Vanishing Viscosity Solutions of Nonlinear Hyperbolic Systems

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(Dedicated to Prof. Constantine Dafermos in the occasion of his 60-th birthday)

Abstract. We consider the Cauchy problem for a strictly hyperbolic, $n \times n$ system in one space dimension: $u_t + A(u)u_x = 0$, assuming that the initial data has small total variation.

We show that the solutions of the viscous approximations $u_t + A(u)u_x = \varepsilon u_{xx}$ are defined globally in time and satisfy uniform BV estimates, independent of ε. Moreover, they depend continuously on the initial data in the $\mathbf{L}^1$ distance, with a Lipschitz constant independent of t, ε. Letting $\varepsilon \to 0$, these viscous solutions converge to a unique limit, depending Lipschitz continuously on the initial data. In the conservative case where $A = Df$ is the Jacobian of some flux function $f : \mathbb{R}^n \mapsto \mathbb{R}^n$, the vanishing viscosity limits are precisely the unique entropy weak solutions to the system of conservation laws $u_t + f(u)_x = 0$.

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1 - Introduction

The Cauchy problem for a system of conservation laws in one space dimension takes the form

$$u_t + f(u)_x = 0, \tag{1.1}$$

$$u(0,x) = \bar u(x). \tag{1.2}$$

Here $u = (u_1, \ldots, u_n)$ is the vector of *conserved quantities*, while the components of $f = (f_1, \ldots, f_n)$ are the *fluxes*. We assume that the flux function $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ is smooth and that the system is *strictly hyperbolic*, i. e., at each point u the Jacobian matrix $A(u) = Df(u)$ has n real, distinct eigenvalues

$$\lambda_1(u) < \cdots < \lambda_n(u). \tag{1.3}$$

One can then select bases of right and left eigenvectors $r_i(u)$, $l_i(u)$, normalized so that

$$|r_i| \equiv 1 \qquad\qquad l_i \cdot r_j = \begin{cases} 1 & \text{if } \ i = j, \\ 0 & \text{if } \ i \neq j. \end{cases} \tag{1.4}$$

Several fundamental laws of physics take the form of a conservation equation. For the relevance of hyperbolic conservation laws in continuum physics we refer to the recent book of Dafermos [D].

A distinguished feature of nonlinear hyperbolic systems is the possible loss of regularity. Even with smooth initial data, it is well known that the solution can develop shocks within finite time. Therefore, global solutions can only be constructed within a space of discontinuous functions. The equation (1.1) must then be interpreted in distributional sense. A vector valued function $u = u(t,x)$ is a *weak solution* of (1.1) if

$$\iint \big[u\,\phi_t + f(u)\,\phi_x\big]\,dxdt = 0 \tag{1.5}$$

for every test function $\phi \in \mathcal{C}^1_c$, continuously differentiable with compact support. When discontinuities are present, weak solutions may not be unique. To single out a unique "good" solution of the Cauchy problem, additional *entropy conditions* must be imposed along shocks [Lx], [L1]. These are often motivated by physical considerations [D].

Toward a rigorous mathematical analysis of solutions, the lack of regularity has always been a considerable source of difficulties. For discontinuous solutions, most of the standard tools of differential calculus do not apply. Moreover, for general $n \times n$ systems, the powerful techniques of functional analysis cannot be used. In particular, solutions cannot be obtained as fixed points of a nonlinear transformation, or in variational form as critical points of a suitable functional. Dealing with vector valued functions, comparison arguments based on upper and lower solutions do not apply either. Up to now, the theory of conservation laws has thus progressed largely by developing *ad hoc* methods. In particular, a basic building block is the so-called *Riemann problem*, where the initial data is piecewise constant with a single jump at the origin:

$$u(0,x) = \begin{cases} u^- & \text{if } \ x < 0\,, \\ u^+ & \text{if } \ x > 0\,. \end{cases}$$

Weak solutions to the Cauchy problem (1.1)-(1.2) were constructed in the celebrated paper of Glimm [G]. This global existence result is valid for small BV initial data and under the additional assumption

