A nonconservative numerical approach for hyperbolic systems with source terms: the Well-Balanced schemes

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Abstract. We present an introduction to the Well-Balanced schemes in both cases of scalar and systems of nonhomogeneous balance laws. Our construction is based on a reformulation of the source terms as nonconservative products. Some simple self-similar Riemann solvers and thus most Godunov-type schemes can be deduced. We provide also a few relevant computational results.

1. Introduction

The purpose of these notes is to provide an easy introduction to the field of Well-Balanced numerical schemes in the sense of [14] for systems of nonhomogeneous balance laws. This class of algorithms has been introduced first for 1D scalar equations in order to fix two problems inherent to classical approaches (method of lines, time-splitting ...). The degradation of performances experienced by these conventional numerical processes in presence of source terms comes mostly from the fact that they do not meet the following requirements:

(i) a stability criterion which does not depend on the *stiffness* of the source term, see *e.g.* [9],
(ii) a good ability to preserve static time-asymptotic solutions with possibly very high spatial gradients.

We propose here a construction based on a nonconservative reformulation of the nonhomogeneous problems which permits to derive a wide class of Riemann-based numerical schemes relying on simple *self-similar* solvers. The structure of these solvers ensures that both properties (i), (ii) should hold on reasonably coarse computational grids. From now on, we assume working on a cartesian space-time discretization which is determined by the parameters $h, \Delta t$. For related works, we refer *e.g.* to [2, 3, 5, 16, 18, 19, 23, 25] and the quotations therein; see also [13] for a study in the special context of a relaxation system.

Partially supported by the European Union TMR project HCL #ERBFMRXCT960033.

2. General source terms in 1D scalar balance laws

We consider the following Cauchy problem:

$$\left\{\begin{array}{c} \partial_t u + \partial_x f(u) = g(x,u), \qquad x \in \mathbb{R}, t > 0 \\ f \in C^2(\mathbb{R}), f'' > 0,\ u \mapsto g(x,u) \in C^1(\mathbb{R}) \text{ and } x \mapsto g(x,u) \in C^1_c(\mathbb{R}) \\ g(.,0) = 0 \text{ and } \exists M \in \mathbb{R}^+ \text{ such that } u.g(.,u) \leq 0 \text{ for } |u| \geq M \\ u(.,0) = u_0 \in L^1 \cap BV(\mathbb{R}). \end{array}\right. \tag{1}$$

Here, $BV(\mathbb{R})$ denotes the space of functions with bounded variation. The existence, uniqueness and stability theory of the entropy solutions of (1) has been established in [17]. Because of the presence of the source term on the right-hand-side, the derivation of a reliable and robust numerical scheme is still a difficult task, [18].

We introduce a function $a \in C^1(\mathbb{R})$ given by $a(x) = x$; (1) reads:

$$\partial_t u + \partial_x f(u) - g(a,u)\partial_x a = 0, \qquad x \in \mathbb{R}, t > 0. \tag{2}$$

It does not seem possible to transform this last equation into an homogeneous one. But we can see it as a very elementary Temple system, [15], if we add the static constraint on a: $\partial_t a = 0$. This system can be solved within the framework of Kružkov as long as a remains at least Lipschitz continuous, in particular, the Riemann problem is highly ambiguous since a so-called *nonconservative product*, [6], appears in the equations.

Because of the structure of the equation (2), it is fortunately possible to give a unique self-similar solution to the Riemann problem for the following singular *homogeneous* system:

$$\left\{\begin{array}{c} \partial_t u + \partial_x f(u) - \left[g(a^h,u)\partial_x a^h\right]_\Phi = 0, \\ \partial_t a^h = 0, \end{array}\right. \tag{3}$$

where a^h, $h > 0$ being given, is some piecewise constant approximation of a:

$$a^h(x) = j.h \text{ if } x \in C_j \stackrel{def}{=} \Big[(j-1/2)h, (j+1/2)h\Big[.$$

The reaction term becomes a Dirac comb and induces some stationary contact discontinuities in the solution of (3). According to the formalism of the *families of paths*, [6], the nonconservative term rendering the localized action of the source term reads:

$$\left[g(a^h,u)\partial_x a^h\right]_\Phi = \sum_{j\in\mathbb{Z}} h \left(\int_0^1 g\big(\Phi(s;V_j,V_{j+1})\big).ds\right) \delta\big(x-(j+1/2)h\big), \tag{4}$$

where $V = (u, a^h)$ and δ stands for the Dirac measure located at $x = 0$. The path Φ can be derived following different methodologies, see *e.g.* [7, 11, 12]; it reads:

$$[0,1] \ni s \mapsto \Phi\left(s; \begin{pmatrix} u \\ a^h \end{pmatrix}(x-0), \begin{pmatrix} u \\ a^h \end{pmatrix}(x+0)\right) = \begin{pmatrix} \tilde{u}(s) \\ \tilde{a}(s) \end{pmatrix}, \tag{5}$$

where

$$\partial_s f(\tilde{u})(s) = h.g(\tilde{a},\tilde{u})(s) \text{ and } \tilde{a}(s) = a((j+s).h). \tag{6}$$

