, see Remark 8.5, Appendix A3.1, below. This gap is remedied by our own, more precise bounds.

Section 2. The Spectral Resolution Formula.

The starting point for our analysis is the spectral resolution, or inverse Laplace Transform formula for the Green's function $G(x,t;y)$ associated with the linearized evolution equations (1.17), defined by:

(i) $(\partial_t - L_x)G = 0$ in the distributional sense, for all $t > 0$; and,

(ii) $G(x,t;y) \rightharpoonup \delta(x-y)$ as $t \to 0$.

Here, G is to be interpreted in (i) as a distribution in the joint variables (x,y,t), and in (ii) as a distribution in (x,y), continuously parametrized by t. We shall see that G, uniquely so defined, by uniqueness of weak solutions of (1.17) within the class of test function initial data, in fact lies in the space of measures and not functions, so should properly be called a Green's distribution rather than a Green's function. Following the convention of [LZe,Ze.2], we shall use the two terms interchangeably.

The Green's distribution may be conveniently constructed using standard semigroup theory, considering the linearized operator L as a closed, densely defined operator on L^2, with domain $\mathcal{H} := \{U : U \in L^2,\ AU \in H^1\}$.

Lemma 2.1. *If $A(x)$ is smooth and symmetrizable, in the sense that there exists a smooth, invertible $S(x)$ such that SAS^{-1} is symmetric, and $Q(x)$ is bounded, then operator $L:\ LU := -(AU)_x + QU$ satisfies the bound*

$$|U|_{L^2} \le |\lambda - \lambda_*|^{-1} |(L-\lambda)U|_{L^2} \tag{2.1}$$

for all $U \in \mathcal{H}$, and real λ greater than some value λ_*.

Proof. Without loss of generality, we may take A symmetric, since we may achieve this by the coordinate change $U \to SU$, without affecting boundedness of Q. Setting

$$(L-\lambda)\phi = f, \tag{2.2}$$

for test functions $\phi \in C_0^\infty$, and performing an elementary energy estimate, we obtain

$$\langle \phi', A\phi\rangle + \langle \phi, Q\phi\rangle = \langle \phi, f\rangle + \lambda|\phi|^2, \tag{2.3}$$

where $\langle \cdot,\cdot\rangle$ and $|\cdot|$ refer to complex L^2 inner product and norm, respectively. Taking real parts in (2.3), we have the Gärding type estimate

$$-\frac{1}{2}\langle \phi, A'\phi\rangle + Re\langle \phi, Q\phi\rangle = Re\langle \phi, f\rangle + Re\lambda|\phi|^2, \tag{2.4}$$

or, rearranging:

$$(Re\lambda - \lambda_*)|\phi|^2 \le |\phi||f|, \tag{2.5}$$

for some real $\lambda_* > 0$ sufficiently large. But, this gives the claimed bound

$$|\phi| \le |\lambda - \lambda_*|^{-1}|f| = |\lambda - \lambda_*|^{-1}|(L-\lambda)\phi| \tag{2.6}$$

for all $\lambda > \lambda_*$ on the real axis, for $\phi \in C_0^\infty$ within the class of test functions. Observing that C_0^∞ is dense within $\mathcal{H}$ (as may be seen by the straightforward computation $|(AU)^\varepsilon - AU^\varepsilon|_{H^1} \to 0$ as $\varepsilon \to 0$, where $f^\varepsilon := f * \eta^\varepsilon$, and $\eta^\varepsilon \in C_0^\infty$ denotes the standard mollifier, together with the standard facts that $|(AU)^\varepsilon - AU|_{H^1} \to 0$ and $|U^\varepsilon - U|_{L^2} \to 0$), we may pass to the limit to obtain the same bound for $\phi \in \mathcal{H}$. ∎

Corollary 2.2. *If the operator $L:\ LU := -(AU)_x + QU$ of Lemma 2.1 has in addition the properties that A is nonsingular, and A and Q are asymptotically constant as $x \to \pm\infty$, then L generates a C^0 semigroup e^{Lt} on L^2, satisfying $|e^{Lt}|_{L^2} \le Ce^{\omega t}$ for some real ω.*

Proof. The bound (2.5) applies also to the limiting, constant-coefficient operators $L_\pm := -A_\pm \partial_x + Q_\pm$ as $x \to \pm\infty$, whence the spectra of these operators is confined to $\mathrm{Re}\,\lambda \le \lambda_*$. Under the assumption that A is nonsingular, we can apply a standard result of Henry ([He], Lemma 2, pp. 138–139) to find that the essential spectrum of L is also confined to this set. Since (2.1) precludes point spectrum for real $\lambda > \lambda_*$, we thus find that all such λ belong to the resolvent set $\rho(L)$, with the resolvent bound

$$|(L-\lambda)^{-1}|_{L^2} \le |\lambda - \lambda_*|^{-1}. \tag{2.7}$$

But, this is a standard sufficient condition that a closed, densely defined operator L generate a C^0 semigroup, with $|e^{Lt}|_{L^2} \le Ce^{\omega t}$ for all real $\omega > \lambda_*$ (see e.g. [Pa], Theorem 5.3, or [Fr,Y]). ∎

An immediate consequence (see, e.g. [Pa], Corollary 7.5) is:

Corollary 2.3. *For L, e^{Lt} as in Corollary 2.2, there holds for all real η greater than some η_0 the inverse Laplace Transform (spectral resolution) formula*

$$e^{Lt}\phi = P.V. \int_{\eta - i\infty}^{\eta + i\infty} e^{ikt}(L-\lambda)^{-1}\phi\, d\lambda, \tag{2.8}$$

for all $\phi \in D(L^2) \subset H^2$, where domain $D(L^2)$ is defined as in [Pa], p. 1.[4]

Remark. Symmetrizability, in the one-dimensional context, is essentially equivalent to hyperbolicity: A has real, semisimple eigenvalues; in particular, it is implied

[4]Note: From the definition in [Pa], we find that the domain $D(L)$ of L satisfies $H^1 \subset \mathcal{H} \subset D(L)$. Likewise, the domain $D(L^2)$, consisting of those functions V such that $LV \in D(L)$, satisfies $H^2 \subset D(L^2)$.

by either strict hyperbolicity, or else nonstrict hyperbolicity with constant multiplicity of eigenvalues, so that all cases considered here fall into this class. Thus, this is no real requirement other than the usual one of hyperbolicity of the principal part of the symbol (Note: this may be satisfied even when the subcharacteristic condition is violated, or when the reduced system is not hyperbolic, as in applications of the Jin–Xin model [JX] to phase transition problems).

With the result of Corollary 2.2, we can express the Green's distribution simply as

$$G(x,t;y) := e^{Lt}\delta(x-y), \tag{2.9}$$

where the evolution $e^{Lt}\phi(x,y)$ of a distribution $\phi(x,y)$ with the property that, for $k \in C_0^\infty$, the distributional pairing $\langle \phi(\cdot,y), k(y)\rangle_y$ is in L^2 for each fixed x, is defined through its action on C_0^∞ test functions $h(x)k(y)$:

$$\langle e^{Lt}\phi(x,y), h(x)k(y)\rangle_{x,y} := \langle h(x), e^{Lt}\langle \phi(x,y), k(y)\rangle_y\rangle_x,$$

or

$$\langle G(x,t;y), h(x)k(y)\rangle_{x,y} := \langle h(x), e^{Lt}k(x)\rangle_x. \tag{2.10}$$

The defining properties (i)–(ii) of the Green's distribution are then consequences of the corresponding properties defining a C^0 semigroup (see, e.g., [Pa], p. 1–4).

Defining the *resolvent kernel*, similarly, through

$$G_\lambda(x,y) := (L-\lambda)^{-1}\delta(x-y), \tag{2.11}$$

where the righthand side is defined again through its action on test functions,

$$\langle (L-\lambda)^{-1}\phi(x,y), h(x)k(y)\rangle_{x,y} := \langle h(x), (L-\lambda)^{-1}\langle \phi(x,y), k(y)\rangle_y\rangle_x,$$

or

$$\langle G_\lambda(x,y), \psi(x,y)\rangle_{x,y} := \langle h(x), (L-\lambda)^{-1}k(x)\rangle_x,$$

we find through (2.8), as a tautology, the fundamental:

Proposition 2.4. *For L, e^{Lt} as in Corollary 2.2, and G, G_λ as in (2.9)–(2.11), there holds the inverse Laplace Transform (spectral resolution) formula*

$$G(x,t;y) = P.V. \int_{\eta-i\infty}^{\eta+i\infty} e^{ikt} G_\lambda(x,y)\, d\lambda, \tag{2.12}$$

valid for all η greater than some η_0.

It is this relation that we shall use to compute the time-evolutionary Green's distribution G. Note that, on the resolvent set $\rho(L)$, the resolvent kernel G_λ likewise has an alternative, intrinsic characterization as the unique distribution satisfying

$$(L_x - \lambda)G_\lambda(x,y) = \delta(x,y) \tag{2.13}$$

and taking $f \in L^2$ to $\langle G_\lambda(x,y), f(y)\rangle_y \in L^2$. This is the characterization that we shall use in computating G_λ.

Remark 2.5. Note that a semigroup on even a still more restricted function class than L^2 would have sufficed in this construction, since we are constructing objects in the very weak class of distributions. Later, by explicit computation of G, we will verify that L in fact generates a C^0 semigroup in any L^p. Note also that (2.10) implies the expected, standard solution formula $e^{Lt}f = \langle G(x,t;y), f(y)\rangle_y$, or, formally:

$$e^{Lt}f = \int G(x,t;y)f(y)dy, \tag{2.14}$$

for f in any underlying Banach space on which e^{Lt} is defined, in this case on L^p, $1 \le p \le \infty$.

Remark 2.6. For general interest, we point out another, more concrete route to (2.12), (2.14), generalizing the approach followed in [ZH] for the parabolic case. Namely, we may observe that, in each finite integral in the approximating sequence defined by the principal value integral (2.12), we may exchange orders of integration and distributional differentiation, using Fubini's Theorem together with the fact, established in the course of our analysis, that G_λ is uniformly bounded on the contours under consideration (in fact, we establish the much stronger result that G_λ decays exponentially in $|x-y|$, with uniform rate), to obtain

$$\begin{aligned}(\partial_t - L_x)\int_{\eta-iK}^{\eta+iK} e^{\lambda t}G_\lambda(x,y)\,d\lambda &= \delta(x-y)\int_{\eta-iK}^{\eta+iK} e^{\lambda t}\,d\lambda\\ &\rightharpoonup \delta(x-y)\delta(t)\end{aligned} \tag{2.15}$$

as $K\to\infty$, for all $t\ge 0$. So, all that we must verify is: (i') the Principal Value integral (2.12) in fact converges to some distribution $G(x,t;y)$ as $K\to\infty$; and, (ii') $G(x,t;y)$ has a limit as $t\to 0$. For, distributional limits and distributions commute (essentially by definition), so that (i')–(ii') together with (2.15) imply (i)–(ii) above. Facts (i')–(ii') will be established by direct calculation in the course of our analysis, thus verifying formula (2.12) at the same time that we use it to obtain estimates on G. Note that we have made no reference in this argument to the semigroup machinery cited above.

Section 3. Construction of the Resolvent kernel.

We next derive explicit representation formulae for the resolvent kernel G_λ, using the classical construction (see, e.g. [CH]) for the Green's function of an ordinary differential operator, in terms of solutions of the homogeneous eigenvalue equation

$$0 = (L-\lambda)W = -(AW)' + (Q-\lambda)W, \tag{3.1}$$

matched across the singularity $x = y$ by appropriate jump conditions, in the process obtaining standard decay estimates on the resolvent kernel (see Proposition 3.4, below) suitable for analysis of intermediate frequencies λ. Here, A and Q are as defined as in (1.17), $W \in \mathbb{R}^N := \mathbb{R}^{n+r}$, and $'$ as usual denotes d/dx. In the details of our treatment, we follow [ZH,Z.4]; see also related analyses of [AGJ,K.1–2] regarding solution of the resolvent equation by variation of constants.

3.1. Basic construction. First, consider the limiting, constant-coefficient eigenvalue equations

$$0 = (L_\pm - \lambda)W := -A_\pm W' + (Q_\pm - \lambda)W. \tag{3.2}$$

Define

$$\Lambda := \cup \Lambda_j^\pm, \quad j = 1, \dots, n + r, \tag{3.3}$$

where $\Lambda_j^\pm$ denote the open sets bounded on the left by the algebraic curves $\lambda_j^\pm(\xi)$ determined by the eigenvalues of $i\xi\binom{df}{dg}(u_\pm, v_\pm) + \binom{0}{dq}(u_\pm, v_\pm)$ as ξ is varied along the real axis.

Lemma 3.1. *The limiting equations (3.2) have no center manifold on the set Λ. Moreover, assuming (H0)–(H1), (H3), we have:*

(i) $\Lambda \subset \{\lambda : \mathrm{Re}\,\lambda > -\eta|\mathrm{Im}\,\lambda|/(1 + |\mathrm{Im}\,\lambda|), \eta > 0$; and

(ii) the stable manifold at $+\infty$ and the unstable manifold at $-\infty$ have dimensions summing to the full dimension $N = n + r$.

Proof. The fundamental modes of (3.2) are of form $e^{\mu x}V$, where μ, V satisfy the *characteristic equation*

$$(-\mu A_\pm + Q_\pm - \lambda)V = 0. \tag{3.4}$$

The existence of a center manifold thus corresponds with existence of solutions $\mu = i\xi$, V of (3.4), ξ real, i.e., solutions of the *dispersion relation*

$$(-i\xi A_\pm + Q_\pm - \lambda)V = 0. \tag{3.5}$$

But, $\lambda \in \sigma(-i\xi A_\pm + Q_\pm)$ implies, by definition (3.3), that λ lies outside of Λ, establishing the first claim. Bound (i), likewise, follows easily from the bound (H3) on the dispersion curves $\lambda_j^\pm$. Finally, nonexistence of a center manifold, together with connectivity of Λ, implies that the dimensions of stable/unstable manifolds at $+\infty/-\infty$ are constant on Λ. Taking $\lambda \to +\infty$ along the real axis, we find from (3.4) that these dimensions are those of the unstable/stable subspaces of A_+/A_-; but, these must sum to N, since, by (H1), the dimensions of unstable/stable subspaces of $A(x)$ are independent of x. This establishes (ii), completing the proof. ∎

For asymptotically constant-coefficient ordinary differential operators, the rightmost component (Λ, in this case) of the set of λ such that the limiting, constant-coefficient eigenvalue equations $(L_\pm - \lambda)W = 0$ are of hyperbolic type, with dimensions of unstable/stable manifolds summing to that of the full phase space is called in the Evans function literature the *domain of consistent splitting* [AGJ]. Its significance (see, e.g. [He], Lemma 2, pp. 138–139) is that it lies in the complement of the essential spectrum of the variable-coefficient operator L, with its boundary lying in the essential spectrum: that is, it is a maximal domain in the essential spectrum complement, consisting of the component containing real, plus infinity. Moreover, provided that the coeffients of L approach their limits at integrable rate, it can be shown (see, e.g. [AGJ,GZ,ZH]) that each connected component consists either entirely of eigenvalues, or else entirely of *normal points*, defined as resolvent points or isolated eigenvalues of finite multiplicity. The latter, and for our purposes more significant fact will be seen directly in the course of our construction.

Recall, (1.14), that coefficients of L converge exponentially to their limiting values in $L_\pm$, in particular at *integrable rate*. In standard fashion (see, for example, [Co,AGJ,ZH,Z.1,ZS,Z.4]), we may thus conclude:

Proposition 3.2. *If (H0)–(H1), (H3) hold, then, about any λ in the domain of consistent splitting Λ, there exist locally analytic (in λ) bases $\phi_1^+, \dots, \phi_k^+(x;\lambda)$ and $\phi_{k+1}^-, \dots, \phi_{n+r}^-(x;\lambda)$ respectively spanning the stable manifold at $+\infty$ and the unstable manifold at $-\infty$ of solutions of the variable-coefficient equation (3.1). Moreover, these manifolds are tangent as $x \to -\infty$, $x \to +\infty$, respectively, to their constant-coefficient counterparts, decaying exponentially for $x \geq 0$, $x \leq 0$, respectively, with uniform rate $Ce^{-\eta|x|}$, $\eta > 0$.*

Likewise, there exist bases $\psi_1^+, \dots, \psi_k^+(x;\lambda)$ and $\psi_{k+1}^-, \dots, \psi_{n+r}^-(x;\lambda)$ respectively spanning the unstable manifold at $+\infty$ and the stable manifold at $-\infty$ of solutions of (3.1), that are locally analytic in λ, tangent as $x \to -\infty$, $x \to +\infty$, respectively, to their constant-coefficient counterparts, and growing exponentially for $x \geq 0$, $x \leq 0$, respectively, with uniform rate $Ce^{\eta|x|}$, $\eta > 0$.

Proof. By spectral separation of stable/unstable subspaces, we may conclude the existence of globally analytic bases of solutions of the limiting equations (3.2), using a standard result of Kato ([Kat], pp. 81–83). The result for the variable-coefficient equations (3.1) then follows by standard, asymptotic theory of ODE [Co,Z.1], using the integrable rate of convergence of coefficients to their limiting values. Alternatively, using the fact that convergence is in fact exponential, we may apply the results of Appendix A2.1 (specifically, Remark 8.4) to obtain a self-contained treatment. ∎

Observation 3.3. To show that λ is in the resolvent set of L, with respect to any L^p, $1 \leq p \leq \infty$, it is enough to: (i) construct a resolvent kernel $G_\lambda(x,y)$ satisfying (2.11) and obeying a uniform decay estimate

$$|G_\lambda(x,y)| \leq Ce^{-\eta|x-y|}; \tag{3.6}$$

and, (ii) show that there are no L^p solutions of $(L-\lambda)W = 0$, i.e., λ is not an eigenvalue of L. (For $p<\infty$, these are necessary as well as sufficient, though we do not show it here [Z.1]). For, then, the Hausdorff–Young inequality yields that the distributional solution formula $(L-\lambda)^{-1}f := \int G_\lambda(x,y)f(y)dy$ yields a bounded right inverse of $(L-\lambda)$ taking L^p to L^p, while nonexistence of eigenvalues implies that this is also a left inverse. On the domain of consistent splitting, Λ, we shall see that (i)–(ii) are equivalent.

Following the strategy of Observation 3.3, we are forced to to choose $G(x,y)$, for fixed y and $x \gtrless y$ from among the decaying solutions at $+\infty/-\infty$, respectively, of the homogeneous eigenvalue equations (3.1), in such a way the G_λ satisfies at $x=y$ the *jump condition*

$$\left[\, G_\lambda \,\right]_{x=y} := G_\lambda(y+0,y) - G_\lambda(y-0,y) = -A^{-1}(y), \tag{3.7}$$

where $[\,h\,]_{x=y}$ denotes the jump of the function h at $x=y$: that is, according to the classical construction of [CH].

Implementing this strategy on the domain Λ of consistent splitting, we seek a solution of form

$$G_\lambda(x,y) = \begin{cases} \Phi^+(x;\lambda)N^+(y;\lambda) & x>y, \\ \Phi^-(x;\lambda)N^-(y;\lambda) & x<y, \end{cases} \tag{3.8}$$

where

$$\Phi^+(x;\lambda) = (\,\phi_1^+(x;\lambda)\ \dots\ \phi_k^+(x;\lambda)\,) \in \mathbb{R}^{N\times k} \tag{3.9}$$

and

$$\Phi^-(x;\lambda) = (\,\phi_{k+1}^-(x;\lambda)\ \dots\ \phi_N^-(x;\lambda)\,) \in \mathbb{R}^{N\times (N-k)}, \tag{3.10}$$

$\phi_j^\pm$ as defined in Proposition 3.2. Here, $N^+ = N^+(y;\lambda)$ and $N^- = N^-(y;\lambda)$ are $k\times N$ and $(N-K)\times N$ matrices, respectively, $N := n+r$, to be determined by imposing the jump condition (3.7), or equivalently

$$(\,\Phi^+(x;\lambda) \quad \Phi^-(x;\lambda)\,) \begin{pmatrix} N^+(y;\lambda) \\ -N^-(y;\lambda) \end{pmatrix} = -A^{-1}(y). \tag{3.11}$$

Inverting (3.11), we obtain the expression

$$\begin{pmatrix} N^+(y;\lambda) \\ -N^-(y;\lambda) \end{pmatrix} = -\,(\,\Phi^+ \quad \Phi^-\,)^{-1}\,(y;\lambda)\,A^{-1}(y). \tag{3.12}$$

for the matrices $N^\pm$, yielding a final expression for the resolvent kernel G_λ, in terms of the matrices $\Phi^\pm$, of

$$G_\lambda(x,y) = \begin{cases} -(\,\Phi^+(x;\lambda) \quad 0\,)\,(\,\Phi^+ \quad \Phi^-\,)^{-1}\,(y;\lambda)\,A^{-1}(y) & x>y, \\ (\,0 \quad \Phi^-(x;\lambda)\,)\,(\,\Phi^+ \quad \Phi^-\,)^{-1}\,(y;\lambda)\,A^{-1}(y) & x<y, \\ \end{cases} \tag{3.13}$$

valid when the matrix

$$\Phi := (\,\Phi^+ \quad \Phi^-\,) \tag{3.14}$$

is invertible. If Φ is singular, on the other hand, then λ is clearly an eigenvalue, since there must be some function lying in the span of both Φ^+ and Φ^-, hence decaying exponentially at both $\pm\infty$; thus, we see that (i) implies (ii) on Λ, partially verifying our claim that they are equivalent.

It will at times be useful to represent the resolvent kernel in a more intrinsic, geometric fashion. Observing that

$$(\,0 \quad \Phi^-(x;\lambda)\,) = \Phi(x;\lambda)\begin{pmatrix} 0 & 0 \\ 0 & I_n \end{pmatrix} = \Phi(x;\lambda)\,\Phi^{-1}(y;\lambda)\,(\,0 \quad \Phi^-(y;\lambda)\,), \tag{3.15}$$

and, similarly,

$$(\,\Phi^+(x;\lambda) \quad 0\,) = \Phi(x;\lambda)\,\Phi^{-1}(y;\lambda)\,(\,\Phi^+(y;\lambda) \quad 0\,), \tag{3.16}$$

we get the expression

$$G_\lambda(x,y) = \begin{cases} -\mathcal{F}^{y\to x}\,\Pi_y^+\,A^{-1}(y) & x > y, \\ \mathcal{F}^{y\to x}\,\Pi_y^-\,A^{-1}(y) & x < y, \end{cases} \tag{3.17}$$

where we have defined the solution operator from y to x of (3.1), denoted by $\mathcal{F}^{y\to x}$, as

$$\mathcal{F}^{y\to x} = \Phi(x;\lambda)\,\Phi^{-1}(y;\lambda) \tag{3.18}$$

and the projections $\Pi_y^\pm$ on the stable manifolds at $\pm\infty$ as

$$\Pi_y^+ = (\,\Phi^+(y;\lambda) \quad 0\,)\,\Phi^{-1}(y;\lambda) \quad \text{and} \quad \Pi_y^- = (\,0 \quad \Phi^-(y;\lambda)\,)\,\Phi^{-1}(y;\lambda). \tag{3.19}$$

Note that in (3.15), there is no longer any reference to the specific choice of bases functions $\phi_j^\pm$, but only to the (uniquely defined) stable and unstable manifolds at $+\infty$, $-\infty$, respectively.

With the above representations in hand, we readily obtain the following basic, intermediate-frequency result. More careful analyses will be required in the large- and small-freqency limits.

Proposition 3.4. *With respect to any L^p, $1 \le p \le \infty$, the domain of consistent splitting, Λ, consists entirely of normal points of L, i.e., resolvent points, or isolated eigenvalues of constant multiplicity. Moreover, the resolvent kernel is meromorphic* *on Λ, satisfying either of formulae (3.13),(3.15). Finally, on any compact subset K of $\rho(L) \cap \Lambda$, there holds the uniform decay estimate*

$$|G_\lambda(x,y)| \le Ce^{-\eta|x-y|}, \tag{3.20}$$

where $C > 0$ and $\eta > 0$ depend only on K, L.

Proof. As noted above, eigenvalues coincide with zeroes of the (locally) analytic function

$$D(\lambda) := \det \Phi, \tag{3.21}$$

which are either isolated of finite multiplicity or else fill all of Λ. For the remaining claims, it is sufficient to verify (3.20) holds for the function G_λ defined by formulae (3.13),(3.15). For, it then follows that this function, off of the set of isolated eigenvalues, indeed represents the resolvent kernel, and that all noneigenvalue points are in the resolvent set. By inspection, (3.13) is meromorphic, with poles of finite order less than the multiplicity of the corresponding zeroes of D; it follows using the spectral resolution formula (see, e.g., [Kat,Y,ZH]) that the eigenprojection of L at each eigenvalue λ_0, defined as the residue at λ_0 of the resolvent $(L-\lambda)^{-1}$, is well-defined, with range contained in the L^p kernel of $(L-\lambda)^k$, where k is the order of the pole. But, the solution set of $(L-\lambda)^k W = 0$ is at most nK, hence finite.

Exponential decay is best seen using the geometric formula (3.15). First, observe that, for $D(\lambda)$ bounded away from zero, the manifold of decaying solutions at $-\infty$ is uniformly transverse at $x = 0$ to the manifold of decaying solutions at $+\infty$, hence is uniformly exponentially growing as $x \to +\infty$: that is, away from eigenvalues of L, this manifold is *globally* uniformly exponentially decaying as x decreases, on the whole real line $-\infty \le x \le +\infty$. Likewise, the manifold of decaying solutions at $+\infty$ is globally uniformly exponentially decaying as x increases. A side-consequence is that the two manifolds are uniformly transverse as $x \to \pm\infty$, approaching the unstable/stable manifolds of the limiting coefficient matrix of (3.1); by continuity, they are uniformly transverse, then, on the whole real line. Thus, projections $\Pi_y^\pm$ are bounded in (3.15), and flows $\mathcal{F}^{y\to x}$ are uniformly exponentially decaying, giving the result (3.20). ∎

The transmission function $D(\cdot)$ defined in (3.21) is known as an *Evans function.* It is possible, with more care, to define a globally analytic Evans function on Λ, and thereby to determine information on location of eigenvalues via topological considerations, see e.g. [E.1–4,J,AGJ,PW,GZ]. However, for our purposes local analyticity is sufficient. Though there exist as many versions of the Evans function as there are analytic bases $\phi_j^\pm$, it is evident that these differ, locally, only by a nonvanishing analytic factor. Thus, location and multiplicity of zeroes agree for all versions of the Evans function, as the following proposition of Gardner and Jones shows that they must.

Proposition 3.5[GJ.1–2]. *On Λ, the zeroes of (any version of) the Evans function $D(\cdot)$ agree in both location and multiplicity with the eigenvalues of L.*

Proof. It is evident that they agree in location, whence the result follows if one can show that zeroes and eigenvalues both split under regular perturbation of L; the existence of such a perturbation (a nontrivial assertion) was shown by Gardner and Jones in [GJ.1]. Alternatively, they showed in [GJ.2] that the correspondence may be seen by explicit computation in a convenient Jordan basis. ∎

Note that Proposition 3.5 gives an alternative proof that eigenvalues of L are of finite multiplicity on Λ.

Remark 3.6. In the constant-coefficient case, the matrices $\Phi^\pm$ above can be calculated (almost) explicitly. To accomplish this aim, we have to find the decaying modes at $\pm\infty$, that is we have to find independent solutions to

$$-AW' + BW - \lambda W = 0 \qquad \Longleftrightarrow \qquad W' = A^{-1}(B-\lambda I)W, \tag{3.22}$$

such that $W(\pm\infty)=0$. So, let $\mu_i^\pm = \mu_i^\pm(\lambda)$ with $i=1,\dots,N$ be the solutions of

$$p(\mu) = \det\,\big(A^{-1}(B-\lambda I) - \mu I\big) = 0, \tag{3.23}$$

with $\mathrm{Re}\ \mu_i^+ < 0 < \mathrm{Re}\ \mu_j^-$ for any $i=1,\dots,k$ and $j=k+1,\dots,N$. Then, assuming $\mu_i^\pm \neq \mu_j^\pm$ for any $i\neq j$, $\phi_i^\pm = r_i^\pm e^{\mu_i^\pm x}$ where $r_i^\pm$ is a right eigenvector of the matrix $A^{-1}(B-\lambda I)$ relative to the eigenvalue $\mu_i^\pm$, so that

$$\Phi(x;\lambda) = (\, r_1^+ e^{\mu_1^+ x} \quad \dots \quad r_k^+ e^{\mu_k^+ x} \quad r_{k+1}^- e^{\mu_{k+1}^- x} \quad \dots \quad r_N^- e^{\mu_N^- x}\,) \in \mathbb{R}^{N\times N}. \tag{3.24}$$

Example 3.7. For the Jin–Xin model, the equations linearized at the stationary solution $(\bar u, \bar v)$ are

$$\begin{cases} u_t - s u_x + v_x = 0, \\ v_t + a^2 u_x - s v_x = \alpha(x) u - v, \end{cases} \tag{3.25}$$

where $\alpha(x) := dh(\bar u(x))$. Hence

$$W = \begin{pmatrix} u \\ v \end{pmatrix}, \quad A = \begin{pmatrix} -s & 1 \\ a^2 & -s \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ \alpha(x) & -1 \end{pmatrix}, \tag{3.26}$$

(here any element in the matrices A and B corresponds to an $n\times n$ block).

If $\alpha(x)\equiv$ constant, then $p(\mu) = \Pi_{j=1}^n \big[(a^2-s^2)\mu^2 + [(1+2\lambda)s - \alpha_j]\mu - \lambda(1+\lambda)\big]$, where α_j denote the eigenvalues of α, so that there are n pairs of eigenvalues

$$\mu_j^\pm(\lambda) = \frac{\alpha_j - (1+2\lambda)s \mp \sigma_j(\lambda)}{2(a^2-s^2)}, \tag{3.27}$$

where $\sigma_j(\lambda) = \sqrt{[\alpha_j - (1+2\lambda)s]^2 + 4\lambda(1+\lambda)(a^2 - s^2)}$. If the subcharacteristic condition holds, then $a^2 - s^2 > 0$ and for any $\lambda > 0$ there holds $\mu_j^+(\lambda) < 0 < \mu_j^-(\lambda)$.

The corresponding eigenvectors (for $\lambda > 0$) are

$$W_j^{\pm} = C \left(\left[(1+\lambda)s + (a^2 - s^2)\mu_j^{\pm}\right] \, r_j \, , \, (\alpha_j s - a^2\lambda) \, r_j \right)^t \qquad (C \neq 0),$$

where r_j denote right eigenvectors of α associated with α_j. Hence,

$$\Phi(x;\lambda) = \Big(W_1^+ e^{\mu_1^+ x} \quad \cdots \quad W_n^+ e^{\mu_n^+ x} \quad W_1^- e^{\mu_1^- x} \quad \cdots \quad W_n^- e^{\mu_n^- x} \Big), \tag{3.28}$$

and the resolvent kernel is given by

$$G_\lambda(x,y) = \begin{cases} -\sum_{i=j}^n W_j^+ V_j^+ A^{-1} e^{\mu_j^+(x-y)} & x < y, \\ \sum_{j=1}^n W_i^- V_j^- A^{-1} e^{\mu_j^-(x-y)} & x > y, \end{cases} \tag{3.29}$$

where $V_j^{\pm}$ are row vector defined by

$$\begin{pmatrix} W_j^+ & W_i^- \end{pmatrix}^{-1} = \begin{pmatrix} V_j^+ \\ V_j^- \end{pmatrix}. \tag{3.30}$$

∎

3.2. Dual formulation. For use in our later treatment of low-frequency behavior, we develop a modified representation of the resolvent kernel involving solutions of the adjoint eigenvalue equation

$$(L^* - \lambda^*)\tilde{W} = 0, \tag{3.31}$$

where

$$L^*\tilde{W} := A^*\tilde{W}' + Q^*\tilde{W} \tag{3.32}$$

denotes the adjoint operator for L, and * for a matrix or vector denotes matrix adjoint, or conjugate transpose. Following [ZH,Z.4], we first point out:

Lemma 3.8. *For any $\tilde{W}$, W such that $(L^* - \lambda^*)\tilde{W} = 0$ and $(L - \lambda)W = 0$, there holds*

$$\langle \tilde{W}, AW \rangle \equiv \text{constant}. \tag{3.33}$$

Proof. By direct calculation,

$$\begin{aligned} \langle \tilde{W}, AW \rangle' &= \langle A^*\tilde{W}', W \rangle + \langle \tilde{W}, (AW)' \rangle \\ &= \langle -(Q^* - \lambda^*)\tilde{W}, W \rangle + \langle \tilde{W}, (Q - \lambda)W \rangle = 0. \end{aligned} \tag{3.34}$$

∎

From (3.34), it follows that if there are k independent solutions $\phi_1^+, \dots, \phi_k^+$ of $(L - \lambda I)W = 0$ decaying at $+\infty$, and $N-k$ independent solutions $\phi_{k+1}^-, \dots, \phi_N^-$ of the same equations decaying at $-\infty$, then there exist $N-k$ independent solutions $\tilde\psi_{k+1}^+, \dots, \tilde\psi_N^+$ of $(L^* - \lambda^* I)\tilde W = 0$ decaying at $+\infty$, and k independent solutions $\tilde\psi_1^-, \dots, \tilde\psi_k^-$ decaying at $-\infty$. More precisely, setting

$$\Psi^+(x;\lambda) = \begin{pmatrix} \psi_{k+1}^+(x;\lambda) & \cdots & \psi_N^+(x;\lambda) \end{pmatrix} \in \mathbb{R}^{N\times(N-k)}, \tag{3.35}$$

$$\Psi^-(x;\lambda) = \begin{pmatrix} \psi_1^-(x;\lambda) & \cdots & \psi_k^-(x;\lambda) \end{pmatrix} \in \mathbb{R}^{N\times k}, \tag{3.36}$$

and

$$\Psi(x;\lambda) = \begin{pmatrix} \Psi^-(x;\lambda) & \Psi^+(x;\lambda) \end{pmatrix} \in \mathbb{R}^{N\times N}, \tag{3.37}$$

where $\psi_j^\pm$ are the exponentially growing solutions described in Proposition 3.2, we may define dual exponentially decaying and growing solutions $\tilde\psi_j^\pm$ and $\tilde\phi_j^\pm$ via

$$\begin{pmatrix} \tilde\Psi & \tilde\Phi \end{pmatrix}_\pm^* A \begin{pmatrix} \Psi & \Phi \end{pmatrix}_\pm \equiv I. \tag{3.38}$$

Recalling (see, e.g. [ZH], Lemma 4.2) the classical duality principle that the transposition $G_\lambda^*(y,x)$ of the Green's function $G_\lambda(x,y)$ associated with operator $(L-\lambda)$ should be the Green's function for the adjoint operator $(L^* - \lambda^*)$, i.e.,

$$(L_y^* - \lambda^*)G_\lambda^*(x,y) = \delta(x-y),$$

we may seek, alternatively, to represent the resolvent kernel in the form

$$G_\lambda(x,y) = \begin{cases} \Phi^+(x;\lambda)M^+(\lambda)\tilde\Psi^{-*}(y;\lambda) & x > y, \\ \Phi^-(x;\lambda)M^-(\lambda)\tilde\Psi^{+*}(y;\lambda) & x < y, \end{cases} \tag{3.39}$$

where $M^+ = M^+(\lambda)$ and $M^- = M^-(\lambda)$ are $k \times k$ and $(N-k)\times(N-k)$ matrices, respectively, to be determined by imposing the jump condition (3.7).

Solving, as before, we obtain

$$M(\lambda) := \begin{pmatrix} -M^+(\lambda) & 0 \\ 0 & M^-(\lambda) \end{pmatrix} = \Phi^{-1}(z;\lambda)A^{-1}(z)\tilde\Psi^{-1*}(z;\lambda), \tag{3.40}$$

where

$$\tilde\Psi := \begin{pmatrix} \tilde\Psi^- & \tilde\Psi^+ \end{pmatrix}. \tag{3.41}$$

Note that the independence of the righthand side with respect to z is a consequence of the previous lemma. Thus,

$$G_\lambda(x,y) = \begin{cases} -(\,\Phi^+(x;\lambda) \quad 0\,)\,M(\lambda)\,(\,\tilde\Psi^-(y;\lambda) \quad 0\,)^* & x > y, \\ (\,0 \quad \Phi^-(x;\lambda)\,)\,M(\lambda)\,(\,0 \quad \tilde\Psi^+(y;\lambda)\,)^* & x < y, \end{cases} \tag{3.42}$$

with M as in (3.40). It is readily seen [ZH,Z.4] by duality, (3.33), that $\tilde\Psi$ is nonsingular if and only if Φ is nonsingular, so that formula (3.42) is indeed valid on the same region as were the previous ones; see also Proposition 3.9, below.

