DECAY OF POSITIVE WAVES FOR $n \times n$ HYPERBOLIC SYSTEMS OF BALANCE LAWS

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ABSTRACT. We prove Oleĭnik-type decay estimates for entropy solutions of $n \times n$ strictly hyperbolic systems of balance laws built out of a wave-front tracking procedure inside which the source term is treated as a nonconservative product localized on a discrete lattice.

1. INTRODUCTION

A classical result proved by Oleĭnik [16] for strictly convex scalar conservation laws in one space dimension shows that the density of positive waves decays in time like $\mathcal{O}(1/t)$, see also [12]. More precisely, if we consider the scalar equation

$$u_t + f(u)_x = 0, \qquad u(t = 0, .) = u_o \in \mathbf{L}^\infty(I\!R), \tag{1.1}$$

with $f''(u) \geq \kappa > 0$, we have that every entropy-admissible solution satisfies

$$u(t, y) - u(t, x) \leq \frac{y - x}{\kappa t} \qquad \text{for all } t > 0,\ x < y. \tag{1.2}$$

and therefore has locally bounded variation (see also [7], Theorem 11.2.2). Conversely, if $u = u(t, x)$ is a weak solution satisfying (1.2), then u is entropy admissible.

The same estimate as in (1.2) has been recovered for the Riemann coordinates of a particular 2×2 system [3] and for $n \times n$ genuinely nonlinear systems belonging to the Temple class [6]. However, one cannot expect such a result to remain valid for general $n \times n$ systems, even assuming all characteristic fields being genuinely nonlinear. Indeed, interactions among existing shocks may generate rarefactions as time increases. Decay estimates must therefore take into account for the generation of new positive waves due to interactions. Results in this direction were proved by Liu [15] in the case of approximate solutions constructed by Glimm's scheme, [8], and by Bressan and Colombo [4] for exact solutions obtained as limits of front tracking appoximations for $n \times n$ homogeneous systems, [17]. This in turn yields uniqueness of solutions satisfying the Oleĭnik entropy condition [5, 6]. From the point of view of practical applications, such one-sided estimates are very useful for instance in the context of multiphase Geometric Optics computations, [10], or local error estimates, [18].

In this paper we are interested in extending Oleĭnik-type estimates on positive waves to quasilinear systems of balance laws. More precisely, we shall deal with the

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Date: September 2001.

1991 *Mathematics Subject Classification.* Primary 35L65; Secondary 35L45.

Key words and phrases. conservation laws, source terms, nonconservative products.

The authors were partially supported respectively by the EC-Marie Curie Individual Fellowship # HPMF-CT-2000-00862 and the EC-TMR grant #ERBFMRXCT970157.

Cauchy problem for the following $n \times n$ system of equations

$$u_t + f(u)_x = g(x,u), \qquad x \in I\!R, t > 0, \tag{1.3}$$

endowed with a (suitably small) initial data $u_o \in \mathbf{L}^1 \cap \mathbf{BV}(I\!R; I\!R^n)$. Here $u(t,x) \in I\!R^n$ is the unknown function, $f : \Omega \to I\!R^n$ is a smooth $\mathcal{C}^2$ vector field defined on an open neighborhood Ω of the origin in $I\!R^n$. We will assume that the system (1.3) is strictly hyperbolic, with each characteristic field either genuinely nonlinear or linearly degenerate in the sense of Lax [13]. Moreover we assume the following Caratheodory-type conditions for the source term g:

(P1) $g : I\!R \times \Omega \to I\!R^n$ is measurable w.r.t. x, for any $u \in \Omega$, and is $\mathcal{C}^2$ w.r.t. u, for any $x \in I\!R$;
(P2) $\|g(x,\cdot)\|_{\mathcal{C}^2}$ is bounded over Ω, uniformly in $x \in I\!R$;
(P3) there exists a function $\omega \in \mathbf{L}^1(I\!R) \cap \mathbf{L}^\infty(I\!R)$ such that $|g(x,u)| \leq \omega(x)$, $\|\nabla_u g(x,u)\| \leq \omega(x)$ for all $(x,u) \in I\!R \times \Omega$.

