Asymptotic Stability of Traveling Waves for Viscous Conservation Laws with Dispersion

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Abstract

This work is concerned with the asymptotic stability of traveling waves for scalar viscous conservation laws with a convex flux function and a dispersion term. First we prove the existence of solutions locally in time of the initial value problem for initial data near a constant solution by Fourier analysis. Using the semigroup method the local existence for initial data that are an L^2-perturbation of a traveling wave profile is proved. We also obtain a regularity property of these solutions. The solution operator generated by the linearized equation plays a crucial role. Using the energy method we estabilish a priori estimates. These estimates, when combined with the local existence, lead to the desired global in time existence as well as the time-asymptotic decay of solutions with initial data close to a monotone traveling wave.

Key words: Traveling waves, asymptotic stability, regularity, existence.
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1 Introduction

Consider viscous conservation laws with dispersion

$$u_t + f(u)_x + u_{xxx} - \mu u_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0 \tag{1.1}$$

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where the flux function $f \in C^3(\mathbb{R})$ is convex. The constant $\mu > 0$ is the viscosity parameter. The initial data

$$u(0,x) = u_0(x) \tag{1.2}$$

are assumed to satisfy $u_0(x) \to u_\pm$ as $x \to \pm\infty$.

We consider monotone traveling wave solutions $\phi(x - st)$ to the equation (1.1). Such a profile ϕ is easily seen to satisfy the following ordinary differential equation

$$\phi_{zzz} - \mu\phi_{zz} + (f(\phi) - s\phi)_z = 0, \quad z = x - st. \tag{1.3}$$

The far-field states $u_\pm \in \mathbb{R}$ and the wave speed $s \in \mathbb{R}$ satisfy the **Rankine-Hugoniot condition**

$$f(u_+) - f(u_-) - s(u_+ - u_-) = 0 \tag{1.4}$$

and the **Oleinik shock condition** [12]

$$f(u) - f(u_-) - s(u - u_-) \begin{cases} < 0 & \text{if} \quad u_+ < u < u_- \\ > 0 & \text{if} \quad u_- < u < u_+, \end{cases} \tag{1.5}$$

for details see Pan [13, Chap. 2]. If $f'' > 0$ then (1.5) implies the **Lax shock condition**

$$f'(u_+) < s < f'(u_-). \tag{1.6}$$

For the special case $f(u) = \frac{u^2}{2}$, the equation (1.1) reduces to the Korteweg-de Vries-Burgers equation

$$u_t + uu_x + u_{xxx} - \mu u_{xx} = 0. \tag{1.7}$$

Grad and Hu [7], as well as Bona and Schonbek [1] showed that (1.7) admits traveling wave solutions $\phi(x - st)$ connecting u_- to u_+, which are monotone if $\mu >> 1$. Moreover, in obtaining solutions to this problem one can look for solutions which are the sum of a traveling wave and a perturbation. It has been shown that this traveling wave is asymptotically stable for small perturbations by Bona et al. [2]. Another proof of the stability was given by Pego [15] based on the energy method due to Goodman [6]. For the case $f(u) = u^3$, Dodd [5] showed that certain traveling wave solutions are asymptotically stable with respect to a weighted norm as perturbations convect away from the wave profile in the direction indicated by the group velocity associated with the linearized perturbation equation. Recently pointwise methods were also developed to investigate the stability of the traveling waves for the equations of type (1.1) by Howard and Zumbrun [8].

In this work we consider a more general equation of the form (1.1). Due to the presence of dispersion, the arguments are much more involved when we replace $\frac{u^2}{2}$ by a general convex flux function $f(u)$. This paper is organized as follows: In Section 2, we establish local existence for the initial value problem with initial data in $L^\infty + H^2$, which is the basis for our stability analysis. To our knowledge, there is no suitable reference containing a direct proof. In Section 3, the asymptotic behavior of the solution is investigated in detail.

2 Existence of local solutions

In this section we discuss the existence of solutions locally in time of the initial value problem (1.1) and (1.2). We denote a generic positive constant by C. Moreover, in the inequalities that follow, the value of the constant C can change from one line to the next. For function spaces, $L^2(\mathbb{R})$ denotes the space of square integrable functions on $\mathbb{R}$ with the norm

$$\|u\| = \left(\int_{\mathbb{R}} |u(x)|^2 dx\right)^{\frac{1}{2}}.$$

By $H^l(\mathbb{R})$ for $l \in \mathbb{N}_0$ we denote the usual l^{th} order Sobolev space with the norm

$$\|u\|_l = \left(\sum_{j=0}^{l} \|\partial_x^j u\|^2\right)^{\frac{1}{2}}.$$

Using the Fourier transform these spaces are defined for arbitrary $l \in \mathbb{R}$, see e.g. Pazy [14, Section 8.5]. This will used in some formulations but is not explicitly needed in the paper. We also note that for $l \geq 0$ one has the dual spaces $H^{-l}(\mathbb{R}) = [H^l(\mathbb{R})]'$.

Let T be a positive constant and $(B, \|\cdot\|_B)$ a Banach space. Here $C^k([0,T];B)$ for any natural number $k \geq 0$ denotes the space of B-valued k-times continuously differentiable functions on $[0,T]$. The space of B-valued L^2 functions on $[0,T]$ is denoted by $L^2([0,T];B)$. The corresponding spaces of B-valued functions on $[0,\infty[$ are denoted by $C^k([0,\infty[;B)$ and $L^2([0,\infty[;B)$, respectively. The following Sobolev Lemma will be often used in the paper.

Lemma 2.1. *(Sobolev Lemma)*
(i) For $t \geq s$, $H^s(\mathbb{R}) \supset H^t(\mathbb{R})$ and $\|u\|_t \geq \|u\|_s$ for any $u \in H^t(\mathbb{R})$.

(ii) For $s > \frac{1}{2}$, $H^s(\mathbb{R}) \subset C(\mathbb{R})$ and for any $u \in H^s(\mathbb{R})$

$$\|u\|_\infty \leq C\|u\|_s$$

where $\|u\|_\infty = \operatorname{ess\,sup}\{|u(x)| : x \in \mathbb{R}\}$.

Proof. For a proof see e.g. Pazy [14, Lemma 8.5.1]). □

Without loss of generality, we assume that the viscosity parameter $\mu = 1$ and consider for initial data u_0 with $u_0(x) \to u_\pm$ as $x \to \pm\infty$ the initial value problem

$$u_t + f(u)_x - u_{xx} + u_{xxx} = 0, \qquad u(0,x) = u_0(x). \tag{2.1}$$

Let ϕ be a steady solution of (2.1) satisfying

$$f(\phi)_x = \phi_{xx} - \phi_{xxx}, \qquad \phi(\pm\infty) = u_\pm. \tag{2.2}$$

Taking $\varpi = u - \phi$ from (2.1) and (2.2) we study the perturbed problem

$$\begin{cases} \varpi_t + \big[f(\varpi + \phi) - f(\phi)\big]_x = \varpi_{xx} - \varpi_{xxx}, \\ \varpi|_{t=0} = \varpi_0. \end{cases} \tag{2.3}$$

2.1 The case: $\phi =$ constant

In this subsection our main idea is to first establish an equivalent integral equation using the Fourier transform. Next we prove that there exists the unique fixed point for this integral equation by using the contraction mapping principle. Finally, this yields the main theorem.

If the steady solution ϕ is a constant, i.e., $\phi(x) \equiv u^0$ the problem (2.3) reduces to

$$\begin{cases} \varpi_t + \big[f(\varpi + u^0) - f(u^0)\big]_x = \varpi_{xx} - \varpi_{xxx}, \\ \varpi|_{t=0} = \varpi_0. \end{cases} \tag{2.4}$$

Let $\rho = f'(u^0)$ and we set

$$\{f(\varpi)\} := f(\varpi + u^0) - f(u^0) - \rho\varpi,$$

then (2.4) can be rewritten as

$$\begin{cases} \varpi_t + \rho\varpi_x + \{f(\varpi)\}_x = \varpi_{xx} - \varpi_{xxx}, \\ \varpi|_{t=0} = \varpi_0. \end{cases} \tag{2.5}$$

We assume that $\varpi_0 \in H^1(\mathbb{R})$. Then trivially $\varpi_0 \in L^2(\mathbb{R})$ and by the Sobolev Lemma 2.1 we have $\varpi_0 \in L^\infty(\mathbb{R})$. Note that the latter implies $\{f(\varpi_0)\} \in L^2(\mathbb{R})$ for any smooth function f. In fact, we have using Taylor expansion

$$\{f(\varpi_0)\} = f(\varpi_0 + u^0) - f(u^0) - f'(u^0)\varpi_0 = f''(u^0 + \theta\varpi_0)\frac{\varpi_0^2}{2},$$

where $0 < \theta(x) < 1$. So

$$\|\{f(\varpi_0)\}\|^2 = \frac{1}{4}\int_{\mathbb{R}} [f''(u^0 + \theta\varpi_0)]^2 \varpi_0^4 dx = \frac{1}{4}\int_{\mathbb{R}} |f''(u^0 + \theta\varpi_0)|^2 |\varpi_0|^2 \varpi_0^2 dx.$$

Since we have that $\varpi_0 \in L^2(\mathbb{R})$, $\varpi_0 \in L^\infty(\mathbb{R})$, f is smooth, and

$$|u^0 + \theta\varpi_0| \leq |u^0| + \sup_{x\in\mathbb{R}} |\varpi_0(x)|,$$

we get $\{f(\varpi_0)\} \in L^2(\mathbb{R})$.