(H) For each $i \in \{1, \ldots, n\}$, the i-th characteristic field is either *linearly degenerate*, so that

$$D\lambda_i(u) \cdot r_i(u) = 0 \qquad \text{for all } \ u\,, \tag{1.6}$$

or else it is *genuinely nonlinear*, i. e.

$$D\lambda_i(u) \cdot r_i(u) > 0 \qquad \text{for all} \ \ u\,. \tag{1.7}$$

In [G], an approximate solution of the general Cauchy problem is obtained by piecing together solutions of several Riemann problems, with a restarting procedure based on random sampling. The key step in Glimm's proof is an a priori estimate on the total variation of the approximate solutions, obtained by introducing a *wave interaction potential.* In turn, the control of the total variation yields the compactness of the family of approximate solutions, and hence the existence of a strongly convergent subsequence. Alternative constructions of approximate solutions, based on front tracking approximations, were subsequently developed in [DP1], [B2], [Ri], [BaJ].

The above existence results are all based on a compactness argument which, by itself, does not guarantee the uniqueness of solutions. The continuous dependence of solutions on the initial data was first proved in [BC1] and [BCP], with a technique based on linearization + homotopy. As a first step, one estimates the distance between a reference solution u and an infinitesimal perturbation. This is achieved by constructing a Lyapunov functional $\Psi(u;\, z)$ which is non-increasing along all solutions z to a linearized system, describing the evolution of a first order perturbation (see [B1], [B4]). In a second step, to compare two solutions u, v, one constructs a one-parameter family of solutions u^θ connecting u with v. For each time t, the distance $\big\|u(t)-v(t)\big\|_{\mathbf{L}^1}$ can then be bounded in terms of the length of the curve $\theta \mapsto u^\theta(t)$. A drawback of this approach comes from the possible loss of regularity of the solutions u^θ. In order to retain the minimal regularity (piecewise Lipschitz continuity) required for the existence of tangent vectors, in [BC1] and [BCP] various approximation and restarting procedures had to be devised. These yield entirely rigorous proofs, but at the price of heavy technicalities.

A quite different approach was introduced in [LY2] by Liu and Yang, defining a functional $\Phi(u, v)$ which is equivalent to the $\mathbf{L}^1$ distance and decreases along couples of solutions of the hyperbolic system. In their construction, a key role is played by a new entropy functional for genuinely nonlinear scalar fields, introduced in [LY1]. This approach was developed into its final form in [BLY]. For yet another proof of continuous dependence, see also [HLF].

Relying on the continuous dependence of limits of front tracking approximations, general uniqueness results for entropy weak solutions could then be proved in [B3], [BLF1], [BG] and [BLe]. The main results can be summarized as follows:

- The solutions obtained as limits of Glimm or front tracking approximations are unique and depend Lipschitz continuously on the initial data, in the $\mathbf{L}^1$ norm.

- Every small BV solution of the Cauchy problem (1.1)-(1.2) which satisfies the Lax entropy conditions coincides with the unique limit of front tracking approximations.

For comprehensive account of the recent uniqueness and stability theory we refer to [B5].

A long standing conjecture is that the entropic solutions of the hyperbolic system (1.1) actually coincide with the limits of solutions to the parabolic system

$$u_t + f(u)_x = \varepsilon\, u_{xx}\,, \tag*{$(1.8)_\varepsilon$}$$

letting the viscosity coefficient $\varepsilon \to 0$. In view of the recent uniqueness results, it looks indeed very plausible that the vanishing viscosity limit should single out the unique "good" solution of the Cauchy problem, satisfying the appropriate entropy conditions. In earlier literature, results in this direction were based on three main techniques:

1 - Comparison principles for parabolic equations. For a *scalar* conservation law, the existence, uniqueness and global stability of vanishing viscosity solutions was first established by Oleinik [O] in one space dimension. The famous paper of Kruzhkov [K] covers the more general class of $\mathbf{L}^\infty$ solutions, in several space dimensions. For an alternative approach based on nonlinear semigroup theory, see also [Cr].