Finally, let us note as before that it is possible to represent the matrix G_λ by means of intrinsic objects as solution operators and projections on stable manifolds

$$G_\lambda(x,y) = \begin{cases} -\mathcal{F}^{z\to x}\,\Pi_z^+\,A^{-1}(z)\,\tilde\Pi_z^-\,\tilde{\mathcal{F}}^{z\to y} & x > y, \\ \mathcal{F}^{z\to x}\,\Pi_z^-\,A^{-1}(z)\,\tilde\Pi_z^+\,\tilde{\mathcal{F}}^{z\to y} & x < y, \\ \end{cases} \tag{3.43}$$

where

$$\tilde{\mathcal{F}}^{z\to y} = \tilde\Psi^{-1}(z;\lambda)\,\tilde\Psi(y;\lambda) \tag{3.44}$$

and

$$\tilde\Pi_z^+ = \tilde\Psi^{-1}(z;\lambda)\begin{pmatrix} 0 \\ \tilde\Psi^+(z;\lambda)\end{pmatrix} \quad \text{and} \quad \tilde\Pi_z^- = \tilde\Psi^{-1}(z;\lambda)\begin{pmatrix} \tilde\Psi^-(z;\lambda) \\ 0\end{pmatrix}. \tag{3.45}$$

Remark 3.9. The exponential decay asserted in Proposition 3.4 is somewhat more straightforward to see in the dual formulation, by judiciously choosing z in (3.43) and using the exponential decay of forward and adjoint flows on $x \gtrless 0$ alone.

3.3. Spectral decomposition. The matrix $M(\lambda)$ in (3.40) depends on the choice of functions $\phi_j^\pm$ and $\psi_j^\pm$. For example, formula (3.42) may be considerably simplified in the constant-coefficient case by the choice of bases $\Psi^\pm = \Phi^\mp$, which yields

$$\tilde\Psi(x;\lambda)A(x)\Phi(x;\lambda) = I \tag{3.46}$$

and $M(\lambda) = I$, or equivalently

$$G_\lambda(x,y) = \begin{cases} -\sum_{j=k+1}^N \phi_j^+(x;\lambda)\tilde\phi_j^{+*}(y;\lambda) & x > y, \\ \sum_{j=1}^k \phi_j^-(x;\lambda)\tilde\phi_j^{-*}(y;\lambda) & x < y, \end{cases} \tag{3.47}$$

where $\phi_j^\pm$, $\tilde\phi_j^\pm$ may be taken as pure exponentials

$$\phi_j^\pm(x)\tilde\phi_j^{\pm*}(y) = e^{\mu_j^\pm(\lambda)(x-y)}V_j^\pm(\lambda)\tilde V_j^{\pm*}(\lambda). \tag{3.48}$$

This appealing formula may be viewed as a *generalized spectral decomposition.* Note that, moving individual modes $\phi_j^\pm \tilde\phi_j^\pm$ in the spectral resolution formula (2.12), using Cauchy's Theorem, to contours $\mu_j(\lambda) \equiv i\xi$ lying along corresponding dispersion curves $\lambda = \lambda_j(\xi)$, we obtain the standard decomposition of e^{Lt} into eigenmodes of continuous spectrum: in this (constant-coefficient) case, just the usual representation obtained by Fourier transform solution.

Example 3.10. For the Jin–Xin model (3.25), the dual bases of $\phi_i^\pm$ can be obtained by inverting the relation $\tilde\Psi(y;\lambda)A\Phi(y;\lambda) = I$. Hence, recalling formula (3.28) and definition of $V_j^\pm$ in (3.30), we obtain

$$\tilde\Psi(y;\lambda) = \begin{pmatrix} V_1^+\, e^{-\mu_1^+ y} \\ \vdots \\ V_n^+\, e^{-\mu_n^+ y} \\ V_1^-\, e^{-\mu_1^- y} \\ \vdots \\ V_n^-\, e^{-\mu_n^- y} \end{pmatrix} A^{-1}$$

$$\frac{1}{(a^2\lambda - \alpha s)\sigma(\lambda)} \begin{pmatrix} (a^2\mu_+ - \alpha)e^{-\mu_+ y} & (\mu_+ s - \lambda)e^{-\mu_+ y} \\ (\alpha - a^2\mu_-)e^{-\mu_- y} & (\lambda - \mu_- s)e^{-\mu_- y} \end{pmatrix}. \tag{3.49}$$

(we used $\sigma(\lambda) = (a^2 - s^2)(\mu_- - \mu_+)$). As noted before, for such a choice of a basis for the space of solutions to the adjoint equation, $M(\lambda) \equiv I$ and the resolvent kernel G_λ can be represented by

$$G_\lambda(x,y) = \begin{cases} (\,0 \quad \Phi^-(x;\lambda)\,)\,(\,0 \quad \tilde\Psi^+(y;\lambda)\,)^* & x<y, \\ -(\,\Phi^+(x;\lambda) \quad 0\,)\,(\,\tilde\Psi^-(y;\lambda) \quad 0\,)^* & x>y, \end{cases} \tag{3.50}$$

or, expanding, (3.47). ∎

In the asymptotically constant-coefficient case, the notion of continuous spectrum is in general not particularly useful outside of the self-adjoint case, see e.g. discussion of [OZ]. However, as pointed out in [ZH], the more primitive relation (3.47) persists in the form of a *scattering decomposition* encoding interactions of scalar modes at $\pm\infty$. This decomposition gives a natural generalization of the Fourier transform to the asymptotically constant-coefficient case, one of the main contributions of [ZH].

The following result of [Z.4], a slight extension of Proposition 7.1 of [ZH], generalizes (3.47) to the asymptotically constant-coefficient case.

Proposition 3.11 [Z.4]. *On $\Lambda \cap \rho(L)$, there hold*

$$G_\lambda(x,y) = \sum_{j,k} M_{jk}^+(\lambda)\phi_j^+(x;\lambda)\tilde\psi_k^-(y;\lambda)^* \tag{3.51}$$

for $y \le 0 \le x$,

$$G_\lambda(x,y) = \sum_{j,k} d^+_{jk}(\lambda)\phi^-_j(x;\lambda)\tilde\psi^-_k(y;\lambda)^* - \sum_k \psi^-_k(x;\lambda)\tilde\psi^-_k(y;\lambda)^* \tag{3.52}$$

for $y \le x \le 0$, and

$$G_\lambda(x,y) = \sum_{j,k} d^-_{jk}(\lambda)\phi^-_j(x;\lambda)\tilde\psi^-_k(y;\lambda)^* + \sum_k \phi^-_k(x;\lambda)\tilde\phi^-_k(y;\lambda)^* \tag{3.53}$$

for $x \le y \le 0$, with

$$M^+ = (-I, 0)\begin{pmatrix}\Phi^+ & \Phi^-\end{pmatrix}^{-1}\Psi^- \tag{3.54}$$

and

$$d^\pm = \big(0, I\big)\begin{pmatrix}\Phi^+ & \Phi^-\end{pmatrix}^{-1}\Psi^-. \tag{3.55}$$

Symmetric representations hold for $y \ge 0$.

Proof. The matrix M^+ in (3.40) may be expanded using duality relation (3.33) as

$$\begin{aligned} M^+ &= (-I, 0)\begin{pmatrix}\Phi^+ & \Phi^-\end{pmatrix}^{-1}A^{-1}\begin{pmatrix}\tilde\Psi^- & \tilde\Phi^-\end{pmatrix}^{*\,-1}\begin{pmatrix} I \\ 0\end{pmatrix}_{|z} \\ &= (-I, 0)\begin{pmatrix}\Phi^+ & \Phi^-\end{pmatrix}^{-1}\begin{pmatrix}\Psi^- & \Phi^-\end{pmatrix}\begin{pmatrix} I \\ 0\end{pmatrix}_{|z} \\ &= (-I, 0)\begin{pmatrix}\Phi^+ & \Phi^-\end{pmatrix}^{-1}\Psi^-_{|z}, \end{aligned} \tag{3.56}$$

yielding (3.54) for $x \ge y$, in particular for $y \le 0 \le x$.

Next, expressing $\phi^\pm_j(x;\lambda)$ as a linear combination of basis elements at $-\infty$, we obtain the preliminary representation

$$G_\lambda(x,y) = \sum_{j,k} d^+_{jk}(\lambda)\phi^-_j(x;\lambda)\tilde\psi^-_k(y;\lambda)^* + \sum_{j,k} e^+_{jk}\psi^-_j(x;\lambda)\tilde\psi^-_k(y;\lambda)^*, \tag{3.57}$$

valid for $y \le x \le 0$. Duality, (3.33), with (3.17), and the fact that $\Pi_+ = I - \Pi_-$, gives

$$\begin{aligned} -\begin{pmatrix} d^+ \\ e^+\end{pmatrix} &= \begin{pmatrix}\tilde\Phi^- & \tilde\Psi^-\end{pmatrix}^* A\Pi_+\Psi^-{}_{|x} \\ &= -\begin{pmatrix}\Phi^- & \Psi^-\end{pmatrix}^{-1}\big[I - \begin{pmatrix}0 & \Phi^-\end{pmatrix}\begin{pmatrix}\Phi^+ & \Phi^-\end{pmatrix}^{-1}\big]\Psi^- \\ &= \begin{pmatrix} 0 \\ -I_k\end{pmatrix} - \begin{pmatrix} 0 & I_{N-k} \\ 0 & 0\end{pmatrix}\begin{pmatrix}\Phi^+ & \Phi^-\end{pmatrix}^{-1}\Psi^-, \end{aligned} \tag{3.58}$$

yielding (3.52) and (3.55) for $y \le x \le 0$. Relations (3.53) and (3.55) follow for $x \le y \le 0$ in similar, but more straightforward fashion from (3.33) and (3.17). ∎

Remarks. 1. Note, in the various scattering decompositions given in Proposition 3.11, that all functions $\phi_j^\pm$, $\tilde\phi_j^\pm$, $\tilde\phi_j^\pm$, $\tilde\psi_j^\pm$ are evaluated on the intervals $[0, \pm\infty]$ on which their behavior is known. Indeed, by Proposition 3.2, they are well-approximated by corresponding solutions of the limiting, constant-coefficient equations at $\pm\infty$, i.e., *exponential modes* generalizing the continuous eigenmodes obtained by Fourier transform in the constant-coefficient case.

2. Letting x, $y \to -\infty$ in (3.53), we find that

$$G_\lambda \sim \sum_k \phi_k^-(x)\tilde\phi_k^-(y)^* \sim \sum_k \bar\phi_k^-(x)\bar{\tilde\phi}_k^-(y)^* = \bar G_\lambda^-,$$

where $\bar\varphi_k^-$, $\bar{\tilde\phi}_k^-$ denote solutions of the asymptotic constant-coefficient ODE of the eigenvalue and adjoint eigenvalue equations at $-\infty$ and $\bar G_\lambda^-$ denotes the associated (constant-coefficient) resolvent kernel. Thus, Proposition 3.11 has the important and intuitively appealing implication that far-field behavior reduces to that of the constant coefficient case.

3. The fact that $d^+ = d^-$ may be deduced without calculation from the fact that the constant-coefficient formula (3.47) satisfies the resolvent kernel equation (2.11), but not the correct boundary conditions of decay at $\pm\infty$. The continuous $d^\pm$ correction satisfies the homogeneous eigenvalue equation, serving only to enforce decay at $+\infty$. As noted already, this term decays at $-\infty$, as do the remaining, constant-coefficient terms $\sum_k \phi_k^- \tilde\phi_k^-$.

Section 4. High frequency expansion.

We now turn to the crucial estimation of the resolvent kernel in the high frequency regime, as $|\lambda| \to 0$ with $\mathrm{Re}\,\lambda \ge -\eta$. According to the usual duality between spatial and frequency variables, large frequency λ corresponds to small time t, or *local behavior* in space and time. Therefore, we begin by examining the frozen-coefficient equations at each value x_0 of x:

$$U_t = L_{x_0} U := -A(x_0)U_x + Q(x_0)U. \tag{4.1}$$

As usual, we shall for clarity state first the result in the case that $(df, dg)^t$ is strictly hyperbolic, indicating afterward the extension to the general case.

Lemma 4.1. *Let $(df, dg)^t(\bar u, \bar v)(x_0)$ have distinct, real eigenvalues. Then, for $|\lambda|$ sufficiently large, the eigenvalue equation $(L_{x_0} - \lambda)W = 0$ associated with the frozen-coefficient operator L_{x_0} at x_0 has a basis of $N = n + r$ solutions*

$$\{\bar\phi_1^+, \dots, \bar\phi_k^+, \bar\phi_{k+1}^-, \dots, \bar\phi_N^-\}(x; \lambda, x_0)$$

that are analytic in $1/\lambda$*, satisfying* $\bar\phi_j^{\pm} = e^{\mu_j(\lambda,x_0)x}V_j(\lambda,x_0)$*, with*

$$\begin{aligned}\mu_j(\lambda,x_0) &= -\lambda/a_j(x_0) - \eta(x_0)/a_j(x_0) + \mathcal{O}(1/\lambda),\\ V_j(\lambda,x_0) &= r_j(x_0)/a_j(x_0) + \mathcal{O}(1/\lambda),\end{aligned} \tag{4.2}$$

$a_j(\cdot)$*,* $\eta_j(\cdot)$*, and* $r_j(\cdot)$ *as defined in Section 1.3. Likewise, the adjoint eigenvalue equation* $(L_{x_0}-\lambda)^*\tilde W = 0$ *has a basis of solutions* $\{\tilde{\bar\phi}_1^-, \dots, \tilde{\bar\phi}_k^-, \tilde{\bar\phi}_{k+1}^+, \dots, \tilde{\bar\phi}_N^+\}(x;\lambda,x_0)$ *satisfying* $\tilde{\bar\phi}_j^{\pm} = e^{-\mu_j(\lambda,x_0)x}\tilde V_j(\lambda,x_0)$*, with*

$$\tilde V_j(\lambda,x_0) = l_j(x_0) + \mathcal{O}(1/\lambda), \tag{4.3}$$

$l_j(\cdot)$ *as defined in Section 1.3.*

The expansions (4.2)–(4.3) hold also in the nonstrictly hyperbolic case, with $\bar\phi_j$*,* $\tilde{\bar\phi}_j$ *and* r_j*,* l_j *now denoting* $n\times m_j$ *blocks, and* μ_j*,* η_j *denoting* $m_j\times m_j$ *matrices,* $j=1,\dots,J$*, where* m_j *is the multiplicity of eigenvalue* a_j *of* $A(x_0)$*.*

Proof. Similarly as in the proof of Lemma 3.1, this follows by inversion of the corresponding expansions about $\xi=\infty$ of the eigenvalues $\lambda_j(\xi)$ and eigenvectors $V_j(\xi)$ of the frozen-coefficient Fourier symbol (1.26), using the basic relation $\mu_j = i\xi$ relating characteristic and dispersion equations (3.4) and (3.5). The expansions for the Fourier symbol are carried out in Appendix A2. ∎

Example 4.2. (Jin–Xin model) For model (1.8), everything is almost explicit and expansion of the eigenvalues and eigenvectors can be computed directly. Recalling that $\mu^\pm = \frac{\alpha}{2a^2} \mp \frac{\sigma(\lambda)}{2a^2}$ where $\sigma(\lambda) = \sqrt{\alpha^2+4a^2\lambda+4a^2\lambda^2}$, $\alpha = dh(\bar u(x_0)) \in \mathbb{R}^{n\times n}$, we find that

$$\mu^\pm(x;\lambda) = \mp\frac{\lambda}{a} + \frac{\alpha(x)\mp a}{2a^2} + O\left(\frac{1}{\lambda}\right) \qquad \lambda\to+\infty,$$

where $\mu_\pm$, $\eta_\pm$ are $n\times n$ blocks. Similar explicit expansions for V_j, $\tilde V_j$ can be computed using the formulae $V_\pm = (\mu_\pm, -\lambda)^*$ and $\tilde V_\pm = (\pm 1/\lambda\sigma(\lambda))(-a^2\mu_\pm, \lambda)^*$ of Examples 3.7 and 3.10. ∎

By (3.47), Lemma 4.1 gives an expression for the constant-coefficient resolvent kernel $G_{\lambda;x_0}$ at x_0 of, to lowest order:

$$G_{\lambda;x_0} \sim \begin{cases} \sum_{j=1}^k a_j(x_0)^{-1} e^{(-\lambda/a_j - \eta_j/a_j)(x_0)(x-y)} r_j(x_0) l_j^t(x_0) & x<y,\\ -\sum_{j=k+1}^N a_j(x_0)^{-1} e^{(-\lambda/a_j - \eta_j/a_j)(x_0)(x-y)} r_j(x_0) l_j^t(x_0) & x>y,\end{cases} \tag{4.4}$$

Our main result of this section, and perhaps the main technical challenge of this paper, will be to establish on an appropriate unbounded subset of the resolvent set

$\rho(L)$ that the variable coefficient solutions $\phi_j^\pm$, $\tilde\phi_j^\pm$ "track" their frozen coefficient analogs to order $1/|\lambda|$ in both direction and (exponential) growth rate, yielding order $1/|\lambda|$ estimates for the resolvent kernel. More precisely, define

$$\Omega := \{\lambda : -\eta_1 \le \text{Re } \lambda\}, \tag{4.5}$$

with $\eta_1 > 0$ sufficiently small that $\Omega \setminus B(0,r)$ is compactly contained in the set of consistent splitting Λ (defined in (3.3)), for some small $r > 0$ to be chosen later; this is possible, by Lemma 3.1(i).

Proposition 4.3. *Assume that there hold (H0)–(H4), plus strict hyperbolicity of $(df, dg)^t$. Then, for any $r > 0$, and $\eta_1 = \eta_1(r) > 0$ chosen sufficiently small, there holds $\Omega \setminus B(0,r) \subset \Lambda \cap \rho(L)$. Moreover, for $R > 0$ sufficiently large, there holds on $\Omega \setminus B(0,R)$ the decomposition*

$$G_\lambda(x,y) = H_\lambda(x,y) + \Theta_\lambda(x,y), \tag{4.6}$$

(4.7)
$$H_\lambda(x,y) := \begin{cases} -\sum_{j=k+1}^N a_j(y)^{-1} e^{\int_y^x (-\lambda/a_j - \eta_j/a_j)(z)\,dz} r_j(x) l_j^t(y) & x > y, \\ \sum_{j=1}^k a_j(y)^{-1} e^{\int_y^x (-\lambda/a_j - \eta_j/a_j)(z)\,dz} r_j(x) l_j^t(y) & x < y, \end{cases}$$

$$\Theta_\lambda(x,y) = \lambda^{-1} B(x,y;\lambda) + \lambda^{-1}(x-y) C(x,y;\lambda) + \lambda^{-2} D(x,y;\lambda), \tag{4.8}$$

where

$$B(x,y;\lambda) = \begin{cases} \sum_{j=k+1}^N e^{-\int_y^x \lambda/a_j(z)\,dz} b_j^+(x,y) & x > y, \\ \sum_{j=1}^k e^{-\int_y^x \lambda/a_j(z)\,dz} b_j^-(x,y) & x < y, \end{cases} \tag{4.9}$$

(4.10)
$$C(x,y;\lambda) = \begin{cases} -\sum_{i,j=k+1}^N \text{mean}_{z\in[x,y]} e^{-\int_y^z \lambda/a_i(s)ds - \int_z^x \lambda/a_j(s)ds} c_{i,j}^+(x,y;z), & x > y, \\ \sum_{i,j=1}^k \text{mean}_{z\in[x,y]} e^{-\int_y^z \lambda/a_i(s)ds - \int_z^x \lambda/a_j(s)ds} c_{i,j}^-(x,y;z), & x < y, \end{cases}$$

with

$$|b_j^\pm|,\ |c_{i,j}^\pm| \le C e^{-\theta|x-y|} \tag{4.11}$$

and

$$D(x,y;\lambda) = \begin{cases} \mathcal{O}(e^{-\int_y^x \text{Re } \lambda/a_{k+1}(z)\,dz} e^{-\theta|x-y|}) & x > y, \\ \mathcal{O}(e^{-\int_y^x \text{Re } \lambda/a_k(z)\,dz} e^{-\theta|x-y|}) & x < y, \end{cases} \tag{4.12}$$

for some uniform $\theta > 0$ independent of x, y, z.

Likewise, there hold derivative bounds

(4.13)
$$(\partial/\partial x)\Theta_\lambda(x,y) = \big(B_x^0(x,y;\lambda) + (x-y)C_x^0(x,y;\lambda)\big) + \lambda^{-1}\big(B_x^1(x,y;\lambda) + (x-y)C_x^1(x,y;\lambda) + (x-y)^2 D_x^1(x,y;\lambda)\big) + \lambda^{-2}E_x(x,y;\lambda)$$

and

(4.14)
$$(\partial/\partial y)\Theta_\lambda(x,y) = \big(B_y^0(x,y;\lambda) + (x-y)C_y^0(x,y;\lambda)\big) + \lambda^{-1}\big(B_y^1(x,y;\lambda) + (x-y)C_y^1(x,y;\lambda) + (x-y)^2 D_y^1(x,y;\lambda)\big) + \lambda^{-2}E_y(x,y;\lambda),$$

where B_β^α and C_β^α satisfy bounds of form (4.9) and (4.10), D_β^1 now denotes the iterated integral

(4.15)
$$D_\beta^1(x,y;\lambda) = \begin{cases} -\sum_{h,i,j=k+1}^N \mathrm{mean}_{y\le w\le z\le x} e^{-\int_y^w \lambda/a_h(s)ds - \int_w^z \lambda/a_i(s)ds - \int_z^x \lambda/a_j(s)ds} \\ \qquad \times d_{h,i,j}^{\beta,+}(x,y;z), & x > y, \\ \sum_{h,i,j=1}^k \mathrm{mean}_{x\le z\le w\le y} e^{-\int_y^w \lambda/a_h(s)ds - \int_w^z \lambda/a_i(s)ds - \int_z^x \lambda/a_j(s)ds} \\ \qquad \times d_{h,i,j}^{\beta,-}(x,y;z), & x < y, \end{cases}$$

with $|d_{h,i,j}^{\beta,-}| \le Ce^{-\theta|x-y|}$, and E_β satisfies a bound of form (4.12).

The bounds (4.8)–(4.14) hold also in the nonstrictly hyperbolic case, with (4.7) replaced by

(4.16)
$$H_\lambda(x,y) = \begin{cases} -\sum_{j=K+1}^J a_j(y)^{-1} e^{-\int_y^x \lambda/a_j(z)\,dz} r_j(x)\tilde\zeta_j(x,y) l_j^t(y) & x > y, \\ \sum_{j=1}^K a_j(y)^{-1} e^{-\int_y^x \lambda/a_j(z)\,dz} r_j(x)\tilde\zeta_j(x,y) l_j^t(y) & x < y, \end{cases}$$

where $\tilde\zeta_j(x,y) \in \mathbb{R}^{m_j\times m_j}$ denotes a dissipative flow similar to ζ_j in (1.44), but with respect to variable x: i.e.,

$$d\tilde\zeta_j/dx = \eta_j(x)\tilde\zeta_j(x,y)/a_j(x), \quad \tilde\zeta_j(y,y) = I, \tag{4.17}$$

or, equivalently, $\tilde\zeta_j(x,y) = \zeta_j(y,\tau)$ for $z_j(y,\tau) = x$, where z_j as in (1.36) denotes the characteristic path associated with a_j; r_j, l_j now denote $n\times m_j$ blocks; and μ_j, η_j denote $m_j \times m_j$ matrices, $j = 1,\dots,K$, $j = K+1,\dots,J$, where m_j is the multiplicity of eigenvalue a_j of $A(x_0)$, with

$$a_1 \le \dots \le a_K < 0 < a_{K+1} \le \dots \le a_J.$$

(Note that we have not here assumed $(\mathcal{D})$).

Remark. The three error terms in (4.8) may be recognized as roughly analogous to the terms that would arise in the constant-coefficient case through expansions

(4.2)–(4.3); see also the related Fourier transform analysis of [LZe,Ze.1–2]. The complicated "mixed" form of the second, averaged term (corresponding to $\mathcal{E}_j$ in Proposition 8.15, Appendix A3.2.3) comes from our search for minimal regularity hypotheses on f, g, q. At the expense of a further order of differentiability, allowing us to approximately diagonalize the eigenvalue equation to one higher order, we could obtain instead a term of form $\lambda^{-1}(x-y)\tilde{C}(x,y;\lambda)$, where, for example,

$$\begin{aligned}\tilde{C}(x,y;\lambda) = \sum_{j=1}^{k} & e^{\int_y^x (-\lambda/a_j - \eta_j/a_j)(z)\,dz} \\ & \times \big(e^{\tilde{c}_j^-(x,y)(x-y)/\lambda} - 1\big)/\big((x-y)/\lambda\big) r_j(x) l_j^t(y)\end{aligned} \tag{4.18}$$

for $x < y$ and symmetrically for $x > y$, precisely mimicking that of the constant-coefficient case. However, this distinction is unimportant for our analysis, in which the precise form of these terms plays no role; indeed, the (variational) form of (4.10) is somewhat more convenient for our purposes.

The proof of Proposition 4.3, and its nonstrictly hyperbolic analog (stated at the end of the section) will be the work of the remainder of the section.

4.1. General Framework. We have to solve, approximately, the eigenvalue equation

$$(AW)' = (Q - \lambda)W$$

as $|\lambda| \to \infty$ within the region Ω defined above: or, setting $Z = AW$, the equation

$$Z' = (Q - \lambda)A^{-1}Z. \tag{4.19}$$

(As noted in [ZH], such an invertible transformation is a trick to reveal the reduced regularity requirements on coefficients of divergence-form operators).

Following standard procedure (see, e.g., [AGJ,GZ,ZH,Z.4]), we perform the rescaling $\tilde{x} = |\lambda|x$ (suggested by the leading order behavior of $\mu(x_0)$, (4.2)) to obtain the perturbation equation

$$Z' = (A_0(\tilde{x}) + |\lambda|^{-1}A_1(\tilde{x}))Z, \tag{4.20}$$

where

$$\begin{aligned} A_0 &:= -\tilde{\lambda}A^{-1}(|\lambda|^{-1}\tilde{x}), \\ A_1(\tilde{x}) &:= QA^{-1}(|\lambda|^{-1}\tilde{x}), \\ \tilde{\lambda} &:= \lambda/|\lambda|. \end{aligned} \tag{4.21}$$

This expansion is to be considered as a continuous family of one-parameter perturbation equations, indexed by $\tilde{\lambda} \in S^1$. We shall have to take some care, therefore, to ensure that all derived bounds depend continuously on the index $\tilde{\lambda}$; this accounts for the rather meticulous description of error terms in the relevant Proposition 8.15 of Appendix A3.2.3, below.

We seek to develop corresponding expansions of the resolvent kernel in orders of $|\xi|^{-1}$, or, equivalently, by representation (3.17), expansions of stable/unstable manifolds of (4.20), and the reduced flows therein. This we will accomplish by a two-stage process. First, we will reduce by a series of coordinate changes to an approximately block-diagonal system segregating spectrally separated (in particular, stable/unstable) modes, with formal error of a suitably small order. Second, we will convert the formal error into rigorous error bounds using a refined version of the Tracking Lemma of [GZ,ZH,Z.1,Z.4].

Both of these steps will be carried out in a general setting suitable for applications to it arbitrary ordinary differential operators of nondegenerate type (i.e., for which the principal part of the symbol has fixed order and type), even including operators that do not have asymptotically constant, but only smooth coefficients. Together, they give an algorithm for the systematic estimation of high-frequency behavior, to an order of accuracy depending only on the smoothness of the coefficients of the operator in question.

4.2. Formal diagonalization procedure. More generally (fixing any indexing parameters such as $\tilde\lambda$ above), consider an ODE

$$W' = A(\varepsilon x, \varepsilon)W \qquad (x \in \mathbb{R}), \tag{4.22}$$

with small parameter ε, where A has formal Taylor expansion

$$A(y, \varepsilon) = \sum_{k=0}^{p} \varepsilon^k A_k(y) + O(\varepsilon^{p+1}). \tag{4.23}$$

(Note that the case considered in (4.20) corresponds to linear dependence with respect to ε.) Assume:

(h1) A is C^{p+1} in y, ε, or, if $A_0, \dots, A_r$ are constant, then $(\partial/\partial\varepsilon)^{r+1}A$ is C^{p-r} in y, ε. In either case, all derivatives considered, in both y and ε, are uniformly bounded for all $y \in \mathbb{R}$, $\varepsilon \le \varepsilon_0$, for some $\varepsilon_0 > 0$.

(h2) $A_0(y)$ is block-diagonalizable to form

$$D_0(y) = \text{diag}\,\{d_{0,1}, \dots, d_{0,s}\}, \tag{4.24}$$

$d_j \in \mathbb{C}^{n_j \times n_j}$, with spectral separation between blocks (i.e., complex distance between their eigenvalues) uniformly bounded below by some constant $\gamma > 0$, for all $\varepsilon \le \varepsilon_0$, $x \in \mathbb{R}$.

We begin by showing that (4.22) may be approximately block-diagonalized to order ε^p.

Proposition 4.4. *Given (h1)–(h2), there exists a uniformly well-conditioned change of coordinates $W = T\tilde{W}$ such that*

$$\tilde{W}' = D(\varepsilon x, \varepsilon)\tilde{W} + O(\varepsilon^{p+1})\tilde{W} \tag{4.25}$$

uniformly for $x \in \mathbb{R}$, $\varepsilon \le \varepsilon_0$, with

$$D(y, \varepsilon) = \sum_{k=0}^{p} \varepsilon^k D_k(y) + O(\varepsilon^{p+1}) \tag{4.26}$$

and

$$T(y, \varepsilon) = \sum_{k=0}^{p} \varepsilon^k T_k(y) + O(\varepsilon^{p+1}), \tag{4.27}$$

where D_j, $T_j \in C^{p+1-j}(y)$ for all $0 \le j \le p$. and each D_j has the same block-diagonal form

$$D_j(y) = \text{diag } \{d_{j,1}, \dots, d_{j,s}\}, \tag{4.28}$$

$d_{j,k} \in \mathbb{C}^{n_k \times n_k}$, as does D_0.

Our proof of this result relies on the following linear algebraic fact.

Lemma 4.5. *Let $d_1 \in \mathbb{C}^{m_1 \times m_1}$ and $d_2 \in \mathbb{C}^{m_2 \times m_2}$ have norm bounded by C_1 and respective spectra separated by $1/C_2 > 0$. Then, the matrix commutator equation*

$$d_1 X - X d_2 = F, \tag{4.29}$$

$X \in \mathbb{C}^{m_1 \times m_2}$, is soluble for all $F \in \mathbb{C}^{m_1 \times m_2}$, with $|X| \le C(C_1, C_2)|F|$.

Proof. Consider (4.29) as a matrix equation $\mathcal{D}\mathcal{X} = \mathcal{F}$ where $\mathcal{X}$ and $\mathcal{F}$ are vectorial representations of the $m_1 \times m_2$ dimensional quantities X and F, and $\mathcal{D}$ is the matrix representation of the linear operator (commutator) corresponding to the lefthand side. The spectrum of $\mathcal{D}$ is readily seen to be the difference

$$\{\alpha_1 - \alpha_2 : \alpha_j \in \sigma(d_j)\},$$

between the spectra of d_1 and d_2, with associated eigenvectors given by $r_1 l_2^*$, where r_1 is the right eigenvector associated with α_1 and l_2 is the left eigenvector associated with α_2. (Here as elsewhere, $*$ denotes adjoint, or conjugate transpose of a matrix or vector). By assumption, therefore, the spectrum of $\mathcal{D}$ has modulus uniformly bounded below by $1/C_2$, whence the result follows. ∎

Proof of Proposition 4.4. Substituting $W = T\tilde{W}$ into (4.25) and rearranging, we obtain

$$\begin{aligned}\tilde{W}' &= (T^{-1}AT - T^{-1}T')\tilde{W}\\ &= (T^{-1}AT - \varepsilon T^{-1}T_y)\tilde{W},\end{aligned}$$

yielding the defining relation

$$(T^{-1}AT - \varepsilon T^{-1}T_y) = D \tag{4.30}$$

for T.

By (h2), there exists a uniformly well-conditioned family of matrices $T_0(x)$ such that $T_0^{-1}A_0T_0 = D_0$, D_0 as in (4.24); moreover, these may be chosen with the same regularity in y as A_0 (i.e., the full regularity of A). Expanding

$$T^{-1}(y,\varepsilon) = \left(I - \big(\sum_{j=2}^{p} \varepsilon^j T_0^{-1}T_j\big) + \big(\sum_{j=2}^{p} \varepsilon^j T_0^{-1}T_j\big)^2 - \dots \right) T_0^{-1} \tag{4.31}$$

by Neumann series, and matching terms of like order ε^j, we obtain a hierarchy of systems of linear equations of form:

$$D_0T_0^{-1}T_j - T_0^{-1}T_jD_0 - F_j = D_j, \tag{4.32}$$

where F_j depends only on A_k for $0 \le k \le j$ and T_k, $(d/dy)T_k$ for $0 \le j-1$.

On off-diagonal blocks (k,l), (4.32) reduces to

$$d_l[T_0^{-1}T_j]^{(k,l)} - [T_0^{-1}T_j]^{(k,l)}d_k = F_j^{(k,l)}, \tag{4.33}$$

uniquely determining $[T_0^{-1}T_j]^{(k,l)}$, by Lemma 4.5. On diagonal blocks (k,k), we are free to set $[T_0^{-1}T_j]^{(k,k)}$, whereupon (4.32) reduces to

$$[D_j]^{(k,k)} = -[F_j]^{(k,k)}, \tag{4.34}$$

determining the remaining unknown $[D_j]^{(k,k)}$.

Thus, we may solve for D_j, T_j at each successive stage in a well-conditioned way. Moreover, it is clear in the general case that $A_0 \not\equiv$ constant that the regularity of D_j, T_j is as claimed, since we lose one order of regularity at each stage, through the dependence of F_j on derivatives of lower order T_k, $k < j$. In the case that $A_0, \dots, A_s$ are constant in y, we may choose also $T_0, \dots, T_s$ to be constant in y, again recovering the claimed regularity. ∎

Remark 4.6. Proposition 4.4 may be verified in more indirect fashion without using Lemma 4.5, by expressing T instead as a product $T_0T_1\cdots T_p$ and solving the resulting succession of nearby diagonalization problems.

Remark 4.7. For relaxation systems, we will need to expand to formal error order ε^2, i.e., $p=1$. Thus, we require A, $Q \in C^2$ in the general case, and C^1 for discrete kinetic models, in which case A_0 above is constant; this translates to our requirements of C^3/C^2 on f, q in (H0).

Refined diagonalization procedure. Now, in addition to (h1)–(h2), assume:
(h3) $A_0(\cdot)$ decays exponentially in y.
(Note: this is the case for operators whose coefficients approach constants at exponential rate as $y=\varepsilon x \to \pm\infty$, in particular for relaxation profiles). In this case, we point out that, by judicious choice of the initial diagonalizing transformation T_0, we may arrange that D is given to first order by the "frozen-coefficient" approximation $T_0^{-1}AT_0$.

Proposition 4.8. *Let (h1)–(h3) hold. Then, in Proposition 4.4, it is possible to choose $T_0(\cdot)$ in such a way that, also, $D_0+\varepsilon D_1$ is determined simply by the block-diagonal part of*

$$T_0^{-1}(A_0+\varepsilon A_1)T_0. \tag{4.35}$$

Reviewing (4.30) and (4.34), we find that (4.35) is equivalent to the requirement that $T_0^{-1}(d/dy)T_0$ vanish on diagonal blocks. The matrix T_0 is given by

$$T_0 = (\,R_1,\dots,R_K\,)\,, \tag{4.36}$$

where the blocks R_j are made up of columns spanning eigenspaces of A_0 corresponding to the various blocks. Let us write suggestively

$$T_0^{-1} = \begin{pmatrix} L_1 \\ \vdots \\ L_K \end{pmatrix}, \tag{4.37}$$

where L_jR_k is a zero block if $j\neq k$ and an identity block if $j=k$. Then, Proposition 4.8 follows from:

Lemma 4.9. *Let (h1)–(h3) hold. Then, given any initial prescription of $R_j(0)$, $L_j(0)$ at $y=0$, there exists a unique choice of R_j, L_j in (4.36)–(4.37) such that*

$$L_jR_j' = L_j'R_j \equiv 0, \tag{4.38}$$

which, moreover, is uniformly smooth and bounded to the same order of differentiability as is A_0.

