On the Boundary Control of Systems of Conservation Laws

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Abstract. The paper is concerned with the boundary controllability of entropy weak solutions to hyperbolic systems of conservation laws. We prove a general result on the asymptotic stabilization of a system near a constant state. On the other hand, we give an example showing that exact controllability in finite time cannot be achieved, in general.

1 - Introduction

Consider an $n \times n$ system of conservation laws on a bounded interval:

$$u_t + f(u)_x = 0 \qquad\qquad t \geq 0, \quad x \in]a, b[\,. \tag{1.1}$$

The system is assumed to be strictly hyperbolic, each characteristic field being either linearly degenerate or genuinely nonlinear in the sense of Lax [8]. We shall also assume that all characteristic speeds are bounded away from zero. More precisely, let $f : \Omega \mapsto \mathbb{R}^n$ be a smooth map, defined on an open set $\Omega \subseteq \mathbb{R}^n$. For each $u \in \Omega$, call $\lambda_1(u) < \cdots < \lambda_n(u)$ the eigenvalues of the Jacobian matrix $Df(u)$. We assume that there exists a minimum speed $c_0 > 0$ and an integer $p \in \{1, \ldots, n\}$ such that

$$\begin{cases} \lambda_i(u) < 0 & \text{if} \quad i \leq p, \\ \lambda_i(u) > 0 & \text{if} \quad i > p, \end{cases} \tag{1.2}$$

$$\big|\lambda_i(u)\big| \geq c_0 > 0 \qquad\qquad u \in \Omega. \tag{1.3}$$

By (1.2), for a solution defined on the strip $t \geq 0, \quad x \in]a, b[\,$, there will be $n - p$ characteristics entering at the boundary point $x = a$, and p characteristics entering at $x = b$. The initial-boundary value problem is thus well posed if we prescribe $n - p$ scalar conditions at $x = a$ and p scalar conditions at $x = b$ [11]. See also [1, 2] for the case of general entropy-weak solutions taking values in the space BV of functions with bounded variation.

In the present paper we study the effect of boundary conditions on the solution of (1.1) from the point of view of control theory. Namely, given an initial condition

$$u(0, x) = \phi(x) \qquad\qquad x \in]a, b[\tag{1.4}$$

with small total variation, we regard the boundary data as *control functions*, and study the family of configurations

$$\mathcal{R}(T) \doteq \big\{u(T, \cdot)\big\} \subset \mathbf{L}^1\big([a, b]\,;\ \mathbb{R}^n\big) \tag{1.5}$$

which can be reached by the system at a given time $T > 0$.

Beginning with the simplest case, consider a strictly hyperbolic system with constant coefficients:

$$u_t + Au_x = 0, \tag{1.6}$$

where A is a $n \times n$ constant matrix, with real distinct eigenvalues

$$\lambda_1 < \cdots < \lambda_p < 0 < \lambda_{p+1} < \cdots < \lambda_n\,.$$

Call

$$\tau \doteq \max_i \frac{b - a}{|\lambda_i|}$$

the maximum time taken by waves to cross the interval $[a, b]$. In this case, it is easy to see that the reachable set in (1.5) is the entire space: $\mathcal{R}(T) = \mathbf{L}^1$ for all $T \geq \tau$. In other words, the system is completely controllable after time τ. Indeed, for any $T \geq \tau$ and initial and terminal data $\phi, \psi \in \mathbf{L}^1\big([a, b]; \mathbb{R}^n\big)$, one can always find a solution of (1.4), defined on the rectangle $[0, T] \times [a, b]$ such that

$$u(0, x) = \phi(x), \qquad\qquad u(T, x) = \psi(x) \qquad\qquad x \in [a, b].$$

Such solution can be constructed as follows. Let $l_1, \ldots, l_n$ and $r_1, \ldots, r_n$ be dual bases of right and left eigenvectors of A so that $l_i \cdot r_j = \delta_{ij}$. For $i = 1, \ldots, n$, let $u_i(t,x)$ be a solution to the scalar Cauchy problem

$$u_{i,t} + \lambda_i u_{i,x} = 0,$$

$$u_i(0,x) = \begin{cases} l_i \cdot \phi(x) & \text{if} \quad x \in [a,b], \\ l_i \cdot \psi(x + \lambda_i T) & \text{if} \quad x \in [a - \lambda_i T,\ b - \lambda_i T], \\ 0 & \text{otherwise.} \end{cases}$$

Then the restriction of

$$u(t,x) = \sum_i u_i(t,x) r_i$$

to the interval $[0,T] \times [a,b]$ satisfies (1.6) and takes the required initial and terminal values. Of course, this corresponds to the solution of an initial-boundary value problem, determined by the n boundary conditions

$$\begin{cases} l_i \cdot u(t,a) = u_i(t,a) & i = p+1, \ldots, n, \\ l_i \cdot u(t,b) = u_i(t,b) & i = 1, \ldots, p. \end{cases}$$

This result on exact boundary controllability has been extended in [9, 10] to the case of general quasilinear systems of the form

$$u_t + A(u) u_x = 0.$$

In this case, the existence of a solution taking the prescribed initial and terminal values is obtained for all sufficiently small data $\phi, \psi \in \mathcal{C}^1$.

Aim of the present paper is to study analogous controllability properties within the context of entropy weak solutions $t \mapsto u(t,\cdot) \in BV$. For the definitions and basic properties of weak solutions we refer to [4]. For general nonlinear systems, it is clear that a complete controllability result within the space BV cannot hold. Indeed, already for a scalar conservation law, it was proved in [3] that the profiles $\psi \in BV$ which can be attained at a fixed time $T > 0$ are only those which satisfy the Oleinik-type conditions

$$\psi'(x) \le \frac{f'\big(\psi(x)\big)}{(x-a) f''\big(\psi(x)\big)} \qquad \text{for a.e. } x \in [a,b].$$

For general $n \times n$ systems, a complete characterization of the reachable set $\mathcal{R}(T)$ does not seem possible, due to the complexity of repeated wave-front interactions.

Our first result is concerned with stabilization near a constant state. Assuming that all characteristic speeds are bounded away from zero, we show that the system can be asymptotically stabilized to any state $u^* \in \Omega$, with quadratic rate of convergence.

Theorem 1. *Let K be a compact, connected subset of the open domain $\Omega \subset \mathbb{R}^n$. Then there exist constants $C_0, \delta, \kappa > 0$ such that the following holds. For every constant state $u^* \in K$ and every initial data $u(0) = \phi : [a,b] \mapsto K$ with Tot.Var.$\{\phi\} < \delta$, there exists an entropy weak solution $u = u(t,x)$ of (1.1) such that, for all $t > 0$,*

$$\text{Tot.Var.}\big\{u(t)\big\} \le C_0\, e^{-2^{\kappa t}}, \tag{1.7}$$

$$\big\| u(t,x) - u^* \big\|_{L^\infty} \le C_0\, e^{-2^{\kappa t}}. \tag{1.8}$$

The proof will be given in Section 2. An interesting question is whether the constant state u^* can be exactly reached, in a finite time T. By the results in [9], this is indeed the case if the initial data has small $\mathcal{C}^1$ norm. On the contrary, in the final part of this paper, we show that exact controllability in finite time cannot be attained in general, if the initial data is only assumed to be small in BV.

