CONTINUOUS DEPENDENCE ESTIMATES FOR VISCOSITY SOLUTIONS OF FULLY NONLINEAR DEGENERATE ELLIPTIC EQUATIONS

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ABSTRACT. Using the maximum principle for semicontinuous functions [3, 4], we prove a general "continuous dependence on the nonlinearities" estimate for bounded Hölder continuous viscosity solutions of fully nonlinear degenerate elliptic equations. Furthermore, we provide existence, uniqueness, and Hölder continuity results for bounded viscosity solutions of such equations. Our results are general enough to encompass Hamilton-Jacobi-Bellman-Isaacs's equations of zero-sum, two-player stochastic differential games. An immediate consequence of the results obtained herein is a rate of convergence for the vanishing viscosity method for fully nonlinear degenerate elliptic equations.

1. INTRODUCTION

We are interested in bounded continuous viscosity solutions of fully nonlinear degenerate elliptic equations of the form

$$F(x, u(x), Du(x), D^2u(x)) = 0 \quad \text{in} \quad \mathbb{R}^N, \tag{1.1}$$

where the usual assumptions on the nonlinearity F are given in Section 2 (see also [4]). We are here concerned with the problem of finding an upper bound on the difference between a viscosity subsolution u of (1.1) and a viscosity supersolution $\bar{u}$ of

$$\bar{F}(x, \bar{u}(x), D\bar{u}(x), D^2\bar{u}(x)) = 0 \quad \text{in} \quad \mathbb{R}^N, \tag{1.2}$$

where $\bar{F}$ is another nonlinearity satisfying the assumptions given in Section 2. The sought upper bound for $u - \bar{u}$ should in one way or another be expressed in terms of the difference between the nonlinearities "$F - \bar{F}$".

A continuous dependence estimate of the type sought here was obtained in [7] for first order time-dependent Hamilton-Jacobi equations. For second order partial differential equations, a straightforward applications of the comparison principle [4] gives a useful continuous dependence estimate when, for example, $\bar{F}$ is of the form $\bar{F} = F + f$ for some function $f = f(x)$. In general, the usefulness of the continuous estimate provided by the comparison principle [4] is somewhat limited. For example, it cannot be used to obtain a convergence rate for the vanishing viscosity method, i.e., an explicit estimate (in terms of $\nu > 0$) of the difference between the viscosity solution u of (1.1) and the viscosity solution u^ν of the uniformly elliptic equation

$$F(x, u^\nu(x), Du^\nu(x), D^2u^\nu(x)) = \nu\Delta u^\nu(x) \quad \text{in} \quad \mathbb{R}^N. \tag{1.3}$$

Continuous dependence estimates for degenerate parabolic equations that imply, among other things, a rate of convergence for the corresponding viscosity method have appeared recently in [2] and [5]. In particular, the results in [5] are general enough to include, among others, the Hamilton-Jacobi-Bellman equation associated with optimal control of a degenerate diffusion process. Continuous dependence estimates for the Hamilton-Jacobi-Bellman equation have up to now been derive via probabilistic arguments, which are entirely avoided in [5].

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The main purpose of this paper is to prove a general continuous dependence estimate for fully nonlinear degenerate elliptic equations. In addition, we establish existence, uniqueness, and Hölder continuity results for bounded viscosity solutions. Although the results presented herein cannot be found in the existing literature, their proofs are (mild) adaptions (as are those in [2, 5]) of the standard uniqueness machinery for viscosity solutions [4], which relies in turn on the maximum principle for semicontinuous functions [3, 4]. In [2, 5], the results are stated for nonlinearities $F, \bar{F}$ with a particular form, and as such the results are not entirely general. In this paper, we avoid this and our main result (Theorem 2.1) covers general nonlinearities $F, \bar{F}$.

We present examples of equations which are covered by our results. In particular, an explicit continuous dependence estimate is stated for the second order Hamilton-Jacobi-Bellman-Isaacs equations associated with zero-sum, two-player stochastic differential games (see, e.g., [8] for a viscosity solution treatment of these equations). For these equations such a result is usually derived via probabilistic arguments, which we avoid entirely here. Also, it is worthwhile mentioning that a continuous dependence estimate of the type derived herein is needed for the proof in [1] of the rate of convergence for approximation schemes for Hamilton-Jacobi-Bellman equations.

The rest of this paper is organized as follows: In Section 2 we state prove our main results. In Section 3 we present examples of equations covered by our results. Finally, in Appendix A we prove some Hölder regularity results needed in section 2.

