ON THE CONVERGENCE RATE OF APPROXIMATION SCHEMES FOR HAMILTON-JACOBI-BELLMAN EQUATIONS

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ABSTRACT. Using systematically a tricky idea of N.V. Krylov, we obtain a general result on the rate of convergence of a certain class of monotone approximation schemes for stationary Hamilton-Jacobi-Bellman Equations with variable coefficients. This result applies in particular to control schemes based on the dynamic programming principle and to finite difference schemes despite, here, we are not able to treat the most general case. General results have been obtained earlier by N.V. Krylov for finite difference schemes in the stationary case with constant coefficients and in the time-dependent case with variable coefficients. In this paper we are able to handle variable coefficients without using control theory and probabilistic methods. In our opinion this way is far simpler and, for the cases we can treat, it yields a better rate of convergence than Krylov obtains in the variable coefficients case.

1. INTRODUCTION

Optimal control problems for diffusion processes have been considered in a great generality recently by using the dynamic programming principle approach and viscosity solution methods: the value-function of such problems was proved to be the unique viscosity solution of the associated Hamilton-Jacobi-Bellman equations under natural conditions on the data. We refer the reader to the articles of Lions [15, 16, 17] and the book by Fleming and Soner [8] for results in this direction and to the User's guide [6] for a detailed presentation of the notion of viscosity solutions.

In order to compute the value function, numerical schemes have been derived and studied for a long time : we refer, for instance, the reader to Lions and Mercier [18], Crandall and Lions [7], and Kushner [13] for the derivation of such schemes (see also the books of Bardi and Capuzzo-Dolcetta [2] and Fleming and Soner [8]), and to Camilli and Falcone [4], Menaldi [19], Souganidis [20] and the recent work of Bonnans and Zidani [3] for the study of their properties, including some proofs of convergence and of rate of convergence.

The convergence can be obtained in a very general setting either by probabilistic methods (see Kushner [13]) or by viscosity solution methods (see Barles and Souganidis [1]). But until recently there were almost no results on the rate of convergence of such schemes in the degenerate diffusion case where the value-function is expected to have only $C^{0,\delta}$ or $W^{1,\infty}$ regularity (see the above references). Viscosity solution methods were providing this rate of convergence only for first-order equations (cf. Souganidis [20]), i.e. for deterministic control problems, or for x-independent coefficients (cf. Krylov [11]). Results in the spirit of our paper but requiring more regularity on the value-functions were anyway obtained by Menaldi [19].

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Progress were made recently by Krylov [11, 12] who obtained general results on the rate of convergence of finite difference schemes by combining analytic and probabilistic methods. Using systematically an idea by Krylov, we present here a completely analytic approach to prove such estimates for a large class of approximation schemes. This approach is, at least in our opinion, much simpler. Unfortunately, for reasons explained below, it can not yet handle finite difference schemes in the most general case.

In order to be more specific, we consider the following type of HJB Equation arising in infinite horizon, discounted, stochastic control problems.

$$F(x, u, Du, D^2u) = 0 \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

with

$$F(x, t, p, M) = \sup_{\vartheta\in\Theta} \Big\{ -\frac{1}{2}\operatorname{tr}[a(x,\vartheta)M] - b(x,\vartheta)p + c(x,\vartheta)t - f(x,\vartheta)\Big\} = 0.$$

where tr denotes the trace of a matrix, Θ, the space of controls, is assumed to be a compact metric space and a, b, c, f are, at least, continuous functions defined on $\mathbb{R}^N \times \Theta$ with values respectively in the space $\mathcal{S}^N$ of symmetric $N \times N$ matrices, $\mathbb{R}^N$ and $\mathbb{R}$. Precise assumptions on these data will be given later on. From now on, for the sake of simplicity of notations and since ϑ plays here only the role of a parameter, we write $\phi^\vartheta(\cdot)$ instead of $\phi(\cdot,\vartheta)$ for $\phi = a$, b, c and f.

Under suitable assumptions on a, b, c and f, it is well-known that the solution of the equation which is also the value-function of the associated stochastic control problem, is bounded, uniformly continuous ; moreover it is also expected to be in $C^{0,\delta}(\mathbb{R}^N)$ for some δ if a, b, c and f satisfy suitable regularity properties.

An approximation scheme for (1.1) can be written as

$$S(h, x, u_h(x), [u_h]^h_x) = 0 \quad \text{for all } x \in \mathbb{R}^N, \tag{1.2}$$

where h is a small parameter which measures typically the mesh size, $u_h : \mathbb{R}^N \to \mathbb{R}$ is the approximation of u and the solution of the scheme, $[u_h]^h_x$ is a function defined at x from u_h. Finally S is the approximation scheme.

The natural and classical idea in order to prove a rate of convergence for S is to look for a sequence of smooth approximate solutions v_ε of (1.1). Indeed, if such a sequence $(v_\varepsilon)_\varepsilon$ exists with a precise bound on $\|u - v_\varepsilon\|_{L^\infty(\mathbb{R}^N)}$ and on the derivatives of v_ε, in order to obtain an estimate of $\|v_\varepsilon - u_h\|_{L^\infty(\mathbb{R}^N)}$ one just has to plug v_ε into S and to use the consistency condition in addition to some comparison properties for S. This estimate immediately yields an estimate of $\|u - u_h\|_{L^\infty(\mathbb{R}^N)}$ which depends on ε and h and the convergence rate's result then follows from optimizing with respect to ε.

Unfortunately, such a program cannot be carried out so easily and, to the best of our knowledge, until now, nobody has been able to prove the existence of such a sequence when the data a, b, c, f depends on x. However Krylov had a very tricky idea in order to build a sequence which is doing "half the job" of the v_ε's above : his key idea was to introduce the solution u^ε of

$$\max_{|e|\le\varepsilon} \big[F(x+e, u^\varepsilon, Du^\varepsilon, D^2u^\varepsilon)\big] = 0 \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

and to regularize it in a suitable way, taking advantage of the convexity of F in u, Du, D^2u. He was getting in this way a sequence of subsolutions (instead of solutions) which provides "half a rate", namely an upper estimate of $u - u_h$. A detailed proof of this estimate is given in Section 2.

The other estimate (a lower estimate of $u - u_h$) is a priori more difficult to obtain and this is where Krylov is using probabilistic estimates, at least in the x-dependent case. In fact it is clear that all the arguments used above are much simpler in the

x-independent case. Our idea to obtain this lower estimate is very simple : to exchange in the above argument the role of the scheme and the equation. This idea was already used by Krylov in the x-independent case. As in the case of the equation, we are lead to introduce the solution of u_h^ε of

$$\max_{|e|\leq\varepsilon} \left[S(h, x+e, u_h^\varepsilon(x), [u_h^\varepsilon]_x^h)\right] = 0 \quad \text{in } \mathbb{R}^N \ . \tag{1.4}$$

At this point we face two main difficulties which explain the limitations of this approach : in order to follow the related proof for the upper bound, we need two key results. First we have to show that there exists $0 < \bar{\delta} \leq 1$ independent of h and ε such that the u_h and u_h^ε are in $C^{0,\bar{\delta}} \cap L^\infty(\mathbb{R}^N)$; moreover we need a rather precise control on their norms in this space and also a rather precise estimate on $\|u_h - u_h^\varepsilon\|_{L^\infty(\mathbb{R}^N)}$. Of course, a natural idea is to copy the proofs of the related results for (1.1). They rely on the doubling of variables method which, unfortunately, does not seem to be extendable to all types of schemes. Roughly speaking, we are able to obtain rates of convergence for approximation schemes for which we can extend this method.

At this point, it is useful to consider a simple 1$-$d example, namely

$$-\frac{1}{2}a(x)u'' + \lambda u = f(x) \quad \text{in} \quad \mathbb{R}\ ,$$

where $a = \sigma^2$ with $\sigma, f \in W^{1,\infty}(\mathbb{R})$ and $\lambda > 0$. We consider two ways of constructing numerical schemes approximating this equation. The first one is to use the stochastic interpretation of the equation and to build what we call a "control scheme"

$$u_h(x) = \frac{1-\lambda h}{2}[u_h(x+\sigma(x)\sqrt{h}) + u_h(x-\sigma(x)\sqrt{h})] + hf(x) \quad \text{in } \mathbb{R}\ .$$

Such schemes are based on the dynamical programming principle and are easily extendable to more general problems (cf. Section 3). For this type of schemes, it is not so difficult (although not completely trivial) to obtain the sought after properties of u_h and u_h^ε.

On the contrary, we do not know how to do it in the second case (at least in a rather general and extendable way), namely for finite difference schemes like

$$-\frac{1}{2}a(x)\left[\frac{u_h(x+h) - 2u_h(x) + u_h(x-h)}{h^2}\right] + \lambda u_h(x) = f(x) \quad \text{in } \mathbb{R}.$$

Indeed we face here the same difficulties as one faced for a long time for the PDEs, but without here the help of the so-called "maximum principle for semicontinuous functions", i.e. Theorem 3.2 in [6].

Since we do not know how to solve this difficulty in a general way, we are going to introduce an assumption on the scheme (Assumption 2.4) which has, unfortunately, to be checked on each example. We do it in Section 3 for control schemes which were studied by classical methods in Menaldi [19] and by viscosity solutions' methods by Camilli and Falcone [4], and in Section 4 for finite difference schemes.

Finally we want to point out that, if the equation and the scheme satisfy symmetrical properties, our approach provides the same order in h for the upper and lower bound on $u - u_h$. This is the case for example if one assumes the discount factors to be large enough compared to the various Lipschitz constants arising in F and S. But, since this rate of convergence relies a lot on the exponent δ of the $C^{0,\delta}$ regularity of u, and also on the possibly different exponent $\bar{\delta}$ of the regularity of u_h and u_h^ε, this symmetry cannot be expected in general.

This paper is organized as follows: in the next section, we state and prove the main result on the convergence rate. In Sections 3 and 4, we study the applications to control schemes and to finite difference schemes. The Appendix contain the proofs of the most technical results of the paper.

2. The main result

We start by introducing the norms and spaces we will use in this article and in particular in this section. We first define the norm denoted by $|\cdot|$ as follows: for any integer $m \geq 1$ and any $z = (z_i)_i \in \mathbb{R}^m$, we set $|z|^2 = \sum_{i_1=1}^m z_i^2$. We identify $N_1 \times N_2$ matrices with $\mathbb{R}^{N_1 \times N_2}$ vectors. For such matrices, $|M|^2 = \mathrm{tr}[M^T M]$ where M^T denotes the transpose of M.

If $f : \mathbb{R}^N \to \mathbb{R}^m$ is a function and $\delta \in (0,1]$, then define the following semi-norms

$$|f|_0 = \sup_{x \in \mathbb{R}^N} |f(x)|, \qquad [f]_\delta = \sup_{\substack{x,y \in \mathbb{R}^N \\ x \neq y}} \frac{|f(x) - f(y)|}{|x-y|^\delta} \qquad \text{and} \qquad |f|_\delta = |f|_0 + [f]_\delta.$$

By $\mathcal{C}^{0,\delta}(\mathbb{R}^N)$ we denote the set of functions $f : \mathbb{R}^N \to \mathbb{R}$ with finite norm $|f|_\delta$. Furthermore for any integer $n \geq 1$ we define $\mathcal{C}^{n,\delta}(\mathbb{R}^N)$ to be the space of n times continuously differentiable functions $f : \mathbb{R}^N \to \mathbb{R}$ with finite norm

$$|f|_{n,\delta} = \sum_{i=0}^n |D^i f|_0 + [D^n f]_\delta,$$

where $D^i f$ denotes the vector of the i-th order partial derivatives of f. Note that $\mathcal{C}^{0,\delta}(\mathbb{R}^N)$ and $\mathcal{C}^{n,\delta}(\mathbb{R}^N)$ are Banach spaces. Finally we denote by $C(\mathbb{R}^N)$, $C_b(\mathbb{R}^N)$ and $C^\infty(\mathbb{R}^N)$ the spaces of continuous functions, bounded continuous functions, and infinitely differentiable functions on $\mathbb{R}^N$. Throughout the paper "C" stands for a positive constant, which may vary from line to line, but which is independent of the small parameters h and ε we use.