Using the L^2-extension of the Fourier transform

$$\widehat{\varpi}(t,\xi) = (\mathcal{F}\varpi)(t,\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ix\xi}\varpi(t,x)dx,$$

for details see Reed and Simon [16], we get from (2.5) the problem

$$\begin{cases} \widehat{\varpi}_t + i\rho\xi\widehat{\varpi} + i\xi\widehat{\{f(\varpi)\}} = -(\xi^2 - i\xi^3)\widehat{\varpi}, \\ \widehat{\varpi}|_{t=0} = \widehat{\varpi}_0. \end{cases}$$

This initial value problem for an ordinary differential equation leads to the equivalent integral equation

$$\widehat{\varpi}(t,\xi) = e^{-(\xi^2 - i\xi^3 - i\rho\xi)t}\widehat{\varpi}_0 + \int_0^t \Big(-i\xi\widehat{\{f(\varpi)\}}\Big) e^{-(\xi^2 - i\xi^3 - i\rho\xi)(t-\tau)} d\tau.$$

After applying the inverse Fourier transform we obtain the following integral equation for ϖ itself

$$\varpi(t,x) = K(t,x) * \varpi_0(x) + \int_0^t K_x(t-\tau,x) * \{f(\varpi)\} d\tau, \tag{2.6}$$

where the kernel K is given as

$$K(t,x) = \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\Big(e^{-(\xi^2 - i\xi^3 - i\rho\xi)t}\Big) \tag{2.7}$$

and the convolution is defined as $(h * g)(x) = \int_{-\infty}^{+\infty} h(x-y)g(y)dy$. The kernel $K(t)$ has the following properties.

Lemma 2.2. *For any $t > 0$, the kernel K decays in time with the following rates*

i) $\|K(t)\| = \frac{1}{2}t^{-1/4}$,

ii) $\|K_x(t)\| = \frac{1}{4}t^{-3/4}$.

Proof. We use the Plancherel identity $\|g\| = \|\widehat{g}\|$ for any $g \in L^2(\mathbb{R})$, see e.g. Reed and Simon [16, page 4, Corollary]. Then we have

$$\begin{aligned}\|K(t)\|^2 &= \int_{\mathbb{R}} |K(t,x)|^2 dx = \int_{\mathbb{R}} |\mathcal{F}[K(t,x)]|^2 dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |e^{-(\xi^2 - i\xi^3 - i\rho\xi)t}|^2 d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-2\xi^2 t} d\xi = \frac{1}{2\sqrt{\pi t}} \int_{\mathbb{R}} e^{-(\sqrt{2t}\xi)^2} d(\sqrt{2t}\xi) = \frac{1}{2}t^{-1/2}\end{aligned}$$

and

$$\begin{aligned}\|K_x(t)\|^2 &= \int_{\mathbb{R}} |K_x(t,x)|^2 dx = \int_{\mathbb{R}} |\mathcal{F}[K_x(t,x)]|^2 dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |i\xi e^{-(\xi^2 - i\xi^3 - i\rho\xi)t}|^2 d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \xi^2 e^{-2\xi^2 t} d\xi = \frac{(2t)^{-3/2}}{\sqrt{2\pi}} \int_{\mathbb{R}} y^2 e^{-y^2} dy = \frac{1}{4}t^{-3/2}.\end{aligned}$$

This proves the lemma. □

We will look for the solution of the initial value problem (2.5) in the convex set

$$X = \Big\{\varpi \in C([0,T], H^1(\mathbb{R})) : \sup_{t\in[0,T]} \|\varpi(t)\|_1 \le 2\|\varpi_0\|_1 := 2M < \infty\Big\}.$$

Here $X = X(T)$ and T depends on the H^1 norm of the initial data and bounds of f''. We consider the operator Φ defined on X by

$$\Phi v(t,x) = K(t,x) * \varpi_0(x) + \int_0^t K_x(t-\tau,x) * \{f(v)\} d\tau, \tag{2.8}$$

where $\{f(v)\} = f(v+u^0) - f(u^0) - \rho v$. Now the integral equation (2.6) is equivalent to the fixed point problem $\varpi(t,x) = \Phi\varpi(t,x)$. We have

Lemma 2.3. *The operator Φ maps X into itself and is contracting.*

Proof. First we claim that Φ maps X into itself. To see this, let $v \in X$, i.e., $\|v(t)\|_1 \le 2M$ for $t \in [0,T]$. From (2.8) one gets

$$\|\Phi v(t)\| \le \|K(t,x) * \varpi_0(x)\| + \int_0^t \|K_x(t-\tau,x) * \{f(v(\tau))\}\| d\tau. \tag{2.9}$$

Using the Plancherel identity, (2.7), and the multiplicative property of the Fourier transform $\mathcal{F}(g * h) = \sqrt{2\pi}\widehat{g}\widehat{h}$, see Reed and Simon [16, Theorem IX.3], we have

$$\begin{aligned}
\|K(t,x) * \varpi_0(x)\| &= \|\mathcal{F}(K(t,x) * \varpi_0(x))\| = \|\sqrt{2\pi}\mathcal{F}(K(t,x)) \cdot \mathcal{F}(\varpi_0(x))\| \\
&= \|e^{-(\xi^2 - i\xi^3 - i\rho\xi)t}\mathcal{F}(\varpi_0(x))\| = e^{-\xi^2 t}\|\mathcal{F}(\varpi_0(x))\| \\
&\leq \|\varpi_0\|. \qquad (2.10)
\end{aligned}$$

In order to estimate the second term of the right hand side in (2.9) we first estimate $\|\{f(v)\}\|_{L^1(\mathbb{R})} = \|f(v+u^0) - f(u^0) - \rho v]\|_{L^1(\mathbb{R})}$. Noting that $\rho = f'(u^0)$, f is smooth, defining the constants

$$r_1 = |u^0| + 6CM \qquad \text{and} \qquad s = \sup_{|u| \leq r_1} |f''(u)|$$

one has

$$\begin{aligned}
\|\{f(v)\}\|_{L^1(\mathbb{R})} &= \int_{\mathbb{R}} |f(v+u^0) - f(u^0) - \rho v| dx \leq \int_{\mathbb{R}} |f'(u^0 + \theta v) - f'(u^0)||v| dx \\
&= \int_{\mathbb{R}} |f''(u^0 + \theta^* v)|\theta \cdot |v|^2 dx \leq s\|v(t)\|^2 \leq 4M^2 s, \qquad (2.11)
\end{aligned}$$

where $0 < \theta(t,x), \theta^*(t,x) < 1$. Combining (2.10), (2.11) with (2.9), using the Young inequality $\|g * h\| \leq \|g\|\|h\|_{L^1(\mathbb{R})}$ and Lemma 2.2 (ii), we have

$$\begin{aligned}
\|\Phi v(t)\| &\leq \|\varpi_0\| + \int_0^t \|K_x(t-\tau,x)\| \cdot \|\{f(v)\}\|_{L^1(\mathbb{R})} d\tau \\
&\leq \|\varpi_0\| + 4M^2 s \int_0^t \|K_x(t-\tau,x)\| d\tau \leq \|\varpi_0\| + 4M^2 s \int_0^t \frac{1}{4}(t-\tau)^{-3/4} d\tau \\
&\leq \|\varpi_0\| + 4M^2 s t^{\frac{1}{4}}. \qquad (2.12)
\end{aligned}$$

Similarly, because

$$(\Phi v)_x(t,x) = K(t,x) * \varpi_{0x}(x) + \int_0^t K_x(t-\tau,x) * \{f(v(\tau))\}_x d\tau,$$

we can estimate

$$\|K(t,\cdot) * \varpi_{0x}\| \leq \|\varpi_{0x}\|$$

and

$$\begin{aligned}
\|\{f(v(\tau))\}_x\|_{L^1(\mathbb{R})} &= \|[f(v+u^0) - f(u^0) - \rho v]_x\|_{L^1(\mathbb{R})} = \int |f'(v+u^0) - f'(u^0)| \cdot |v_x| dx \\
&= \int |f''(u^0 + \bar{\theta} v)| \cdot |v v_x| dx \leq s \int |v v_x| dx \leq s\|v(\tau)\| \cdot \|v_x(\tau)\| \\
&\leq 4M^2 s.
\end{aligned}$$

Therefore we obtain

$$
\begin{aligned}
\|(\Phi v)_x(t)\| &\le \|\varpi_{0x}\| + \int_0^t \|K_x(t-\tau,x)\| \cdot \|\{f(v)\}_x\|_{L^1(\mathbb{R})} d\tau \\
&\le \|\varpi_{0x}\| + 4M^2 s \int_0^t \frac{1}{4}(t-\tau)^{-3/4} d\tau \\
&\le \|\varpi_{0x}\| + 4M^2 s t^{\frac{1}{4}}. \qquad (2.13)
\end{aligned}
$$

For $T = (8Ms)^{-4}$ and $0 < t \le T$, combining (2.12) with (2.13) we have

$$
\|\Phi v(t)\|_1 \le \|\varpi_0\|_1 + 8M^2 s t^{\frac{1}{4}} \le 2M.
$$

This proves our claim.