Notation. Let $|\cdot|$ be defined as follows: $|x|^2 = \sum_{i=1}^m |x_i|^2$ for any $x \in \mathbb{R}^m$ and any $m \in \mathbb{N}$. We also let $|\cdot|$ denote the matrix norm defined by $|M| = \sup_{e\in\mathbb{R}^p} \frac{|Me|}{|e|}$, where $M \in \mathbb{R}^{m\times p}$ is a $m \times p$ matrix and $m, p \in \mathbb{N}$. We denote by $\mathbb{S}^N$ the space of symmetric $N \times N$ matrices, and let B_R and $\mathbb{B}_R$ denote balls of radius R centered at the origin in $\mathbb{R}^N$ and $\mathbb{S}^N$ respectively. Finally, we let $\leq$ denote the natural orderings of both numbers and square matrices.

Let $USC(U)$, $C(U)$ and $C_b(U)$ denote the spaces of upper semicontinuous functions, continuous functions, and bounded continuous functions on the set U. If $f : \mathbb{R}^N \to \mathbb{R}^{m\times p}$ is a function and $\mu \in (0, 1]$, then define the following (semi) norms:

$$|f|_0 = \sup_{x\in\mathbb{R}^N} |f(x)|, \qquad [f]_\mu = \sup_{\substack{x,y\in\mathbb{R}^N \\ x\neq y}} \frac{|f(x) - f(y)|}{|x-y|^\mu}, \qquad \text{and} \qquad |f|_\mu = |f|_0 + [f]_\mu.$$

By $C_b^{0,\mu}(\mathbb{R}^N)$ we denote the set of functions $f : \mathbb{R}^N \to \mathbb{R}$ with finite norm $|f|_\mu$.

2. The Main Result

We consider the fully nonlinear degenerate elliptic equation in (1.1). The following assumptions are made on the nonlinearity $F : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N \to \mathbb{R}$:

(C1) For every $R > 0, F \in C(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N)$ is uniformly continuous on $\mathbb{R}^N \times [-R, R] \times B_R \times \mathbb{B}_R$.

(C2) For every $x, r, p,$ if $X, Y \in \mathbb{S}^N, X \leq Y,$ then $F(x, r, p, X) \geq F(x, r, p, Y)$.

(C3) For every $x, p, X,$ and for $R > 0,$ there is $\gamma_R > 0$ such that
$$F(x, r, p, X) - F(x, s, p, X) \geq \gamma_R(r - s), \text{ for } -R \leq s \leq r \leq R.$$

Our main result is stated in the following theorem:

Theorem 2.1 (Continuous Dependence Estimate). *Let F and $\bar{F}$ be functions satisfying assumptions (C1) – (C3). Moreover, let the following assumption hold for some $\eta_1, \eta_2 \geq 0$, $\mu \in (0, 1]$, and $M, K > 0$:*

$$\begin{aligned} &F(x, r, \alpha(x-y) + z, X) - \bar{F}(y, r, \alpha(x-y) + z, Y) \\ &\leq K\Big(|x-y|^\mu + \eta_1 + \alpha\left(|x-y|^2 + \eta_2^2\right) + |z| + \varepsilon\left(1 + |x|^2 + |y|^2\right)\Big), \end{aligned} \tag{2.1}$$

for $\alpha, \varepsilon > 0,\ x, y, z \in \mathbb{R}^N,\ r \in \mathbb{R},\ |z|, |r| \leq M,$ and $X, Y \in \mathbb{S}^N$ satisfying

$$\frac{1}{K}\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \varepsilon \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{2.2}$$

If $u, \bar{u} \in C_b^{0,\mu_0}(\mathbb{R}^N)$, $\mu_0 \in (0, \mu]$, satisfy in the viscosity sense $F[u] \le 0$ and $\bar{F}[\bar{u}] \ge 0$, then there is a constant $C > 0$ such that:

$$\sup_{\mathbb{R}^N}(u - \bar{u}) \le \frac{C}{\gamma}\left(\eta_1 + \eta_2^{\mu_0}\right).$$

Remark 2.2. For simplicity, we consider only equations without boundary conditions. However, the techniques used herein can be applied to the classical Dirichlet and Neumann problems. The classical Dirichlet boundary condition can be handled in the same way as the initial condition is in [5]. The Neumann boundary condition can be analyzed as in [2]. On the other hand, we are not able to treat so-called boundary conditions in the viscosity sense [4, section 7C].