Our counterexample is concerned with a class of strictly hyperbolic, genuinely nonlinear 2×2 systems of the form (1.1). More precisely, we assume

(H) The eigenvalues $\lambda_i(u)$ of the Jacobian matrix $A(u) = Df(u)$ satisfy

$$-\lambda^* < \lambda_1(u) < -\lambda_* \; < 0 \; < \lambda_* < \lambda_2(u) < \lambda^* . \tag{1.9}$$

Moreover, the right eigenvectors $r_1(u)$, $r_2(u)$ satisfy the inequalities

$$D\lambda_1 \cdot r_1 > 0, \qquad\qquad D\lambda_2 \cdot r_2 > 0, \tag{1.10}$$

$$r_1 \wedge r_2 < 0, \qquad r_1 \wedge (Dr_1 \cdot r_1) < 0, \qquad r_2 \wedge (Dr_2 \cdot r_2) < 0. \tag{1.11}$$

A partucular system which satisfies the above assumptions is the one studied by DiPerna [7]:

$$\begin{cases} \rho_t + (u\rho)_x = 0\,, \\ u_t + \left(\dfrac{u^2}{2} + \dfrac{K^2}{\gamma - 1}\rho^{\gamma-1}\right)_x = 0\,, \end{cases}$$

with $1 < \gamma < 3$. Here $\rho > 0$ and u denote the density and the velocity of a gas, respectively.

The last two inequalities in (1.11) imply that the rarefaction curves (i.e. the integral curves of the vector fields r_1, r_2) in the (u_1, u_2) plane turn clockwise (fig. 1). In such case, the interaction of two shocks of the same family generates a shock in the other family.

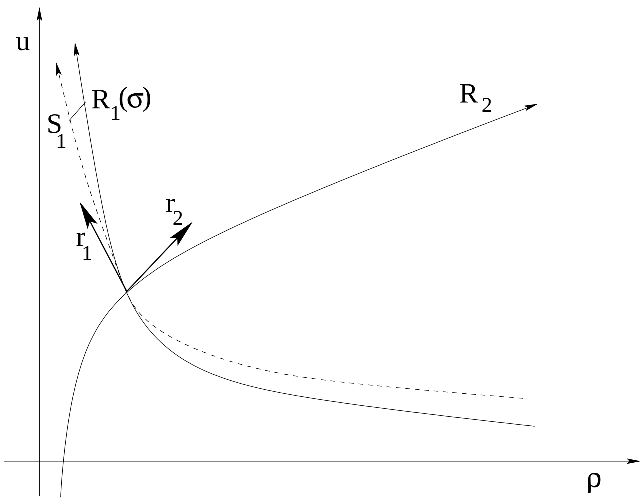

figure 1

Theorem 2. *Consider a 2×2 system satisfying the assumption (H). Then there exist initial data $\phi : [a,b] \mapsto I\!R^2$ having arbitrarily small total bounded variation for which the following holds. For every entropy weak solution u of (1.1), (1.4), with Tot.Var.$\{u(t,\cdot)\}$ remaining small for all t, the set of shocks in $u(t,\cdot)$ is dense on $[a,b]$, for each $t>0$. In particular, $u(t,\cdot)$ cannot be constant.*

As a preliminary, in Section 3 we establish an Oleinik-type estimate on the decay of positive waves. This bound is of independent interest, and sharpens the results in [5], for systems satisfying the additional conditions (H).

As a consequence, this implies that positive waves are "weak", and cannot completely cancel a shock within finite time. The proof of Theorem 2 is then achieved by an induction argument. We show that, if the set of 1-shocks is dense on $[0,T]\times[a,b]$, then the set of points $P_j = (t_j, x_j)$ where two 1-shocks interact and create a new 2-shock is also dense on the same domain. Therefore, new shocks are constantly generated, and the solution can never be reduced to a constant. Details of the proof will be given in Section 4.

As in [9], all of the above results refer to the case where total control on the boundary values is available. As a consequence, the problem is reduced to proving the existence (or nonexistence) of an entropy weak solution defined on the open strip $t>0$, $x\in\,]a,b[\,$, satisfying the required conditions. This is a first step toward the analysis of more general controllability problems, where the control acts only on some of the boundary conditions. We thus leave open the case where a subset of indices $I\subset\{1,\ldots,n\}$ is given, and one requires

$$l_i\cdot u(t,a) = \begin{cases} \alpha_i(t) & \text{if} \quad i\in I,\\ 0 & \text{if} \quad i\notin I,\end{cases} \qquad i = p+1,\ldots,n,$$

$$l_i\cdot u(t,b) = \begin{cases} \alpha_i(t) & \text{if} \quad i\in I,\\ 0 & \text{if} \quad i\notin I,\end{cases} \qquad i = 1,\ldots,p,$$

for some control functions α_i acting only on the components $i\in I$.

Throughout the following, we denote by $r_i(u)$, $l_i(u)$ the right and left i-eigenvectors of the Jacobian matrix $A(u)\doteq Df(u)$. As in [4], we write $\sigma\mapsto R_i(\sigma)(u_0)$ for the parametrized i-rarefaction curve through the state u_0, so that

$$\frac{d}{d\sigma}R_i(\sigma) = r_i\big(R_i(\sigma)\big), \qquad R_i(0) = u_0.$$

The i-shock curve through u_0 is denoted by $\sigma\mapsto S_i(\sigma)(u_0)$. It satisfies the Rankine-Hugoniot equations

$$f\big(S_i(\sigma)\big) - f(u_0) = \lambda_i(\sigma)\,\big(S_i(\sigma)-u_0\big)$$

for some shock speed λ_i. We recall (see [4], Chapter 5) that the general Riemann problem is solved in terms of the composite curves

$$\Psi_i(u_0)(\sigma) = \begin{cases} R_i(u_0)(\sigma), & \text{if } \sigma\geq 0,\\ S_i(u_0)(\sigma), & \text{if } \sigma<0.\end{cases} \tag{1.12}$$

2 - Proof of Theorem 1

The proof relies on the two following two lemmas.

