FRACTIONAL RATE OF CONVERGENCE FOR VISCOUS APPROXIMATION TO NONCONVEX CONSERVATION LAWS

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Abstract. The paper studies the viscous approximations to conservation laws with non-convex flux function. It is shown that when the entropy solutions are piecewise smooth the rate of L^1-convergence is a *fractional* number α satisfying $1/2 < \alpha \leq 1$. Numerical computations indicate that the theoretical estimate for the convergence order is optimal. This is in contrast to the corresponding result for the convex conservation laws.

Keywords: L^1-convergent rate, error estimate, viscous approximation, conservation law

AMS Subject Classification: 35L65, 49M25, 34A65

1. Introduction

In this paper we consider the initial value problem for *non-convex* conservation laws

$$u_t + f(u)_x = 0, \qquad t > 0\,, \quad x \in \mathbf{R}, \tag{1.1}$$

which is subject to the initial condition prescribed at $t = 0$,

$$u(x, 0) = u_0(x), \tag{1.2}$$

where $f \in C^2$. We shall investigate viscous approximations to the entropy solution of (1.1):

$$\partial_t u^\epsilon + \partial_x f(u^\epsilon) = \epsilon \partial_{xx} u^\epsilon \tag{1.3}$$

subject to the initial data

$$u^\epsilon(x, 0) = u_0(x). \tag{1.4}$$

In this work, we assume that $f''(u)$ vanishes at *a finite number* of points. We also assume that the entropy solution of (1.1) and (1.2) is *piecewise smooth*, with finitely many shock discontinuities. The existence and uniqueness of the solutions to (1.1) in the class of piecewise smooth weak solutions are studied by Ballou [1].

When f is convex, a detailed description of the structure for (1.1) and (1.2) has been obtained, see e.g. Lax [10] and Dafermos [2]. When $f(u)$ has inflection points, the situation is more complicated. Some deeper understanding for the structures and asymptotic behavior of solutions in this case has been obtained by, e.g., Dafermos [3], Liu [11] and Zumbrun [32]. However, we

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are still far from having a complete description of the general case. This is because the geometric structure of the solution when f changes convexity is much more complicated due to the presence of contact discontinuities, and there is a larger variety of asymptotic states.

The asymptotic convergence of solutions to the viscous problem (1.3) and (1.4) to the corresponding discontinuous solutions of the inviscid problem (1.1) and (1.2) has been the main driving force for the mathematical theory of shock waves from both theoretical and numerical point of view. Substantial progress has been made in the past in this regard (see [29], [20] and the references therein), pioneered by Hopf, Lax [10], Oleinik [18] and Krushkov [7] etc.. For BV entropy solutions, Kuznetsov [8] was the first to establish the half-order rate of L^1-convergence for viscosity approximation and monotone schemes. It was proved by Tang and Teng that this half-order rate of convergence is optimal in BV solution class [25, 19]. However, for convex conservation laws with piecewise smooth solutions the L^1-convergence rate can be improved to first-order, see e.g., Teng and Zhang [27] for monotone scheme, Tang and Teng [24] for viscosity approximation and Teng [26] for relaxation method. The basic method in obtaining the first order rate of convergence is the matching asymptotic method developed by Goodman and Xin [5] and Liu and Xin [14]. One of key ingredients in this method is the nonlinear large asymptotic stability of viscous shock profiles. For systems of viscous conservation laws, this stability theory has been extensively studied in the past decade. Important progress has been made by Goodman [4], Matsumura and Nishihara [16], Liu [12], and Szepessy and Xin [21], see also some recent new approaches by Howard and Zumbrun [6], Liu [13], and Kreiss and Kreiss [9]. In particular, convergence with a rate to viscous shock profiles has been obtained by Liu [13] by using pointwise estimate of approximate Green's functions. Even in the case of non-convex fluxes, the nonlinear large time asymptotic stability has been established for some special systems, see [15] and [17]. The convergence of viscous solutions to piecewise smooth solutions for general systems has been established by Goodman and Xin [5], see also [31] for recent improvement. For the convergence of viscous solutions in the presence of physical boundaries, we refer to [30] and the references therein. We also point out that there have some first-order *pointwise* convergence results for viscous approximations to convex conservation laws, see e.g., Tadmor and Tang [22, 23] where they use the energy method with some bootstrap extrapolation technique. It is proved in [24] that for convex conservation laws whose entropy solution consists of finitely many discontinuities, the L^1-error between the viscosity solution, u^ϵ, and its inviscid limit, u, is bounded by $\mathcal{O}(\epsilon|\ln\epsilon|)$. If neither central rarefaction waves nor spontaneous shocks occur, the error bound is improved to $\mathcal{O}(\epsilon)$, see also [28]. In this work, we will show that for *nonconvex* conservation laws, the L^1-error between the viscosity solution and its inviscid limit is bounded by $\mathcal{O}(\epsilon^\alpha|\ln\epsilon|)$, where $\frac{1}{2} < \alpha \le 1$, even in the piecewise smooth solution class. The constant α is determined by the index numbers of shock curves to be defined in the next section. Based on the form of the flux function, the rate α can be any number between $\frac{1}{2}$ and 1. This indicates that for the viscous approximations the L^1-convergence rate of the nonconvex conservation laws is substantially different from that of the convex ones. We finish the introduction by outlining the rest of the paper . In the next section, we give some preliminaries, define index number for a shock discontinuity, and list some properties of the index number. In Section 3 we state our main convergence theorem, whose proof occupies Section 4 -Section 8. Finally, some numerical experiments, which indicates that the rate of convergence obtained in this paper is optimal, are presented in Section 9.

2. Piecewise smooth solution

In this paper we assume that the entropy solution of (1.1) and (1.2) is *piecewise smooth*, with finitely many shock discontinuities. More precisely, we can divide the given time interval $[0,T]$ into finite intervals $\{[t_{m-1},t_m]\}_{m=1}^M$ such that in each interval $[t_{m-1},t_m]$ the entropy solution is a finite combination of the cases plotted in Figures 1-3 (demonstrated in the case of three inflection points of $f(u)$). Thus, if we denote $S(t)$ the set of the discontinuous curve of $u(\cdot,t)$ in the time interval $[t_{m-1},t_m]$, then it consists of finitely many shocks:

$$S(t) := \{(x,t)\,|\,x = X_k(t),\ 1\le k\le K;\ t_{m-1}\le t\le t_m\}$$

where $X_k(t) < X_{k+1}(t)$ for $t\in(t_{m-1},t_m)$. It is understood that u is smooth with bounded limits $u(X_k(t)\pm 0,t)$ (denote by $u_k^{\pm}(t)$) and $u_x(X_k(t)\pm 0,t)$. For simplicity, we will not consider the new formed shock wave here; although this case has been investigated extensively in [24]. As a result we always have $u_k^+(t) - u_k^-(t) \neq 0$. For ease of notation we omit the dependence of $S(t)$, $X_k(t)$ and K on m. Each of the *non-contact* shock $X_k(t)$, plotted in Figure 1, satisfies the Rankine-Hugoniot and the Lax conditions

$$X_k'(t) = \sigma(u_k^+(t),u_k^-(t)) := \frac{f(u_k^+(t)) - f(u_k^-(t)}{u_k^+(t)-u_k^-(t)}, \tag{2.1}$$

$$a(u_k^-(t)) > X_k'(t) > a(u_k^+(t)); \quad \text{where} \quad a(v) := f'(v) \tag{2.2}$$

Each of the *contact shock* $X_k(t)$, plotted in Figures 2-3, satisfies the Rankine-Hugoniot and the contact conditions

$$X_k'(t) = \sigma(u_k^+(t),u_k^-(t)), \tag{2.3}$$

$$a(u_k^-(t)) > X_k'(t) = a(u_k^+(t)) \quad \text{or/and} \tag{2.4}$$

$$a(u_k^-(t)) = X_k'(t) > a(u_k^+(t)). \tag{2.5}$$

We now define *index numbers* $\beta_k^{\pm}$ for a shock curve $x = X_k(t)$. Since $X_k'(t) = a(u_k^+(t))$ (or/and $X_k'(t) = a(u_k^-(t))$) for a contact shock $x = X_k(t)$, there exists a positive number $\beta_k^+ > 0$ (or/and $\beta_k^- > 0$) such that

$$|\sigma(u_k^+,u_k^-) - a(u)| = |a(u_k^+) - a(u)| \sim |u_k^+ - u|^{\beta_k^+} \quad \text{as} \quad u\to u_k^+ + 0, \tag{2.6}$$

where $g(u)\sim h(u)$ means that there exists a constant $D>0$ such that $D^{-1}h \le g \le Dh$ as $u\to a$. The β_k^+ satisfying the above equation is called the index number for the contact shock. Similarly we can define index number β_k^- for a contact shock. For a non-contact shock $x = X_k(t)$, $\beta_k^{\pm}$ are defined as zero. Some calculation on (2.6) shows that

Theorem 1. *If $f(u)\in C^2(\mathbf{R})$, then*

$$\beta_k^+ = \begin{cases} 0 & \text{for } \sigma(u_k^+,u_k^-) \neq a(u_k^+), \\ \lim\limits_{u\to u_k^+} \dfrac{a'(u)(u-u_k^+)}{a(u)-a(u_k^+)} & \text{for } \sigma(u_k^+,u_k^-) = a(u_k^+). \end{cases} \tag{2.7}$$

The formulations for β_k^- can be obtained in a similar way.

In the remaining of this section, we will provide some discussions on the properties of the index numbers.

Lemma 1. *If $f''(u_k^+)\neq 0$, then $\beta_k^+ = 0$ or 1.*

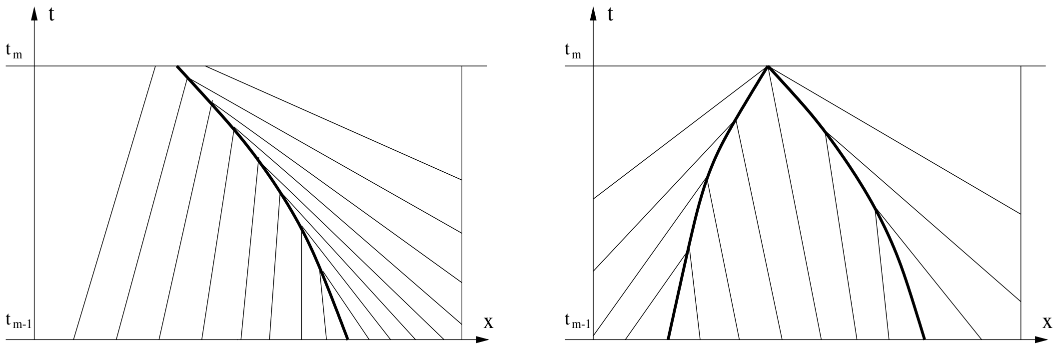

FIGURE 1. Illustration of non-contact shocks: Thin lines are characteristics and thick solid lines are *non-contact* shock curves, where characteristics come into shocks from both sides.