In addition, we require that a *non-resonance* condition holds, that is, the characteristic speeds are bounded away from zero: for some $p \in \{1, \dots, n\}$ and some $c > 0$ one has

$$\begin{cases} \lambda_i(u) \leq -c & \text{if } i \leq p, \\ \lambda_i(u) \geq c & \text{if } i > p, \end{cases} \tag{1.4}$$

for all $u \in \Omega$, where $\lambda_i(u)$ denote the eigenvalues of the Jacobian matrix $Df(u)$.

Under these assumptions, it was proved in [1] that there exists a family of entropy weak solutions to (1.3) continuously depending on the initial data. More precisely, if the $\mathbf{L}^1$-norm of ω is sufficiently small, there exist a closed domain $\mathcal{D} \subset \mathbf{L}^1(I\!R; I\!R^n)$ of functions with sufficiently small total variation, a constant L and a unique semigroup $P : [0, +\infty) \times \mathcal{D} \to \mathcal{D}$ with the properties:

(i) For all $u, v \in \mathcal{D}$ and $t, s \geq 0$ one has $\|P_s u - P_t v\|_{\mathbf{L}^1} \leq L\big(|t-s| + \|u-v\|_{\mathbf{L}^1}\big)$.
(ii) For all $u_o \in \mathcal{D}$ the function $u(t,\cdot) = P_t u_o$ is a weak entropy solution of the Cauchy problem (1.3), $u(t=0,.) = u_o$.

Under the above assumptions we aim at showing that, for genuinely nonlinear characteristic fields, an Oleĭnik type estimate on the decay of positive waves holds, which takes into account not only new waves generated by interactions but even the contribution of the source term. A careful statement of these results requires some notations, [2, 4, 5].

As usual, let $A(u) = Df(u)$ be the Jacobian matrix of f, and call $\lambda_i(u)$, $l_i(u)$, $r_i(u)$ respectively the eigenvalues and the left and right eigenvctors of $A(u)$. Let $u : I\!R \to \Omega$ have bounded variation in x and satisfy (1.3) with $g \equiv 0$. The distributional derivative $\mu \doteq D_x u$ is a vector measure. For $i = 1, \dots, n$ we can now define μ^i as

$$\int \phi d\mu^i = \int \phi \tilde{l}_i \cdot D_x u, \qquad \phi \in \mathcal{C}_c^0,$$

where $\tilde{l}_i(x) = l_i(u(x))$ at points where u is continuous, while $\tilde{l}_i(x_\alpha)$ is some vector which satisfies

$$|\tilde{l}_i(x_\alpha) - l_i(u(x_\alpha))| = \mathcal{O}(1) \cdot |u(x_\alpha+) - u(x_\alpha-)|$$
$$\tilde{l}_i(x_\alpha) \cdot \big(u(x_\alpha+) - u(x_\alpha-)\big) = \sigma_\alpha^i,$$

where with σ_α^i we denote the strength of the i-th wave generated by the resolution of the corresponding discontinuity in x_α.

We denote by μ^{i+}, μ^{i-} the positive and negative parts of μ, then we have

$$\mu^i = \mu^{i+} - \mu^{i-}, \qquad |\mu^i| = \mu^{i+} + \mu^{i-}.$$

The *total strength of waves* in u is defined as

$$\mathbf{V}(u) \doteq \sum_{i=1}^{n} \mathbf{V}_i(u), \qquad \mathbf{V}_i(u) \doteq |\mu^i|(I\!R), \tag{1.5}$$

while the *interaction potential* is defined in terms of product measures on $I\!R^2$:

(1.6)

$$\mathbf{Q}(u) \doteq \sum_{i<j} \left(|\mu^j| \times |\mu^i|\right)\left(\{(x,y) : x < y\}\right) + \sum_{i \in \mathcal{GN}} \left(\mu^{i-} \times |\mu^i|\right)\left(\{(x,y) : x \neq y\}\right),$$

where $\mathcal{GN}$ denotes the set of genuinely nonlinear families.