Next, we prove that Φ is a contraction on X. If $v_1, v_2 \in X$, using again the Young inequality $\|g * h\| \le \|g\|\|h\|_{L^1(\mathbb{R})}$ and Lemma 2.2, one has

$$
\begin{aligned}
\|\Phi v_1(t) - \Phi v_2(t)\| &\le \int_0^t \|K_x(t-\tau,\cdot)\| \cdot \|f(v_1+u^0) - f(v_2+u^0) - \rho(v_1 - v_2)\|_{L^1(\mathbb{R})} d\tau \\
&\le \frac{1}{4}\int_0^t (t-\tau)^{-\frac{3}{4}} \Big\| [f'(u^0 + v_2 + \bar{\theta}(v_1 - v_2)) - \rho](v_1 - v_2)(\tau) \Big\|_{L^1(\mathbb{R})} d\tau.
\end{aligned}
$$

Similar to (2.11), using the Cauchy-Schwarz inequality we have

$$
\begin{aligned}
\Big\| \Big[f'(u^0 + v_2 + \bar{\theta}(v_1 - v_2)) - \rho \Big] (v_1 - v_2) \Big\|_{L^1(\mathbb{R})} &\le 2s\big(\|v_1\| + \|v_2\|\big)\|v_1 - v_2\| \\
&\le 8Ms\|v_1(\tau) - v_2(\tau)\|.
\end{aligned}
$$

Choose $0 < \alpha < 1$ and take $T > 0$ such that $8MsT^{1/4} = \alpha < 1$, we have $T = (\frac{\alpha}{8Ms})^4$ and

$$
\begin{aligned}
\|\Phi v_1(t) - \Phi v_2(t)\| &\le 8Ms \sup_{t\in[0,T]} \|v_1(t) - v_2(t)\| \int_0^t \frac{1}{4}(t-\tau)^{-\frac{3}{4}} d\tau \\
&\le 8Mt^{1/4} \sup_{t\in[0,T]} \|v_1(t) - v_2(t)\| \\
&\le \alpha \sup_{t\in[0,T]} \|v_1(t) - v_2(t)\|.
\end{aligned}
$$

Thereby Φ is a contraction on X. This proves the lemma. □

By the Lemma 2.3, applying contracting mapping principle one can immediately conclude

Lemma 2.4. *The operator Φ has a unique fixed point in X. Therefore the integral equation (2.6) has a unique solution in X.*

Now we obtain the following theorem of existence locally in time for (2.1):

Theorem 2.5. *We consider the constant solution u^0 and a perturbation u_0 such that $u_0 - u^0 \in H^1(\mathbb{R})$, then there exists $T > 0$, where T depends only on f and $\|u_0 - u^0\|_1$, such that (2.1) has a unique solution u satisfying*

$$u - u^0 \in C([0,T]; H^1(\mathbb{R})) \quad \text{and} \quad \|u - u^0\|_1 \leq 2\|u_0 - u^0\|_1.$$

Moreover, if the flux function f is sufficiently smooth, then $u - u^0 \in C(]\tau, T[; H^l(\mathbb{R}))$ for any $\tau > 0$ and any integer $l > 0$.

Proof. The existence part of this theorem is directly implied by Lemma 2.4. Next, we prove the regularity part of the theorem. Here we only assume that $\varpi_0 \in H^1(\mathbb{R})$ and the existence part gives us that $\varpi(t,\cdot) \in H^1(\mathbb{R})$ for all $t \in [0,T]$.

Consider the solution of (2.5) for any fixed τ with $0 < \tau < t < T$

$$\varpi(t,x) = K(t-\tau,x) * \varpi(\tau,x) + \int_\tau^t K_x(t-s,x) * \{f(v)\} ds.$$

We have

$$\varpi_{xx}(t,x) = K_x(t-\tau,x) * \varpi_x(\tau,x) + \int_\tau^t K_x(t-s,x) * \{f(v)\}_{xx} ds.$$

Thus,

$$\|\varpi_{xx}(t)\| \leq \|K_x(t,x) * \varpi_x(\tau,x)\| + \int_\tau^t \|K_x(t-s,x) * \{f(v)\}_{xx}\| ds. \tag{2.14}$$

In order to obtain the desired result, we estimate the two terms on the right hand side separately.
First, since $\xi^2 e^{-2\xi^2(t-\tau)} \leq C(t-\tau)^{-1/2}$ for any $\xi \in \mathbb{R}$ we have

$$\begin{aligned}
\|K_x(t,x) * \varpi_x(\tau,x)\| &= \|\sqrt{2\pi}\widehat{K}_x(t,x)\widehat{\varpi}_x(\tau,x)\| \\
&= \Big(\int_{\mathbb{R}} |i\xi e^{-(\xi^2 - i\xi^3)(t-\tau)}\widehat{\varpi}_x(\tau,x)|^2 d\xi\Big)^{\frac{1}{2}} \\
&= \Big(\int_{\mathbb{R}} \xi^2 e^{-2\xi^2(t-\tau)}|\widehat{\varpi}_x(\tau,x)|^2 d\xi\Big)^{\frac{1}{2}} \leq C(t-\tau)^{-\frac{1}{2}}\Big(\int_{\mathbb{R}} |\widehat{\varpi}_x(\tau,x)|^2 d\xi\Big)^{\frac{1}{2}} \\
&= C(t-\tau)^{-\frac{1}{2}}\|\varpi_x(\tau)\| \leq 2MC(t-\tau)^{-\frac{1}{2}}.
\end{aligned} \tag{2.15}$$

Next, using Lemma 2.2 (ii) we get

$$\int_\tau^t \|K_x(t-s,x) * \{f(v)\}_{xx}\| ds \le \int_\tau^t \|K_x(t-s,x)\| \cdot \|\{f(v)\}_{xx}\|_{L^1(\mathbb{R})} ds$$
$$\le \int_\tau^t \frac{1}{4}(t-s)^{-\frac{3}{4}} \|\{f(v)\}_{xx}\|_{L^1(\mathbb{R})} ds.$$

Similarly to (2.11), we can show that

$$\|\{f(v)\}_{xx}\|_{L^1(\mathbb{R})} \le C\Big(1 + \|\varpi_{xx}(t)\|\Big).$$

Therefore,

$$\int_\tau^t \|K_x(t-s,x) * \{f(v)\}_{xx}\| ds \le C(t-\tau)^{\frac{1}{4}} + C\int_\tau^t \frac{1}{4}(t-s)^{-\frac{3}{4}} \|\varpi_{xx}(s)\| ds. \qquad (2.16)$$

Combining (2.15), (2.16) with (2.14) we have using a generic constant C

$$\|\varpi_{xx}(t)\| \le C\Big\{(t-\tau)^{-\frac{1}{2}} + (t-\tau)^{\frac{1}{4}} + \int_\tau^t (t-s)^{-\frac{3}{4}} \|\varpi_{xx}(s)\| ds\Big\}. \qquad (2.17)$$

From (2.17) we can get the following estimate, which will be proven in Lemma 2.7 below,

$$\|\varpi_{xx}(t)\| \le C(t-\tau)^{-\frac{1}{2}} \qquad \text{for} \quad \tau < t < T.$$

This proves $\varpi(t,\cdot) \in H^2(\mathbb{R})$ for $0 < \tau < t \le T$.

Repeating this procedure we get by a bootstrapping argument $\varpi(t,\cdot) \in H^l(\mathbb{R})$ for any $l \in \mathbb{N}$ on $]0,T]$. This finishes the proof of the theorem. □

The inequality (2.17) is a weakly singular Gronwall-type inequality. In order to estimate $\|w_{xx}(t)\|$ from this inequality we cite the following lemma by Dixon and McKee [4, Theorem 3.1 and inequality (3.6)].

Lemma 2.6. *Let g be continuous and non-negative on $[0,T]$. Suppose that*

$$g(t) \le \psi(t) + M\int_0^t (t-s)^{-\alpha} g(s) ds, \qquad 0 \le t \le T,$$

where $0 \le \alpha < 1$and M is a positive constant.