Lemma 1. *In the setting of Theorem 1, there exists a time $T > 0$ such that the following holds. For every pair of states $\omega, \omega' \in K$ there exists an entropic solution $u = u(t, x)$ of (1.1) such that*

$$u(0, x) \equiv \omega, \qquad u(T, x) \equiv \omega' \qquad \text{for all } x \in [a, b]. \tag{2.1}$$

Proof. Consider the function

$$\Phi(\sigma_1, \ldots, \sigma_n \,;\; v,\, v') \doteq \Psi_n(\sigma_n) \circ \cdots \circ \Psi_{p+1}(\sigma_{p+1})(v') - \Psi_p(\sigma_p) \circ \cdots \circ \Psi_1(\sigma_1)(v). \tag{2.2}$$

Observe that, whenever $v = v'$, the $n \times n$ Jacobian matrix $\partial\Phi/\partial\sigma_1 \cdots \sigma_n$ computed at $\sigma_1 = \sigma_2 = \cdots = \sigma_n = 0$ has full rank. Indeed, the columns of this matrix are given by the linearly independent vectors $-r_1(v), \ldots, -r_p(v), r_{p+1}(v), \ldots, r_n(v)$. By the Implicit Function Theorem and a compactness argument we can find $\delta > 0$ such that the following holds. For every $v, v' \in K$, with $|v - v'| \le \delta$, there exist unique values $\sigma_1, \ldots, \sigma_n$ such that

$$v'' \doteq \Psi_n(\sigma_n) \circ \cdots \circ \Psi_{p+1}(\sigma_{p+1})(v') = \Psi_p(\sigma_p) \circ \cdots \circ \Psi_1(\sigma_1)(v)\,. \tag{2.3}$$

Defining the time

$$\tau \doteq \max_{1 \le i \le n} \sup_{u \in \Omega} \frac{b - a}{\big|\lambda_i(u)\big|}\,, \tag{2.4}$$

we claim that there exists an entropy weak solution $u : [0, 2\tau] \times [a, b] \mapsto \Omega$ such that

$$u(0, x) \equiv v, \qquad u(2\tau, x) \equiv v'. \tag{2.5}$$

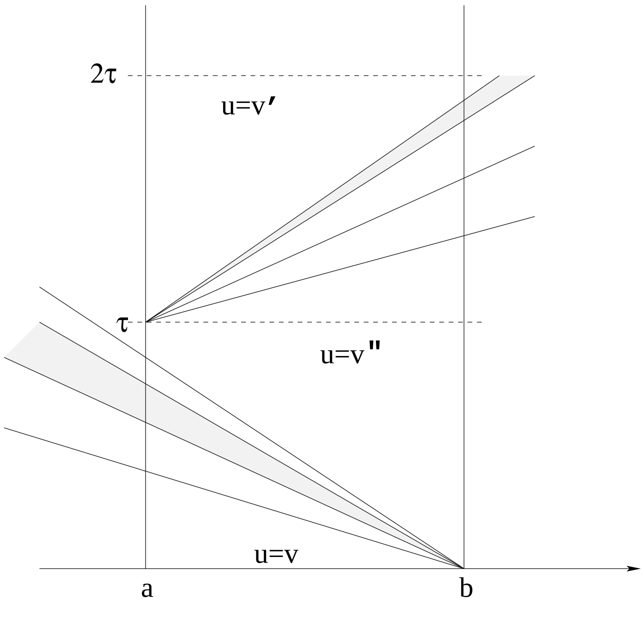

figure 2

The function u is constructed as follows (fig. 2). For $t \in [0, \tau]$ we let u be the solution of the Riemann problem

$$u(0, x) = \begin{cases} v & \text{if} \quad x < b, \\ v'' & \text{if} \quad x > b. \end{cases} \tag{2.6}$$

Moreover, for $t \in [\tau,\ 2\tau]$, we define u as the solution of the Riemann problem

$$u(\tau, x) = \begin{cases} v' & \text{if} \quad x < a, \\ v'' & \text{if} \quad x > a. \end{cases} \tag{2.7}$$

It is now clear that the restriction of u to the domain $[0,\ 2\tau] \times [a, b]$ satisfies the conditions (2.5). Indeed, by (2.3), on $[0, \tau]$ the solution u contains only waves of families $\leq p$, originating at the point $(0, b)$. By (2.4) these waves cross the whole interval $[a, b]$ and exit from the boundary point a before time τ. Hence $u(\tau, x) \equiv v''$. Similarly, still by (2.3), for $t \in [\tau,\ 2\tau]$ the function u contains only waves of families $\geq p + 1$, originating at the point (τ, a). By (2.4) these waves cross the whole interval $[a, b]$ and exit from the boundary point b before time 2τ. Hence $u(2\tau, x) \equiv v'$.

Next, given any two states $\omega, \omega' \in K$, by the connectedness assumption we can find a chain of points $\omega_0 = \omega, \omega_1, \ldots, \omega_N = \omega'$ in K such that $|\omega_i - \omega_{i-1}| < \delta$ for every $i = 1, \ldots, N$. Repeating the previous construction in connection with each pair of states (ω_{i-1}, ω_i), we thus obtain an entropy weak solution $u : [0,\ 2N\tau] \times [a, b] \mapsto \Omega$ that satisfies the conclusion of the lemma, with $T = 2N\tau$. □

In the following, we shall construct the desired solution $u = u(t, x)$ as limit of a sequence of front tracking approximations. Roughly speaking, an ε-approximate front tracking solution is a piecewise constant function u^ε, having jumps along a finite set of straight lines in the t-x plane say $x = x_\alpha(t)$, which approximately satisfies the Rankine-Hugoniot equations:

$$\sum_\alpha \Big| f\big(u(t, x_\alpha+)\big) - f\big(u(t, x_\alpha-)\big) - \dot{x}_\alpha \,\big(u(t, x_\alpha+) - u(t, x_\alpha-)\big)\Big| < \varepsilon$$

for all $t > 0$. For details, see [4], p.125.

Lemma 2. *In the setting of Theorem 1, for every state $u^* \in \Omega$ there exist constants $C, \delta_0 > 0$ for which the following holds. For any $\varepsilon > 0$ and every piecewise constant function $\bar{u} : [a, b] \mapsto \Omega$ such that*

$$\rho \doteq \sup_{x \in [a,b]} \big|\bar{u}(x) - u^*\big| \leq \delta_0, \qquad\qquad \delta \doteq \textit{Tot.Var.}\{\bar{u}\} \leq \delta_0, \tag{2.8}$$

there exists an ε-approximate front tracking solution $u = u(t, x)$ of (1.1), with $u(0, x) = \bar{u}(x)$, such that

$$\sup_{x \in [a,b]} \big|u(3\tau,\ x) - u^*\big| \leq C\delta^2, \qquad\qquad \text{Tot.Var.}\big\{u(3\tau)\big\} \leq C\delta^2\,. \tag{2.9}$$