This provides a way to solve in a unique manner the Riemann problem for (3) as long as f' does not vanish inside the domain. This pathological situation has been investigated by several authors, see *e.g.* [15, 21], and goes beyond the scope of these notes.

Therefore, under the assumption $f' \neq 0$, we can construct a sequence of piecewise constant approximations u^h to u, the entropy solution of (1) by means of any Riemann-based numerical scheme. This class contains the ones of Glimm, Godunov, Murman-Roe *etc* ... One advantage is that the reaction term is going to be treated directly inside the Riemann solver and the error coming from the projection step will be reduced. And it is well-known that the Godunov scheme is **exact** at steady-state for *homogeneous* problems. This entails also very interesting stability properties for the schemes since the source creates no time-step restriction (see [7, 12] for the case of the Godunov scheme). The last and very good point is that all these schemes will preserve the piecewise constant approximations of the static solutions derived in [22].

To be more precise, the Godunov scheme for (1) relying on (3), (5) consists in solving Riemann problems at each endpoint of the cells C_j, $j \in \mathbb{Z}$. Assuming that the condition $f' > 0$ holds true, it reads in standard notations:

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{h}\Big[f(u_j^n) - f(u_{j-\frac{1}{2}}^n)\Big]. \tag{7}$$

And as a consequence of (4), (6), the following jump relation is of use:

$$u_{j-\frac{1}{2}}^n = \tilde{u}(1),\ \tilde{u}(0) = u_{j-1}^n. \tag{8}$$

Taking advantage of (6), one can rewrite the scheme (7) in an integral form

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{h}\Big[f(u_j^n) - f(u_{j-1}^n)\Big] + \Delta t \int_0^1 g(\tilde{a}, \tilde{u})(s).ds. \tag{9}$$

which emphasizes its ability to preserve **exactly** static regimes satisfying (6). In contrast, one can observe that the *interface method* in [16] corresponds to a mid-point rule approximation of the integral term in (9). We can define u^h, a sequence of approximate solutions of (1) depending on h

$$u^h(x,t) = u_j^n \text{ for } x \in C_j,\ t \in [n\Delta t, (n+1)\Delta t[, \tag{10}$$

which is uniformly stable in $L^1 \cap BV(\mathbb{R})$ and $L^1(\mathbb{R})$-equicontinuous in time under the homogeneous CFL condition *only*, [7, 8, 12]. More precisely, we have the result:

Theorem 1. *Let u be the entropy solution in the sense of Kružkov of (1). Under the further assumptions $f'(u) > 0$ and $\sup_{j,n} |f'(u_j^n)|\Delta t \leq h$, the following error estimate holds for all $t \in \mathbb{R}^+$:*

$$\|u^h(.,t) - u(.,t)\|_{L^1(\mathbb{R})} \leq \exp(Nt)\mathbf{O}\left(\sqrt{h}\right),$$

where $N = \sup_{x\in\mathbb{R};u,u^h}\left\{\partial_u g(x,u)\right\}$.

Another nice feature of this construction lies in its ability to be applied also to rather general strictly hyperbolic *systems* of balance laws.

3. Extension to hyperbolic systems of balance laws

We consider now the more difficult problem of finding a good numerical process to approximate the solutions of the following $n \times n$ strictly hyperbolic system:

$$\left\{ \begin{array}{c} \partial_t U + \partial_x f(U) = g(x,U), \qquad U \in B(0,\varrho)_{\mathbb{R}^n},\ x \in \mathbb{R},\ t > 0, \\ U(0,.) = U_0 \in \left(L^1 \cap BV(\mathbb{R})\right)^n, \\ |\lambda_i(U)| \geq c > 0 \text{ for all } i \in \{1,\dots,n\}, \\ \sup_U \left\{ \|g(.,U)\|_{\mathbb{R}^n} + \|\nabla_U g(.,U)\|_{M_n(\mathbb{R})} \right\} \leq \omega \in L^1 \cap L^\infty(\mathbb{R}), \end{array} \right. \tag{11}$$

each characteristic field being genuinely nonlinear or linearly degenerate. Of course, f and g are assumed the same smoothness than in the preceding section. This meets essentially the hypotheses of the existence theorem in [20] provided $TV(U_0)$ and $\|\omega\|_{L^1(\mathbb{R})}$ are small enough.