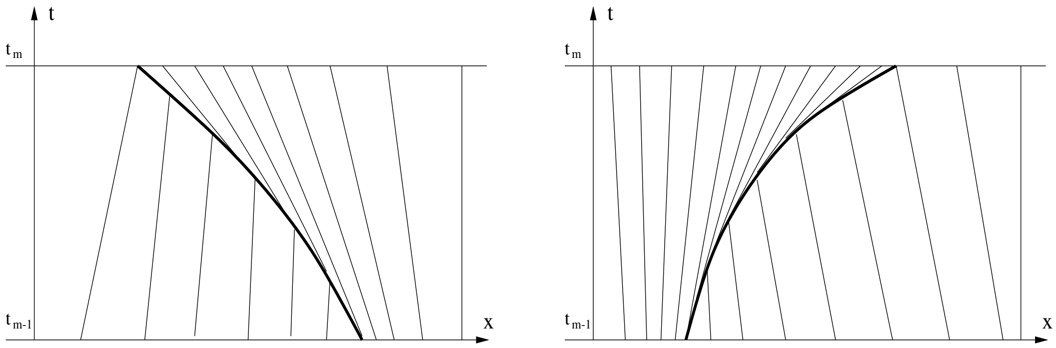

FIGURE 2. Illustration of contact shocks: Thin lines are characteristics and thick solid lines are *contact* shock curves, where characteristics are tangent to shocks at least in one side.

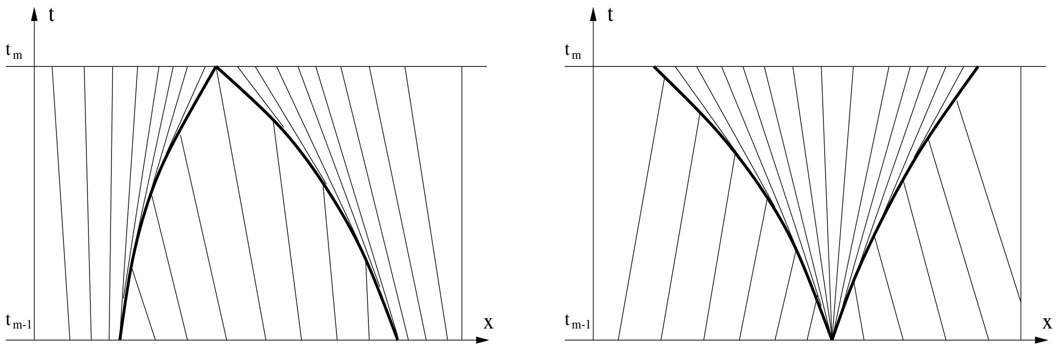

FIGURE 3. Illustration of contact shocks: same as Fig. 2.

Proof. If $\beta_k^+ \neq 0$, then it satisfies the second equation of (2.7). This shows that

$$\beta_k^+ = \lim_{u \to u_k^+} \frac{a'(u)(u - u_k^+)}{a(u) - a(u_k^+)} = \lim_{u \to u_k^+} \frac{a'(u)}{a'(u_k^+ + \theta(u - u_k^+))} = 1,$$

where $a'(u_k^+) = f''(u_k^+) \neq 0$. $\square$

The following result is an immediate consequence of the above lemma.

Corollary 2.1. *If $f''(u)$ has only a finite number of zero points, then the index number can only take a finite number of values.*

The following result gives a rule for calculating the index number.

Lemma 2. *If $f(u) \in C^r(\mathbf{R})$, $\sigma(u_k^+, u_k^-) = a(u_k^+)$, the derivative of $a(u)$ is zero at $u = u_k^+$ up to $(r-1)$-th order but $a^{(r)}(u_k^+) \neq 0$, then $\beta_k^+ = r$.*

Proof. Using the L'Hospital's rule on the second equation of (2.7) gives

$$\begin{aligned} \beta_k^+ &= \lim_{u \to u_k^+} \frac{a'(u)(u - u_k^+)}{a(u) - a(u_k^+)} = 1 + \lim_{u \to u_k^+} \frac{a''(u)(u - u_k^+)}{a'(u)} \\ &= \cdots = r + \lim_{u \to u_k^+} \frac{a^{(r+1)}(u)(u - u_k^+)}{a^{(r)}(u)} = r, \end{aligned} \tag{2.8}$$

where we have used the fact $a^{(r)}(u_k^+) \neq 0$. $\square$

Corollary 2.2. *If $\beta_k^+ > 1$ and $a(u)$ is of the form*

$$a(u) = \prod_{s=1}^{l} (u - u_s)^{\alpha_s} R(u),$$

where $R(u)$ has no real roots, $\{u_s\}_{s=1}^l$ are all of the distinct real roots for $a(u)$ and $\{\alpha_s\}_{s=1}^l$ are their respective multiples, then there exists an s such that $u_k^+ = u_s$ and $\beta_k^+ = \alpha_s$.

Example 2.1. *Let $f(u) = u^{2m+1}$. Assume the entropy solution is of the form*

$$u(x,t) = \begin{cases} u_- & \text{for} \quad x < \sigma(u_+, u_-)t \\ u_+ & \text{for} \quad x \geq \sigma(u_+, u_-)t, \end{cases} \tag{2.9}$$

where $u_- > 0$ is a given number, and $u_+ < 0$ is the solution of the equation $\sigma(u_+, u_-) = a(u_+)$. In other words, u_+ is determined by

$$\sum_{s=0}^{2m} (u_+)^{2m-s} (u_-)^s = (2m+1)(u_+)^{2m}.$$

It is easy to show that $a(u_-) > \sigma(u_+, u_-) = a(u_+)$, and hence $\beta^- = 0$. Since $a'(u_+) > 0$, it follows from Lemma 2 that $\beta^+ = 1$.

Example 2.2. *If $f(u) = (1-u)^p(1+u)^q$ with $p \geq 1$ and $q \geq 1$ and an entropy solution is given by*

$$u(x,t) = \begin{cases} -1, & x < 0 \\ +1, & x \geq 0, \end{cases} \tag{2.10}$$

then the curve $x = X(t) = 0$ is a contact shock with $\sigma(1,-1) = a(1) = a(-1)$ and the index numbers are $\beta^+ = p - 1$ and $\beta^- = q - 1$.

3. Main theorem

In this section, we first state the main result of this paper and its proof will be splited into several sections below.

Theorem 2. *Let $f \in C^2$ and assume that f changes its sign only at a finite number of points. Let u be the piecewise smooth entropy solution of (1.1)-(1.2) with finitely many shock discontinuities, and let u^ϵ be viscosity solutions of (1.3)-(1.4). Then the following error estimates hold for any $0 < t \le T$:*

$$\|u^\epsilon(\cdot,t) - u(\cdot,t)\|_{L^1(\mathbf{R})} \le \begin{cases} C(T)\epsilon|\ln\epsilon| & \text{for } \bar\beta < 1 \\ C(T)\epsilon^{(1+1/\bar\beta)/2}|\ln\epsilon| & \text{for } \bar\beta \ge 1 \end{cases} \tag{3.1}$$

where $\bar\beta = \max_{0\le t\le T}\beta(t)$, $\beta = \max\{\beta^+,\beta^-\}$, $\beta^\pm = \max_k\{\beta_k^\pm\}$ and $\beta_k^\pm$ are index numbers defined by (2.6).

The above theorem will be established by using a matched asymptotic analysis, a stability lemma and some detailed analysis for the traveling wave solution. The following stability lemma is only valid for the scalar conservation laws, and therefore makes the analysis much simpler than the system case. For the hyperbolic system, Goodman and Xin [5] constructed high-order approximations in obtaining a local first order rate of convergence for viscous approximations.

In the analysis of this work, we have to deal with the L^1 estimate of piecewisely continuous functions, some of which involve derivatives of some other piecewisely continuous functions, see e.g. (7.9). Therefore, special attention is paid to take care of these derivatives. In order to avoid confusion, we define the L^1-norm for a piecewisely smooth function q by

$$\|q(\cdot)\|_{\text{pis}(\mathbf{R})} = \sum_{i=1}^{I+1} \|q(\cdot)\|_{L^1(Y_{i-1},\,Y_i)}$$

where Y_i, with $Y_0 := -\infty$ and $Y_{I+1} := \infty$, are all possible discontinuous points of $q(x)$. The proof of the following stability lemma can be found in [24].

Lemma 3. *Let u^ϵ be the viscous solution of (1.3)-(1.4). Let $v^\epsilon \in C(\mathbf{R}\times[0,T])$ be a piecewisely smooth function having derivative jumps in the set $\mathcal{A} = \{(x,t)\,|\,x = Y_i(t),\ 1 \le i \le I\}$. If v^ϵ satisfies*

$$\partial_t v^\epsilon + \partial_x f(v^\epsilon) = \epsilon\partial_{xx}v^\epsilon + g(x,t) \tag{3.2}$$

everywhere except on the set $\mathcal{A}$, then for any $0 \le \tau < t \le T$

$$\|u^\epsilon(\cdot,t) - v^\epsilon(\cdot,t)\|_{L^1(\mathbf{R})} \le \|u^\epsilon(\cdot,\tau) - v^\epsilon(\cdot,\tau)\|_{L^1(\mathbf{R})}$$
$$+\epsilon\sum_{i=1}^{I}\int_\tau^t \Big|\,[\partial_x v^\epsilon(x,t)]\,|_{x=Y_i(t)}\Big|dt + \int_\tau^t \|g(\cdot,t)\|_{\text{pis}(\mathbf{R})}dt,$$

where the jumps are defined by

$$[w(x,t)]|_{x=Y(t)} := w(Y(t)+0,t) - w(Y(t)-0,t)$$

Remark 3.1. It will be seen in Sections 7 and 8 that $g(x,t)$ in (3.2) may involve some derivatives of a discontinuous function so we use the norm $\|\bullet\|_{\text{pis}(\mathbf{R})}$ to define its L^1-norm.

Using Theorem 2 we can obtain the following results, indicating that both the first-order and fractional-order of convergence rate can be true for non-convex conservation laws, which is in contrast with that for the convex conservation laws.

Example 3.1. *Let $f(u) = u^{2m+1}$. If the entropy solution $u(x,t)$ is defined by (2.9), then it follows from Example 2.1 that*

$$\|u^\epsilon(\cdot,t) - u(\cdot,t)\|_{L^1(\mathbf{R})} \le C(T)\epsilon|\ln\epsilon|. \tag{3.3}$$

Example 3.2. *Let $f(u) = (1-u)^p(1+u)^q$, with $s := \max(p,q) \geq 1$. If the entropy solution $u(x,t)$ is defined by (2.10), then it follows from Example 2.2 that*

$$\|u^\epsilon(\cdot,t) - u(\cdot,t)\|_{L^1(\mathbf{R})} \leq \begin{cases} C(T)\epsilon|\ln\epsilon| & \text{for} \quad 1 \leq s \leq 2 \\ C(T)\epsilon^{\frac{s}{2(s-1)}}|\ln\epsilon| & \text{for} \quad s > 2. \end{cases} \tag{3.4}$$

4. Traveling wave solution of viscous equation

Our construction of approximation solution is based on some detailed properties of viscous shock profiles for (1.3), whose nonlinear asymptotic stability has been studied by Matsumura and Nishihara [15]. We will summarize some of their results in this section, which will be used in our error analysis. Some results not obtained in [15] can be derived by using the techniques developed in [24]. Let

$$u^\epsilon(x,t) = V^\epsilon(x - \sigma t; u_+, u_-), \tag{4.1}$$

which is subject to the boundary conditions:

$$V^\epsilon(\xi; u_+, u_-) \to u_\pm \quad \text{as} \quad \xi \to \pm\infty.$$

If $V^\epsilon(x-\sigma t; u_+, u_-)$ satisfies (1.3), then it is called a *traveling wave solution* of (1.3). Substituting (4.1) into (1.3) gives

$$\epsilon V^\epsilon_{\xi\xi} = -\sigma V^\epsilon_\xi + f(V^\epsilon)_\xi. \tag{4.2}$$