2 - Singular perturbations. Let u be a piecewise smooth solution of the $n \times n$ system (1.1), with finitely many non-interacting, entropy admissible shocks. In this special case, using a singular perturbation technique, Goodman and Xin [GX] were able to construct a sequence of solutions u^ε to $(1.8)_\varepsilon$, with $u^\varepsilon \to u$ as $\varepsilon \to 0$. See also [Yu] for further results in this direction.

3 - Compensated compactness. If, instead of a BV bound, only a uniform bound on the $\mathbf{L}^\infty$ norm of solutions of $(1.8)_\varepsilon$ is available, one can still construct a weakly convergent subsequence $u^\varepsilon \rightharpoonup u$. In general, we cannot expect that this weak limit satisfies the nonlinear equations (1.5). However, for a class of 2×2 systems, in [DP2] DiPerna showed that this limit u is indeed a weak solution of (1.1). The proof relies on a compensated compactness argument, based on the representation of the weak limit in terms of Young measures, which must reduce to a Dirac mass due to the presence of a large family of entropies. We remark that the solution is here found in the space $\mathbf{L}^\infty$. Since the known uniqueness results apply only to BV solutions, the uniqueness of solutions obtained by the compensated compactness method remains a difficult open problem.

In our point of view, to develop a satisfactory theory of vanishing viscosity limits, the heart of the matter is to establish a priori BV bounds on solutions $u(t,\cdot)$ of $(1.8)_\varepsilon$, uniformly valid for all $t \in [0, \infty[$ and $\varepsilon > 0$. This is indeed what we will accomplish in the present paper. Our results apply, more generally, to strictly hyperbolic $n \times n$ systems with viscosity, not necessarily in conservation form:

$$u_t + A(u)u_x = \varepsilon\, u_{xx}\,. \tag{1.9$_\varepsilon$}$$

As a preliminary, we observe that the rescaling of coordinates $s = t/\varepsilon,\ y = x/\varepsilon$ transforms the Cauchy problem $(1.9)_\varepsilon$, (1.2) into

$$u_s + A(u)u_y = u_{yy}, \qquad\qquad u(0,y) = \bar u^\varepsilon(y) \doteq \bar u(\varepsilon y)\,.$$

Clearly, the total variation of the initial data $\bar u^\varepsilon$ does not change with ε. To obtain a priori BV bounds and stability estimates for solutions of $(1.9)_\varepsilon$, it thus suffices to consider the system

$$u_t + A(u)u_x = u_{xx}\,, \tag{1.10}$$

and derive estimates uniformly valid for all times $t \geq 0$, depending only on the total variation of the initial data $\bar u$.

The first step in our proof is a decomposition of the gradient $u_x = \sum v_i \tilde r_i$ into scalar components. In the purely hyperbolic case without viscosity, it is natural to decompose u_x along a basis $\{r_1, \ldots, r_n\}$ of eigenvectors of the matrix $A(u)$. Remarkably, this choice does not work here. Instead, we will decompose u_x as a sum of gradients of viscous travelling waves, selected by a center manifold technique.

As a second step, we study the evolution of each component v_i, which is governed by a scalar conservation law with a source term, accounting for nonlinear wave interactions. Uniform bounds on these source terms are achieved by means of a *transversal interaction* functional, controlling the interaction between waves of different families, and suitable *swept area* and *curve length* functionals, controlling the interaction of waves of the same family. All these can be regarded as "viscous" counterparts of the *wave interaction potential*, introduced by Glimm [G] in the purely hyperbolic case. Indeed, our "area functional" is closely related to the interaction potential used by Liu in [L4]. Finally, on regions where the diffusion is dominant, the strength of the source term is bounded by an *energy* functional. All together, these estimates yield the desired a priori bound on $\|u_x(t,\cdot)\|_{\mathbf{L}^1}$, independent of $t \in [0, \infty[$.