(1) If ψ is a non-negative, monotonic increasing function on $[0,T]$, then using the Γ-function we have the estimate

$$g(t) \le \psi(t) E_{1-\alpha}\left(M\Gamma(1-\alpha)t^{1-\alpha}\right), \quad 0 \le t \le T,$$

where $E_{1-\alpha}(z)$ is the Mittag-Leffler function defined for all $\alpha < 1$ by

$$E_{1-\alpha}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma\left(n(1-\alpha)+1\right)}.$$

(2) For an integrable function ψ, the sharp estimate is given by

$$g(t) \leq \frac{d}{dt}\int_0^t \psi(s) E_{1-\alpha}\left(M\Gamma(1-\alpha)(t-s)^{1-\alpha}\right) ds, \quad 0 \leq t \leq T.$$

Lemma 2.7. *For $0 < \tau < t < T$ we have that the inequality (2.17) implies*

$$\|\varpi_{xx}(t)\| \leq C(t-\tau)^{-\frac{1}{2}}.$$

Proof. Set $g(t) := \|w_{xx}(t+\tau)\|$ for $\tau < t \leq T$, then the inequality (2.17) can be written as

$$g(t) \leq Ct^{-1/2} + \int_0^t (t-s)^{-3/4} g(s) ds.$$

Using (2) of the Lemma 2.6 we obtain

$$g(t) \leq \frac{d}{dt}\int_0^t Cs^{-1/2} E_{1/4}\left(C\Gamma(1/4)(t-s)^{1/4}\right) ds$$

which gives

$$g(t) \leq Ct^{-1/2} + \frac{C}{4}\Gamma(1/4)\int_0^t s^{-1/2}(t-s)^{-3/4} E'_{1/4}\left(C\Gamma(1/4)(t-s)^{1/4}\right) ds.$$

Using the fact that $t \leq T$ and the definition of $E_{1/4}(z)$ we have that the second term on the right side is bounded by a constant depending on T. Thus $g(t)$ is bounded from above by $Ct^{-1/2}$ for any $0 < t \leq T$, here C may depend on T. This completes the lemma. □

2.2 The case: ϕ non-constant

If the traveling wave solution ϕ is non-constant, the above argument does not work any more. To establish the desired local existence let us first recall the local existence theory of the abstract quasilinear initial value problem for the evolution equation

$$\frac{du}{dt} + A(u)u = 0, \qquad u(0) = \phi \tag{2.18}$$

in which $A(u)$ is a linear operator in a Hilbert space X with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|_X$ for any u belonging to a subset of X to be defined below.

Let X and Y be real, Hilbert spaces such that Y is continuously and densely imbedded in X; and $S : Y \to X$ be a linear isomorphism.

Take the open ball $B_r := \{y \in Y : \|y\|_Y \le r,\ r > 0\} \subset Y$; and suppose we are given constants $\beta \in \mathbb{R}$, $C_1, C_2, C_3 > 0$ depending only on r.

We define the following conditions:

(H1) For every $y \in Y$ the operator $-A(y) - \lambda I$ is the infinitesimal generator of a C^0 semigroup of contractions on X for $\lambda > \beta$.

(H2) For all $y, z, v \in Y$ with $y, z \in B_r$,

$$\left\| \Big(A(y) - A(z)\Big) v \right\|_X \le C_1 \|y - z\|_X \|v\|_Y.$$

We denote by $B(X)$ the space of bounded linear operators on X. Now we assume that

(H3) There is a mapping $C : B_r \to B(X)$ such that for every $y \in B_r$,

$$SA(y)S^{-1} = A(y) + C(y)$$

and $\|C(y)\|_{B(X)} \le C_2$.

(H4) For all $y, z, v \in Y$ with $y, z \in B_r$,

$$\left\| \Big(C(y) - C(z)\Big) v \right\|_X \le C_3 \|y - z\|_Y \|v\|_X.$$

Remarks:

1) The assumption (H1) can be replaced by the following equivalent conditions:

(a) The operator $-A(y)$ is the infinitesimal generator of a C^0 semigroup e^{-tA} on X satisfying

$$\|e^{-tA(y)}\| \le e^{\beta t}.$$

(b) For every $y \in B_r$ the operator $A(y)$ satisfies

(i) the range of $A(y) + \lambda I$ is X for some $\lambda > \beta$,

(ii) for all $v \in D(A(y))$,

$$\langle A(y)v, v\rangle_X \ge -\beta \|v\|_X^2.$$

(c) For every $y \in B_r$, $-A(y)$ is *stable* in the sense that,

$$\left\| \left(\lambda + A(y)\right)^{-1} \right\| \leq (\lambda - \beta)^{-1}, \forall \lambda > \beta.$$

It is easy to see that (H1) is equivalent to (a). By the Hille-Yosida Theorem, see Pazy [14, Theorem 1.3.1], (c) is equivalent to (a). By the Lumer-Phillips Theorem, see Pazy [14, Theorem 1.4.3 and Corollary 3.3.3], (H1) is equivalent to (b).

2) For hyperbolic-type equations, (H1) can be replaced by the following condition, see e.g. Okazawa [11],

($\tilde{H}_1$) For every $y \in B_r$ and $v \in Y$

$$|\langle A(y)v, v\rangle| \leq \beta \|v\|^2.$$

The operator $A(y)$ associated with our problem does not satisfy this inequality due to the presence of the diffusion term.

3) The assumptions (H1)-(H3) will suffice to guarantee the existence of a local solution of (2.18), see e.g. Pazy [14]. Condition (H4) was used by Kato in [9] to establish stronger results concerning the dependence of the solution of (2.18) on A and ϕ, and the role of (H4) in Crandall and Souganidis [3] is also related to this. We will show that the operator $A(y)$ associated with our problem also satisfies (H4), see Lemma 2.16. □

Using the assumptions introduced above one may formulate the following theorem due to Kato.

Theorem 2.8 (Abstract Existence Theorem). *Suppose that for any $r > 0$, there exist constants $\beta \in \mathbb{R}$, $C_1, C_2, C_3 > 0$ depending only on r such that the conditions (H1)-(H3) are satisfied. Given any $\phi \in Y$ with $\phi \in B_r$, there exists a time $T > 0$ such that the abstract Cauchy problem (2.18) has a unique solution*

$$u \in C([0,T], Y) \cap C^1([0,T], X).$$

Moreover, if (H4) is also satisfied, then u depends continuously on $u(0) = \phi$ in the $Y-$norm.

Proof. This theorem was shown by Kato in [9] with (H1) replaced by 1)-(b) in the remark above. Also it was proved by Crandall and Souganidis [3, Theorem 1] by using difference methods with (H1) replaced by 1)-(c). $\square$

To apply Theorem 2.8 we rewrite (2.3) as an abstract quasi-linear evolution equation of the form

$$\frac{d\varpi}{dt} + A(\varpi)\varpi = 0, \qquad \varpi|_{t=0} = \varpi_0. \tag{2.19}$$

The operator $A(y)$ is given by

$$A(y) := A_0 + B(y),$$

where using the mean value theorem for appropriate $\theta(t,x) \in]0,1[$

$$\begin{aligned} A_0 &:= \partial_x^3 - \partial_x^2, \\ B(y) &:= A_1(y) + A_2(y), \\ A_1(y) &:= f''(\phi + \theta y)\phi_x, \\ A_2(y) &:= f'(\phi + y)\partial_x. \end{aligned}$$

Let $X = L^2(\mathbb{R})$ and $Y = H^3(\mathbb{R})$. Our theorem of local existence is:

Theorem 2.9. *Fix a constant $r > 0$. Then there exists some time $T > 0$ depending only on r such that whenever $\|\varpi_0\|_3 < r$, the initial value problem (2.19) has a unique solution*

$$\varpi \in C([0,T], H^3(\mathbb{R})) \cap C^1([0,T], L^2(\mathbb{R})),$$

and $\varpi(t,\cdot)$ depends continuously on ϖ_0 in the space $H^3(\mathbb{R})$.

The proof of this theorem is broken into several basic lemmas and will be given as a summary of these lemmas after Lemma 2.16.

Lemma 2.10. *Let T_0 be the infinitesimal generator of a C_0 semigroup of contractions on X. Let T be a dissipative operator on X, i.e. $\langle Tv, v\rangle_X \le 0$ and satisfy $D(T) \supset D(T_0)$ and*

$$\|Tv\|_X \le a\|T_0 v\|_X + b\|v\|_X \qquad \text{for} \quad v \in D(T_0)$$

where $0 \le a < 1$ and $b \ge 0$. Then $T_0 + T$ is the infinitesimal generator of a C_0 semigroup of contractions.

Proof. See Pazy [14, Corollary 3.3.3]. $\square$

We define the commutator $[A, B] = AB - BA$ and the operators $\Lambda^s = (I - \partial_x^2)^{s/2}$ using the Fourier transform. Then for any $s \in \mathbb{R}$ we have that $u \in H^s(\mathbb{R})$ iff

$$\Lambda^s u = \mathcal{F}^{-1}\Big([1 + \xi^2]^{\frac{s}{2}} \hat{u}\Big) \in L^2(\mathbb{R}).$$

We have

Lemma 2.11. *For any $l > 3/2$, if $b \in L^\infty(\mathbb{R})$ and $\partial_x b \in H^{l-1}(\mathbb{R})$, then*

$$\|[\Lambda^l, b]u\| \leq C\|\partial_x b\|_{l-1} \cdot \|u\|_{l-1}$$

for any $u \in H^{l-1}(\mathbb{R})$.