Now we are ready to state our main result in the case $g \not\equiv 0$:

Theorem 1.1. **(Decay of positive waves)** *Let the system* (1.3) *be strictly hyperbolic and let the i-th characteristic field be genuinely non-linear. Then there exists a constant C depending solely on f such that for every $0 \leq s < t$ and every solution u with small total variation obtained as limit of wave-front tracking approximations, the measure μ_t^{i+} of i-waves in $u(t,\cdot)$ satisfies*

$$\mu_t^{i+}(J) \leq C \cdot \left(\frac{meas(J)}{t-s} + \mathbf{Q}(s) - \mathbf{Q}(t) + \mathbf{V}(u_o) \cdot \|\omega\|_{\mathbf{L}^1} \right) \tag{1.7}$$

for every Borel set $J \subset I\!R$.

Of course, we tacitly assume (P1)–(P3) and (1.4) throughout all the text.

2. Wave-front tracking with zero-waves

In this section we briefly recall the construction of wave-front tracking approximations as stated in [1]. We start with the definition of the h-Riemann solver. For small $h > 0$ we introduce the map

$$\Phi_h(x_o, u) \doteq f^{-1}\left[f(u) + \int_0^h g(x_o + s, u)\,ds \right]$$

(note that f is invertible due to (1.4)), that approximates the flow of the stationary equation associated to (1.3). Consider now the Riemann problem with initial states

$$u(0,x) = \begin{cases} u_l & \text{if } x < x_o, \\ u_r & \text{if } x > x_o. \end{cases} \tag{2.1}$$

To locally render the source term's effects, a stationary discontinuity is introduced along the line $x = x_o$, that is, a wave whose speed is equal to zero; it will be referred to as a **zero-wave**. An h-Riemann solver for (1.3) – (2.1) has been defined in [1] as a self-similar function $u(t,x) = R_h((x-x_o)/t; u_l, u_r)$ as follows:

(a) there exist two states u^-, u^+ which satisfy $u^+ = \Phi_h(x_o, u^-)$;
(b) $u(t,x)$ coincides, on the set $\{t \geq 0, x < x_o\}$, with the solution to the homogeneous Riemann problem with initial values u_l, u^- and, on the set $\{t \geq 0, x > x_o\}$, with the solution to the homogeneous Riemann problem with initial values u^+, u_r;
(c) the Riemann problem between u_l and u^- is solved only by waves with negative speed (i.e. of the families $1, \dots, p$);

(d) the Riemann problem between u^+ and u_r is solved only by waves with positive speed (i.e. of the families $p+1,\dots,n$).

This clearly shares a lot of common features with the nonconservative Riemann problems studied in [14]. Let now $\epsilon, h > 0$ be given: an ϵ, h-approximate solution of (1.3) is constructed as follows. First of all, the source term is localized by means of a Dirac comb along zero-waves located on the lattice $x = jh$, $j \in (-\frac{1}{h\epsilon}, \frac{1}{h\epsilon}) \cap \mathbb{Z}$:

$$u_t + f(u)_x = h \sum_j g(x,u).\delta(x - jh), \tag{2.2}$$

where δ stands for the Dirac measure concentrated on $x = 0$.

Given the initial data u_o, we deduce a piecewise constant approximation $u(0,\cdot)$ without increasing its $\mathbf{BV}$-norm and $u(t,x)$ is constructed, for small t, by applying the h-Riemann solver at every point $x = jh$, and by solving the remaining discontinuities in $u(0,\cdot)$ using a classical homogeneous Riemann solver (rarefaction waves are discretized following [2]: for a fixed small parameter ν, each rarefaction of size σ is divided, at its starting time, into $m = \left[\frac{\sigma}{\nu}\right] + 1$ wave fronts of size $\sigma/m \le \nu$).

At every interaction point, a new Riemann problem arises. Notice that because of their null speed, zero-waves cannot interact among each other. In order to keep finite the total number of wave-fronts, two distinct procedures are used for solving a Riemann problem: an accurate method, which possibly creates several new fronts, and a simplified method , which minimizes the number of new wave-fronts. For a detailed description, as well as the proof of the stability of the algorithm, we refer the reader to [1, 2].