Similar techniques can also be applied to a solution $z = z(t,x)$ of the variational equation

$$z_t + \big[DA(u) \cdot z\big] u_x + A(u) z_x = z_{xx}\,, \tag{1.11}$$

which describes the evolution of a first order perturbation to a solution u of (1.10). Assuming that the total variation of u remains small, we shall establish an estimate of the form

$$\|z(t,\cdot)\|_{\mathbf{L}^1} \leq L\,\|z(0,\cdot)\|_{\mathbf{L}^1} \qquad \text{for all } \ t \geq 0\,, \tag{1.12}$$

valid for all solutions of (1.11). As soon as this estimate is proved, as in [B1] a standard homotopy argument yields the Lipschitz continuity of the flow of (1.10) w.r.t. the initial data, uniformly in time.

By the simple rescaling of coordinates $t \mapsto \varepsilon t$, $x \mapsto \varepsilon x$, all of the above estimates remain valid for solutions u^ε of the system $(1.9)_\varepsilon$. By a compactness argument, these BV bounds imply the existence of a strong limit $u^{\varepsilon_m} \to u$ in $\mathbf{L}^1_{\rm loc}$, at least for some subsequence $\varepsilon_m \to 0$. In the conservative case where $A = Df$, it is now easy to show that this limit u provides a weak solution to the Cauchy problem (1.1)-(1.2).

At this intermediate stage of the analysis, since we are using a compactness argument, it is not yet clear whether the vanishing viscosity limit is unique. In principle, different subsequences $\varepsilon_m \to 0$ may yield different limits. Toward a uniqueness result, in [B3] the second author introduced a definition of *viscosity solution* for the hyperbolic system of conservation laws (1.1), based on local integral estimates. Roughly speaking, a function u is a *viscosity solution* if

- In a forward neighborhood of each point of jump, the function u is well approximated by the self-similar solution of the corresponding Riemann problem.

- On a region where its total variation is small, u can be accurately approximated by the solution of a linear system with constant coefficients.

For a strictly hyperbolic system of conservation laws satisfying the standard assumptions (H), the analysis in [B3] proved that the viscosity solution of a Cauchy problem is unique, and coincides with the limit of Glimm and front tracking approximations. The definition given in [B3] was motivated by a natural conjecture. Namely, the viscosity solutions (characterized in terms of local integral estimates) should coincide precisely with the limits of vanishing viscosity approximations.

In the present paper we adopt an entirely similar definition of viscosity solutions and prove that the above conjecture is indeed true. Our results apply to the more general case of (possibly non-conservative) quasilinear strictly hyperbolic systems. In particular, we obtain the uniqueness of the vanishing viscosity limit.

As in [B3], [BLFP], the underlying idea is that a semigroup is entirely determined by its local behavior on piecewise constant initial data. Namely, if two semigroups yield the same solution to each Riemann problem, then they coincide. In our proof of uniqueness, a basic step is thus the analysis of the vanishing viscosity solution to a general Riemann problem. The construction given here extends the previous results by Lax and by Liu to general, non-conservative hyperbolic systems. As in the cases considered in [Lx], [L1], for a given left state u^- there exists a Lipschitz continuous curve of right states u^+ which can be connected to u^- by i-waves. These right states are here obtained by looking at the fixed point of a suitable contractive transformation. Remarkably, our center manifold plays again a key role, in defining this transformation.

Our main results are as follows.

Theorem 1. *Consider the Cauchy problem for the hyperbolic system with viscosity*

$$u_t + A(u)u_x = \varepsilon\, u_{xx} \qquad\qquad u(0,x) = \bar u(x)\,. \tag{1.13$_\varepsilon$}$$