The assumptions we use on the Hamilton-Jacobi-Bellman Equation (1.1) are the following

(A1) For any $\vartheta \in \Theta$, there exists a $N \times P$ matrix σ^ϑ such that $a^\vartheta = \sigma^\vartheta \sigma^{\vartheta T}$. Moreover there exists $M > 0$ and $\delta \in (0,1]$ such that, for any $\vartheta \in \Theta$,

$$|\sigma^\vartheta|_1, |b^\vartheta|_1, |c^\vartheta|_\delta, |f^\vartheta|_\delta \leq M.$$

(A2) There exists $\lambda > 0$ such that, for any $x \in \mathbb{R}^N$ and $\vartheta \in \Theta$, $c^\vartheta(x) \geq \lambda$.

We will also use the following quantity

(2.1)
$$\lambda_0 := \sup_{\substack{x \neq y \\ \vartheta \in \Theta}} \left\{ \frac{1}{2} \frac{\mathrm{tr}\left[\big(\sigma^\vartheta(x) - \sigma^\vartheta(y)\big)\big(\sigma^\vartheta(x) - \sigma^\vartheta(y)\big)^T \right]}{|x-y|^2} + \frac{\big(b^\vartheta(x) - b^\vartheta(y), x-y\big)}{|x-y|^2} \right\}.$$

By assumption (A1), we have $0 \leq \lambda_0 < 3M/2$. The next two (almost) classical results recall that, under assumptions (A1) and (A2), we have existence, uniqueness, and Hölder regularity of viscosity solutions of (1.1).

Theorem 2.1. *Under assumptions (A1) and (A2) there exists a unique bounded continuous viscosity solution of* (1.1). *Moreover for $u, v \in C_b(\mathbb{R}^N)$, if u and v are viscosity sub- and supersolutions of* (1.1) *respectively, then $u \leq v$ in $\mathbb{R}^N$.*

The proof of this result is classical and left to the reader. The second result is

Theorem 2.2. *Assume that (A1) and (A2) hold, and assume that u is the (unique) bounded viscosity solution of* (1.1). *Then $u \in \mathcal{C}^{0,\bar\delta}(\mathbb{R}^N)$, where $\bar\delta$ is defined as follows: (i) when $\lambda < \delta\lambda_0$ then $\bar\delta = \frac{\lambda}{\lambda_0}$, (ii) when $\lambda = \delta\lambda_0$ then $\bar\delta$ is any number in $(0,\delta)$, and (iii) when $\lambda > \delta\lambda_0$ then $\bar\delta = \delta$.*

This result is proved in [15, 16, 17] in the case $\delta = 1$. The case $\delta < 1$ follows after easy modifications in this proof. We now state the assumptions on the approximation scheme (1.2).

(C1) (Monotony) There exists $\bar\lambda > 0$ such that, for every $h \geq 0$, $x \in \mathbb{R}^N$, $t \in \mathbb{R}$, $m \geq 0$ and bounded functions u, v such that $u \leq v$ in $\mathbb{R}^N$ then

$$S(h, x, t+m, [u+m]^h_x) \geq S(h, x, t, [v]^h_x) + \bar\lambda m \ .$$

(C2) (Regularity) For every $h > 0$ and $\phi \in C_b(\mathbb{R}^N)$, $x \mapsto S(h, x, \phi(x), [\phi]^h_x)$ is bounded and continuous in $\mathbb{R}^N$ and the function $t \mapsto S(h, x, t, [\phi]^h_x)$ is uniformly continuous for bounded t, uniformly with respect to $x \in \mathbb{R}^N$.

To state the next assumption, we use a sequence of mollifiers $(\rho_\varepsilon)_\varepsilon$ defined as follows

$$(2.2) \quad \rho_\varepsilon(x) = \frac{1}{\varepsilon^N}\rho(\frac{x}{\varepsilon}) \ \text{ where } \ \rho \in C^\infty(\mathbb{R}^N), \int_{\mathbb{R}^N} \rho = 1, \ \text{and } \mathit{supp}\{\rho\} = \bar B(0,1).$$

The next assumption is

(C3) (Convexity) For any $\hat\delta \in (0,1]$ and $v \in \mathcal{C}^{0,\hat\delta}(\mathbb{R}^N)$, there exists a constant $K > 0$ such that for $h > 0$ and $x \in \mathbb{R}^N$

$$\int_{\mathbb{R}^N} S(h, x, v(x-e), [v(\cdot - e)]^h_x)\rho_\varepsilon(e)de \geq S(h, x, (v*\rho_\varepsilon)(x), [v*\rho_\varepsilon]^h_x) - K\varepsilon^{\hat\delta} \ .$$

(C4) (Consistency) There exist $n \in \mathbb{N}$, $\delta_0 \in (0,1]$, and $k > 0$ such that for every $v \in \mathcal{C}^{n,\delta_0}(\mathbb{R}^N)$, there is a constant $\bar K > 0$ such that for $h \geq 0$ and $x \in \mathbb{R}^N$

$$|F(x, v, Dv, D^2 v) - S(h, x, v(x), [v]^h_x)| \leq \bar K |v|_{n,\delta_0} h^k \ .$$

Condition (C1) is a monotonicity condition stating that $S(h, x, t, [u]^h_x)$ is nondecreasing in $t \in \mathbb{R}$ and non-increasing in $[u]^h_x$ for bounded (possibly discontinuous) functions u equipped with the usual partial ordering. In the schemes we are going to consider in this article $\bar\lambda = \lambda$, but it is also natural to consider schemes where $\bar\lambda \neq \lambda$. Condition (C3) is satisfied with $K = 0$ by Jensen's inequality if S is convex in t and $[u]^h_x$. Finally, condition (C4) implies that smooth solutions of the scheme (1.2) will converge towards the solution of equation (1.1).

In the sequel, we say that a function $u \in C_b(\mathbb{R}^N)$ is a subsolution (resp. supersolution) to the scheme if

$$S(h, x, u(x), [u]^h_x) \leq 0 \ (\text{resp. } \geq 0) \quad \text{for all } x \in \mathbb{R}^N \ .$$

Condition (C1) and (C2) imply a comparison result for continuous solutions of (1.2).

Lemma 2.3. *Let $u, v \in C_b(\mathbb{R}^N)$. If u and v are sub- and supersolutions of* (1.2) *respectively, then $u \leq v$ in $\mathbb{R}^N$.*

Proof. Let $\{x_n\}_n$ be a sequence in $\mathbb{R}^N$ such that $u(x_n) - v(x_n) =: \delta_n \to m := \sup_{\mathbb{R}^N}(u - v)$ as $n \to \infty$. Then (C1) and (C2) yield

$$\begin{aligned} 0 &\geq S(h, x_n, u(x_n), [u]^h_{x_n}) - S(h, x_n, v(x_n), [v]^h_{x_n}) \\ &\geq S(h, x_n, v(x_n) + \delta_n, [v+m]^h_{x_n}) - S(h, x_n, v(x_n), [v]^h_{x_n}) \\ &\geq \bar\lambda\delta_n + \omega(m - \delta_n) \ , \end{aligned}$$

where $\omega(t) \to 0$ when $t \to 0^+$ is given by (C2). Letting $n \to \infty$, we get $m \leq 0$ and the proof is complete. $\square$

The uniqueness of continuous solutions of (1.2) is a consequence of the previous lemma. Now, in order to follow Krylov's method, we have to consider the existence and regularity of solutions, not only for (1.2) but also for a perturbed version of it, namely equation (1.4).

In our approach, we need the solution of (1.4) to exist, to have a suitable regularity and to be close to the solution of (1.2). Unfortunately, as mentioned in the introduction, we are unable to prove that such results follow from (C1) – (C4) and we are lead to the following assumption:

Assumption 2.4. *For $h > 0$ small enough and $0 \leq \varepsilon \leq 1$, the scheme* (1.4) *has a solution $u_h^\varepsilon \in C_b(\mathbb{R}^N)$. Moreover there exists a $\tilde{\delta} \in (0, \bar{\delta}]$ ($\bar{\delta}$ defined in Theorem 2.2), independent of h and ε, such that*

$$|u_h^\varepsilon|_{\tilde{\delta}} \leq C \quad \textit{and} \quad |u_h^0 - u_h^\varepsilon|_0 \leq C\varepsilon^{\tilde{\delta}} .$$

Note that u_h^0 is the solution of (1.2). This assumption is a key assumption and, at least for the moment, this is the limiting step in our approach. In Section 3 and 4, we verify it for each of the examples that we have in mind.

We need a last assumption on the scheme

(C5) (Commutation with translations) For any $h > 0$ small enough, $0 \leq \varepsilon \leq 1$, $y \in \mathbb{R}^N$, $t \in \mathbb{R}$, $v \in C_b(\mathbb{R}^N)$ and $|e| \leq \varepsilon$, we have

$$S(h, y, t, [v]_{y-e}^h) = S(h, y, t, [v(\cdot - e)]_y^h) .$$

Our main result is

Theorem 2.5 (Convergence rate for HJB)**.** *Assume that (A1) and (A2) hold, and that the scheme* (1.2) *satisfies (C1) – (C5) and Assumption 2.4. Let $u \in \mathcal{C}^{0,\bar{\delta}}(\mathbb{R}^N)$ and $u_h \in \mathcal{C}^{0,\tilde{\delta}}(\mathbb{R}^N)$ be the viscosity solution of* (1.1) *and the solution of* (1.2) *respectively. Then the following two bounds hold*

$$(i) \quad u - u_h \leq Ch^{\frac{\bar{\delta}k}{n+\delta_0}} \qquad \textit{and} \qquad (ii) \quad u - u_h \geq Ch^{\frac{\tilde{\delta}k}{n+\delta_0}}.$$

As we already mentioned, the bounds (i) and (ii) do not need to coincide. We proceed by proving Theorem 2.5. We start by proving the bound (i) using mostly properties of the equation (1.1). Then we prove the bound (ii) using mainly properties of the scheme (1.2).

Proof of bound (i) in Theorem 2.5.

As we mentioned in the Introduction, this bound was proved by Krylov [11, 12] ; we provide a proof for the sake of completeness and for the reader's convenience.

1. We first consider the approximate HJB equations (1.3) : the existence and the properties of the solutions of (1.3) are given in the following lemma whose proof is given in the Appendix.

Lemma 2.6. *Assume that (A1) and (A2) hold and let $0 \leq \varepsilon \leq 1$. Equation* (1.3) *where F is given by* (1.1) *has a unique bounded viscosity solution $u^\varepsilon \in \mathcal{C}^{0,\bar{\delta}}(\mathbb{R}^N)$ satisfying $|u^\varepsilon|_{\bar{\delta}} \leq C$ and $|u^\varepsilon - u|_0 \leq C\,\varepsilon^{\bar{\delta}}$, where $\bar{\delta}$ is defined in Theorem 2.2.*

2. Because of the definition of equation (1.3), it is clear, after the change of variables $y = x + e$, that $u^\varepsilon(\cdot - e)$ is a subsolution of (1.1) for every $|e| \leq \varepsilon$, i.e. that, for every $|e| \leq \varepsilon$, $u^\varepsilon(\cdot - e)$ satisfies in the viscosity sense

$$F(y, u^\varepsilon(\cdot - e), Du^\varepsilon(\cdot - e), D^2 u^\varepsilon(\cdot - e)) \leq 0 \quad \text{ in } \mathbb{R}^N .$$

3. In order to regularize u^ε, we consider the function u_ε defined in $\mathbb{R}^N$ by

$$u_\varepsilon(x) := \int_{\mathbb{R}^N} u^\varepsilon(x - e)\rho_\varepsilon(e) de .$$

where $(\rho_\varepsilon)_\varepsilon$ are the standard mollifiers defined in (2.2). We have

Lemma 2.7. *The function u_ε is a viscosity subsolution of* (1.1).

The proof of this lemma is also postponed to the Appendix.