The approximate solution can have four types of jumps: shocks (or contact discontinuities), rarefaction fronts, non-physical waves and zero-waves: $\mathcal{J} = \mathcal{S} \cup \mathcal{R} \cup \mathcal{NP} \cup \mathcal{Z}$. A priori bounds on the functions $u^{\epsilon,h}$ are obtained by modifying slightly the Glimm's functionals [8] in order to keep track of the zero-waves,

$$V(t) = \sum_{\alpha \in \mathcal{J}} |\sigma_\alpha| = \sum_{\alpha \in \mathcal{S}\cup\mathcal{R}\cup\mathcal{NP}\cup\mathcal{Z}} |\sigma_\alpha|, \tag{2.3}$$

$$Q(u(t)) = \sum_{\alpha,\beta \in \tilde{\mathcal{A}}} |\sigma_\alpha \sigma_\beta| \le V(t)^2, \tag{2.4}$$

measuring respectively the total wave strengths and the interaction potential in $u(t,\cdot)$. In particular, if $\alpha \in \mathcal{Z}$ then the strength of the wave located in $x_\alpha = j_\alpha h$ can be measured by means of

$$\sigma_\alpha = \int_0^h \omega(j_\alpha h + s)ds. \tag{2.5}$$

In (2.4) we have denoted by $\tilde{\mathcal{A}}$ an extended set of *approaching waves*. More precisely, a couple of wave-fronts, located at $x_\alpha < x_\beta$, belongs to $\tilde{\mathcal{A}}$ in any of the following cases:

- if none of the two is a zero-wave, either $k_\alpha > k_\beta$, or else $k_\alpha = k_\beta$ and at least one of them is a genuinely nonlinear shock, [8];
- if α is a zero-wave and β is a physical one, $k_\beta \le p$;
- if β is a zero-wave and α is a physical one, $k_\alpha > p$.

Notice that for some $C > 1$ there holds

$$\frac{1}{C}\Big[\|\omega\|_{\mathbf{L}^1(I)} + \text{Tot.Var.}u(t,\cdot)\Big] \le V(t) \le C\Big[\|\omega\|_{\mathbf{L}^1(I)} + \text{Tot.Var.}u(t,\cdot)\Big],$$

where the interval I is defined by

$$I \doteq \bigcup_{\alpha \in \mathcal{Z}} [j_\alpha h, (j_\alpha + 1)h] = \bigcup_{j \in \left(-\frac{1}{\epsilon h}, \frac{1}{\epsilon h}\right) \cap \mathbb{Z}} [jh, (j+1)h].$$

Passing to the limit as $\epsilon \to 0$, $h > 0$ fixed, one has:

(i) the total variation of $u^{\epsilon,h}(t,\cdot)$ remains uniformly bounded;
(ii) the maximum size of rerefaction fronts approaches zero;
(iii) the total strength of all non-physical waves approaches zero.

By (i), Helly's theorem guarantees the existence of a subsequence strongly convergent in $\mathbf{L}^1{}_{loc}$. By (ii) and (iii), this limit provides a weak solution to (2.2) in agreement with nonconservative theories, [14]. At this stage, we can extract again a subsequence u^{h_i} which converges to some function u in $\mathbf{L}^1{}_{loc}$ and solves (1.3) in the usual weak distributional sense.

In the sequel, we will need a semicontinuity property of Glimm's functionals. For $h > 0$ fixed, let the total strength of waves $\mathbf{V}_h$ and the interaction potential $\mathbf{Q}_h$ be as in [1], Section 4.1, that is, they are defined as in (1.5)-(1.6), but including the zero-waves. We have

$$\mathbf{V}_h(u) = \mathbf{V}(u) + \|\omega\|_{\mathbf{L}^1}, \tag{2.6}$$

$$\mathbf{Q}(u) \le \mathbf{Q}_h \le \mathbf{Q}(u) + \|\omega\|_{\mathbf{L}^1} \cdot \mathbf{V}(u). \tag{2.7}$$

Proceeding as in [2] we recover the lower semicontinuity of the functionals $\mathbf{Q}_h$ and $\Upsilon_h(u) \doteq \mathbf{V}_h(u) + C_o \mathbf{Q}_h(u)$, $C_o > 0$, on a domain $\mathcal{D}$ of the form

$$\mathcal{D} \doteq \Big\{ u \in \mathbf{L}^1 \cap \mathbf{BV}(\mathbb{R}; \mathbb{R}^n), \Upsilon_h(u) \le \gamma \Big\}, \qquad \gamma \text{ small enough},$$