Assume that the matrices $A(u)$ are strictly hyperbolic, smoothly depending on u in a neighborhood of a compact set $K \subset I\!\!R^n$. Then there exist constants C, L, L' and $\delta > 0$ such that the following holds. If

$$Tot.Var.\{\bar u\} < \delta\,, \qquad\qquad \lim_{x\to-\infty} \bar u(x) \in K\,, \tag{1.14}$$

then for each $\varepsilon > 0$ the Cauchy problem (1.13)$_\varepsilon$ has a unique solution u^ε, defined for all $t \geq 0$. Adopting a semigroup notation, this will be written as $t \mapsto u^\varepsilon(t,\cdot) \doteq S^\varepsilon_t \bar u$. In addition, one has:

BV bounds :
$$Tot.Var.\big\{S^\varepsilon_t \bar u\big\} \leq C\ Tot.Var.\{\bar u\}\,. \tag{1.15}$$

$\mathbf{L}^1$ **stability** :
$$\big\|S^\varepsilon_t \bar u - S^\varepsilon_t \bar v\big\|_{\mathbf{L}^1} \leq L\,\big\|\bar u - \bar v\big\|_{\mathbf{L}^1}\,, \tag{1.16}$$

$$\big\|S^\varepsilon_t \bar u - S^\varepsilon_s \bar u\big\|_{\mathbf{L}^1} \leq L'\left(|t-s| + \big|\sqrt{\varepsilon t} - \sqrt{\varepsilon s}\,\big|\right). \tag{1.17}$$

Convergence: *As $\varepsilon \to 0+$, the solutions u^ε converge to the trajectories of a semigroup S such that*

$$\big\|S_t \bar u - S_s \bar v\big\|_{\mathbf{L}^1} \leq L\,\|\bar u - \bar v\|_{\mathbf{L}^1} + L'\,|t-s|\,. \tag{1.18}$$

These vanishing viscosity limits can be regarded as the unique **vanishing viscosity solutions** *of the hyperbolic Cauchy problem*

$$u_t + A(u)u_x = 0, \qquad\qquad u(0,x) = \bar u(x)\,. \tag{1.19}$$

In the conservative case $A(u) = Df(u)$, every vanishing viscosity solution is a weak solution of

$$u_t + f(u)_x = 0, \qquad\qquad u(0,x) = \bar u(x)\,, \tag{1.20}$$

satisfying the Liu admissibility conditions.

Assuming, in addition, that each field is genuinely nonlinear or linearly degenerate, the vanishing viscosity solutions coincide with the unique limits of Glimm and front tracking approximations.

Notice that in the above theorem the only key assumptions are the strict hyperbolicity of the system and the small total variation of the initial data. It is interesting to compare this result with previous literature.

1. Concerning the global existence of weak solutions, Glimm's proof requires the additional assumption (H) of genuine nonlinearity or linear degeneracy of each characteristic field. This assumption has been greatly relaxed in subsequent works by Liu [L4] and Liu and Yang [LY3], and eventually removed in the forthcoming paper [ILF], but at the price of considerable technicalities. The underlying reason is the following. In all papers based on the Glimm scheme (or front tracking), the construction of approximate solutions as well as the BV estimates rely on a careful analysis of the Riemann problem and of interactions between elementary waves. In this connection, the hypothesis (H) is a simplifying assumption, which guarantees that every Riemann problem can be solved in terms of n elementary waves (shocks, centered rarefactions or contact discontinuities), one for each characteristic field $i = 1, \ldots, n$. When this assumption fails, constructing a solution to each Riemann problem and deriving interaction estimates is still possible, but far more complicated.

On the other hand, our present approach based on vanishing viscosity marks the first time where uniform BV estimates are obtained without any reference to Riemann problems. Global existence is obtained for the whole class of strictly hyperbolic systems.

2. Concerning the uniform stability of entropy weak solutions, the results previously available for $n \times n$ hyperbolic systems [BC1], [BCP], [BLY] always required the assumption (H). For 2×2 systems, this condition was somewhat relaxed in [AM]. Again, we remark that the present result makes no reference to the assumption (H).

3. For the viscous system (1.10), previous results in [L5], [SX], [SZ], [Yu] have established the stability of special types of solutions, such as travelling viscous shocks or viscous rarefactions, w.r.t. suitably small perturbations. Taking $\varepsilon = 1$, our present theorem yields at once the uniform Lipschitz stability of *all* viscous solutions with sufficiently small total variation, w.r.t. the $\mathbf{L}^1$ distance.