Proof. Let $\tilde R_j$, $\tilde L_j$ denote any smooth choice, and define

$$R_j := \tilde R_j\alpha_j \qquad L_j := \alpha_j^{-1}\tilde L_j, \tag{4.39}$$

where α_j are nonsingular $k_j \times k_j$ matrices to be determined, k_j denoting the dimension of the jth block (i.e., R_j dimension $N \times k_j$). Defining α_j to solve the linear ODE

$$\alpha_j' = -\tilde{L}_j \tilde{R}_j' \, \alpha_j, \qquad \alpha_j(0) = I_{k_j}, \tag{4.40}$$

where $'$ in this single instance denotes d/dy, we obtain (4.38) by direct calculation:

$$L_j R_j' = L_j(\tilde{R}_j'\alpha_j + \tilde{R}_j\alpha_j') = L_j(\tilde{R}_j'\alpha_j - \tilde{R}_j\tilde{L}_j\tilde{R}_j'\alpha_j) = L_j(I - \tilde{R}_j\tilde{L}_j)\tilde{R}_j'\alpha_j = 0. \tag{4.41}$$

The other relation comes from $L_jR_j = I$, from which $L_j'R_j + L_jR_j' = 0$. Uniform global bounds and smoothness then follow by linear theory, and the fact that $\tilde{R}_j$, $\tilde{L}_j$ could be chosen in the first place so that $\tilde{L}_j\tilde{R}_j'$ decays exponentially in y at spatial infinity. ∎

Remark 4.10. The key result of Lemma 4.9 was first pointed out in the diagonalizable case by Goodman, in [Go.1]. There, the careful choice of diagonalizing coordinates was essential in his analysis of stability by energy methods. It is essential for us here, as well, in this different context. Note that we do not need this result in the case that A_0 is identically constant, corresponding to the case that the principal part of the symbol of the associated operator L has constant coefficients, for which we may simply choose L, R constant as well. This was done, for example, in the treatment of scalar dispersive–diffusive systems in [HZ.2], for which the dispersion coefficient was assumed constant, and can be done also in the case of discrete kinetic (relaxation) models. *Thus, the role of Lemma 4.9 in our analysis is to give sharp estimates in the case of a variable-coefficient principal part.*

Remark 4.11. An important observation for us is that, in both Propositions 4.4 and 4.8, any additional regularity of A with respect to ε translates directly to regularity of the error term $\mathcal{O}(\varepsilon^{p+1})$. More precisely, if $A \in C^q(\varepsilon \to C^{p+1}(y))$, $q \geq p+1$, then the error term has regularity $C^{q-p-1}(\varepsilon \to C^0(y))$. In case $A \in C^\omega(\varepsilon \to C^{p+1}(y))$, $q \geq p+1$, as often happens (for example, for relaxation profiles), the error term has regularity $C^\omega(\varepsilon \to C^0(y))$. This observation is used in conjunction with Proposition 8.15 below; specifically, it gives regularity with respect to δ (in this case ε^k) in the error term Θ_{11}.

Remark 4.12. A second important observation is that even though we have used in (4.20) the (standard) homogeneous rescaling $\tilde{x} = |\lambda| x$, which introduces terms such as $\tilde{\lambda} = \lambda/|\lambda|$ that are singular at $\lambda = \infty$, any associated singularities at infinity in the resulting expansion are only apparent. That is, they are *coordinate singularities* that disappear when the diagonalized system is converted back to original coordinates. More precisely, since $|\lambda| A$ is analytic at ∞ with respect to $\lambda^{-1} = \varepsilon/\tilde{\lambda}$, we may choose the initial diagonalizing transformation T_0 to be analytic in λ^{-1} as well (indeed, it may be chosen independent of λ; see (4.21), with the effect that $|\lambda|D_0$ is also analytic in λ^{-1}. If we make this choice, then it is readily verified that all further T_j, D_j also enjoy these properties, since they are preserved by the (uniquely determined, from this point) diagonalization procedure. Clearly, this observation is not unique to the case of relaxation systems, but holds for general equations and rescalings. An immediate consequence is that the rigorous bounds obtained from the formal diagonalization series by the fixed point iteration described in Appendix A3.2, when expressed in original coordinates, must also be analytic in λ^{-1}; for, the fixed point iteration, being scale invariant, may just as well be carried out in the original (i.e., unrescaled) coordinates to begin with, whereupon we obtain analyticity as usual from the uniform convergence of iterates.

We point out that we could alternatively have performed the formal diagonalization to begin with in variable $\varepsilon := \lambda^{-1}$, using the modified perturbation formulation

$$W' = \tilde{\lambda}(-A^{-1} + \lambda^{-1}QA^{-1}),$$

and taking A_j to depend on $(\tilde{\lambda}^{-1}\varepsilon x, \varepsilon)$.

4.3. Error bounds/application to relaxation systems. We now return to our original problem, that of approximating the stable/unstable manifolds at $+\infty/-\infty$ of ODE (4.22), and the reduced flows therein. In Appendix A3.2, we present a systematic method for generating such approximations, given a system in the approximately diagonal form (4.34). The main requirement in the theory is that the approximate flows $\mathcal{F}_j^{y\to x}$ generated by the blocks $d_j := D^{(j,j)}$ be "exponentially separated" to some order ε^k, $k \le p$, in the sense that, as x increases, they are either uniformly exponentially decaying or else uniformly exponentially growing, i.e.:

$$\begin{cases} |\mathcal{F}_j^{y\to x}| \le Ce^{\varepsilon^k\theta_1(x-y)}, \\ |\mathcal{F}_j^{x\to y}| \le Ce^{\varepsilon^k\theta_2(y-x)}, \end{cases} \tag{4.42}$$

for $x \ge y$, with

$$\theta_1 < 0 < \theta_2. \tag{4.43}$$

Observation 4.13. In the simplest case, that A_0 is diagonalizable, the verification of (4.43) is straightforward. For, in this case, the block flows are scalar, and exponential separation is equivalent to the integral spectral condition

$$\begin{cases} \int_y^x \mathrm{Re}\ d_j(z)\,dz \le \varepsilon^k\theta_1(x-y) + \mathcal{O}(1), \\ \int_y^x \mathrm{Re}\ d_j(z)\,dz \ge \varepsilon^k\theta_2(x-y) + \mathcal{O}(1), \end{cases} \tag{4.44}$$

or, roughly, the satisfaction on average of a spectral gap Re $d_j < \varepsilon^k\theta_1$/Re $d_j > \varepsilon^k\theta_2$. Moreover, this is clearly equivalent to the satisfaction of the corresponding condition

$$\begin{cases} \int_y^x \mathrm{Re}\ (\sum_{l=0}^k \varepsilon^l d_{l,j}(z)\,dz \le \varepsilon^k\theta_1(x-y) + \mathcal{O}(1), \\ \int_y^x \mathrm{Re}\ (\sum_{l=0}^k \varepsilon^l d_{l,j}(z)\,dz \ge \varepsilon^k\theta_2(x-y) + \mathcal{O}(1), \end{cases} \tag{4.45}$$

at the kth stage of the diagonalization process; thus, we need only check (4.45) for each $1 \leq k \leq p$. In particular, it is sufficient that there hold the pointwise spectral gap

$$\begin{cases} \text{Re } (\sum_{l=0}^{k} \varepsilon^l d_{l,j} \leq \varepsilon^k \theta_1, \\ \text{Re } (\sum_{l=0}^{k} \varepsilon^l d_{l,j} \geq \varepsilon^k \theta_2, \end{cases} \tag{4.46}$$

on all except a set of measure $1/\alpha$, where it is violated by at most α; for, the corresponding error can be absorbed in the multiplying constant C of (4.44)–(4.45).

In the case of strictly hyperbolic relaxation systems satisfying (H0)–(H4), (4.46) holds on Ω for $k = p = 1$ for all $|y| \geq M$, or equivalently $|x| \geq M/\varepsilon$, by Lemma 4.8 together with Lemma 4.1, (1.27), and the exponential approach in y of coefficients to their constant states, Lemma 1.2, while on $|y| \leq M$, or $|x| \leq \varepsilon$, it is violated by at most order ε. Thus, the entire theory of Appendix A3.2 applies, yielding the error bounds reported below. ∎

Observation 4.14. When A_0 is only block-diagonalizable, the verification of (4.43) is more involved. Since d_j is now a matrix, (4.44) is neither necessary nor sufficient. Moreover, the expansion (4.26) must be considered as a matrix perturbation problem, and so the dependence of spectrum on coefficients D_k is not necessarily monotone in k. Thus, (4.42)–(4.43) are in general not equivalent (as in the scalar case) to the corresponding conditions on the kth order approximate flow, unless there hold additional properties such as diagonalizability of $\sum_{l \leq k-1} \varepsilon^l d_{l,j}$ (as implied, in particular, by *symmetry*) or special structure of $d_{l,j}$ relative to $d_{k-1,j}$, for all $k \leq l \leq p$.

On the other hand, suppose as in the present case that $d_{l,j}$ are *scalar* until step k, where splitting occurs. Then, the contribution of these lower order terms to the approximate flow induced by $d_{k,j}$ may be split off as an exponential factor

$$e^{\int_y^x \sum_{l=0}^{k-1} \varepsilon^l d_{l,j}(z)\, dz} \tag{4.47}$$

times the approximate flow induced by $\varepsilon^k d_{k,j}$ alone, simplifying the computation somewhat. In particular, for nonstrictly hyperbolic relaxation systems satisfying (H0)–(H4), with $k = p = 1$, we obtain just $e^{-\int_y^x \tilde{\lambda}/a_j(z)\, dz}$ times the flow induced by $\varepsilon\eta(z)$, η as defined in Section 1.3. The latter flow may be seen to have an order ε spectral separation, by (1.46) plus exponential convergence of η_j to $\eta_j^{\pm}$ as $y \to \pm\infty$[5], while the former has exponential separation (in agreeing sense) of order

[5]For example, we may apply the Gap Lemma of Appendix A3.1 separately to cases x, $y \in (-\infty, 0]$ and x, $y \in [0, +\infty)$, treating the case $y < 0 < x$ by the semigroup property as the concatenation of cases $y < 0 = x$ and $y = 0 < x$. Alternatively, we may proceed as in Appendix A3.2 by applying a change of coordinates $w_j = P\tilde{w}_j$, $P = I + \mathcal{O}(e^{-\theta\varepsilon x})$, converting η_j to positive/negative definite form for all $x \geq M/\varepsilon$).

Re $\tilde\lambda/\min_{j,y}|a_j(y)|$. Combining these observations, we find that there is spectral separation on all of Ω. Thus, the theory of Appendix A3.2 applies again, yielding the desired error bounds. ∎

Remark 4.15. More generally, we point out, in the case relevant to traveling waves that all A_j converge exponentially in y to constants $A_j^\pm$ as $y \to \pm\infty$, that, provided that $d_{1,j}$ are scalar, and there is neutral separation to zeroth order

$$\begin{cases} \text{Re } d_{1,j} \geq 0, \\ \text{Re } d_{1,j} \leq 0, \end{cases}$$

for all y (as is then a *necessary* condition for spectral separation along rays pointing into the domain of consistent splitting), then a sufficient condition for spectral separation of order ε^k is *spectral gap* of order ε^k in the limiting blocks $d_j(\pm\infty)$ (also a necessary condition, by the Gap Lemma of Appendix A3.1). For, the diagonalizing construction described in Proposition 4.4 clearly preserves the property of exponential decay to constant states. Thus, making a constant change of coordinates such that $d_j^\pm$ becomes positive/negative definite of order ε^k, we have that d_j is positive/negative definite to order $\varepsilon^k + \mathcal{O}(\varepsilon e^{\varepsilon x})$, readily yielding (4.42). More precise accounting of the separation, as in the two observations just above, requires more detailed consideration of the flows of different orders. We remark that $d_{1,j}$ scalar is roughly equivalent to diagonalizability/constant multiplicity of A_0, since it may then be arranged that each block d_j correspond to a single eigenvalue of A_0. *Thus, the requirement of spectral separation is in practice no restriction for applications to stability of traveling waves.*

Proof of Proposition 4.3. We will carry out in detail the basic estimate (4.8), indicating the extension to derivative bounds (4.13)–(4.14) by a few brief remarks. First, observe that in the modified coordinates $Z = AW$, (3.17) just becomes

$$G_\lambda(x,y) = \begin{cases} A^{-1}(x)\mathcal{F}^{y\to x}\,\Pi_y^- & x < y, \\ \\ -A^{-1}(x)\mathcal{F}^{y\to x}\,\Pi_y^+ & x > y, \end{cases} \tag{4.48}$$

where $\Pi_\pm$, $\mathcal{F}^{y\to x}$ now denote the corresponding objects in Z-coordinates.

Applying the formal diagonalization procedure of Proposition 4.4 to (4.20), and choosing the special initialization of Lemma 4.9, at the same time taking care as described in Remark 4.12 to preserve analyticity with respect to λ^{-1} in original coordinates, we obtain the approximately diagonalized system

$$\begin{aligned} z_j' &= (\tilde\lambda a_j + \varepsilon\eta_j)z_j + \varepsilon^2\sum_{j,k}\tilde\Theta_{jk}z_k \\ &= |\lambda|^{-1}\left((\lambda a_j + \eta_j)z_j + \lambda^{-1}\sum_{j,k}\Theta_{j,k}z_k\right), \end{aligned}$$

where error terms $\Theta_{j,k}$ are bounded and analytic in λ^{-1} as λ goes to infinity, or, converting back to the original spatial coordinates:

$$z_j' = \left((\lambda a_j + \eta_j) z_j + \lambda^{-1} \sum_{j,k} \Theta_{j,k} z_k \right). \tag{4.49}$$

Moreover, the approximatedly diagonalizing transformation T is given by

$$\begin{aligned} T &= T_0 + \lambda^{-1} T_1 \\ &= (r_1, \dots, r_J) + \lambda^{-1} T_1, \end{aligned}$$

where r_j are as described in Proposition 4.3.

As described in Observations 4.13–4.14 just above, the approximate, block-diagonal flows $z_j' = ((\lambda a_j + \eta_j) z_j$ are exponentially separated, with uniform rates

$$\begin{cases} C^{\int_y^x \mathrm{Re}\ \lambda / a_K(z)\, dz} e^{-\theta|x-y|}, \\ Ce^{-\int_y^x \mathrm{Re}\ \lambda / a_{K+1}(z)\, dz} e^{-\theta|x-y|}, \end{cases} \tag{4.50}$$

where $a_K < 0 < a_{K+1}$. Applying the fixed point construction of Appendix A3.2.2(ii), we find that the stable/unstable manifolds of (4.49) are *uniformly transverse* for $\lambda \in \Omega$, $|\lambda|$ sufficiently large, with associated projections $\Pi_\pm$ given by

$$\Pi_+ = \sum_{j=1}^{K} r_j l_j^* + \lambda^{-1} E_+ + \mathcal{O}(\lambda^{-2}) \tag{4.51}$$

and

$$\Pi_- = \sum_{j=K+1}^{J} r_j l_j^* + \lambda^{-1} E_- + \mathcal{O}(\lambda^{-2}), \tag{4.52}$$

for some continuous matrices $E_\pm(x)$.

From transversality, we obtain immediately that $\Omega \setminus B(0, R) \subset \rho(L) \cap \Lambda$, for R sufficiently large, as claimed. Moreover, approximating the reduced flows on stable/unstable manifolds using Proposition 8.15 of Appendix A3.2.3, and recalling representation (4.48), we obtain after rearrangement the claimed estimates on the resolvent kernel G_λ, where error terms B correspond to error terms $E\pm$ in (4.51)–(4.52), C to term $\mathcal{E}_1^\pm$ in (8.61)–(8.62) of Proposition 8.15, and D contains all remaining, $\mathcal{O}(\lambda^{-2})$ errors: for example, the third, $\mathcal{O}(\delta^2/\eta^2) = \mathcal{O}(\lambda^{-2})$ term in the righthand side of (8.61), the $\mathcal{O}(\lambda^{-2})$ terms in (4.51)–(4.52), and cross-terms coming from the product of different order λ^{-1} terms.

Derivative estimates. To obtain derivative estimates (4.13)–(4.14), we may follow precisely the same procedure, but carrying out to one further order the fixed point iteration at the stage of the reduced flow estimate described in Proposition 8.15: that is, to order λ^{-2}, with error term $\mathcal{O}(\lambda^{-3}$. (Note: this leads to the second-order iterated integral (4.15)). This yields a Taylor expansion to the same higher order for the x and y derivatives of the reduced flow, within the *rescaled* spatial coordinates, and thus, in original coordinates, to the desired λ^{-1} order, with $\mathcal{O}(\lambda^{-2})$ error. ∎

Section 5. Low frequency bounds.

Our next object is the estimation of the resolvent kernel in the remaining critical regime $|\lambda| \to 0$, corresponding to large time behavior of the Green's function G, or *global behavior* in space and time. Here, the details of the system are not important, and we may follow essentially the same treatment as in the strictly parabolic case, combining aspects of [ZH,Z.2,Z.4]. From a global perspective, the structure of the linearized equations is that of two nearly constant-coefficient regions separated by a thin shock layer near $x = 0$; accordingly, behavior is essentially governed by the two, limiting far-field equations:

$$U_t = L_\pm U, \tag{5.1}$$

coupled by an appropriate transmission relation at $x = 0$.

Similarly as in the previous section, we begin by examining these limiting systems. For simplicity of exposition, we make the provisional assumption:

(A1) The $r \times r$ matrices $dqA_\pm^{-1}(0, I_r)^t$ have distinct eigenvalues.

(Recall, (1.12), that these matrices govern behavior near $(u_\pm, v_\pm)$ in traveling wave ODE (1.10).) Since none of our estimates will depend on the distance between eigenvalues, this assumption is easily removed by a limiting argument.

Lemma 5.1. *Assume that there hold (H1)–(H3) and (A1). Then, for $|\lambda|$ sufficiently small, the eigenvalue equation $(L_\pm - \lambda)W = 0$ associated with the limiting, constant-coefficient operator $L_\pm$ has a basis of $n + r$ solutions*

$$\bar{W}_j^\pm = e^{\mu_j^\pm(\lambda)x} V_j(\lambda), \tag{5.2}$$

$\mu_j^\pm$, $V_j^\pm$, analytic in λ, consisting of r "fast" modes

$$\begin{aligned} \mu_j^\pm &= \gamma_j^\pm + \mathcal{O}(\lambda), \\ V_j^\pm &= A_\pm^{-1} S_j^\pm + \mathcal{O}(\lambda), \end{aligned} \tag{5.3}$$

$S_j^\pm = (0, s_j^{\pm t})^t$ where $\gamma_j^\pm$, $s_j^\pm$ are eigenvalues and associated right eigenvectors of $dqA_\pm^{-1}(0, I_r)^t$ (equivalently, $\gamma_j^\pm$, $S_j^\pm$ are nonzero eigenvalues and associated right eigenvectors of QA^{-1}), and n "slow" modes

$$\begin{aligned} \mu_{r+j}^\pm(\lambda) &:= -\lambda / a_j^{*\pm} + \lambda^2 \beta_j^{*\pm} / {a_j^{*\pm}}^3 + \mathcal{O}(\lambda^3), \\ V_{r+j}^\pm(\lambda) &:= R_j^{*\pm} + \mathcal{O}(\lambda), \end{aligned} \tag{5.4}$$

where $a_j^{*\pm}$, $R_j^{*\pm}$, and $\beta_j^*\pm$ are as in Proposition 1.9. Likewise, the adjoint eigenvalue equation $(L_\pm - \lambda)^* Z = 0$ has a basis of solutions $\tilde{\bar{W}}_1^{\pm} = e^{-\mu_j^\pm(\lambda)x}\tilde{V}_j(\lambda)$, with

$$\tilde{V}_j(\lambda) = \tilde{T}_j^\pm + \mathcal{O}(\lambda) \tag{5.5}$$

and

$$\tilde{V}_{r+j}(\lambda) = L_j^* + \mathcal{O}(\lambda), \tag{5.6}$$

$\tilde{V}$ analytic in λ, where $\tilde{T}_j$ are the left eigenvectors of $QA_\pm^{-1}$ associated with the nonzero eigenvalues $-\mu_j^\pm$, and $L_j^{*\pm}$ are as in Proposition 1.9.

Proof. Equivalently, we must show that $\mu_j^\pm$, $V_j^\pm$ describe the eigenvalues and associated eigenvectors of coefficient matrix $A^{-1}(Q-\lambda)_\pm$, and (see Lemma 3.8) that $\tilde{V}_j^\pm$ describe the left eigenvectors of $(Q-\lambda)A_\pm^{-1}$. At $\lambda = 0$, there are r distinct nonzero eigenvalues $\mu_j^\pm = \gamma_j^\pm$, by assumption (A1), and an n-fold eigenvalue $\mu = 0$ corresponding to the n-dimensional kernel of Q. By spectral separation, the former vary analytically as λ is varied, with Taylor expansions (5.3) and (5.5).

The latter may be seen to vary analytically, with expansions (5.4) and (5.6), by inversion of the expansions

$$\lambda_j(i\xi) = -ia_j^{*\pm}\xi - \beta_j^{*\pm}\xi^2 + \mathcal{O}(\xi^3)$$

carried out in Appendix A2 for the the dispersion curves near $\xi = 0$, together with the fundamental relation $\mu = i\xi$ between roots of characteristic and dispersion equations (recall (3.4)–(3.5) of Section 3.1). Alternatively, one can carry out directly the associated matrix bifurcation problem as in [ZH] to obtain the same result. ∎

Our main result of this section is then:

Proposition 5.2. *Assume that there holds (H0)–(H4) and ($\mathcal{D}$) (equivalently, (D1)–(D2)). Then, for $r > 0$ sufficiently small, the resolvent kernel G_λ has a meromorphic extension onto $B(0,r)$, which may in the Lax or overcompressive case be decomposed as*

$$G_\lambda = E_\lambda + S_\lambda + R_\lambda \tag{5.7}$$

where, for $y \le 0$:

$$E_\lambda(x,t;y) := \lambda^{-1} \sum_{a_k^{*-}>0} [c_{k,-}^{j,0}] \frac{\partial}{\partial \delta_j} \begin{pmatrix} \bar{u}^\delta(x) \\ \bar{v}^\delta(x) \end{pmatrix} L_k^{*-t} e^{(\lambda/a_k^{*\pm} - \lambda^2\beta_k^{*\pm}/a_k^{*\pm 3})y}, \tag{5.8}$$

(5.9)
$$S_\lambda(x,t;y) := \sum_{a_k^{*-}>0,\, a_j^{*+}>0} [c_{k,-}^{j,+}] R_j^{*+} L_k^{*-t} e^{(-\lambda/a_j^{*+} + \lambda^2\beta_k^{*+}/a_k^{*+3})x + (\lambda/a_k^{*-} - \lambda^2\beta_k^{*-}/a_k^{*-3})y}$$

for $y \le 0 \le x$,

(5.10)
$$S_\lambda(x,t;y) := \sum_{a_k^{*-}>0} R_k^{*-} L_k^{*-t} e^{(-\lambda/a_k^{*-} + \lambda^2\beta_k^{*-}/a_k^{*-3})(x-y)} + \sum_{a_k^{*-}>0,\, a_j^{*-}<0} [c_{k,-}^{j,-}] R_j^{*-} L_k^{*-t} e^{(-\lambda/a_j^{*-} + \lambda^2\beta_j^{*-}/a_j^{*-3})x + (\lambda/a_k^{*-} - \lambda^2\beta_k^{*-}/a_k^{*-3})y}$$

for $y \le x \le 0$, *and*

(5.11)
$$S_\lambda(x,t;y) := \sum_{a_k^{*-}<0} R_k^{*-} L_k^{*-t} e^{(-\lambda/a_k^{*-} + \lambda^2\beta_k^{*-}/a_k^{*-3})(x-y)} + \sum_{a_k^{*-}>0,\, a_j^{*-}<0} [c_{k,-}^{j,-}] R_j^{*-} L_k^{*-t} e^{(-\lambda/a_j^{*-} + \lambda^2\beta_j^{*-}/a_j^{*-3})x + (\lambda/a_k^{*-} - \lambda^2\beta_k^{*-}/a_k^{*-3})y}$$

for $x \le y \le 0$, *and* $R_\lambda = R_\lambda^E + R_\lambda^S$ *is a faster-decaying residual satisfying*

$$(\partial/\partial x)^\gamma (\partial/\partial y)^\alpha R_\lambda^E = \mathcal{O}(e^{-\theta|x-y|}) + \sum_{a_k^{*-}>0} e^{-\theta|x|} e^{(\lambda/a_k^{*\pm} - \lambda^2\beta_k^{*\pm}/a_k^{*\pm 3})y} \times \left(\lambda^{\alpha-1}\mathcal{O}(e^{\mathcal{O}(\lambda^3)y} - 1) + \lambda^{\alpha-1}\mathcal{O}(e^{\mathcal{O}(\lambda^3)x} - 1) + \mathcal{O}(\lambda^\alpha)\right), \tag{5.12}$$

(5.13)
$$(\partial/\partial x)^\gamma (\partial/\partial y)^\alpha R_\lambda^S(x,t;y) = \sum_{a_k^{*-}>0,\, a_j^{*+}>0} e^{(-\lambda/a_j^{*+} + \lambda^2\beta_k^{*+}/a_k^{*+3})x + (\lambda/a_k^{*-} - \lambda^2\beta_k^{*-}/a_k^{*-3})y} \times \left(\lambda^{\gamma+\alpha}\mathcal{O}(e^{\mathcal{O}(\lambda^3)y} - 1) + \lambda^{\gamma+\alpha}\mathcal{O}(e^{\mathcal{O}(\lambda^3)x} - 1) + \lambda^\alpha \mathcal{O}(e^{-\theta|x|}) + \mathcal{O}(\lambda^{1+\gamma})\right)$$

for $y \le 0 \le x$,

(5.14)
$$\begin{aligned}(\partial/\partial x)^\gamma (\partial/\partial y)^\alpha R_\lambda^S(x,t;y) = &\sum_{a_k^{*-}>0} e^{(-\lambda/a_k^{*-} + \lambda^2\beta_k^{*-}/a_k^{*-3})(x-y)} \\ &\times \left(\lambda^{\gamma+\alpha}\mathcal{O}(e^{\mathcal{O}(\lambda^3)y} - 1) + \lambda^{\gamma+\alpha}\mathcal{O}(e^{\mathcal{O}(\lambda^3)x} - 1) + \lambda^\alpha \mathcal{O}(e^{-\theta|x|}) + \mathcal{O}(\lambda^{1+\gamma})\right) \\ &+ \sum_{a_k^{*-}>0,\, a_j^{*-}<0} e^{(-\lambda/a_j^{*-} + \lambda^2\beta_j^{*-}/a_j^{*-3})x + (\lambda/a_k^{*-} - \lambda^2\beta_k^{*-}/a_k^{*-3})y} \\ &\times \left(\lambda^{\gamma+\alpha}\mathcal{O}(e^{\mathcal{O}(\lambda^3)y} - 1) + \lambda^{\gamma+\alpha}\mathcal{O}(e^{\mathcal{O}(\lambda^3)x} - 1) + \lambda^\alpha \mathcal{O}(e^{-\theta|x|}) + \mathcal{O}(\lambda^{1+\gamma})\right)\end{aligned}$$

for $y \le x \le 0$, and

(5.15)
$$
\begin{aligned}
(\partial/\partial x)^\gamma(\partial/\partial y)^\alpha R^S_\lambda(x,t;y) = &\sum_{a_k^{*-}<0} e^{(-\lambda/a_k^{*-}+\lambda^2\beta_k^{*-}/a_k^{*-3})(x-y)}\\
&\times\Big(\lambda^{\gamma+\alpha}\mathcal{O}(e^{\mathcal{O}(\lambda^3)y}-1)+\lambda^{\gamma+\alpha}\mathcal{O}(e^{\mathcal{O}(\lambda^3)x}-1)+\lambda^\alpha\mathcal{O}(e^{-\theta|x|})+\mathcal{O}(\lambda^{1+\gamma})\Big)\\
&+\sum_{a_k^{*-}>0,\, a_j^{*-}<0} e^{(-\lambda/a_j^{*-}+\lambda^2\beta_j^{*-}/a_j^{*-3})x+(\lambda/a_k^{*-}-\lambda^2\beta_k^{*-}/a_k^{*-3})y}\\
&\times\Big(\lambda^{\gamma+\alpha}\mathcal{O}(e^{\mathcal{O}(\lambda^3)y}-1)+\lambda^{\gamma+\alpha}\mathcal{O}(e^{\mathcal{O}(\lambda^3)x}-1)+\lambda^\alpha\mathcal{O}(e^{-\theta|x|})+\mathcal{O}(\lambda^{1+\gamma})\Big)
\end{aligned}
$$

for $x \le y \le 0$, $\alpha+\beta\le 1$, with each $\mathcal{O}(\cdot)$ term separately analytic in λ, and where $[c^{j,0}_{k,\alpha}]$, $\alpha = -,0,+$, are constants to be determined later. Symmetric bounds hold for $y \ge 0$. Similar, but more complicated formulae hold in the undercompressive case (see Remark 6.2, below).

5.1. Normal modes (behavior in x). We begin by relating the normal modes of the variable coefficient eigenvalue equation (3.1) to those of the limiting, constant-coefficient equations (5.1).

Lemma 5.3. *Let (H0)–(H3) hold. Then, for $\lambda \in B(0,r)$, r sufficiently small, there exist solutions $W_j^\pm(x;\lambda)$ of (3.1), C^2 in x and analytic in λ, satisfying*

(5.16)
$$
\begin{aligned}
W_j^\pm(x;\lambda) &= V_j^\pm(x;\lambda)e^{\mu_j^\pm x}\\
(\partial/\partial\lambda)^k V_j^\pm(x;\lambda) &= (\partial/\partial\lambda)^k V_j^\pm(\lambda)+\mathcal{O}(e^{-\tilde\theta|x|}|V_j^\pm(\lambda)|), \quad x \gtrless 0,
\end{aligned}
$$

for any $k\ge 0$ and $0<\tilde\theta<\theta$, where θ is the rate of decay given in (1.14), $\mu_j^\pm(\lambda)$, $V_j^\pm(\lambda)$ are as in Lemma 5.1 above, and $\mathcal{O}(\cdot)$ depends only on k, $\tilde\theta$.

Proof. With the exception of the regularity C^2 asserted in x, this is a direct consequence of Lemma 5.1 together with Proposition 8.1 of Appendix A3.1 (the Gap Lemma). The additional regularity of C^2 vs. the C^1 asserted in the Proposition is a consequence of the divergence form of the eigenvalue equation, and may be deduced as in the previous section by the change of coordinates (4.19). ∎

The bases $\phi_j^\pm$, $\psi_j^\pm$ of Section 3.1 may evidently be chosen from among $W_j^\pm$, yielding an analytic choice of bases in λ, with the detailed description (5.16). It follows that the dual bases $\tilde\phi_j^\pm$, $\tilde\psi_j^\pm$ defined in Section 3.2 are also analytic in λ and satisfy corresponding bounds with respect to the dual solutions (5.5)–(5.6). With this observation, we have immediately, from Definition (3.21) and Proposition 3.11:

Corollary 5.4. *Let (H0)–(H3) hold. Then, the Evans function $D(\lambda)$ admits an analytic extension onto $B(0,r)$, for r sufficiently small, and the resolvent kernel*

$G_\lambda(x,y)$ admits a meromorphic extension, with an isolated pole of finite multiplicity at $\lambda = 0$.

Moreover, we have close control on the (x, y) behavior of G_λ through the spectral decomposition formulae of Proposition 3.11. For *slow, dual modes*, the bounds (5.16) (in particular, the consequent bounds on first spatial derivatives) can be considerably sharpened, provided that we appropriately initialize our bases at $\lambda = 0$. This observation will be quite significant in the Lax or overcompressive case. Likewise, *fast-decaying forward modes* can be well-approximated near $\lambda = 0$ by their representatives at $\lambda = 0$, using only the basic bounds (5.16). These two categories comprise the modes determining behavior of the Green's function to lowest order.

Specifically, due to the special, partially conservative structure of the underlying evolution equations, the adjoint eigenvalue equation (3.31) at $\lambda = 0$ admits an n-dimensional subspace of constant solutions

$$\tilde{W} = (\tilde{w}_1^t, \tilde{w}_2^t)^t \equiv (\text{constant}, 0)^t; \tag{5.17}$$

this is equivalent to the observation [L.2] that integral quantities in variable u are conserved under time evolution for relaxation systems of form (1.1). Thus, at $\lambda = 0$, we may choose, by appropriate change of coordinates if necessary, that slow-decaying dual modes $\tilde{\phi}_j^\pm$ and slow-growing dual modes $\tilde{\psi}_j^\pm$ be identically constant, Note that this does not interfere with our previous choice in Lemma 5.3, since that concerned only the choice of limiting solutions $\overline{W}_j^\pm$ of the asymptotic, constant coefficient equations at $x \to \pm\infty$, and not the particular representatives $W_j^\pm$ that approach them (which, in the case of slow modes, are specified only up to the addition of an arbitrary fast-decaying mode).

Remark 5.5. The prescription of constant dual bases just described requires that we reverse our previous approach, choosing dual bases first using the Gap Lemma, then defining forward bases using duality. Alternatively, rewriting the forward eigenvalue equation at $\lambda = 0$ as

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}' = QA^{-1}Z = \begin{pmatrix} 0 \\ dqA^{-1}Z \end{pmatrix}, \tag{5.18}$$

where $Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} := AW$, we may observe that fast-decaying modes satisfy

$$\begin{aligned} &z_1 \equiv 0 \\ &z_2' = dqA^{-1}\begin{pmatrix} 0 \\ z_2 \end{pmatrix}, \end{aligned} \tag{5.19}$$

i.e., the linearization of the reduced traveling wave ode (1.11). By duality relation (3.33), requiring slow dual modes to be constant is equivalent to choosing fast-growing (forward) modes as well as fast-decaying modes from among the solutions of (5.19). (Recall: though fast-decaying modes are uniquely determined as a subspace, fast-growing modes are only determined up to the addition of faster decaying modes.)

Lemma 5.6. *Assuming (H0)–(H3), with the above choice of bases at $\lambda = 0$, and for $\lambda \in B(0,r)$ and r sufficiently small, slow modes $\tilde W_j^\pm$ satisfy*

$$\tilde W_j^\pm(y;\lambda) = e^{-\mu(\lambda)y}\bar{\tilde V}_j^\pm(0) + \lambda\tilde\Theta_j^\pm(y;\lambda), \tag{5.20}$$

where

$$\begin{aligned} |\tilde\Theta_j^\pm| &\le C|e^{-\mu(\lambda)y}| \\ |(\partial/\partial y)\tilde\Theta_j^\pm| &\le C|e^{-\mu(\lambda)y}|(|\lambda| + e^{-\theta|y|}), \end{aligned} \tag{5.21}$$

$\theta > 0$, as $y \to \pm\infty$, and $\bar{\tilde V}_j^\pm \equiv$ constant.[6] *Similarly, fast-decaying (forward) modes $W_j^\pm$ satisfy*

$$W_j^\pm(x;\lambda) = W_j^\pm(x;0) + \lambda\Theta_j^\pm(x;\lambda), \tag{5.22}$$

where

$$|\Theta_j^\pm|,\ |(\partial/\partial x)\Theta_j^\pm| \le Ce^{-\theta|x|} \tag{5.23}$$

as $x \to \pm\infty$, for some $\theta > 0$.