Proof. On the domain $(t, x) \in [0, \tau] \times [a, b]$, we construct u as an ε-approximate front tracking solution in such a way that, whenever a front hits one of the boundaries $x = a$ or $x = b$, no reflected front is ever created (fig. 3). Since all fronts emerging from the initial data $\bar{u}$ at time $t = 0$ exit from $[a, b]$ within time τ, it is clear that $u(\tau)$ can contain only fronts of second or higher generation order. In other words, the only fronts that can be present in $u(\tau, \cdot)$ are the new ones, generated by interactions at times $t > 0$ (the dotted lines in fig. 3). Therefore, using the interaction estimate (7.69) in [4] we obtain

$$\sup_{x \in [a,b]} \big|u(\tau,\ x) - u^*\big| = \mathcal{O}(1) \cdot (\rho + \delta) \qquad\qquad \text{Tot.Var.}\big\{u(\tau)\big\} = \mathcal{O}(1) \cdot \delta^2\,. \tag{2.10}$$

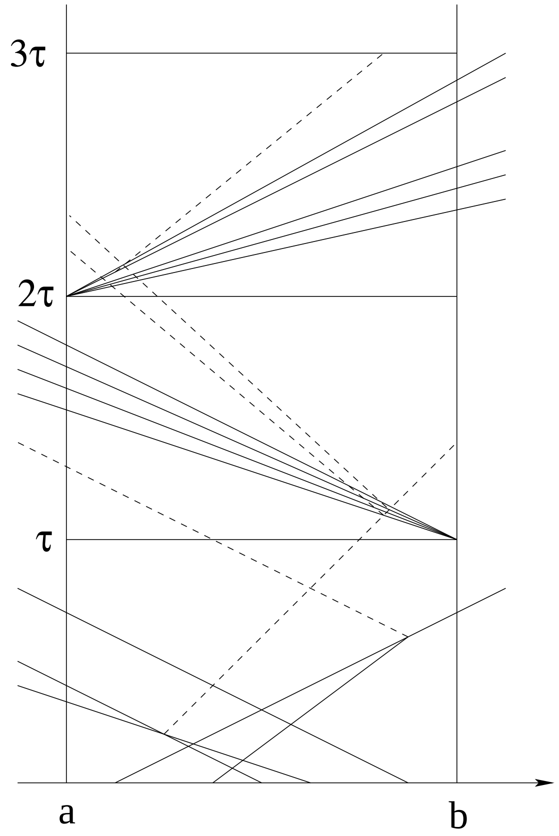

figure 3

We now apply a similar procedure as in the proof of Lemma 1, and construct a solution on the interval $[\tau,\ 3\tau]$ in such a way that $u(3\tau) \approx u^*$. More precisely, to construct u on the domain $[\tau,\ 2\tau] \times [a,b]$, consider the state v'' implicitly defined by (2.2), with $v \doteq u(\tau, b-)$, $v' \doteq u^*$. On a forward neighborhood of the point (τ, b) we let u coincide with (a front-tracking approximation of) the solution to the Riemann problem

$$u(\tau, x) = \begin{cases} u(\tau, b-) & \text{if} \quad x < b, \\ v'' & \text{if} \quad x > b. \end{cases}$$

This procedure will introduce at the point (τ, b) a family of wave-fronts of families $i = 1, \ldots, p$, whose total strength is $\mathcal{O}(1) \cdot (\rho + \delta)$. Because of (2.4), all these fronts will exit from the boundary $x = a$ within time 2τ. Of course, they can interact with the other fronts present in $u(\tau, \cdot)$. In any case, the total strength of fronts in $u(2\tau, \cdot)$ is still estimated as

$$\text{Tot.Var.}\big\{u(2\tau)\big\} = \mathcal{O}(1) \cdot \delta^2 \,. \tag{2.11}$$

Next, to define u for $t \in [2\tau,\ 3\tau]$, consider the state v''' implicitly defined by

$$\begin{cases} u(2\tau, a+) = \Psi_n(\sigma_n) \circ \cdots \circ \Psi_{p+1}(\sigma_{p+1})(v'''), \\ \qquad\quad\ u^* = \Psi_p(\sigma_p) \circ \cdots \circ \Psi_1(\sigma_1)(v'''). \end{cases} \tag{2.12}$$

On a forward neighborhood of the point $(2\tau, a)$ we let u coincide with (a front-tracking approximation of) the solution to the Riemann problem

$$u(2\tau,x)=\begin{cases} u(2\tau,\ a+) & \text{if} \quad x>a,\\ v''' & \text{if} \quad x<a.\end{cases}$$

This procedure introduces at the point $(2\tau,\ a)$ a family of wave-fronts of families $i=p+1,\ldots,n$, whose total strength is $\mathcal{O}(1)\cdot(\rho+\delta)$. Because of (2.4), all these fronts will exit from the boundary $x=b$ within time 3τ. Of course, they can interact with the other fronts present in $u(2\tau,\cdot)$. In any case, the total strength of fronts in $u(3\tau,\cdot)$ is still estimated as

$$\text{Tot.Var.}\big\{u(3\tau)\big\}=\mathcal{O}(1)\cdot\delta^2\,. \tag{2.13}$$

Moreover, the difference between the values $u(3\tau,x)$ and u^* will be of the same order of the total strength of waves in $u(\tau,\cdot)$, so that the first inequality in (2.9) will also hold. $\square$

Proof of Theorem 1. Using the same arguments as in the proof of Lemma 1.1, for every $\varepsilon>0$ we can construct an ε-approximate front tracking solution $u=u(t,x)$ on $[0,\ 2N\tau]\times[a,b]$ such that

$$\sup_{x\in[a,b]}\big|u(2N\tau,x)-u^*\big|=\mathcal{O}(1)\cdot\delta,\qquad\qquad \text{Tot.Var.}\big\{u(2N\tau)\big\}=\mathcal{O}(1)\cdot\delta\,. \tag{2.14}$$

Choosing $\delta>0$ sufficiently small, we can assume that, in (2.14), $\mathcal{O}(1)\cdot\delta<\delta_0<1/C$, the constant in Lemma 2. Calling $T\doteq 2N\tau$, we can now repeat the construction described in Lemma 2 on each interval $\big[T+3k\tau,\ T+3(k+1)\tau\big]$. This yields

$$\sup_{x\in[a,b]}\big|u(T+3k\tau,\ x)-u^*\big|\leq\delta_k,\qquad\qquad \text{Tot.Var.}\big\{u(T+3k\tau)\big\}\leq\delta_k\,, \tag{2.15}$$

where the constants δ_k satisfy the inductive relations

$$\delta_{k+1}\leq C\delta_k^2. \tag{2.16}$$

Choosing a sequence of ε-approximate front tracking solutions u_ε satisfying (2.15)-(2.16) and taking the limit as $\varepsilon\to 0$, we obtain an entropy weak solution u which still satisfies the same estimates. The bounds (1.7)-(1.8) are now a consequence of (2.15)-(2.16), with a suitable choice of the constants C_0,κ. $\square$

3 - Decay of positive waves

Throughout the following, we consider a 2×2 system of conservation laws

$$u_t+f(u)_x=0\,, \tag{3.1}$$

satisfying the assumptions (H). Following [6], p. 128, we construct a set of Riemann coordinates (w_1,w_2) . One can then choose the right eigenvectors of $Df(u)$ so that

$$r_i(u)=\frac{\partial u}{\partial w_i}\,,\qquad\qquad \frac{\partial\lambda_i}{\partial w_i}=D\lambda_i\cdot r_i>0\qquad\qquad i=1,2. \tag{3.2}$$

It will be convenient to perform most of the analysis on a special class of solutions: piecewise Lipschitz functions with finitely many shocks and no compression waves. Due to the geometric structure of the system, this set of functions turns out to be positively invariant for the flow generated by the hyperbolic system. We first derive several a priori estimate concerning these solutions, in particular on the strength and location of the shocks. We then observe that any BV solution can be obtained as limit of a sequence of piecewise Lipschitz solutions in our special class. Our estimates can thus be extended to general BV solutions.