It is thus completely straightforward to follow the same canvas in order to derive a Well-Balanced scheme. We introduce a function $a(x) = x$ at the level of the continuous problem (11) and the equations read:

$$\partial_t U + \partial_x f(U) - g(a,U)\partial_x a = 0, \qquad \partial_t a = 0.$$

We go one step beyond and replace a by a piecewise constant approximation a^h as it is of use in any formally first-order scheme for (11); we are once again led to tackle a singular nonconservative system:

$$\left\{ \begin{array}{c} \partial_t U + \partial_x f(U) - \left[g(a^h,U)\partial_x a^h\right]_\Phi = 0 \\ \partial_t a^h = 0. \end{array} \right. \tag{12}$$

The nonconservative products still have the form (4) and the family of paths extends very naturally, [7, 10, 11]:

$$[0,1] \ni s \mapsto \Phi\left(s; \begin{pmatrix} U \\ a^h \end{pmatrix}(x-0), \begin{pmatrix} U \\ a^h \end{pmatrix}(x+0)\right) = \begin{pmatrix} \tilde{U}(s) \\ \tilde{a}(s) \end{pmatrix}, \tag{13}$$

where

$$\partial_s f(\tilde{U})(s) = h.g(\tilde{a},\tilde{U})(s) \text{ and } \tilde{a}(s) = a((j+s).h). \tag{14}$$

The theorem of Lax generalized in [6] or a direct construction, [1], provide a unique self-similar solution to the Riemann problem for (12), (13), (14). Therefore, we are in position to use any Riemann-based numerical scheme as long as no eigenvalue of the Jacobian matrix of f vanishes. This last hypothesis is called the "non-resonance assumption" and appears also *e.g.* in [1, 20]. At this point, it is theoretically possible to derive a Godunov scheme, but a severe drawback will be the computational cost of the elementary Riemann problems resolution. A robust alternative based on flux-vector splitting has been presented for one- or two-dimensional problems in [10]. Here, we propose to derive a very simple linearized Well-Balanced Roe-type scheme for (11) following the ideas of [11, 24].

We want to avoid solving the Riemann problem (12) exactly for initial states (W_L, W_R). One idea is to introduce a local linearized problem whose simple self-similar solution shall be a reasonable approximation. Let us denote $A_\Phi(W_L, W_R)$

such a generalized Roe matrix and consider:

$$\partial_t W + A_\Phi(W_L, W_R)\partial_x W = 0, \tag{15}$$

where $W = (U, a^h)$ is somehow an "augmented unknowns vector". Then, according to the Roe conditions, a matrix A_Φ has the following form:

$$A_\Phi(W_L, W_R) = \begin{pmatrix} A(U_L, U_R) & -G_\Phi(W_L, W_R) \\ 0 & 0 \end{pmatrix}$$

where $A(U_L, U_R)$ is a usual Roe matrix for the homogeneous problem, *i.e.* satisfies the jump relation $A(U_L, U_R).(U_R - U_L) = f(U_R) - f(U_L)$. Therefore, we just have to find a reliable average value for the source terms; according to [11, 24], it reads:

$$G_\Phi(W_L, W_R) = \int_0^1 G\big(\Phi(s; W_L^*, W_R^*)\big).ds = \int_0^1 G(\tilde{a}, \tilde{U})(s).ds$$

where Φ is of course the path (13), (14). This choice still requires the complete solution of the Riemann problem in order to know the states (W_L^*, W_R^*) located at the endpoints of the steady nonconservative contact discontinuity. One simple possibility is thus to decide to fix $W_L^* = W_L$ and this is enough to ensure the Well-Balanced property.

A last problem could be the resolution of the steady differential system (14). Despite the fact that powerful O.D.Es solvers exist in the literature, one can decide to use very simple average values coming from piecewise-constant or -linear approximations of $\tilde{U}$. In the simplest case, one recovers the well-known and widely used value, [23]:

$$G_\Phi(W_L, W_R) = \frac{G(\tilde{a}(0), U_L) + G(\tilde{a}(1), U_R)}{2}.$$

We close this section mentioning that it is possible to use this approach also for theoretical purposes. Indeed, in [1], a ε, h-wave-front tracking algorithm relying on the Riemann solver (12), (13), (14) is studied. Some compactness properties are established by means of a Glimm-type functional's decay under similar assumptions than in [20]. Moreover, the limit solutions as $\varepsilon, h \to 0$ determine a uniformly Lipschitz semigroup defined on a closed domain of integrable functions with small total variation.