Before giving the proof, we state and prove the following technical lemma:

Lemma 2.3. *Let $f \in USC(\mathbb{R}^N)$ be bounded and define $m, m_\varepsilon \ge 0$, $x_\varepsilon \in \mathbb{R}^n$ as follows:*

$$m_\varepsilon = \max_{x \in \mathbb{R}^n}\{f(x) - \varepsilon|x|^2\} = f(x_\varepsilon) - \varepsilon|x_\varepsilon|^2, \qquad m = \sup_{x \in \mathbb{R}^n} f(x).$$

Then as $\varepsilon \to 0$, $m_\varepsilon \to m$ and $\varepsilon|x_\varepsilon|^2 \to 0$.

Proof. Choose any $\eta > 0$. By the definition of supremum there is an $x' \in \mathbb{R}^N$ such that $f(x') \ge m - \eta$. Pick an ε' so small that $\varepsilon'|x'|^2 < \eta$, then the first part follows since

$$m \ge m_{\varepsilon'} = f(x_{\varepsilon'}) - \varepsilon'|x_{\varepsilon'}|^2 \ge f(x') - \varepsilon'|x'|^2 \ge m - 2\eta.$$

Now define $k_\varepsilon = \varepsilon|x_\varepsilon|^2$. This quantity is bounded by the above calculations since f is bounded. Pick a converging subsequence $\{k_\varepsilon\}_\varepsilon$ and call the limit k (≥ 0). Note that $f(x_\varepsilon) - k_\varepsilon \le m - k_\varepsilon$, so going to the limit yields $m \le m - k$. This means that $k \le 0$, that is $k = 0$. Now we are done since if every subsequence converges to 0, the sequence has to converge to 0 as well. □

Proof of Theorem 2.1. Assume that F satisfies (C1) – (C3) and that u is Hölder continuous as in the statement of the theorem. Now define the following quantities

$$\begin{aligned}
\phi(x, y) &:= \frac{\alpha}{2}|x - y|^2 + \frac{\varepsilon}{2}\left(|x|^2 + |y|^2\right), \\
\psi(x, y) &:= u(x) - \bar{u}(y) - \phi(x, y), \\
\sigma &:= \sup_{x,y \in \mathbb{R}^N} \psi(x, y) := \psi(x_0, y_0),
\end{aligned}$$

where the existence of $x_0, y_0 \in \mathbb{R}^N$ is assured by the continuity of ψ and precompactness of sets of the type $\{\phi(x, y) > k\}$ for k close enough to σ. We shall derive a positive upper bound on σ, so we may assume that $\sigma > 0$.

We can now apply the maximum principle for semicontinuous functions [4, Theorem 3.2] to conclude that there are symmetric matrices $X, Y \in \mathbb{S}^N$ such that $(D_x\phi(x_0, y_0), X) \in \overline{\mathcal{J}}^{2,+}u(x_0)$, $(-D_y\phi(x_0, y_0), Y) \in \overline{\mathcal{J}}^{2,-}\bar{u}(y_0)$, where X and Y satisfy inequality (2.2) for some constant K. So by the definition of viscosity sub- and supersolutions we get

$$0 \le \bar{F}(y_0, \bar{u}(y_0), -D_y\phi(x_0, y_0), Y) - F(x_0, u(x_0), D_x\phi(x_0, y_0), X). \tag{2.3}$$

Since $\sigma > 0$ it follows that $u(x_0) \ge \bar{u}(y_0)$. We can now use (C3) (on F) and the fact that $u(x_0) - \bar{u}(y_0) = \sigma + \phi(x_0, y_0) \ge \sigma$ to introduce σ and to rewrite (2.3) in terms of $\bar{u}(y_0)$:

$$F(x_0, u(x_0), D_x\phi(x_0, y_0), X) - F(x_0, \bar{u}(y_0), D_x\phi(x_0, y_0), X) \ge \gamma(u(x_0) - \bar{u}(y_0)) \ge \gamma\sigma, \tag{2.4}$$

so that (2.3) becomes

$$\gamma\sigma \le \bar{F}(y_0, \bar{u}(y_0), -D_y\phi(x_0, y_0), Y) - F(x_0, \bar{u}(y_0), D_x\phi(x_0, y_0), X). \tag{2.5}$$

Furthermore, by Lemma 2.3 there is a continuous nondecreasing function $m : [0, \infty) \to [0, \infty)$ satisfying $m(0) = 0$ and

$$|x_0|, |y_0| \le \varepsilon^{-1/2} m(\varepsilon). \tag{2.6}$$

This implies that $|D_x\phi(x_0,y_0)+D_y\phi(x_0,y_0)| = \varepsilon|x_0+y_0| \le m(\varepsilon)$. So by (C1) we may replace $D_x\phi$ by $-D_y\phi$ in (2.4) such that

$$F(x_0,\bar u(y_0),D_x\phi(x_0,y_0),X) \ge F(x_0,\bar u(y_0),-D_y\phi(x_0,y_0),X) - \omega_F(m(\varepsilon)), \tag{2.7}$$

where ω_F denotes the modulus of continuity of F. Hence we may replace (2.5) by

$$\gamma\sigma \le \bar F(y_0,\bar u(y_0),-D_y\phi(x_0,y_0),Y) - F(x_0,\bar u(y_0),-D_y\phi(x_0,y_0),X) + \omega_F(m(\varepsilon)). \tag{2.8}$$