Definition 1. *We call $\mathcal{U}$ the set of all piecewise Lipschitz functions $u : \mathbb{R} \mapsto \mathbb{R}^2$ with finitely many jumps, such that:*

(i) at every jump, the corresponding Riemann problem is solved only in terms of shocks (no centered rarefactions);

(ii) no compression waves are present, i.e.: $w_{i,x}(x) \geq 0$ at almost every $x \in \mathbb{R}$, $i = 1, 2$.

The next lemma establishes the forward invariance of the set $\mathcal{U}$.

Lemma 3. *Consider the 2×2 system of conservation laws (3.1), satisfying the assumptions (H). Let $u = u(t, x)$ be the solution to a Cauchy problem, with small total variation, satisfying $u(0, \cdot) \in \mathcal{U}$. Then*

$$u(t, \cdot) \in \mathcal{U} \qquad \text{for all} \quad t \geq 0. \tag{3.3}$$

Proof. We have to show that, as time progresses, the total number of shocks does not increase and no compression wave is ever formed. This will be the case provided that

(i) The interaction of two shocks of the same family produces an outgoing shock of the other family.

(ii) The interaction of a shock with an infinitesimal rarefaction wave of the same family produces a rarefaction wave in the other family.

Both of the above conditions can be easily checked by analysing the relative positions of shocks and rarefaction curves. We will do this for the first family, leaving the verification of the other case to the reader.

Call $\sigma \mapsto R_1(\sigma)$ the rarefaction curve through a state u_0,parametrized so that

$$\lambda_1\big(R_1(\sigma)\big) = \lambda_1(u_0) + \sigma .$$

It is well known that the shock curve through u_0 has a second order tangency with this rarefaction curve. Hence there exists a smooth function $c_1(\sigma)$ such that the point

$$S_1(\sigma) \doteq R_1(\sigma) + c_1(\sigma)\frac{\sigma^3}{6} r_2(u_0)$$

lies on this shock curve, for all σ in a neighborhood of zero. From the Rankine-Hugoniot equations it now follows

$$\chi(\sigma) \doteq \Big(f\big(R_1(\sigma) + c_1(\sigma)(\sigma^3/6)\, r_2(u_0)\big) - f(u_0)\Big) \wedge \Big(R_1(\sigma) + c_1(\sigma)(\sigma^3/6)\, r_2(u_0) - u_0\Big) = 0\,. \tag{3.4}$$

Differentiating the wedge product (3.4) four times at $\sigma = 0$ and denoting derivatives with upper dots, we obtain

$$\begin{aligned}
\frac{d^4\chi}{d\sigma^4}(0) &= 4\big[\lambda_1(u_0)\,\dddot{R}_1(0) + 2\ddot{R}_1(0) + \lambda_2(u_0)\,c_1(0) r_2(u_0)\big] \wedge \dot{R}_1(0) \\
&\qquad + 6\big[\lambda_1(u_0)\ddot{R}_1(0) + \dot{R}_1(0)\big] \wedge \ddot{R}_1(0) + 4\lambda_1(u_0)\dot{R}_1(0) \wedge \big[\,\dddot{R}_1(0) + c(0) r_2(u_0)\big] \\
&= 4\big(\lambda_2(u_0) - \lambda_1(u_0)\big) c_1(0)\, r_2(u_0) \wedge r_1(u_0) + 2(Dr_1 \cdot r_1)(u_0) \wedge r_1(u_0) \\
&= 0\,.
\end{aligned}$$

Hence

$$c_1(0) = \frac{(Dr_1 \cdot r_1) \wedge r_1}{2(\lambda_2 - \lambda_1)(r_1 \wedge r_2)} < 0\,. \tag{3.5}$$

By (3.5), the relative position of 1-shock and 1-rarefaction curves is as depicted in fig. 1. By the geometry of wave curves, the properties (i) and (ii) are now clear. Figure 4a illustrates the interaction of two 1-shocks, while fig. 4b shows the interaction between a 1-shock and a 1-rarefaction. By u_l, u_m, u_r we denote the left, middle and right states before the interaction, while u_m' is the middle state after the interaction. In the two cases, the solution of the Riemann problem contains a 2-shock and a 2-rarefaction, respectively. □

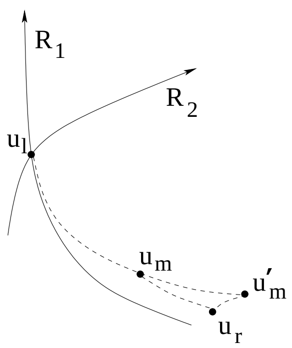

figure 4a

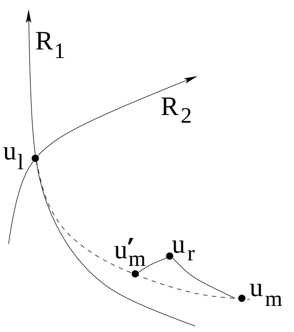

figure 4b

The next lemma shows the decay of positive waves for solutions with small total variation, taking values inside $\mathcal{U}$.

Lemma 4. *Let $u = u(t,x)$ be a solution of the Cauchy problem for the 2×2 system (3.1) satisfying (H). Assume that*

$$u(t,\cdot) \in \mathcal{U} \qquad\qquad t \geq 0. \tag{3.6}$$

Then there exist $\kappa, \delta > 0$ such that if Tot.Var.$(u(t,\cdot)) < \delta$ for all t, then its Riemann coordinates (w_1, w_2) satisfy

$$0 \leq w_{i,x}(t,x) \leq \frac{\kappa}{t}, \qquad\qquad t > 0,\ i = 1,2. \tag{3.7}$$

Proof. We consider the case $i = 1$. Fix any point $(\bar{t},\ \bar{x})$. Since centered rarefaction waves are not present, there exists a unique 1-characteristic through this point, which we denote as $t \mapsto x_1(t;\ \bar{t},\ \bar{x})$. It is the solution of the Cauchy problem

$$\dot{x}(t) = \lambda_1\big(u(t, x(t))\big), \qquad\qquad x(\bar{t}) = \bar{x}. \tag{3.8}$$