4. By properties of mollifiers, since the u^ε are uniformly bounded in $\mathcal{C}^{0,\bar\delta}$, we have $u_\varepsilon \in \mathcal{C}^{n,\delta_0}(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$ with $|u_\varepsilon|_{n,\delta_0} \le C\varepsilon^{\bar\delta-n-\delta_0}$. Then using the consistency property (C4), we obtain

$$F(y,u_\varepsilon(y),Du_\varepsilon(y),D^2u_\varepsilon(y)) \ge S(h,y,u_\varepsilon(y),[u_\varepsilon]^h_y) - \bar K|u_\varepsilon|_{n,\delta_0}h^k \quad \text{ in } \mathbb{R}^N \ .$$

From Lemma 2.7, we deduce that $S(h,y,u_\varepsilon(y),[u_\varepsilon]^h_y) \le Ch^k\varepsilon^{\bar\delta-n-\delta_0}$ in $\mathbb{R}^N$.

5. By (C1) we see that $u_\varepsilon - Ch^k\varepsilon^{\bar\delta-n-\delta_0}/\bar\lambda$ is a subsolution of the scheme (1.2). Hence by the comparison principle for (1.2) (cf. Lemma 2.3)

$$u_\varepsilon - u_h \le Ch^k\varepsilon^{\bar\delta-n-\delta_0} \quad \text{ in } \mathbb{R}^N \ .$$

6. The properties of mollifiers and the uniform boundedness in $\mathcal{C}^{0,\bar\delta}$ of the u^ε's imply $|u^\varepsilon - u_\varepsilon|_0 \le C\varepsilon^{\bar\delta}$. Moreover from Lemma 2.6 it follows that $|u-u^\varepsilon|_0 \le C\varepsilon^{\bar\delta}$. All in all we conclude that

$$|u-u_\varepsilon|_0 \le C\varepsilon^{\bar\delta}.$$

7. Finally, gathering the information obtained in step 5 and 6 yields

$$u - u_h \le Ch^k\varepsilon^{\bar\delta-n-\delta_0} + C\varepsilon^{\bar\delta} \quad \text{ in } \mathbb{R}^N \ .$$

The conclusion follows by choosing an optimal ε, namely $\varepsilon^{n+\delta_0} = h^k$. And the proof is complete.

Proof of bound (ii) in Theorem 2.5.

We follow exactly the same method as for the bound (i), changing the role of the equation and the scheme.

1. Let u_h^ε be the $\mathcal{C}^{0,\tilde\delta}$ solution of the scheme (1.4) provided by Assumption 2.4. From the scheme (1.4), by performing the change of variables $y = x+e$, and using (C5), we see that $S(h,y,u_h^\varepsilon(y-e),[u_h^\varepsilon(\cdot-e)]^h_y) \le 0$ for all $|e| \le \varepsilon$ and $y \in \mathbb{R}^N$.

2. Let $(\rho_\varepsilon)_\varepsilon$ be the standard mollifiers defined in (2.2). Multiplying the above inequality by $\rho_\varepsilon(e)$, integrating with respect to e and using (C3) yield

$$\begin{aligned} 0 &\ge \int_{\mathbb{R}^N} \rho_\varepsilon(e)S(h,y,u_h^\varepsilon(y-e),[u_h^\varepsilon(\cdot-e)]^h_y)de \\ &\ge S(h,y,(u_h^\varepsilon * \rho_\varepsilon)(y),[u_h^\varepsilon * \rho_\varepsilon]^h_y) - K\varepsilon^{\tilde\delta} \ , \end{aligned}$$

where

$$u_h^\varepsilon * \rho_\varepsilon(x) := \int_{\mathbb{R}^N} u_h^\varepsilon(x-e)\rho_\varepsilon(e)de \ .$$

Note that all the above integrals are well-defined since the integrand is continuous by (C2).

3. Because of the properties of u_h^ε given in Assumption 2.4 and the properties of mollifiers, $u_h^\varepsilon * \rho_\varepsilon \in \mathcal{C}^{n,\delta_0}(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$ with $|u_h^\varepsilon * \rho_\varepsilon|_{n,\delta_0} \le C\varepsilon^{\tilde\delta-n-\delta_0}$. By (C4) we then have

$$\begin{aligned} &S(h,y,(u_h^\varepsilon * \rho_\varepsilon)(y),[u_h^\varepsilon * \rho_\varepsilon]^h_y) \\ &\ge F(y,u_h^\varepsilon * \rho_\varepsilon, D(u_h^\varepsilon * \rho_\varepsilon), D^2(u_h^\varepsilon * \rho_\varepsilon)) - \bar K|u_h^\varepsilon * \rho_\varepsilon|_{n,\delta_0}h^k \ . \end{aligned}$$

4. Gathering all this information, we have

$$F(y,u_h^\varepsilon * \rho_\varepsilon, D(u_h^\varepsilon * \rho_\varepsilon), D^2(u_h^\varepsilon * \rho_\varepsilon)) \le C(\varepsilon^{\tilde\delta} + h^k\varepsilon^{\tilde\delta-n-\delta_0}) \quad \text{ in } \mathbb{R}^N \ .$$

5. By (A2) we see that $u_h^\varepsilon * \rho_\varepsilon - C(\varepsilon^{\tilde\delta} + h^k\varepsilon^{\tilde\delta-n-\delta_0})/\lambda$ is subsolution of (1.1), and by the comparison principle for (1.1) (cf. Theorem 2.1)

$$u_h^\varepsilon * \rho_\varepsilon - u \le C(\varepsilon^{\tilde\delta} + h^k\varepsilon^{\tilde\delta-n-\delta_0}) \quad \text{in } \mathbb{R}^N .$$

6. Again by the properties of mollifiers and the $\mathcal{C}^{0,\tilde\delta}$ regularity of u_h^ε we get that $|u_h^\varepsilon - u_h^\varepsilon * \rho_\varepsilon|_0 \le C\varepsilon^{\tilde\delta}$. Moreover, by Assumption 2.4, it follows that $|u_h - u_h^\varepsilon|_0 \le C\varepsilon^{\tilde\delta}$. All in all we conclude that

$$|u_h - u_h^\varepsilon * \rho_\varepsilon|_0 \le C\varepsilon^{\tilde\delta} \quad \text{in } \mathbb{R}^N .$$

7. Finally, we deduce from step 5 and 6 that

$$u_h - u \le C(\varepsilon^{\tilde\delta} + h^k\varepsilon^{\tilde\delta-n-\delta_0}) \quad \text{in } \mathbb{R}^N .$$

In order to conclude, we choose again an optimal ε, namely $\varepsilon^{n+\delta_0} = h^k$. And the proof is complete.

3. Application 1 : Control-schemes.

In this section, we consider general so-called control schemes. Such schemes were introduced for first-order Hamilton-Jacobi equations (in the viscosity solutions setting) by Capuzzo-Dolcetta [5] and for second-order equations (in a classical setting) by Menaldi [19]. We will consider the schemes as they were defined in Camilli and Falcone [4]. Actually, we will consider a slight generalization where c^ϑ is not assumed to be constant. We also consider an other extension: In [4] there is the condition that $\lambda > \delta\lambda_0$. We treat the general case where λ is only assumed to be positive.

The scheme is defined in the following way

$$u_h(x) = \min_{\vartheta\in\Theta}\Big\{(1 - hc^\vartheta(x))\Pi_h^\vartheta u_h(x) + hf^\vartheta(x)\Big\}, \tag{3.1}$$

where Π_h^ϑ is the operator:

$$\Pi_h^\vartheta\phi(x) = \frac{1}{2N}\sum_{m=1}^N \Big(\phi(x + hb^\vartheta(x) + \sqrt{h}\sigma_m^\vartheta(x)) + \phi(x + hb^\vartheta(x) - \sqrt{h}\sigma_m^\vartheta(x))\Big),$$

and σ_m^ϑ is the m-th row of σ^ϑ. We note that this is not yet a fully discrete method because the placement of the nodes varies with x. In [4] a fully discrete method is derived from (3.1) and analyzed in the case $c^\vartheta(x) = \lambda$. The authors also provide the rate of convergence for the convergence of the solution of the fully discrete method to the solution of the scheme (3.1). We now complete this work by giving the rate of the convergence of the solution of the scheme (3.1) to the solution of the equation (1.1) as $h \to 0$.

To do so, we first rewrite the scheme (3.1) in a different way. Indeed, on one hand, because of Assumption 2.4 and (C5), the role of the different x-dependences in the scheme need to be defined precisely. On the other hand, the consistency requirement has to be satisfied. Therefore, we are going to define $S(h, y, t, [\phi]_x^h)$ where ϕ is a bounded, continuous function in $\mathbb{R}^N$. First, for any $x, z \in \mathbb{R}^N$, we set $[\phi]_x^h(z) = \phi(x + z)$ and then

$$S(h, y, t, [\phi]_x^h) = \sup_{\vartheta\in\Theta}\left\{-\frac{1}{h}(A(h, \vartheta, y, [\phi]_x^h) - t) + c^\vartheta(y)t - f^\vartheta(y)\right\} , \tag{3.2}$$

where A is given by

$$A(h, \vartheta, y, [\phi]_x^h) := \frac{1}{2N}\sum_{m=1}^N \Big([\phi]_x^h(hb^\vartheta(y) + \sqrt{h}\sigma_m^\vartheta(y)) + [\phi]_x^h(hb^\vartheta(y) - \sqrt{h}\sigma_m^\vartheta(y))\Big) .$$

It is easy to see that S defines a scheme which is equivalent to (3.1) and, in the sequel, we will use one or the other indifferently.

We start by checking that conditions (C1) – (C5) hold.

Proposition 3.1. *Assume that (A1) and (A2) hold. Then the scheme* (3.2) *satisfy conditions (C1) – (C5) with* $\bar\lambda = \lambda$, $K = 0$, $k = 1$, $n = 3$, *and* $\delta_0 = 1$.

Proof. First, conditions (C1) and (C2) follow easily from conditions (A2) and (A1) respectively. It is worth noticing that we have here $\bar\lambda = \lambda$. Condition (C3) holds with $K = 0$ because for any function $g(x, \vartheta)$,

$$\rho_\varepsilon * g(\cdot,\vartheta)(x) \le \rho_\varepsilon * \sup_{\vartheta\in\Theta} g(\cdot,\vartheta)(x) \implies \sup_{\vartheta\in\Theta} \rho_\varepsilon * g(\cdot,\vartheta)(x) \le \rho_\varepsilon * \sup_{\vartheta\in\Theta} g(\cdot,\vartheta)(x).$$

The consistency condition (C4) takes the following form:

$$|F(x, v, Dv, D^2v) - S(h, x, v(x), [v]^h_x)| \le \bar K |v|_{3,1} h\ ,$$

for any $v \in \mathcal{C}^{3,1}(\mathbb{R}^N)$. And finally (C5) holds since, for any bounded, continuous function ϕ, $[\phi]^h_{x-e} = [\phi(\cdot - e)]^h_x$. □

We have the following result on existence, uniqueness, and regularity of solutions of (3.1).

Theorem 3.2. *Assume that (A1) and (A2) hold. Then there exists a unique bounded solution of the scheme* (3.1) *satisfying the following bound*

$$|u_h|_0 \le \sup_{\vartheta\in\Theta} \left\{ \frac{|f^\vartheta|_0}{\lambda} \right\}.$$

Moreover, if $\lambda > \delta\bar\lambda_0$ *where* $\bar\lambda_0 = \sup_\vartheta([\sigma^\vartheta]_1^2/2 + [b^\vartheta]_1)$, *then* $u_h \in \mathcal{C}^{0,\delta}(\mathbb{R}^N)$ *and the following bound holds*

$$[u_h]_\delta \le \sup_{\vartheta\in\Theta} \left\{ \frac{[c^\vartheta]_\delta |u_h|_0 + [f^\vartheta]_\delta}{\lambda - \delta\bar\lambda_0} \right\}.$$

This result was proved in [4] in the case where $c^\vartheta(x) \equiv \lambda$. The extension to non-constant $c^\vartheta(x)$ is easy. We proceed by using an iteration technique due to Lions [14] to obtain regularity in the case $\lambda \le \delta\bar\lambda_0$.