Proof. Applying the Gap Lemma to the augmented variables

$$\begin{aligned} \tilde{\mathbb{W}}_j^\pm(y;\lambda) := \begin{pmatrix} \tilde W_j^\pm \\ \tilde W_j^{\pm\prime} \end{pmatrix}(y;\lambda) &=: e^{-\mu(\lambda)y}\tilde{\mathbb{V}}_j^\pm(y;\lambda)\begin{pmatrix} \tilde W_j^\pm \\ \tilde W_j^{\pm\prime} \end{pmatrix}(y;\lambda) \\ &= e^{-\mu(\lambda)y}\begin{pmatrix} \tilde V_j^\pm \\ -\mu_j^\pm\tilde V_j^\pm + \tilde V_j^{\pm\prime} \end{pmatrix}(y;\lambda) \end{aligned}$$

and

$$\begin{aligned} \mathbb{W}_j^\pm(x;\lambda) := \begin{pmatrix} W_j^\pm \\ W_j^{\pm\prime} \end{pmatrix}(x;\lambda) &=: e^{\mu(\lambda)x}\mathbb{V}_j^\pm(x;\lambda) \\ &= e^{-\mu(\lambda)y}\begin{pmatrix} \tilde V_j^\pm \\ \mu_j^\pm V_j^\pm + V_j^{\pm\prime} \end{pmatrix}(y;\lambda) \end{aligned}$$

we obtain bounds

$$\begin{aligned} \tilde{\mathbb{W}}_j^\pm(x;\lambda) &= \tilde{\mathbb{V}}_j^\pm(x;\lambda)e^{-\mu_j^\pm(\lambda)x} \\ (\partial/\partial\lambda)^k\tilde{\mathbb{V}}_j^\pm(x;\lambda) &= (\partial/\partial\lambda)^k\tilde{\mathbb{V}}_j^\pm(\lambda) + \mathcal{O}(e^{-\tilde\theta|x|}|\tilde{\mathbb{V}}_j^\pm(\lambda)|), \quad x \gtrless 0, \end{aligned} \tag{5.24}$$

---

[6] In particular, this includes $|(\partial/\partial y)\tilde W_j^\pm(y;\lambda)| \le C|\lambda e^{\mu_j^\pm(\lambda)y}|$ in place of the general spatial-derivative bound $|(\partial/\partial y)\tilde W_j^\pm(y;\lambda)| \le C(|\mu\tilde V_j^\pm| + |(\partial/\partial y)\tilde V_j^\pm|)e^{\mu_j^\pm(\lambda)y} \sim C|e^{\mu_j^\pm(\lambda)y}|$.

and

$$\begin{aligned} &\mathbb{W}_j^{\pm}(x;\lambda) = \mathbb{V}_j^{\pm}(x;\lambda)e^{\mu_j^{\pm}(\lambda)x} \\ &(\partial/\partial\lambda)^k\mathbb{V}_j^{\pm}(x;\lambda) = (\partial/\partial\lambda)^k\mathbb{V}_j^{\pm}(\lambda) + \mathcal{O}(e^{-\tilde\theta|x|}|\tilde{\mathbb{V}}_j^{\pm}(\lambda)|), \quad x \gtrless 0, \end{aligned} \tag{5.25}$$

$\tilde\theta > 0$, analogous to (5.16), valid for $\lambda \in B(0,r)$, where

$$\tilde{\mathbb{V}}_j^{\pm}(\lambda) = \begin{pmatrix} \tilde V(\lambda) \\ -\mu_j^{\pm}\tilde V(\lambda) \end{pmatrix}$$

and

$$\mathbb{V}_j^{\pm}(\lambda) = \begin{pmatrix} V(\lambda) \\ \mu_j^{\pm}V(\lambda). \end{pmatrix}$$

By Taylor's Theorem with differential remainder, applied to $\tilde{\mathbb{V}}$, we have:

$$\tilde{\mathbb{W}}_j^{\pm}(y,\lambda) = e^{-\mu_j^{\pm}(\lambda)y}\left(\tilde{\mathbb{V}}_j^{\pm}(y;0) + \lambda(\partial/\partial\lambda)\tilde{\mathbb{V}}_j^{\pm}(y;0) + \frac{1}{2}\lambda^2(\partial/\partial\lambda)^2\tilde{\mathbb{V}}_j^{\pm}(y;\lambda_*)\right), \tag{5.26}$$

for some λ_* on the ray from 0 to λ, where, recall, $(\partial/\partial\lambda)\tilde{\mathbb{V}}_j^{\pm}(y;\cdot)$ and $(\partial/\partial\lambda)^2\tilde{\mathbb{V}}_j^{\pm}(y;\cdot)$ are uniformly bounded in $L^\infty[0,\pm\infty]$ for $\lambda \in B(0,r)$. Together with the choice $\tilde V_j^{\pm}(y,0) \equiv$ constant, this immediately gives the first (undifferentiated) bound in (5.21).

Applying now the bound (5.25) with $k=1$, we may expand the second coordinate of (5.26) as

$$\begin{aligned} (\partial/\partial y)\tilde W_j^{\pm}(y,\lambda) &= e^{-\mu_j^{\pm}(\lambda)y}\Big(-\mu_j^{\pm}\tilde V_j^{\pm}(y;0) + \tilde V_j^{\pm\,\prime}(y;0) \\ &\qquad - \lambda\big((\partial/\partial\lambda)(\mu_j^{\pm}\tilde V_j^{\pm})(0) + \mathcal{O}(e^{-\theta|y|})\big) + \mathcal{O}(\lambda^2)\Big) \\ &= e^{-\mu_j^{\pm}(\lambda)y}\left(-\lambda\big((\partial/\partial\lambda)\mu_j^{\pm}(0)\tilde V_j^{\pm}(0) + \mathcal{O}(e^{-\theta|y|})\big) + \mathcal{O}(\lambda^2)\right), \end{aligned} \tag{5.27}$$

and subtracting off the corresponding Taylor expansion

$$\begin{aligned} &(\partial/\partial y)\big(e^{-\mu_j^{\pm}(\lambda)y}\tilde V_j^{\pm}(y,0)\big) \\ &\quad = \mu_j^{\pm}(\lambda)e^{-\mu_j^{\pm}(\lambda)y}\tilde V_j^{\pm}(y,0) \\ &\quad = e^{-\mu_j^{\pm}(\lambda)y}\left(-\mu_j^{\pm}(0)\tilde V_j^{\pm}(y;0) - \lambda(\partial/\partial\lambda)\mu_j^{\pm}(0)\tilde V_j^{\pm}(y;0) + \mathcal{O}(\lambda^2)\right) \\ &\quad = e^{-\mu_j^{\pm}(\lambda)y}\left(-\lambda(\partial/\partial\lambda)\mu_j^{\pm}(0)\tilde V_j^{\pm}(y;0) + \mathcal{O}(\lambda^2)\right), \end{aligned} \tag{5.28}$$

we obtain

$$(\partial/\partial y)\tilde\Theta_j^{\pm}(y,\lambda) = e^{-\mu_j^{\pm}(\lambda)y}\left(\lambda\mathcal{O}(e^{-\theta|y|}) + \mathcal{O}(\lambda^2)\right), \tag{5.29}$$

as claimed.

Similarly, we may obtain (5.22)–(5.23) by Taylor's Theorem with differential remainder applied to $W = e^{\mu(\lambda)x}$, and the Leibnitz calculation

$$(\partial/\partial\lambda)W = (d\mu/d\lambda)xe^{\mu x}V + e^{\mu x}(\partial/\partial\lambda)V,$$

together with the observation that $|xe^{-\theta|x|} \le Ce^{-\theta|x|/2}$. ∎

Remark 5.7. For a more explicit derivation of (5.20)–(5.21) in the scalar case, see Lemma 2.2, (2.6)–(2.7) in [HoZ.2]. (Note: in [HoZ.2], the one-dimensional case corresponds to $\xi \equiv 0$).

This leaves only the problem of determining behavior in λ through the study of coefficients M, $d^{\pm}$. To this end, we make the following further observations in the Lax or overcompressive case, generalizing the corresponding observation of Lemma 4.30, [Z.4], in the strictly parabolic case:

Lemma 5.8. *For transverse ($\gamma \neq 0$) Lax and overcompressive shocks, with the above-specified choice of basis at $\lambda = 0$, fast-growing modes ψ_j^+, ψ_j^- are fast-decaying at $-\infty$, $+\infty$, respectively. Equivalently, fast-decaying modes $\tilde\psi_j^+$, $\tilde\psi_j^-$ are fast-growing at $-\infty$, $+\infty$: i.e., the only bounded solutions of the adjoint eigenvalue equation are constant solutions.*

Proof. Noting that the manifold of solutions of the r-dimensional ODE (5.19) that decay at either $x \to +\infty$ or $x \to -\infty$ is by transversality, together with (1.15), exactly $d_+ + d_- - \ell = r$, we see that *all* solutions of this ODE in fact decay at least at one infinity. This implies the first assertion, by the alternative characterization of our bases described in Remark 5.5 above; the second follows by duality, (3.33). Since the manifold of fast-decaying dual modes is uniquely determined, independent of the choice of basis, this implies that nonconstant dual modes that are bounded at one infinity must blow up at the other, hence bounded solutions must be constant as claimed. ∎

Remark 5.9. As noted in [LZ.2,ZH,Z.2,Z.4], the property of Lax and overcompressive shocks that the adjoint eigenvalue equation has only constant solutions has the interpretation that the only L^1 time-invariants of the evolution of the one-dimensional linearized equations about $(\bar u, \bar v)(\cdot)$ are given by *conservation of mass*, see discussion [LZ.2]. This distinguishes then from undercompressive shocks, which *do* have additional L^1 time-invariants. The presence of additional time-invariants for undercompressive shocks has significant implications for their behavior, (see discussions [LZ.2] Section 3 and [ZH], Section 10): in particular, the time-asymptotic location of a perturbed undercompressive shock (generically) evolves nonlinearly, and is not determinable by any linear functional of the initial perturbation. By contrast, the time-asymptotic location of a perturbed (stable) Lax or overcompressive shock may be determined by the mass of the initial perturbation alone.

5.2. Scattering coefficients (behavior in λ). We next turn to the estimation of scattering coefficients M_{jk}, $d^{\pm}_{jk}$. Consider coefficient M_{jk}. Expanding (3.56) using Kramer's rule, and setting $z=0$, we obtain

$$M^{\pm}_{jk} = D^{-1}C^{\pm}_{jk}, \tag{5.30}$$

where

$$C^{+} := (I,0)\begin{pmatrix} \Phi^{+} & \Phi^{-} \\ \Phi^{+\prime} & \Phi^{-\prime} \end{pmatrix}^{\mathrm{adj}} \begin{pmatrix} \Psi^{-} \\ \Psi^{-\prime} \end{pmatrix}_{|z=0} \tag{5.31}$$

and a symmetric formula holds for C^{-}. Here, P^{adj} denotes the *adjugate matrix* of a matrix P, i.e. the transposed matrix of minors. As the adjugate is polynomial in the entries of the original matrix, it is evident that $|C^{\pm}|$ is uniformly bounded and therefore

$$|M^{\pm}_{jk}| \le C_1|D^{-1}| \le C_2\lambda^{-\ell} \tag{5.32}$$

by $(\mathcal{D})$, where C_1, $C_2>0$ are uniform constants.

However, the crude bound (5.32) hides considerable cancellation, a fact that will be crucial in our analysis. Relabel the $\{\varphi^{\pm}\}$ so that, at $\lambda=0$,

$$\varphi^{+}_{N-j+1} \equiv \varphi^{-}_{j} = (\partial/\partial\delta_j)\begin{pmatrix} \bar u^{\delta} \\ \bar v^{\delta} \end{pmatrix}, \quad j=1,\cdots,\ell. \tag{5.33}$$

With convention (5.33), we have the sharpened bounds:

Lemma 5.10. *Let (H0)–(H4) and ($\mathcal{D}$) (equivalently, (D1)–(D2)) hold, and let $\phi^{\pm}_j$ be labeled as in (5.33). Then, for $|\lambda|$ sufficiently small, there hold*

$$|M^{\pm}_{jk}|, |d^{\pm}_{jk}|, |\bar d^{\pm}_{jk}| \le C\begin{cases} \lambda^{-1} & \text{for } j=1,\cdots,\ell, \\ 1 & \text{otherwise,} \end{cases} \tag{5.34}$$

where $M^{\pm}$, $d^{\pm}$ are as defined in Proposition 3.11. Moreover,

$$\text{Residue}_{\,\lambda=0}\, M^{+}_{N-j+1,k} = \text{Residue}_{\,\lambda=0}\, d^{\pm}_{j,k}, \tag{5.35}$$

for $1\le j\le \ell$, all k.

That is, blowup in M_{jk} occurs to order $\lambda^{\ell-1}|D^{-1}|$ rather than $|D^{-1}|$, and, more importantly, only in (fast-decaying) *stationary modes* $(\partial/\partial\delta_j)(\bar u^{\delta},\bar v^{\delta})$. Moreover, each stationary mode $(\partial/\partial\delta_j)(\bar u^{\delta},\bar v^{\delta})$ has consistent scattering coefficients with dual mode $\tilde\psi^{-}_{k}$, to lowest order, across all of the various representations of G_λ given in Proposition 3.11.

Proof. Formula (5.31) may be rewritten as

$$C_{jk}^{+} = \det \begin{pmatrix} \varphi_1^+, \cdots, \varphi_{j-1}^+, \psi_k^-, \varphi_j^+, \cdots, \varphi_n^+, & \Phi^- \\ \varphi_1^{+\prime}, \cdots, \varphi_{j-1}^{+}{}', \psi_k^{-\prime}, \varphi_j^{+\prime}, \cdots, \varphi_n^+, & \Phi^{-\prime} \end{pmatrix}_{|z=0}, \tag{5.36}$$

from which we easily obtain the desired cancellation in $M^+ = C^+D^{-1}$. For example, for $j > \ell$, we have

$$\begin{aligned} C_{jk}^{+} &= \det \begin{pmatrix} \varphi_1^+ + \lambda\varphi_{1_\lambda}^+ + \cdots, & \cdots, & \varphi_n^+ + \lambda\varphi_{n_\lambda}^+ + \cdots \\ \cdots, & ,\cdots, & \varphi_n^{+\prime} + \lambda{\varphi_{n_\lambda}^+}' + \cdots \end{pmatrix} \\ &= \mathcal{O}(\lambda^\ell) \le C|D|, \end{aligned} \tag{5.37}$$

yielding $|M_{jk}| = |C_{jk}||D|^{-1} \le C$ as claimed, by elimination of ℓ zero-order terms, using linear dependency among fast modes at $\lambda = 0$. For $1 \le j \le \ell$, there is only an $\ell - 1$ order dependency, and we obtain instead the bound $|C_{jk}| \le C|\lambda|^{\ell-1}$, or $|M_{jk}| = |C_{jk}||D|^{-1} \le C|\lambda|^{-1}$. The bounds on $|d_{jk}|$ and $|\bar{d}_{jk}|$ follow similarly; likewise, (5.35) follows easily from the observation that, since columns ψ_j^+ and ψ_{N-j+1}^- agree, and $\tilde{\psi}_j^-$ is held fixed, the expansions of the various representations by Kramer's rule yield determinants which at λ^{-1} order differ only by a transposition of columns. ∎

In the Lax or overcompressive case, we can say a bit more:

Lemma 5.11. *Let (H0)–(H4) and ($\mathcal{D}$) (equivalently, (D1)–(D2)) hold, with $|\lambda|$ sufficiently small. Then, for Lax and overcompressive shocks, with appropriate basis at $\lambda = 0$ (i.e. slow dual modes taken identically constant), there hold*

$$|M_{jk}|, |d_{jk}|, |\bar{d}_{jk}| \le C \tag{5.38}$$

if $\tilde{\psi}_k$ is a fast mode, and

$$|M_{jk}|, |d_{jk}|, |\bar{d}_{jk}| \le C|\lambda| \tag{5.39}$$

if, additionally, φ_j is a slow mode.

As we shall see in the following section, this result has the consequence that only slow, constant dual modes play a role in long-time behavior of G.

Proof. Transversality, $\gamma \neq 0$, follows from (D2), so that the results of Lemma 5.8 hold. We first establish the bound (5.38). By Lemma 5.10, we need only consider $j = 1, \ldots, \ell$, for which $\phi_j^- = \phi_j^+$. By Lemma 5.8, all fast-growing modes ψ_k^+, ψ_k^- lie in the fast-decaying manifolds Span $(\varphi_1^-, \cdots, \varphi_{i_-}^-)$, Span $(\varphi_1^+, \cdots, \varphi_{i_+}^+)$ at $-\infty$,

$+\infty$ respectively. It follows that in the righthand side of (5.36), there is at $\lambda = 0$ a linear dependency between columns

$$\varphi_1^+, \cdots, \varphi_{j-1}^+, \psi_k^-, \varphi_{j+1}^+, \varphi_{i_+}^+ \text{ and } \varphi_j^- = \varphi_j^+ \tag{5.40}$$

i.e. an ℓ-fold dependency among columns

$$\varphi_1^+, \cdots, \psi_k^-, \cdots, \varphi_{i+}^+ \text{ and } \varphi_1^-, \cdots, \varphi_\ell^-. \tag{5.41}$$

It follows as in the proof of Lemma 5.10 that $|C_{jk}| \le C\lambda^\ell$ for λ near zero, giving bound (5.38) for M_{jk}. If φ_j^+ is a slow mode, on the other hand, then the same argument shows that there is a linear dependency in columns

$$\varphi_1^+, \cdots, \varphi_{j-1}^+, \psi_k^-, \varphi_{j+1}^+, \varphi_{i_+}^+ \tag{5.42}$$

and an $(\ell+1)$-fold dependency in (5.41), since the omitted slow mode φ_j^+ plays no role in either linear dependence; thus, we obtain the bound (5.39), instead. Analogous calculations yield the result for $d_{jk}^\pm$, $\bar{d}_{jk}$ as well. ∎

5.3. Proof of Proposition 5.2. The proof of Proposition 5.2 is now just a matter of collecting the bounds of Lemmas 5.3–5.11, and substituting in the representations of Proposition 3.11. More precisely, approximating fast-decaying dual modes $\phi_1^+, \ldots, \phi_\ell^+$ and $\phi_{N-\ell+1}^-, \ldots, \phi_N^-$ by stationary modes $(\partial/\partial\delta_j)(\bar{u}^\delta, \bar{v}^\delta)$, and slow dual modes by $e^{-\mu(\lambda)y}\tilde{\bar{V}}(\lambda)$ as described in Lemma 5.6, truncating $\mu(\lambda)$ at second order, and keeping only the lowest order terms in the Laurent expansions of scattering coefficients $M^\pm$, $d^\pm$ we obtain E_λ and S_λ, respectively, as order λ^{-1} and order one terms, except for negligible $\mathcal{O}(e^{-\theta|x-y|})$ terms which we have accounted for in error term R_λ^E.

Besides the latter, we have accounted in term R_λ^E for: (i) the truncation errors involved in the aforementioned approximations (difference between stationary modes and actual fast modes, and between constant-coefficient approximant and actual slow dual modes, plus truncation errors in exponential rates); and, (ii) neglected fast-decaying forward/slow-decaying dual mode combinations (of order λ, by Lemma 5.10). A delicate point is the fact that term $\mathcal{O}(e^{-\theta|x-y|})\mathcal{O}(e^{-\theta|y|})$ arising in the dual truncation of E through the $\lambda\mathcal{O}(e^{-\theta|y|})$ portion of estimate (5.21) for $(\partial/\partial y)\tilde{\Theta}_j^\pm$, absorbs in the (leading) $\mathcal{O}(e^{-\theta|x-y|})$ term of (5.12). It was to obtain this key reduction that we carried out expansion (5.21) to such high order.

In R_λ^S, we have accounted for truncation errors in the slow/slow pairings approximated by term S_λ (difference between constant-coefficient approximants and actual slow forward modes, corresponding to an additional $e^{-\theta|x|}$ term in the truncation error factor, and between constant-coefficient approximants and actual slow dual modes, plus truncation errors in exponential rates). We have also accounted for the remaining, slow-decaying forward/fast-decaying dual modes, which according to Lemma 5.11 have scattering coefficient of order λ and therefore may be grouped with the difference between dual modes and their constant-coefficient approximants (of smaller order by a factor of $e^{-\theta|y|}$) in the term $\mathcal{O}(\lambda)$. ∎

5.4. Extended spectral theory. Finally, we cite without proof the extended spectral theory of [ZH] that we shall need in order to establish necessity of condition $(\mathcal{D})$. In the case that $(\mathcal{D})$ holds, this reduces to simple relations (5.35).

Let $C^\infty_{\exp}$ denote the space of C^∞ functions decaying exponentially in all derivatives at some sufficiently high rate.

Definition 5.12. *Let L be a linear ordinary differential operator with bounded, C^∞ coefficients (so that $L : C^\infty_{\exp} \to C^\infty_{\exp}$), and let G_λ denote the Green's function of $L - \lambda I$. Further, let Ω be an open, simply connected domain intersecting the resolvent set of L, on which G_λ has a (necessarily unique) meromorphic extension. Then, for $\lambda_0 \in \Omega$, we define the* **effective eigenprojection** $\mathcal{P}_{\lambda_0} : C^\infty_{\exp} \to C^\infty$ *by*

$$\mathcal{P}_{\lambda_0} f(x) = \int_{-\infty}^{+\infty} P_{\lambda_0}(x,y) f(y)\, dy,$$

where

$$P_{\lambda_0}(x,y) = \mathrm{Res}_{\lambda_0} G_\lambda(x,y) \tag{5.43}$$

and Res_{λ_0} *denotes residue at λ_0. We will refer to $P_{\lambda_0}(x,y)$ as the* **projection kernel**. *Likewise, we define the* **effective eigenspace** $\Sigma'_{\lambda_0}(L)$ *by*

$$\Sigma'_{\lambda_0}(L) = \mathrm{Range}\ (\mathcal{P}_{\lambda_0}),$$

and the **effective point spectrum** $\sigma'_p(L)$ *of L in Ω to be the set of $\lambda \in \Omega$ such that* $\dim \Sigma'_{\lambda_0}(L) \neq 0$.

The above definition is the natural one from the point of view of the spectral resolution of the identity, $I = \int_\Gamma (L - \lambda I)^{-1} d\lambda$, hence also from the point of view of asymptotic behavior of solutions of PDE (see Section 6, below). Away from the essential spectrum of L, it corresponds with the usual definition [Kat,Y].

Definition 5.13. *Let L, Ω, λ_0 be as above, and K be the order of the pole of $(L-\lambda I)^{-1}$ at λ_0. For $\lambda_0 \in \Omega$, and k any integer, we define $\mathcal{Q}_{\lambda_0,k} : C^\infty_{\exp} \to C^\infty$ by*

$$\mathcal{Q}_{\lambda_0,k} f(x) = \int_{-\infty}^{+\infty} Q_{\lambda_0,k}(x,y) f(y)\, dy,$$

where

$$Q_{\lambda_0,k}(x,y) = \mathrm{Res}_{\lambda_0} (\lambda - \lambda_0)^k G_\lambda(x,y).$$

For $0 \le k \le K$, we define the **effective eigenspace of ascent** k *by*

$$\Sigma'_{\lambda_0,k}(L) = \mathrm{Range}\ (\mathcal{Q}_{\lambda_0,K-k})$$

With the above definitions, we obtain the following, modified Fredholm Theory.

Proposition 5.14 [ZH]. *Let L, λ_0, Ω be as in Definition 5.12, and K be the order of the pole of G_λ at λ_0. Then,*

(i) The operators $\mathcal{P}_{\lambda_0}$, $\mathcal{Q}_{\lambda_0,k} : C^\infty_{\text{exp}} \to C^\infty$ are L-invariant, with

$$\mathcal{Q}_{\lambda_0,k+1} = (L - \lambda_0 I)\mathcal{Q}_{\lambda_0,k} = \mathcal{Q}_{\lambda_0,k}(L - \lambda_0 I) \tag{5.44}$$

for all $k \neq -1$, and

$$\mathcal{Q}_{\lambda_0,k} = (L - \lambda_0 I)^k \mathcal{P}_{\lambda_0} \tag{5.45}$$

for $k \geq 0$.

(ii) The effective eigenspace of ascent k satisfies

$$\Sigma'_{\lambda_0,k}(L) = (L - \lambda_0 I)\Sigma'_{\lambda_0,k+1}(L). \tag{5.46}$$

for all $0 \leq k \leq K$, with

$$\{0\} = \Sigma'_{\lambda_0,0}(L) \subset \Sigma'_{\lambda_0,1}(L) \subset \cdots \subset \Sigma'_{\lambda_0,K}(L) = \Sigma'_{\lambda_0}(L). \tag{5.47}$$

Moreover, each containment in (5.47) is strict.

(iii) On $\mathcal{P}^{-1}_{\lambda_0}(C^\infty_{\text{exp}})$, $\mathcal{P}_{\lambda_0}$, $\mathcal{Q}_{\lambda_0,k}$ all commute ($k \geq 0$), and $\mathcal{P}_{\lambda_0}$ is a projection. More generally, $\mathcal{P}_{\lambda_0} f = f$ for any $f \in \Sigma_{\lambda_0}(L : C^\infty_{exp})$, hence $\Sigma_{\lambda_0,k}(L : C^\infty_{exp}) \subset \Sigma'_{\lambda_0,k}(L)$ for all $0 \leq k \leq K$.

(iv) The multiplicity of the eigenvalue λ_0, defined as $\dim\ \Sigma'_{\lambda_0}(L)$, is finite and bounded by Kn. Moreover, for all $0 \leq k \leq K$,

$$\dim\ \Sigma'_{\lambda_0,k}(L) = \dim\ \Sigma'_{\lambda_0^*,k}(L^*). \tag{5.48}$$

Further, the projection kernel can be expanded as

$$P_{\lambda_0} = \sum_j \varphi_j(x)\pi_j(y), \tag{5.49}$$

where $\{\varphi_j\}$, $\{\pi_j\}$ are bases for $\Sigma'_{\lambda_0}(L)$, $\Sigma'_{\lambda_0^*}(L^*)$, respectively.

(v) (Restricted Fredholm alternative) For $g \in C^\infty_{\text{exp}}$,

$$(L - \lambda_0 I)f = g$$

is soluble in C^∞ **if** *and soluble in C^∞_{exp}* **only if** *$Q_{\lambda_0,K-1}\, g = 0$, or equivalently*

$$g \in \Sigma'_{\lambda_0^*,1}(L^*)^\perp. \tag{5.50}$$

Proof: See [ZH]. ∎

Remarks: For λ_0 in the resolvent set, $\mathcal{P}_{\lambda_0}$ agrees with the standard definition, hence $\Sigma'_{\lambda_0,k}(L)$ agrees with the usual L^p eigenspace of generalized eigenfunctions of ascent $\leq k$, for all $p < \infty$, since C^∞_{exp} is dense in L^p, $p > 1$, and $\Sigma'_{\lambda_0,k}(L)$ is closed. In the context of stability of traveling waves (more generally, whenever the coefficients of L exponentially approach constant values at $\pm\infty$), $\Sigma'_{\lambda_0,k}(L)$ lies *between* the L^p subspace $\Sigma_{\lambda_0,k}(L)$ and the corresponding L^p_{Loc} subspace.

The modification from standard Fredholm Theory is primarily that here conclusions (iii) and (v) apply on restricted domains. This restriction is clearly necessary, since the various expressions occurring in their statements would otherwise not be defined. In this regard, note that $\mathcal{P}_{\lambda_0}$ is, strictly speaking, not a projection in the case that $\lambda \in \sigma_{ess}(L)$, since its domain does not then match its range. However, it maintains the projective structure (5.49), somewhat justifying our abuse of notation.

Now, suppose further (as holds in the case under consideration) that, on Ω: (i) L is a nondegenerate operator of some order s, i.e. the coefficient matrix of the principal part of the symbol is nonsingular for all x[7]; and, (ii) there exists an analytic choice of bases for stable/unstable manifolds at $+\infty/-\infty$ of the associated eigenvalue equation.

In this case, we can define an analytic Evans function $D(\lambda)$ as in (3.21), which we assume does not vanish identically. It is easily verified that the order to which D vanishes at any λ_0 is independent of the choice of analytic bases $\Phi^\pm$. The following result, established in [ZH], generalizes to the extended spectral framework the standard result of Gardner and Jones [GJ.1–2] in the classical setting.

Proposition 5.15 [ZH]. *Let L, λ_0 be as above. Then,*

(i) dim $\Sigma'_{\lambda_0}(L)$ *is equal to the order d to which the Evans function D_L vanishes at λ_0.*

(ii) $P_{\lambda_0}(L) = \sum_j \varphi_j(x)\pi_j(y)$, where φ_j and π_j are in $\Sigma'_{\lambda_0}(L)$ and $\Sigma'_{\lambda_0}(L^*)$, respectively, with ascents summing to $\leq K+1$, where K is the order of the pole of G_λ at λ_0.

Remark 5.16. Evidently, the theory of this section goes through also, with suitable modifications regarding regularity, in the case (as for relaxation shocks) that the coefficients of L have finite regularity C^s_{exp}, i.e. decay exponentially in up to s derivatives, in which case $L : C^s_{\text{exp}} \to C^0_{\text{exp}}$.

Section 6. Pointwise Green's function bounds.

[7] This assumption was made in [ZH] for convenience in calculation; the result holds in considerably more generality (for example for systems with real viscosity).

In this section, we prove the main results of the paper, establishing pointwise bounds on the Green's function G and consequently sufficiency and necessity of condition $(\mathcal{D})$ for linearized orbital stability.

6.1. Proof of Theorem 1.10. We will establish Theorem 1.10 as a corollary of the following, more detailed result:

Proposition 6.1. *Assuming (H0)–(H4) and the spectral stability criterion $(\mathcal{D})$ (equivalently, (D1)–(D2)), the Green's function $G(x,t;y)$ associated with (1.17) may be decomposed as in Theorem 1.10, where, for $y \le 0$:*

(6.1)

$$\begin{aligned} R(x,t;y) &= \mathcal{O}(e^{-\eta(|x-y|+t)}) \\ &\quad + \sum_{k=1}^{n} \mathcal{O}\left((t+1)^{-1/2} e^{-\eta x^+} + e^{-\eta|x|}\right)(t+1)^{-1/2} e^{-(x-y-a_k^{*-}t)^2/Mt} \\ &\quad + \sum_{a_k^{*-}>0,\, a_j^{*-}<0} \chi_{\{|a_k^{*-}t|\ge|y|\}} \mathcal{O}((t+1)^{-1} e^{-(x-a_j^{*-}(t-|y/a_k^{*-}|))^2/Mt} e^{-\eta x^+}, \\ &\quad + \sum_{a_k^{*-}>0,\, a_j^{*+}>0} \chi_{\{|a_k^{*-}t|\ge|y|\}} \mathcal{O}((t+1)^{-1} e^{-(x-a_j^{*+}(t-|y/a_k^{*-}|))^2/Mt} e^{-\eta x^-}, \end{aligned}$$

(6.2)

$$\begin{aligned} R_x(x,t;y) &= \sum_{j=1}^{J} \mathcal{O}(e^{-\eta t})\delta_{x-\bar{a}_j t}(-y) + \sum_{j=1}^{J} \mathcal{O}(e^{-\eta(|x-y|+t)}) \\ &\quad + \sum_{k=1}^{n} \mathbf{O}\left((t+1)^{-1} e^{-\eta x^+} + e^{-\eta|x|}\right)(t+1)^{-1/2} e^{-(x-y-a_k^{*-}t)^2/Mt} \\ &\quad + \sum_{a_k^{*-}>0,\, a_j^{*-}<0} \chi_{\{|a_k^{*-}t|\ge|y|\}} \mathbf{O}((t+1)^{-3/2} e^{-(x-a_j^{*-}(t-|y/a_k^{-}|))^2/Mt} e^{-\eta x^+} \\ &\quad + \sum_{a_k^{*-}>0,\, a_j^{*+}>0} \chi_{\{|a_k^{*-}t|\ge|y|\}} \mathbf{O}((t+1)^{-3/2} e^{-(x-a_j^{*+}(t-|y/a_k^{*-}|))^2/Mt} e^{-\eta x^-}, \end{aligned}$$

and
(6.3)
$$\begin{aligned}R_y(x,t;y) &= \sum_{j=1}^{J} \mathcal{O}(e^{-\eta t})\delta_{x-\bar a_j t}(-y) + \sum_{j=1}^{J} \mathcal{O}(e^{-\eta(|x-y|+t)})\\ &+ \sum_{k=1}^{n} \mathcal{O}\left((t+1)^{-1/2}e^{-\eta x^+} + e^{-\eta|x|}\right)(t+1)^{-1}e^{-(x-y-a_k^{*-}t)^2/Mt}\\ &+ \sum_{a_k^{*-}>0,\, a_j^{*-}<0} \chi_{\{|a_k^{*-}t|\ge|y|\}}\mathcal{O}((t+1)^{-e/2}e^{-(x-a_j^{*-}(t-|y/a_k^{*-}|))^2/Mt}e^{-\eta x^+}\\ &+ \sum_{a_k^{*-}>0,\, a_j^{*+}>0} \chi_{\{|a_k^{*-}t|\ge|y|\}}\mathcal{O}((t+1)^{-3/2}e^{-(x-a_j^{*+}(t-|y/a_k^{*-}|))^2/Mt}e^{-\eta x^-},\end{aligned}$$

for some η, $M>0$, where $\bar a_j$ are as in Theorem 1.10, $x^\pm$ denotes the positive/negative part of x, and indicator function $\chi_{\{|a_k^{*-}t|\ge|y|\}}$ is one for $|a_k^{*-}t|\ge|y|$ and zero otherwise. Symmetric bounds hold for $y\ge 0$.

Proof. Our starting point is the representation

$$G(x,t;y) = \frac{1}{2\pi i}\mathrm{P.V.}\int_{\eta-i\infty}^{\eta+i\infty} e^{\lambda t}G_\lambda(x,y)d\lambda, \tag{6.4}$$

valid by Proposition 2.3 for any η sufficiently large.

Case I. $|x-y|/t$ large. We first treat the simpler case that $|x-y|/t\ge S$, S sufficiently large, for which G is known a priori to vanish, by the general property of finite propagation speed for systems of hyperbolic type. More precisely, $G\equiv 0$ for $x\notin[z_1(y,t),z_J(y,t)]$, where $z_j(\cdot,\cdot)$ are the standard hyperbolic characteristic paths defined in (1.36), $a_1\le\cdots\le a_J$ denoting the eigenvalues of $A:=df(\bar u,\bar v)$. This may be seen, for example, by standard energy estimates as in, e.g., [Ev]; however, it is instructive to recover this result by direct calculation.

Suppose, then, that $x\notin[z_1,z_J]$. Fixing x, y, t, set $\lambda=\eta+i\xi$. Applying Proposition 4.3, we obtain, for $\eta>0$ sufficiently large, the decomposition

$$G(x,t;y) = \frac{1}{2\pi i}\mathrm{P.V.}\int_{\eta-i\infty}^{\eta+i\infty} e^{\lambda t}H_\lambda(x,y)d\lambda + \frac{1}{2\pi i}\mathrm{P.V.}\int_{\eta-i\infty}^{\eta+i\infty} e^{\lambda t}\Theta_\lambda(x,y)d\lambda, \tag{6.5}$$

where
(6.6)
$$H_\lambda(x,y) = \begin{cases} -\sum_{j=K+1}^{J} a_j(y)^{-1}e^{-\int_y^x \lambda/a_j(z)\,dz}r_j(x)\tilde\zeta_j(x,y)l_j^t(y) & x>y,\\ \sum_{j=1}^{K} a_j(y)^{-1}e^{-\int_y^x \lambda/a_j(z)\,dz}r_j(x)\tilde\zeta_j(x,y)l_j^t(y) & x<y,\end{cases}$$

and

$$\Theta_\lambda(x,y) = \lambda^{-1}B(x,y;\lambda) + \lambda^{-1}(x-y)C(x,y;\lambda) + \lambda^{-2}D(x,y;\lambda) \tag{6.7}$$

with ζ_j, B, C, D as defined in Proposition 4.3.

For definiteness, take $x > y$. Then, the first term of (6.5) contributes to integral (6.4) the explicitly evaluable quantity

(6.8)

$$\begin{aligned}\frac{1}{2\pi i}\mathrm{P.V.}\int_{\eta-i\infty}^{\eta+i\infty} e^{\lambda t}H_\lambda(x,y)d\lambda &= \sum_{j=K+1}^{J}\Big(a_j(y)^{-1}\mathrm{P.V.}\int_{-\infty}^{+\infty} e^{i\xi\left(t-\int_y^x (1/a_j(z))\,dz\right)}\,d\xi\Big)\\ &\quad\times e^{\eta\left(t-\int_y^x (1/a_j(z))\,dz\right)} r_j(x)\tilde\zeta_j(x,y)l_j^t(y)\\ &= \sum_{j=K+1}^{J}\Big(a_j(y)^{-1}\delta\big(t-\int_y^x (1/a_j(z))\,dz\big)\Big) r_j(x)\tilde\zeta_j(x,y)l_j^t(y)\\ &= \sum_{j=K+1}^{J}\delta_{x-\bar a_j t}(-y)r_j(x)\zeta_j(y,t)l_j^t(y)\\ &= H(x,t;y),\end{aligned}$$

where $\bar a_j$, ζ_j, and H are as defined in Theorems 1.10 and 1.11. Note: in the final equality of (6.8) we have used the general fact that

$$f_y(x,y,t)\delta\big(f(x,y,t)\big) = \delta_{h(x,t)}(y), \tag{6.9}$$

provided $f_y \neq 0$, where $h(x,t)$ is defined by $f(x,h(x,t),t) \equiv 0$. This term clearly vanishes for x outside $[z_1(y,t), z_J(y,t)]$, as claimed.