(see [2], Theorem 10.1):

Theorem 2.1. (Lower semicontinuity of the Glimm functionals) *There exists a choice of the constants $C_o, \gamma > 0$ such that, if $\Upsilon_h(u) = \mathbf{V}_h(u) + C_o \mathbf{Q}_h(u) < \gamma$, then for any sequence of functions $u_\nu \in \mathcal{D}$, $u_\nu \to u$ in $\mathbf{L}^1$ as $\nu \to \infty$, one has*

$$\mathbf{Q}_h(u) \le \liminf_{\nu \to \infty} \mathbf{Q}_h(u_\nu), \tag{2.8}$$

$$\Upsilon_h(u) \le \liminf_{\nu \to \infty} \Upsilon_h(u_\nu). \tag{2.9}$$

Moreover, for every finite union of open intervals $J = I_1 \cup \ldots \cup I_S$ there holds:

$$\mu^{i\pm}(J) + C_o \mathbf{Q}_h(u) \le \liminf_{\nu \to \infty} \big(\mu_\nu^{i\pm}(J) + C_o \mathbf{Q}_h(u_\nu) \big), \qquad i = 1, \ldots, n. \tag{2.10}$$

3. Proof of Theorem 1.1

I. By Lipschitz continuous dependence of the trajectories it is not restrictive to assume $s = 0, t = T$. We will consider a particular converging sequence $u^{\nu,h}$ of ϵ_ν, h-approximate solutions with the following properties:

(i) each rarefaction front x_α travels with the characteristic speed of the state on the right:
$$\dot{x}_\alpha = \lambda_{k_\alpha}(u(x_\alpha+));$$
(ii) each shock x_α travels with a speed strictly contained between the right and the left characteristic speeds:
$$\lambda_{k_\alpha}(u(x_\alpha+)) < \dot{x}_\alpha < \lambda_{k_\alpha}(u(x_\alpha-));$$

(iii) the number N_ν of jumps in $u_o^{\nu,h} = u^{\nu,h}(0,\cdot)$ satisfies

$$\epsilon_\nu N_\nu \to 0 \qquad \text{as } \nu \to \infty; \tag{3.1}$$

(iv) the interaction potential satisfies

$$\mathbf{Q}_h(u^{\nu,h}(0,\cdot)) \to \mathbf{Q}_h(u_o) \qquad \text{as } \nu \to \infty. \tag{3.2}$$

Let $u = u(t,x)$ be a piecewise constant ϵ, h-approximate solution constructed via front-tracking approximation (we shall drop the ϵ, h superscripts since there is no ambiguity). As usual, by *(generalized) i-characteristic* we mean an absolutely continuous curve $x = x(t)$ such that, [7],

$$\dot{x}(t) \in [\lambda_i(u(t,x+)), \lambda_i(u(t,x-))] \qquad \text{a.e. } t \geq 0$$

By $t \mapsto y^i(t;\bar{x})$ we denote the minimal i-characteristic passing through $\bar{x}$ at time T. Because of (1.4) the presence of zero-waves does not affect the usual construction.

Let now $I \doteq [a,b[$ be any half-open interval, and define

$$I(t) \doteq [y^i(t;a), y^i(t;b)[\doteq [a(t), b(t)[.$$

We seek an estimate of the amount of positive i-waves in the the approximate solution $u(T,\cdot)$ contained in I. As usual we call k_α the family of the front located at x_α, with size σ_α. We recall that, in the case of a genuinely nonlinear family, the size of the jump can be measured like

$$\sigma_\alpha \doteq \lambda_{k_\alpha}(u(x_\alpha+)) - \lambda_{k_\alpha}(u(x_\alpha-)), \tag{3.3}$$

while the size of a zero-wave is still given by (2.5). Define

$$m(t) \doteq b(t) - a(t).$$

By (3.3) and the Lipschitz continuity of the map $u \mapsto \lambda_i(u)$ we deduce that