Theorem 3.3. *Assume that (A1) and (A2) hold and that* $0 < \lambda < \delta\bar\lambda_0$. *If* u_h *is the solution of* (3.1), *then* $u_h \in \mathcal{C}^{0,\frac{\lambda}{\bar\lambda_0}}(\mathbb{R}^N)$.

Proof. Let $\gamma > 0$ be such that $\lambda + \gamma > \delta\bar\lambda_0$ and let $v \in \mathcal{C}^{0,\delta}(\mathbb{R}^N)$ be in the set $X := \{w \in C(\mathbb{R}^N) : |w|_0 \le M/\lambda\}$. Consider the following equation

$$S(h, x, u(x), [u]^h_x) + \gamma u(x) = \gamma v(x) \quad \text{in } \mathbb{R}^N. \tag{3.3}$$

Let T denote the operator taking v to the viscosity solution u of (3.3). It is well-defined because by replacing $c^\vartheta, f^\vartheta, \lambda$ by $c^\vartheta + \gamma, f^\vartheta - \gamma v, \lambda + \gamma$, Theorem 3.2 yield existence and uniqueness of a solution $u \in \mathcal{C}^{0,\delta}(\mathbb{R}^N)$ of equation (3.3).

Now we note that by (A1), (A2), and the definition of v, $\pm M/\lambda$ are semisolutions of (3.3) as well as (3.1). By comparison, Lemma 2.3, this implies that $|u|_0 \le M/\lambda$. So we see that $T : \mathcal{C}^{0,\delta}(\mathbb{R}^N) \cap X \to \mathcal{C}^{0,\delta}(\mathbb{R}^N) \cap X$. For $v, w \in \mathcal{C}^{0,\delta}(\mathbb{R}^N) \cap X$ we note that $Tw - |w - v|_0\gamma/(\lambda + \gamma)$ and $Tv - |w - v|_0\gamma/(\lambda + \gamma)$ are subsolutions of (3.3) with right hand sides γv and γw respectively. So by using the comparison principle Lemma 2.3 twice we get

$$|Tw - Tv|_0 \le \frac{\gamma}{\lambda + \gamma} |w - v|_0 \qquad \forall w, v \in \mathcal{C}^{0,\delta}(\mathbb{R}^N) \cap X. \tag{3.4}$$

Let $u^0_h(x) = M/\lambda$ and $u^n_h(x) = Tu^{n-1}_h(x)$. Since X is a Banach space and T a contraction mapping (3.4) on this space, the contraction mapping theorem yields the existence and uniqueness of $u_h \in X$ where $u_h^n \to u_h \in X$ and u_h solves (3.1). Since $|u_h - u_h^n|_0 \le |u_h - u_h^{n+k}|_0 + \sum_{i=1}^k |u_h^{n+i} - u_h^{n+i-1}|_0$, using (3.4) and sending $k \to \infty$, and then using (3.4) again, we obtain

$$
\begin{aligned}
|u_h - u_h^n|_0 &\le \frac{1}{1-\frac{\gamma}{\lambda+\gamma}} |u_h^{n+1} - u_h^n|_0 \\
&\le \frac{\lambda+\gamma}{\gamma}\Big(\frac{\gamma}{\lambda+\gamma}\Big)^n |u_h^1 - u_h^0|_0 \\
&\le \frac{2M}{\lambda}\Big(\frac{\gamma}{\lambda+\gamma}\Big)^{n-1}.
\end{aligned} \tag{3.5}
$$

Furthermore since $\lambda + \gamma \ge \delta\bar\lambda_0$, Theorem 3.2 yield the following estimate on the Hölder seminorm of u_h^n

$$
[u_h^n]_\delta \le \frac{K + \gamma[u_h^{n-1}]_\delta}{\lambda+\gamma-\delta\bar\lambda_0} \le \left(\frac{\gamma}{\lambda+\gamma-\delta\bar\lambda_0}\right)^{n-1}\left([u_h^0]_\delta + \frac{K}{\lambda+\gamma-\delta\bar\lambda_0}\right), \tag{3.6}
$$

where the constant K does not depend on n or γ. Since $\gamma \ge \delta\bar\lambda_0 - \lambda$, we can replace the last parenthesis in (3.6) by a constant not depending on n or γ. Now let $m = n-1$, $x, y \in \mathbb{R}^N$, and note that $|u_h(x) - u_h(y)| \le |u_h(x) - u_h^n(x)| + |u_h^n(x) - u_h^n(y)| + |u_h^n(y) - u_h(y)|$. Using (3.5) and (3.6) we get the following expression

$$
|u_h(x) - u_h(y)| \le C\left\{\left(\frac{\gamma}{\lambda+\gamma}\right)^m + \left(\frac{\gamma}{\lambda+\gamma-\delta\bar\lambda_0}\right)^m |x-y|^\delta\right\}. \tag{3.7}
$$

Let $t = |x-y|$ and ω be the modulus of continuity of u. Fix $t \in (0,1)$ and define γ in the following way

$$
\gamma := \frac{m\bar\lambda_0}{\log\frac{1}{t}}.
$$

Note that if m_t is sufficiently large, then $m \ge m_t$ implies that $\gamma \ge \delta\bar\lambda_0$. Using this new notation, we can rewrite (3.7) the following way

$$
\omega(t) \le C\left\{\left(1 + \frac{\lambda}{\bar\lambda_0}\log\left(\frac{1}{t}\right)\frac{1}{m}\right)^{-m} + \left(1 + \frac{\lambda-\delta\bar\lambda_0}{\bar\lambda_0}\log\left(\frac{1}{t}\right)\frac{1}{m}\right)^{-m} t^\delta\right\},
$$

and letting $m \to \infty$ we obtain $\omega(t) \le C\{t^{\lambda/\bar\lambda_0} + t^{\lambda/\bar\lambda_0 - \delta} t^\delta\}$. Now we can conclude since this inequality must hold for any $t \in (0,1)$. □

Finally we need a continuous dependence type of result to bound the difference between u_h of (3.1) and solution u_h^ε of (1.4). The "direct" method used in the proof of Theorem 3.2 to prove Hölder regularity seems not to work so well here. In order to overcome this difficulty, we use "discrete viscosity methods". That is, we double the variables and replace the solution by a test-function. The difficulty is to work without the maximum principle for semicontinuous functions. This is done by constructing schemes for the doubling of variable problem in $\mathbb{R}^{2N}$. Let us state the result corresponding to Assumption 2.4.

Theorem 3.4. *Assume that (A1) and (A2) hold and let* $0 \le \varepsilon \le 1$ *and* $h \le 1$. *Then the scheme* (1.4) *has a unique bounded solution* $u_h^\varepsilon \in C^{0,\tilde\delta}(\mathbb{R}^N)$ *satisfying* $|u_h^\varepsilon|_{\tilde\delta} \le C$ *and* $|u_h^\varepsilon - u_h|_0 \le C\varepsilon^{\tilde\delta}$, *where* $u_h = u_h^0$ *is the solution of* (3.1), *and where* $\tilde\delta := \lambda/\bar\lambda_0$ *when* $\lambda < \bar\lambda_0\delta$, $\tilde\delta := \delta$ *when* $\lambda > \delta\bar\lambda_0$, *and* $\tilde\delta$ *is any number in* $(0,\delta)$ *when* $\lambda = \bar\lambda_0\delta$.

Proof. We write $S_\varepsilon(h, x, u(x), [u]_x^h) := \sup_{|e|\le\varepsilon} S(h, x+e, u(x), [u]_x^h)$, and note that (C1) holds for this scheme with the same constant λ. By replacing ϑ by (ϑ, e), we see that existence, uniqueness and the Hölder norm bound follow from Theorems 3.2 and 3.3.

We turn to the bound on $u_h^\varepsilon - u_h$. First notice that because of the very definition of the scheme (1.4), u_h^ε is a subsolution for the S–scheme and Lemma 2.3 implies that $u_h^\varepsilon \leq u_h$ in $\mathbb{R}^N$.

Therefore we have just to prove that $u_h - u_h^\varepsilon \leq C\varepsilon^{\tilde{\delta}}$ and, to do so, we consider the $\mathbb{R}^{2N}$–scheme which can be written either as

$$w(x,y) = \sup_{\substack{\vartheta\in\Theta\\ |e|\leq\varepsilon}} \Big\{ (1 - hc^\vartheta(x))\Pi_h^{\vartheta,e} w(x,y) \Big\},$$

where $\Pi_h^{\vartheta,e}$ is the operator:

$$\Pi_h^{\vartheta,e}\psi(x,y) = \frac{1}{2N}\sum_{m=1}^{N} \Big\{ \psi\big(x + hb^\vartheta(x) + \sqrt{h}\sigma_m^\vartheta(x), y + hb^\vartheta(y+e) + \sqrt{h}\sigma_m^\vartheta(y+e)\big) + \psi\big(x + hb^\vartheta(x) - \sqrt{h}\sigma_m^\vartheta(x), y + hb^\vartheta(y+e) - \sqrt{h}\sigma_m^\vartheta(y+e)\big) \Big\},$$

or, equivalently, in the following way

$$\inf_{\substack{\vartheta\in\Theta\\ |e|\leq\varepsilon}} \Big\{ -\frac{1}{h}(\Pi_h^{\vartheta,e} w(x,y) - w(x,y)) + c^\vartheta(x) w(x,y) \Big\} = 0 .$$

We denote by $D_\varepsilon(h,x,y,w(x,y),[w]_{x,y}^h)$ the right-hand side of this equation with $[w]_{x,y}^h(z_1,z_2) = w(x+z_1, y+z_2)$ for any $x,y,z_1,z_2 \in \mathbb{R}^N$.

We first remark that this scheme satisfies the $\mathbb{R}^{2N}$ version of (C1), even with the same constant λ, and (C2).

Then we consider the function $w : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ defined by $w(x,y) := u_h(x) - u_h^\varepsilon(y)$. By the definitions of S, S_ε, D_ε and using the inequality $\inf\{\cdots - \cdots\} \leq \sup\{\cdots\} - \sup\{\cdots\}$, we obtain

$$D_\varepsilon(h,x,y,w(x,y),[w]_{x,y}^h) \leq S(h,x,u_h(x),[u_h]_x^h) - S_\varepsilon(h,y,u_h^\varepsilon(y),[u_h^\varepsilon]_y^h) + (|x-y|+\varepsilon)^\delta \max_{\vartheta\in\Theta} \big\{ [c^\vartheta]_\delta |u_h^\varepsilon| + [f^\vartheta]_\delta \big\} . \tag{3.8}$$

and since u_h, u_h^ε are respectively the solutions of the S and S_ε schemes, we have

$$D_\varepsilon(h,x,y,w(x,y),[w]_{x,y}^h) \leq (|x-y|+\varepsilon)^\delta \max_{\vartheta\in\Theta} \big\{ [c^\vartheta]_\delta |u_h^\varepsilon| + [f^\vartheta]_\delta \big\} \quad \text{in } \mathbb{R}^N \times \mathbb{R}^N . \tag{3.9}$$

Next we introduce $\phi(x,y) := \alpha|x-y|^2 + \eta(|x|^2 + |y|^2)$. (Here and below we drop any dependence in α and η for the sake of simplicity of notations.) By straightforward computations and using (A1), it is easy to show that

$$D_\varepsilon(h,x,y,\phi(x,y),[\phi]_{x,y}^h) \geq -C\Big(\alpha(|x-y|^2 + \varepsilon^2) + \eta(|x|^2 + |y|^2 + |\varepsilon|^2)\Big) . \tag{3.10}$$

Finally we consider $\psi(x,y) := u_h(x) - u_h^\varepsilon(y) - \phi(x,y)$. Since u_h and u_h^ε are bounded, there exists $x_0, y_0 \in \mathbb{R}^N$ such that $m_{\alpha,\eta} := \sup_{x,y\in\mathbb{R}^N} \psi(x,y) = \psi(x_0,y_0)$. We note that $w - m_{\alpha,\eta} \leq \phi$ with equality holding at (x_0,y_0). Moreover, from the inequality $2\psi(x_0,y_0) \geq \psi(x_0,x_0) + \psi(y_0,y_0)$ and the Hölder regularity of u_h and u_h^ε (which is uniform w.r.t h and ε) we see that

$$2\alpha|x_0 - y_0|^2 \leq [u_h]_{\tilde{\delta}} |x_0 - y_0|^{\tilde{\delta}} + [u_h^\varepsilon]_{\tilde{\delta}} |x_0 - y_0|^{\tilde{\delta}}.$$

and therefore we can conclude that $|x_0 - y_0| \leq C\alpha^{-1/(2-\tilde{\delta})}$, which again implies that

$$\alpha|x_0 - y_0|^2 \leq C\alpha^{-\frac{\tilde{\delta}}{2-\tilde{\delta}}} \quad \text{and} \quad |x_0 - y_0|^\delta \leq C\alpha^{-\frac{\delta}{2-\tilde{\delta}}} . \tag{3.11}$$

Furthermore for fixed α, Lemma A.2 yields $\lim_{\eta\to 0}\eta(|x_0|^2+|y_0|^2)=0$ and $\lim_{\eta\to 0} m_{\alpha,\eta}\geq m$, where $m=\sup_{\mathbb{R}^N}\{u_h-u_h^\varepsilon\}$.