Integrating the above equation over $(-\infty, \xi)$ gives

$$\epsilon V^\epsilon_\xi = -\sigma(V^\epsilon - u_-) + f(V^\epsilon) - f(u_-). \tag{4.3}$$

It is easy to show by re-scaling $\eta = \xi/\epsilon$ that $V^\epsilon(\xi; u_+, u_-) = V^1(\xi/\epsilon; u_+, u_-)$. In the following we will use the notation $V(\eta; u_+, u_-)$ to denote $V^1(\eta; u_+, u_-)$. We also denote by $V(\eta; a, b)_a$ and $V(\eta; a, b)_b$ the partial derivatives of V with respect to a and b, respectively. Note that $V^\epsilon(\xi; u_+, u_-) = V(\xi/\epsilon; u_+, u_-)$ where V satisfies

$$V' = -\sigma(V - u_-) + f(V) - f(u_-). \tag{4.4}$$

Then it is well known that a necessary and sufficient condition for the existence of traveling solution is that the constants $u_\pm$ and σ satisfy the Rankine-Hugoniot condition

$$-\sigma(u_+ - u_-) + f(u_+) - f(u_-) = 0 \tag{4.5}$$

and the entropy condition

$$\Phi(u; u_+, u_-) =: -\sigma(u - u_\pm) + f(u) - f(u_\pm) \begin{cases} < 0 & \text{if} \quad u_+ < u < u_- \\ > 0 & \text{if} \quad u_- < u < u_+. \end{cases} \tag{4.6}$$

Lemma 4. Let $u_\pm$ and σ satisfy (4.5)-(4.6) and

$$|\Phi(u; u_+, u_-)| \sim |u - u_\pm|^{1+\beta_\pm} \quad \text{as} \quad u \to u_\pm \tag{4.7}$$

with $\beta_\pm \geq 0$. Then there exists $V(\eta; u_+, u_-)$, unique up to a shift, which is determined by the ordinary differential equation (4.4). Moreover, for $k = 1, 2$:

(1) if $\beta_\pm = 0$ then $f'(u_+) < \sigma < f'(u_-)$ and for $\eta \to \pm\infty$

$$\begin{aligned} &|V(\eta; u_+, u_-) - H(\eta; u_+, u_-)| \sim \exp(-c|\eta|), \quad |V^{(k)}(\eta; u_+, u_-)| \sim \exp(-c|\eta|), \\ &|V(\eta; u_+, u_-)_{u_\pm} - 1| \sim \exp(-c|\eta|), \end{aligned} \tag{4.8}$$

where $H(\eta; u_+, u_-)$ is the Heaviside function satisfying $H = u_+$ for $\eta > 0$ and $H = u_-$ for $\eta < 0$;

(2) if $\beta_+ > 0$ then $\sigma = f'(u_+)$ and for $\eta \to +\infty$

$$\begin{aligned} &|V(\eta; u_+, u_-) - u_+| \sim |\eta|^{-1/\beta_+}, \quad |V^{(k)}(\eta; u_+, u_-)| \sim |\eta|^{-1/\beta_+ - k}, \\ &|V(\eta; u_+, u_-)_{u_+} - 1| \sim |\eta|^{-1/\beta_+}; \end{aligned} \tag{4.9}$$

(3) if $\beta_- > 0$ then $\sigma = f'(u_-)$ and for $\eta \to -\infty$

$$\begin{aligned} &|V(\eta; u_+, u_-) - u_-| \sim |\eta|^{-1/\beta_-}, \quad |V^{(k)}(\eta; u_+, u_-)| \sim |\eta|^{-1/\beta_- - k}, \\ &|V(\eta; u_+, u_-)_{u_-} - 1| \sim |\eta|^{-1/\beta_-}. \end{aligned} \tag{4.10}$$

Proof. For completeness, we indicate the proof of this lemma. It follows from (4.4) that $V(\eta; u_+, u_-)$ can be defined implicitly by

$$\eta = \int_{(u_+ + u_-)/2}^{V} \Phi(v; u_+, u_-)^{-1} dv. \tag{4.11}$$

All the proofs of this lemma are based on the above expression and the assumption of (4.7). Here we only show some of the estimate in (4.9). The rest of the estimates is similar. (4.7) means that there is a constant $D > 0$ such that

$$D^{-1}|v - u_-|^{1+\beta_-}|v - u_+|^{1+\beta_+} \le |\Phi(v; u_+, u_-)| \le D|v - u_-|^{1+\beta_-}|v - u_+|^{1+\beta_+}.$$

It follows from this and (4.11) that for $\eta > 0$, it holds:

$$\begin{aligned} D^{-1}2^{-(1+\beta_-)} &\left|\frac{2}{u_- - u_+}\right|^{(1+\beta_-)} \left|\int_{(u_+ + u_-)/2}^{V} (v - u_+)^{-1-\beta_+} dv\right| \\ &\le \eta \le D \left|\frac{2}{u_- - u_+}\right|^{(1+\beta_-)} \left|\int_{(u_+ + u_-)/2}^{V} (v - u_+)^{-1-\beta_+} dv\right|. \end{aligned}$$

Solving the above inequalities for V gives

$$\begin{aligned} &\left|\frac{u_+ - u_-}{2}\right| \left(1 + D2^{(1+\beta_-)} \left|\frac{u_+ - u_-}{2}\right|^{1+\beta_+ + \beta_-} \beta_+ \eta\right)^{-1/\beta_+} \\ &\qquad \le |V - u_+| \le \left|\frac{u_+ - u_-}{2}\right| \left(1 + D^{-1} \left|\frac{u_+ - u_-}{2}\right|^{1+\beta_+ + \beta_-} \beta_+ \eta\right)^{-1/\beta_+}. \end{aligned}$$

This proves the first estimate in (4.9). It is easy to show that as $\eta \to +\infty$

$$|V'(\eta; u_+, u_-)| = |\Phi(V; u_+, u_-)| \sim |V - u_+|^{1+\beta_+} \sim |\eta|^{-1/\beta_+(1+\beta_+)} = |\eta|^{-1/\beta_+ - 1}.$$

Thus the second estimate in (4.9) follows. □

Corollary 4.1. If (4.7) holds and $\beta_\pm > 0$, then

$$|a(u) - a(u_\pm)| \sim |u - u_\pm|^{\beta_\pm} \quad \text{as} \quad u \to u_\pm. \tag{4.12}$$

Corollary 4.2. Under the same assumptions as in Lemma 4, the following results hold:

(1) If $\beta_\pm = 0$, then $\forall \eta \in \mathbf{R}$ and $k = 1, 2$

$$\begin{aligned} &|V(\eta; u_+, u_-) - H(\eta; u_+, u_-)| \le C \exp(-c|\eta|), \\ &|V^{(k)}(\eta; u_+, u_-)| \le C \exp(-c|\eta|); \end{aligned}$$

(2) If $\beta_+ > 0$, then $\forall \eta \in \mathbf{R}^+$ and $k = 1, 2$

$$|V(\eta; u_+, u_-) - u_+| \le C(1 + |\eta|)^{-1/\beta_+},$$
$$|V^{(k)}(\eta; u_+, u_-)| \le C(1 + |\eta|)^{-1/\beta_+ - k}; \tag{4.13}$$

(3) If $\beta_- > 0$, then $\forall \eta \in \mathbf{R}^-$ and $k = 1, 2$

$$|V(\eta; u_+, u_-) - u_-| \le C(1 + |\eta|)^{-1/\beta_-},$$
$$|V^{(k)}(\eta; u_+, u_-)| \le C(1 + |\eta|)^{-1/\beta_- - k}; \tag{4.14}$$

(4) If $\beta_\pm > 0$, then $\forall \eta \in \mathbf{R}$

$$\begin{aligned} & \left|V(\eta; u_+, u_-)_{u_+} \dot{u}_+ + V(\eta; u_+, u_-)_{u_-} \dot{u}_- - H(\eta; \dot{u}_+, \dot{u}_-)\right| \\ \le \quad & C(1 + |\eta|)^{-1/\beta}, \end{aligned} \tag{4.15}$$

where $\beta = \max\{\beta_-, \beta_+\}$ and $\dot{w} = w'(t)$.

Remark 4.1. It should be noted that the constant C in the above inequalities is depending on $|u_+ - u_-|^{-1}$. But in our applications it is assumed that $u_k^+(t) - u_k^-(t) \neq 0$ on all $[t_{m-1}, t_m]$, and hence in what follows C can be considered as uniform constant with respect to both $t \in [0, T]$ and $k = 1, \dots, K$.

5. Construction of an approximate solution

In this section we construct an approximate solution $\widehat{u}^\epsilon$, to u and u^ϵ by using the method of matching asymptotic expansions. As in [5] and [24], the main idea of constructing $\widehat{u}^\epsilon$ is that $\widehat{u}^\epsilon$ is small perturbation of u in the smooth region while posses a viscous shock profile in places of discontinuities. We start with the simpler case that there is only one single shock wave curve in u.

5.1. **The approximate solution to u and u^ϵ with one shock curve.** Assume that there is only one shock curve $x = X_1(t)$ in the entropy solution $u(x, t)$ in the time interval $[t_{m-1}, t_m]$. We construct an *continuous* approximate solution, $\widehat{u}^\epsilon$, to u and u^ϵ in $[t_{m-1}, t_m]$

$$\widehat{u}^\epsilon(x, t) = m\left(\frac{x - X_1(t)}{\epsilon^\gamma}\right) I(x, t) + \left(1 - m\left(\frac{x - X_1(t)}{\epsilon^\gamma}\right)\right) O(x, t), \tag{5.1}$$

where

$$I(x, t) = u(x, t) + V\left(\frac{x - X_1(t)}{\epsilon}; u_+(t), u_-(t)\right) - H\left(\frac{x - X_1(t)}{\epsilon}; u_+(t), u_-(t)\right) \tag{5.2}$$

and

$$O(x, t) = u(x, t) \tag{5.3}$$

are called first order *inner* and *outer* solutions, $u_\pm(t) = u(X_1(t) \pm 0, t)$, $H(\xi; u_+, u_-)$ is the Heaviside function, γ is a constant to be determined later and $m(\xi) \in C^\infty(\mathbf{R})$ satisfying $0 \le m(\xi) \le 1$, and

$$m(\xi) = \begin{cases} 1, & |\xi| \le 1 \\ 0, & |\xi| \ge 2. \end{cases} \tag{5.4}$$

The approximate solution can also be written in an equivalent form:

$$\widehat{u}^\epsilon(x, t) = u(x, t) + m\left(\frac{x - X_1(t)}{\epsilon^\gamma}\right)(V - H)\left(\frac{x - X_1(t)}{\epsilon}; u_+(t), u_-(t)\right). \tag{5.5}$$

The following lemma shows that $\widehat{u}^\epsilon$ approximates u well in L^1 space.

Lemma 5. For any $t \in [t_{m-1}, t_m]$

$$\|\widehat{u}^\epsilon(\cdot,t) - u(\cdot,t)\|_{L^1(\mathbf{R})} \le \begin{cases} C\epsilon, & \bar\beta < 1 \\ C\epsilon^{\gamma+(1-\gamma)/\bar\beta}|\ln\epsilon|, & \bar\beta \ge 1, \end{cases} \tag{5.6}$$

where $\bar\beta = \max_t \beta(t)$ and $\beta = \max\{\beta_1^+, \beta_1^-\}$.