Since $-D_y\phi(x_0,y_0) = \alpha(x_0-y_0)-\varepsilon y_0$, we set $z=-\varepsilon y_0$ in (2.1). Then by (2.1) and (2.8), the following estimate holds:

(2.9)
$$\gamma\sigma \le \text{Const}\left[|x_0-y_0|^{\mu_0} + \eta_1 + \alpha\left(|x_0-y_0|^2+\eta_2^2\right) + m(\varepsilon) + \varepsilon\left(1+|x_0|^2+|y_0|^2\right)\right] + \omega_F(m(\varepsilon)).$$

By considering the inequality $2\psi(x_0,y_0)\ge\psi(x_0,x_0)+\psi(y_0,y_0)$, and Hölder continuity of u and $\bar u$, we find

$$\alpha|x_0-y_0|^2 \le u(x_0)-u(y_0)+\bar u(x_0)-\bar u(y_0) \le \text{Const}\,|x_0-y_0|^{\mu_0},$$

which means that $|x_0-y_0| \le \text{Const}\,\alpha^{-1/(2-\mu_0)}$. Using this estimate and (2.6), we see that (2.9) is equivalent to

$$\gamma\sigma \le \text{Const}\left[\alpha^{-\frac{\mu}{2-\mu_0}} + \eta_1 + \alpha^{-\frac{\mu_0}{2-\mu_0}} + \alpha\eta_2^2\right] + \omega(\varepsilon), \tag{2.10}$$

for some modulus ω. Without loss of generality, we may assume $\eta_2^2<1$. Now we choose α such that $\alpha^{-\mu_0/(2-\mu_0)} = \alpha\eta_2^2$, and observe that this implies that $\alpha>1$, which again means that $\alpha^{-\mu/(2-\mu_0)} < \alpha^{-\mu_0/(2-\mu_0)}$. Thus we can bound the the smaller term by the larger term. By the definition of σ, $u(x)-\bar u(x)-\varepsilon 2|x|^2 \le \sigma$ for any $x\in\mathbb{R}^N$, so substituting our choice of α into (2.10), leads to the following expression

$$\gamma(u(x)-\bar u(x)) \le \text{Const}\,(\eta_1+\eta_2^{\mu_0}) + \omega(\varepsilon) + \gamma\varepsilon 2|x|^2,$$

and we can conclude by sending ε to 0. □

Next we state results regarding existence, uniqueness, and Hölder continuity of bounded viscosity solutions of (1.1). To this end, make the following natural assumptions:

(C4) There exist $\mu\in(0,1]$, $K>0$, and $\gamma_{0R},\gamma_{1R},K_R>0$ for any $R>0$ such that for any $\alpha,\varepsilon>0$, $x,y\in\mathbb{R}^N$, $-R\le r\le R$, $X,Y\in\mathbb{S}^N$ satisfying (2.2)
$$|F(x,r,\alpha(x-y),X)-F(y,r,\alpha(x-y),Y)| \le \gamma_{0R}|x-y|^\mu + \gamma_{1R}\alpha|x-y|^2 + K_R\varepsilon\left(1+|x|^2+|y|^2\right),$$

(C5) $M_F := \sup_{\mathbb{R}^N}|F(x,0,0,0)| < \infty$.

Theorem 2.4. *Assume that (C1) – (C5) hold and that $\gamma_R=\gamma$ is independent of R. Then there exists a unique bounded viscosity solution u of (1.1) satisfying $\gamma|u|_0\le M_F$.*

Proof. Under conditions (C1) – (C4) we have a strong comparison principle for bounded viscosity solutions of (1.1) (see also [4]). By assumptions (C3) and (C5) we see that M_F/γ and $-M_F/\gamma$ are classical supersolution and subsolution respectively of (1.1). Hence existence of a continuous viscosity solution satisfying the bound $\gamma|u|_0\le M_F$ follows from Perron's method, see [4]. Uniqueness of viscosity solutions follows from the comparison principle. □

Remark 2.5. The condition that γ_R be independent of R and condition (C5) are not necessary for having strong comparison and uniqueness.