Now we use the information given by (3.9) and (3.10) at (x_0,y_0) together with (C1) : since $\max_{\vartheta\in\Theta}([c^\vartheta]_\delta|u_h^\varepsilon|_0+[f^\vartheta]_\delta)$ is bounded independently of h and ε, we have

$$\begin{aligned} C(|x_0-y_0|^\delta+\varepsilon^\delta) &\geq D_\varepsilon(h,x_0,y_0,w(x_0,y_0),[w]^h_{x_0,y_0})\\ &\geq D_\varepsilon(h,x_0,y_0,w(x_0,y_0)-m_{\alpha,\eta},[w-m_{\alpha,\eta}]^h_{x_0,y_0})+\lambda m_{\alpha,\eta}\\ &\geq D_\varepsilon(h,x_0,y_0,\phi(x_0,y_0),[\phi]^h_{x_0,y_0})+\lambda m_{\alpha,\eta}\\ &\geq \lambda m_{\alpha,\eta}+C\Big(\alpha\big(|x_0-y_0|^2+\varepsilon^2\big)-\eta\big(|x_0|^2+|y_0|^2+\varepsilon^2\big)\Big). \end{aligned}$$

We can therefore conclude that

$$\lambda m_{\alpha,\eta}\leq C\big(|x_0-y_0|^\delta+\alpha|x_0-y_0|^2+\varepsilon^\delta+\alpha\varepsilon^2+\eta(|x_0|^2+|y_0|^2+\varepsilon^2)\big). \tag{3.12}$$

Finally, using the estimates (3.11) into (3.12) and passing to the limit $\eta\to 0$ for α fixed, we get

$$\lambda m\leq C(\alpha\varepsilon^2+\varepsilon^\delta+\alpha^{-\frac{\tilde\delta}{2-\tilde\delta}}+\alpha^{-\frac{\delta}{2-\delta}}). \tag{3.13}$$

For $k_1,k_2>0$, by optimization with respect to α, we obtain

$$k_1\alpha+k_2\alpha^{-\frac{\tilde\delta}{2-\tilde\delta}}\leq \bar c(\tilde\delta,\tilde\delta)k_1^{\frac{\tilde\delta}{2}}k_2^{\frac{2-\tilde\delta}{2}}, \tag{3.14}$$

and

$$k_1\alpha+k_2\alpha^{-\frac{\delta}{2-\delta}}\leq \bar c(\tilde\delta,\delta)k_1^{\frac{\delta}{2-\tilde\delta+\delta}}k_2^{\frac{2-\tilde\delta}{2-\tilde\delta+\delta}}, \tag{3.15}$$

where $\bar c(s,t)$ is positive and finite for $0\leq s\leq t\leq 1$. We note that for $0\leq\tilde\delta\leq\delta\leq 1$, $\frac{\tilde\delta}{2}\leq\frac{\delta}{2-\tilde\delta+\delta}$. So with $k_1=\varepsilon^2\leq 1$ we get $k_1^{\delta/(2-\tilde\delta+\delta)}\leq k_1^{\tilde\delta/2}$. Combining (3.13), (3.14) and (3.15), then yield $\lambda\sup_{\mathbb{R}^N}(u_h-u_h^\varepsilon)=\lambda m\leq C\varepsilon^{\tilde\delta}$. And the proof is complete. ☐

From Definition 2.1 of λ_0, we see that $\bar\lambda_0\geq\lambda_0$. Assumption 2.4 holds by Theorem 3.4. Hence we can conclude from Proposition 3.1 and Theorem 2.5 that the following result holds.

Theorem 3.5. *Assume that (A1) and (A2) hold. Let $\bar\lambda_0$ be defined in Theorem 3.2 and define $\tilde\delta$ as follows : (i) when $\lambda>\delta\bar\lambda_0$ then $\tilde\delta=\delta$, (ii) when $\lambda<\delta\bar\lambda_0$ then $\tilde\delta=\frac{\lambda}{\bar\lambda_0}$, (iii) when $\lambda=\bar\lambda_0\delta$ then $\tilde\delta\in(0,\delta)$ (any number). Let u and u_h be the solutions of* (1.1) *and* (3.1) *respectively, then*

$$|u-u_h|_0\leq Ch^{\tilde\delta/4}.$$

Remark 3.6. We remark that $\bar\delta$ defined in Theorem 2.2 is greater than or equal to $\tilde\delta$. This means that for the scheme (3.1) the bound (i) in Theorem 2.5 is always at least as good as the bound (ii). When $\lambda>\delta\bar\lambda_0$ where $\bar\lambda_0$ is defined in Theorem 3.2, then the upper and lower bounds coincide.

Next, we consider a deterministic optimal control problem ($a^\vartheta\equiv 0$ for any ϑ). In this case, condition (C4) takes the following form

$$|F(x,v,Dv)-S(h,x,v(x),[v]^h_x)|\leq\bar K|v|_{1,1}h\ ,$$

for $v\in\mathcal{C}^{1,1}(\mathbb{R}^N)$. It is then clear that Theorem 2.5 yields the following result.

Theorem 3.7. *Assume that (A1) and (A2) hold and that $\sigma^\vartheta \equiv 0$ for any ϑ. Let $\bar\lambda_0$ be defined in Theorem 3.2 and $\tilde\delta$ as in Theorem 3.5. Let u and u_h be the solutions of* (1.1) *and* (3.1) *respectively, then*

$$|u - u_h|_0 \le Ch^{\tilde\delta/2}.$$

When $\delta = 1$ and $\lambda > \bar\lambda_0 = \sup_\vartheta [b^\vartheta]_1$, this result is in agreement with [2, Appendix 1].

4. Application 2: Finite difference schemes

In this section we consider a finite difference scheme proposed by Kushner [13, 8] for the N-dimensional Hamilton-Jacobi-Bellman equation (1.1). We use the notation for these schemes introduced in the books [13, 8].

In this section, we assume that (A1) and (A2) hold, that a^ϑ is independent of x, and that the following two assumptions hold

$$a_{ii}^\vartheta - \sum_{j\neq i} |a_{ij}^\vartheta| \ge 0, \quad i = 1, \dots, N, \tag{4.1}$$

$$\sum_{i=1}^N \Big\{ a_{ii}^\vartheta - \sum_{j\neq i} |a_{ij}^\vartheta| + |b_i^\vartheta(x)| \Big\} \le 1 \quad \text{in } \mathbb{R}^N . \tag{4.2}$$

Condition (4.1) is standard (see [13, 8]) : it implies that the Kushner scheme is monotone. We also refer to Lions and Mercier [18] and to Bonnans and Zidani [3] for a discussion of this condition. Conditions (4.2) may be viewed as normalization of the coefficients in (1.1). We can always have this assumption satisfied by multiplying equation (1.1) by an appropriate positive constant.

In order to simplify matters, in this section we make the additional assumption that (A1) holds with $\delta = 1$. Contrarily to assumption (4.1) which we cannot remove, to treat the case $0 < \delta < 1$ is a little bit more tedious but does not present any real additional difficulty. Roughly speaking, the $0 < \delta < 1$ case can be deduce from the $\delta = 1$ case by using the continuous dependence (w.r.t the sup-norm) of u and u_h in the c^ϑ's and f^ϑ's and a suitable regularizing argument.

The difference operators we use are defined in the following way

$$\begin{aligned}
\Delta_{x_i}^{\pm} w(x) &= \pm\frac{1}{h}\{w(x \pm e_i h) - w(x)\},\\
\Delta_{x_i}^{2} w(x) &= \frac{1}{h^2}\{w(x + e_i h) - 2w(x) + w(x - e_i h)\},\\
\Delta_{x_i x_j}^{+} w(x) &= \frac{1}{2h^2}\{2w(x) + w(x + e_i h + e_j h) + w(x - e_i h - e_j h)\}\\
&\quad - \frac{1}{2h^2}\{w(x + e_i h) + w(x - e_i h) + w(x + e_j h) + w(x - e_j h)\},\\
\Delta_{x_i x_j}^{-} w(x) &= \frac{1}{2h^2}\{w(x + e_i h) + w(x - e_i h) + w(x + e_j h) + w(x - e_j h)\}\\
&\quad - \frac{1}{2h^2}\{2w(x) + w(x + e_i h - e_j h) + w(x - e_i h + e_j h)\}.
\end{aligned}$$

Let $b^+ = \max\{b, 0\}$ and $b^- = (-b)^+$. Note that $b = b^+ - b^-$. For each x, t, $p_i^\pm$, A_{ii}, $A_{ij}^\pm$, $i, j = 1, \dots, N$, let

$$\begin{aligned}
\tilde F(x, t, p_i^\pm, A_{ii}, A_{ij}^\pm) = \sup_{\vartheta\in\Theta} \Big\{ \sum_{i=1}^N \Big[-\frac{a_{ii}^\vartheta}{2} A_{ii} + \sum_{j\neq i} \Big(-\frac{a_{ij}^{\vartheta+}}{2} A_{ij}^+ + \frac{a_{ij}^{\vartheta-}}{2} A_{ij}^- \Big)\\
- b_i^{\vartheta+}(x) p_i^+ + b_i^{\vartheta-}(x) p_i^- \Big] + c^\vartheta(x) t - f^\vartheta(x) \Big\}.
\end{aligned}$$

Now we can write the Kushner scheme in the following way

$$\tilde{F}(x, u_h(x), \Delta^{\pm}_{x_i} u_h(x), \Delta^2_{x_i} u_h(x), \Delta^{\pm}_{x_i x_j} u_h(x)) = 0. \tag{4.3}$$

We remark that this is a monotone finite difference scheme which is consistent with (1.1). Before we check conditions (C1) – (C5), we shall derive an equivalent scheme to the scheme (4.3). This new scheme will be better suited to proving existence, regularity and continuous dependence results. We are going to rewrite (4.3) as a "discrete dynamical programming principle". In this way, it will appear under, essentially, the same form as the scheme presented in Section 3. This point of view was introduced by Kushner, see eg. [13]. But, as opposed to Kushner, we use purely analytical methods in the following. Let $h \le 1$ and define the following "one step transition probabilities"

$$\begin{aligned}
p^\vartheta(x,x) &= 1 - \sum_{i=1}^N \Big\{ a^\vartheta_{ii} - \sum_{j\neq i} |a^\vartheta_{ij}| + h|b^\vartheta_i(x)| \Big\},\\
p^\vartheta(x, x \pm e_i h) &= \frac{a^\vartheta_{ii}}{2} - \sum_{j\neq i} \frac{|a^\vartheta_{ij}|}{2} + h b^{\vartheta\pm}_i(x),\\
p^\vartheta(x, x + e_i h \pm e_j h) &= \frac{a^{\vartheta\pm}_{ij}}{2},\\
p^\vartheta(x, x - e_i h \pm e_j h) &= \frac{a^{\vartheta\mp}_{ij}}{2},
\end{aligned}$$

and $p^\vartheta(x,y) = 0$ for all other y. Note that by (4.1) and (4.2), $0 \le p^\vartheta(x,y) \le 1$ for all ϑ, x, y. Furthermore $\sum_{z\in h\mathbb{Z}^N} p^\vartheta(x, x+z) = 1$ for all ϑ, x.