Proof. It follows from (5.5) and (5.4) that

$$\begin{aligned} & \|\widehat{u}^\epsilon(\cdot,t) - u(\cdot,t)\|_{L^1(\mathbf{R})} \\ = \; & \int_{-2\epsilon^\gamma}^0 + \int_0^{2\epsilon^\gamma} m\left(\frac{x}{\epsilon^\gamma}; u_+(t), u_-(t)\right) \left|(V-H)\left(\frac{x}{\epsilon}; u_+(t), u_-(t)\right)\right| dx \\ =: \; & I_- + I_+. \end{aligned}$$

Using change of variable $\xi = x/\epsilon$ and the estimate (4.13), we have for $0 < \beta_1^+ \le 1$ that

$$\begin{aligned} I_+ &= \epsilon \int_0^{2\epsilon^{-1+\gamma}} m(\epsilon^{1-\gamma}\xi)|V(\xi;u_+,u_-) - H(\xi;u_+,u_-)|d\xi \\ &= C\epsilon \int_0^{2\epsilon^{-1+\gamma}} (1+|\xi|)^{-1/\beta_1^+} d\xi \le \begin{cases} C\epsilon, & 0 < \beta_1^+ < 1 \\ C\epsilon|\ln\epsilon|, & \beta_1^+ = 1. \end{cases} \end{aligned} \tag{5.7}$$

On the other hand for $\beta_1^+ > 1$ using variable substitution $\xi^{\beta_1^+} = x/\epsilon$ we have

$$\begin{aligned} I_+ &= \beta_1^+ \epsilon \int_0^{2^{1/\beta_1^+}\epsilon^{-(1-\gamma)/\beta_1^+}} m\left(\epsilon^{1-\gamma}\xi^{\beta_1^+}\right)\left|(V-H)\left(\xi^{\beta_1^+}; u_+, u_-\right)\right|\xi^{\beta_1^+-1} d\xi \\ &\le C\epsilon^{1-(1-\gamma)(\beta_1^+-1)/\beta_1^+} \int_0^{2^{1/\beta_1^+}\epsilon^{-(1-\gamma)/\beta_1^+}} (1+|\xi|^{\beta_1^+})^{-1/\beta_1^+} d\xi \\ &\le C\epsilon^{1-(1-\gamma)(\beta_1^+-1)/\beta_1^+}|\ln\epsilon|. \end{aligned} \tag{5.8}$$

It follows from the above result that

$$I_+ \le \begin{cases} C\epsilon, & 0 < \beta_1^+ < 1 \\ C\epsilon|\ln\epsilon|, & \beta_1^+ = 1 \\ C\epsilon^{1-(1-\gamma)(\beta_1^+-1)/\beta_1^+}|\ln\epsilon|, & 1 < \beta_1^+. \end{cases} \tag{5.9}$$

Similarly, we can obtain the estimates for I_-:

$$I_- \le \begin{cases} C\epsilon, & 0 < \beta_1^- < 1 \\ C\epsilon|\ln\epsilon|, & \beta_1^- = 1 \\ C\epsilon^{1-(1-\gamma)(\beta_1^--1)/\beta_1^-}|\ln\epsilon|, & 1 < \beta_1^-. \end{cases} \tag{5.10}$$

Combining the estimates for I_+ and I_- gives the desired result (5.6). □

We can also establish an estimate of the difference between $\widehat{u}^\epsilon$ and u^ϵ. Its proof will be postponed to Section 7.

Lemma 6. *For any* $t \in [t_{m-1}, t_m]$

$$\begin{aligned} & \|\widehat{u}^\epsilon(\cdot,t) - u^\epsilon(\cdot,t)\|_{L^1(\mathbf{R})} \\ \le \; & \|\widehat{u}^\epsilon(\cdot,t_{m-1}) - u^\epsilon(\cdot,t_{m-1})\|_{L^1(\mathbf{R})} + C\epsilon^{(1-\gamma)(\bar\beta+1)/\bar\beta} + C\epsilon^{\gamma+(1-\gamma)/\bar\beta}. \end{aligned} \tag{5.11}$$

5.2. **The approximate solution to u and u^ϵ with two shock curves.** Assume that there exist two shock curves $x = X_1(t)$ and $x = X_2(t)$ in the entropy solution $u(x,t)$ in the time interval $[t_{m-1}, t_m]$ and they are either interacting at $t = t_m$, i.e., $X_1(t_m) = X_2(t_m)$ or started from a same point at $t = t_{m-1}$, i.e., $X_1(t_{m-1}) = X_2(t_{m-1})$. We construct an *continuous* approximate solution, $\widehat{u}^\epsilon$, to u and u^ϵ in $[t_{m-1}, t_m]$ by using the method of matching asymptotic expansions as follows

$$\begin{aligned}\widehat{u}^\epsilon(x,t) =& m\left(\frac{x - X_1(t)}{\epsilon^\gamma}\right) I_1(x,t) + \left(1 - m\left(\frac{x - X_1(t)}{\epsilon^\gamma}\right)\right) O_1(x,t)\\ &+ m\left(\frac{x - X_2(t)}{\epsilon^\gamma}\right) I_2(x,t) + \left(1 - m\left(\frac{x - X_2(t)}{\epsilon^\gamma}\right)\right) O_2(x,t)\end{aligned} \tag{5.12}$$

where

$$I_i(x,t) = u(x,t) + (V - H)\left(\frac{x - X_i(t)}{\epsilon}; u_{i+}(t), u_{i-}(t)\right) \tag{5.13}$$

for $i = 1, 2$, and

$$O_i(x,t) = u(x,t) \tag{5.14}$$

are the first order inner and outer solutions, $u_{i\pm}(t) = u(X_i(t) \pm 0, t)$, $H(\xi; u_+, u_-)$ is the Heaviside function, γ is a constant to be determined later. The approximate solution can also be written in another form

$$\begin{aligned}\widehat{u}^\epsilon(x,t) \;\; = \;\; & u(x,t) + m\left(\frac{x - X_1(t)}{\epsilon^\gamma}\right)(V - H)\left(\frac{x - X_1(t)}{\epsilon}; u_{1+}(t), u_{1-}(t)\right)\\ &+ m\left(\frac{x - X_2(t)}{\epsilon^\gamma}\right)(V - H)\left(\frac{x - X_2(t)}{\epsilon}; u_{2+}(t), u_{2-}(t)\right).\end{aligned} \tag{5.15}$$

We will show the following lemma.

Lemma 7. *For any $t \in [t_{m-1}, t_m]$*

$$\|\widehat{u}^\epsilon(\cdot,t) - u(\cdot,t)\|_{L^1(\mathbf{R})} \le \begin{cases} C\epsilon, & \bar\beta < 1\\ C\epsilon^{\gamma + (1-\gamma)/\bar\beta}\,|\ln\epsilon|, & \bar\beta \ge 1,\end{cases} \tag{5.16}$$

where $\bar\beta = \max_t \beta(t)$ and $\beta = \max\{\beta_1^+, \beta_1^-, \beta_2^+, \beta_2^-\}$

Proof. The proof is similar to that of Lemma 5. □

We can also show that $\widehat{u}^\epsilon$ is an approximation to u^ϵ.

Lemma 8. *For any $t \in [t_{m-1}, t_m]$*

$$\begin{aligned}&\|\widehat{u}^\epsilon(\cdot,t) - u^\epsilon(\cdot,t)\|_{L^1(\mathbf{R})}\\ &\quad \le \|\widehat{u}^\epsilon(\cdot,t_{m-1}) - u^\epsilon(\cdot,t_{m-1})\|_{L^1(\mathbf{R})}\\ &\quad\quad + C\epsilon^{(1-\gamma)(\bar\beta+1)/\bar\beta} + C\epsilon^{\gamma+(1-\gamma)/\bar\beta} + C\epsilon^{2\gamma}.\end{aligned} \tag{5.17}$$

We will postpone the proof to Section 8.

6. Proof of main theorem

In this paper we will only prove for $\bar\beta \ge 1$, i.e., the non-convex case. (The convex result has been established in [24]). Recall that we assume in last section in each $[t_{m-1}, t_m]$ the entropy solution u is a finite combination of some non-contact shocks, contact shocks etc. Theorem 2 will be proved by induction on m, i.e., we will prove

$$\|u(\cdot,t_m)-u^\epsilon(\cdot,t_m)\|_{L^1(\mathbf{R})}\le\begin{cases}C(T)\epsilon|\ln\epsilon| & \text{for } \bar\beta<1\\ C(T)\epsilon^{(1+1/\bar\beta)/2}|\ln\epsilon| & \text{for } \bar\beta\ge 1,\end{cases}$$

under the induction assumption of

$$\|u(\cdot,t_{m-1})-u^\epsilon(\cdot,t_{m-1})\|_{L^1(\mathbf{R})}\le\begin{cases}C(T)\epsilon|\ln\epsilon| & \text{for } \bar\beta<1\\ C(T)\epsilon^{(1+1/\bar\beta)/2}|\ln\epsilon| & \text{for } \bar\beta\ge 1.\end{cases}$$

The induction assumption holds for $m=1$, this because of $u(x,0)=u^\epsilon(x,t)=u_0(x)$ for $|x|\le\infty$.

Observe that

$$\begin{aligned}&\|u(\cdot,t_m)-u^\epsilon(\cdot,t_m)\|_{L^1(\mathbf{R})}\\ \le\ &\|u(\cdot,t_m)-\hat v^\epsilon(\cdot,t_m)\|_{L^1(\mathbf{R})}+\|\hat v^\epsilon(\cdot,t_m)-u^\epsilon(\cdot,t_m)\|_{L^1(\mathbf{R})}.\end{aligned}$$

It follows from Lemma 5 and Lemma 7 that

$$\|u(\cdot,t_m)-\hat v^\epsilon(\cdot,t_m)\|_{L^1(\mathbf{R})}\le C\epsilon^{\gamma+(1-\gamma)/\bar\beta}|\ln\epsilon|.$$

On the other hand Lemma 6 and Lemma 8 indicate that

$$\begin{aligned}\|\hat v^\epsilon(\cdot,t_m)-u^\epsilon(\cdot,t_m)\|_{L^1(\mathbf{R})}&\le\|\hat v^\epsilon(\cdot,t_{m-1})-u^\epsilon(\cdot,t_{m-1})\|_{L^1(\mathbf{R})}\\ &\quad+C\epsilon^{(1-\gamma)(\bar\beta+1)/\bar\beta}+C\epsilon^{\gamma+(1-\gamma)/\bar\beta}+C\epsilon^{2\gamma}.\end{aligned}\tag{6.1}$$

By using induction assumption and the above estimate we have

$$\begin{aligned}&\|\hat v^\epsilon(\cdot,t_{m-1})-u^\epsilon(\cdot,t_{m-1})\|_{L^1(\mathbf{R})}\\ &\qquad\le\|\hat v^\epsilon(\cdot,t_{m-1})-u(\cdot,t_{m-1})\|_{L^1(\mathbf{R})}+\|u(\cdot,t_{m-1})-u^\epsilon(\cdot,t_{m-1})\|_{L^1(\mathbf{R})}\\ &\qquad\le C\epsilon^{\gamma+(1-\gamma)/\bar\beta}|\ln\epsilon|+C\epsilon^{(1+1/\bar\beta)/2}|\ln\epsilon|\\ &\qquad\quad+C\epsilon^{(1-\gamma)(\bar\beta+1)/\bar\beta}+C\epsilon^{\gamma+(1-\gamma)/\bar\beta}+C\epsilon^{2\gamma}.\end{aligned}$$

Setting $\gamma=1/2$ in the above estimate leads to the induction conclusion.