More precisely, we assume the following Caratheodory-type conditions for the source term g in (11):

- (P1) $g : \mathbb{R} \times \Omega \mapsto \mathbb{R}^n$ is measurable w.r.t. x, for any $U \in \Omega$, and is C^2 w.r.t. U, for any $x \in \mathbb{R}$;
- (P2) $\|g(x, \cdot)\|_{C^2}$ is bounded over $\Omega \subset B(0, \varrho)_{\mathbb{R}^n}$, uniformly in $x \in \mathbb{R}$;
- (P3) there exists a function $\omega \in L^1 \cap L^\infty(\mathbb{R})$ such that: $\|g(., U)\|_{\mathbb{R}^n} \leq \omega, \quad \|\nabla_U g(., U)\|_{M_n(\mathbb{R})} \leq \omega$ for all $(x, U) \in \mathbb{R} \times \Omega$.

In addition, we require that a *non-resonance* condition holds, that is, the characteristic speeds are bounded away from zero: for all i and some **fixed** p in $\{1, \ldots, n\}$,

$$\forall U \in \Omega, \qquad \lambda_{i \leq p}(U) \leq -c < 0 \;\text{ and }\; \lambda_{i > p}(U) \geq c > 0, \tag{16}$$

where $\lambda_i(U)$ denote the eigenvalues of the Jacobian matrix $\nabla f(U)$. This expresses but a strict hyperbolicity assumption for the nonconservative system (12). Under these assumptions, it was proved in [1] that there exists a family of entropy weak solutions to (11) continuously depending on the initial data. That is to say, if the L^1-norm of ω is small enough, there exist a closed domain $\mathcal{D} \subset L^1(\mathbb{R};\mathbb{R}^n)$ of functions with sufficiently small total variation, a constant L and a unique semigroup $P : [0,+\infty) \times \mathcal{D} \to \mathcal{D}$ endowed with the properties:

(i) For all $U, V \in \mathcal{D}$ and $t, s \geq 0$ one has:
$\|P_s U - P_t V\|_{L^1} \leq L\big(|t-s| + \|U - V\|_{L^1}\big)$.

(ii) For all $U_0 \in \mathcal{D}$ the function $U(\cdot,t) = P_t U_0$ is a weak entropy solution of the Cauchy problem (11), $U(.,t=0) = U_0$.

At last, the trajectories of P can be characterized by means of local integral estimates in the spirit of [4], see [1].

4. Numerical results

a. An initial boundary value problem.

As an illustrative test-case, we consider the following equation:

$$\begin{cases} \partial_t u + \partial_x \left(\frac{u^2}{2}\right) = k(x) u(1-u) \\ x \in [0,1] \text{ and } 0 < t \leq 3.5 \\ u(0,t) = 0.5; \ u(x,0) = 0 \end{cases} \tag{17}$$

As a consequence of the time-stabilization results proven in [22], the steady-state for (17), independently of the initial datum, is given by:

$$v(x) = 1 - \frac{1}{2}.\exp\left(-\int_0^x k(s).ds\right). \tag{18}$$

We compare the accuracy of the numerical stationary curves for the proposed nonconservative scheme (7), (8) and a classical time-splitting algorithm which corresponds approximately to a Godunov scheme for the "time-localized" equation, [13]:

$$\partial_t u + \partial_x \left(\frac{u^2}{2}\right) = k(x) \sum_{n \in \mathbb{N}^*} \Delta t.u(1-u).\delta(t - n\Delta t),$$

and reads (compare with (9)):

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2h}\Big[(u_j^n)^2 - (u_{j-1}^n)^2\Big] + \Delta t.k(jh).u_j^n(1-u_j^n). \tag{19}$$

Thus, when reaching the numerical steady-state, the discretization (19) achieves only an upwind first-order approximation of the static relation (6) for (17). We selected the following "delta-like" function:

$$k(x) = 20 \exp\left(-\frac{(x-0.5)^2}{\varepsilon}\right), \qquad \varepsilon = 0.01. \tag{20}$$

At time $T = 3.5$, all the waves are out of the domain $x \in [0, 1]$. We display the numerical solutions in Figure 1. The advantages of this nonconservative approach are clearly noticeable. We took for both schemes (7), (19) $h = 0.03$ and $\Delta t = 0.02$.