Tedious but straightforward computations show that u_h satisfies the following equation which is equivalent to (4.3)

$$u_h(x) = \inf_{\vartheta\in\Theta} \left\{ \frac{1}{1 + h^2 c^\vartheta(x)} \Big(\sum_{z\in h\mathbb{Z}^N} p^\vartheta(x, x+z) u_h(x+z) + h^2 f^\vartheta(x) \Big) \right\}. \tag{4.4}$$

It is worth noticing that this formulation is analogous to (3.1).

Analogously to what we did in Section 3, we now define the scheme S. For $\phi \in C_b(\mathbb{R}^N)$, we set $[\phi]^h_x(\cdot) := \phi(x + \cdot)$ and S is given by

$$S(h, y, t, [\phi]^h_x) := \sup_{\vartheta\in\Theta} \left\{ -\frac{1}{h^2} \left[\sum_{z\in h\mathbb{Z}^N} p^\vartheta(y, y+z)[\phi]^h_x(z) - t \right] + c^\vartheta(x)t - f^\vartheta(y) \right\}.$$

It is easy to see that S defines a scheme which is equivalent to (4.4), note also the similarities with (3.2). Using this new notation, let us now check that conditions (C1) – (C5) are satisfied.

Proposition 4.1. *Assume that (A1) with $\delta = 1$ and (A2) hold. Then the scheme* (4.3) *satisfy conditions (C1) – (C5) with $\bar\lambda = \lambda$, $K = 0$, $k = 1$, $n = 2$, and $\delta_0 = 1$.*

Proof. With S in this form is not difficult to see that conditions (C1) (with $\bar\lambda = \lambda$) and (C2) follow from (A2) and (A1). Condition (C3) holds with $K = 0$ because for any function $g(x, \vartheta)$,

$$\rho_\varepsilon * g(\cdot,\vartheta)(x) \le \rho_\varepsilon * \sup_{\vartheta\in\Theta} g(\cdot,\vartheta)(x) \implies \sup_{\vartheta\in\Theta} \rho_\varepsilon * g(\cdot,\vartheta)(x) \le \rho_\varepsilon * \sup_{\vartheta\in\Theta} g(\cdot,\vartheta)(x).$$

The consistency condition for (4.3) reads

$$|F(x, v, Dv, D^2 v) - S(h, x, v(x), [v]^h_x)| \le \bar{K} |v|_{2,1} h\,,$$

for any $v \in \mathcal{C}^{2,1}(\mathbb{R}^N)$. Finally, (C5) follows directly from the above definition of $[\phi]^h_x$. □

We use fix point arguments to prove Assumption 2.4 in the case $\delta = 1$.

Proposition 4.2. *Assume that (A1) with $\delta = 1$ and (A2) hold. Then there exists a unique solution $u_h \in C_b(\mathbb{R}^N)$ of the scheme (1.2). Moreover if $\lambda > \bar\lambda_0 := 2\sqrt{N}\sup_\vartheta [b^\vartheta]_1$, then $|u_h|_1 \le C$.*

Proof. Let $T_h : C_b(\mathbb{R}^N) \to C_b(\mathbb{R}^N)$ be the map defined by in the following way: for any $v \in C_b(\mathbb{R}^N)$

$$T_h v(x) := \inf_{\vartheta\in\Theta} \left\{ \frac{1}{1+h^2 c^\vartheta(x)} \Big(\sum_{z\in h\mathbb{Z}^N} p^\vartheta(x,x+z)v(x+z) + h^2 f^\vartheta(x) \Big) \right\}.$$

We first prove that T_h is a contraction in $C_b(\mathbb{R}^N)$ equipped with the sup-norm. For $u, v \in C_b(\mathbb{R}^N)$, we subtract the expressions for $T_h u$ and $T_h v$. After we use the inequality $\inf(\cdots) - \inf(\cdots) \le \sup(\cdots - \cdots)$, the probability interpretation of p^ϑ, and (A2), we obtain

$$\begin{aligned} T_h u(x) - T_h v(x) &\le \frac{1}{1+\lambda h^2} \sup_\vartheta \left[\sum_{z\in h\mathbb{Z}^N} p^\vartheta(x,x+z)|u(x+z) - v(x+z)| \right] \\ &\le \frac{1}{1+\lambda h^2} |u-v|_0. \end{aligned}$$

Combining this inequality and the inequality obtained by reversing the roles of u and v, we have a contraction. Since $C_b(\mathbb{R}^N)$ is a Banach space, the contraction mapping theorem yields the existence and uniqueness of a $u_h \in C_b(\mathbb{R}^N)$ solving (4.4).

We proceed by proving that u_h has a bounded Lipschitz constant. First we make the simplifying assumption that $c^\vartheta(x) \equiv \lambda$. Given $v \in \mathcal{C}^{0,1}(\mathbb{R}^N)$ we prove that $T_h v \in \mathcal{C}^{0,1}(\mathbb{R}^N)$. Subtracting the expressions for $T_h v(x)$ and $T_h v(y)$, we obtain

$$\begin{aligned} T_h v(x) - T_h v(y) \le \frac{1}{1+\lambda h^2} \sup_\vartheta \Big\{ \sum_{z\in h\mathbb{Z}^N} \Big[& p^\vartheta(x,x+z)(v(x+z) - v(y+z)) \\ & + v(y+z)\big(p^\vartheta(x,x+z) - p^\vartheta(y,y+z)\big)\Big] + h^2(f^\vartheta(x) - f^\vartheta(y))\Big\}. \end{aligned}$$

In the right-hand side, the first sum is bounded by $[v]_1|x-y|$, and by using the definition of p^ϑ, the second sum is equivalent to

$$\begin{aligned} & h\sum_{i=1}^N \Big[\big(b_i^{\vartheta+}(x) - b_i^{\vartheta+}(y)\big)\Delta_{x_i}^+ v(y) - \big(b_i^{\vartheta-}(x) - b_i^{\vartheta-}(y)\big)\Delta_{x_i}^- v(y) \Big] \\ & \le 2\sqrt{N} h^2 |b^\vartheta(x) - b^\vartheta(y)| [v]_1. \end{aligned}$$

By the above expressions, and by exchanging the roles of x and y, we obtain the following estimate

$$|T_h v(x) - T_h v(y)| \le \frac{1}{1+\lambda h^2} \Big[(1+\bar\lambda_0 h^2)[v]_1 + h^2 \sup_\vartheta [f^\vartheta]_1 \Big] |x-y|. \tag{4.5}$$

By assumption $\lambda > \bar\lambda_0$, if $[v]_1 \le M/(\lambda - \bar\lambda_0)$ with M defined in (A1), then $[T_h v]_1$ satisfies the same inequality. In particular, for any $n \in \mathbb{N}$, $[T_h^n 0]_1 \le M/(\lambda - \bar\lambda_0)$ and since, by the contraction mapping theorem, the sequence $(T_h^n 0)_n$ converges uniformly to u_h, this means that $[u_h]_1 \le M/(\lambda - \bar\lambda_0)$, and the proposition is proved in the case $c^\vartheta(x) \equiv \lambda$.

In the case of non-constant $c^\vartheta(x)$ we would obtain an expression like (4.5) with $\sup_\vartheta [f^\vartheta]_1$ replaced by $\sup_\vartheta([f^\vartheta]_1 + [c^\vartheta]_1(|v|_0 + h^2|f^\vartheta|_0))$, hence the lemma would hold again. ∎

Now let us consider the scheme (1.4). In the expressions defining p^ϑ, replace $b_i^\pm(x)$ by $b_i^\pm(x+e)$ and call the resulting functions for $p^{\vartheta,e}$. Then it is clear that (1.4) is equivalent with the following "dynamic programming principle"

(4.6)
$$u_h^\varepsilon(x) = \inf_{\substack{\vartheta\in\Theta\\ |e|\le\varepsilon}} \left\{ \frac{1}{1+h^2c^\vartheta(x+e)} \left(\sum_{z\in h\mathbb{Z}^N} p^{\vartheta,e}(x,x+z)u_h^\varepsilon(x+z) + h^2 f^\vartheta(x+e) \right) \right\}.$$

Now by arguing as in the proof of Proposition 4.2, we obtain the following proposition.

Proposition 4.3. *Assume that (A1) with $\delta = 1$ and (A2) hold. Then for any $\varepsilon \ge 0$ there exists a unique solution $u_h^\varepsilon \in C_b(\mathbb{R}^N)$ of the scheme* (4.6)*. Moreover if $\lambda > \bar\lambda_0$ (defined in Proposition 4.2), then $|u_h^\varepsilon|_1 \le C$.*

Using the same technique as in the proof of Proposition 4.2, we now prove that $|u_h - u_h^\varepsilon|_0 \le C\varepsilon$.

Proposition 4.4. *Assume that (A1) with $\delta = 1$ and (A2) hold and that $\lambda > \bar\lambda_0$ (defined in Proposition 4.2), then $|u_h - u_h^\varepsilon|_0 \le C\varepsilon$.*

Proof. We only give the proof in the case where $c^\vartheta(x) \equiv \lambda$.

As in the proof of Theorem 3.4, we first notice that, because of the very definition of the scheme (4.6), u_h^ε is a subsolution for the S–scheme and Lemma 2.3 implies that $u_h^\varepsilon \le u_h$ in $\mathbb{R}^N$. Hence, again, we only need to have an upper estimate of $u_h - u_h^\varepsilon$.

Let T_h^ε be the operator for (4.6) corresponding to T_h. After similar manipulations as in the previous proofs we obtain the following inequality

$$\begin{aligned} T_h u_h(x) - T_h^\varepsilon u_h^\varepsilon(x) \le \frac{1}{1+\lambda h^2} \sup_{\vartheta,e} \Big\{ \sum_{z\in h\mathbb{Z}^N} & \Big[p^\vartheta(x,x+z)(u_h(x+z) - u_h^\varepsilon(x+z)) \\ & + u_h^\varepsilon(x+z)(p^\vartheta(x,x+z) - p^{\vartheta,e}(x,x+z)) \Big] \\ & + h^2(f^\vartheta(x) - f^\vartheta(x+e)) \Big\}. \end{aligned}$$

Since the p^ϑ's are positive and sum up to 1, the first sum is bounded by $|u_h - u_h^\varepsilon|_0$. The second sum is equivalent to the following expression

$$h\sum_{i=1}^N \Big[\big(b_i^{\vartheta+}(x) - b_i^{\vartheta+}(x+e)\big)\Delta_{x_i}^+ u_h^\varepsilon(x) - \big(b_i^{\vartheta-}(x) - b_i^{\vartheta-}(x+e)\big)\Delta_{x_i}^- u_h^\varepsilon(x) \Big].$$

By Proposition 4.3, $|u_h^\varepsilon|_1$ is bounded independent of h and ε. Combining this fact with (A1), we see that the above expression can be bounded by $Ch^2\varepsilon$. All in all we have obtained

$$T_h u_h(x) - T_h^\varepsilon u_h^\varepsilon(x) \le \frac{1}{1+\lambda h^2}\Big[|u_h - u_h^\varepsilon|_0 + C\varepsilon h^2 \Big].$$

We can now conclude the proof using the fact that $T_h u_h = u_h$ and $T_h^\varepsilon u_h^\varepsilon = u_h^\varepsilon$. □

From Definition 2.1 of λ_0, we see that $\bar\lambda_0 > \lambda_0$. Therefore when (A1) and (A2) hold with $\delta = 1$ and $\lambda > \bar\lambda_0$ (defined in Proposition 4.2), by Theorem 2.2, we have $\bar\delta = 1$. Under the same conditions, Propositions 4.3 and 4.4 yield that Assumption 2.4 is satisfied with $\tilde\delta = 1$. Therefore we can conclude from Proposition 4.1 and Theorem 2.5 that the following result holds

Theorem 4.5. *Assume that (A1) with $\delta = 1$ and (A2) hold, that, for any ϑ, a^ϑ is independent of x, and that $\lambda > \bar{\lambda}_0$ (defined in Proposition 4.2). If u and u_h are solutions of* (1.1) *and* (4.3) *respectively, then*

$$|u - u_h|_0 \le C h^{1/3}.$$

Remark 4.6. It is worth noticing that, in this case, we obtain the same exponent in the upper and lower bounds on $u - u_h$. This, and the value $1/3$, is in agreement with Krylov's paper on constant coefficients [11]. In his paper on variable coefficient parabolic equations (including x-dependence in a^ϑ), he gets different exponents for the upper and lower bound on $u - u_h$, the one being $1/3$ and the other being $1/27$.