Theorem 2.6. *Assume that (C1) – (C5) hold and that $\gamma_R=\gamma$ is independent of R. Then the bounded viscosity solution u of (1.1) is Hölder continuous with exponent $\mu_0\in(0,\mu]$.*

Proof. This theorem is consequence Lemmas A.1 and A.3, which are stated and proved in the appendix. □

The final result in this section concerns the rate of convergence for the vanishing viscosity method, which considers the uniformly elliptic equation (1.3). Existence, uniqueness, boundedness, and Hölder regularity of viscosity solutions of (1.3) follows from Theorems 2.4 and 2.6 under the same assumptions as for (1.1).

Theorem 2.7. *Assume that (C1) – (C5) hold and that $\gamma_R = \gamma$ is independent of R. Let u and u^ν be $C_b^{0,\mu_0}(\mathbb{R}^N)$ viscosity solutions of* (1.1) *and* (1.3) *respectively. Then* $|u - u^\nu|_0 \leq \text{Const}\,\nu^{\mu_0/2}$.

Proof. It is clear from Theorem 2.4, Lemma A.1, and the proof of Lemma A.3 that $|u^\nu|_{\mu_0}$ can be bounded independently of ν. Now we use Theorem 2.1 with $\bar{F}[u] = F[u] - \nu\Delta u$. This means that

$$\begin{aligned}&\bar{F}(x, r, \alpha(x-y), Y) - F(y, r, \alpha(a-y), X)\\&\quad\leq -\nu trY + \gamma_{0R}|x-y|^\mu + \gamma_{1R}\alpha|x-y|^2 + \varepsilon K_R\left(1 + |x|^2 + |y|^2\right),\end{aligned}$$

with $R = M_F/\gamma$. From (2.2) it follows that if e_i is a standard basis vector in $\mathbb{R}^N$, then $-e_i Y e_i \leq K(\alpha+\varepsilon)$, so $-trY \leq NK(\alpha+\varepsilon)$. This means that (2.1) is satisfied with $\eta_1 = 0$ and $\eta_2^2 = NK\nu$. Now Theorem 2.1 yield $u - u^\nu \leq \text{Const}\,\nu^{\mu_0/2}$. Interchanging u, F and $u^\nu, \bar{F}$ in the above argument yields the other bound. □

3. Applications

In this section, we give three typical examples of equations handled by our assumptions. It is quite easy to verify (C1) – (C5) for these problems. We just remark that in order to check (C4), it is necessary to use a trick by Ishii and the matrix inequality (2.2), see [4, Example 3.6].

Example 3.1 (Quasilinear equations).

$$-tr[\sigma(x,Du)\sigma(x,Du)^T D^2u] + f(x,u,Du) + \gamma u = 0 \quad \text{in} \quad \mathbb{R}^N,$$

where $\gamma > 0$, for any $R > 0$, σ and f are uniformly continuous on $\mathbb{R}^N \times B_R$ and $\mathbb{R}^N \times [-R, R] \times B_R$ respectively, and for any $R > 0$ there are $K, K_R > 0$ such that the following inequalities hold:

$$\begin{gathered}\sigma(x,p) \geq 0, \quad |\sigma(x,p) - \sigma(y,p)| \leq K|x-y|,\\ |f(x,t,p) - f(y,t,p)| \leq K_R\left(|p||x-y| + |x-y|^\mu\right), \quad \text{for } |t| \leq R,\\ f(x,t,p) \leq f(x,s,p) \text{ when } t \leq s, \quad |f(x,0,0)| \leq K,\end{gathered}$$

for any $x, y, p \in \mathbb{R}^N$ and $t, s \in \mathbb{R}$.

Example 3.2 (Hamilton-Jacobi-Bellman-Isaacs equations).

$$\sup_{\alpha\in\mathcal{A}}\inf_{\beta\in\mathcal{B}}\left\{-\,tr\left[\sigma^{\alpha,\beta}(x)\sigma^{\alpha,\beta}(x)^T D^2u\right] - b^{\alpha,\beta}(x)Du + c^{\alpha,\beta}(x)u + f^{\alpha,\beta}(x)\right\} = 0 \quad \text{in} \quad \mathbb{R}^N, \tag{3.1}$$

where $\mathcal{A}, \mathcal{B}$ are compact metric spaces, $c \geq \gamma > 0$, and $[\sigma^{\alpha,\beta}]_1, [b^{\alpha,\beta}]_1, [c^{\alpha,\beta}]_\mu, [f^{\alpha,\beta}]_\mu + |f^{\alpha,\beta}|_0$ are bounded independent of α, β.