Remark 1.1. The vanishing viscosity approach is based on a different building block, namely the viscous travelling waves. This appears to be more basic, and yields more general results. However, the earlier point of view based on piecewise constant approximations and the analysis of the Riemann problem retains some advantages. In particular, it gives a better geometrical intuition and provides additional results on the qualitative structure and asymptotic properties of solutions as in [L2], [L3], [L4], [BLF2], [B5].

Remark 1.2. It remains an important open problem to establish the convergence of vanishing viscosity approximations of the form

$$u_t + A(u)u_x = \varepsilon\big(B(u)u_x\big)_x \tag{1.21$_\varepsilon$}$$

for more general viscosity matrices B. In the present paper we are exclusively concerned with the case where B is the identity matrix. For systems which are not in conservative form, we expect that the limit of solutions of $(1.21)_\varepsilon$, as $\varepsilon \to 0$, will be heavily dependent on the choice of the matrix B.

Remark 1.3. In the present paper we only consider initial data with small total variation. This is a convenient setting, adopted in much of the current literature, which guarantees the global existence of BV solutions of (1.1) and captures all basic features of the problem. A recent example of Jenssen [J] shows that, for initial data with large total variation, the solution can blow up in finite time. In this more general setting, one expects that the existence and uniqueness of weak solutions, together with the convergence of vanishing viscosity approximations, will hold locally in time as long as the total variation remains bounded. For the hyperbolic system (1.1), results on the existence and stability of solutions with large BV data can be found in [S] and [BC2].

Remark 1.4. For initial data in $\mathbf{L}^\infty$, on the other hand, one cannot expect to have any general theorem on uniqueness and stability of vanishing viscosity solutions. A simple example of non-uniqueness was given in [BS].

The plan of the paper is as follows. Section 2 collects those estimates which can be obtained by standard parabolic techniques. In particular, we show that the solution of (1.10) with initial data $\bar{u} \in BV$ is well defined on an initial time interval $[0, \hat{t}]$ where the $\mathbf{L}^\infty$ norms of all derivatives decay rapidly. Moreover, for large times, as soon as an estimate on the total variation is available, one immediately obtains a bound on the $\mathbf{L}^1$ norms of all higher order derivatives. Our basic strategy for obtaining the BV estimate is outlined in Section 3. The decomposition of u_x as a sum of gradients of viscous travelling profiles is performed in Section 5. This decomposition will depend pointwise on the second order jet (u_x, u_{xx}), involving $2n$ scalar parameters. To fit these data, we must first select n smooth families of viscous travelling waves, each depending on 2 parameters. This preliminary construction is achieved in Section 4, relying on the center manifold theorem. In Section 6 we derive the evolution equation satisfied by the gradient components and analize the form of the various source terms. As in [G], our point of view is that these source terms are the result of interactions between viscous waves, and can thus be controlled by suitable *interaction functionals*. In Sections 7 to 9 we introduce various Lyapounov functionals, which eventually allow us to estimate the integral of all source terms. The proof of the uniform BV bounds is then completed in Section 10.

In Section 11 we study the linearized evolution equation (1.11) for an infinitesimal perturbation z, and derive the key estimate (1.12). In turn, this yields the Lipschitz continuity of the flow, stated in (1.16). Some of the estimates here require lengthy calculations, which are postponed to the Appendices. Section 12 contains an additional estimate for solutions of (1.11), showing that, even in the parabolic case, the bulk of a perturbation propagates at a finite speed. This estimate is crucial because, passing to the limit $\varepsilon \to 0$, it implies that the values of a vanishing viscosity solution $u(t, \cdot)$ on an interval $[a, b]$ depend only on the values of the initial data $u(0, \cdot)$ on a bounded interval $[a - \beta t,\ b + \beta t]$. In Section 13 we study the existence and various properties of a semigroup obtained as vanishing viscosity limit: $S = \lim S^{\varepsilon_m}$. At this stage, we only know that the limit exists for a suitable subsequence $\varepsilon_m \to 0$. In the case of a system of conservation laws satisfying the standard assumptions (H), we can show that every limit solution satisfies the *Lax shock conditions* and the *tame oscillation property*. Hence, by the uniqueness theorem in [BG], the limit is unique and does not depend on the subsequence $\{\varepsilon_m\}$. This already achieves a proof of Theorem 1 valid for this special case.