Remark 4.7. In order to have $u \in \mathcal{C}^{0,1}(\mathbb{R}^N)$, by Theorem 2.2 we need $\lambda > \lambda_0$. But to handle the scheme, we needed the stronger condition $\lambda > \bar{\lambda}_0$. From their definitions we see that $\bar{\lambda}_0 \ge 2\sqrt{N}\lambda_0$.

Next, we consider first-order equations ($a^\vartheta \equiv 0$ for any ϑ). Condition (C4) then takes the following form $|F(x, v, Dv) - S(h, x, v(x), [v]_x^h)| \le \bar{K}|v|_{1,1}h$ for $v \in \mathcal{C}^{1,1}(\mathbb{R}^N)$. It is now clear that Theorem 2.5 yields the following result.

Theorem 4.8. *Assume that (A1) with $\delta = 1$ and (A2) hold, that $\sigma^\vartheta \equiv 0$ for any ϑ, and that $\lambda > \bar{\lambda}_0$ (defined in Proposition 4.2). If u and u_h are solutions of* (1.1) *and* (4.3) *respectively, then*

$$|u - u_h|_0 \le C h^{1/2}.$$

This is the expected rate. The same rate was obtained in e.g. [7, 20] for time-dependent problems.

Remark 4.9. It is possible to handle certain type of equations and schemes in the case of non-constant a^ϑ provided they are equivalent to equations and schemes with constant a^ϑ. Here is a typical example we have in mind.

Let $k^\vartheta : \mathbb{R}^N \to \mathbb{R}$ be functions such that $|k^\vartheta|_1 \le M$ (independent of ϑ) and, for each ϑ, either k^ϑ is a nonnegative constant or k^ϑ satisfies $k^\vartheta(x) \ge k > 0$ in $\mathbb{R}^N$. Furthermore assume that a^ϑ is constant for any ϑ and that (A1) and (A2) hold. We consider the following equation

$$\sup_{\vartheta \in \Theta} \Big\{ -\frac{1}{2} k^\vartheta(x)\, \mathrm{tr}[a^\vartheta D^2 u] - b^\vartheta(x) Du + c^\vartheta(x) u - f^\vartheta(x) \Big\} = 0. \tag{4.7}$$

Since for every ϑ where k^ϑ is non-constant, $0 < k \le k^\vartheta \le M$, we may divide *inside* the supremum by $K^\vartheta(x)$, where $K^\vartheta(x)$ is equal to 1 for every ϑ where k^ϑ is constant, and otherwise equal to k^ϑ. We then obtain new coefficients which still satisfies (A1) and (A2), but with new constants $\min(\lambda/k, \lambda)$ and $\max(M/k, M)$. The new coefficients in front of the second-order terms are now constants.

More important, since the problem comes mainly from the scheme, we can do the same for the scheme (4.3) corresponding to (4.7), that is the solution u_h to (4.3) is the solution of an other finite difference scheme which can be handled directly by Theorem 4.5.

A simple special case of (4.7) is the following 1-dimensional problem

$$\max \Big\{ - a(x)u'' - b(x)u' + c(x)u - f(x), -\bar{b}(x)u' + \bar{c}(x)u - \bar{f}(x) \Big\} = 0,$$

where $a(x) \ge k > 0$ and (A1) and (A2) hold.

Appendix A. Results needed in the proof of bound (i) in Theorem 2.5.

In this section we will prove Lemmas 2.6 and 2.7 which were stated in the proof of bound (i) in Theorem 2.5. In order to prove Lemma 2.6, we use the following continuous dependence result.

Theorem A.1. *For $\mu \in (0,1]$, let $u, v \in \mathcal{C}^{0,\mu}(\mathbb{R}^N)$ be solutions of (1.1) with coefficients $\{a^\vartheta,\ b^\vartheta,\ c^\vartheta,\ f^\vartheta\}$ and $\{\bar{a}^\vartheta,\ \bar{b}^\vartheta,\ \bar{c}^\vartheta,\ \bar{f}^\vartheta\}$ respectively. Moreover assume that (A1) and (A2) hold for both sets of coefficients with constants $M, \bar{M}$ and $\lambda = \bar{\lambda}$. If $\mu \leq \delta$, then there is a constant $\bar{C}$ depending only on M, $\bar{M}$, λ, μ, and δ such that*

$$\lambda|u - v|_0 \leq \bar{C} \sup_{\vartheta\in\Theta} \left\{|\sigma^\vartheta - \bar{\sigma}^\vartheta|_0^\mu + |b^\vartheta - \bar{b}^\vartheta|_0^\mu\right\} + \sup_{\vartheta\in\Theta}\left\{|u|_0 \wedge |v|_0|c^\vartheta - \bar{c}^\vartheta|_0 + |f^\vartheta - \bar{f}^\vartheta|_0\right\}.$$

Here $a \wedge b = \max(a, b)$. Before giving the proof, we prove the following classical lemma.

Lemma A.2. *Let f be a bounded upper-semicontinuous function in $\mathbb{R}^N$ and define $m, m_\varepsilon \geq 0$ and $x_\varepsilon \in \mathbb{R}^n$ as follows : $m_\varepsilon = \max_{x\in\mathbb{R}^n}\{f(x) - \varepsilon|x|^2\} = f(x_\varepsilon) - \varepsilon|x_\varepsilon|^2$ and $m = \sup_{x\in\mathbb{R}^n} f(x)$. Then as $\varepsilon \to 0$, $m_\varepsilon \to m$ and $\varepsilon|x_\varepsilon|^2 \to 0$.*

Proof. Take an arbitrary $\eta > 0$. By the definition of the supremum, there exists $x' \in \mathbb{R}^N$ such that $f(x') \geq m - \eta$. If ε is small enough in order to have $\varepsilon|x'|^2 < \eta$, then the first part follows since

$$m \geq m_\varepsilon = f(x_\varepsilon) - \varepsilon|x_\varepsilon|^2 \geq f(x') - \varepsilon|x'|^2 \geq m - 2\eta.$$

Now define $k_\varepsilon = \varepsilon|x_\varepsilon|^2$. This quantity is bounded by the above calculations since f is bounded. We consider a converging subsequence $\{k_{\varepsilon'}\}_{\varepsilon'}$ and call the limit k (which is non-negative by definition). We remark that $f(x_{\varepsilon'}) - k_{\varepsilon'} \leq m - k_{\varepsilon'}$ and passing to the limit yields $m \leq m - k$. This means that $k \leq 0$, that is $k = 0$. Now we are done since if every subsequence converges to 0, the sequence converges to 0 as well. □

Proof of Theorem A.1. Define $m := \sup_{\mathbb{R}^N}(u - v)$, $\phi(x, y) := \alpha|x - y|^2 + \varepsilon(|x|^2 + |y|^2)$, and $\psi(x, y) := u(x) - v(y) - \phi(x, y)$ in $\mathbb{R}^N \times \mathbb{R}^N$. Then we set $m_{\alpha,\varepsilon} := \sup_{x,y\in\mathbb{R}^N} \psi(x, y)$. By classical arguments, there exists $x_0, y_0 \in \mathbb{R}^N$ such that $m_{\alpha,\varepsilon} = \psi(x_0, y_0)$. Here and below we drop any dependence in α and ε when there is no possible ambiguity.

By the maximum principle for semicontinuous functions, Theorem 3.2 in [6], there are $X, Y \in \mathcal{S}^N$ such that $(D_x\phi(x_0, y_0), X) \in \bar{\mathcal{J}}^{2,+}u(x_0)$ and $(-D_y\phi(x_0, y_0), Y) \in \bar{\mathcal{J}}^{2,-}v(y_0)$. Moreover, the following inequality holds for some constant $k > 0$

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq k\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + k\varepsilon \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{A.1}$$

Subtracting the viscosity solutions' inequalities we obtain after using the definitions of viscosity sub- and supersolutions, and using the inequality $\sup(\cdots) - \sup(\cdots) \leq \sup(\cdots - \cdots)$

$$\begin{aligned} 0 \leq \sup_{\vartheta\in\Theta} \Big\{ &- \frac{1}{2}\,\mathrm{tr}[\bar{a}^\vartheta(y_0)Y - a^\vartheta(x_0)X] \\ &- \bar{b}^\vartheta(y_0)\big(2\alpha(x_0 - y_0) - 2\varepsilon y_0\big) + b^\vartheta(x_0)\big(2\alpha(x_0 - y_0) + 2\varepsilon x_0\big) \\ &+ \bar{c}^\vartheta(y_0)v(y_0) - c^\vartheta(x_0)u(x_0) - \bar{f}^\vartheta(y_0) + f^\vartheta(x_0)\Big\}. \end{aligned} \tag{A.2}$$

By the computations given in Ishii and Lions [9, p. 35] and (A.1), and the inequality $(s + t)^2 \leq 2(s^2 + t^2)$ for $s, t \in \mathbb{R}$, we get

$$\begin{aligned} -\,\mathrm{tr}[\bar{a}^\vartheta(y_0)Y - a^\vartheta(x_0)X] \leq\ & 2k\alpha\big\{|\bar{\sigma}^\vartheta(y_0) - \sigma^\vartheta(y_0)|^2 + |\sigma^\vartheta(y_0) - \sigma^\vartheta(x_0)|^2\big\} \\ &+ k\varepsilon\big\{|\sigma^\vartheta(x_0)|^2 + |\bar{\sigma}^\vartheta(y_0)|^2\big\}. \end{aligned}$$

Furthermore the following estimates hold

$$\begin{aligned}
&-\big(\bar b^\vartheta(y_0)-b^\vartheta(x_0)\big)(x_0-y_0)\\
&\quad\le 2|\bar b^\vartheta(y_0)-b^\vartheta(y_0)|^2+2|x_0-y_0|^2+|b^\vartheta(y_0)-b^\vartheta(x_0)||x_0-y_0|,\\
&\bar c^\vartheta(y_0)v(y_0)-c^\vartheta(x_0)u(x_0)\\
&\quad\le |v(y_0)||\bar c^\vartheta(y_0)-c^\vartheta(y_0)|+|u(x_0)||c^\vartheta(y_0)-c^\vartheta(x_0)|-\lambda m_{\alpha,\varepsilon}.
\end{aligned}$$

In the second estimate we used that $u(x_0)=v(y_0)+\phi(x_0,y_0)+m_{\alpha,\varepsilon}\ge v(y_0)+m_{\alpha,\varepsilon}$ and (A2). Inserting all these estimates into (A.2) and using (A1) yield

(A.3)
$$\begin{aligned}
\lambda m_{\alpha,\varepsilon}\le{}& 2k\alpha\sup_{\vartheta\in\Theta}\big\{|\sigma^\vartheta-\bar\sigma^\vartheta|_0^2+|b^\vartheta-\bar b^\vartheta|_0^2\big\}+\sup_{\vartheta\in\Theta}\big\{|v|_0|c^\vartheta-\bar c^\vartheta|_0+|f^\vartheta-\bar f^\vartheta|_0\big\}\\
&+k_1\alpha|x_0-y_0|^2+k_2|x_0-y_0|^\delta+\varepsilon\, C(1+|x_0|^2+|y_0|^2)
\end{aligned}$$

where $k_1=\sup_{\vartheta\in\Theta}\{k[\sigma^\vartheta]_1^2+4+2[b^\vartheta]_1\}$ and $k_2=\sup_{\vartheta\in\Theta}\{|u|_0[c^\vartheta]_\delta+[f^\vartheta]_\delta\}$.