Similar calculations show that the second, error term in (6.5) also vanishes. For example, the term $e^{\lambda t}\lambda^{-1}B(x,y;\lambda)$ contributes

(6.10)

$$\frac{1}{2\pi i}\sum_{j=K+1}^{K}\Big(\mathrm{P.V.}\int_{-\infty}^{+\infty}(\eta+i\xi)^{-1}e^{i\xi\left(t-\int_y^x (1/a_j(z))\,dz\right)}\,d\xi\Big)\, e^{\eta\left(t-\int_y^x (1/a_j(z))\,dz\right)} b_j^+(x,y).$$

The factor

$$\mathrm{P.V.}\int_{-\infty}^{+\infty}(\eta+i\xi)^{-1}e^{i\xi\left(t-\int_y^x (1/a_j(z))\,dz\right)}\,d\xi,$$

though not absolutely convergent, is integrable and uniformly bounded as a principal value integral, for all $\eta\in\mathbb{R}$, by explicit computation. On the other hand,

$$e^{\eta\left(t-\int_y^x (1/a_j(z))\,dz\right)} \le Ce^{-\eta\, d(x,[z_1(y,t),z_J(y,t)])/\min_{j,x}|a_j(x)|} \to 0 \tag{6.11}$$

as $\eta\to+\infty$, for each $K+1\le j\le J$, since $a_j>0$ on this range. Thus, taking $\eta\to\infty$, we find that the product (6.10) goes to zero, giving the result.

Similarly, the contributions of terms $e^{\lambda t}C(x,y;\lambda)$ and $e^{\lambda t}D(x,y;\lambda)$ split into the product of a convergent, uniformly bounded integral in ξ, a (constant) factor depending only on (x,y), and a factor $\alpha(x,y,t,\eta)$ going to zero as $\eta\to 0$ at rate

(6.11). Thus, taking $\eta \to +\infty$, we find that each of these terms vanishes, and thus G vanishes also, as claimed.

Case II. $|x-y|/t$ **bounded.** We now turn to the critical case that $|x-y|/t \le S$ for some fixed S. In this regime, note that any contribution of order $e^{\theta t}$, $\theta > 0$, may be absorbed in the residual (error) term R (resp. R_x, R_y); we shall use this observation repeatedly.

Decomposition of the contour. We begin by converting contour integral (6.4) into a more convenient form decomposing high, intermediate, and low frequency contributions.

Observation 6.2. *Assuming (H0)–(H4), there holds the representation*

$$
\begin{aligned}
G(x,t;y) &= I_a + I_b + II_a + II_b \\
&:= \frac{1}{2\pi i}\text{P.V.}\int_{\eta - i\infty}^{\eta + i\infty} e^{\lambda t} H_\lambda(x,y)\,d\lambda \\
&+ \frac{1}{2\pi i}\text{P.V.}\left(\int_{-\eta_1 - i\infty}^{-\eta_1 - iR} + \int_{-\eta_1 + iR}^{-\eta_1 + i\infty}\right) e^{\lambda t}(G_\lambda - H_\lambda)(x,y)\,d\lambda \\
&+ \frac{1}{2\pi i}\oint_\Gamma e^{\lambda t} G_\lambda(x,y)\,d\lambda - \frac{1}{2\pi i}\int_{-\eta_1 - iR}^{-\eta_1 + iR} e^{\lambda t} H_\lambda(x,y)\,d\lambda,
\end{aligned}
\tag{6.12}
$$

$$
\Gamma := [-\eta_1 - iR, \eta - iR] \cup [\eta - iR, \eta + iR] \cup [\eta + iR, -\eta_1 + iR],
\tag{6.13}
$$

for any $\eta > 0$ such that (6.4) holds, R sufficiently large, and $-\eta_1 < 0$ as in (3.3). (Note that we have not here assumed $(\mathcal{D})$).

Proof. We first observe that, by Proposition 4.3, L has no spectrum on the portion of $\Gamma := \{\lambda : \text{Re }\lambda \ge -\eta_1 < 0\}$ lying outside of the rectangle

$$
\mathcal{R} := \{\lambda : -\eta_1 \le \text{Re }\lambda \le \eta,\ -R \le \text{Im }\lambda \le R\}
\tag{6.14}
$$

for $\eta > 0$, $R > 0$ sufficiently large, hence G_λ is analytic on this region. Since, also, H_λ is analytic on the whole complex plane, contours involving either one of these contributions may be arbitrarily deformed within $\Gamma \setminus \mathcal{R}$ without affecting the result, by Cauchy's Theorem.

Recalling, further, that

$$
|G_\lambda - H_\lambda| = \mathcal{O}(\lambda^{-1})
$$

by Proposition 4.3, we find that

$$
\frac{1}{2\pi i}\text{P.V.}\int_{\eta - i\infty}^{\eta + i\infty} e^{\lambda t}(G_\lambda - H_\lambda)(x,y)\,d\lambda
$$

may be deformed to

$$\frac{1}{2\pi i}\text{P.V.}\oint_{\partial(\Lambda\setminus\mathcal{R})} e^{\lambda t}(G_\lambda - H_\lambda)(x,y)d\lambda$$

Noting, finally, that

$$+\frac{1}{2\pi i}\oint_{\partial\mathcal{R}} e^{\lambda t}H_\lambda(x,y)d\lambda = 0,$$

by Cauchy's Theorem, we obtain, finally,

(6.15)

$$\begin{aligned}
&\frac{1}{2\pi i}\text{P.V.}\int_{\eta-i\infty}^{\eta+i\infty} e^{\lambda t}G_\lambda(x,y)d\lambda = \\
&\quad \frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty} e^{\lambda t}H_\lambda(x,y)d\lambda + \frac{1}{2\pi i}\text{P.V.}\int_{\eta-i\infty}^{\eta+i\infty} e^{\lambda t}(G_\lambda - H_\lambda)(x,y)d\lambda \\
&\quad = \frac{1}{2\pi i}\text{P.V.}\int_{\eta-i\infty}^{\eta+i\infty} e^{\lambda t}H_\lambda(x,y)d\lambda + \frac{1}{2\pi i}\text{P.V.}\oint_{\partial(\Lambda\setminus\mathcal{R})} e^{\lambda t}(G_\lambda - H_\lambda)(x,y)d\lambda \\
&\qquad + \frac{1}{2\pi i}\oint_{\partial\mathcal{R}} e^{\lambda t}H_\lambda(x,y)d\lambda,
\end{aligned}$$

from which the result follows by combining the second and third contour integrals along their common edges Γ. ∎

Remark 6.3. We have already shown in (6.8) that $\frac{1}{2\pi i}\text{P.V.}\int_{\eta-i\infty}^{\eta+i\infty} e^{\lambda t}H_\lambda(x,y)d\lambda = H(x,t;y)$ independent of η. Thus, we may deform the the entire contour integral

$$\frac{1}{2\pi i}\text{P.V.}\int_{\eta-i\infty}^{\eta+i\infty} e^{\lambda t}G_\lambda(x,y)d\lambda$$

to

$$\frac{1}{2\pi i}\text{P.V.}\int_{\partial(\Lambda\setminus\mathcal{R})} e^{\lambda t}G_\lambda(x,y)d\lambda$$

to obtain (6.12) by a different route.

Observation 6.4. *Assuming ($\mathcal{D}$) together with (H0)–(H4), we may replace (6.12) by*

$$G(x,t;y) = I_a + I_b + II_{\tilde a} + II_b + III,$$

where I_a, I_b and II_b are as in (6.12), and

$$II_{\tilde a} := \frac{1}{2\pi}\left(\int_{-\eta_1-iR}^{-\eta_1-ir/2} + \int_{-\eta_1+ir/2}^{-\eta_1+iR}\right) e^{\lambda t}G\lambda(x,y)dxi,$$

$$III := \frac{1}{2\pi i} \oint_{\tilde\Gamma} e^{\lambda t} G_\lambda(x,y) d\lambda, \tag{6.16}$$

and

$$\tilde\Gamma := [-\eta_1 - ir/2, \eta - ir/2] \cup [\eta - ir/2, \eta + ir/2] \cup [\eta + ir/2, -\eta_1 + ir/2], \tag{6.17}$$

for any η, $r > 0$, *and* η_1 *sufficiently small with respect to* r.

Proof. By assumption $(\mathcal{D})$, L has no spectrum on the region between contour Γ and the union of contour $\tilde\Gamma$ and the contour of term $II_{\tilde a}$, hence G_λ is analytic on that region, and

$$II_a = II_{\tilde a} + III$$

by Cauchy's Theorem, giving the result. ∎

Using the final decomposition (6.16), we shall estimate in turn the high frequency contributions I_a and I_b, the intermediate frequency contributions $II_{\tilde a}$ and II_b, and the low frequency contributions III.

High frequency contribution. We first carry out the straightforward estimatation of the high-frequency terms I_a and I_b. The principal term I_a has already been computed in (6.8) to be $H(x,t;y)$. Likewise, calculations similar to those of (6.10)–(6.11) show that the error term

$$\begin{aligned} I_b &= \frac{1}{2\pi i} \text{P.V.} \left(\int_{-\eta_1 - i\infty}^{-\eta_1 - R} \int_{-\eta_1 + iR}^{-\eta_1 + i\infty} \right) e^{\lambda t} \Theta_\lambda(x,y) d\lambda \\ &= \frac{1}{2\pi i} \text{P.V.} \left(\int_{-\eta_1 - i\infty}^{-\eta_1 - R} \int_{-\eta_1 + iR}^{-\eta_1 + i\infty} \right) e^{\lambda t} \big(\lambda^{-1} B(x,y;\lambda) \\ &\qquad + \lambda^{-1}(x-y) C(x,y;\lambda) + \lambda^{-2} D(x,y;\lambda) \big) d\lambda \end{aligned}$$

is time-exponentially small.

For example, for $x > y$ the term $e^{\lambda t} \lambda^{-1} B(x,y;\lambda)$ contributes

$$\begin{aligned} \sum_{j=K+1}^{K} \frac{1}{2\pi i} \text{P.V.} \left(\int_{-\infty}^{-R} \int_{R}^{+\infty} \right) (-\eta_1 + i\xi)^{-1} e^{i\xi \left(t - \int_y^x (1/a_j(z))\, dz \right)} \, d\xi \\ \times e^{-\eta_1 \left(t - \int_y^x (1/a_j(z))\, dz \right)} b_j^+(x,y), \end{aligned} \tag{6.18}$$

where

$$\frac{1}{2\pi i} \text{P.V.} \left(\int_{-\infty}^{-R} \int_{R}^{+\infty} \right) (\eta + i\xi)^{-1} e^{i\xi \left(t - \int_y^x (1/a_j(z))\, dz \right)} \, d\xi < \infty, \tag{6.19}$$

and

$$e^{\eta_1 \int_y^x (1/a_j(z))\,dz} b(x,y) \le C_1 e^{\eta_1 \int_y^x (1/a_j(z))\,dz - \theta|x-y|} \le C_2, \tag{6.20}$$

for η_1 sufficiently small. This may be absorbed in the first term of R, (4.36).

Likewise, the contributions of terms $e^{\lambda t}C(x,y;\lambda)$ and $e^{\lambda t}D(x,y;\lambda)$ split into the product of a convergent, uniformly bounded integral in ξ, a bounded factor analogous to (6.20), and the factor $e^{-\eta_1 t}$, giving the result.

Derivative bounds. Derivatives $(\partial/\partial x)I_b$ (resp. $(\partial/\partial y)I_b$) may be treated in identical fashion using (4.13) (resp. (4.14), to show that they are absorbable in the estimates given for R_x (resp. R_y). We point out that the integral arising from term B_x^0 (resp. B_y^0) of (4.13) (resp. (4.14)) corresponds to the first, delta-function term in (6.2) (resp. (6.3)), while the integral arising from term $(x-y)C_x^0$ (resp. $(x-y)C_y^0$) vanishes, as the product of $(x-y)$ and a delta function $\delta_z(y)$ with $z \ne x$.

Intermediate frequency contribution. Error term II_b is time-exponentially small for η_1 sufficiently small, by the same calculation as in (6.18)–(6.20), hence negligible. Likewise, term $II_{\tilde a}$ by the basic estimate (3.20) is seen to be time-exponentially small of order $e^{-\eta_1 t}$ for any $\eta_1 > 0$ sufficiently small that the associated contour lies in the resolvent set of L.

Low frequency contribution. It remains to estimate the low-frequency term III, which is of essentially the same form as the low-frequency contribution analyzed in [ZH,Z.2,Z.4] in the strictly parabolic case, in that the contour is the same and the resolvent kernel G_λ satisfies identical bounds in this regime. Thus, we may conclude from these previous analyses that III gives contribution $E+S+R$, as claimed, exactly as in the strictly parabolic case. For completeness, we indicate the main features of the argument here.

Case $t \le 1$. First observe that estimates in the short-time regime $t \le 1$ are trivial, since then $|e^{\lambda t}G_\lambda(x,y)|$ is uniformly bounded on the compact set $\tilde\Gamma$, and we have $|G(x,t;y)| \le C \le e^{-\theta t}$ for $\theta > 0$ sufficiently small. But, likewise, $|E|$ and S are uniformly bounded in this regime, hence time-exponentially decaying. As observed previously, all such terms are negligible, being absorbable in the error term R. Thus, we may add $E+S$ and subtract G to obtain the result

Case $t \ge 1$. Next, consider the critical (long-time) regime $t \ge 1$. For definiteness, take $y \le x \le 0$; the other two cases are similar. Decomposing

$$G(x,t;y) = \frac{1}{2\pi i}\oint_{\tilde\Gamma} e^{\lambda t}E_\lambda(x,y)d\lambda + \frac{1}{2\pi i}\oint_{\tilde\Gamma} e^{\lambda t}S_\lambda(x,y)d\lambda + \frac{1}{2\pi i}\oint_{\tilde\Gamma} e^{\lambda t}R_\lambda(x,y)d\lambda, \tag{6.21}$$

with E_λ, S_λ, and R_λ as defined in Proposition 5.2, we consider in turn each of the three terms on the righthand side.

E_λ *term.* Let us first consider the dominant term

$$\frac{1}{2\pi i}\oint_{\tilde\Gamma} e^{\lambda t}E_\lambda(x,y)d\lambda, \tag{6.22}$$

which, by (5.8), is given by

$$\sum_{a_k^{*-}>0}[c_{k,-}^{j,0}]\frac{\partial}{\partial\delta_j}\begin{pmatrix}\bar u^\delta(x)\\ \bar v^\delta(x)\end{pmatrix}L_k^{*-t}\alpha_k(x,t;y), \tag{6.23}$$

where

$$\alpha_k(x,t;y):=\frac{1}{2\pi i}\oint_{\tilde\Gamma}\lambda^{-1}e^{\lambda t}e^{(\lambda/a_k^{*-}-\lambda^2\beta_k^{*-}/a_k^{*-3})y}\,d\lambda. \tag{6.24}$$

Using Cauchy's Theorem, we may move the contour $\tilde\Gamma$ to obtain

$$\begin{aligned}\alpha_k(x,t;y)&=\frac{1}{2\pi}\text{P.V.}\int_{-r/2}^{+r/2}(i\xi)^{-1}e^{i\xi t}e^{(i\xi/a_k^{*-}+\xi^2\beta_k^{*-}/a_k^{*-3})y}\,d\xi\\&\quad+\frac{1}{2\pi i}\left(\int_{-\eta_1-ir/2}^{-ir/2}+\int_{ir/2}^{-\eta_1+ir/2}\right)\lambda^{-1}e^{\lambda t}e^{(\lambda/a_k^{*-}-\lambda^2\beta_k^{*-}/a_k^{*-3})y}\,d\lambda\\&\quad+\frac{1}{2}\text{Residue }_{\lambda=0}\,e^{\lambda t}e^{(\lambda/a_k^{*-}-\lambda^2\beta_k^{*-}/a_k^{*-3})y},\end{aligned} \tag{6.25}$$

or, rearranging and evaluating the final, residue term:

$$\begin{aligned}\alpha_k(x,t;y)&=\left(\frac{1}{2\pi}\text{P.V.}\int_{-\infty}^{+\infty}(i\xi)^{-1}e^{i\xi(t+y/a_k^{*-})}e^{\xi^2(\beta_k^{*-}/a_k^{*-3})y}\,d\xi+\frac{1}{2}\right)\\&\quad-\frac{1}{2\pi}\left(\int_{-\infty}^{-r/2}+\int_{r/2}^{+\infty}\right)(i\xi)^{-1}e^{i\xi(t+y/a_k^{*-})}e^{\xi^2(\beta_k^{*-}/a_k^{*-3})y}\,d\xi\\&\quad+\frac{1}{2\pi i}\left(\int_{-\eta_1-ir/2}^{-ir/2}+\int_{ir/2}^{-\eta_1+ir/2}\right)\lambda^{-1}e^{\lambda t}e^{(\lambda/a_k^{*-}-\lambda^2\beta_k^{*-}/a_k^{*-3})y}\,d\lambda.\end{aligned} \tag{6.26}$$

The first term in (6.26) may be explicitly evaluated to give

$$\text{errfn }\left(\frac{y+a_k^{*-}t}{\sqrt{4\beta_k^{*-}|y/a_k^{*-}|}}\right), \tag{6.27}$$

where

$$\text{errfn }(z):=\frac{1}{2\pi}\int_{-\infty}^{z}e^{-y^2}\,dy, \tag{6.28}$$

whereas the second and third terms are clearly time-exponentially small for $t\le C|y|$ and η_1 sufficiently small relative to r (see detailed discussion of similar calculations below, under R_λ *term*). In the trivial case $t \geq C|y|$, $C > 0$ sufficiently large, we can simply move the contour to $[-\eta_1 - ir/2, -\eta_1 + ir/2]$ to obtain (complete) residue 1 plus a time-exponentially small error corresponding to the shifted contour integral, which result again agrees with (6.27) up to a time-exponentially small error.

Expression (6.27) may be rewritten as

$$\text{errfn}\left(\frac{y + a_k^{*-}t}{\sqrt{4\beta_k^{*-}|y/a_k^{*-}|}}\right) \tag{6.29}$$

plus error

$$\begin{aligned}
&\text{errfn}\left(\frac{y + a_k^{*-}t}{\sqrt{4\beta_k^{*-}|y/a_k^{*-}|}}\right) - \text{errfn}\left(\frac{y + a_k^{*-}t}{\sqrt{4\beta_k^{*-}t}}\right)\\
&\sim \text{errfn}\,'\left(\frac{y + a_k^{*-}t}{\sqrt{4\beta_k^{*-}t}}\right)\left(-\frac{4\beta_k^{*-}}{2}(y + a_k^{*-}t)^2(4\beta_k^{*-}t)^{-3/2}\right)\\
&= \mathcal{O}(t^{-1}e^{(y+a_k^{*-}t)^2/Mt}),
\end{aligned} \tag{6.30}$$

for $M > 0$ sufficiently large, and similarly for x- and y-derivatives. Multiplying by

$$[c_{k,-}^{j,0}]\frac{\partial}{\partial\delta_j}\begin{pmatrix}\bar{u}^\delta(x)\\ \bar{v}^\delta(x)\end{pmatrix}L_k^{*-t} = \mathcal{O}(e^{-\theta|x|}),$$

we find that term (6.29) gives contribution

$$[c_{k,-}^{j,0}]\frac{\partial}{\partial\delta_j}\begin{pmatrix}\bar{u}^\delta(x)\\ \bar{v}^\delta(x)\end{pmatrix}L_k^{*-t}\text{errfn}\left(\frac{y + a_k^{*-}t}{\sqrt{4\beta_k^{*-}|y/a_k^{*-}|}}\right), \tag{6.31}$$

whereas term (6.30) gives a contribution absorbable in R (resp. R_x, R_y); see Remark 6.5 below for detailed discussion of a similar calculation.

Finally, observing that

$$[c_{k,-}^{j,0}]\frac{\partial}{\partial\delta_j}\begin{pmatrix}\bar{u}^\delta(x)\\ \bar{v}^\delta(x)\end{pmatrix}L_k^{*-t}\text{errfn}\left(\frac{y - a_k^{*-}t}{\sqrt{4\beta_k^{*-}t}}\right) \tag{6.32}$$

is time-exponentially small for $t \geq 1$, since $a_k^{*-} > 0$, $y < 0$, and $(\partial/\partial\delta_j)(\bar{u}^\delta, \bar{v}^\delta) \leq Ce^{-\theta|x|}$, $\theta > 0$, we may subtract and add this term to (6.31) to obtain a total of $E(x,t;y)$ plus terms absorbable in R (resp. R_x, R_y).

Remark 6.5. There is a slight discrepancy between the form of E used here, and the form

$$\begin{aligned} E(x,t;y) &:= \sum_{a_k^- > 0} [c_{k,-}^{j,0}] \frac{\partial}{\partial \delta_j} \begin{pmatrix} \bar u^\delta(x) \\ \bar v^\delta(x) \end{pmatrix} L_k^{*-t} \left(\text{errfn} \left(\frac{x-y+a_k^{*-}t}{\sqrt{4\beta_k^{*-}t}} \right) \right. \\ &\qquad \left. - \text{errfn} \left(\frac{x-y-a_k^{*-}t}{\sqrt{4\beta_k^{*-}t}} \right) \right), \\ &= \sum_{a_k^- > 0} [c_{k,-}^{j,0}] \frac{\partial}{\partial \delta_j} \begin{pmatrix} \bar u^\delta(x) \\ \bar v^\delta(x) \end{pmatrix} L_k^{*-t} \left(\text{errfn} \left(\frac{-x+y+a_k^{*-}t}{\sqrt{4\beta_k^{*-}t}} \right) \right. \\ &\qquad \left. - \text{errfn} \left(\frac{-x+y-a_k^{*-}t}{\sqrt{4\beta_k^{*-}t}} \right) \right), \end{aligned} \tag{6.33}$$

given in [Z.2]; however, this is only cosmetic, since the two expressions are equal modulo terms absorbable in the error terms R (resp. R_x, R_y). To see this, first note that the difference is time-exponentially small outside the critical region

$$x \in [-C\log(t+1), +C\log(t+1)],$$

for $C > 0$ sufficiently large. Then, estimating

$$\begin{aligned} \left| \text{errfn} \left(\frac{-x+y\pm a_k^{*-}t}{\sqrt{4\beta_k^{*-}t}} \right) - \text{errfn} \left(\frac{y\pm a_k^{*-}t}{\sqrt{4\beta_k^{*-}t}} \right) \right| &\le |x| \left| \text{errfn}\,' \left(\frac{-x+y\pm a_k^{*-}t}{\sqrt{4\beta_k^{*-}t}} \right) \right| \\ &\le |x|\, t^{-1/2} e^{(x-y\mp a_k^{*-}t)^2/4\beta_k^{*-}t}, \end{aligned}$$

and recalling that $(\partial/\partial\delta_j)(\bar u^\delta, \bar v^\delta) = \mathcal{O}(e^{-\theta|x|})$, we find that the error is bounded by

$$C|x|e^{-\theta|x|} t^{-1/2} e^{(x-y-a_k^{*-}t)^2/4\beta_k^{*-}t} \le C_2 e^{-\theta|x|/2} t^{-1/2} e^{(x-y-a_k^{*-}t)^2/4\beta_k^{*-}t}, \tag{6.34}$$

and thus absorbable in R. The calculation for the y-derivative is similar. We have chosen to present the result in the form used here, since it is more straightforward to prove, and is more convenient for the final stability analysis (in fact, this was the form that was actually used in the stability analysis of [Z.2], as well). It was presented as (6.33) in [Z.2] in order to capture the parabolic property of almost-finite propagation speed; here, we have dealt with propagation speed separately, which seems the preferable approach.

Remark 6.6. As described in [Z.2], the time-exponentially small terms (6.32) that we have (artificially) included in E are useful for linear and nonlinear stability analyses, serving to regularize behavior at $t = 0$. Note, also, that the resulting "balanced" form of E is the one that appears naturally in the explicit Green's function formula for viscous shock solutions of Burgers equation; see equation (2.13), [Z.2], and surrounding example.

S_λ *term.* Next, consider the second-order term

$$\frac{1}{2\pi i}\oint_{\tilde\Gamma} e^{\lambda t} S_\lambda(x,y)d\lambda, \tag{6.35}$$

which, by (5.10), is given by

$$\sum_{a_k^{*-}>0} R_k^{*-} L_k^{*-t}\alpha_k(x,t;y) + \sum_{a_k^{*-}>0,\, a_j^{*-}<0} [c_{k,-}^{j,-}] R_j^{*-} L_k^{*-t}\alpha_{jk}(x,t;y) \tag{6.36}$$

where

$$\alpha_k(x,t;y) := \frac{1}{2\pi i}\oint_{\tilde\Gamma} e^{\lambda t} e^{(-\lambda/a_k^{*-} + \lambda^2\beta_k^{*-}/a_k^{*-3})(x-y)}\, d\lambda. \tag{6.37}$$

and

$$\alpha_{jk}(x,t;y) := \frac{1}{2\pi i}\oint_{\tilde\Gamma} e^{\lambda t} e^{(-\lambda/a_j^{*-} + \lambda^2\beta_j^{*-}/a_j^{*-3})x + (\lambda/a_k^{*-} - \lambda^2\beta_k^{*-}/a_k^{*-3})y}\, d\lambda. \tag{6.38}$$

Similarly as in the treatment of the E_λ term, just above, by deforming the contour $\tilde\Gamma$ to

$$\Gamma' := [-\eta_1 - ir/2, -ir/2] \cup [-ir/2, +ir/2] \cup [+ir/2, -\eta_1 + ir/2], \tag{6.39}$$

these may be transformed modulo time-exponentially decaying terms to the elementary Fourier integrals

$$\begin{aligned} &\frac{1}{2\pi}\text{P.V.} \int_{-\infty}^{+\infty} e^{i\xi(t-(x-y)/a_k^{*-})} e^{\xi^2(-\beta_k^{*-}/a_k^{*-3})(x-y)}\, d\xi \\ &\qquad = (4\pi\beta_k^- t)^{-1/2} e^{-(x-y-a_k^{*-}t)^2/4\beta_k^{*-}t} \end{aligned} \tag{6.40}$$

and

$$\begin{aligned} &\frac{1}{2\pi}\text{P.V.} \int_{-\infty}^{+\infty} e^{i\xi\left((t-|y/a_k^{*-}|)-x/a_j^{*-}\right)} e^{-\xi^2\left((\beta_k^{*-}/a_k^{*-3})|y| + (\beta_j^{*-}/a_j^{*-3})|x|\right)}\, d\xi \\ &\qquad = (4\pi\bar\beta_{jk}^{*-} t)^{-1/2} e^{-(x-z_{jk}^{*-})^2/4\bar\beta_{jk}^{*-}t}, \end{aligned} \tag{6.41}$$

respectively, where $\bar\beta_{jk}{-}^{+-}$ and z_{jk}^* are as defined in (1.38) and (1.37). These correspond to the first and third terms in expansion (1.35), the latter of which has an additional factor $e^{-x}/(e^x+e^{-x})$. Noting that the second and fourth terms of (1.35) are time-exponentially small for $t\ge 1$, $y\le x\le 0$, and that

$$\begin{aligned}|(4\pi\bar\beta_{jk}^{*-}t)^{-1/2}e^{-(x-z_{jk}^{*-})^2/4\bar\beta_{jk}^{*-}t}(1-e^{-x}/(e^x+e^{-x}))|\\ \le |(4\pi\bar\beta_{jk}^{*-}t)^{-1/2}e^{-(x-z_{jk}^{*-})^2/4\bar\beta_{jk}^{*-}t}e^{-\theta|x|}\end{aligned}$$

for some $\theta>0$, so is absorbable in error term R, we find that the total contribution of this term, modulo terms absorbable in R, is S.

R_λ term. Finally, we briefly discuss the estimation of error term

$$\frac{1}{2\pi i}\oint_\Gamma e^{\lambda t}R_\lambda(x,y)d\lambda, \tag{6.42}$$

which decomposes into the sum of integrals involving the various terms of R_λ^E and R_λ^S given in (5.12) and (5.14). Since each of these are separately analytic, they may be split up and estimated sharply via the *Riemann saddlepoint method* (method of steepest descent), as described at great length in [ZH,HoZ.2]. That is, for each summand $\alpha_\lambda(x,y)\sim e^{\beta(\lambda)x+\gamma(\lambda)y}$ in R_λ we deform the contour $\tilde\Gamma$ to a new, contour in Λ that is a mini-max contour for the modulus

$$m_\alpha(x,y,\lambda):=|e^{\lambda t}\alpha_\lambda(x,y)|=e^{\text{Re }\lambda t+\text{Re }\beta x+\text{Re }\gamma y},$$

passing through an appropriate saddlepoint/critical point of $m_\alpha(x,y,\cdot)$: necessarily lying on the real axis, by the underlying complex symmetry resulting from reality of operator L.

Since terms of each type appearing in R have been sharply estimated in [ZH], we shall omit the details, only describing two sample calculations to illustrate the method:

Example 1: $e^{-\theta|x-y|}$. Contour integrals of form

$$\frac{1}{2\pi i}\oint_{\tilde\Gamma} e^{\lambda t}e^{-\theta|x-y|}d\lambda, \tag{6.43}$$

arising through the pairing of fast forward and fast dual modes, may be deformed to

$$\frac{1}{2\pi i}\int_{-\eta_1-ir/2}^{-\eta_1+ir/2} e^{\lambda t}e^{-\theta|x-y|}d\lambda \tag{6.44}$$

and estimated as $\mathcal{O}(e^{-\eta_1 t-\theta|x-y|})$, a negligible, time-exponentially decaying contribution. ∎

Example 2: $\lambda^r e^{(-\lambda/a_k^{*-}+\lambda^2\beta_k^{*-}/a_k^{*-3})(x-y)}$. Contour integrals of form

$$\frac{1}{2\pi i}\oint_{\tilde\Gamma} e^{\lambda t}\lambda^r e^{(-\lambda/a_k^{*-}+\lambda^2\beta_k^{*-}/a_k^{*-3})(x-y)}d\lambda, \tag{6.45}$$

$a_k^{*-} > 0$, arising through the pairing of slow forward and slow dual modes, may be deformed to contour

$$\Gamma' := [-\eta_1 - ir/2, \eta_* - ir/2] \cup [\eta_* - ir/2, \eta_* + ir/2] \cup [\eta_* + ir/2, -\eta_1 + ir/2], \tag{6.46}$$

where saddlepoint η_* is defined as

$$\eta_*(x,y,t) := \begin{cases} \frac{\bar\alpha}{p} & \text{if } |\frac{\bar\alpha}{p}| \le \varepsilon \\ \pm\varepsilon & \text{if } \frac{\bar\alpha}{p} \gtrless \varepsilon, \end{cases} \tag{6.47}$$

with

$$\bar\alpha := \frac{x - y - a_k^- t}{2t}, \quad p := \frac{\beta_k^-(x-y)}{(a_k^-)^2 t} > 0, \tag{6.48}$$

and η_1, $\varepsilon > 0$ are chosen sufficiently small with respect to r, to yield, modulo time-exponentially decaying terms, the estimate

$$e^{-(x-y-a_k^- t)^2/4\beta^{*-}t} \int_{-\infty}^{+\infty} \mathcal{O}(|\eta_*|^r + |\xi|^r) e^{-\theta\xi^2 t}\, d\xi = \mathcal{O}(t^{-(r+1)/2} e^{-(x-y-a_k^- t)^2/Mt}) \tag{6.49}$$

if $|\frac{\bar\alpha}{p}| \le \varepsilon$, and

$$e^{-\varepsilon t/M} \int_{-\infty}^{+\infty} \mathcal{O}(|\eta_*|^r + |\xi|^r) e^{-\theta\xi^2 t}\, d\xi = \mathcal{O}(t^{-(r+1)/2} e^{-\eta t}). \tag{6.50}$$

if $|\frac{\bar\alpha}{p}| \ge \varepsilon$, $\theta > 0$. In either case, the main contribution lies along the central portion $[\eta_* - ir/2, \eta_* + ir/2]$ of contour Γ'.

To see this, note that $|x-y|$ and t are comparable when $|\bar\alpha/p|$ is bounded, whence the evident spatial decay $e^{-\theta\xi^2|x-y|}$ of the integrand along contour $\lambda = \eta_* + i\xi$ may be converted to the $e^{-\xi^2 t}$ decay displayed in (6.49)–(6.50); likewise, temporal growth in $e^{\lambda t}$ on the horizontal portions of the contour, of order $\le e^{(|\eta_1|+|\varepsilon|)t}$, is dominated by the factor $e^{-\theta\xi^2} = e^{-\theta r^2/4}$, provided $|\eta_1| + |\varepsilon|$ is sufficiently small with respect to r^2. For $|\bar\alpha/p|$ large, on the other hand, η_* is uniformly negative, and also $|x-y|$ is negligible with respect to t, whence the estimate (6.50) holds trivially.

We point out that η_* is easily determined, as the minimal point on the real axis of the quadratic function

$$f_{x,y,t}(\lambda) := \lambda t(-\lambda/a_k^{*-} + \lambda^2\beta_k^{*-}/a_k^{*-3})(x-y), \tag{6.51}$$

the argument of the integrand of (6.45). ∎

Other terms may be treated similarly: All "constant-coefficient" terms $\phi_k\tilde\phi_{k*}$ or $\psi_k\tilde\psi_{k*}$ are of either the form treated in Example 1 (fast modes) or in example 2 (slow modes). Scattering pairs involving slow forward and slow dual modes from different families (i.e., terms with coefficients M_{jk}, $d^{\pm}_{jk}$) may be treated similarly as in Example 2. Scattering pairs involving two fast modes are of the form already treated in Example 1, since both modes of a scattering pair are decaying. Scattering pairs involving one fast and one slow mode may be factored as the product of a term of the form treated in example 2 and a term that is uniformly exponentially decaying in either x or y; factoring out the exponentially decay, we may treat such terms as in Example 2.

Remark 6.7. In [ZH,HoZ.2], the saddlepoint contours followed were hyperbolae through η_*, rather than the simpler, vertical contours used here. Any reasonable contour through the saddlepoint will suffice, yielding the same estimates.

Scattering coefficients. The relation (1.39) may now be deduced, a posteriori, from conservation of mass in the u variable of the linearized flow (1.17). For, by inspection, all terms save E and S in the decomposition (1.32) of G decay in L^1, whence these terms must, time-asymptotically, carry exactly the mass in u of the initial perturbation $\delta_y(x)I_n$, i.e.,

$$\lim_{t\to\infty}(I_n,0)\int_{-\infty}^{+\infty}\big(E(x,t;y)+S(x,t;y)\big)\,dx = I_{n+r}. \tag{6.52}$$

Taking $y\le 0$ for definiteness, and right-multiplying both sides of (6.52) by $(r_k^{*-t},0)^t$, we thus obtain the result from definitions (1.34)–(1.35).

Moreover, rewriting (1.39) as

$$(r_1^{*-},\dots,r_{n-i_-}^{*-},r_{i_+}^{*+},\dots,r_n^{*+},m_1,\dots,m_\ell)\begin{pmatrix}[c_{k,-}^{1,-}]\\ \vdots\\ [c_{k,-}^{n-i_-,-}]\\ [c_{k,-}^{i_+,+}]\\ \vdots\\ [c_{k,-}^{n,+}]\\ [c_{k,-}^{1,0}]\\ \vdots\\ [c_{k,-}^{\ell,0}]\end{pmatrix} = r_k^{*-}, \tag{6.53}$$

where

$$m_j := \int_{-\infty}^{+\infty}(\partial/\partial\delta_j)\bar u^{\delta}(x)\,dx, \tag{6.54}$$

we find that, under assumption $(\mathcal{D})$, it uniquely determines the scattering coefficients $[c_{k,-}^{j,\pm}]$. For, the determinant

$$\det\,(r_1^{*-},\dots,r_{n-i_-}^{*-},r_{i_+}^{*+},\dots,r_n^{*+},m_1,\dots,m_\ell) \tag{6.55}$$

of the matrix on the righthand side of (6.53) is exactly the inviscid stability coefficient Δ defined in [ZS,Z.4], hence is nonvanishing by the equivalence of $(\mathcal{D}2)$ and (D2) (recall discussion of Section 1.2, just below Theorem 1.4). Note that (6.55) reduces to the Liu–Majda determinant (1.21) in the Lax case, $\ell = 1$, with $\bar u^\delta(x)$ parametrized as $\bar u^\delta(x) := \bar u(x - \delta_1 x)$, for which m_1 reduces to $[u]$.