The evolution of $w_{1,x}$ along this characteristic is described by

$$\frac{d}{dt}w_{1,x}\big(t,x_1(t)\big) = w_{1,xt} + \lambda_1 w_{1,xx} = -(\lambda_1 w_{1,x})_x + \lambda_1 w_{1,xx} = -\frac{\partial\lambda_1}{\partial w_1}w_{1,x}^2 - \frac{\partial\lambda_1}{\partial\omega_2}w_{1,x}w_{2,x}.$$

Since the system is genuinely nonlinear there exists $k_1 > 0$ such that $\partial\lambda_1/\partial w_1 \geq k_1 > 0$, hence

$$\frac{d}{dt}w_{1,x}\big(t,x_1(t)\big) \leq -k_1 w_{1,x}^2 + \mathcal{O}(1)\cdot w_{1,x}w_{2,x}. \tag{3.9}$$

Moreover, at each time t_α where the characteristic crosses a 2-shock of strength $|\sigma_\alpha|$ we have the estimate

$$w_{1,x}(t_\alpha+) \leq \big(1+\mathcal{O}(1)\cdot|\sigma_\alpha|\big)w_{1,x}(t_\alpha-). \tag{3.10}$$

Let $Q(t)$ be the total interaction potential at time t (see for example [4], p. 202) and let $V_2(t)$ be the total amount of 2-waves approaching our 1-wave located at $x_1(t)$. Repeating the arguments in [4], p.139, we can find a constant $C_0 > 0$ such that the quantity

$$\Upsilon(t) \doteq V_1(t) + C_0\,Q(t), \qquad\qquad t > 0,$$

is non-increasing. Moreover, for a.e. t one has

$$\dot\Upsilon(t) \leq -\big|\lambda_2 - \lambda_1\big|\big|w_{2,x}\big|\big(t,x_1(t)\big)\,,$$

while at times t_α where x_1 crosses a 2-shock of strength $|\sigma_\alpha|$ there holds

$$\Upsilon(t_\alpha-) \leq \Upsilon(t_\alpha+) - |\sigma_\alpha|\,.$$

Call $W(t) \doteq w_{1,x}\big(t,x_1(t)\big)$. By the previous estimates, from (3.9) and (3.10) it follows form

$$\dot W(t) \leq -k_1 W^2(t) - C\,\dot\Upsilon(t)W(t), \tag{3.9}$$

$$W(t_\alpha+) - W(t_\alpha-) \leq C\big[\Upsilon(t_\alpha+) - \Upsilon(t_\alpha-)\big]W(t_\alpha-), \tag{3.11}$$

for a suitable constant C. We now observe that

$$y(t) \doteq \frac{e^{-C\Upsilon(t)}}{\displaystyle\int_0^t k_1 e^{-C\Upsilon(s)}ds}$$

is a distributional solution of the equation

$$\dot y = -k_1 y^2 - C\,\dot\Upsilon(t)y\,,$$

with $y(t)\to\infty$ as $t\to 0+$. A comparison argument now yields $W(t)\leq y(t)$. Since Υ is positive and decreasing, we have

$$W(t) \leq \bar W(t) \leq \frac{1}{k_1}\frac{1}{\displaystyle\int_0^t e^{-C\Upsilon(s)}ds} \leq \frac{e^{C\Upsilon(0)}}{k_1 t},$$

for all $t > 0$. This establishes (3.7) for $i = 1$, with $\kappa \doteq e^{C\Upsilon(0)}/k_1$. The case $i = 2$ is identical. $\square$

We conclude this section by proving a decay estimate for positive waves, valid for general BV solutions of the system (3.1). For this purpose, we need to recall some definitions introduced in [5]. See also p. 201 in [4].

Let $u : I\!R \mapsto I\!R^2$ have bounded variation. By possibly changing the values of u at countably many points, we can assume that u is right continuous. The distributional derivative $\mu \doteq D_x u$ is a vector measure, which can be decomposed into a continuous and an atomic part: $\mu = \mu_c + \mu_a$. For $i = 1, 2$, the scalar measures $\mu^i = \mu^i_c + \mu^i_a$ are defined as follows. The continuous part of μ^i is the Radon measure μ^i_c such that

$$\int \phi \, d\mu^i_c = \int \phi \, l_i(u) \cdot d\mu_c \tag{3.12}$$

for every scalar continuous function ϕ with compact support. The atomic part of μ^i is the measure μ^i_a concentrated on the countable set $\{x_\alpha;\ \alpha = 1, 2, \ldots\}$ where u has a jump, such that

$$\mu^i_a\big(\{x_\alpha\}\big) = \sigma_{\alpha,i} \doteq E_i\big(u(x_\alpha-),\ u(x_\alpha+)\big) \tag{3.13}$$

is the size of the i-th wave in the solution of the corresponding Riemann problem with data $u(x_\alpha\pm)$. We regard μ^i as the *measure of i-waves* in the solution u. It can be decomposed in a positive and a negative part, so that

$$\mu^i = \mu^{i+} - \mu^{i-}, \qquad\qquad |\mu^i| = \mu^{i+} + \mu^{i-}. \tag{3.14}$$

The decay estimate in (3.7) can now be extended to general BV solutions. Indeed, we show that the density of positive i-waves decays as κ/t. By meas(J) we denote here the Lebesgue measure of a set J.

Lemma 5. *Let $u = u(t, x)$ be a solution of the Cauchy problem for the 2×2 system (3.1) satisfying (H). Then there exist $\kappa, \delta > 0$ such that if Tot.Var.$(u(t,\cdot)) < \delta$ for all t, then the measures μ^{1+}_t, μ^{2+}_t of positive waves in $u(t,\cdot)$ satisfy*

$$\mu^{i+}_t(J) \leq \frac{\kappa}{t}\, meas\,(J) \tag{3.15}$$

for every Borel set $J \subset I\!R$ and every $t > 0$, $i = 1, 2$.

Proof. For every BV solution u of (3.1) we can construct a sequence of solutions u_ν with $u_\nu \to u$ as $\nu \to \infty$ and such that $u_\nu(t,\cdot) \in \mathcal{U}$ for all t. Calling $(w^\nu_1,\, w^\nu_2)$ the Riemann coordinates of u_ν, by Lemma 4 we have

$$0 \leq w^\nu_{i,x}(t,x) \leq \frac{\kappa}{t}, \qquad\qquad t > 0,\ i = 1, 2, \qquad \nu \geq 1\,. \tag{3.16}$$