Example 3.3 (Sup and inf of quasilinear operators).

$$\sup_{\alpha\in\mathcal{A}}\inf_{\beta\in\mathcal{B}}\left\{-\,tr\left[\sigma^{\alpha,\beta}(x,Du)\sigma^{\alpha,\beta}(x,Du)^T D^2u\right] + f^{\alpha,\beta}(x,u,Du) + \gamma u\right\} = 0 \quad \text{in} \quad \mathbb{R}^N,$$

where $\mathcal{A}, \mathcal{B}$ are as above, $\gamma > 0$, and σ, f continuous satisfies the same assumptions as in Example 3.1 uniformly in α, β.

We end this section by giving an explicit continuous dependence result for second order Hamilton-Jacobi-Bellman-Isaacs equations associated with zero-sum, two-player stochastic differential games with controls and strategies taking values in $\mathcal{A}$ and $\mathcal{B}$ (see Example 3.2).

We refer to [8] for an overview of viscosity solution theory and its application to the partial differential equations of deterministic and stochastic differential games.

Theorem 3.4. *Let u and $\bar u$ be viscosity solutions to* (3.1) *with coefficients* (σ, b, c, f) *and* $(\bar\sigma, \bar b, \bar c, \bar f)$ *respectively. Moreover, assume that both sets of coefficients satisfy the assumptions stated in Example 3.2. Then there is a* $\mu_0 \in (0, \mu]$ *such that* $u, \bar u \in C_b^{0,\mu_0}(\mathbb{R}^N)$ *and*

$$|u - \bar u|_0 \le C\Big(\sup_{\mathcal{A}\times\mathcal{B}} \Big[|\sigma^{\alpha,\beta} - \bar\sigma^{\alpha,\beta}|_0^{\mu_0} + |b^{\alpha,\beta} - \bar b^{\alpha,\beta}|_0^{\mu_0}\Big] + \sup_{\mathcal{A}\times\mathcal{B}} \Big[|c^{\alpha,\beta} - \bar c^{\alpha,\beta}|_0 + |f^{\alpha,\beta} - \bar f^{\alpha,\beta}|_0\Big]\Big),$$

for some constant C.

Proof. With

$$\eta_1 = \sup_{\mathcal{A}\times\mathcal{B}} \Big[|c^{\alpha,\beta} - \bar c^{\alpha,\beta}|_0 + |f^{\alpha,\beta} - \bar f^{\alpha,\beta}|_0\Big], \qquad \eta_2^2 = \sup_{\mathcal{A}\times\mathcal{B}} \Big[|\sigma^{\alpha,\beta} - \bar\sigma^{\alpha,\beta}|_0^2 + |b^{\alpha,\beta} - \bar b^{\alpha,\beta}|_0^2\Big],$$

we apply Theorem 2.1 to $u - \bar u$ and then to $\bar u - u$ to obtain the result. □

Appendix A. Hölder Regularity

We consider the two cases $\gamma > 2\gamma_1$ and $0 < \gamma < 2\gamma_1$ separately.

Lemma A.1. *Assume that (C1) – (C5) hold and that u is a bounded viscosity solution of* (1.1). *Let $R = |u|_0$, define $\gamma := \gamma_R$, and similarly define γ_0, γ_1, K. If $\gamma > 2\gamma_1$ then $u \in C_b^{0,\mu}$, and for all $x, y \in \mathbb{R}^N$,*

$$|u(x) - u(y)| \le \frac{\gamma_0}{\gamma - 2\gamma_1}|x - y|^\mu.$$

Proof. This proof is very close to the proof of Theorem 2.1, and we will only indicate the differences. Let σ, ϕ, x_0, y_0 be defined as in Theorem 2.1 when $\psi(x, y) = u(x) - u(y) - 2\phi(x, y)$. Note the factor 2 multiplying ϕ. We need this factor to get the right form of our estimate! A consequence of this is that we need to change α, ε to $2\alpha, 2\varepsilon$ whenever we use (C4) and (2.2). Now we proceed as in the proof of Theorem 2.1: We use the maximum principle and the definition of viscosity sub- and supersolutions (u is both!), we use the uniform continuity (C1) to get rid of unwanted terms in the gradient slot of F, we use (C3) together with

$$u(x_0) - u(y_0) = \sigma + \alpha|x_0 - y_0|^2 + \varepsilon\left(|x_0|^2 + |y_0|^2\right) \ge \sigma + \alpha|x_0 - y_0|^2,$$

and finally we use (C4) and all the above to conclude that

$$\gamma\sigma \le \gamma_0|x_0 - y_0|^\mu - (\gamma - 2\gamma_1)\alpha|x_0 - y_0|^2 + \omega(\varepsilon), \tag{A.1}$$

for some modulus ω. Here we have also used the bounds (2.6) on x_0, y_0. Compare with (2.9).