Relation (1.40) follows from the observation that

$$\begin{aligned} P_\lambda(x,y) &:= \text{Residue }_{\lambda=0} G_\lambda(x,y) \\ &= \sum_{j=1}^{\ell} (\partial/\partial\delta_j) \begin{pmatrix} \bar u^\delta \\ \bar v^\delta \end{pmatrix} (x)\, \pi_j, \end{aligned}$$

with π_j defined by the first, or second expressions appearing in (1.40), according as y is less than or equal to/greater than or equal to zero. For, the extended spectral theory of Section 5.4 then implies that π_j are the left effective eigenfunctions associated with right eigenfunctions $(\partial/\partial\delta_j)(\bar u^\delta, \bar v^\delta)$. Alternatively, (1.40) may be deduced by linear-algebraic manipulation directly from (6.53) and its counterpart for $y \ge 0$ (the same identity with $k, -$ everywhere replaced by $k, +$).

Remark 6.8. As pointed out in [LZ.2], the left eigenfunctions (constant vectors) $\pi_j = (\pi_{j,u}, 0)$ may themselves be determined by the relation

$$\pi_{j,u}^t (r_1^{*-}, \dots, r_{n-i_-}^{*-}, r_{i_+}^{*+}, \dots, r_n^{*+}, m_1, \dots, m_\ell) = (0, e_j^t), \tag{6.56}$$

from which (1.40) is easily deduced, where e_j here denotes the jth standard basis element in $\mathbb{R}^\ell$. This may be deduced a priori through the extended spectral theory of Section 5.4; it would interesting also to derive (6.53) a priori, directly through the relations (3.54)–(3.55).

∎

Remark 6.9 (The undercompressive case). In the undercompressive case, the result of Lemma 5.8 is false, and consequently the estimates of Lemma 5.11 do not hold. This fact has the implication that shock dynamics are not governed solely by conservation of mass, as in the Lax or overcompressive case, but by more complicated dynamics of front interaction; for related discussion, see [LZ.2,Z.2]. At the level of Proposition 5.2, it means that the simple representations of E_λ and S_λ in terms of slow dual modes alone (corresponding to equilibrium characteristics that are incoming to the shock) are no longer valid in the undercompressive case, and there appear new terms involving rapidly decaying dual modes $\sim e^{-\theta|y|}$ related to inner layer dynamics. Though precise estimates can nonetheless be carried out, we have not found a similarly compact representation of the resulting bounds as that of the Lax/overcompressive case, and so we shall not state them here. We mention only that this rapid variation in the y-coordinate precludes the L^p stability arguments used here and in [Z.2], requiring instead detailed pointwise bounds as in [HZ.2,Z.5]. See [LZ.1–2,ZH,Z.2,Z.5] for further discussion of this interesting case.

6.2. Inner layer dynamics. Similarly as in [ZH], we now investigate dynamics of the inner shock layer, in the case that $(\mathcal{D})$ does not necessarily hold, in the process establishing the necessity of $(\mathcal{D})$ for linearized orbital stability.

Proposition 6.10. *Let (H0)–(H4) hold. Then, there exists $\eta > 0$ such that, for x, y restricted to any bounded set, and t sufficiently large,*

$$G(x,y;t) = \sum_{\lambda \in \sigma'_p(L) \cap \{Re(\lambda) \geq 0\}} e^{\lambda t} \sum_{k \geq 0} t^k (L - \lambda I)^k P_\lambda(x,y) + \mathcal{O}(e^{-\eta t}), \tag{6.57}$$

where $P_\lambda(x,y)$ is the effective projection kernel described in Definition 5.12, and $\sigma'_-(L)$ the effective point spectrum.

Proof. Similarly as in the proof of Proposition 6.1, decompose G into terms I_a, I_b, II_a, and II_b of (6.12). Then, the same argument (Case II. $|x-y|/t$ bounded) yields that $I_a = H(x,t;y)$, while I_b and II_b are time-exponentially small. Observing that $H = 0$ for x, y bounded, and t sufficiently large, we have reduced the problem to the study of

$$II_a := \frac{1}{2\pi i} \oint_\Gamma e^{\lambda t} G_\lambda(x,y) d\lambda, \tag{6.58}$$

where, recall,

$$\Gamma := [-\eta_1 - iR, \eta - iR] \cup [\eta - iR, \eta + iR] \cup [\eta + iR, -\eta_1 + iR]. \tag{6.59}$$

Note that the high-frequency bounds of Proposition 4.3 imply that L has no point spectrum in Ω outside of the rectangle $\mathcal{R}$ enclosed by $\Gamma \cup [-\eta_1 - iR, -\eta + iR]$. Choosing η_1 sufficiently small, therefore, we may ensure that no effective point spectrum lies within the strip $-\eta_1 \leq \text{Re } \lambda < 0$, by compactness of $\mathcal{R}$ together with the fact that effective eigenvalues are isolated from one another, as zeroes of the analytic Evans function.

Recall that G_λ is meromorphic on Ω. Thus, we may express (6.58), using Cauchy's Theorem, as

$$\frac{1}{2\pi i} \int_{-\eta_1 - iR}^{-\eta_1 + iR} e^{\lambda t} G_\lambda(x,y) d\lambda + \text{Residue }_{\lambda \in \mathcal{R}} e^{\lambda t} G_\lambda(x,t;y). \tag{6.60}$$

By Definition 5.12 and Proposition 5.14,

$$\begin{aligned} &\text{Res}_{\lambda \in \{\text{Re } \lambda \geq 0\}} e^{\lambda t} G_\lambda(x,y) \\ &\quad = \sum_{\lambda_0 \in \sigma'_p(L) \cap \{\text{Re } \lambda \geq 0\}} e^{\lambda_0 t} \text{Res}_{\lambda_0} e^{(\lambda - \lambda_0)t} G_\lambda(x,y) \\ &\quad = \sum_{\lambda_0 \in \sigma'_p(L) \cap \{\text{Re } \lambda \geq 0\}} e^{\lambda_0 t} \sum_{k \geq 0} (t^k/k!) \text{Res}_{\lambda_0} (\lambda - \lambda_0)^k G_\lambda(x,y) \\ &\quad = \sum_{\lambda_0 \in \sigma'_p(L) \cap \{\text{Re } \lambda \geq 0\}} e^{\lambda_0 t} \sum_{k \geq 0} (t^k/k!) Q_{\lambda_0,k}(x,y) \\ &\quad = \sum_{\lambda_0 \in \sigma'_p(L) \cap \{\text{Re } \lambda \geq 0\}} e^{\lambda_0 t} \sum_{k \geq 0} (t^k/k!) (L - \lambda_0 I)^k P_{\lambda_0}(x,y)). \end{aligned} \tag{6.61}$$

On the other hand, for x, y bounded and t sufficiently large, t dominates $|x|$ and $|y|$ and we obtain from Propositions 3.4 and 5.2 that

$$|G_\lambda(x,t;y)| \le C$$

for all $\lambda \in \Omega$, for η_1 sufficiently small, hence

$$\frac{1}{2\pi i}\int_{-\eta_1 - iR}^{-\eta_1 + iR} e^{\lambda t} G_\lambda(x,y)d\lambda \le 2CRe^{-\eta_1 t}. \tag{6.62}$$

Combining (6.60), (6.61), and (6.62), we obtain the result. ∎

Corollary 6.11. *Let (H0)–(H4) hold. Then, ($\mathcal{D}$) is necessary for linearized orbital stability with respect to compactly supported initial data, as measured in any L^p norm, $1 \le p \le \infty$.*

Proof: From (6.57), we find that $(\bar u, \bar v)(\cdot)$ is linearly orbitally stable only if $P_\lambda = 0$ for all $\lambda \in \{\text{Re}\ \lambda \ge 0\} \setminus \{0\}$ and Range $\mathcal{P}_0 = \text{Span}\ \{(\partial/\partial\delta_j)(\bar u^\delta, \bar v^\delta\}$. By Propositions 5.14 and 5.15, this is equivalent to ($\mathcal{D}$). ∎

Section 7. Stability.

We now establish our remaining results on linearized and nonlinear stability.

7.1. Linearized stability. Linearized orbital stability follows immediately from the pointwise bounds we have established. Similarly as in [Z.2], define the *linear instantaneous projection*:

$$\begin{aligned}\varphi(x,t) &:= \int_{-\infty}^{+\infty} E(x,t;y)v_0(y)\,dy \\ &=: \delta(t)(\partial/\partial\delta)\begin{pmatrix}\bar u^\delta(x)\\ \bar v^\delta(x)\end{pmatrix}.\end{aligned} \tag{7.1}$$

The coordinates $\delta \in \mathbb{R}^{1\times\ell}$ may be expressed, alternatively, as

$$\delta(t) = \int_{-\infty}^{+\infty} e(y,t)v_0(y)\,dy,$$

where

$$E(x,t;y) =: \begin{pmatrix}\bar u^\delta(x)\\ \bar v^\delta(x)\end{pmatrix} e(y,t), \tag{7.2}$$

i.e.,

$$e(y,t) := \sum_{a_k^{*-}} \left(\text{errfn}\ \left(\frac{y + a_k^{*-}t}{\sqrt{4\beta_k^{*-}t}}\right) - \text{errfn}\ \left(\frac{y - a_k^{*-}t}{\sqrt{4\beta_k^{*-}t}}\right)\right) L_k^{*-} \tag{7.3}$$

for $y \le 0$, and symmetrically for $y \ge 0$.

Then

$$v(x,t) - \phi(x,t) = \int_{-\infty}^{+\infty} (H + \tilde{G})(x,t;0) v_0(y)\, dy, \tag{7.4}$$

where

$$\tilde{G} := S + R \tag{7.5}$$

is the regular part and H the singular part of the time-decaying portion of the Green's function G.

Lemma 7.1. *Under assumptions (H0)–(H4) and ($\mathcal{D}$), there hold:*

$$|\int_{-\infty}^{+\infty} \tilde{G}(\cdot,t;y) f(y) dy|_{L^p} \le C(1+t)^{-\frac{1}{2}(1-1/r)} |f|_{L^q}, \tag{7.6}$$

$$|\int_{-\infty}^{+\infty} \tilde{G}(\cdot,t;y)(0,I_r)^t f(y) dy|_{L^p} \le C(1+t)^{-\frac{1}{2}(1-1/r)-1/2} |f|_{L^q}, \tag{7.7}$$

$$|\int_{-\infty}^{+\infty} \tilde{G}_y(\cdot,t;y) f(y) dy|_{L^p} \le C(1+t)^{-\frac{1}{2}(1-1/r)-1/2} |f|_{L^q}, \tag{7.8}$$

and

$$|\int_{-\infty}^{+\infty} H(\cdot,t;y) f(y) dy|_{L^p} \le C e^{-\eta t} |f|_{L^p}, \tag{7.9}$$

for all $t \ge 0$, some C, $\eta > 0$, for any $1 \le r \le p$ and $f \in L^q$ (resp. L^p), where $1/r + 1/q = 1 + 1/p$.

Proof. Bounds (7.6)–(7.8) follow by the Hausdorff-Young inequality together with bounds (1.35) and (6.1)–(6.3). Bound (7.9) follows by direct computation and the fact that particle paths $z_j(y,t)$ satisfy uniform bounds

$$1/C \le |(\partial/\partial y) z_j| < C,$$

for all y, t, by the fact that characteristic speeds $a_j(x)$ converge exponentially as $x \to \pm\infty$ to constant states. ∎

Corollary 7.2. *Let (H0)–(H4) hold. Then, ($\mathcal{D}$) is sufficient for $L^1 \cap L^p \to L^p$ linearized orbital stability, for any $p > 1$. More precisely, for initial data $v_0 \in L^1 \cap L^p$, we have the linear decay bound*

$$|v(\cdot, t) - \varphi(\cdot, t)|_{L^p} \le C(1+t)^{-\frac{1}{2}(1-1/p)}(|v_0|_{L^1} + |v_0|_{L^p}). \tag{7.10}$$

Proof. Immediate, from (7.4) and bounds (7.6) and (7.9), with $q = p$. ∎

Proof of Theorem 1.4. Theorem 1.4 now follows immediately from Proposition 6.11 together with Corollary 7.2. ∎

7.2. Nonlinear stability (Proof of Theorem 1.13). We conclude by establishing nonlinear stability of relaxation profiles for simultaneously symmetrizable and discrete kinetic models, as described in Theorem 1.13.

7.2.1. Simultaneously symmetrizable models. We first establish the result of Theorem 1.13 for relaxation profiles of simultaneously symmetrizable relaxation models, under the weak shock assumption

$$|(\bar{u}, \bar{v})'|_{L^\infty} \le \varepsilon, \tag{7.11}$$

$\varepsilon > 0$ sufficiently small with respect to the parameters of system (1.1). Before beginning, we recall some needed preliminaries regarding the class of simultaneously symmetrizable systems:

Definition 7.3. A relaxation model (1.1) is called *simultaneously symmetrizable*, if there exists a smooth $A^0(u,v)$ symmetric, positive definite, such that $A^0(u,v)A(u,v)$ and $A^0(u,v)Q(u,v)$ are symmetric, and $A^0(u,v)Q(u,v)$ is negative semidefinite, where, as in the introduction, $A := (df, dg)^t$ and $Q := (0, dq)^t$.

Lemma 7.4 ([SK]). *Assuming simultaneous symmetrizability, condition (H3) is equivalent to either of:*

(K1) *There exists a smooth skew-symmetric matrix $K(u)$ such that*

$$\mathrm{Re}\,(KA - A^0Q)(u) \ge \theta > 0. \tag{7.12}$$

A^0 as in Definition 7.3.

(K2) *There is no eigenvector of $A(u)$ lying in the kernel of $Q(u)$.*

Proof. These and other useful equivalent formulations are established in [SK]. ∎

Next, we recall the basic iteration scheme of [Z.2]; for precursors of this scheme, see [Go.2,K.1–2,LZ.1–2,ZH,HZ.1–2]. Define the nonlinear perturbation

$$U(x,t) := \begin{pmatrix} u \\ v \end{pmatrix}(x + \delta(t), t) - \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}(x), \tag{7.13}$$

where $\delta(t)$ (estimating shock location) is to be determined later; for definiteness, fix $\delta(0)=0$. Then,

$$U_t - LU = N_1(U,U)_x + (0,I_r)^t N_2(U,U) + \dot\delta(t)(\bar U_x + U_x), \tag{7.14}$$

$$\bar U := \begin{pmatrix} \bar u \\ \bar v \end{pmatrix}, \tag{7.15}$$

where

$$N_j(U,U) = \mathcal{O}(|U|^2) \tag{7.16}$$

so long as $|U|$ remains bounded. By Duhamel's principle, and the fact that

$$\int_{-\infty}^{\infty} G(x,t;y)\bar U_x(y)dy = e^{Lt}\bar U_x(x) = \bar U_x(x), \tag{7.17}$$

we have

$$\begin{aligned} U(x,t) = &\int_{-\infty}^{\infty} G(x,t;y)U_0(y)dy \\ &- \int_0^t \int_{-\infty}^{\infty} G_y(x,t-s;y)(N_1(U,U)+\dot\delta U)(y,s)dyds \\ &+ \int_0^t \int_{-\infty}^{\infty} G(x,t-s;y)(0,I_r)^t N_2(U,U) \\ &+ \delta(t)\bar U_x. \end{aligned} \tag{7.18}$$

Defining, by analogy with the linear case, the *nonlinear instantaneous projection*:

$$\begin{aligned} \varphi(x,t) &:= -\delta(t)\bar U_x \\ &:= \int_{-\infty}^{\infty} E(x,t;y)U_0(y)dy \\ &\quad - \int_0^t \int_{-\infty}^{\infty} E_y(x,t-s;y)(N_1(U,U)+\dot\delta U)(y,s)dy, \end{aligned} \tag{7.19}$$

or equivalently, the *instantaneous shock location*:

$$\begin{aligned} \delta(t) = &- \int_{-\infty}^{\infty} e(y,t)U_0(y)dy \\ &+ \int_0^t \int_{-\infty}^{+\infty} e_y(y,t-s)(N_1(U,U)+\dot\delta U)(y,s)dyds, \end{aligned} \tag{7.20}$$

where E, e are defined as in (1.34), (7.3), and recalling (7.5), we thus obtain the *reduced equations*:

(7.21)
$$\begin{aligned}
U(x,t) = &\int_{-\infty}^{\infty} (H+\tilde G)(x,t;y)U_0(y)dy\\
&+\int_0^t\int_{-\infty}^{\infty} H(x,t-s;y)(N_1(U,U)_x + (0,I_r)^t N_2(U,U) + \dot\delta U_x)(y,s)dy\,ds\\
&-\int_0^t\int_{-\infty}^{\infty} \tilde G_y(x,t-s;y)(N_1(U,U) + \dot\delta U)(y,s)dy\,ds\\
&+\int_0^t\int_{-\infty}^{\infty} \tilde G(x,t-s;y)(0,I_r)^t N_2(U,U)dy\,ds,
\end{aligned}$$

and, differentiating (7.20) with respect to t,

$$\begin{aligned}
\dot\delta(t) = &-\int_{-\infty}^{\infty} e_t(y,t)U_0(y)dy\\
&+\int_0^t\int_{-\infty}^{+\infty} e_{yt}(y,t-s)(N_1(U,U)+\dot\delta U)(y,s)dyds.
\end{aligned}\tag{7.22}$$

Notes: In deriving (7.22), we have used the fact that $e_y(y,s) \rightharpoonup 0$ as $s\to 0$, as the difference of approaching heat kernels, in evaluating the boundary term

$$\int_{-\infty}^{+\infty} e_y(y,0)(N_1(U,U)+\dot\delta U)(y,t)dy = 0.\tag{7.23}$$

(Indeed, $|e_y(\cdot,s)|_{L^1}\to 0$, see Remark 2.6, below). We have also used the fact that $E(0,I_r)\equiv 0$, to eliminate the term corresponding to nonlinear source $(0,I_r)N_2$ in (7.19)–(7.20). Moreover, we have in the first place used the fact that the dimension ℓ of the stationary manifold is one, a property of Lax and undercompressive shocks, in order to factor out $\bar U(x)$ in the first line of (7.19); for this reason, our analysis is inherently limited to the case of Lax shocks (recall that the favorable Green's function bounds of Theorem 1.10 are available for Lax and overcompressive shocks).

The defining relation $\delta(t)\bar u_x := -\varphi$ in (7.19) can be motivated heuristically by

$$\begin{aligned}
\tilde U(x,t)-\varphi(x,t) \sim U = \begin{pmatrix} u\\ v\end{pmatrix}(x+\delta(t),t) - \begin{pmatrix}\bar u\\ \bar v\end{pmatrix}(x)\\
\sim \tilde U(x,t) + \delta(t)\bar U_x(x),
\end{aligned}$$

where $\tilde U$ denotes the solution of the linearized perturbation equations, and $\bar U$ the background profile. Alternatively, it can be thought of as the requirement that the instantaneous projection of the shifted (nonlinear) perturbation variable U be zero, [HZ.1–2].

Lemma 7.5 [Z.2]. *The kernel e satisfies*

$$|e_y(\cdot,t)|_{L^p}, |e_t(\cdot,t)|_{L^p} \le Ct^{-\frac{1}{2}(1-1/p)}, \tag{7.24}$$

$$|e_{ty}(\cdot,t)|_{L^p} \le Ct^{-\frac{1}{2}(1-1/p)-1/2}, \tag{7.25}$$

for all $t>0$. Moreover, for $y \le 0$ we have the pointwise bounds

$$|e_y(y,t)|, |e_t(y,t)| \le Ct^{-\frac{1}{2}} e^{-\frac{(y+a_-t)^2}{Mt}}, \tag{7.26}$$

$$|e_{ty}(y,t)| \le Ct^{-1} e^{-\frac{(y+a_-t)^2}{Mt}}, \tag{7.27}$$

for $M>0$ sufficiently large (i.e. $> 4b_\pm$), and symmetrically for $y \ge 0$.

Proof. For definiteness, take $y \le 0$. Then, (7.3) gives

$$e_y(y,t) = \left(\frac{1}{u_+ - u_-}\right)\left(K(y+a_-t,t) - K(y-a_-t,t)\right), \tag{7.28}$$

$$e_t(y,t) = \left(\frac{1}{u_+ - u_-}\right)\left((K+K_y)(y+a_-t,t) - (K+K_y)(y-a_-t,t)\right), \tag{7.29}$$

$$e_{ty}(y,t) = \left(\frac{1}{u_+ - u_-}\right)\left((K_y+K_{yy})(y+a_-t,t) - (K_y+K_{yy})(y-a_-t,t)\right), \tag{7.30}$$

where

$$K(y,t) := \frac{e^{-y^2/4b_-t}}{\sqrt{4\pi b_- t}} \tag{7.31}$$

denotes an appropriate heat kernel. The pointwise bounds (7.26)–(7.27) follow immediately for $t \ge 1$ by properties of the heat kernel, in turn yielding (7.24)–(7.25) in this case. The bounds for small time $t \le 1$ follow from estimates

$$\begin{aligned} |K_y(y+a_-t,t) - K_y(y-a_-t,t)| &= \Big|\int_{y+a_-t}^{y-a_-t} K_{yy}(z,t)\,dz\Big| \\ &\le Ct^{-3/2}\int_{y+a_-t}^{y-a_-t} e^{\frac{-z^2}{Mt}}\,dz \\ &\le Ct^{-1/2} e^{-\frac{(y+a_-t)^2}{Mt}}, \end{aligned} \tag{7.32}$$

and, similarly,

$$
\begin{aligned}
|K_{yy}(y+a_-t,t)-K_{yy}(y-a_-,t)| &= |\int_{-a_-t}^{a_-t} K_{yyy}(z,t)dz| \\
&\le Ct^{-2}\int_{y+a_-t}^{y-a_-t} e^{\frac{-z^2}{Mt}}dz, \\
&\le Ct^{-1}e^{-\frac{(y+a_-t)^2}{Mt}}.
\end{aligned}
\tag{7.33}
$$

The bounds for $|e_y|$ are again immediate. Note that we have taken crucial account of cancellation in the small time estimates of e_t, e_{ty}. $\square$

Remark 7.6: For $t \le 1$, a calculation analogous to that of (7.32) yields $|e_y(y,t)| \le Ce^{-\frac{(y+a_-t)^2}{Mt}}$, and thus $|e(\cdot,s)|_{L^1} \to 0$ as $s \to 0$.

With these preparations, we are ready to carry out our analysis:

Preliminary result. To illustrate the method, we first carry out the analysis completely for the simplest case of discrete kinetic models with the minimal regularity assumptions

$$
|U_0|_{L^1\cap H^1} \le \zeta_0, \tag{7.34}
$$

(still keeping the weak shock assumption (7.11)). That is, we establish the second assertion of Theorem 1.13. Define

$$
\begin{aligned}
\zeta(t) := \sup_{0\le s\le t} \Big(&|U(\cdot,s)|_{L^2}(1+s)^{1/4} + |\dot\delta(s)|(1+s)^{\frac{1}{2}} + |\delta(s)| \\
&+ |U_x(\cdot,s)|_{L^2}(1+s)^{-1/4} + |U(\cdot,s)|_{L^\infty}\Big).
\end{aligned}
\tag{7.35}
$$

We shall establish:

Claim. For all $t \ge 0$ for which a solution exists with ζ uniformly bounded by some fixed, sufficiently small constant, there holds

$$
\zeta(t) \le C_2(\zeta_0 + \zeta(t)^2). \tag{7.36}
$$

From this result, it follows by continuous induction that, provided $\zeta_0 < 1/4C_2$, there holds

$$
\zeta(t) \le 2C_2\zeta_0 \tag{7.37}
$$

for all $t \ge 0$ such that ζ remains small. By standard short-time theory/continuation, we find that the solution (unique, in this regularity class) in fact remains in $L^2 \cap H^1$ for all $t \ge 0$, with bound (7.37), yielding existence and the claimed L^p bounds. Thus, it remains only to establish the claim above.

Proof of Claim. We must establish bounds on $|U|_{L^2}$, $|\dot\delta|$, $|\delta|$, $|U_x|_{L^2}$, and $|U|_{L^\infty}$. Provided that we can establish the others, the L^∞ bound follows by Sobolev estimate

$$|U|_{L^\infty} \le |U|_{L^2}^{1/2}|U_x|_{L^2}^{1/2}.$$

Likewise, $|U_x|_{L^2}$ may be estimated in by now routine fashion using a nonstandard energy estimate of the type formalized by Kawashima [Kaw]. The origin of this approach goes back to [Ka,MN] in the context of gas dynamics; see, e.g., [HoZ.2] for further discussion/references.

We first perform a standard, "Friedrichs-type" estimate for symmetrizable hyperbolic systems. Expanding and differentiating (7.14), and using the key fact that $N_1(U,U) \equiv 0$ in the case of discrete kinetic models, we obtain

$$U_{xt} - (AU)_{xx} - (QU)_x = (0, I_r)^t N_2(U,U)_x + \dot\delta(t)(\bar U_{xx} + U_{xx}), \tag{7.38}$$

where

$$N_2(U,U)_x = \mathcal{O}(|U|^2 + |U||U_x|) \tag{7.39}$$

provided $|U|_{L^\infty}$ remains uniformly bounded as we have assumed. Let $A^0 = A^0(x)$ denote the simultaneous symmetrizer of A, Q. Taking the L^2 inner product of $\tilde A^0(x)U_x$ against (7.38), we obtain after rearrangement/integration by parts, and applications of Young's inequality, the energy estimate

$$\frac12\langle A^0U_x, U_x\rangle_t \le \langle U_x, A^0QU_x\rangle + \varepsilon|U_x|_{L^2}^2 + \varepsilon^{-1}\mathcal{O}(|U|_{L^2}^2 + |\dot\delta(t)|^2), \tag{7.40}$$

where $\varepsilon > 0$ is as in (7.11). Here, we have freely used assumption (7.11) and the consequent higher order bound $|\bar U''| \le C\varepsilon$, plus the bounds $|\bar U_x|_{L^p}$, $|\bar U_{xx}|_{L^p} \le C$.

Next, we perform a nonstandard, "Kawashima-type" derivative estimate. Let $K = K(x)$ denote the skew-symmetric matrix described in Lemma 7.4, associated with A, Q. Taking the L^2 inner product of U_x against K times the undifferentiated equation

$$U_t - (AU)_x - QU = (0, I_r)^t N_2(U,U) + \dot\delta(t)(\bar U_x + U_x), \tag{7.41}$$

and noting that (integrating by parts, and using skew-symmetry of K)

$$\begin{aligned}\frac12\langle U_x, KU\rangle_t &= \frac12\langle U_x, KU_t\rangle + \frac12\langle U_{xt}, KU\rangle\\ &= \frac12\langle U_x, KU_t\rangle + -\frac12\langle U_t, KU_x\rangle - \frac12\langle U_t, K_xU\rangle\\ &= \langle U_x, KU_t\rangle + \frac12\langle U, K_xU_t\rangle,\end{aligned} \tag{7.42}$$

we obtain after rearrangement the auxiliary energy estimate:

$$\frac{1}{2}\langle U_x, KU\rangle_t \le -\langle U_x, KAU_x\rangle + \varepsilon|U_x|_{L^2}^2 + \varepsilon^{-1}\mathcal{O}(|U|_{L^2}^2 + |\dot\delta(t)|^2). \tag{7.43}$$

Adding (7.40) and (7.43), and recalling (7.12), we obtain, finally:

$$\begin{aligned}\frac{1}{2}(\langle A^0U_x, U_x\rangle + \langle U_x, KU\rangle)_t &\le \frac{1}{2}\langle U_x, (A^0Q - KA)U_x\rangle + \varepsilon^{-1}\mathcal{O}(|U|_{L^2}^2 + |\dot\delta(t)|^2)\\ &\le C(|U|_{L^2}^2 + |\dot\delta(t)|^2),\end{aligned} \tag{7.44}$$

provided ε is sufficiently small that $\varepsilon|U_x|_{L^2}^2$ may be absorbed in the negative definite term $\frac{1}{2}\langle U_x, (A^0Q - KA)U_x\rangle$.

Integrating (7.44) from 0 to t, and recalling definition (7.35), thus yields

$$(\langle A^0U_x, U_x\rangle + \langle U_x, KU\rangle)|_0^t \le C\int_0^t (|U|_{L^2}^2 + |\dot\delta|^2)(s)\,ds. \tag{7.45}$$

Assuming, then, that claim (7.36) holds when restricted to $|U|_{L^2}$ and $|\dot\delta|$, i.e.,

$$|U|_{L^2}(s) \le C(\zeta_0 + \zeta(t)^2)(1+s)^{-1/4} \tag{7.46}$$

and

$$|\dot\delta|(s) \le C(\zeta_0 + \zeta(t)^2)(1+s)^{-1/2}, \tag{7.47}$$

for all $0 \le s \le t$, we obtain

$$\begin{aligned}(\langle A^0U_x, U_x\rangle + \langle U_x, KU\rangle)|_0^t &\le C(\zeta(0) + \zeta(t)^2)^2\int_0^t (1+s)^{-1/2}\,ds\\ &\le C(\zeta(0) + \zeta(t)^2)^2(1+t)^{1/2}.\end{aligned} \tag{7.48}$$

Using Young's inequality to bound

$$\begin{aligned}\langle U_x, KU\rangle(t) &\le \epsilon|U_x|_{L^2}^2(t) + C\epsilon^{-1}|U|_{L^2}^2(t)\\ &\le \epsilon|U_x|_{L^2}^2(t) + C\epsilon^{-1}(\zeta(0) + \zeta(t)^2)^2(1+t)^{-1/2},\end{aligned} \tag{7.49}$$

and recalling, by assumption, that $(\langle A^0U_x, U_x\rangle + \langle U_x, KU\rangle)(0) \le C\zeta_0^2$, we may rearrange (7.48) to obtain

$$\langle A^0U_x, U_x\rangle \le C(\zeta(0) + \zeta(t)^2)(1+t)^{1/2},$$

which, by uniform positive definiteness of A^0, verifies (7.36) for $|U_x|_{L^2}$ as well.

Thus, in order to establish the result, we have only to verify claim (7.36) when restricted to $|U|_{L^2}$, $|\dot\delta|$, and $|\delta|$: that is, to establish (7.46), (7.47), and

$$|\delta|(s) \le C(\zeta_0 + \zeta(t)^2), \tag{7.50}$$

for all $0 \le s \le t$. By (7.21)–(7.22), and $N_1 \equiv 0$, we have

$$\begin{aligned} \big|U\big|_{L^2}(t) \le & \,\Big|\int_{-\infty}^{\infty} (H+\tilde G)(x,t;y)U_0(y)\Big|_{L^2} dy \\ &+ \Big|\int_0^t \int_{-\infty}^{\infty} H(x,t-s;y)(0,I_r)^t N_2(U,U)dy\,ds\Big| \\ &+ \Big|\int_0^t \int_{-\infty}^{\infty} H(x,t-s;y)\dot\delta U_x(y,s)dy\,ds\Big|_{L^2} \\ &+ \Big|\int_0^t \int_{-\infty}^{\infty} \tilde G_y(x,t-s;y)\dot\delta U(y,s)dy\,ds\Big|_{L^2} \\ &+ \Big|\int_0^t \int_{-\infty}^{\infty} \tilde G(x,t-s;y)(0,I_r)^t N_2(U,U)dy\,ds\Big|_{L^2}, \\ &=: I_a + I_b + I_c + I_d + I_e, \end{aligned} \tag{7.51}$$

$$\begin{aligned} \big|\dot\delta\big|(t) = & \,\Big|\int_{-\infty}^{\infty} e_t(y,t)U_0(y)dy\Big| \\ &+ \Big|\int_0^t \int_{-\infty}^{+\infty} e_{yt}(y,t-s)\dot\delta U(y,s)dyds\Big| \\ &=: II_a + II_b. \end{aligned} \tag{7.52}$$

and

$$\begin{aligned} \big|\delta\big|(t) = & \,\Big|\int_{-\infty}^{\infty} e(y,t)U_0(y)dy\Big| \\ &+ \Big|\int_0^t \int_{-\infty}^{+\infty} e_y(y,t-s)\dot\delta U(y,s)dyds\Big| \\ &=: III_a + III_b. \end{aligned} \tag{7.53}$$

We estimate each term in turn, following the approach of [Z.2].

The linear term I_a satisfies bound

$$I_a \le C\zeta_0(1+t)^{-1/4}, \tag{7.54}$$

as already shown in the proof of Corollary 7.2. Likewise, applying the bounds of Lemma 7.1 together with definition (7.35), we have:

$$
\begin{aligned}
II_b &= \Big|\int_0^t \int_{-\infty}^{\infty} H(x,t-s;y)(0,I_r)^t N_2(U,U)\,dy\,ds\Big|_{L^2} \\
&\le C\int_0^t e^{-\eta(t-s)}|U|_{L^\infty}|U|_{L^2}(s)ds \\
&\le C\zeta(t)^2\int_0^t e^{-\eta(t-s)}(1+s)^{-1/4}ds \\
&\le C\zeta(t)^2(1+t)^{-1/4},
\end{aligned}
\tag{7.55}
$$

$$
\begin{aligned}
II_c &= \Big|\int_0^t \int_{-\infty}^{\infty} H(x,t-s;y)\dot\delta U_x(y,s)\,dy\,ds\Big|_{L^2} \\
&\le C\int_0^t e^{-\eta(t-s)}|\dot\delta||U_x|_{L^2}(s)ds \\
&\le C\zeta(t)^2\int_0^t e^{-\eta(t-s)}(1+s)^{-1/4}ds \\
&\le C\zeta(t)^2(1+t)^{-1/4},
\end{aligned}
\tag{7.56}
$$

$$
\begin{aligned}
II_d &= \Big|\int_0^t \int_{-\infty}^{\infty} \tilde G_y(x,t-s;y)\dot\delta U(y,s)\,dy\,ds\Big|_{L^2} \\
&\le C\int_0^t \big(1+(t-s)\big)^{-1/2}|\dot\delta||U|_{L^2}(s)ds \\
&\le C\zeta(t)^2\int_0^t \big(1+(t-s)\big)^{-1/2}(1+s)^{-3/4}ds \\
&\le C\zeta(t)^2(1+t)^{-1/4},
\end{aligned}
\tag{7.57}
$$

and

$$
\begin{aligned}
II_e &= \Big|\int_0^t \int_{-\infty}^{\infty} \tilde G(x,t-s;y)(0,I_r)^t N_2(U,U)\,dy\,ds\Big|_{L^2}, \\
&\le C\int_0^t \big(1+(t-s)\big)^{-3/4}|U|_{L^2}^2(s)ds \\
&\le C\zeta(t)^2\int_0^t \big(1+(t-s)\big)^{-3/4}(1+s)^{-1/2}ds \\
&\le C\zeta(t)^2(1+t)^{-1/4},
\end{aligned}
\tag{7.58}
$$

Summing bounds (7.54)–(7.58), we obtain (7.46), as claimed.

Similarly, applying the bounds of Lemma 7.5 together with definition (7.35), we find that

$$
\begin{aligned}
II_a &= \Big|\int_{-\infty}^{\infty} e_t(y,t)U_0(y)dy\Big| \\
&\le |e_t(y,t)|_{L^\infty}(t)|U_0|_{L^1} \\
&\le C\zeta_0(1+t)^{-1/2}
\end{aligned}
\tag{7.59}
$$

and

$$
\begin{aligned}
II_b &= \Big|\int_0^t \int_{-\infty}^{+\infty} e_{yt}(y,t-s)\dot{\delta}U(y,s)dyds\Big| \\
&\le \int_0^t |e_{yt}|_{L^2}(t-s)|\dot{\delta}||U|_{L^2}(s)ds \\
&\le C\zeta(t)^2 \int_0^t (t-s)^{-3/4}(1+s)^{-3/4}ds \\
&\le C\zeta(t)^2(1+t)^{-1/2},
\end{aligned}
\tag{7.60}
$$

while

$$
\begin{aligned}
III_a &= \Big|\int_{-\infty}^{\infty} e(y,t)U_0(y)dy\Big| \\
&\le |e(y,t)|_{L^\infty}(t)|U_0|_{L^1} \\
&\le C\zeta_0
\end{aligned}
\tag{7.61}
$$

and

$$
\begin{aligned}
III_b &= \Big|\int_0^t \int_{-\infty}^{+\infty} e_y(y,t-s)\dot{\delta}U(y,s)dyds\Big| \\
&\le \int_0^t |e_y|_{L^2}(t-s)|\dot{\delta}||U|_{L^2}(s)ds \\
&\le C\zeta(t)^2 \int_0^t (t-s)^{-1/4}(1+s)^{-3/4}ds \\
&\le C\zeta(t)^2.
\end{aligned}
\tag{7.62}
$$

Summing (7.59)–(7.60) and (7.61)–(7.62), we obtain (7.47) and (7.50), as claimed.