Proof. See Kato [10]. □

We remind that $X = L^2(\mathbb{R})$, $Y = H^3(\mathbb{R})$ and take $S = \Lambda^3$. To prove Theorem 2.9 by using Theorem 2.8, we need only to check that the conditions (H1)-(H4) hold for the initial value problem (2.19). First we show the following

Lemma 2.12. *The operator $-A_0$ is the generator of a C_0 contractions on $L^2(\mathbb{R})$ with domain $D(-A_0) = H^3(\mathbb{R})$.*

Proof. Note that $A_0 = \partial_x^3 - \partial_x^2$. According to the remark 1)(b) after the introduction of conditions (H1-4), it suffices to show that A_0 satisfies for $\beta = 0$

i) The range of $A_0 + \lambda I$ is $L^2(\mathbb{R})$ for some $\lambda > 0$.

ii) For all $v \in H^3(\mathbb{R})$,

$$\langle A_0 v, v\rangle_{L^2(\mathbb{R})} \geq 0.$$

To show i) we need to prove

(1) For any $v \in H^3(\mathbb{R})$, we have $(A_0 + \lambda I)v \in L^2(\mathbb{R})$. This is obvious.

(2) For any $u \in L^2(\mathbb{R})$ there exists a $v \in H^3(\mathbb{R})$ such that $u = (A_0 + \lambda I)v$. This fact can be ensured by the invertibility of the operator $B_0 := A_0 + \lambda I$ on $L^2(\mathbb{R})$ and the fact $v = B_0^{-1}u \in H^3(\mathbb{R})$.

First, we prove the invertibility of B_0, i.e., for any $v \in H^3(\mathbb{R})$, $v \neq 0$, $\langle B_0 v, v\rangle > 0$. Actually, noting that $A_0 = \partial_x^3 - \partial_x^2$, integration by parts leads to

$$\langle B_0 v, v\rangle = \langle v_{xxx} - v_{xx} + \lambda v, v\rangle = \langle v_x, v_x\rangle + \lambda\langle v, v\rangle > 0.$$

Next, we prove for any $u \in L^2(\mathbb{R})$, $v = B_0^{-1}u \in H^3(\mathbb{R})$. By the definition of general Sobolev spaces on $\mathbb{R}$ using the Fourier transform, in order to prove that $v \in H^3(\mathbb{R})$ it suffices to show $(i\xi)^\alpha \widehat{v} \in L^2(\mathbb{R})$ for $0 \le \alpha \le 3$ and $\xi \in \mathbb{R}$. Noting that

$$\widehat{B_0} = (i\xi)^3 - (i\xi)^2 + \lambda = -i\xi^3 + \xi^2 + \lambda$$

we then have

$$(i\xi)^\alpha \widehat{v} = (i\xi)^\alpha \widehat{B_0^{-1}u} = (i\xi)^\alpha \hat{B}_0^{-1}\hat{u}.$$

Since $\hat{u} \in L^2(\mathbb{R})$, by the Plancherel theorem $\|u\| = \|\hat{u}\|$, and for $0 \le \alpha \le 3$ the following rational function is uniformly bounded in ξ

$$|(i\xi)^\alpha \hat{B}_0^{-1}| = |\frac{(i\xi)^\alpha}{-i\xi^3 + \xi^2 + \lambda}| \le C$$

we have $(i\xi)^\alpha \hat{v} \in L^2(\mathbb{R})$ thereby $v \in H^3(\mathbb{R})$.

It's easy to prove ii). In fact, we have

$$\langle A_0 v, v\rangle = \langle v_{xxx} - v_{xx}, v\rangle = \langle v_x, v_x\rangle \ge 0.$$

□

Now we check (H1) for $A(y) = A_0 + B(y)$ in the following lemma.

Lemma 2.13. *Choose $r > 0$. Then for every $y \in B_r$ the operator $-A(y) - \lambda I$ is the infinitesimal generator of a C^0 semigroup of contractions on $L^2(\mathbb{R})$ for any*

$$\lambda \ge \beta = \left(\frac{3}{2}\|\phi_x\|_\infty + Cr\right) \sup_{|u| \le \|\phi\|_\infty + r} |f''(u)|.$$

Proof. We note first that for $y \in H^3(\mathbb{R})$ it follows from Lemma 2.1 that

$$\|y\|_\infty \le C\|y\|_3, \quad \|y_x\|_\infty \le C\|y\|_3. \tag{2.20}$$

Now, for every $v \in H^3(\mathbb{R})$ using integration by parts we have

$$\begin{aligned}\langle B(y)v, v\rangle &= \langle f''(\phi + \theta y)\phi_x v + f'(\phi + y)v_x, v\rangle \\ &= \langle f''(\phi + \theta y)\phi_x, v^2\rangle - \langle f'(\phi + y)_x, \frac{v^2}{2}\rangle.\end{aligned}$$

Take $r_2 = \|\phi\|_\infty + 3Cr$, and set $\bar{s} = \sup_{|u|\le r_2} |f''(u)|$. For every $y \in H^3(\mathbb{R})$ with $\|y\|_3 \le r$, noting that $f \in C^3(\mathbb{R})$ and using (2.20) we have

$$f''(\phi+\theta y)\phi_x \ge -\bar{s}\|\phi_x\|_\infty$$

and

$$-f'(\phi+y)_x = -f''(\phi+y)(\phi_x+y_x) \ge -\bar{s}(\|\phi_x\|_\infty + \|y_x\|_\infty) \ge -\bar{s}(\|\phi_x\|_\infty + Cr).$$

Setting $\beta = \big(\frac{3}{2}\|\phi_x\|_\infty + Cr\big)\bar{s}$, we have

$$\langle B(y)v, v\rangle \ge -\beta\|v\|^2. \tag{2.21}$$

From (2.21) we know that $-(B(y)+\lambda I)$ is dissipative for all $\lambda \ge \beta$. Moreover, setting $C_1(r) = \bar{s}\|\phi_x\|_\infty + \lambda$, and $C_2(r) = \sup_{|u|\le r_2}|f'(u)|$ one gets

$$\begin{aligned}\Big\|\Big(B(y)+\lambda I\Big)v\Big\| &\le \|f''(\phi+\theta y)\phi_x v\| + \|f'(\phi+y)v_x\| + \lambda\|v\| \\ &\le C_1(r)\|v\| + C_2(r)\|v_x\|. \end{aligned}\tag{2.22}$$

On the other hand, for any $v \in H^3(\mathbb{R})$ we have the following interpolation inequalities

$$\|\partial_x^j v\| \le \epsilon\|\partial_x^3 v\| + C(\epsilon)\|v\| \qquad \text{for} \qquad j = 0,1,2. \tag{2.23}$$

Since $\partial_x^3 = A_0 + \partial_x^2$ we get

$$\|\partial_x^3 v\| \le \|A_0 v\| + \epsilon\|\partial_x^3 v\| + C(\epsilon)\|v\|.$$

Hence

$$\|\partial_x^3 v\| \le \frac{1}{1-\epsilon}\|A_0 v\| + \frac{C(\epsilon)}{1-\epsilon}\|v\|. \tag{2.24}$$

Combining (2.24) with $\|v_x\| \le \epsilon\|\partial_x^3 v\| + C(\epsilon)\|v\|$, taking $j=1$ in (2.23), gives

$$\|v_x\| \le \frac{\epsilon}{1-\epsilon}\|A_0 v\| + \frac{C(\epsilon)}{1-\epsilon}\|v\|. \tag{2.25}$$

Substituting (2.25) into (2.22) yields

$$\Big\|\Big(B(y)+\lambda I\Big)v\Big\| \le \frac{\epsilon C_2(r)}{1-\epsilon}\|A_0 v\| + \Big(C_1(r) + \frac{C(\epsilon)C_2(r)}{1-\epsilon}\Big)\|v\|.$$

Choosing $0 < \epsilon < \frac{1}{1+C_2(r)}$ and letting $a = \frac{\epsilon C_2(r)}{1-\epsilon} < 1$, $b = C_1(r) + \frac{C(\epsilon)C_2(r)}{1-\epsilon}$, we have

$$\Big\|\Big(B(y)+\lambda I\Big)v\Big\| \le a\|A_0 v\| + b\|v\| \qquad \text{for} \quad v \in H^3(\mathbb{R}).$$

By Lemma 2.12 and Lemma 2.10, $-A_0 - B(y) - \lambda I = -A(y) - \lambda I$ is the infinitesimal generator of a C_0 semigroup of contractions of $L^2(\mathbb{R})$ for every $\lambda \ge \beta$. $\square$

Lemma 2.14. *Choose $r > 0$. The operator $A(y)$ satisfies the condition (H2), i.e. for all $y, z, v \in H^3(\mathbb{R})$ with $\|y\|_3$, $\|z\|_3 \le r$, we have*

$$\Big\| \Big(A(y) - A(z)\Big) v \Big\| \le C_1 \|y - z\| \cdot \|v\|_3. \tag{2.26}$$