7. Proof of Lemma 6

The main tool for establishing Lemma 6 is the stability lemma, i.e., Lemma 3. Let $v^\epsilon=\widehat{u}^\epsilon$, where $\widehat{u}^\epsilon$ is defined by (5.1), then v^ϵ satisfies the equation (3.2) on its smooth region $\{(x,t):x\neq X_1(t)\}$ with

$$\begin{aligned}g(x,t)&=u_t+(m(V-H))_t+f(u+m(V-H))_x-\epsilon(u_{xx}+(m(V-H)_{xx})\\ &=-a(u)u_x+m_t(V-H)-mV'\frac{\dot X_1(t)}{\epsilon}+m(V-H)_{u_+}\dot u_++m(V-H)_{u_-}\dot u_-\\ &\quad+(a(u+m(V-H))(u_x+m_x(V-H)+mV'\frac{1}{\epsilon})\\ &\quad-\epsilon\left(u_{xx}+m_{xx}(V-H)+2m_xV'\frac{1}{\epsilon}+mV''\frac{1}{\epsilon^2}\right).\end{aligned}$$

It follows from Lemma 3 that for for any $t \in [t_{m-1}, t_m]$

$$\begin{aligned}
&\|\widehat{u}^\epsilon(\cdot,t) - u^\epsilon(\cdot,t)\|_{L^1(\mathbf{R})} \le \|\widehat{u}^\epsilon(\cdot,t_{m-1}) - u^\epsilon(\cdot,t_{m-1})\|_{L^1(\mathbf{R})} \\
&\quad + \epsilon \int_{t_{m-1}}^t \Big| [\partial_x \widehat{u}^\epsilon(x,t)]\,|_{x=X_1(t)} \Big| dt + \int_{t_{m-1}}^t \|g(\cdot,t)\|_{\mathrm{pis}(\mathbf{R})} dt \\
\le\ & \|\widehat{u}^\epsilon(\cdot,t_{m-1}) - u^\epsilon(\cdot,t_{m-1})\|_{L^1(\mathbf{R})} + C\epsilon + \int_{t_{m-1}}^t \|g(\cdot,t)\|_{\mathrm{pis}(\mathbf{R})} dt,
\end{aligned} \tag{7.1}$$

where we have used the fact that

$$[\partial_x \widehat{u}^\epsilon(x,t)]\,|_{x=X_1(t)} = [u_x(x,t)]\,|_{x=X_1(t)}$$

and the limits of $u_x(X(t) \pm 0, t)$ are uniformly bounded on $[t_{m-1}, t_m]$. We now claim that g is suitably small so that $\widehat{u}^\epsilon$ satisfies (1.3) approximately. To this end, we rewrite g as follows:

$$\begin{aligned}
g(x,t) &= m\left(\big(a(u+m(V-H)) - \dot{X}_1(t)\big)V' - V''\right)\epsilon^{-1} \\
&\quad + \big(a(u+m(V-H)) - a(u)\big)u_x \\
&\quad + \left(\big(a(u+m(V-H)) - \dot{X}_1(t)\big)m' - m''\epsilon^{1-\gamma}\right)\epsilon^{-\gamma}(V-H) \\
&\quad - 2m'\epsilon^{-\gamma}V' - \epsilon u_{xx} + m(V-H)_{u_+}\dot{u}_+ + m(V-H)_{u_-}\dot{u}_- \\
&= \big(a(u+m(V-H)) - \dot{X}_1(t)\big)\left(mV'\epsilon^{-1} + u_x + m'(V-H)\epsilon^{-\gamma}\right) \\
&\quad - mV''\epsilon^{-1} + (\dot{X}_1(t) - a(u))u_x - m''(V-H)\epsilon^{1-2\gamma} - 2m'\epsilon^{-\gamma}V' - \epsilon u_{xx} \\
&\quad + m\left(V_{u_+}\dot{u}_+ + V_{u_-}\dot{u}_- - H\big(x - X_1(t); \dot{u}_+(t), \dot{u}_-(t)\big)\right).
\end{aligned}$$

Using the traveling wave equation $V'' = (a(V) - \dot{X}_1(t))V'$ further gives:

$$\begin{aligned}
g(x,t) &= \big(a(u+m(V-H)) - a(V)\big)mV'\epsilon^{-1} \\
&\quad + \big(a(u+m(V-H)) - \dot{X}_1(t)\big)\left(u_x + m'(V-H)\epsilon^{-\gamma}\right) \\
&\quad + (\dot{X}_1(t) - a(u))u_x - m''(V-H)\epsilon^{1-2\gamma} - 2m'\epsilon^{-\gamma}V' - \epsilon u_{xx} \\
&\quad + m\left(V_{u_+}\dot{u}_+ + V_{u_-}\dot{u}_- - H\big(x - X_1(t); \dot{u}_+(t), \dot{u}_-(t)\big)\right).
\end{aligned}$$

Without loss of generality, we will only consider contact shock curves, i.e., $\beta_1^\pm \ge 1$. The assumption $\beta_1^\pm \ge 1$ implies $\dot{X}_1(t) = a(u_\pm)$, which leads to

$$\begin{aligned}
g(x,t) &= \big(a(u+m(V-H)) - a(V)\big)mV'\epsilon^{-1} \\
&\quad + \big(a(u+m(V-H)) - a(u_\pm)\big)\left(u_x + m'(V-H)\epsilon^{-\gamma}\right) \\
&\quad + (a(u_\pm) - a(u))u_x - m''(V-H)\epsilon^{1-2\gamma} - 2m'\epsilon^{-\gamma}V' - \epsilon u_{xx} \\
&\quad + m\left(V_{u_+}\dot{u}_+ + V_{u_-}\dot{u}_- - H\big(x - X_1(t); \dot{u}_+(t), \dot{u}_-(t)\big)\right).
\end{aligned} \tag{7.2}$$

We now split $\|g(\cdot,t)\|_{\mathrm{pis}(\mathbf{R})}$ to the following three parts:

$$\begin{aligned}
\|g(\cdot,t)\|_{\mathrm{pis}(\mathbf{R})} &= \int_{0<|x-X_1(t)|\le\epsilon^\gamma} + \int_{\epsilon^\gamma\le|x-X_1(t)|\le 2\epsilon^\gamma} + \int_{|x-X_1(t)|\ge 2\epsilon^\gamma} |g(x,t)|dt \\
&= I + II + III.
\end{aligned} \tag{7.3}$$

7.1. **Piecewise constant solution.** In order to estimate I, II and III, we first consider a simple but important case of piecewise constant solution, i.e.,

$$u(x,t) = \begin{cases} u_+, & x > X_1(t) \\ u_-, & x \le X_1(t), \end{cases} \tag{7.4}$$

where u_+ and u_- are constants, and $\dot{X}_1(t) = (f(u_-) - f(u_+))/(u_- - u_+)$. It is easy to show that $g(x,t) = 0$ for $0 < |x - X_1(t)| \le \epsilon^\gamma$ and $|x - X_1(t)| > 2\epsilon^\gamma$. Therefore $I = III = 0$ and what we need to estimate is II only. Let

$$II = II_+ + II_-, \tag{7.5}$$

where

$$\begin{aligned} II_\pm =: \pm \int_{X_1(t)\pm\epsilon^\gamma}^{X_1(t)\pm 2\epsilon^\gamma} &\Big|\big(a(u_\pm + m(V - u_\pm)) - a(V)\big) mV'\epsilon^{-1} \\ &+ \big(a(u_\pm + m(V-u_\pm)) - a(u_\pm)\big) m'(V - u_\pm)\epsilon^{-\gamma} \\ &- m''(V - u_\pm)\epsilon^{1-2\gamma} - 2m'\epsilon^{-\gamma}V'\Big| dx \\ = \pm \int_{X_1(t)\pm\epsilon^\gamma}^{X_1(t)\pm 2\epsilon^\gamma} &\Big|\big(a(u_\pm + m(V-u_\pm)) - a(u_\pm)\big)\big(mV'\epsilon^{-1} + m'(V-u_\pm)\epsilon^{-\gamma}\big) \\ &- (a(V) - a(u_\pm))mV'\epsilon^{-1} - m''(V-u_\pm)\epsilon^{1-2\gamma} - 2m'\epsilon^{-\gamma}V'\Big| dx. \end{aligned} \tag{7.6}$$

Using change of variable $\xi = (x - X_1(t))/\epsilon^\gamma$ gives

$$\begin{aligned} II_\pm \;=\; \pm \int_{\pm 1}^{\pm 2} &\Big|\big(a(u_\pm + m(V-u_\pm)) - a(u_\pm)\big)\big(mV'\epsilon^{-1+\gamma} + m'(V-u_\pm)\big) \\ &-(a(V) - a(u_\pm))mV'\epsilon^{-1+\gamma} - m''(V-u_\pm)\epsilon^{1-\gamma} - 2m'V'\Big| d\xi, \end{aligned} \tag{7.7}$$

where in the above expression $m = m(\xi)$ and $V = V(\xi/\epsilon^{1-\gamma})$. The following estimates can be obtained from Lemma 4 and Corollary 4.1:

$$\begin{aligned} &|V - u_\pm| \le C\epsilon^{(1-\gamma)/\beta_1^\pm} \\ &|V'| \le C\epsilon^{(1-\gamma)(1+1/\beta_1^\pm)} \\ &|a(u_\pm + m(V-u_\pm)) - a(u_\pm)| \le C|V - u_\pm|^{\beta_1^\pm} \le C\epsilon^{1-\gamma} \\ &|a(V) - a(u_\pm)| \le C|V-u_\pm|^{\beta_1^\pm} \le C\epsilon^{1-\gamma}, \end{aligned}$$

as $\epsilon \ll 1$. Substituting the above estimates into (7.7) gives

$$II_\pm \le C\epsilon^{(1-\gamma)(1+1/\beta_1^\pm)}. \tag{7.8}$$

The above results, together with the facts $I = III = 0$, give the desired upper bound for $\|g(\cdot,t)\|_{\mathrm{pis}(\mathbf{R})}$. Therefore, Lemma 6 is established in the piecewise constant solution case.

7.2. **Piecewise smooth solution.** We now consider a more general case, i.e. u is piecewise smooth. It follows from (7.2) that $g(x,t) = -\epsilon u_{xx}$ for $|x - X_1(t)| > 2\epsilon^\gamma$, and

$$\begin{aligned} g(x,t) =& \big(a(u + (V-H)) - a(V)\big)V'\epsilon^{-1} \\ &+ \big(a(u + (V-H)) - a(u)\big)u_x - \epsilon u_{xx} \\ &+ V_{u_+}\dot{u}_+ + V_{u_-}\dot{u}_- - H(x - X_1(t); \dot{u}_+, \dot{u}_-) \end{aligned} \tag{7.9}$$

for $0 < |x - X_1(t)| \le \epsilon^\gamma$. It is easy to see from (7.3) that

$$III = \epsilon \int_{2\epsilon^\gamma}^{\infty} |u_{xx}| dx \le C\epsilon, \tag{7.10}$$

where $u_{xx}(\cdot, t) \in L^1(-\infty, 0) \cup L^1(0, \infty)$ is assumed. It follows from (7.9) that

$$I \le I_1 + I_2 + I_3 + I_4, \tag{7.11}$$

where

$$\begin{aligned}
I_1 &= \int_{0<|x-X_1(t)|\le\epsilon^\gamma} -|a(u + (V - H)) - a(V)| V' \epsilon^{-1} dx \qquad (7.12)\\
I_2 &= \int_{0<|x-X_1(t)|\le\epsilon^\gamma} \big|\big(a(u + (V - H)) - a(u)\big) u_x\big| dx\\
I_3 &= \int_{0<|x-X_1(t)|\le\epsilon^\gamma} \epsilon |u_{xx}| dx\\
I_4 &= \int_{0<|x-X_1(t)|\le\epsilon^\gamma} \big|V_{u_+}\dot{u}_+ + V_{u_-}\dot{u}_- - H(x - X_1(t); \dot{u}_+, \dot{u}_-)\big| dx.
\end{aligned}$$

We now estimate $I_i, 1 \le i \le 4$. Since u is piecewise smooth and $u - H \to 0$ as $x \to X_1(t) \pm 0$, we have

$$|a(u + (V - H)) - a(V)| = |a'(V + \theta(u - H))(u - H)| \le C|x - X_1(t)|.$$

Therefore we can find a positive function $A(\xi)$ such that $\forall x \in (-\infty, \infty)$

- $|a(u + (V - H)) - a(V)| \le A|x - X_1(t)| \le M$;
- $A(\xi) \le C|\xi|$;
- $|A'(\xi)| \le M$.