This completes the proof of the claim, and the result. ∎

Remark 7.7. The hypotheses $U_0 \in L^1 \cap H^1$ made in this subsection on initial perturbations are similar to but slightly weaker than those for the stability result announced in [S,SX] for weak profiles of the discrete kinetic Broadwell model. We obtain sharp rates of decay in L^2 and smallness in L^∞, with nonsharp rates of decay in L^p, $2 < p < \infty$, whereas they assert decay in L^p, all $p \geq 2$, with no rates. (Recall that spectral stability holds for weak Broadwell shocks, by the results of [KM], so that we obtain a complete stability result in this case).

The general case. We next sharpen our analysis for simultaneously symmetrizable systems under the strengthened regularity assumption

$$|U_0|_{L^1 \cap H^2} \leq \zeta_0, \tag{7.63}$$

and assuming that $q \in C^3$ for discrete kinetic models as well as general systems. That is, we establish the first assertion of Theorem 1.13.

Similarly as in (7.35), define

$$\begin{aligned} \zeta(t) := \sup_{0 \leq s \leq t} \Big(& |U(\cdot, s)|_{H^1} (1+s)^{1/4} + |\dot\delta(s)|(1+s)^{1/2} + |\delta(s)| \\ & + |U_{xx}(\cdot, s)|_{L^2}(1+s)^{-1/4} + |U(\cdot, s)|_{L^\infty}(1+s)^{1/2} + |U_x(\cdot, s)|_{L^\infty} \Big). \end{aligned} \tag{7.64}$$

We shall establish, as before, the claim (7.36), from which the desired sharp L^2 and L^∞ bounds immediately follow, and thus, by interpolation, the claimed sharp bounds in all L^p such that $2 \leq p \leq \infty$.

Proof of claim. As in the previous proof, we first observe that the bound on $|U_x|_{L^\infty}$ follows by Sobolev embedding, provided that we can establish the claimed H^1 and H^2 bounds on U.

As before, we next show by energy estimates that the H^2 estimate follows provided we can establish the claimed bounds on $|U|_{H^1}$, $|\dot\delta|$, and $|U|_{L^\infty}$. However, we must now take a bit more care in this step, due to the presence of the higher-order nonlinearity term $N_1(U,U)_x$ in (7.14) in the general case. To avoid this term, we rewrite (7.14) more strategically, using the quadratic Leibnitz relation

$$A_2U_2 - A_1U_1 = A_2(U_2 - U_1) + (A_2 - A_1)U_1, \tag{7.65}$$

in the quasilinear form

$$U_t - \tilde A U_x - \tilde Q U = M_1(U)\bar U_x + (0, I_r)^t M_2(U)\bar U + \dot\delta(t)(\bar U_x + U_x), \tag{7.66}$$

where

$$\begin{aligned} \tilde A &:= (df, dg)^t(u,v)(x+\delta(t), t), \\ \tilde Q &:= (0, dq)^t(u,v)(x+\delta(t), t), \end{aligned} \tag{7.67}$$

and

$$\begin{aligned} M_1(U) = \mathcal{O}(|U|) &:= \tilde A(x,t) - A(x), \\ M_2(U) = \mathcal{O}(|U|) &:= \tilde Q(x,t) - Q(x), \end{aligned} \tag{7.68}$$

$A(x)$ and $Q(x)$ defined as usual as $(df,dg)^t(\bar u,\bar v)$ and $(0,dq)^t(\bar u,\bar v)$, or, differentiating twice:

(7.69)
$$U_{xxt}-(\tilde A U_x)_{xx}-(\tilde Q U)_{xx}=(M(U)\bar U_x)_{xx}+(0,I_r)^t(M_2(U)\bar U)_{xx}+\dot\delta(t)(\bar U_{xxx}+U_{xxx}).$$

Let $\tilde A^0$ denote the simultaneous symmetrizer of $\tilde A$, $\tilde Q$. Taking the L^2 inner product of $\tilde A^0(x)U_{xx}$ against (7.69), we obtain after rearrangement/integration by parts, and several applications of Young's inequality, the energy estimate

$$\langle \tilde A^0 U_{xx},U_{xx}\rangle_t\le \langle U_{xx},\tilde A^0\tilde Q U_{xx}\rangle+\varepsilon|U_{xx}|^2_{L^2}+\varepsilon^{-1}\mathcal{O}(|U|^2_{H^1}+|\dot\delta(t)|^2),$$

where $\varepsilon>0$ is as in (7.11). Here, we have freely used assumption (7.11) and the consequent higher order bounds $|\bar U''|\le C\varepsilon$, $|\bar U'''|\le C\varepsilon$, plus the bounds $|\bar U_x|_{L^p}$, $|\bar U_{xx}|_{L^p}$. $|\bar U_{xxx}|_{L^p}\le C$, and also used the fact that $(|U|_{L^\infty}+|U_x|_{L^\infty})\le C\zeta(t)$ is assumed to remain small, to bound the new term

$$\langle \tilde A^0_t U_{xx},U_{xx}\rangle=\mathcal{O}(|U_{xx}|_{L^2}|U_t|_{L^\infty})=\mathcal{O}(|U_{xx}|_{L^2}(|U|_{L^\infty}+|U_x|_{L^\infty}) \tag{7.70}$$

arising from the fact that $\tilde A^0$ is now time-dependent.

Similarly, taking the L^2 inner product of U_{xx} against $\tilde K$ times the singly differentiated equation

$$U_{xt}-(\tilde A U_x)_x-(\tilde Q U)_x=(M(U)\bar U_x)_x+(0,I_r)^t(M_2(U)\bar U)_x+\dot\delta(t)(\bar U_{xx}+U_{xx}), \tag{7.71}$$

where $\tilde K$ denotes the skew-symmetric matrix described in Lemma 7.4, associated with $\tilde A$, $\tilde Q$, and proceeding as before, we obtain the auxiliary energy estimate:

$$\frac12\langle \tilde U_{xx},KU_x\rangle_t\le -\langle U_{xx},\tilde K A U_{xx}\rangle+\varepsilon|U_{xx}|^2_{L^2}+\varepsilon^{-1}\mathcal{O}(|U|^2_{H^1}+|\dot\delta(t)|^2). \tag{7.72}$$

From here, we may proceed as in the previous case to obtain the desired bound on $|U|_{H^2}$, assuming the claimed bounds on $|U|_{H^1}$ and $|\dot\delta|$.

But, the bounds on $|U|_{H^1}$, $|\dot\delta|$, and $|\delta|$ follow exactly as in the previous case, once we observe that $\tilde G_x$ breaks up into the sum of a time-exponentially decaying delta-function term H_1 analogous to H and a kernel $\tilde G_1$ obeying the same bounds as $\tilde G$. This term therefore gives the same bounds as in the L^2 estimates of the previous case. The term involving H, on the other hand, may be estimated (see comment just beneath (7.75), below) as

$$\begin{aligned}
&\Big|\int_0^t\int_{-\infty}^{\infty}H(x,t-s;y)\mathcal{O}(|U_{xx}||U|+|U_x|^2)(y,s)dy\,ds\Big|_{L^2}\\
&\quad\le C\int_0^t e^{-\eta(t-s)}||(|U_{xx}|_{L^2}|U|_{L^\infty}+|U_x|_{L^2}|U_x|_{L^\infty})(s)ds\\
&\quad\le C\zeta(t)^2\int_0^t e^{-\eta(t-s)}(1+s)^{-1/4}ds\\
&\quad\le C\zeta(t)^2(1+t)^{-1/4},
\end{aligned}$$

again giving the desired result.

Finally the claimed bound on $|U|_{L^\infty}$ follows, for terms involving the regular kernel $\tilde G$, by the exactly the computations used to estimate $|\dot\delta|$, and for terms involving the singular kernel H, using the new bound $|U_x|_{L^\infty} \le C\zeta(t)$ coming from definition (7.64), and proceeding similarly as in the estimation of U_{H^1}. We omit the details, which are tedious but straightforward. ∎

7.2.2. Discrete kinetic models. Finally, we briefly indicate the extension of our arguments to general L^p, $1 \le p \le \infty$, and strong relaxation profiles, for discrete kinetic models with sufficiently regular coefficients $q \in C^3$, not necessarily simultaneously symmetrizable, and perturbations that are sufficiently small in $W^{3,1} \cap W^{3,\infty}$. That is, we establish the third, and last assertion of Theorem 1.13, completing the proof of the theorem.

The main idea in this subsection is to take advantage of the special property of discrete kinetic models that nonlinear terms, in the original, unshifted equations (1.1), appear without derivatives. Likewise, the basic, unshifted perturbation variable

$$\tilde U := (u,v)(x,t) - (\bar u, \bar v)(x) \tag{7.73}$$

satisfies an equation with this same property:

$$\tilde U_t - (A\tilde U)_x - Q\tilde U = (0, I_r)^t \tilde N_2(\tilde U, \tilde U), \tag{7.74}$$

where $\tilde N(\tilde U, \tilde U) = \mathcal{O}(|\tilde U|^2)$, and, more generally,

$$|(d/dx)^k \tilde N(\tilde U, \tilde U)| \le C \sum_{0 \le \alpha+\beta \le k:\, \alpha,\beta \ge 0} |(d/dx)^\alpha \tilde U (d/dx)^\beta \tilde U|, \tag{7.75}$$

$k = 0, \dots, 3$, provided $|\tilde U|_{W^{1,\infty}}$ remains bounded, as we shall assume it does throughout our argument. (Note: a straightforward Leibnitz calculation shows that, provided coefficients q are sufficiently regular, bound (7.75) extends to $k = 3K+2$, provided $|\tilde U|_{W^{K,\infty}}$ remains bounded).

Using this observation, and the previously observed fact that x-derivative bounds on the Green's function, modulo time-exponentially decaying singular terms (i.e., delta-functions and their derivatives), are essentially independent of the order of differentiation, we may use Green's function bounds instead of energy estimates to bound derivatives of the solution. That is, the integral equations for $\tilde U$ close, with no loss of derivatives, in the special case of discrete kinetic models. On the other hand, the variable $\tilde U$ is not expected to decay, since it includes a possible stationary component corresponding to translation of the original profile. Thus, the bounds we obtain by this technique are quite crude, namely

$$|U_{xxx}|_{L^p} \le C|\tilde U_{xxx}|_{L^p} \sim t, \tag{7.76}$$

where U as usual denotes the shifted perturbation variable $U(x,t) := (u,v)^t(x+\delta(t),t) - (\bar u, \bar v)^t(x)$. However, this is sufficient for our argument, since as we have seen in previous subsections, $|(d/dx)^k U|_{L^p}$ bounds improve as k decreases, by order $t^{-1/2}$ for each successive derivative. At level $|U_x|_{L^p}$, we therefore obtain a uniform $\mathcal{O}(1)$ bound, which, as in the proof of the previous case, is sufficient to establish sharp bounds in all L^p, $1\le p\le\infty$. Note that the resulting order-one $|U|_{W^{1,\infty}}$ bound, which implies a corresponding bound on $|\tilde U|_{W^{1,\infty}}$, is consistent with the assumption under which we obtained bound (7.75), an important point.

Remark 7.8. The equations for the more favorable (i.e., decaying) shifted perturbation U feature the nonlinear forcing term $\dot\delta(t)U_x$, which does involve a derivative. Thus, the integral equations for U do not close, involving a loss of one derivative, and so we cannot directly estimate $|U_{xxx}|_{L^p}$ by this approach.

We now describe the argument in somewhat further detail. Define

(7.77)
$$\zeta(t) := \sup_{0\le s\le t}\Big(|U(\cdot,s)|_{L^\infty}(1+s)^{1/2} + |U(\cdot,s)|_{L^1} + |\dot\delta(s)|(1+s)^{\frac12} + |\delta(s)| \\ + |U_x(\cdot,s)|_{L^1\cap L^\infty} + |U_{xx}(\cdot,s)|_{L^1\cap L^\infty}(1+s)^{-1/2} + |U_{xxx}(\cdot,s)|_{L^1\cap L^\infty}(1+s)^{-1}\Big).$$

As usual, our goal is to establish

$$\zeta(t)\le C_2(\zeta_0+\zeta(t)^2) \tag{7.78}$$

for all $t\ge 0$ for which a solution exists with $\zeta(t)$ sufficiently small, from which we may conclude as before both global existence and the bound

$$\zeta(t)\le C\zeta_0 \tag{7.79}$$

for ζ_0 sufficiently small. From (7.79) and the definition (7.77), we then immediately obtain the claimed sharp bounds in L^1 and L^p, and, by interpolation, sharp bounds for all $1\le p\le\infty$, completing the proof.

Observation 7.9. Assuming $q\in C^3$, (H0)–(H4), and $(\mathcal{D})$, there holds for any $0\le k\le 3$ the decomposition:

$$(d/dx)^k(G-E) = \tilde G_k + \sum_{j=0}^k H_j, \tag{7.80}$$

where E is as defined in (1.34); kernel $\tilde G_k$ is a measurable function in $C^1(y)$, satisfying the same bounds asserted for $\tilde G$ in Lemma 7.1; and

$$H_k(x,t;y) = \mu_k(x,t;y)(d/dy)^k \tag{7.81}$$

where $\mu_k(x,t;\cdot)$ are measures satisfying

$$|\mu_k(x,t;\cdot| = \mathcal{O}(e^{-\eta t}). \tag{7.82}$$

Proof. The regularity $q \in C^3$ yields $Q \in C^2$ for the coefficients of the linearized operator L, whence we obtain differentiability of order $C^3(x)$ (resp. $C^3(y)$) for solutions of the eigenvalue (resp. adjoint eigenvalue) ODE from which the resolvent kernel G_λ is constructed: more precisely, the functions $V(x;\lambda)$ (resp. $\tilde V(y;\lambda)$) from which modes $e^{\mu\lambda x}V(x;\lambda)$ (resp. $e^{-\mu\lambda x}\tilde V(x;\lambda)$) are constructed are uniformly bounded in $C^3(x)$ (resp. $C^3(y)$).

With this observation, the claimed bounds follow in straightforward fashion, by the same procedure used to estimate G, G_x, G_y in previous sections. Indeed, the proof is somewhat simpler, since we only require modulus bounds here, so need not carry out the detailed bookkeeping of previous sections. That bounds for the regular part $\tilde G_k$ do not degrade with increasing k is clear from the elementary calculation

$$(d/dx)e^{\mu\lambda x}V(x;\lambda) = \mu e^{\mu\lambda x}V(x;\lambda) + e^{\mu\lambda x}(d/dx)V(x;\lambda),$$

together with the fact that, modulo negligible error terms, bounds on the regular part come entirely from a bounded set of the complex plane, on which μ is bounded. Likewise, the singular terms H_k derive from large-λ bounds by explicit computation, similarly as in previous sections. ∎

We first apply Observation 7.9 in crude fashion to obtain the crucial bound $|U|_{W^{3,1}\cap W^{3,\infty}}$ that will be used to close our iteration; this step replaces the energy estimates of the previous subsections. Clearly, it is sufficient to bound instead $|\tilde U|_{W^{3,1}\cap W^{3,\infty}}$, $\tilde U$ defined as in (7.73) above, using

$$U(x-\delta(t),t) - \tilde U(x,t) = \bar U(x) - \bar U(x-\delta(t)) \sim \delta(t)\bar U(x) \tag{7.83}$$

and assuming the desired order one bound on $\delta(t)$.

By (7.74), Observation 7.9, and Duhamel's formula, we have the representation:
(7.84)

$$\begin{aligned}\tilde U_{xxx}(x,t) = &\int_{-\infty}^{\infty}\Big((E_{xxx}+\tilde G_3)(x,t;y)\tilde U_0(y) + \sum_{k=0}^{3} H_k(x,t;y)(d/dy)^k\tilde U_0(y)\Big)dy\\ &+\int_0^t\int_{-\infty}^{\infty}(E_{xxx}+\tilde G_3)(x,t-s;y)(0,I_r)^t\tilde N_2(\tilde U,\tilde U)(y,s)\,dy\,ds\\ &+\sum_{k=0}^{3}\int_0^t\int_{-\infty}^{\infty}H_k(x,t;y)(d/dy)^k(0,I_r)^t\tilde N_2(\tilde U,\tilde U)(y,s)\,dy\,ds,\end{aligned}$$

from which we readily obtain the desired bound using the estimates of Observation 7.9, together with lower derivative bounds

$$|(d/dx)^k\tilde U|_{L^p\cap L^1}(s) \le \begin{cases} C\zeta(t)(1+s)^{(k-1)/2} & 1\le k\le 2,\\ C\zeta(t) & k=0,\end{cases}$$

$0 \le s \le t$, which follow from the U bounds of definition (7.77) by an argument similar to that given in (7.83) above.

For example, we may bound the term

$$|\int_0^t \int_{-\infty}^{\infty} H_3(x,t;y)\mathcal{O}(|\tilde{U}||\tilde{U}_{xxx}|)(y,s)dy\,ds,|_{L^1\cap L^\infty}$$

arising in the expansion via (7.75) of the worst-case singular term

$$\int_0^t \int_{-\infty}^{\infty} H_3(x,t;y)(d/dy)^3(0,I_r)^t\tilde{N}_2(\tilde{U},\tilde{U})(y,s)dy\,ds, \tag{7.85}$$

by

$$\begin{aligned} C\int_0^t e^{-\eta(t-s)}|\tilde{U}|_{L^\infty}|\tilde{U}_{xxx}|_{L^1\cap L^\infty}(y,s)dy\,ds &\le C\zeta(t)^2\int_0^t e^{-\eta(t-s)}(1+s)ds\\ &\le C\zeta(t)^2(1+t), \end{aligned}$$

and similarly for all other terms arising in (7.85), since by homogeneity of our derivative estimates the bound on the nonlinear term depends only on the total number of derivatives.

Likewise, the worst-case regular term

$$|\int_0^t \int_{-\infty}^{\infty} E_{xxx}(x,t-s;y)(0,I_r)^t\tilde{N}_2(\tilde{U},\tilde{U})(y,s)dy\,ds|_{L^1\cap L^\infty} \tag{7.86}$$

may be bounded using the triangle inequality by

$$\begin{aligned} C\int_0^t |E_{xxx}|_{L^1\cap L^\infty}(t-s)|\tilde{N}|_{L1}(s)ds &\le C\zeta(t)^2\int_0^t ds\\ &\le C\zeta(t)^2(1+t), \end{aligned}$$

completing the verification of the claimed bound for $|U_{xxx}|_{L^p\cap L^\infty}$.

From this highest-derivative bound, the lower-derivative bounds on the shifted perturbation U may be obtained in routine fashion from the corresponding integral representations for U, using the bounds of Observation 7.9 and estimates precisely analogous to those of the previous subsections. We omit these lengthy but straightforward calculations. This completes the proof of the claim, and the Theorem ∎

Remark 7.10. In this subsection, we have made little attempt to minimize regularity requirements. It seems likely that the assumption $W^{3,1}\cap W^{3,\infty}$ on initial perturbation could be substantially reduced.

A. Appendices: Auxiliary Results/Calculations.

Appendix A1. Structure of Relaxation Profiles.

In this appendix, we prove the results cited in the introduction concerning structure of relaxation profiles.

Proof of Lemma 1.2. First, note that matrix (1.13) can have no nonzero purely imaginary eigenvalues $i\xi$, by (H3). Observing that

$$\begin{aligned}\det\,(q_u,q_v)\begin{pmatrix} f_u & f_v \\ g_u & g_v\end{pmatrix}^{-1}\begin{pmatrix}0\\1\end{pmatrix} &= \det\,\begin{pmatrix} f_u & f_v \\ q_u & q_v\end{pmatrix}\begin{pmatrix} f_u & f_v \\ g_u & g_v\end{pmatrix}^{-1}\\ &= \det\, df^* \det\, q_v \det\,\begin{pmatrix} f_u & f_v \\ g_u & g_v\end{pmatrix}^{-1}\end{aligned}$$

is nonzero, by (H1), (H2), and (1.2), we find that $g+$ is a hyperbolic rest point of (1.11), from which (1.14) follows. ∎

Proof of Lemma 1.3. In contrast to the analogous result of [MP] in the viscous case, which can be phrased as $i_\pm = d_\pm$, $n=r$, *independently* for each state $u_\pm$, the present result must be stated jointly, and in fact depends in an essential way on the fact that the two states are joined by an admissible connecting profile. Moreover, the proof emerges in a natural way from the investigation of stability properties through the study of the linearized eigenvalue ODE. A similar argument was sketched in Remark 2.3, [ZH], for the parabolic case.

Precisely, recall from Lemma 3.1, Section 3, that the $(n+r)$-dimensional eigenvalue ODE satisfies the *consistent splitting* hypothesis at spatial plus/minus infinity, as a consequence of (H3): that is, the dimensions of the stable/unstable subspaces of the "frozen," limiting coefficient matrices at plus and minus infinity have common values, for all frequencies Re $\lambda > 0$. On the other hand, by Lemma 5.1, Section 5, the eigenvalues of the frozen coefficient matrices at $\lambda = 0$ consist of r "fast" modes Re $\mu_j \neq 0$, corresponding to the eigenvalues of the reduced traveling wave ODE linearized at $(u_\pm, v_\pm)$, and n "slow" modes $\mu_j \equiv 0$ corresponding to the n constants of motion of the full traveling wave ODE. At plus infinity, the former contribute d_+ decaying values, while the latter bifurcate as Re λ increases into i_+ growing modes and $(n-i_+)$ decaying modes, accounting for a total of $d_+ + (n-i_+)$ negative real part eigenvalues of the coefficient matrix for small λ with Re $\lambda > 0$. Likewise, at minus infinity, we find that there are $d_- + (n-i_-)$ positive real part eigenvalues for small λ with Re $\lambda > 0$. Invoking consistent splitting, we have that $d_+ + (n-i_+) + d_- + (n-i_-) = n + r$, yielding the result. ∎

Remark. In the strictly parabolic case, the argument above yields instead that $d_\pm + (n - i_\pm) = n$, or $d_\pm = i_\pm$, the result of Majda and Pego [MP].

Appendix A2. Expansion of the Fourier Symbol.

In this appendix, we carry out the Taylor expansions about $\xi = 0$ and $\xi = \infty$ of the Fourier symbol $-i\xi A(x) + Q(x)$ of the frozen, constant-coefficient operator at a given x.

High frequency expansion. By straightforward matrix perturbation theory [Kat], the first-order expansion at infinity of $P(i\xi)$ with respect to $(i\xi)^{-1}$ is just

$$P = -i\xi A + R\,\mathrm{diag}\,\{\eta_1, \dots, \eta_J\}L,$$

$$\mathrm{diag}\,\{\eta_1, \dots, \eta_J\} := LQR,$$

where L, R are composed of left, right eigenvectors of A, as described in the introduction.

Low frequency expansion. We next carry out the expansion of P in ξ about zero. Recall that

$$Q := \begin{pmatrix} 0 & 0 \\ q_u & q_v \end{pmatrix}, \quad A := \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}. \tag{8.1}$$

Claim. To second order, "slow" dispersion relations

$$\lambda(i\xi) = \sigma(Q - i\xi A), \quad \lambda(0) = 0 \tag{8.2}$$

are given by

$$\lambda_j(\xi) = -i\xi a_j^* - \beta_j^* \xi^2 + \dots, \quad j = 1, \cdots n, \tag{8.3}$$

with corresponding eigenvectors

$$R_j^*(i\xi) = \begin{pmatrix} r_j^* \\ -q_v^{-1} q_u r_j^* \end{pmatrix} + \dots,$$

and

$$\beta_j^* := l_j^* B^* r_j^*, \tag{8.4}$$

$$B^* := f_v \Big[g_u - g_v q_v^{-1} q_u + f_u^* q_v^{-1} q_u \Big], \tag{8.5}$$

where a_j^*, r_j^*, l_j^* are eigenvalues and associated right and left eigenvectors of $A^* := f_u^*$. The remaining r "fast", or strictly stable roots are $\sim \sigma(q_v)$.

Remark. This expansion may be viewed as a rigorous justification of the Chapman–Enskog expansion in the constant-coefficient case.

Proof. Setting

$$(Q - iA - (-i\alpha_j - \beta_j \xi^2)\cdots)(V_0 + V_1\xi + V_2\xi^2 + \cdots) = 0,$$

and collecting terms of successive orders in ξ, we obtain:

(Order 1):

$$QV_0 = 0 \qquad \Rightarrow V_0 = \begin{pmatrix} r_j \\ -q_v^{-1} q_u r_j \end{pmatrix} \tag{8.6}$$

(Order ξ):

$$QV_1 - i(A - \alpha_j)V_0 = 0, \tag{8.7}$$

$\Rightarrow$ (looking at first coordinate)

$$(A_* - \alpha_j)r_j = 0, \qquad A^* := f_u^* = f_u - f_v q_v^{-1} q_u \tag{8.8}$$

$\Rightarrow$ α_j, r_j are eigenvalue a_j^*, right eigenvector r_j^* of A^*.

Further, comparing with the second coordinate in (8.7), we have

$$\frac{1}{\tau}(q_u, q_v)V_1 = i(g_u, g_v - a_j^*)V_0, \tag{8.9}$$

giving

$$V_1 = \begin{pmatrix} s \\ q_v^{-1}(-q_u s + \tau i \Big[g_u - (g_v - a_j^*) q_v^{-1} q_u \Big] r_j) \end{pmatrix}. \tag{8.10}$$

(Order ξ^2):

$$QV_2 - i(A - \alpha_j)V_1 + \beta_j V_0 = 0.$$

Examining the first coordinate, we have

$$\begin{aligned} \beta_j r_j &= i(f_u - a_j^*, f_v)V_1 \\ &= i(A^* - a_j^*)s - \tau f_v q_v^{-1}[g_u - (g_v - a_j^*)q_v^{-1}q_u]r_j^*. \end{aligned} \tag{8.11}$$

Taking inner product with l_j^*, left e–vector of A_*, we obtain

$$\begin{aligned} \beta_j^* &= -\tau l_j^* f_v q_v^{-1}[g_u - (g_v - a_j^*)q_v^{-1}q_u]r_j^* \\ &= -\tau l_j^* f_v q_v^{-1}[g_u - g_v q_v^{-1} q_u f_u^*]r_j^*, \end{aligned} \tag{8.12}$$

completing the result. ∎

Remark. The "strict asymptotic parabolicity" condition

$$\mathrm{Re}\ \beta_j^* > 0, \quad j = 1, \ldots, n, \tag{8.13}$$

reduces in the case $n = r = 1$ treated by Liu [L.2] to the classical subcharacteristic condition [Wh]; however, these are not equivalent. As discussed in [Yo.4], the relation between the subcharacteristic condition and stability is quite analogous to that between the CFL condition and stability in numerical analysis. In the symmetrizable case, condition (8.13) is equivalent (see [Kaw,Ze.2]) to $Q \leq 0$ and no vector in $ker(Q)$ is an eigenvector of A.

Appendix A3. The Gap and Tracking Lemmas.

For completeness, we present here two technical lemmas referred to throughout the text, relating behavior of variable- and constant-coefficient ODE in, respectively, the *asymptotically constant* and the *slowly varying* coefficient case. The resulting bounds are crucial in our study of solutions of the generalized eigenvalue equations, in the former case on bounded domains in frequency space λ, and in the latter as $|\lambda| \to \infty$.

A3.1. The Gap Lemma.

A common problem arising in the asymptotic study of eigenvalue ODE is to relate behavior near $x = \pm\infty$ of solutions of an asymptotically constant-coefficient eigenvalue equation to that of solutions of the corresponding limiting, constant-coefficient equations, in a manner that is smooth with respect to spectral parameters. More generally, consider a general ODE with parameter

$$W' = \mathbb{A}(\lambda, x)W, \tag{8.14}$$

"$'$" denoting (d/dx), where the coefficient $\mathbb{A}$ is C^0 in the evolution variable x and analytic (resp. C^r) in parameter λ, and converges as $x \to \pm\infty$ to limiting values $\mathbb{A}_\pm$. It is well known (see [Co], Thm. 4, p. 94) that, provided that

$$\int_0^{\pm\infty} |\mathbb{A} - \mathbb{A}_\pm| dx < +\infty, \tag{8.15}$$

there is a one-to-one correspondence between the normal modes $V_j^\pm e^{\mu_j^\pm x}$ of the asymptotic systems

$$W' = \mathbb{A}_\pm(\lambda)W, \tag{8.16}$$

where $V_j^\pm$, $\mu_j^\pm$ are eigenvector and eigenvalue of $\mathbb{A}_-$ (alternatively, $V_j^\pm x^\ell e^{\mu_j^\pm x}$, if $V_j^\pm$ is a generalized eigenvector of order ℓ) and certain solutions $W_j^\pm$ of (8.14) having the same asymptotic behavior, i.e.,

$$W_j^\pm(\lambda, x) = V_j^\pm e^{\mu_j^\pm x}(1 + o(1)) \text{ as } x \to \pm\infty \tag{8.17}$$

(alternatively, $W_j^\pm(\lambda, x) = V_j^\pm x^\ell e^{\mu_j^\pm x}(1 + o(1))$. That is, the flows near $\pm\infty$ of (8.14) and (8.16) are homeomorphic.

Such a correspondence is of course highly nonunique, since (8.17) determines $W_j^\pm$ only up to faster decaying modes. However, provided that Re $(\mu_j^\pm)$ is *strictly separated* from all other Re $(\mu_k^\pm)$, i.e., that there is a *spectral gap*, the choice defined in [Co], Theorem 4 by fixed point iteration is in fact analytic (resp. C^r) in λ, as the uniform limit of an analytic (resp. C^r) sequence of iterates. The argument breaks down at points λ_0 where Re (μ_j) = Re (μ_k) for some $k \neq j$, since in this case (Re (μ_j) − Re (μ_k)) does not have a definite sign, and the definition of the fixed point iteration is determined by the signs of all (Re (μ_j) − Re (μ_k)).

The purpose of the present section is to point out that analyticity (resp. smoothness) in λ can be recovered in the absence of a spectral gap, by virtually the same argument as in [Co] if we substitute for (8.15) the stronger hypothesis:

$$|\mathbb{A} - \mathbb{A}_\pm| = \mathcal{O}(e^{-\alpha|x|}) \text{ as } x \to \pm\infty. \tag{8.18}$$

This observation is a special case of the "Gap Lemma of [GZ], also proved independently in [KS]. The original version was phrased in terms of the projectivized flow associated with (8.14). Here, we give an alternative statement and derivation directly in terms of (8.14), a form more convenient for our needs.

Proposition 8.1. *In (8.14), let $\mathbb{A}$ be C^0 in x and analytic (resp. C^r) in λ, with $|\mathbb{A} - \mathbb{A}_-(\lambda)| = \mathcal{O}(e^{-\alpha|x|})$ as $x \to -\infty$ for $\alpha > 0$, and $\bar\alpha < \alpha$. If $V^-(\lambda)$ is an eigenvector of $\mathbb{A}_-$ with eigenvalue $\mu(\lambda)$, both analytic (resp. C^r) in λ, then there exists a solution $W(\lambda, x)$ of (8.14) of form*

$$W(\lambda, x) = V(x; \lambda)e^{\mu(\lambda)x},$$

where V (hence W) is C^1 in x and locally analytic (resp. C^r) in λ and for each $j = 0, 1, \dots$ satisfies

$$(\partial/\partial\lambda)^j V(x; \lambda) = (\partial/\partial\lambda)^j V^-(\lambda) + \mathcal{O}(e^{-\bar\alpha|x|}|V^-(\lambda)|), \quad x < 0, \tag{8.19}$$

Moreover, if Re $\mu(\lambda) >$ Re $\widetilde{\mu}(\lambda) - \alpha$ for all (other) eigenvalues $\widetilde{\mu}$ of $\mathbb{A}_-$, then W is uniquely determined by (8.19), and (8.19) holds for $\bar\alpha = \alpha$.

Proof. Setting $W(x) = e^{\mu x}V(x)$, we can rewrite $W' = \mathbb{A}W$ as

$$\begin{aligned} V' &= (\mathbb{A}_- - \mu I)V + \theta V, \\ \theta &:= (\mathbb{A} - \mathbb{A}_-) = \mathcal{O}(e^{-\alpha|x|}), \end{aligned} \tag{8.20}$$

and seek a solution $V(x; \lambda) \to V^-(x)$ as $x \to \infty$.

Set

$$\bar\alpha = \bar\alpha_1 < \alpha_1 < \alpha_2 < \bar\alpha_2 < \alpha. \tag{8.21}$$

Fixing a base point λ_0, we can define on some neighborhood of λ_0 to the complementary $\mathbb{A}_-$-invariant projections $P(\lambda)$ and $Q(\lambda)$ where P projects onto the direct sum of all eigenspaces of $\mathbb{A}_-$ with eigenvalues $\widetilde{\mu}$ satisfying

$$\text{Re } (\widetilde{\mu}) < \text{Re } (\mu) + \alpha_2, \tag{8.22}$$

and Q projects onto the direct sum of the remaining eigenspaces, with eigenvalues satisfying

$$\text{Re }(\tilde{\mu}) \geq \text{Re }(\mu) + \alpha_2 > \text{Re }(\mu) + \alpha_1. \tag{8.23}$$

By basic matrix perturbation theory (eg. [Kat]) it follows that P and Q are analytic (resp. C^r) in a neighborhood of λ_0, with

$$\begin{aligned} \left| e^{(\mathbb{A}_- - \mu I)x} P \right| &= \mathcal{O}(e^{\bar{\alpha}_2 x}), \quad x > 0, \\ \left| e^{(\mathbb{A}_- - \mu I)x} Q \right| &= \mathcal{O}(e^{\bar{\alpha}_1 x}), \quad x < 0. \end{aligned} \tag{8.24}$$

Thus, for $M > 0$ sufficiently large, the map $\mathcal{T}$ defined by

$$\mathcal{T}V(x) = V^- + \int_{-\infty}^x e^{(\mathbb{A}_- - \mu I)(x-y)} P\theta(y)V(y)dy - \int_x^{-M} e^{(\mathbb{A}_- - \mu I)(x-y)} Q\theta(y)V(y)dy \tag{8.25}$$

is a contraction on $L^\infty(-\infty, -M]$; for, applying (8.24), we have

$$\begin{aligned} |\mathcal{T}V_1 - \mathcal{T}V_2|_{(x)} &\leq \mathcal{O}(1)|V_1 - V_2|_\infty \left(\int_{-\infty}^x e^{\bar{\alpha}_2(x-y)} e^{\alpha y} dy + \int_x^{-M} e^{\bar{\alpha}_1(x-y)} e^{\alpha y} dy \right) \\ &= \mathcal{O}(1)|V_1 - V_2|_\infty \left(e^{\bar{\alpha}_2 x} e^{(\alpha - \bar{\alpha}_2)y}|_{-\infty}^x + e^{\bar{\alpha}_1 x} e^{(\alpha - \bar{\alpha}_1)y}|_x^{-M} \right) \\ &= \mathcal{O}(1)|V_1 - V_2|_\infty (e^{\alpha x} + e^{\bar{\alpha} x}) \\ &= \mathcal{O}(1)|V_1 - V_2|_\infty e^{-\bar{\alpha} M} < \frac{1}{2}. \end{aligned} \tag{8.26}$$

By iteration, we thus obtain a solution $V \in L^\infty(-\infty, -M]$ of $V = \mathcal{T}V$ with $V = \mathcal{O}(|V^-|)$; since $\mathcal{T}$ clearly preserves analyticity (resp. smoothness), $V(\lambda, x)$ is analytic (resp. C^r) in λ as the uniform limit of analytic (resp. smooth) iterates (starting with $V_0 = 0$). Differentiation shows that V is a bounded solution of $V = \mathcal{T}V$ iff it is a bounded solution of (8.20). Further, taking $V_1 = V$, $V_2 = 0$ in (8.26), we obtain from the second to last equality that

$$|V - V^-| = |\mathcal{T}(V) - \mathcal{T}(0)| = \mathcal{O}(1)e^{\bar{\alpha}x}|V| = \mathcal{O}(e^{\bar{\alpha}x})|V^-|, \tag{8.27}$$

giving (8.19) for $j = 0$. Derivative bounds, $j > 0$, follow by standard interior estimates, or, alternatively, by differentiating (8.25) with respect to λ and repeating the same argument. Analyticity (resp. smoothness), and the bounds (8.19), extend to $x < 0$ by standard analytic dependence for the initial value problem at $x = -M$.