Toward a proof of uniqueness in the general case, in Section 14 we construct a self-similar solution $\omega(t, x) = \tilde{\omega}(x/t)$ to the non-conservative Riemann problem, and show that it provides the unique vanishing viscosity limit. A definition of *viscosity solution* in terms of local integral estimates is introduced in Section 15. By a minor modification of the arguments in [B3], [B5] we prove that these viscosity solutions are unique and coincide with the trajectories of any semigroup $S = \lim S^{\varepsilon_m}$ obtained as limit of vanishing viscosity approximations. Since this result is independent of the subsequence $\{\varepsilon_m\}$, we obtain the convergence to a unique limit of the whole family of viscous approximations $S_t^\varepsilon \bar u \to S_t \bar u$, over all real values of ε. This completes the proof of Theorem 1.

Finally, in Section 16 we derive two easy estimates. One is concerned with the dependence of the limit semigroup S on the coefficients of the matrix A in (1.19). The other estimate describes the asymptotic limit of solutions of the parabolic system (1.10) as $t \to \infty$.

2 - Parabolic estimates

In classical textbooks, the local existence and regularity of solutions to the parabolic system (1.10) are derived by regarding the hyperbolic term $A(u)u_x$ as a first order perturbation of the heat equation. This leads to the definition of *mild solutions*, characterized by the representation

$$u(t) = G(t) * u(0) - \int_0^t G(t-s) * A\big(u(s)\big)u_x(s)\,ds$$

in terms of convolutions with the standard heat kernel G.

In this initial section we collect all the relevant estimates which can be achieved by this approach. In particular, we prove various decay and regularity results for solutions of (1.10) as well as (1.11). Given a BV solution $u = u(t,x)$ of (1.10), consider the state

$$u^* \doteq \lim_{x\to-\infty} u(t,x)\,, \tag{2.1}$$

which is clearly independent of time. We then define the matrix $A^* \doteq A(u^*)$ and let λ_i^*, r_i^*, l_i^* be the corresponding eigenvalues and right and left eigenvectors, normalized as in (1.4). It will be convenient to use "$\bullet$" to denote a directional derivative, so that $z \bullet A(u) \doteq DA(u)\cdot z$ indicates the derivative of the matrix valued function $u \mapsto A(u)$ in the direction of the vector z. We can now rewrite the systems (1.10) and (1.11) respectively as

$$u_t + A^* u_x - u_{xx} = \big(A^* - A(u)\big)u_x\,, \tag{2.2}$$

$$z_t + A^* z_x - z_{xx} = \big(A^* - A(u)\big)z_x - \big(z \bullet A(u)\big)u_x\,. \tag{2.3}$$

In both cases, we regard the right hand side as a perturbation of the linear parabolic system with constant coefficients

$$w_t + A^* w_x - w_{xx} = 0\,. \tag{2.4}$$

We denote by G^* the Green kernel for (2.4), so that

$$w(t,x) = \int G^*(t,\,x-y)\,w(0,y)\,dy\,.$$

The matrix valued function G^* is easily computed. Indeed, if w solves (2.4), then its i-th component $w_i \doteq l_i^* \cdot w$ satisfies the scalar equation

$$w_{i,t} + \lambda_i^* w_{i,x} - w_{i,xx} = 0\,.$$

Therefore $w_i(t) = G_i^*(t) * w_i(0)$, where

$$G_i^*(t,x) = \frac{1}{2\sqrt{\pi t}} \exp\left\{-\frac{(x-\lambda_i^* t)^2}{4t}\right\}.$$