By using the above auxiliary function A and the estimates (4.9) and (4.10) we have

$$\begin{aligned}
I_1 &\le \int_{|x-X_1(t)|\le\epsilon^\gamma} -A(x - X_1(t)) V'((x - X_1(t))/\epsilon) \epsilon^{-1} dx\\
&= \int_{-\epsilon^{\gamma-1}}^{\epsilon^{\gamma-1}} -A(\epsilon\xi) V'(\xi) d\xi\\
&= \int_{-\epsilon^{\gamma-1}}^{\epsilon^{\gamma-1}} -A(\epsilon\xi) \big(V(\xi) - H(\xi)\big)' d\xi \quad \text{(using the fact } A(0) = 0\text{)}\\
&\le A(\epsilon^\gamma)|V(\epsilon^{\gamma-1}) - H(\epsilon^{\gamma-1})| + A(-\epsilon^\gamma)|V(-\epsilon^{\gamma-1}) - H(-\epsilon^{\gamma-1})|\\
&\quad + M\epsilon \int_{-\epsilon^{\gamma-1}}^{\epsilon^{\gamma-1}} |V(\xi) - H(\xi)| d\xi\\
&\le C\epsilon^{\gamma+(1-\gamma)/\beta} + C\epsilon^{1-(1-\gamma)(1-1/\beta)} \le \epsilon^{\gamma+(1-\gamma)/\beta}.
\end{aligned} \tag{7.13}$$

Observe that $|a(u+(V-H))-a(u)| \le C|V-H|$, which if being applied to I_2 gives

$$
\begin{aligned}
I_2 &\le C\int_{|x-X_1(t)|\le\epsilon^\gamma} |V-H|dx \\
&\le C\epsilon\int_{-\epsilon^{\gamma-1}}^{\epsilon^{\gamma-1}} |V(\xi)-H(\xi)|d\xi \\
&\le C\epsilon\int_{-\epsilon^{\gamma-1}}^{\epsilon^{\gamma-1}} (1+|\xi|)^{-1/\beta}d\xi \\
&\le C\epsilon^{1-(1-\gamma)(1-1/\beta)} = C\epsilon^{\gamma+(1-\gamma)/\beta}.
\end{aligned} \tag{7.14}
$$

Also, by the definition of I_3 we can easily obtain $I_3 \le C\epsilon^{1+\gamma}$. Using change of variables $\xi = (x-X_1(t))/\epsilon$ and the estimate (4.15) gives

$$
I_4 \le C\epsilon\int_{-\epsilon^{\gamma-1}}^{\epsilon^{\gamma-1}} (1+|\xi|)^{-1/\beta}d\xi \le C\epsilon^{1-(1-\gamma)(1-1/\beta)} = C\epsilon^{\gamma+(1-\gamma)/\beta}. \tag{7.15}
$$

Combining the above estimates yields

$$
I \le C\left(\epsilon^{\gamma+(1-\gamma)/\beta} + \epsilon^{\gamma(1+\beta)} + \epsilon|\ln\epsilon| + \epsilon^{(1+\gamma)}\right). \tag{7.16}
$$

What left is to estimate II. It follows from (7.2) that

$$
II \le II_+ + II_- + \sum_{i=1}^{5} II_+^{(i)} + \sum_{i=1}^{5} II_-^{(i)},
$$

where $II_\pm$ is defined by (7.6) and

$$
\begin{aligned}
II_\pm^{(1)} &= \pm\int_{X_1(t)\pm\epsilon^\gamma}^{X_1(t)\pm 2\epsilon^\gamma} |(a(u+m(V-H)) - a(u_\pm+m(V-H)))mV'|\epsilon^{-1}dx, \\
II_\pm^{(2)} &= \pm\int_{X_1(t)\pm\epsilon^\gamma}^{X_1(t)\pm 2\epsilon^\gamma} |(a(u+m(V-H)) - a(u))u_x|dx, \\
II_\pm^{(3)} &= \pm\int_{X_1(t)\pm\epsilon^\gamma}^{X_1(t)\pm 2\epsilon^\gamma} |(a(u+m(V-H)) - a(u_\pm+m(V-H)))m'(V-H)|\epsilon^{-\gamma}dx, \\
II_\pm^{(4)} &= \pm\int_{X_1(t)\pm\epsilon^\gamma}^{X_1(t)\pm 2\epsilon^\gamma} \epsilon|u_{xx}|dx, \\
II_\pm^{(5)} &= \pm\int_{X_1(t)\pm\epsilon^\gamma}^{X_1(t)\pm 2\epsilon^\gamma} m|V_{u_+}\dot{u}_+ + V_{u_-}\dot{u}_- - H(x-X_1(t);\dot{u}_+,\dot{u}_-)\big|dx.
\end{aligned}
$$

It follows from (7.8) that

$$
II_\pm \le C\epsilon^{(1-\gamma)(1+1/\beta_1^\pm)}. \tag{7.17}
$$

The estimate of $II_\pm^{(1)}$ is similar to that for I_1, with the same error bound as (7.13), namely

$$
II_\pm^{(1)} \le C\epsilon^{\gamma+(1-\gamma)/\beta}. \tag{7.18}
$$

The estimate of $II_\pm^{(2)}$ is similar to that for I_2, with the same error bound as (7.14), namely

$$
II_\pm^{(2)} \le C\left(\epsilon^{\gamma(1+\beta)} + \epsilon|\ln\epsilon|\right). \tag{7.19}
$$

The estimate of $II_\pm^{(5)}$ is similar to that for I_4, with the same error bound as (7.15), namely

$$
II_\pm^{(5)} \le C\epsilon^{1-(1-\gamma)(1-1/\beta)} = C\epsilon^{\gamma+(1-\gamma)/\beta}. \tag{7.20}
$$

Using the facts that $a(u+m(V-H))-a(u_\pm+m(V-H)) = a'(\bullet)(u-u_\pm)$ and $|u-u_\pm| \le C|x-X_1(t)|$ away from $x = X_1(t)$ we obtain

$$II_\pm^{(3)} \le \pm C\epsilon^\gamma \int_{X_1(t)\pm\epsilon^\gamma}^{X_1(t)\pm 2\epsilon^\gamma} |(V-H)|\epsilon^{-\gamma}dx.$$

Using change of variables $(x - X_1(t))/\epsilon = \xi$ to the above integration gives

$$II_\pm^{(3)} \le \pm C\epsilon \int_{\pm\epsilon^{\gamma-1}}^{\pm 2\epsilon^{\gamma-1}} |V(\xi) - H(\xi)|d\xi.$$

It then follows from (4.13) and (4.14) that

$$\begin{aligned} II_\pm^{(3)} &\le \pm C\epsilon \int_{\pm\epsilon^{\gamma-1}}^{\pm 2\epsilon^{\gamma-1}} (1+|\xi|)^{-1/\beta_1^\pm} d\xi \le C\beta^{1-(1-\gamma)(1-1/\beta_1^\pm)} \\ &= C\epsilon^{\gamma+(1-\gamma)/\beta_1^\pm}. \end{aligned} \tag{7.21}$$

By its definition, we have

$$II_\pm^{(4)} \le \pm C \int_{X_1(t)\pm\epsilon^\gamma}^{X_1(t)\pm 2\epsilon^\gamma} \epsilon|u_{xx}|dx \le C\epsilon^{1+\gamma}. \tag{7.22}$$

Combining the above estimates, namely (7.17)-(7.22), leads to

$$\begin{aligned} II &\le C\Big(\epsilon^{(1-\gamma)(1+1/\beta)} + \epsilon^{\gamma+(1-\gamma)/\beta)} + \epsilon^{\gamma(1+\beta)} + \epsilon|\ln\epsilon| + \epsilon^{1+\gamma}\Big) \\ &\le C\Big(\epsilon^{(1-\gamma)(1+1/\beta)} + \epsilon^{\gamma+(1-\gamma)/\beta)}\Big). \end{aligned} \tag{7.23}$$

Adding up the estimates for I, II and III gives

$$\begin{aligned} \|g(\cdot,t)\|_{\mathrm{pis}(\mathbf{R})} &\le C\left(\epsilon^{\gamma+(1-\gamma)/\beta} + \epsilon^{(1-\gamma)(1+1/\beta)}\right) \\ &\le C\left(\epsilon^{\gamma+(1-\gamma)/\bar\beta} + \epsilon^{(1-\gamma)(1+1/\bar\beta)}\right). \end{aligned} \tag{7.24}$$

This proves Lemma 6 for piecewise smooth solution case.

8. Proof of Lemma 8

The main difference between Lemma 6 and Lemma 8 is that Lemma 6 deals with only one shock while the latter deals with two interacting shocks. The main tool for proofing Lemma 8 is still the stability lemma, namely Lemma 3. Let $v^\epsilon = \widehat{u}^\epsilon$, where $\widehat{u}^\epsilon$ is defined by (5.15). Then v^ϵ satisfies the equation (3.2) on its smooth region $\{(x,t) : x \ne X_i(t),\ i = 1,2\}$, with

$$\begin{aligned} g(x,t) =& \left(a(u + m_1(V_1 - H_1) + m_2(V_2 - H_2)) - a(u)\right)u_x \\ &+ m_1'\left(a(u + m_1(V_1 - H_1) + m_2(V_2 - H_2)) - a(H_1)\right)(V_1 - H_1)\epsilon^{-\gamma} \\ &+ m_2'(a(u + m_1(V_1 - H_1) + m_2(V_2 - H_2)) - a(H_2))(V_2 - H_2)\epsilon^{-\gamma} \\ &+ m_1(a(u + m_1(V_1 - H_1) + m_2(V_2 - H_2)) - a(V_1))V_1'\epsilon^{-1} \\ &+ m_2(a(u + m_1(V_1 - H_1) + m_2(V_2 - H_2)) - a(V_2))V_2'\epsilon^{-1} \\ &- \epsilon u_{xx} - \left(m_1''(V_1 - H_1) + m_2''(V_2 - H_2)\right)\epsilon^{1-2\gamma} - 2\left(m_1'V_1' + m_2'V_2'\right)\epsilon^{-\gamma} \\ &+ m_1(V_{1u_+}\dot u_{1+} + V_{1u_-}\dot u_{1-} - \dot H_1) + m_2(V_{2u_+}\dot u_{2+} + V_{2u_-}\dot u_{2-} - \dot H_2) \\ :=& g_1 + g_2 + g_3 + g_4 + g_5 + g_6 + g_7, \end{aligned}$$

where for $i = 1, 2$

$$m_i = m((x - X_i(t))/\epsilon^\gamma), \qquad u_{i\pm}(t) = \lim_{x\to X_i\pm 0} u(x,t),$$
$$\dot u_{i\pm}(t) = \frac{d}{dt}u_{i\pm}(t), \qquad V_i = V((x - X_i(t))/\epsilon; u_{i+}, u_{i-}),$$
$$H_i = H((x - X_i(t))/\epsilon; u_{i+}, u_{i-}), \qquad \dot H_i = H((x - X_i(t))/\epsilon; \dot u_{i+}, \dot u_{i-}).$$