$$\begin{aligned} \dot{m}(t) &= \lambda_i\big(u(t,b(t))\big) - \lambda_i\big(u(t,a(t))\big) \\ &= M(t) + \mathcal{O}(1)(\epsilon + K(t)) \end{aligned} \tag{3.4}$$

for a.e. t. The Landau symbol stands for a quantity whose modulus is uniformly bounded. We see that

$$M(t) \doteq \sum_{k_\alpha = i, x_\alpha \in I(t)} \sigma_\alpha = \mu_t^i\Big(I(t)\Big)$$

is the total amount of (signed) i-waves in $u(t,\cdot)$ contained in $I(t)$, while

$$K(t) \doteq \sum_{k_\alpha \neq i, x_\alpha \in I(t)} |\sigma_\alpha| = \sum_{k \neq i} |\mu_t^k|\Big(I(t)\Big) + \int_{I(t)} \omega(x).dx$$

stands for the total strength of waves of families $\neq i$ inside $I(t)$, zero-waves included. To estimate the contribution of the term $K(t)$ in (3.4) we introduce

$$\Phi(t) \doteq \sum_{k_\alpha \neq i} \phi_{k_\alpha}(t, x_\alpha(t)) \cdot |\sigma_\alpha| \leq \mathbf{V}_h(u(t)),$$

where

$$\phi_j(t,x) \doteq \begin{cases} 1 & \text{if } x < a(t) \\ \frac{b(t)-x}{m(t)} & \text{if } x \in [a(t), b(t)[\\ 0 & \text{if } x \geq b(t) \end{cases}$$

or

$$\phi_j(t,x) \doteq \begin{cases} 0 & \text{if } x < a(t) \\ \frac{x-a(t)}{m(t)} & \text{if } x \in [a(t), b(t)[\\ 1 & \text{if } x \geq b(t) \end{cases}$$

in the cases $j < i$ or $j > i$ respectively. Roughly speaking, $\Phi(t)$ represents the cumulated strength of the waves which do **not** approach the interval $I(t)$. By strict hyperbolicity, we can expect it to grow with time. Observe that Φ is piecewise Lipschitz continuous with a finite number of discontinuities occurring at interaction times, where it may decrease at most of:

$$\Phi(\tau+) - \Phi(\tau-) = \mathcal{O}(1)[Q(u(\tau-)) - Q(u(\tau+))]. \tag{3.5}$$

Following [2], we assume that there holds for some $c_o > 0$,

$$|\lambda_i(u) - \lambda_i(v)| \leq c_o, \qquad |\lambda_i(u) - \lambda_{k_\alpha}(v)| \geq 2c_o, \tag{3.6}$$

for every couple of states u, v and every $k_\alpha \neq i$. Outside interaction times Φ is non-decreasing; indeed, we have

$$\begin{aligned} \dot{\Phi}(t) &= \sum_{k_\alpha \neq i} |\sigma_\alpha| \cdot \frac{d}{dt}\phi_{k_\alpha}(t, x_\alpha(t)) \\ &= \sum_{k_\alpha < i, x_\alpha \in I(t)} |\sigma_\alpha| \cdot \left(\frac{\dot{b} - \dot{x}_\alpha}{m} - \frac{(b - x_\alpha)\dot{m}}{m^2} \right) \\ &\quad + \sum_{k_\alpha > i, x_\alpha \in I(t)} |\sigma_\alpha| \cdot \left(\frac{\dot{x}_\alpha - \dot{a}}{m} - \frac{(x_\alpha - a)\dot{m}}{m^2} \right) \\ &\geq \sum_{k_\alpha \neq i} |\sigma_\alpha| \cdot \frac{c_o}{m(t)} \end{aligned}$$

thanks to the system's strict hyperbolicity and the non-resonance condition (1.4). In particular, observe that $c_o \leq c$ in (3.6). The above estimate yields the bound valid for all but finitely many times t:

$$K(t) \leq \frac{1}{c_o}\dot{\Phi}(t)m(t). \tag{3.7}$$

We notice again, as in [1], that there's a need for a completely different theory in order to tackle resonant cases, see also [11].