For a fixed $t > 0$, observe that the map $x \mapsto w^\nu_1(t,x)$ has upward jumps precisely at the points x_α where $u(t,\cdot)$ has a 2-shock. Define $\tilde\mu_\nu$ as the positive, purely atomic measure, concentrated on the finitely many points x_α where $u(t,\cdot)$ has a 2-shock, such that

$$\tilde\mu_\nu\big(\{x_\alpha\}\big) = w^\nu_1(t,\, x_\alpha+) - w^\nu_1(t,\, x_\alpha-) \leq C\,|\sigma_\alpha|^3 \tag{3.17}$$

for some constant C. By possibly taking a subsequence, we can assume the existence of a weak limit $\tilde\mu_\nu \rightharpoonup \tilde\mu$. Because of the estimate in (3.17), the measure $\tilde\mu$ is purely atomic, and is concentrated on the set of points x_β which are limits as $\nu \to \infty$ of a sequence of points x^ν_α where $u_\nu(t,\cdot)$ has a 2-shock of uniformly positive strength $|\sigma_\nu| \geq \delta > 0$. Therefore, $\tilde\mu$ is concentrated on the set of points where the limit solution $u(t,\cdot)$ has a 2-shock, and makes no contribution to the positive part of μ^{1+}_t. We thus conclude that the positive part of μ^{1+}_t is absolutely continuous w.r.t. Lebesgue measure, with density $\leq \kappa/t$. An analogous argument holds for μ^{2+}_t. □

Corollary 1. *Let $u = u(t,x)$ be a solution of the 2×2 system (1.1). Let the assumptions (H) hold. Fix $\varepsilon > 0$ and consider the subinterval $[a',b'] \doteq [a+\varepsilon,\ b-\varepsilon]$. Assume that, at time $t = 0$, the measures μ^{1+}, μ^{2+} of positive waves in $u(0,\cdot)$ on $[a,b]$ vanish identically. Then, for every $t > 0$ one has*

$$\mu_t^{i+}(J) \leq \frac{\kappa\lambda^*}{\varepsilon}\, meas\,(J) \tag{3.18}$$

for every Borel set $J \subset [a',b']$ and every $t > 0$, $i = 1, 2$.

Indeed, recalling (1.9), the values of $u(t,\cdot)$ restricted to the interval $[a',b']$ can be obtained by solving a Cauchy problem, with initial data assigned on the whole interval $[a,b]$ at time $t - \varepsilon/\lambda^*$.

4 - Proof of Theorem 2

Lemma 6. *In the same setting as Lemma 4, assume that there exists $\kappa' > 0$ such that*

$$0 \leq w_{i,x}(t,x) \leq \kappa' \qquad\qquad t \in [0,T], \quad i = 1, 2\,. \tag{4.1}$$

Let $t \mapsto x(t)$ be the location of a shock, with strength $\big|\sigma(t)\big|$. There exists a constant $0 < c < 1$ such that

$$\big|\sigma(t)\big| \geq c\big|\sigma(s)\big|\,, \qquad\qquad 0 \leq s < t \leq T\,. \tag{4.2}$$

Proof. To fix the ideas, let $u(t,\cdot)$ have a 1-shock located at $x(t)$, with strength $\big|\sigma(t)\big|$. Outside points of interaction with other shocks, the strength satisfies an inequality of the form

$$\frac{d}{dt}\big|\sigma(t)\big| \geq -C \cdot \Big(w_{1,x}\big(t,x(t)+\big) + w_{1,x}\big(t,x(t)-\big)w_{2,x}\big(t,x(t)+\big) + w_{2,x}\big(t,x(t)-\big)\Big)\big|\sigma(t)\big|\,. \tag{4.3}$$

At times where our 1-shock interacts with other 1-shocks, its strength increases. Moreover, at each time t_α where our 1-shock interacts with a 2-shock, say of strength $|\sigma_\alpha|$, one has

$$\big|\sigma(t_\alpha+)\big| \geq \big|\sigma(t_\alpha-)\big|\,\big(1 - C'|\sigma_\alpha|\big)\,. \tag{4.4}$$

for some constant C'. Assuming that the total variation remains small, the total amount of 2-shocks which cross any given 1-shock is uniformly small. Hence, (4.3)-(4.4) together imply (4.2). □

Lemma 7. *Let $t \mapsto u(t,\cdot) \in \mathcal{U}$ be a solution of the Cauchy problem for a genuinely nonlinear 2×2 system satisfying (1.11). Assume that there exists $\kappa' > 0$ such that*

$$w_{i,x}(t,x) \leq \kappa' \qquad\qquad t \in [0,T], \quad i = 1, 2\,. \tag{4.5}$$

Since no centered rarefactions are present, any two i-characteristics, say $x(t) < y(t)$, can uniquely be traced backward up to time $t = 0$. There exists a constant $L > 0$ such that

$$y(t) - x(t) \leq L\,\big(y(s) - x(s)\big) \qquad\quad 0 \leq s < t \leq T\,. \tag{4.6}$$

Proof. Consider the case $i = 2$. By definition, the characteristics are solutions of

$$\dot{x}(t) = \lambda_2\big(u(t,x(t))\big), \qquad \dot{y}(t) = \lambda_2\big(u(t,y(t))\big).$$

Since the characteristic speed λ_2 decreases across 2-shocks, we can write

$$\dot{y}(t) - \dot{x}(t) \leq C \int_{x(t)}^{y(t)} \big|w_{1,x}(t,\xi)\big| + \big|w_{2,x}(t,\xi)\big|\, d\xi + C \sum_{\alpha\in\mathcal{S}_1[x,y]} \big|\sigma_\alpha(t)\big|\,, \tag{4.7}$$

where $\mathcal{S}_1[x,y]$ denotes the set of all 1-shocks located inside the interval $\big[x(t),\ y(t)\big]$. Introduce the function

$$\phi(t,x) \doteq \begin{cases} 0 & \text{if} \quad x \leq x(t), \\ \frac{x-x(t)}{y(t)-x(t)} & \text{if } x(t) < x < y(t), \\ 1 & \text{if} \quad x \geq y(t). \end{cases}$$

Moreover, define the functional

$$\Phi(t) \doteq \sum_{\alpha\in\mathcal{S}_1} \phi\big(t, x_\alpha(t)\big)\, \big|\sigma_\alpha(t)\big| + C_0\, Q(t)\,,$$

where the summation now refers to all 1-shocks in $u(t,\cdot)$ and Q is the usual interaction potential. Observe that the map $t \mapsto \Phi(t)$ is non-increasing. By (4.5) and (4.7) we can now write

$$\dot{y}(t) - \dot{x}(t) \leq C' \,\big(1 - \dot{\Phi}(t)\big)\, \big(y(t) - x(t)\big)$$

for some constant C'. This implies (4.6) with $L = \exp\big\{C'T + C'\Phi(0)\big\}$. □

The next result is the key ingredient toward the proof of Theorem 2. It provides the density of the set of interaction points where new shocks are generated.