Proof. From the definition of $A(y)$, we have

$$\Big(A(y) - A(z)\Big) v = \big[f''(\phi + \theta y) - f''(\phi + \theta z)\big] \phi_x v + \big[f'(\phi + y) - f'(\phi + z)\big] v_x.$$

So

$$\begin{aligned} \Big\| \Big(A(t,y) - A(t,z)\Big) v \Big\| \le & \big\| \big[f''(\phi + \theta y) - f''(\phi + \theta z)\big] \phi_x v \big\| \\ & + \big\| \big[f'(\phi + y) - f'(\phi + z)\big] v_x \big\|. \end{aligned}$$

Next, we estimate respectively the two terms on the right side of the above inequality. We again consider $r_2 = \|\phi\|_\infty + 3Cr$ as in the proof of Lemma 2.13. Set $\widetilde{s} = \sup_{|u| < r_2} |f'''(u)|$. Note that for all $y, z, v \in H^3(\mathbb{R})$ with $\|y\|_3, \|z\|_3 \le r$ using that $\|v\|_\infty$, $\|v_x\|_\infty \le C\|v\|_3$ by Lemma 2.1 and that ϕ, ϕ_x are bounded, see Pan [13, Corollary 2.3.2], we obtain

$$\begin{aligned} \big\| [f''(\phi + \theta y) - f''(\phi + \theta z)] \phi_x v \big\| &= \big\| f'''\big(\phi + \theta z + \theta\theta_1 (y - z)\big)(y - z) \phi_x v \big\| \\ &\le \widetilde{s} \|v\|_\infty \|y - z\| \|\phi_x\|_\infty \le C \widetilde{s} \|y - z\| \|v\|_3. \end{aligned}$$

The second term can be handled similarly. In fact,

$$\begin{aligned} \big\| [f'(\phi + y) - f'(\phi + z)] v_x \big\| &= \big\| f''\big(\phi + z + \theta_2 (y - z)\big)(y - z) v_x \big\| \\ &\le \bar{s} \|v_x\|_\infty \|y - z\| \le C \bar{s} \|y - z\| \|v\|_3. \end{aligned}$$

Therefore, we have (2.26) with $C_1 = C(\bar{s} + \widetilde{s})$. □

Lemma 2.15. *Choose $r > 0$ and set $S = \Lambda^3 = (I - \partial_x^2)^{3/2}$ and $C(y) = SA(y)S^{-1} - A(y)$. The operator $A(y)$ satisfies the condition (H3), i.e. for $\|y\|_3 \le r$ the linear Operator $C(y)$ is bounded and $\|C(y)\|_{B(L^2(\mathbb{R}))} \le C_2$.*

Proof. We need to estimate

$$\|SA(y)S^{-1} y^* - A(y) y^*\| \le K(r) \|y^*\|$$

for all $y^* \in H^3(\mathbb{R})$, here $K(r)$ is a constant depending only on r. This can be rewritten as

$$\|SA(y)S^{-1} y^* - A(y) S S^{-1} y^*\| \le K(r) \|S S^{-1} y^*\|$$

for all $y^* \in H^3(\mathbb{R})$. Setting $z = S^{-1}y^*$ and noting that $[A, B] = AB - BA$ one has to show

$$\|[S, A(y)]z\| \leq K(r)\|Sz\| \leq CK(r)\|z\|_3$$

for all $z \in S^{-1}Y = \Lambda^{-3}[H^3(\mathbb{R})] = H^6(\mathbb{R})$. Note that we can also choose the norm in $H^3(\mathbb{R})$ as $\|z\|_3 := \|Sz\|$ such that S becomes an isometry. Since $A(y) = A_0 + B(y)$ and the operator S commutes with A_0, we need only show

$$\|[S, B(y)]z\| \leq K(r)\|z\|_3. \tag{2.27}$$

From the definitions of S and $B(y)$, we have

$$[S, B(y)] = [\Lambda^3, f''(\phi + \theta y)\phi_x] + [\Lambda^3, f'(\phi + y)\partial_x].$$

Noting that $\phi_x \in H^l(\mathbb{R})$ for $l \geq 0$, see Pan [13, Collary 2.3.2], and using Lemma 2.11 we have

$$\big\| \big[\Lambda^3, f''(\phi + \theta y)\phi_x\big] z\big\| \leq C\big\|\partial_x\big(f''(\phi + \theta y)\phi_x\big)\big\|_2\|z\|_2.$$

For $y \in H^3(\mathbb{R})$ with $\|y\|_3 \leq r$, the term $\|\partial_x(f''(\phi + \theta y)\phi_x)\|_2$ can be bounded by some $K(r)$ because of the smoothness of the flux function. Therefore we get

$$\big\|[\Lambda^3, f''(\phi + \theta y)\phi_x]z\big\| \leq K(r)\|z\|_2 \leq K(r)\|z\|_3. \tag{2.28}$$

Similarly, since

$$\begin{aligned}
\big[\Lambda^3, f'(\phi + y)\partial_x\big] z &= \Lambda^3 f'(\phi + y)z_x - f'(\phi + y)\partial_x\Lambda^3 z \\
&= \Lambda^3 f'(\phi + y)z_x - f'(\phi + y)\Lambda^3 z_x \\
&= [\Lambda^3, f'(\phi + y)]z_x,
\end{aligned}$$

one has by Lemma 2.11

$$\|[\Lambda^3, f'(\phi + y)]z_x\| \leq C\|\partial_x f'(\phi + y)\|_2\|z_x\|_2 \leq K(r)\|z\|_3. \tag{2.29}$$

Combining (2.28) with (2.29) we get (2.27). This proves the lemma. $\square$

Lemma 2.16. *Choosing $r > 0$, the operator $C(y) = SA(y)S^{-1} - A(y)$ satisfies the condition (H4): for all $y, z, v \in H^3(\mathbb{R})$ with $\|y\|_3$, $\|z\|_3 \leq r$,*

$$\Big\|\Big(C(y) - C(z)\Big)v\Big\| \leq C_3\|y - z\|_3\|v\|.$$

Proof. Noting that

$$C(y) = SA(y)S^{-1} - A(y) = [S, A(y)]S^{-1}$$

we have

$$\Big(C(y) - C(z)\Big)v = \big[S, A(y) - A(z)\big]S^{-1}v.$$

It suffices to show that for any $v \in H^3(\mathbb{R})$,

$$\big\|\big[S, A(y) - A(z)\big]S^{-1}v\big\| \leq C_3\|y - z\|_3\|v\|$$

or equivalently that for $w = S^{-1}v \in \Lambda^{-3}[H^3(\mathbb{R})] = H^6(\mathbb{R})$,

$$\big\|\big[\Lambda^3, A(y) - A(z)\big]w\big\| \leq C_3\|y - z\|_3\|w\|_3. \tag{2.30}$$

To prove this we write

$$\begin{aligned}
[\Lambda^3, A(y) - A(z)]w = & \big[\Lambda^3\big(f''(\phi + \theta y) - f''(\phi + \theta z)\big)\phi_x \\
& - \big(f''(\phi + \theta y) - f''(\phi + \theta z)\big)\phi_x\Lambda^3\big]w \\
& + \big[\Lambda^3\big(f'(\phi + y) - f'(\phi + z)\big) \\
& \quad - (f'(\phi + y) - f'(\phi + z))\Lambda^3\big]w_x \\
= & \big[\Lambda^3, (f''(\phi + \theta y) - f''(\phi + \theta z))\phi_x\big]w \\
& + \big[\Lambda^3, f'(\phi + y) - f'(\phi + z)\big]w_x.
\end{aligned}$$

Similarly as in the proof of Lemma 2.15 we can then show (2.30). ☐

Now we summarize the proof of Theorem 2.9 as follows:

Proof of Theorem 2.9. Combining Lemmas 2.13 - 2.16 to prove the assumptions of Theorem 2.8 we can immediately deduce the statement of Theorem 2.9. ☐

In an analogous manner one may weaken the assumption on ϖ_0 and the conclusions of Theorem 2.9 as follows.

Theorem 2.17. *Fix $r > 0$. Then there exists some $T > 0$ depending only on r such that whenever $\|\varpi_0\|_2 < r$, the initial value problem (2.19) has a unique solution*

$$\varpi \in C([0, T], H^2(\mathbb{R})) \cap C^1([0, T], H^{-1}(\mathbb{R})),$$

and $\varpi(t, \cdot)$ depends continuously on ϖ_0 in the $H^2(\mathbb{R})$.

Proof. To apply Theorem 2.8 we need to choose the spaces $X = H^{-1}(\mathbb{R})$ and $Y = H^2(\mathbb{R})$. Similarly as in the proof of Theorem 2.9 we can check the assumptions (H1)-(H4). The details are omitted. $\square$

Set $w(t,x) = \int_{-\infty}^{x} \varpi(t,y)dy$ to be the primitive in space for any solution $\varpi = u - \phi$ obtained in Theorem 2.17, then (2.3) reduces to

$$w_t + \Big[f(\phi + w_x) - f(\phi)\Big] = w_{xx} - w_{xxx} \tag{2.31}$$

with initial data $w_0 = \int_{-\infty}^{x} \varpi_0(x)dx$. This is a Hamilton-Jacobi type equation with diffusion and dispersion.