Note that for any $k_1, k_2 > 0$,

$$\max_{r\ge 0}\left\{k_1 r^\mu - k_2\alpha r^2\right\} = \bar c_1 k_1^{\frac{2}{2-\mu}} (\alpha k_2)^{-\frac{\mu}{2-\mu}} \quad \text{where} \quad \bar c_1 = \Big(\frac{\mu}{2}\Big)^{\frac{\mu}{2-\mu}} - \Big(\frac{\mu}{2}\Big)^{\frac{2}{2-\mu}}. \tag{A.2}$$

Furthermore for fixed α, Lemma 2.3 yields

$$\lim_{\varepsilon\to 0}\sigma = \sup_{x,y\in\mathbb{R}^N}\left(u(x) - u(y) - \alpha|x - y|^2\right) := m.$$

So let $k_1 = \gamma_0$ and $k_2 = \gamma - 2\gamma_1$ (> 0 by assumption), and go to the limit $\varepsilon \to 0$ for α fixed in (A.2). The result is

$$m \le \frac{k_1^{\frac{2}{2-\mu}}}{\gamma k_2^{\frac{\mu}{2-\mu}}}\, \bar c_1 \alpha^{-\frac{\mu}{2-\mu}} \le \frac{\gamma - 2\gamma_1}{\gamma}\Big(\frac{\gamma_0}{\gamma - 2\gamma_1}\Big)^{\frac{2}{2-\mu}} \bar c_1 \alpha^{-\frac{\mu}{2-\mu}} \le k\alpha^{-\frac{\mu}{2-\mu}}, \tag{A.3}$$

where $k = \left(\frac{\gamma_0}{\gamma - 2\gamma_1}\right)^{\frac{2}{2-\mu}}\bar c_1$. Since, in view of (A.3),

$$u(x) - u(y) \le m + \alpha|x - y|^2 \le k\alpha^{-\frac{\mu}{2-\mu}} + \alpha|x - y|^2,$$

we can minimize with respect to α obtain

$$u(x) - u(y) \le \min_{\alpha\ge 0}\Big\{k\alpha^{-\frac{\mu}{2-\mu}} + \alpha|x - y|^2\Big\} = \bar c_2 k^{\frac{2-\mu}{2}}|x - y|^\mu,$$

where $\bar{c}_2 = \left(\frac{\mu}{2-\mu}\right)^{\frac{2-\mu}{2}} + \left(\frac{2-\mu}{\mu}\right)^{\frac{\mu}{2}}$.

Now we can conclude by substituting for k and observing that $\bar{c}_2 \bar{c}_1^{\frac{2-\mu}{2}} \equiv 1$. □

Remark A.2. Lemma A.1 is not sharp. It is possible to get sharper results using test functions of the type $\phi(x) = L|x-y|^\delta + \varepsilon\left(|x|^2 + |y|^2\right)$ and playing with all three parameters L, δ, ε. However assumption (C4) is adapted to the test functions used in this paper, so changing the test functions, require us to change assumption (C4) too.

We will now use the previous result and an iteration technique introduced in [6] (for first order equations) to derive Hölder continuity for solutions of (1.1) for $0 < \gamma < 2\gamma_1$. Note that since Lemma A.1 is not sharp, our next result will not be sharp either. We also note that in the case $\gamma = 2\gamma_1$ the Hölder exponent is of course at least as good as for $\gamma = 2\gamma_1 - \varepsilon$, $\varepsilon > 0$ small.

Lemma A.3. *Assume that (C1) – (C5) hold and that u is a bounded viscosity solution of* (1.1). *Let $R = |u|_0$, define $\gamma := \gamma_R$, and similarly define γ_0, γ_1, K. If $0 < \gamma < 2\gamma_1$ then $u \in C_b^{0,\mu_0}(\mathbb{R}^N)$ where $\mu_0 = \mu\frac{\gamma}{2\gamma_1}$.*

Proof. Let $\lambda > 0$ be such that $\gamma + \lambda > 2\gamma_1 + 1$ and let $v \in C_b^{0,\mu}(\mathbb{R}^N)$ be in the set

$$X := \left\{f \in C(\mathbb{R}^N) : |f|_0 \le M_F/\gamma\right\}.$$

Then note that $\pm M_F/\gamma$ are respectively super- and subsolutions of the following equation:

$$F(x, u(x), Du(x), D^2u(x)) + \lambda u(x) = \lambda v(x) \qquad \forall x \in \mathbb{R}^N. \tag{A.4}$$