Finally, if Re $(\mu(\lambda)) >$ Re $(\tilde{\mu}(\lambda)) - \frac{\alpha}{2}$ for all other eigenvalues, then $P = I$, $Q = 0$, and $V = \mathcal{T}V$ must hold for any V satisfying (8.19), by Duhamel's principle.

Further, the only term appearing in (8.26) is the first integral, giving bound (8.27) for $\bar\alpha = \alpha$. ∎

Proposition 8.1 extends also to subspaces of solutions. This can be seen most easily by associating to a k-plane of solutions, Span $\{W_1(x), \dots, W_k(x)\}$, the corresponding k-form $\eta = W_1 \wedge \cdots \wedge W_k$. The equations (8.14) induce a linear flow

$$\eta' = \mathbb{A}^{(k)}(x;\lambda)\eta, \tag{8.28}$$

on the space of k-forms via the Leibnitz rule,

$$\mathbb{A}^{(k)}(W_1 \wedge \cdots \wedge W_k) = (\mathbb{A}W_1 \wedge \cdots \wedge W_k) + \cdots + (W_1 \wedge \cdots \wedge \mathbb{A}W_k). \tag{8.29}$$

The evolution of the k-plane of solutions of (8.14) is clearly determined by that of $\eta(x;\lambda)$. It is easily seen that for a given (constant) matrix $\mathbb{A}$, the eigenvectors of $\mathbb{A}^{(k)}$ are of form $V_1 \wedge \cdots \wedge V_k$, where Span $\{V_1, \cdots, V_k\}$ is an invariant subspace of $\mathbb{A}$, and that the corresponding eigenvalue is the trace of $\mathbb{A}$ on that subspace.

Definition 8.2. *Let $C =$ Span $\{V_{k+1}, \dots, V_N\}$ and $E =$ Span $\{V_1, \dots, V_k\}$ be complementary $\mathbb{A}$-invariant subspaces. We define their* **spectral gap** *to be the difference β between the real part of the eigenvalue of minimal real part of $\mathbb{A}$ restricted to C and the real part of the eigenvalue of maximal real part of $\mathbb{A}$ restricted to E.*

If η is a k-form associated with an A-invariant subspace E as in the definition above, then the spectral gap β is the minimum difference between the real part of the eigenvalue μ of $\mathbb{A}^{(k)}$ associated with η and the real part of the eigenvalue associated with any other eigenvector of $\mathbb{A}^{(k)}$. Combining these observations with the result of Proposition 8.1, we obtain a complete version of the Gap Lemma of [GZ]:

Corollary 8.3 (The Gap Lemma). *Let $\mathbb{A}(x;\lambda)$ be C^0 in x, analytic (resp. C^r) in λ, with $\mathbb{A}(x;\lambda) \to \mathbb{A}_\pm(\lambda)$ as $x \to \pm\infty$ at exponential rate $e^{-\alpha|x|}$, $\alpha > 0$, and let $\eta^-(\lambda)$ and ζ^- be analytic (resp. C^r) k and $n-k$-forms associated to complementary $\mathbb{A}_-(\lambda)$-invariant subspaces C^- and E^- as in Definition 3.2, with arbitrary spectral gap β, and let $\tau_{C^-}(\lambda)$ be analytic, where $\tau_{C^-}(\lambda)$ is the trace of $\mathbb{A}_-(\lambda)$ restricted to C^-. Then, there exists a solution $\mathcal{W}(\lambda, x)$ of (8.28) of form*

$$\mathcal{W}(\lambda, x) = \eta(\lambda, x)e^{\tau_{C^-}x}$$

where η (hence $\mathcal{W}$) is C^1 in x locally analytic (resp. C^r) in λ, and for each $j = 0, 1, \dots$ satisfies

$$(\partial/\partial\lambda)^j\eta(x;\lambda) = (\partial/\partial\lambda)^j\eta^-(\lambda) + \mathcal{O}(e^{-\bar\alpha|x|}|\eta^-(\lambda)|), \quad x < 0, \tag{8.30}$$

for all $\bar\alpha < \alpha$. Moreover, if $\beta > -\alpha$, then η is uniquely determined by (8.30) and (8.30) holds for $\bar\alpha = \alpha$.

Remark 8.4. Note that the construction of Corollary 8.3 determines at the same time analytic subspaces $C(x;\lambda)$, $E(x;\lambda)$ asymptotic to C^-, E^-, and thereby corresponding complementary analytic projections $P_C(x;\lambda)$, $P_E(x;\lambda)$. Fixing x_0 and λ_0, and applying projections $P_C(x_0, \lambda)$, $P_E(x_0, \lambda)$ to fixed bases of $C(x_0, \lambda_0)$, $E(x_0, \lambda_0)$, we may thus obtain a *locally analytic choice* of individual solutions $W_j^-(x; \lambda)$ spanning $C(x; \lambda)$, $E(x; \lambda)$, in a neighborhood of any λ_0. Moreover, if the eigenvalues associated with C^- have real parts lying strictly between η_1 and η_2, then the standard homeomorphism between solutions of limiting, constant coefficient equations and their variable-coefficient counterparts (described, e.g. in Section 3) yields that the flow on $C(\lambda, \cdot)$ decays/grows exponentially with $|x|$ on $x \le 0$, with rate uniformly bounded between η_1 and η_2. Alternatively, such an analytic choice may be obtained in more straightforward fashion by a standard "stable-manifold type" construction, as described in [Z.1], from which exponential decay/growth may be deduced at the same time.

Remark 8.5. Bounds (8.19), (8.30) assert no information about the direction of $V(\cdot; \lambda)$, $\eta(\cdot; \lambda)$ at $x = 0$. A review of our fixed-point construction reveals that in general no such information is available, i.e., $\mathcal{T}$ is a contraction only on $L^\infty(-\infty, -M]$, for $M > 0$ sufficiently large, and not on $L^\infty(-\infty, 0]$. Specifically, the second integral in definition (8.25) becomes for $x > -M$ an expansive term with expansion coefficient growing exponentially in $x + M$ with rate at least $\bar{\alpha}_1$ and possibly arbitrarily larger; at $x = 0$, this becomes at least order $e^{\bar{\alpha}_1 M}$, and typically greater than one.

In particular, we obtain from the bounds of the Gap Lemma no lower bound on the Evans function $|D(\lambda)|$, i.e., no information on existence or nonexistence of point spectrum at λ. Indeed, this could not be so, since we know a priori in many cases that $D(0) = 0$ even though the Gap Lemma estimates are uniformly valid up to $\lambda = 0$. To obtain lower bounds on the Evans function, for example in the high-frequency limit, we will use instead an alternative construction described in the following subsection, based on the *projectivized flow* of the eigenvalue ODE. By factoring out variations in modulus, this allows more accurate estimates of direction; in particular, the mapping analogous to $\mathcal{T}$ in projectivized coordinates ((8.54) below) will be seen to be a *global* L^∞-contraction on the whole real line.[8]

A3.2. The Tracking Lemma.

Another general situation that arises in the asymptotic study of eigenvalue equations is an ODE

$$w' = (\mathbb{A}(x, \delta) + \delta\Theta(x, \delta))w, \quad w \in \mathbb{C}^N, \tag{8.31}$$

[8]In this regard, we point out a key error in the proof of Lemma 5 of [R], specifically of lower bound (20) asserted on the Evans function (bounds (18)–(19) clearly extend from x_0 to $x = 0$ by Abel's formula). Namely, the claimed bounds on operator N of (34), following directly from contractivity of the mapping T_λ (our $\mathcal{T}$, above) are valid only on the interval $[x_0, \infty)$ where T_λ is a contraction, and not all of $[0, +\infty)$ as claimed. The error results from an incorrect citation of the Gap Lemma construction of [GZ,ZH], stating that mapping T_λ itself is a contraction on $[0, \infty)$). As discussed in Remark 8.5 above, such information cannot be obtained by the Gap Lemma construction, but rather requires the projectivized version described below (indeed, the required lower bound in [R] is a trivial consequence of the machinery developed here, in Section 4 and Appendix A3.2 below).

with a small parameter $\delta \to 0$, satisfying

$$|\mathbb{A}| + |\Theta| \le C \tag{8.32}$$

and

$$|\mathbb{A}'| \le C\delta, \tag{8.33}$$

where "$'$" denotes differentiation with respect to x: that is, an ODE with slowly varying coefficients. This situation arises in the limit as the frequency rather than the spatial variable goes to infinity, after rescaling to a length scale on which the resulting rapid oscillations in the solution have period of order one with $\delta \to 0$ as $|\text{frequency}| \to \infty$. Thus, it is dual to the previous case.

A3.2.1. The basic estimate. To introduce ideas, we first present an analysis of the "standard" case treated in [GZ,ZH,Z.1,Z.4]. Suppose in addition to (8.32)–(8.33) that the spectrum of the matrix $\mathbb{A}(x,\delta)$ divides into two spectral groups,

$$\alpha_1(x,\delta),\dots\alpha_\ell(x,\delta) \le \underline{\alpha}(x) < \bar{\alpha}(x) \le \alpha_{\ell+1}(x,\delta),\dots\alpha_N(x,\delta), \tag{8.34}$$

where $<$ denotes ordering with respect to real parts, with a *uniform spectral gap*

$$0 < 2\eta \le \bar{\alpha}(x) - \underline{\alpha}(x); \tag{8.35}$$

in the setting of asymptotic eigenvalue equations, this is achieved by restricting to an appropriately small subset of the region of consistent splitting, or normal set, of the operator L under investigation. By standard matrix perturbation theory [Kat], therefore, we have $\mathbb{A}(\cdot)$-invariant projections $P(x)$ and $Q(x)$ onto the eigenspaces associated with $(\alpha_1,\dots,\alpha_\ell)$ and $(\alpha_{\ell+1},\dots\alpha_v)$, respectively, satisfying

$$|P'| + |Q'| \le C_2\delta,\ |P|, |Q| \le C; \tag{8.36}$$

we assume that these bounds hold uniformly over all $-\infty < x < +\infty$ (this can be guaranteed, for example, by the assumption that $\mathbb{A}$ varies within a compact set, as is the case in the applications we have in mind).

Under these general assumptions, we will show that the "stable"/"unstable" manifolds of solutions of (8.31), decaying at $+\infty/-\infty$ with rates $\sim e^{\underline{\alpha}x}/e^{\bar{\alpha}x}$, respectively, approximately *track* the corresponding subspaces of the principal coefficient matrix $\mathbb{A}(x,\delta)$ as they vary with x, lying always within angle $\mathcal{O}(\delta/\eta)$ and decreasing/increasing with uniform rate $\sim e^{\underline{\tilde{\alpha}}x}/e^{\bar{\tilde{\alpha}}x}$ for any $\underline{\tilde{\alpha}} > \max_x \underline{\alpha}$, $\bar{\tilde{\alpha}} < \min_x \bar{\alpha}$.

Remark 8.6. The most common applications of this result concern the situation that

$$\underline{\alpha}(x) \equiv \underline{\alpha} < 0 < \bar{\alpha} \equiv \bar{\alpha}(x),$$

in which case the manifolds described are truly stable/unstable and $\bar{\alpha} = \min_x \bar{\alpha}$, $\underline{\alpha} = \max_x \underline{\alpha}$. However, neither our results nor their useful application are limited to this case.

We will follow an approach based on energy estimates/invariant cones, though other approaches are certainly possible; see [AGJ,GZ], for example, for a revealing alternative formulation of Proposition 8.22 (just below) in terms of projectivized differential forms. For further discussion, we refer the reader to [ZH], Section 7.

Proposition 8.7. *For C sufficiently large, δ/η sufficiently small, the cone*

$$\mathbb{K}_-(x) := \{w : \frac{|P(x)w|}{|Q(x)w|} \le C\delta/\eta\}$$

is **positively invariant** *under the flow of (8.31) (i.e., invariant in forward time $x \nearrow$), and* **exponentially attracting** *on $\mathbb{J}_-(x) := \{w : \frac{|P(x)w|}{|Q(x)w|} \le \eta/C\delta\}$. Here, C depends only on the bounds (8.32)–(8.33) and (8.36) and η as defined in (8.35) measures spectral gap.*

The motivation behind Proposition 8.7 is clear: the P-component of w is in some sense growing exponentially slower than the Q-component, by the amount of the spectral gap. To quantify this observation, we use the following standard linear algebraic result:

Lemma 8.8. *For matrices M s.t. $(M) \le C$ and* Re $\delta(M) \ge \eta > 0$, *and for any $0 < \tilde\eta < \eta$, there exists a C^∞ choice of coordinate transformation $S(M)$ such that $\widetilde{M} := SMS^{-1}$ is* **real positive definite**, *satisfying*

$$\text{Re }(\widetilde{M}) := \frac{1}{2}(\widetilde{M} + \widetilde{M}^*) \ge \tilde\eta > 0.$$

Proof. See, e.g., Proposition A.9, p. 361 of [St].∎

It is worthwhile to consider the import of the Lemma in the context of a constant coefficient ODE

$$w' = Mw, \tag{8.37}$$

namely, the existence of a coordinate change $\tilde w = Sw$ for which the flow of (8.37), now of form

$$\tilde w' = \widetilde{M}\tilde w, \tag{8.38}$$

strictly increases $|\tilde w|$. For,

$$\begin{aligned}(\frac{1}{2}|\tilde w|^2)' = \frac{1}{2}(\langle \tilde w, \widetilde{M}\tilde w\rangle + \langle M\tilde w, \tilde w\rangle) &= \langle \tilde w, \text{Re }\widetilde{M}\tilde w\rangle \\ &\ge \frac{\eta}{2}|\tilde w|^2.\end{aligned} \tag{8.39}$$

Related topics are Lyapunov theory and the Kreiss matrix theorem.

Proof of Proposition 8.7. By rescaling if necessary by $y = e^{-1/2(\underline{\alpha}+\bar\alpha)x}W$, we can reduce without loss of generality to the symmetric case

$$\alpha_1, \ldots, \alpha_\ell \le -\eta < 0 < \eta \le \alpha_{\ell+1}, \ldots, \alpha_N. \tag{8.40}$$

By (8.33), there exists a change of coordinates $w \to S(x)w$ reducing (8.31) further to the case that $\mathbb{A}(x,\delta)$ has block diagonal form,

$$\mathbb{A}(x,\delta) = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}. \tag{8.41}$$

For, by standard matrix theory [Kat], (8.36) implies the existence of bases spanning the left and right eigenspaces associated with projections P, Q, and depending smoothly on P, Q; it follows that these bases, and thus the corresponding diagonalizing transformation S have derivatives of order δ, whence error term $S^{-1}S'w$ may be absorbed in Θ. Finally, using Lemma 8.8, we can reduce by a further coordinate change $w \to Tw$ to the case that

$$\text{Re } M_1 < -\widetilde{\eta}, \quad \text{Re } M_2 > \widetilde{\eta}. \tag{8.42}$$

Expressing (8.31) coordinate-wise, we have

$$w_j' = M_j w_j + \delta\Theta_j w, \tag{8.43}$$

where $|\Theta_j| \le C$, from which we obtain the growth/decay estimates

$$\begin{aligned} |w_j|' &= \text{Re } \langle w_j/|w_j|, w_j'\rangle \\ &= \langle w_j/|w_j|, \text{Re } (M_j)w_j\rangle + \text{Re } \langle w_j/|w_j|, \delta\Theta_j w\rangle \\ &\gtrless \mp\widetilde{\eta}|w_j| + C\delta|w|. \end{aligned} \tag{8.44}$$

Defining $r := \frac{|w_1|}{|w_2|}$, we thus find, after some simplification, that

$$\begin{aligned} r' &= \frac{|w_2|'|w_1| - |w_2||w_1|'}{|w_2|^2} \\ &< \frac{-2\widetilde{\eta}|w_2||w_1| + C\delta|w|^2}{|w_2|^2}. \\ &= -2\widetilde{\eta}r + C\delta(1+r^2). \end{aligned} \tag{8.45}$$

It is thus clear that $\mathbb{K}_- := \{w : r \le \frac{\widetilde{C}\delta}{4\widetilde{\eta}}\}$ is invariant, provided δ is small enough that $\frac{C\delta}{4\widetilde{\eta}} \le 1$. More generally, we have, for $\frac{C\delta}{4\widetilde{\eta}} \le r \le \frac{\widetilde{\eta}}{4C\delta}$, that $r' \le -\widetilde{\eta}r$. Thus, $\mathbb{K}_-$ is exponentially attracting on $\mathbb{J}_- := \{w : r \le \frac{\widetilde{\eta}}{4C\delta}\}$. ∎

Corollary 8.9. *For C sufficiently large, δ/η sufficiently small, solutions in*

$$\mathbb{J}_-(x) := \{w : \frac{|P(x)w|}{|Q(x)w|} \le \eta/C\delta\}$$

increase as $x \to +\infty$ at exponential rate $e^{\tilde\alpha x}$, for any $\tilde\alpha < \liminf_{x\to+\infty} \bar\alpha$ ($\bar\alpha$ as defined in (8.34)).

Proof. By Proposition 8.7, solutions in $\mathbb{J}_-(x)$ eventually enter cone

$$\mathbb{K}_- := \{w : \frac{|P(x)w|}{|Q(x)w|} \le C\delta/\eta\}.$$

Thus, (8.44), translated back to original coordinates, gives

$$\begin{aligned} |w_2|' &> \tilde{\tilde{\alpha}}\, |w_2| - (C^2\delta^2/\eta)|w_2| \\ &\ge \tilde\alpha |w_2|, \end{aligned} \tag{8.46}$$

for any $\tilde{\tilde{\alpha}} < \alpha$, $\tilde\alpha < \tilde{\tilde{\alpha}} - (C^2\delta^2/\eta)$, from which the result follows. ∎

Corollary 8.10 (The Tracking Lemma). *For δ/η sufficiently small, solutions w^+/w^- of (8.31) decaying at $+\infty/-\infty$ at rate $e^{\underline{\tilde\alpha} x}/e^{\tilde{\bar\alpha} x}$ lie always within cones*

$$\mathbb{K}_-(x) := \{w : \frac{|P(x)w|}{|Q(x)w|} \le C\delta/\eta\}, \tag{8.47}$$

$$\mathbb{K}_+(x) := \{w : \frac{|Q(x)w|}{|P(x)w|} \le C\delta/\eta\}, \tag{8.48}$$

respectively, for any $\underline{\tilde\alpha} < \liminf_{x\to+\infty} \bar\alpha$, $\tilde{\bar\alpha} > \limsup_{x\to-\infty} \underline{\alpha}$. Moreover, there hold the uniform decay/growth rates

$$\begin{aligned} \frac{|w^+(x)|}{|\bar w^+(y)|} &\le Ce^{\underline{\tilde\alpha}|x-y|}, \\ \frac{|w^-(x)|}{|\bar w^-(y)|} &\ge C^{-1}e^{\tilde{\bar\alpha}|x-y|}, \end{aligned} \tag{8.49}$$

for all $x > y$, and symmetrically for $x < y$, for any $\underline{\tilde\alpha} > \max_x \underline{\alpha}$, $\tilde{\bar\alpha} < \min_x \bar\alpha$. (Note: C depends in part upon the choice of $\underline{\tilde\alpha}$ and $\tilde{\bar\alpha}$).

Proof. Without loss of generality, consider the $+\infty$ case. Bounds (8.47)–(8.48) follow immediately by contradiction from the previous Corollary, whence (8.49) follows from (8.46). ∎

Remark 8.11. In the "standard" case that $\underline{\alpha} < 0 < \bar\alpha$, $\underline{\alpha}$ and $\bar\alpha$ constant, the statement of Corollary 8.10 considerably simplifies; in particular, the conclusion applies to solutions merely *decaying* at $+\infty/-\infty$, with no specified rate. In the case of varying $\underline{\alpha}$ and $\bar{\alpha}$, the rates given in (8.49) can clearly be sharpened to $e^{\int_y^x \tilde{\underline{\alpha}}(z)dz}$ and $e^{\int_y^x \tilde{\bar{\alpha}}(z)dz}$, without any change in the argument.

A3.2.2. Extensions. (i) *The case* $\eta \to 0$. The treatment above is essentially that given in [ZH,Z.1,Z.4], and suffices for the sectorial operators studied there. It suffices also for the nonsectorial, dispersive–diffusive operators studied in [HZ.2], or even purely dispersive operators, since in either case a uniform spectral gap is maintained up to the essential spectrum boundary. However, it does not directly apply to the nonsectorial operators arising in the study of relaxation or real viscosity models, for which $\underline{\alpha}$, $\bar{\alpha}$, and η typically vary with δ, with η, $\delta/\eta \to 0$ as $\delta \to 0$, and derivative A' is typically of order $\eta >> \delta$.

Here, we only point out that all conclusions of the above subsection remain valid also in this more general setting, provided that we can find a coordinate change reducing (8.31) to form (8.41)–(8.42). Such a reduction may in fact be carried out under rather general circumstances, as discussed in Section 4, above: for example, if the distance between *all* eigenvalues of A remains uniformly bounded from below even as the difference between their real parts goes to zero (as occurs for strictly hyperbolic relaxation models), or if the distance between *groups* $\{\alpha_1, \dots \alpha_\ell\}(x,\delta)$ and $\{\alpha_{\ell+1}, \dots \alpha_N\}(x,\delta)$ remains uniformly bounded from below, while at the same time $|A'| \le \eta/C$ with $C > 0$ sufficiently large, (as occurs for general relaxation models, away from the shock layer).

(ii) *Contraction mapping formulation.* The analysis of this subsection can equivalently be phrased as an integral equation/contraction mapping argument giving existence of stable/unstable manifolds at the same time that estimates are obtained. This has the advantage of slightly further generality: for example, one can obtain complete short-time theory/pointwise bounds (which depend only on high-frequency estimates) by this technique with no assumptions other than smoothness on the coefficients of L. Moreover, it disassociates regularity in λ from smoothness in x, allowing high accuracy in λ with minimal regularity assumptions on coefficients. For example, in the application to relaxation systems in Section 4, it allows us to terminate our diagonalization process at the first order, with $\delta \sim O(|\lambda|^{-2})$, $\eta \sim |\lambda|^{-1}$, while still achieving the necessary regularity in λ to order λ^{-1}, with error estimates to order $|\lambda|^{-2}$, whereas with the invariant cone approach we should have had to diagonalize to one further order, $\delta \sim \mathcal{O}(|\lambda|^{-3}$, to obtain the needed estimates. This would have increased our smoothness requirements in hypothesis (H0) to f, g, $q \in C^4$ for general, strictly hyperbolic relaxation systems, C^3 for strictly hyperbolic discrete kinetic systems.

Specifically, after first reducing to form (8.41)–(8.42), where M_j are assumed further to be *diagonal*, we may consider the "lifted equations" to the space of ℓ- (resp. $(N-\ell)$-) forms, as in (8.28), Section A3.1, in this case appearing as

$$\eta' = \mathbb{A}^{(\ell)}(x;\lambda)\eta + \delta\Theta\eta, \tag{8.50}$$

where $\mathcal{E}_1 := e_1 \wedge \cdots \wedge e_\ell$ is the unique eigenvector of $\mathbb{A}^{(k)}$ corresponding to the eigenvalue of minimum real part, which moreover is separated from all other eigenvalues with spectral gap 2η, e_j as usual denoting the standard jth basis element in $\mathbb{C}^n$.

Enumerating the standard basis elements of the space of ℓ-forms as $\mathcal{E}_j$, and defining η_j to be the coordinate representation of η with respect to this basis, we obtain from structure (8.41)–(8.42) equations for η_j' with the same, block-diagonal structure, where the first block, $\mathcal{M}_1$, corresponding to η_1 is scalar, with Re $\mathcal{M}_1 < -\eta$, and (by a straightforward calculation using definition (8.29)) the second block again satisfies Re $\mathcal{M}_2 > \eta$. Thus, defining $\hat\eta_j := \eta_j/\eta_1$ for $j \geq 2$, $\hat\eta := (\hat\eta_2, \dots)$, we obtain projectivized equations of form

$$\hat\eta = (\mathcal{M}_2 - \mathcal{M}_1 I)(x)\hat\eta + \delta\mathcal{Q}(\hat\eta, x) \tag{8.51}$$

where

$$\text{Re } (\mathcal{M}_2 - \mathcal{M}_1 I) < -2\eta \tag{8.52}$$

for all $j \geq 2$ and $\mathcal{Q}(r, x)$ is a quadratic polynomial in r, with uniformly bounded coefficients, similarly as in (8.45).

Define, similarly as in Section A3.1, the map

$$\mathcal{T}\hat\eta(x) := \delta \int_x^\infty \mathcal{F}^{z\to x}\mathcal{Q}(\hat\eta(z), z)\, dz, \tag{8.53}$$

where $\mathcal{F}^{z\to x}$ denotes the flow from z to x of the approximate equation

$$\hat\eta = (\mathcal{M}_2 - \mathcal{M}_1 I)(x)\hat\eta. \tag{8.54}$$

From (8.52), we obtain the bound

$$|\mathcal{F}^{z\to x}| \leq Ce^{-2\eta(z-x)} \tag{8.55}$$

for all $z > x$, similarly as in (8.44). From this, together with the quadratic form of $\mathcal{Q}$, we readily find that $\mathcal{T}$ is a contraction on any ball in L^∞, with constant $C\delta/\eta$, provided that δ/η is sufficiently small. From this observation, we obtain both existence of the desired fixed point $\hat\eta = \mathcal{T}(\hat\eta)$ and the bound

$$|\hat\eta - \mathcal{T}(0)| = |\mathcal{T}(\hat\eta) - \mathcal{T}(0)| \leq (C\delta/\eta)|\hat\eta|,$$

yielding $|\hat\eta| \leq C_2|\mathcal{T}(0)| \leq C_3\delta/\eta$: the same rate of tracking as obtained before using the invariant cone approach. On the other hand, we have at the same time a means to generate the solution $\hat\eta$ to any desired order in δ/η, by fixed point iteration: in particular, we have the first-order expansion

$$\hat\eta = \mathcal{T}(0) + \mathcal{O}(|\delta/\eta|^2), \tag{8.56}$$

where the first order term is explicitly given as an (finite) exponential integral against the constant term of $\mathcal{Q}$.

Remark 8.12. Note that block-diagonal structure of $\mathbb{A}$ is necessary in order that the approximate flow (8.54) be linear; in general, it could be an arbitrary quadratic function, in which case Duhamel's principle would not apply.

Remark 8.13. The estimate (8.56) in terms of ℓ- (resp. $(N-\ell)$) forms yields a corresponding estimate in terms of subspaces: that is, a description of the stable manifold at $+\infty$ (resp. unstable manifold at $-\infty$) as a graph over $w_2 \equiv 0$, smooth in x and analytic in δ/η.

Remark 8.14. Clearly, the global decay bound (8.55) suffices for the above construction. The pointwise decay afforded by negativity of $\mathcal{M}_2 - \mathcal{M}_1 I$ is not strictly necessary, and may be allowed to fail by order α on intervals with total length of order $1/\alpha$. This observation allows us to treat strong relaxation shocks, for which negativity may fail by order η on a shock layer of order $1/\eta$. Likewise, in the tracking construction of A3.2.1 and A3.2.2(i), negativity/positivity of M_1/M_2 may be allowed to fail by order α on intervals with total length of order $1/\alpha$, during which solutions may move to a larger cone (by a bounded multiple), which may be chosen to be invariant on "good" intervals.

A3.2.3. The reduced flow. Finally, suppose that we have successfully carried out the reduction (8.41) of (8.31) to block-diagonal form in such a way that (8.32) remains valid; that we have verified positivity condition (8.42) (except perhaps on a finite collection of intervals of total length order $1/\alpha$, where positivity may be allowed to fail to order α, see Remark 8.14), or by some other means established uniform exponential decay/growth

$$|\mathcal{F}_j^{y\to x}| \lessgtr Ce^{\mp\theta\eta|x-y|}, \tag{8.57}$$

$\theta > 0$, of flows $\mathcal{F}_j$ associated with the approximate (decoupled) equations

$$w_j' = M_j w_j; \tag{8.58}$$

and that $\delta/\eta \to 0$ as $\delta \to 0$, so that the entire stable/unstable manifold construction described in Sections A3.2.2 (i)–(ii) can be carried out (alternatively, the estimates of Sections A3.2.1-A3.2.2(i)). Then, the "reduced flow" restricted to stable/unstable manifolds takes the simple form

$$w_j' = M_j w_j + \delta\Theta_{jj} w_j + (\delta^2/\eta)\mathcal{R}_j, \tag{8.59}$$

where Θ_{jj}, $j = 1, 2$ are the diagonal blocks of error Θ, and terms $\mathcal{R}_j = \mathcal{O}(1)$ corresponding to off-diagonal blocks, arise through the basic estimate $|w_{\tilde{j}}|/|w_j| \leq \delta/\eta$, $\tilde{j} \neq j$.

Proposition 8.15. *The flows (i.e., solution operator) $\bar{\mathcal{F}}_j^{y\to x}$ of the reduced equations (8.59) satisfy*

$$\bar{\mathcal{F}}_j^{y\to x} = \mathcal{F}_j^{y\to x} + (\delta/\eta)\mathcal{E}_j(x, y, \delta) + \mathcal{O}(\delta^2/\eta^2)e^{-\tilde{\theta}\eta|x-y|}, \tag{8.60}$$

for $x \gtrless y$, *for any* $0 < \tilde\theta < \theta$, *where*

$$\mathcal{E}_1 := \eta \int_y^x \mathcal{F}_1^{z\to x}\Theta_{11}(z,\delta)\mathcal{F}_1^{y\to z}\, dz \tag{8.61}$$

satisfies the uniform exponential decay estimate

$$|\mathcal{E}_j(x,y,\delta)| \le C_1\eta|x-y|e^{-\theta\eta|x-y|} \le C_2 e^{-\tilde\theta\eta|x-y|}, \tag{8.62}$$

is smooth in x *and* y *and as smooth in* δ *as is* Θ_{11}, *and* $\mathcal{F}_j^{y\to x}$ *are the approximate flows associated with (8.58)* C_j *and* $\mathcal{O}(\cdot)$, *depending only on* $\tilde\theta$ *and the bounds (8.57), are of order* $C(\theta-\tilde\theta)^{-1}$, *where* C *is the constant of (8.57).*

Proof. Restrict to the case of the stable manifold, $j=1$; the case $j=2$ may be treated in symmetric fashion. Fixing y, define similarly as in previous subsections the map

$$\mathcal{T}\bar{\mathcal{F}}_1^{y\to x} := \mathcal{F}_1^{y\to x} + \int_y^x \mathcal{F}_1^{z\to x}\big(\delta\Theta_{11} + (\delta^2/\eta)\mathcal{R}_1\big)(z,\delta)\bar{\mathcal{F}}_1^{y\to z}\, dz, \tag{8.63}$$

where $\mathcal{F}_1^{z\to x}$ denotes the flow from z to x of the approximate equation (8.58). Then, from the decay estimate (8.57), it is readily established as in the argument of Section A3.2.2(ii) that $\mathcal{T}$ is a contraction on $L^\infty[y,+\infty)$, with constant of order δ/η, establishing the existence of a unique fixed-point solution. Carrying out two iterations of the fixed point iteration, we obtain the estimate

$$\begin{aligned}
\bar{\mathcal{F}}_1^{y\to x} &= \mathcal{T}^2(0) + \mathcal{O}(\delta^2/\eta^2)\\
&= \mathcal{F}_1^{y\to x} + \int_y^x \mathcal{F}_1^{z\to x}\big(\delta\Theta_{11} + (\delta^2/\eta)\mathcal{R}_1\big)(z,\delta)\mathcal{F}_1^{y\to z}\, dz + \mathcal{O}(\delta^2/\eta^2)\\
&= \mathcal{F}_1^{y\to x} + \delta\int_y^x \mathcal{F}_1^{z\to x}\Theta_{11}(z,\delta)\mathcal{F}_1^{y\to z}\, dz + \mathcal{O}(\delta^2/\eta^2),
\end{aligned} \tag{8.64}$$

yielding (8.60) with $\tilde\theta = 0$ and (8.61).

The term $\mathcal{E}_1$ may be estimated as

$$\begin{aligned}
|\mathcal{E}_1| &\le \eta\int_y^x |\mathcal{F}_1^{z\to x}||\Theta_{11}(z,\delta)|\,|\mathcal{F}_1^{y\to z}|\, dz \le C\eta\int_y^x e^{-\theta\eta|x-z|}e^{-\theta\eta|y-z|}\, dz\\
&= C\eta|x-y|e^{-\theta\eta|x-y|} \le C\eta(|x-y|e^{-(\theta-\tilde\theta)\eta|x-y|})e^{-\tilde\theta\eta|x-y|},\\
&\le C(\theta-\tilde\theta)^{-1}e^{-\tilde\theta\eta|x-y|},
\end{aligned}$$

for any $0<\tilde\theta<\theta$, and likewise the term involving $\mathcal{R}_1$ in the second line of (8.64) may be estimated as $\mathcal{O}(\delta^2/\eta^2)e^{-\tilde\theta\eta|x-y|})$. By a similar calculation, we can show that $\mathcal{T}$ is actually a contraction in the exponentially weighted norm

$$\|f\|_{\tilde\theta} := \|f(\cdot)e^{\tilde\theta\eta(\cdot-y)}\|_{L^\infty[y,+\infty)},$$

yielding the improved bound $\mathcal{O}(\delta^2/\eta^2)e^{-\tilde\theta\eta|x-y|}$, on the third, $\mathcal{O}(\cdot)$ term in the second line of (8.64). Specifically, we have

$$
\begin{aligned}
&\|\mathcal{T}\mathcal{F}_1 - \mathcal{T}\mathcal{F}_2\|_{\tilde\theta}\\
&\qquad \le e^{\tilde\theta\eta|x-y|}\|\mathcal{F}_1-\mathcal{F}_2\|_{\tilde\theta}\delta\int_y^x |\mathcal{F}_1^{z\to x}||(\Theta_{11}+(\delta/\eta)\mathcal{R}_1)(z,\delta)|\,e^{-\tilde\eta|z-y|}\,dz\\
&\qquad \le C\delta\|\mathcal{F}_1-\mathcal{F}_2\|_{\tilde\theta}e^{\tilde\theta\eta|x-y|}\int_y^x e^{-\theta\eta|x-z|}e^{-\tilde\theta\eta|y-z|}\,dz\\
&\qquad = C\delta\|\mathcal{F}_1-\mathcal{F}_2\|_{\tilde\theta}\int_y^x e^{-(\theta-\tilde\theta)\eta|x-z|}\,dz\\
&\qquad \le C(\theta-\tilde\theta)^{-1}(\delta/\eta)\|\mathcal{F}_1-\mathcal{F}_2\|_{\tilde\theta}.
\end{aligned}
$$

Combining these observations, we obtain (8.60) as claimed. ∎

Remark 8.16. Represention (8.60) becomes particularly simple in the case that M_j are *diagonal*, and $\mathcal{F}_j^{y\to x} = e^{\int_y^x M_j(z)}dz$. This occurs, for example, in the case of *strictly hyperbolic* relaxation systems, or in the case of *scalar* (third and) higher order equations considered in [HZ.2]. However, for more general applications, it is important that we do not limit ourselves to this case. For *nonstrictly hyperbolic* relaxation systems, and for higher order *systems*, M_j are block diagonal, with blocks of form

$$m_{j,k} = \beta_{j,k} + \mathcal{O}(\delta),$$

$\beta_{j,k}$ scalar. Thus, it is possible again to express the dominant (typically highly oscillatory) effects in each block as a simple exponential integral factor multiplying a higher-order, coupled flow. We describe such a calculation in detail in Section 4, above.

Note that we do not require in Proposition 8.15 any spectral gap between different modes (blocks) of M_j in order to obtain an approximately block-diagonal flow. This is important, for example, in the case of dispersive–diffusive equations considered in [HZ.2]. Indeed, Proposition 8.15 and the contraction mapping construction of Section A3.2.2(ii) together repair an omission in the analogous constructions carried out in [HZ.2]: specifically in the proof of Lemma 3.2, [HZ.2], wherein a nonexistent gap was implicitly assumed.[9]

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