Concerning the term $M(t)$, observe that it can change only when an interaction occurs within the interval $[a(t), b(t)]$. In this case, one has

$$M(\tau+) - M(\tau-) = \mathcal{O}(1)[Q(u(\tau-)) - Q(u(\tau+))].$$

This yields an estimate of the form

$$M(T) - M(t) = \mathcal{O}(1) \sum_{\tau \in \mathcal{T}} [Q(u(\tau-)) - Q(u(\tau+))] \tag{3.8}$$

where the summation extends over all times $\tau \in]0, T]$ where an interaction occurs inside $[a(\tau), b(\tau)]$. Inserting the estimates (3.7) – (3.8) in (3.4) we obtain

$$\dot{m}(t) + C\dot{\Phi}(t)m(t) \geq M(T) - C\left(\epsilon + \sum_{\tau \in \mathcal{T}} |\Delta Q(\tau)|\right), \tag{3.9}$$

for some constant C and a.e. t. We now observe that m is a continuous, piecewise linear function of t, and Φ is uniformly bounded. It can decrease only at interaction times, where (3.5) holds. Hence its total variation in uniformly bounded and for some constant K_o we have the estimate

$$\int_0^T \dot{\Phi}(t)dt \leq K_o. \tag{3.10}$$

As in [2], from (3.9) we deduce the decay estimate:

$$M(T) \leq 2e^{CK_o}\left(\frac{b-a}{T}\right) + 2C\epsilon + 2C\sum_{\tau\in\mathcal{T}} |\Delta Q(\tau)|. \tag{3.11}$$

II. Repeating the above process for any finite number S of disjoint half-open intervals $I_s \doteq [a_s, b_s[$ we obtain

$$\sum_{s=1}^{S} M_s(T) \leq C'\left(\sum_{s=1}^{S}\frac{b_s-a_s}{T} + S\epsilon + [Q(u(0)) - Q(u(T))]\right), \tag{3.12}$$

for some constant C' independent of S and of the particular ϵ, h-approximate solution. Here we have used the notation

$$M_s(T) \doteq \sum_{k_\alpha = i, x_\alpha \in [a_s, b_s[} \sigma_\alpha = \mu_T^i([a_s, b_s[)$$

to denote the sum of (signed) strength of all i-waves in $u(T,\cdot)$ contained in the interval $[a_s, b_s[$.

III. Let us consider now any open interval $]a,b[$. Let N be the number of i-shocks of the first generation in the front-tracking ϵ, h-approximate solution u, as defined in [2], Chapter 7. We can thus construct half-open intervals $I_s \doteq [a_s, b_s[$, $s = 1, \ldots, S \leq N+1$, such that the following holds (see [2], Chapter 10):

- Every i-rarefaction front in $u(T,\cdot)$ contained in $]a,b[$ falls inside one of the intervals I_s.
- No i-shock front of the first generation fall inside any of the intervals I_s.

Calling μ_T^{i+} the measure of positive i-waves in $u(T,\cdot)$, the above properties imply

$$\mu_T^{i+}(]a,b[) = \sum_s M_s(T) + \mathcal{O}(1)\cdot[\mathbf{Q}_h(u(0)) - \mathbf{Q}_h(u(T))] + \mathcal{O}(1)\cdot r(\epsilon) \tag{3.13}$$

where $\mathbf{Q}_h$ is the interaction potential introduced in [1], Section 4.1.

Indeed the only negative i-waves contained in $\bigcup_s I_s$ must have generation order ≥ 2, originating from interactions during the time interval $]0,T]$. The total strength of these negative i-waves is bounded by the decrease in the interaction potential Q. The last term on the right hand side of (3.13) tends to zero as ϵ does and comes from the difference between $Q(u)$ and $\mathbf{Q}_h(u)$: in the latter there are no non-physical fronts and all the countable h-Riemann problems are solved (whereas before only zero-waves inside the interval $(-1/\epsilon, 1/\epsilon)$ were considered).

Altogether, (3.12) and (3.13) yield

$$\mu_T^{i+}(]a,b[) \leq C''\left(\frac{b-a}{T} + (N+1)r(\epsilon) + [\mathbf{Q}_h(u(0)) - \mathbf{Q}_h(u(T))]\right) \tag{3.14}$$

for some constant C'' independent of ϵ.