Similarly, it follows from Lemma 3 that for any $t \in [t_{m-1}, t_m]$

$$\begin{aligned} &\|\widehat{u}^\epsilon(\cdot,t) - u^\epsilon(\cdot,t)\|_{L^1(\mathbf{R})} \\ \le\ & \|\widehat{u}^\epsilon(\cdot,t_{m-1}) - u^\epsilon(\cdot,t_{m-1})\|_{L^1(\mathbf{R})} + \epsilon\sum_{i=1}^{2}\int_{t_{m-1}}^{t}\Big|[\partial_x\widehat{u}^\epsilon(x,t)]\,|_{x=X_i(t)}\Big|dt + \int_{t_{m-1}}^{t}\|g(\cdot,t)\|_{\mathrm{pis}(\mathbf{R})}dt \\ \le\ & \|\widehat{u}^\epsilon(\cdot,t_{m-1}) - u^\epsilon(\cdot,t_{m-1})\|_{L^1(\mathbf{R})} + C\epsilon + \int_{t_{m-1}}^{t}\|g(\cdot,t)\|_{\mathrm{pis}(\mathbf{R})}dt. \end{aligned} \tag{8.1}$$

In order to estimate the last term above we need to estimate $\int_{t_{m-1}}^{t}\|g_i(\cdot,t)\|_{\mathrm{pis}(\mathbf{R})}dt$ for $i = 1, \ldots, 7$. It is observed that the estimates for $\|g_6\|_{\mathrm{pis}(\mathbf{R})}$ and $\|g_7\|_{\mathrm{pis}(\mathbf{R})}$ are similar to those in one shock case, so they can be upper bounded by the right-hand side of (7.24). Therefore, we just need to estimate $\|g_i\|_{\mathrm{pis}(\mathbf{R})}$ for $1 \le i \le 5$.

In what follows we assume that the two shock curves are started at a same point i.e.,

$$X_1(t_{m-1}+0) = X_2(t_{m-1}+0), \tag{8.2}$$

but not tangent to each other at this point, namely

$$\dot X_1(t_{m-1}+0) < \dot X_2(t_{m-1}+0). \tag{8.3}$$

This implies that there is a constant $c > 0$ such that

$$\delta(t) := X_2(t) - X_1(t) \ge c(t - t_{m-1}) \quad \text{for} \quad t \in [t_{m-1}, t_m]. \tag{8.4}$$

Note that the support of $m((\cdot - X_i(t))/\epsilon^\gamma)$ is $[X_i(t) - 2\epsilon^\gamma, X_i(t) + 2\epsilon^\gamma]$. Hence, if $X_2(t) - X_1(t) \ge 4\epsilon^\gamma$ then for each $x \in \mathbf{R}$ only one of m_1 and m_2 appears in $g(x,t)$. Therefore, when $X_2(t) - X_1(t) \ge 4\epsilon^\gamma$ the estimates for $\|g_i\|_{L^1}$, $1 \le i \le 5$, are similar to those for the one shock case. Let $\tau \in (t_{m-1}, t_m)$ such that

$$X_2(\tau) - X_1(\tau) = 4\epsilon^\gamma \text{ and } X_2(t) - X_1(t) > 4\epsilon^\gamma \text{ for } t \in (\tau, t_m]. \tag{8.5}$$

The above analysis implies that we only need to estimate $\|g_i\|_{L^1}$, $1 \le i \le 5$, for $t \in [t_{m-1}, \tau]$.

We first estimate $\|g_1\|_{L^1}$ for $t \in [t_{m-1}, \tau]$. Observe

$$\begin{aligned} \|g_1(\cdot,t)\|_{\mathrm{pis}(\mathbf{R})} &= \int_{\{x \le X_1(t) - 2\epsilon^\gamma\}\cup\{x \ge X_2(t) + 2\epsilon^\gamma\}} |g_1|dx + \int_{X_1(t) - 2\epsilon^\gamma < x < X_2(t) + 2\epsilon^\gamma} |g_1|dx \\ &:= G_{11} + G_{12}. \end{aligned} \tag{8.6}$$

Since only one of m_1 and m_2 appears in G_{11}, the estimate of G_{11} is similar to that of $II_\pm^{(2)}$ in the one shock case and it follows from (7.19) that

$$G_{11} \le C\Big(\epsilon^{\gamma(1+\beta)} + \epsilon|\ln\epsilon|\Big), \tag{8.7}$$

where $\beta = \max\{\beta_1, \beta_2\}$. The integrand of G_{12} is bounded, therefore

$$G_{12} \le C\Big(X_2(t) - X_1(t) + \epsilon^\gamma\Big).$$

It follows from (8.5) and (8.2) that $\tau - t_{m-1} \le C\epsilon^\gamma$ and $X_2(t) - X_1(t) \le C(t - t_{m-1})$ as $t \to t_{m-1}+0$. Therefore

$$\int_{t_{m-1}}^{\tau} G_{12} dt \le C\epsilon^{2\gamma},$$

which, together with (8.7), gives

$$\int_{t_{m-1}}^{\tau} \|g_1(\cdot,t)\|_{\mathrm{pis}(\mathbf{R})} dt \le C\Big(\epsilon^{2\gamma} + \epsilon^{\gamma(2+\beta)} + \epsilon^{1+\gamma}|\ln\epsilon|\Big). \tag{8.8}$$

Next we estimate $\|g_2\|_{\mathrm{pis}(\mathbf{R})}$. Observe

$$\begin{aligned}
\|g_2\|_{\mathrm{pis}(\mathbf{R})} &= \int_{2\epsilon^\gamma > |x - X_1(t)| \ge \epsilon^\gamma} |g_2(x,t)| dx \le C \int_{2\epsilon^\gamma > |x - X_1(t)| \ge \epsilon^\gamma} |V_1 - H_1|\epsilon^{-\gamma} dx \\
&= \int_{2 > |\xi| > 1} |V_1(\xi/\epsilon^{1-\gamma}) - H_1(\xi/\epsilon^{1-\gamma})| d\xi \le C\epsilon^{(1-\gamma)/\beta_1}.
\end{aligned}$$

Therefore,

$$\int_{t_{m-1}}^{\tau} \|g_2(\cdot,t)\|_{\mathrm{pis}(\mathbf{R})} dt \le C\epsilon^{(1-\gamma)/\beta_1 + \gamma}. \tag{8.9}$$

Similarly, we can prove that

$$\int_{t_{m-1}}^{\tau} \|g_3(\cdot,t)\|_{\mathrm{pis}(\mathbf{R})} dt \le C\epsilon^{(1-\gamma)/\beta_2 + \gamma}. \tag{8.10}$$

We now estimate $\|g_4\|_{\mathrm{pis}(\mathbf{R})}$. Observe

$$\begin{aligned}
\|g_4(\cdot,t)\|_{\mathrm{pis}(\mathbf{R})} &= \int_{0 < |x - X_1(t)| < \epsilon^\gamma} + \int_{|x - X_1(t)| \ge \epsilon^\gamma} |g_4(x,t)| dx \\
&:= G_{41} + G_{42}.
\end{aligned} \tag{8.11}$$

The integrand in G_{41} satisfies the following inequality:

$$\begin{aligned}
|g_4| \text{ in } G_{41} &= |a(u + (V_1 - H_1) + m_2(V_2 - H_2)) - a(V_1))|(-V_1')\epsilon^{-1} \\
&\le |a(u + (V_1 - H_1) + m_2(V_2 - H_2)) - a(V_1 + m_2(V_2 - H_2))|(-V_1')\epsilon^{-1} \\
&\quad + |a(V_1 + m_2(V_2 - H_2)) - a(V_1)|(-V_1')\epsilon^{-1} \\
&:= |g_{41}| + |g_{42}|.
\end{aligned}$$

The integration of $|g_{41}|$ is similar to that of (7.12), and it follows from (7.13) that

$$\int_{\{0 < |x - X_1| \le \epsilon^\gamma\}} |g_{41}| dx \le C\epsilon^{\gamma + (1-\gamma)/\beta_1}. \tag{8.12}$$

We estimate the integral of g_{42}:

$$\begin{aligned}
&\int_{\{0 < |x - X_1| \le \epsilon^\gamma\}} |g_{42}| dx \\
=\ & \int_{\{|x - X_1| \le \epsilon^\gamma\} \cap \{|x - X_2| \le \delta(t)/2\}} |g_{42}| dx + \int_{\{|x - X_1| \le \epsilon^\gamma\} \cap \{|x - X_2| > \delta(t)/2\}} |g_{42}| dx \\
:=\ & G_{411} + G_{412},
\end{aligned}$$

where $\delta(t) := X_2(t) - X_1(t)$. Since $|g_{42}| \le -CV_1'\epsilon^{-1}$, we have

$$\begin{aligned}G_{411} &\le C\epsilon^{-1}\int_{|x-X_2(t)|\le\delta(t)/2} -V_1'((x-X_1(t))/\epsilon)dx \\ &\le C\epsilon^{-1}\left(\frac{2\epsilon}{\delta(t)}\right)^{1/\beta_1+1}\delta(t) \le C\left(\frac{\epsilon}{\delta(t)}\right)^{1/\beta_1},\end{aligned}$$

which leads to

$$\begin{aligned}\int_{t_{m-1}}^{\tau} G_{411}dt &\le C\epsilon^{1/\max_t\beta_1}(t-t_{m-1})^{1-1/\max_t\beta_1}\Big|_{t_{m-1}}^{\tau} \\ &= C\epsilon^{1/\max_t\beta_1}(\tau-t_{m-1})^{1-1/\max_t\beta_1},\end{aligned}$$

where we have applied the inequality (8.4) to the above integration. Since $\tau - t_{m-1} \le C\epsilon^{\gamma}$ we have

$$\int_{t_{m-1}}^{\tau} G_{411}dt \le C\epsilon^{\gamma+(1-\gamma)/\max_t\beta_1}. \tag{8.13}$$

On the other hand, using the Mean Value Theorem to g_{42} gives

$$|g_{42}| \le C|V_2 - H_2|(-V_1')\epsilon^{-1}.$$

Substituting this inequality into G_{412} yields

$$\begin{aligned}G_{412} &= \int_{\{|x-X_1|\le\epsilon^{\gamma}\}\cap\{|x-X_2|>\delta(t)/2\}} |g_{42}|dx \\ &\le C\left|V_2\left(\frac{\delta(t)}{2\epsilon}\right) - H_2\left(\frac{\delta(t)}{2\epsilon}\right)\right|\int_{-\infty}^{\infty}(-V_1'((x-X_1)/\epsilon)\epsilon^{-1}dx \\ &\le C\left(\frac{2\epsilon}{\delta(t)}\right)^{1/\beta_2}\int_{-\infty}^{\infty} -V_1'(\xi)d\xi \le C\left(\frac{2\epsilon}{\delta(t)}\right)^{1/\beta_2}.\end{aligned}$$

Therefore, on account of $\delta(t) \ge c(t-t_{m-1})$ and $\tau - t_{m-1} \le C\epsilon^{\gamma}$, we have

$$\int_{t_{m-1}}^{\tau} G_{412}dt \le C\epsilon^{1/\max_t\beta_2}\epsilon^{\gamma(1-1/\max_t\beta_2)} = C\epsilon^{\gamma+(1-\gamma)/\max_t\beta_2}. \tag{8.14}$$

This result, together with (8.13) and (8.12), yields

$$\int_{t_{m-1}}^{\tau} G_{41}dt \le C\epsilon^{\gamma+(1-\gamma)/\bar{\beta}}, \tag{8.15}$$

where $\bar{\beta} = \max\{\max_t\beta_1, \max_t\beta_2\}$.