For later use in proving the stability theorem we proceed to establish the following theorem.

Theorem 2.18. *(Local existence). For any $\epsilon_0 > 0$, there exists a positive constant T_0, depending on ϵ_0 and f, such that if $\|w_0\|_3 \le \epsilon_0/2$, then the problem(2.31) with initial data w_0 has a unique solution $w \in C^0([0,T_0], H^3(\mathbb{R}))$ satisfying*

$$\|w(t)\|_3 \le 2\|w_0\|_3 \le \epsilon_0 \tag{2.32}$$

for $0 \le t \le T_0$.

Proof. Because $\varpi_0 = w_{0x} \in H^2(\mathbb{R}) \subset H^{-1}(\mathbb{R})$, applying Theorem 2.17 we obtain that $w_x(t) = \varpi(t) \in H^2(\mathbb{R})$. In order to conclude the theorem we need only to prove $w(t) \in L^2(\mathbb{R})$ as well as the inequality (2.32).

Multiplying (2.31) by $2w$ and integrating the obtained result over $\mathbb{R}$, we have

$$\frac{d}{dt}\int_{\mathbb{R}} w^2 dx + 2\int_{\mathbb{R}} w_x^2 dx = -2\int_{\mathbb{R}} w\Big[f(\phi + w_x) - f(\phi)\Big]dx.$$

Since

$$|f(\phi + w_x) - f(\phi)| = |f'(\phi + \theta w_x)w_x| \le C|w_x|,$$

using the inequality $2|ab| \le \frac{a^2}{C} + Cb^2$ we have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} w^2 dx + 2\int_{\mathbb{R}} w_x^2 dx &\le 2C\int_{\mathbb{R}} |ww_x| dx \\ &\le \int_{\mathbb{R}} w_x^2 dx + C^2\int_{\mathbb{R}} w^2 dx,\end{aligned}$$

i.e.

$$\frac{d}{dt}\int_{\mathbb{R}} w^2 dx + \int_{\mathbb{R}} w_x^2 dx \le C^2 \int_{\mathbb{R}} w^2 dx. \tag{2.33}$$

Multiplying (2.33) by $e^{-C^2 t}$ and integrating over $]0,t[$ one has

$$e^{-C^2 t}\|w(t)\|^2 - \|w_0\|^2 + \int_0^t e^{-C^2 s}\|w_x(s)\|^2 ds \le 0.$$

It follows that

$$\|w(t)\|^2 \le e^{C^2 t}\|w_0\|^2 \le 4\|w_0\|^2$$

if $e^{C^2 t} \le 4$ or $t \le \frac{\ln 4}{C^2} =: T_0$. Then we have shown $w(t) \in L^2(\mathbb{R})$ and $\|w(t)\| \le 2\|w_0\|$.

Similarly, for the higher order derivative we can also get $\|w(t)\|_3 \le 2\|w_0\|_3$. This proves the theorem. □

3 Asymptotic Stability of Traveling Waves

To investigate the stability of the traveling wave solution ϕ, we assume that the initial perturbation $u_0 - \phi$ is integrable over $\mathbb{R}$. It is known that the solutions with the initial data near ϕ will typically approach a translate of ϕ rather than ϕ itself, but the shift can usually be determined by the mass carried by the initial perturbation. We define the point x_0 by

$$\int_{\mathbb{R}} \big[u_0(x) - \phi(x)\big] dx = x_0(u_+ - u_-), \tag{3.1}$$

where u_0 are the initial data given in (1.2). Let us set

$$I(y) := \int_{\mathbb{R}} \big[\phi(x+y) - \phi(x)\big] dx.$$

Then we have $I(0) = 0$ and due to the asymptotic behaviour of ϕ at infinity $I'(0) = u_+ - u_-$ as well as $I^{(k)}(0) = 0$ for all $k > 1$. This gives $I(x_0) = x_0(u_+ - u_-)$. Therefore (3.1) is equivalent to

$$\int_{\mathbb{R}} \big[u_0(x) - \phi(x + x_0)\big] dx = 0.$$

Without loss of generality let us assume that $x_0 = 0$, i.e.,

$$\int_{\mathbb{R}} \big[u_0(x) - \phi(x)\big] dx = 0. \tag{3.2}$$

Note that for given initial data u_0 condition (3.2) selects a unique traveling wave ϕ from the whole family of shifted traveling waves satisfying (1.3).

We use again the primitive of the the perturbation

$$w_0(x) = \int_{-\infty}^{x} \big[u_0(y) - \phi(y)\big]dy.$$

Then our stability results are given as follows.

Theorem 3.1. *Assume that the Rankine-Hugoniot condition (1.4) and the Lax shock condition (1.6) hold. Let ϕ be a monotonically decreasing traveling wave solution of (1.1) uniquely determined by (3.2). Suppose $u_0 - \phi$ is integrable over $\mathbb{R}$ and $w_0 \in H^3(\mathbb{R})$. Then there exists a constant $\epsilon > 0$ such that if $\|w_0\|_3 < \epsilon$ the Cauchy problem (1.1), (1.2) has a unique global solution u satisfying*

$$u - \phi \in C([0,\infty[; H^2(\mathbb{R})) \cap L^2([0,\infty[; H^3(\mathbb{R}))$$

and

$$\sup_{x\in\mathbb{R}} |u(t,x) - \phi(x - st)| \to 0 \qquad \text{as} \quad t \to \infty.$$

To prove this theorem, we set

$$w(t,x) := \int_{-\infty}^{x} \big[u(t, y + st) - \phi(y)\big]dy$$

which satisfies $\lim_{x\to\pm\infty} w(t,x) = 0$ by (3.2). Now we decompose the solution to equation (1.1) into the form

$$u(t,x) = \phi(z) + w_z(t,z), \qquad z = x - st$$

where ϕ is a monotone traveling wave solution to (1.1) and w_z denotes the perturbation. Upon substitution into (1.1) the problem (1.1), (1.2) is reduced to

$$w_t - sw_z + f(\phi + w_z) - f(\phi) + w_{zzz} - \mu w_{zz} = 0, \tag{3.3}$$

with

$$w(0,z) = w_0(z) = \int_{-\infty}^{z} \big[u_0(y) - \phi(y)\big]dy. \tag{3.4}$$

Using

$$Q(u) = f(u) - f(u_-) - s(u - u_-),$$

gives $Q'(\phi) = f'(\phi) - s$. Equation (3.3) may be rewritten as

$$w_t + Q'(\phi)w_z + w_{zzz} - \mu w_{zz} = F(\phi, w_z) \tag{3.5}$$

with

$$F(\phi, w_z) = -\Big[f(\phi + w_z) - f(\phi) - f'(\phi)w_z\Big]$$

where s is the shock speed depending on u_+ and u_-, see (1.4). We note that

$$F(\phi, w_z) = -\frac{f''(\phi + \theta w_z)}{2} w_z^2$$

for suitable $\theta(t, z) \in]0, 1[$. This will be used later.

The existence of local solution to the Cauchy problem (3.5), (3.4) is guaranteed by the local existence Theorem 2.18 since the present case can be reduced to the case $s = 0$ by simply replacing $f(u)$ by $f(u) - su$.

Theorem 3.1 is the immediate consequence of the following result for the primitive of the perturbation.

Theorem 3.2. *(Global existence). Suppose $w_0 \in H^3(\mathbb{R})$. Then there exists a positive constant ϵ such that if $\|w_0\|_3 < \epsilon$, we obtain a solution $w \in C([0, \infty[, H^3(\mathbb{R}))$ to (3.5),(3.4) satisfying*

$$\|w(t)\|_3^2 + \int_0^t \|w_z(\tau)\|_3^2 d\tau \leq C\|w_0\|_3^2$$

and moreover

$$\|w_z(\cdot, t)\|_2^2 \to 0 \qquad \text{as} \quad t \to \infty.$$

This global existence and decay for w will be derived from the local existence given by Theorem 2.18 for w combined with the following a priori estimate.