From the inequality $2\psi(x_0,y_0)\ge\psi(x_0,x_0)+\psi(y_0,y_0)$ and Hölder regularity of u and v, we see that

$$2\alpha|x_0-y_0|^2\le[u]_\mu|x_0-y_0|^\mu+[v]_\mu|x_0-y_0|^\mu.$$

And we can conclude that $|x_0-y_0|\le C\alpha^{-1/(2-\mu)}$, which again implies that

$$\text{(A.4)}\qquad \alpha|x_0-y_0|^2\le C\alpha^{-\frac{\mu}{2-\mu}}\qquad\text{and}\qquad |x_0-y_0|^\delta\le C\alpha^{-\frac{\delta}{2-\mu}}.$$

Furthermore for fixed α, Lemma A.2 yield $\lim_{\varepsilon\to0}\varepsilon(|x_0|^2+|y_0|^2)=0$ and $\lim_{\varepsilon\to0}m_{\alpha,\varepsilon}\ge m$. Hence if we insert (A.4) into (A.3) and pass to the limit $\varepsilon\to0$ for α fixed, we get

$$\text{(A.5)}\qquad\begin{aligned}
\lambda m\le{}&2k\alpha\sup_{\vartheta\in\Theta}\big\{|\sigma^\vartheta-\bar\sigma^\vartheta|_0^2+2|b^\vartheta-\bar b^\vartheta|_0^2\big\}+C(\alpha^{-\frac{\mu}{2-\mu}}+\alpha^{-\frac{\delta}{2-\mu}})\\
&+\sup_{\vartheta\in\Theta}\big\{|v|_0|c^\vartheta-\bar c^\vartheta|_0+|f^\vartheta-\bar f^\vartheta|_0\big\}.
\end{aligned}$$

Let $k_1,k_2>0$ and note that by optimization with respect to α, we obtain

(A.6)
$$k_1\alpha+k_2\alpha^{-\frac{\mu}{2-\mu}}\le\bar c(\mu,\mu)k_1^{\frac{\mu}{2}}k_2^{\frac{2-\mu}{2}}\quad\text{and}\quad k_1\alpha+k_2\alpha^{-\frac{\delta}{2-\mu}}\le\bar c(\mu,\delta)k_1^{\frac{\delta}{2-\mu+\delta}}k_2^{\frac{2-\mu}{2-\mu+\delta}},$$

where $\bar c(s,t)$ is positive and finite for $0\le s\le t\le1$. We note that for $0\le\mu\le\delta\le1$, $\frac{\mu}{2}\le\frac{\delta}{2-\mu+\delta}$. Therefore, assuming $k_1\le1$ we get $k_1^{\delta/(2-\mu+\delta)}\le k_1^{\mu/2}$. Now let $k_1=2k\sup_{\vartheta\in\Theta}\{|\sigma^\vartheta-\bar\sigma^\vartheta|_0^2+2|b^\vartheta-\bar b^\vartheta|_0^2\}/C$, where by boundedness of the coefficients, the constant $C>0$ is chosen so big that $k_1\le1$. Combining (A.5) and (A.6) then yield

$$\lambda m\le C\sup_{\vartheta\in\Theta}\big\{|\sigma^\vartheta-\bar\sigma^\vartheta|_0^2+|b^\vartheta-\bar b^\vartheta|_0^2\big\}^{\frac{\mu}{2}}+\sup_{\vartheta\in\Theta}\big\{|v|_0|c^\vartheta-\bar c^\vartheta|_0+|f^\vartheta-\bar f^\vartheta|_0\big\}.$$

Note that we could have arrived at the above inequality interchanging $|v|_0$ by $|u|_0$. Finally we can conclude since $(s^2+t^2)^{\mu/2}\le|t|^\mu+|s|^\mu$ for any $s,t\in\mathbb{R}$, and since the argument is symmetric in u and v. □

For a more detailed proof of a similar result, see [10]. Now we give the

Proof of Lemma 2.6. Equation (1.3) can be considered as a special case of equation (1.1) by replacing the control parameter ϑ by $\vartheta'=(\vartheta,e)$. Now the corresponding conditions (A1) and (A2) hold with the same constants as in the unperturbed problem. So existence, uniqueness and regularity follows from Theorems 2.1 and 2.2. The second part is a direct consequence of Theorem A.1 and (A1). □

Finally we prove Lemma 2.7. The proof relies on the following lemma.

Lemma A.3. *Assume that (A1) and (A2) hold and that, for $u^1, \cdots, u^n \in C_b(\mathbb{R}^N)$ are viscosity subsolutions of* (1.1). *If $\lambda_1, \cdots, \lambda_n$ are positive numbers such that $\sum_{i=1}^n \lambda_i = 1$, then $\sum_{i=1}^n \lambda_i u^i$ is still a viscosity subsolution of* (1.1).

Proof. We first show the result in the linear case and when $n = 2$. This means that all coefficients in (1.1) are independent of ϑ.

We consider a function $\chi \in C^2(\mathbb{R}^N)$ and assume that $\lambda_1 u^1 + \lambda_2 u^2 - \chi$ has a strict local maximum at some point $\bar{x} \in \mathbb{R}^N$, let's say in $\overline{B}$ where B is a ball centered at $\bar{x}$.

We introduce $\psi(x,y) := \lambda_1 u^1(x) + \lambda_2 u^2(y) - \lambda_1\chi(x) - \lambda_2\chi(y) - \phi(x,y)$ where $\phi(x,y) = \alpha|x-y|^2$, and let $m_\alpha = \sup_{x,y\in\overline{B}} \psi(x,y)$. Since $\overline{B}$ is compact, this supremum is attained at some point $(x_\alpha, y_\alpha) \in \overline{B}\times\overline{B}$ and, by classical arguments using mainly that $\bar{x}$ is a strict maximum point of $\lambda_1 u^1 + \lambda_2 u^2 - \chi$ in $\overline{B}$, it is easy to show that $x_\alpha, y_\alpha \to \bar{x}$ and $\alpha|x_\alpha - y_\alpha|^2 \to 0$ (see Lemma 3.1 in [6]). In particular, $x_\alpha, y_\alpha \in B$ for α large enough and from now on we assume that we are in this case.

By the maximum principle for semi-continuous functions (Theorem 3.2 in [6]), we get the existence of $X, Y \in \mathcal{S}^N$ such that $(D_x\phi(x_\alpha,y_\alpha) + \lambda_1 D\chi(x_\alpha), X) \in \bar{\mathcal{J}}^{2,+}\lambda_1 u^1(x_\alpha)$ and $(D_y\phi(x_\alpha,y_\alpha) + \lambda_2 D\chi(y_\alpha), Y) \in \bar{\mathcal{J}}^{2,+}\lambda_2 u^2(y_\alpha)$. Moreover the following inequality holds for some constant $k > 0$:

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq k\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \begin{pmatrix} \lambda_1 D^2\chi(x_\alpha) & 0 \\ 0 & \lambda_2 D^2\chi(y_\alpha) \end{pmatrix}. \tag{A.7}$$

Now using the definition of viscosity subsolutions for both u^1 and u^2 and adding the obtained inequalities yield

$$\begin{aligned} 0 \geq &-\frac{1}{2}\operatorname{tr}[a(x_\alpha)X + a(y_\alpha)Y] \\ &- b(x_\alpha)(D\phi_x(x_\alpha) + \lambda_1 D\chi(x_\alpha)) - b(y_\alpha)(D\phi_y(y_\alpha) + \lambda_2 D\chi(y_\alpha)) \\ &+ c(x_\alpha)\lambda_1 u^1(x_\alpha) + c(y_\alpha)\lambda_2 u^2(y_\alpha) - \lambda_1 f(x_\alpha) - \lambda_2 f(y_\alpha). \end{aligned} \tag{A.8}$$

By the argument of Ishii and Lions [9, p. 35] and (A.7) we are lead to

$$\begin{aligned} \operatorname{tr}[a(x_\alpha)X + a(y_\alpha)Y] \leq\ & \operatorname{tr}[\lambda_1 a(x_\alpha)D^2\chi(x_\alpha) + \lambda_2 a(y_\alpha)D^2\chi(y_\alpha)] \\ &+ k\alpha\big|\sigma(x_\alpha) - \sigma(y_\alpha)\big|^2. \end{aligned} \tag{A.9}$$

By (A.9) and the Lipschitz continuity of σ and b, we can rewrite the (A.8) in the following way

$$\begin{aligned} &-\frac{1}{2}\operatorname{tr}[\lambda_1 a(x_\alpha)D^2\chi(x_\alpha) + \lambda_2 a(y_\alpha)D^2\chi(y_\alpha)] \\ &- \lambda_1 b(x_\alpha)D\chi(x_\alpha) - \lambda_2 b(y_\alpha)D\chi(y_\alpha) \\ &+ c(x_\alpha)\lambda_1 u^1(x_\alpha) + c(y_\alpha)\lambda_2 u^2(y_\alpha) - \lambda_1 f(x_\alpha) - \lambda_2 f(y_\alpha) \\ &\quad \leq C\alpha|x_\alpha - y_\alpha|^2. \end{aligned} \tag{A.10}$$

We let α tend to ∞ in this inequality, using the properties of x_α and y_α together with the continuity of u^1, u^2, χ and the coefficients. We obtain the following

$$-\frac{1}{2}\operatorname{tr}[a(\bar{x})D^2\chi(\bar{x})] - b(\bar{x})D\chi(\bar{x}) + c(\bar{x})(\lambda_1 u^1(\bar{x}) + \lambda_2 u^2(\bar{x})) - f(\bar{x}) \leq 0.$$

This completes the proof in the linear case.

To treat the case where the coefficients depend on ϑ, we just notice that (1.1) is equivalent to

$$-\frac{1}{2}\operatorname{tr}[a^\vartheta(x)D^2u(x)] - b^\vartheta(x)Du(x) + c^\vartheta(x)u(x) - f^\vartheta(x) \leq 0 \quad \text{in } \mathbb{R}^N, \tag{A.11}$$

for all $\vartheta \in \Theta$. We can therefore argue by fixing ϑ : $\lambda_1 u^1 + \lambda_2 u^2$ is a subsolution of (A.11) by the linear case. Now this holds for all $\vartheta \in \Theta$, so $\lambda_1 u^1 + \lambda_2 u^2$ must be a subsolution of (1.1).

Finally, the general result follows by induction. To convince ourselves of this, we consider the case $n = 3$. Consider the following convex combination of 3 subsolutions of (1.1):

$$\begin{aligned} &\lambda_1 u^1 + \lambda_2 u^2 + (1 - \lambda_1 - \lambda_2) u^3 \\ &= (\lambda_1 + \lambda_2) \left(\frac{\lambda_1}{\lambda_1 + \lambda_2} u^1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} u^2 \right) + (1 - \lambda_1 - \lambda_2) u^3. \end{aligned} \tag{A.12}$$

Let w denote what is inside the big parenthesis. Note that w is a convex combination of two subsolutions of (1.1). So by the result for the case $n = 2$, w is a viscosity subsolution of (1.1). This means that (A.12) is in fact a convex combination of *two* subsolutions w and u^3, so we can conclude using once more the results for the case $n = 2$. This completes the proof of Lemma A.3. □

We can now complete the

Proof of Lemma 2.7. Let $Q_h^e := e + [-h/2, h/2)^N$, $\bar{\rho}_\varepsilon(e,h) = \int_{Q_h^e} \rho_\varepsilon(y) dy$ and $I_h(x) := \sum_{e \in h\mathbb{Z}^N} u^\varepsilon(x - e) \bar{\rho}_\varepsilon(e,h)$. By a classical result, the function I_h, obtained through a discretization of the convolution integral, converges uniformly to u^ε. On the other hand, I_h is a convex combination of subsolutions of (1.1) and therefore, by Lemma A.3, I_h is itself a viscosity subsolution of (1.1).

We can conclude that u_ε is a viscosity subsolution of (1.1) using the stability result for viscosity solutions of second-order PDEs (Lemma 6.1 in [6]). □

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