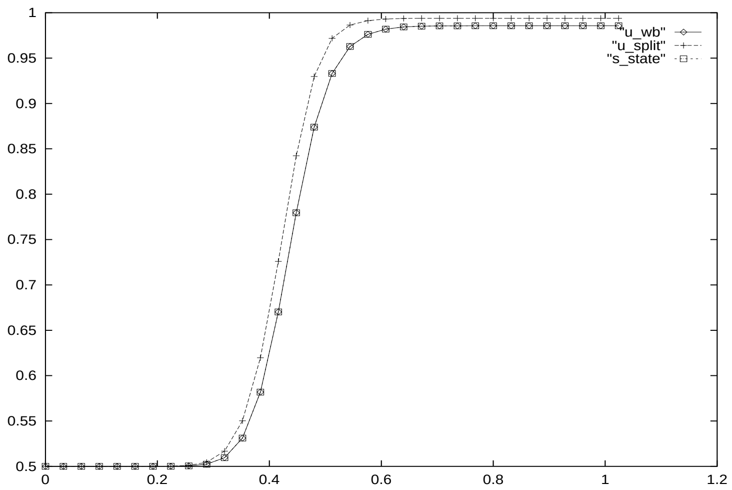

FIGURE 1. Exact and numerical solutions for (17), (20) at $T = 3.5$.

b. A supersonic flow in a divergent nozzle.

Let us consider a perfect gas flow in a cylindric smooth and slowly-varying variable cross-section duct. The variation of section is denoted by $x \mapsto S(x)$ with $S \in C^1([0, 1]; \mathbb{R})$. We get a one-dimensional system written in conservative form with a source term:

$$\left\{ \begin{array}{c} \partial_t(\rho S) + \partial_x(\rho u S) = 0 \\ \partial_t(\rho u S) + \partial_x(S(\rho u^2 + p)) = pS'(x) \\ \partial_t(\rho e S) + \partial_x(Su(\rho e + p)) = 0 \end{array} \right. \tag{21}$$

As usual, ρ, u, i are respectively the density, velocity, internal energy of the fluid; $e = i + \frac{u^2}{2}$, $p = (\gamma - 1)\rho i$ being the total energy and the pressure. It is convenient to introduce new variables for (21): $R = \rho S$, $P = pS$, $h = i + \frac{P}{R}$, $H = e + \frac{P}{R} = h + \frac{u^2}{2}$ and we denote W the augmented unknown vector: $W = (R, Ru, Re, S)$. The test is carried out with a supersonic divergent nozzle introduced in [26]. The cross-section of the nozzle is given by $S(x) = 1.398 + 0.347 \tanh(8x - 4)$, $x \in [0, 1]$. The initial data are those of a gas at rest: $\rho_0 = 0.125$, $u = 0$, $e_0 = i_0 = 2.5$, $\gamma = 1.4$, and we put a supersonic left boundary condition: $\rho_l = 0.502$, $u_l = 1.299$, $p_l = 0.381$. The flow is highly transient, but a stationary regime appears for both methods around the time $t \simeq 3.6$: see Figure 2 for a comparison between the present approach and a more conventional algorithm of the type (19). The stopping criterion was simply given by the decay of the residues in the L^∞ norm as in [10]. The parameters are fixed initially to $h = 0.02$ and $\Delta t = 0.007$; this corresponds to a maximal CFL number close to 1.

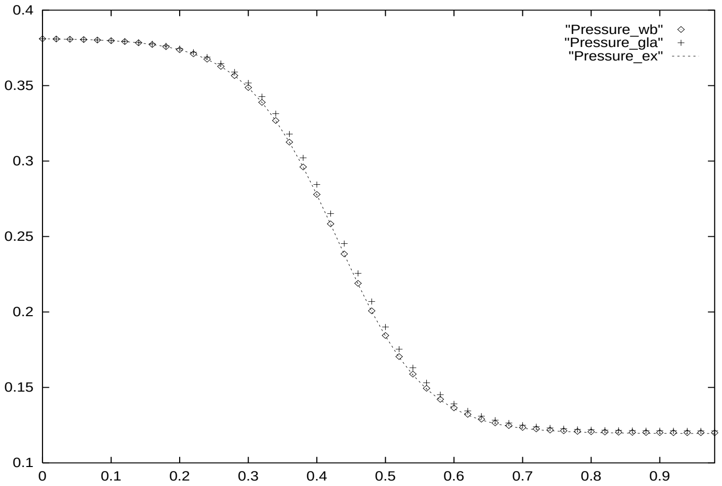

FIGURE 2. Theoretical and numerical values of the pressure for (21) at steady state.

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