Lemma 3.3. *Let T be a positive constant. Suppose that the problem (3.5),(3.4) has a solution $w \in C([0, T], H^3(\mathbb{R}))$. Then there exist positive constants ϵ_1 and C_1, which are independent of T, such that if $\sup_{0 \leq t \leq T} \|w(t)\|_3 \leq \epsilon_1$ we have the estimate*

$$\|w(t)\|_3^2 + \int_0^t \|w_z(\tau)\|_3^2 d\tau \leq C_1\|w_0\|_3^2 \quad \text{for} \quad 0 \leq t < T. \tag{3.6}$$

Proof. Let

$$N(T) = \sup_{0 \leq t \leq T} \|w(t)\|_3.$$

One has by the Sobolev Lemma 2.1

$$\sup_{z \in \mathbb{R}} |w(t, z), w_z(t, z), w_{zz}(t, z)| \leq CN(T). \tag{3.7}$$

We multiply (3.5) by $2w$, integrate over $\mathbb{R}$ and integrate by parts to obtain

$$\begin{aligned}\int_{\mathbb{R}} \left[(w^2)_t - Q''(\phi)\phi_z^2 w^2 - (w_z^2)_z + 2\mu w_z^2\right] dz &= 2\int_{\mathbb{R}} F(\phi, w_z) w dz \\ &= \int_{\mathbb{R}} f''\big(\phi + \theta w_z\big) w_z^2 w dz.\end{aligned}$$

Note that ϕ is bounded, see Pan [13, Corollary 2.3.2], that $\theta(t,z) \in]0,1[$ and by (3.7) $|w_z| \le CN(T)$. Therefore, f'' is bounded and (3.7) applied to w gives

$$\int_{\mathbb{R}} f''\big(\phi + \theta w_z\big) w_z^2 w dz \le C_N N(T) \int_{\mathbb{R}} w_z^2 dz.$$

Here the constant $C_N = C \cdot \sup_{u \in [u_+ - CN(T), u_- + CN(T)]} f''(u)$ remains bounded when $N(T)$ becomes small. Also $Q''(\phi) = f''(\phi) > 0$ and $\phi_z < 0$. This gives

$$\frac{d}{dt}\int_{\mathbb{R}} w^2 dz + (2\mu - CN(T)) \int_{\mathbb{R}} w_z^2 dz < 0. \tag{3.8}$$

Here we need to take $\epsilon_1 < \frac{2\mu}{C}$ such that $2\mu - CN(T) > 0$ for $N(T) \le \epsilon_1$. Then (3.8) implies that $\|w(t)\|^2$ is strictly decreasing in time. We now estimate higher derivatives. Let $v = w_z$, we have

$$v_t + Q'(\phi) v_z + v_{zzz} - \mu v_{zz} = F_z - Q'(\phi)_z v. \tag{3.9}$$

Multiplying (3.9) by $2v$ and integrating over $\mathbb{R}$ by parts, we get

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} w_z^2 dz + 2\mu \int_{\mathbb{R}} w_{zz}^2 dz - C_\phi \int_{\mathbb{R}} \phi_z w_z^2 dz &= \int_{\mathbb{R}} F_z w_z dz - \int_{\mathbb{R}} Q'(\phi)_z w_z^2 dz \\ &\le CN(T)\Big[\int_{\mathbb{R}} w_z^2 dz + \int_{\mathbb{R}} w_{zz}^2 dz\Big] + C_\phi \int_{\mathbb{R}} w_z^2 dz,\end{aligned} \tag{3.10}$$

where we have used the fact that $|F_z| \le C(|w_z|^2 + |w_{zz}|^2)$. Similarly, we can obtain the following estimates

$$\frac{d}{dt}\int_{\mathbb{R}} w_{zz}^2 dz + (2\mu - CN(T)) \int_{\mathbb{R}} w_{zzz}^2 dz \le CN(T) \int_{\mathbb{R}} w_{zz}^2 dz + C_\phi\Big(\int_{\mathbb{R}} w_z^2 dz + \int_{\mathbb{R}} w_{zz}^2 dz\Big) \tag{3.11}$$

and

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}} w_{zzz}^2 dz + (2\mu - CN(T)) \int_{\mathbb{R}} w_{zzzz}^2 dz \\ &\quad \le CN(T) \int_{\mathbb{R}} w_{zzz}^2 dz + C_\phi\Big(\int_{\mathbb{R}} w_z^2 dz + \int_{\mathbb{R}} w_{zz}^2 dz + \int_{\mathbb{R}} w_{zzz}^2 dz\Big).\end{aligned} \tag{3.12}$$

Denoting $D^j w = \frac{\partial^j w}{\partial z^j}$ we rewrite (3.10),(3.11),(3.12) as

$$(\mathrm{E}_j) \qquad \frac{d}{dt}\int_{\mathbb{R}} |D^j w|^2 dz + (2\mu - CN(T)) \int_{\mathbb{R}} |D^{j+1} w|^2 dz \\ \leq CN(T) \int_{\mathbb{R}} |D^j w|^2 dz + C_\phi \sum_{k=1}^{j} \int_{\mathbb{R}} |D^k w|^2 dz$$

for $j = 1, 2, 3$. Take $\bar{\mu} \leq 1$, to be fixed in a moment. Multiply (E_j) by $\bar{\mu}^j$, $j = 1, 2, 3$, and add these inequalities to (3.8). We then obtain

$$\frac{d}{dt}\sum_{j=0}^{3} \bar{\mu}^j \int_{\mathbb{R}} |D^j w|^2 dz + (2\mu - CN(T)) \sum_{j=0}^{3} \bar{\mu}^j \int_{\mathbb{R}} |D^{j+1} w|^2 dz \\ \leq \bar{\mu}\big(CN(T) + 3C_\phi\big) \sum_{j=1}^{3} \bar{\mu}^{j-1} \int_{\mathbb{R}} |D^j w|^2 dz. \tag{3.13}$$

Now fix $\bar{\mu} < \frac{\mu}{3C_\phi}$. As long as

$$CN(T)(1 + \bar{\mu}) \leq \mu - 3\bar{\mu}C_\phi, \tag{3.14}$$

we get from inequality (3.13)

$$\frac{d}{dt}\sum_{j=0}^{3} \bar{\mu}^j \int_{\mathbb{R}} |D^j w|^2 dz + \mu \sum_{j=0}^{3} \bar{\mu}^j \int_{\mathbb{R}} |D^{j+1} w|^2 dz < 0. \tag{3.15}$$

Inequality (3.14) is true if we take $\epsilon_1 = \frac{\mu - 3\bar{\mu}C_\phi}{C(1+\bar{\mu})}$. Thus, integration of (3.15) in t establishes the a priori estimate (3.6). □

Remark In the proof of a priori estimate we used the additional regularity of the form $w_z \in L^2([0,T], H^3(\mathbb{R}))$, which was already proved in the local existence Theorem 3.2.17 by the energy method. In the a priori estimate parts this regularity should be of course assumed. □

Proof of Theorem 3.2 Taking $\epsilon = \min\{\frac{\epsilon_1}{2}, \frac{\epsilon_1}{2\sqrt{C_1}}\}$ we obtain the global existence by combining the a priori estimate (3.6) with the local existence given by Theorem 2.18. In fact, for the initial data w_0 with $\|w_0\|_3 \leq \epsilon$ we have $\|w_0\| \leq \frac{\epsilon_1}{2}$. By Theorem 2.18 there exists a $T(\epsilon_1)$ such that the problem (3.5),(3.4) has a solution w in $C^0([0, T(\epsilon_1)], H^3(\mathbb{R}))$ satisfying

$$\|w(t)\|_3 \leq 2\|w_0\|_3 \leq \epsilon_1 \quad \text{for} \quad 0 \leq t \leq T(\epsilon_1).$$

Therefore the a priori estimate (3.6) gives

$$\|w(T(\epsilon_1))\|_3 \le \sqrt{C_1}\|w_0\|_3 \le \frac{\epsilon_1}{2},$$

where we have used $\|w_0\|_3 \le \epsilon \le \frac{\epsilon_1}{2\sqrt{C_1}}$. Applying again Theorem 2.18 with initial time $t = T(\epsilon_1)$ we obtain a solution w in $[T(\epsilon_1), 2T(\epsilon_1)]$ satisfying $\|w(t)\|_3 \le 2\|w(T(\epsilon_1))\|_3 \le \epsilon_1$ for $T(\epsilon_1) \le t \le 2T(\epsilon_1)$, thereby in $C([0, 2T(\epsilon_1)], H^3(\mathbb{R}))$ with $\sup_{0\le t\le 2T(\epsilon_1)} \|w(t)\|_3 \le \epsilon_1$. Hence we can use the a priori estimate Lemma 3.3.3 on $[0, 2T(\epsilon_1)]$ to get

$$\|w(2T(\epsilon_1))\|_3 \le \sqrt{C_1}\|w_0\|_3 \le \epsilon_1/2.$$

In the same way we can extend the solution to the interval $[0, nT(\epsilon_1)]$ successively, $n = 1, 2, \cdots$, and get a global solution.

Now we show that $v = w_z$ decays to zero in $H^2(\mathbb{R})$ as $t \to \infty$. By the priori estimate (3.6) we know that $\|w(t)\|^2$ and $\|w(t)\|^2 + \sum_{j=1}^3 \|D^j w(t)\|^2$ are strictly decreasing in time; hence, they approach limits as $t \to \infty$. Then $\sum_1^3 \|D^j w(t)\|^2$ approaches a limit as $t \to \infty$, and since this quantity is integrable over $t \in [0, \infty[$ by (3.6), it follows that the limit is zero, so

$$\|w_z\|_2^2 \le \sum_{j=1}^{3} \|D^j w(t)\|^2 \to 0 \qquad \text{as} \quad t \to \infty.$$

□

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