In order to estimate G_{42} in (8.11), we first split it into two parts:

$$\begin{aligned}G_{42} &= \int_{\{\epsilon^{\gamma}<|x-X_1|\le2\epsilon^{\gamma}\}\cap\{|x-X_2|\le\delta(t)/2\}} + \int_{\{\epsilon^{\gamma}<|x-X_1|\le2\epsilon^{\gamma}\}\cap\{|x-X_2|>\delta(t)/2\}} |g_4|dx \\ &:= G_{421} + G_{422}\end{aligned} \tag{8.16}$$

Since the integrand $|g_4|$ in G_{422} has the following estimates:

$$\begin{aligned}
|g_4| \text{ in } G_{422} &= m_1|a(u+m_1(V_1-H_1)+m_2(V_2-H_2))-a(V_1)|(-V_1')\epsilon^{-1}\\
&\le m_1|a(u+m_1(V_1-H_1)+m_2(V_2-H_2))\\
&\quad -a(H_1+m_1(V_1-H_1)+m_2(V_2-H_2))|(-V_1')\epsilon^{-1}\\
&\quad + m_1\big(|a(H_1+m_1(V_1-H_1)+m_2(V_2-H_2))\\
&\quad -a(H_1)|+|a(H_1)-a(H_1+V_1-H_1)|\big)(-V_1')\epsilon^{-1}\\
&\le C\Big(|u-H_1|+|V_1-H_1|^{\beta_1}+|V_2-H_2|^{\beta_1}\Big)(-V_1')\epsilon^{-1},
\end{aligned}$$

where we have used the facts that $|V_1-H_1|\to 0$ and $|V_2-H_2|\to 0$ as $\epsilon\to 0$ for $\{\epsilon^\gamma<|x-X_1|\le 2\epsilon^\gamma\}\cap\{|x-X_2|>\delta(t)/2\}$. Therefore

$$\begin{aligned}
G_{422} &\le C\int_{\{\epsilon^\gamma<|x-X_1|\le 2\epsilon^\gamma\}\cap\{|x-X_2|>\delta(t)/2\}}|u-H_1|(-V_1')\epsilon^{-1}dx\\
&\quad + C\int_{\{\epsilon^\gamma<|x-X_1|\le 2\epsilon^\gamma\}\cap\{|x-X_2|>\delta(t)/2\}}|V_1-H_1|^{\beta_1}(-V_1')\epsilon^{-1}dx\\
&\quad + C\int_{\{\epsilon^\gamma<|x-X_1|\le 2\epsilon^\gamma\}\cap\{|x-X_2|>\delta(t)/2\}}|V_2-H_2|^{\beta_1}(-V_1')\epsilon^{-1}dx\\
&=: J_1+J_2+J_3.
\end{aligned}$$

Here the estimation for J_1 is similar to that of I_1, with an upper bound as same as (7.13), so

$$J_1 \le C\epsilon^{\gamma+(1-\gamma)/\beta_1}. \tag{8.17}$$

It follows from (4.9) and (4.10) that

$$J_2 \le C\epsilon^{1-\gamma}\epsilon^{(1-\gamma)(1+1/\beta_1)}\epsilon^{-1}\epsilon^{\gamma} \le C\epsilon^{(1-\gamma)(1+1/\beta_1)}. \tag{8.18}$$

Observe

$$\begin{aligned}
J_3 &= C\int_{\{|x-X_1|\le 2\epsilon^\gamma\}\cap\{|x-X_2|>\delta(t)/2\}}|V_2-H_2|^{\beta_1}(-V_1')\epsilon^{-1}dx\\
&\le C\left|V_2\left(\frac{\delta(t)}{2\epsilon}\right)-H_2\left(\frac{\delta(t)}{2\epsilon}\right)\right|^{\beta_1}\int_{-\infty}^{\infty}(-V_1'((x-X_1)/\epsilon)\epsilon^{-1}dx\\
&\le C\left(\frac{2\epsilon}{\delta(t)}\right)^{\beta_1/\beta_2}\int_{-\infty}^{\infty}-V_1'(\xi)d\xi\\
&\le C\left(\frac{2\epsilon}{\delta(t)}\right)^{\beta_1/\beta_2}\le C\left(\frac{2\epsilon}{\delta(t)}\right)^{1/\beta_2}.
\end{aligned}$$

Therefore, on account of $\delta(t)\ge c(t-t_{m-1})$ and $\tau-t_{m-1}\le C\epsilon^\gamma$, we have

$$\int_{t_{m-1}}^{\tau} J_3dt \le C\epsilon^{1/\max_t\beta_2}\epsilon^{\gamma(1-1/\max_t\beta_2)} = C\epsilon^{\gamma+(1-\gamma)/\max_t\beta_2}. \tag{8.19}$$

This, together with (8.17) and (8.18), yields

$$\int_{t_{m-1}}^{\tau} G_{422}dt \le C\Big(\epsilon^{\gamma+(1-\gamma)/\bar\beta}+\epsilon^{(1-\gamma)(1+1/\bar\beta)}\Big), \tag{8.20}$$

where $\bar{\beta} = \max\{\max_t \beta_1, \max_t \beta_2\}$. Since $|g_4| \le -CV_1'\epsilon^{-1}$, we have

$$\begin{aligned} G_{421} &\le C\epsilon^{-1} \int_{|x-X_2(t)|\le\delta(t)/2} -V_1'((x-X_1(t))/\epsilon)dx \\ &\le C\epsilon^{-1}\left(\frac{2\epsilon}{\delta(t)}\right)^{1/\beta_1+1}\delta(t) \le C\left(\frac{\epsilon}{\delta(t)}\right)^{1/\beta_1}. \end{aligned}$$

Thus

$$\begin{aligned} \int_{t_{m-1}}^{\tau} G_{421}dt &\le C\epsilon^{1/\max_t\beta_1}(t-t_{m-1})^{1-1/\max_t\beta_1}\Big|_{t_{m-1}}^{\tau} \\ &= C\epsilon^{1/\max_t\beta_1}(\tau-t_{m-1})^{1-1/\max_t\beta_1}, \end{aligned}$$

where we have used the inequality (8.4) to the above integration. Since $\tau - t_{m-1} \le C\epsilon^{\gamma}$, we have

$$\int_{t_{m-1}}^{\tau} G_{421}dt \le C\epsilon^{\gamma+(1-\gamma)/\max_t\beta_1}. \tag{8.21}$$

This result, together with (8.20), gives

$$\int_{t_{m-1}}^{\tau} G_{42}dt \le C\Big(\epsilon^{\gamma+(1-\gamma)/\bar{\beta}} + \epsilon^{(1-\gamma)(1+1/\bar{\beta})}\Big). \tag{8.22}$$

Combining (8.15) and (8.22) gives

$$\int_{t_{m-1}}^{\tau} \|g_4(\cdot,t)\|_{\mathrm{pis}(\mathbf{R})}dt \le C\Big(\epsilon^{\gamma+(1-\gamma)/\bar{\beta}} + \epsilon^{(1-\gamma)(1+1/\bar{\beta})}\Big). \tag{8.23}$$

Similarly, it can be shown that

$$\int_{t_{m-1}}^{\tau} \|g_5(\cdot,t)\|_{\mathrm{pis}(\mathbf{R})}dt \le C\Big(\epsilon^{\gamma+(1-\gamma)/\bar{\beta}} + \epsilon^{(1-\gamma)(1+1/\bar{\beta})}\Big). \tag{8.24}$$

Therefore, we have proved that

$$\int_{t_{m-1}}^{\tau} \|g(\cdot,t)\|_{\mathrm{pis}(\mathbf{R})}dt \le C\Big(\epsilon^{\gamma+(1-\gamma)/\bar{\beta}} + \epsilon^{(1-\gamma)(1+1/\bar{\beta})}\Big). \tag{8.25}$$

Substituting this into (8.1) proves Lemma 8.

9. Numerical experiments

To verify the theoretical results obtained in this work, we make some numerical computations in this section. The main purpose is to demonstrate the existence of the fractional rate of convergence. It is generally believed that monotone schemes have the same order of convergence as that for the viscosity approximation. Therefore, to make the numerical verification available, we consider the following (generalized) Lax-Friedrichs scheme,

$$u_j^{n+1} = u_j^n - \frac{\lambda}{2}\left(f(u_{j+1}^n) - f(u_{j-1}^n)\right) + \frac{\mu}{2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right) \tag{9.1}$$

to approximate the conservation law (1.1), where u_j^n is an approximation of $u(x_j, t_n)$, $x_j = j\Delta x$ and $t_n = n\Delta t$, with Δx and Δt being the spatial and temporal grid sizes; μ is a constant satisfying $0 < \mu < 1$, and the temporal and spatial grid ratio $\lambda = \Delta t/\Delta x$ satisfies a Courant-Friedrichs-Levy condition,

$$\lambda \sup_{|u|\le\|u_0\|_\infty} |f'(u)| \le \mu.$$

The theoretical properties of the scheme (9.1) has been investigated by Liu and Xin [14].

Example 9.1. *In the first example, we approximate*

$$u_t + f(u)_x = 0, \qquad\qquad f(u) = (1-u^2)^3$$

with the initial data $u_0(x) = \mathrm{sgn}(x)$*, by using the Lax-Fridrichs scheme (9.1).*

The entropy solution for the above Riemann problem is $u(x,t) = u_0(x)$. It can be verified that $\max_{|u|\le 1} |f'(u)| \le 6/\sqrt{5}$. We then choose $\mu = 0.5, T = 1$ and $\lambda = \sqrt{5}\mu/6$. It follows from Theorem 2 that the rate of convergence should be $(1+\frac{1}{2})/2 = \frac{3}{4}$. It is observed from Table 1 that the numerical result on convergence rate agrees very well with the theoretical prediction.

TABLE 1. The L^1-error and the convergence order for Example 9.1.

mesh Δx	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$
L^1-error	3.22×10^{-1}	2.01×10^{-1}	1.23×10^{-1}	7.44×10^{-2}	4.46×10^{-2}
order		0.680	0.709	0.725	0.738

TABLE 2. The L^1-error and the convergence order for Example 9.2.

mesh Δx	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$
L^1-error	2.79×10^{-1}	1.89×10^{-1}	1.25×10^{-1}	8.05×10^{-2}	5.11×10^{-2}
order		0.562	0.597	0.635	0.656

Example 9.2. *In the second example, we approximate*

$$u_t + f(u)_x = 0, \qquad\qquad f(u) = (1-u)^3(1+u)^4$$

with the initial data $u_0(x) = \mathrm{sgn}(x)$*, by using the Lax-Fridrichs scheme (9.1).*

The entropy solution for the above Riemann problem is again $u(x,t) = u_0(x)$. It can be verified that $\max_{|u|\le 1} |f'(u)| \le 2$. We then choose $\mu = 0.5, T = 1$ and $\lambda = \mu/2$. It follows from Theorem 2 that the rate of convergence should be $(1+\frac{1}{3})/2 = \frac{2}{3}$. It is observed from Table 2 that the numerical result for the convergence rate is again in excellent agreement with the theoretical prediction.

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