Lemma 8. *Fix $\varepsilon > 0$ and define $a'' = a + 2\varepsilon$, $b'' = b - 2\varepsilon$. Consider a 2×2 system of the form (1.1), satisfying (H). Let u be an entropy weak solution defined on $[0,\tau] \times [a,b]$, with $\tau \doteq \varepsilon/4\lambda^*$. Let (3.18) hold for all $t \in [0,\tau]$, and assume that $u(0,\cdot)$ has a dense set of 1-shocks on the interval $[a'',\, b'']$. Then, for $0 \leq t \leq \tau$, the solution $u(t,\cdot)$ has a set of 1-shocks which is dense on $[a'',\, b' - \lambda^* t]$ and a set of 2-shocks which is dense on $[a'',\, b'']$.*

Proof. By the assumptions of the lemma, there exists a sequence of piecewise Lipschitz solutions $t \mapsto u_\nu(t) \in \mathcal{U}$ such that $u_\nu \to u$ in $\mathbf{L}^1$,

$$0 \leq w^\nu_{i,x}(t,x) \leq \frac{2\kappa\lambda^*}{\varepsilon} \qquad\qquad i = 1,2, \quad \nu \geq 1\,,$$

and moreover the following holds. For every $\rho > 0$, there exists $\delta > 0$ such that each $u_\nu(0,\cdot)$ (with ν large enough) contains at least one 1-shock of strength $\big|\sigma_\nu(0)\big| \geq \delta$ on every subinterval $J \subset [a'',\, b'']$ having length $\geq \rho$.

To prove the first statement in Lemma 8, fix $t \in [0,\tau]$ and consider any non-trivial interval $[p,q] \subseteq [a'',\ b'' - t\lambda^*]$. Call $s \mapsto p_\nu(s)$, $s \mapsto q_\nu(s)$ the backward characteristics through these points, relative to the solution u_ν. We thus have

$$\begin{cases} \dot{p}_\nu(s) = \lambda_1\big(u_\nu(s,\, p_\nu(s))\big), \\ \dot{q}_\nu(s) = \lambda_1\big(u_\nu(s,\, q_\nu(s))\big), \end{cases} \qquad\qquad \begin{cases} p_\nu(t) = p, \\ q_\nu(t) = q. \end{cases}$$

By Lemma 7, $q_\nu(0) - p_\nu(0) \geq \rho$ for some $\rho > 0$ independent of ν. Hence, each solution u_ν contains a shock of strength $\big|\sigma_\nu(s)\big| \geq \delta$ located inside the interval $\big[p_\nu(0),\ q_\nu(0)\big]$. Lemma 5 now yields

$\big|\sigma_\nu(t)\big| \geq c\delta$. By possibly taking a subsequence, we conclude that the limit solution $u(t,\cdot)$ contains a 1-shock of positive strength at the point $x(t) = \lim x_\nu(t) \in [p,q]$.

To prove the second statement, we will show that the set of points where two 1-shocks in u interact and produce a new 2-shock is dense on the triangle

$$\Delta \doteq \big\{(t,x)\,;\quad t\in[0,\tau],\quad a'' < x < b'' - \lambda^* t\big\}.$$

Indeed, let $t\in[0,\tau]$ and $p<q$ be as before. For each ν sufficiently large, let $t\mapsto x_\nu(t)$ be the location of a 1-shock in u_ν, with strength $\big|\sigma_\nu(t)\big| \geq \delta > 0$. Assume $x_\nu(\cdot)\to x(\cdot)$ as $\nu\to\infty$, and $x_\nu(t)\in[p,q]$, so that $x(t)$ is the location of a 1-shock of the limit solution u, say with strength $\big|\sigma(t)\big| > 0$.

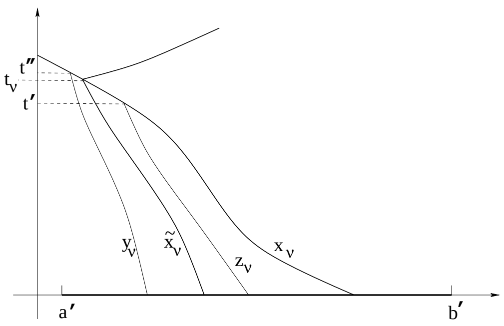

figure 5

We claim that the set of times $\hat t$ where some other 1-shock σ' impinges on σ and generates a new 2-shock is dense on $[0,t]$. To see this, fix $0<t'<t''<t$. For each ν sufficiently large, consider the backward 1-characteristics $y_\nu,\ z_\nu$ impinging from the left on the shock x_ν at times t'',t' respectively (fig. 5). These provide solutions to the Cauchy problems

$$\dot y_\nu(t) = \lambda_1\big(u_\nu(t,y_\nu(t))\big), \qquad\qquad y_\nu(t'') = x_\nu(t''),$$

$$\dot z_\nu(t) = \lambda_1\big(u_\nu(t,z_\nu(t))\big), \qquad\qquad z_\nu(t') = x_\nu(t'),$$

respectively. Observe that

$$z_\nu(0) - y_\nu(0) \geq \rho$$

for some $\rho>0$ independent of ν. Indeed, the genuine nonlinearity of the system implies

$$\lambda_1\big(u_\nu(t,x_\nu(t)-)\big) - \dot x_\nu(t) \geq \kappa\Big|u_\nu\big(t,\,x_\nu(t)+\big) - u_\nu\big(t,\,x_\nu(t)-\big)\Big| \geq \kappa\delta.$$

Therefore,

$$x_\nu(t') - y_\nu(t') \geq \rho' > 0,$$

for some constant $\rho'>0$ independent of ν. By Lemma 6, the interval $\big[y_\nu(0),\ z_\nu(0)\big]$ has uniformly positive length. Hence it contains a 1-shock of $u_\nu(0,\cdot)$ with uniformly positive strength $\big|\sigma_\nu(0)\big| \geq \delta > 0$. By Lemma 5, every u_ν has a 1-shock with strength $\big|\sigma_\nu(t)\big| \geq c\delta$ located along some curve $t\mapsto \tilde x_\nu(t)$ with

$$y_\nu(t) < \tilde x_\nu(t) < z_\nu(t) \qquad t\in[0,t'].$$

Clearly, this second 1-shock impinges on the shock x_ν at some time $t_\nu\in[t',t'']$, creating a new 2-shock with uniformly large strength. Letting $\nu\to\infty$ we obtain the result. $\square$

Proof of Theorem 2. Let $\delta_0 > 0$ be given. We can then construct an initial condition $u(0, \cdot) = \phi$, with Tot.Var.$\{\phi\} < \delta_0$, having a dense set of 1-shocks on the interval $[a, b]$, and no other waves. As a consequence, for any $\varepsilon > 0$ by Corollary 1 we have the estimate (3.18) on the density of positive waves away from the boundary.

Fix $\tau = \varepsilon/4\lambda^*$, and consider again the subinterval $[a'', b''] = [a + 2\varepsilon,\ b - 2\varepsilon]$. We can apply Lemma 8 first on the time interval $[0, \tau]$, obtaining the density of 2-shocks on the region $[0, \tau] \times [a'', b'']$. Then, by induction on m, the same argument is repeated on each time interval $t \in \big[m\tau,\ (m+1)\tau\big]$, proving the theorem. $\square$

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