Let T denote the operator taking v to the viscosity solution u of (A.4). It is well-defined because by Theorem 2.4 there exists a unique viscosity solution u of equation (A.4). Furthermore by Theorem A.1 and the fact that $\pm M_F/\gamma$ are respectively super- and subsolutions of (A.4), we see that

$$T : C_b^{0,\mu}(\mathbb{R}^N) \cap X \to C_b^{0,\mu}(\mathbb{R}^N) \cap X.$$

For $v, w \in C_b^{0,\mu}(\mathbb{R}^N) \cap X$ we note that $Tw - |w-v|_0\lambda/(\gamma+\lambda)$ and $Tv - |w-v|_0\lambda/(\gamma+\lambda)$ are both subsolutions of (A.4) but with different right hand sides, namely λv and λw respectively. So by using the comparison principle Theorem 2.4 twice (comparing with Tv and Tw respectively) we get:

$$|Tw - Tv|_0 \le \frac{\lambda}{\gamma+\lambda}|w-v|_0 \qquad \forall w, v \in C_b^{0,\mu}(\mathbb{R}^N) \cap X. \tag{A.5}$$

Let $u^0(x) = M_F/\gamma$ and $u^n(x) = Tu^{n-1}(x)$ for $n = 1, 2, \dots$. Since $C_b^{0,\mu}(\mathbb{R}^N) \cap X$ is a Banach space and T a contraction mapping (A.5) on this space, Banach's fix point theorem yields $u^n \to u \in C_b^{0,\mu}(\mathbb{R}^N) \cap X$. By the stability result for viscosity solutions of second order PDEs, see Lemma 6.1 and Remark 6.3 in [4], u is the viscosity solution of (1.1). Since

$$|u - u^n|_0 \le |u - u^{n+k}|_0 + \sum_{i=1}^{k} |u^{n+i} - u^{n+i-1}|_0,$$

using (A.5), sending $k \to \infty$, and then using (A.5) again, we obtain

$$|u - u^n|_0 \le \frac{1}{1 - \frac{\lambda}{\gamma+\lambda}}|u^{n+1} - u^n|_0 \le \frac{\gamma+\lambda}{\lambda}\Big(\frac{\lambda}{\gamma+\lambda}\Big)^n |u^1 - u^0|_0 \le \frac{2M_F}{\gamma}\Big(\frac{\lambda}{\gamma+\lambda}\Big)^{n-1}. \tag{A.6}$$

Furthermore by Theorem A.1 we have the following estimate on the Hölder seminorm of u^n:

$$[u^n]_\mu \le \frac{\gamma_0 + \lambda[u^{n-1}]_\mu}{\gamma + \lambda - 2\gamma_1} \le \left(\frac{\lambda}{\gamma + \lambda - 2\gamma_1}\right)^{n-1}\left([u^0]_\mu + K\right), \tag{A.7}$$

where the constant K does not depend on n or $\lambda(\ge 1)$. Now let $m = n-1$, $x, y \in \mathbb{R}^N$, and note that

$$|u(x) - u(y)| \le |u(x) - u^n(x)| + |u^n(x) - u^n(y)| + |u^n(y) - u(y)|.$$

Using (A.6) and (A.7) we get the following expression:

$$|u(x)-u(y)| \leq \mathrm{Const}\left\{\left(\frac{\lambda}{\gamma+\lambda}\right)^m + \left(\frac{\lambda}{\gamma+\lambda-2\gamma_1}\right)^m |x-y|^\mu\right\}. \tag{A.8}$$

Now let $t=|x-y|$ and ω be the modulus of continuity of u. Fix $t \in (0,1)$ and define λ in the following way:

$$\lambda := \frac{2\gamma_1}{\mu}\frac{m}{\log\frac{1}{t}}.$$

Note that if m_t is sufficiently large, then $m \geq m_t$ implies that $\lambda \geq \gamma_1$. Using this new notation, we can rewrite (A.8) the following way:

$$\omega(t) \leq \mathrm{Const}\left\{\left(1+\frac{\mu\gamma}{2\gamma_1}\log\left(\frac{1}{t}\right)\frac{1}{m}\right)^{-m} + \left(1+\mu\frac{\gamma-2\gamma_1}{2\gamma_1}\log\left(\frac{1}{t}\right)\frac{1}{m}\right)^{-m} t^\mu\right\}.$$

Letting $m \to \infty$, we obtain

$$\omega(t) \leq \mathrm{Const}\left\{t^{\mu\gamma/2\gamma_1} + t^{\mu\gamma/2\gamma_1-\mu}t^\mu\right\}.$$

Now we can conclude since this inequality must hold for any $t \in (0,1)$. □

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