IV. For $h > 0$ fixed, we now consider a sequence of ϵ_ν, h-approximate solutions satisfying the properties $(i) - (iv)$ stated at the beginning of this section. It is clearly not restrictive to take $C'' \geq C_o$ in (3.14), where C_o is the (big) constant in Theorem 2.1. Using (2.8), (2.10), (3.14), (3.1) and (3.2) we obtain

$$\begin{aligned}
&\mu_T^{i+}(]a,b[) \\
&\leq \liminf_{\nu\to\infty} \left(\mu_{\nu,T}^{i+}(]a,b[) + C''\mathbf{Q}_h(u_\nu(T))\right) - C''\mathbf{Q}_h(u(T)) \\
&\leq C'' \cdot \liminf_{\nu\to\infty} \left(\frac{b-a}{T} + (N_\nu+1)r(\epsilon) + \mathbf{Q}_h(u_\nu(0))\right) - C''\mathbf{Q}_h(u(T)) \\
&\leq C'' \cdot \frac{b-a}{T} + C''\Big[\mathbf{Q}_h(u(0)) - \mathbf{Q}_h(u(T))\Big] \\
&\leq C'' \cdot \frac{b-a}{T} + C''\Big[\mathbf{Q}(u(0)) - \mathbf{Q}(u(T)) + \|\omega\|_{\mathbf{L}^1} \cdot \mathbf{V}(u(0))\Big].
\end{aligned}$$

Since the last term is independent of h this proves (1.7) in case J is an open interval. The same arguments can be used in the case where J is a finite collection of open intervals. Since μ^i is a bounded Radon measure, the estimate (1.7) holds for every Borel set J and we are done. □

4. A short comment

All these computations heavily rely on the restrictive assumption that the source term is dominated by a function $\omega \in \mathbf{L}^1(\mathbb{R})$ (the $\mathbf{L}^\infty(\mathbb{R})$ bound being just a consequence of the smoothness of g). In the case of convex scalar law, [9],

$$u_t + f(u)_x = g(u), \qquad u_o \in \mathbf{L}^1 \cap \mathbf{BV}(\mathbb{R}),$$

the stability of the approximation procedure for (1.3) can be obtained requiring only $\omega \in \mathbf{L}^\infty(\mathbb{R})$. Hence one can follow the same canvas and (3.4) simplifies a lot thanks to the obvious bound $K(t) \leq \|\omega\|_{\mathbf{L}^\infty} m(t)$. Thus (3.9) boils down to

$$\dot m(t) + \|\omega\|_{\mathbf{L}^\infty} m(t) \geq M(T),$$

which leads in sharp contrast to an exponential bound of the type ($\mu^1 = u_x$):

$$\mu_t^{1+}(J) \leq C e^{\|\omega\|_{\mathbf{L}^\infty} t} \cdot \frac{meas(J)}{t}, \qquad t > 0. \tag{4.1}$$

This agrees of course with simple computations, since in this context Oleĭnik's estimates can be derived from the Riccati differential equation:

$$z_t - g'(u)z + f''(u)z^2 = 0, \qquad z(0) = \sup_{\mathbb{R}} \Big\{ \max\big(0, (u_o)_x\big)\Big\} \in \bar{\mathbb{R}}.$$

Therefore one recovers (4.1) in the case $C = 1/\kappa$ and $\omega \equiv Lip(g)$ since we have:

$$z(t) = \frac{e^{Lip(g).t}}{\frac{1}{z(0)} + \kappa\left(\frac{e^{Lip(g).t}-1}{Lip(g)}\right)} \leq \frac{e^{Lip(g).t}}{\kappa t}, \qquad t > 0.$$

All in all, the $\mathbf{L}^1(\mathbb{R})$ bound on ω expresses somehow the fact that the source has negligible effects outside a compact interval in $\mathbb{R}$ as pointed out in [7] p.329. Hence, by the strict hyperbolicity ensured by (1.4)–(3.6), waves exit this region after some time and then are ruled only by the convective process.

Acknowledgments

The first author would like to thank the Mathematics Department of the University of Pavia for its hospitality. We also thank Prof. A. Bressan for constructive comments.

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