Looking at the explicit form of its components, it is clear that the Green kernel $G^* = G^*(t,x)$ satisfies the bounds

$$\big\|G^*(t)\big\|_{\mathbf{L}^1} \leq \kappa\,, \qquad \big\|G^*_x(t)\big\|_{\mathbf{L}^1} \leq \frac{\kappa}{\sqrt{t}}\,, \qquad \big\|G^*_{xx}(t)\big\|_{\mathbf{L}^1} \leq \frac{\kappa}{t}\,, \tag{2.5}$$

for some constant κ and all $t > 0$. It is important to observe that, if u is a solution of (2.2), then $z = u_x$ is a particular solution of the variational equation (2.3). Hence all the estimates proved for z_x, z_{xx} are certainly valid also for the corresponding derivatives u_{xx}, u_{xxx}. Assuming that the initial data $u(0,\cdot)$ has small total variation, we now derive some estimates on higher derivatives. In particular, we will show that

- The solution is well defined on some initial interval $[0,\,\hat t]$, where the $\mathbf{L}^\infty$ norm of all derivatives decays rapidly.
- As long as the total variation remains small, the solution can be prolonged in time. In this case, all higher order derivatives remain small. Indeed, waiting a long enough time, one has $\big\|u_{xxx}(t)\big\|_{\mathbf{L}^1} << \big\|u_{xx}(t)\big\|_{\mathbf{L}^1} << \big\|u_x(t)\big\|_{\mathbf{L}^1} = \text{Tot.Var.}\big\{u(t)\big\}$.

Proposition 2.1. *Let u, z be solutions of the systems (2.2)-(2.3), satisfying the bounds*

$$\big\|u_x(t)\big\|_{\mathbf{L}^1} \leq \delta_0, \qquad \big\|z(t)\big\|_{\mathbf{L}^1} \leq \delta_0, \tag{2.6}$$

for some constant $\delta_0 < 1$ and all $t \in [0,\hat t\,]$, where

$$\hat t \doteq \left(\frac{1}{400\kappa\,\kappa_A\,\delta_0}\right)^2, \qquad \kappa_A \doteq \sup_u \big(\|DA\| + \|D^2A\|\big) \tag{2.7}$$

and κ is the constant in (2.5). Then for $t \in [0,\hat t]$ the following estimates hold:

$$\big\|u_{xx}(t)\big\|_{\mathbf{L}^1}\,,\ \big\|z_x(t)\big\|_{\mathbf{L}^1} \leq \frac{2\kappa\delta_0}{\sqrt{t}}\,, \tag{2.8}$$

$$\big\|u_{xxx}(t)\big\|_{\mathbf{L}^1}\,,\ \big\|z_{xx}(t)\big\|_{\mathbf{L}^1} \leq \frac{5\kappa^2\delta_0}{t}\,, \tag{2.9}$$

$$\big\|u_{xxx}(t)\big\|_{\mathbf{L}^\infty}\,,\ \big\|z_{xx}(t)\big\|_{\mathbf{L}^\infty} \leq \frac{16\kappa^3\delta_0}{t\sqrt{t}}\,. \tag{2.10}$$

Proof. The function z_x can be represented as

$$z_x(t) = G^*_x(t) * z(0) + \int_0^t G^*_x(t-s) * \Big\{\big(A^* - A(u)\big)z_x(s) - \big(z \bullet A(u)\big)u_x(s)\Big\}\,ds\,. \tag{2.11}$$

Using (2.5) and (2.6) we obtain

$$\begin{aligned}
&\left\|\int_0^t G^*_x(t-s) * \Big\{\big(A^* - A(u)\big)z_x(s) - \big(z \bullet A(u)\big)u_x(s)\Big\}\,ds\right\|_{\mathbf{L}^1}\\
&\leq \int_0^t \big\|G^*_x(t-s)\big\|_{\mathbf{L}^1} \cdot \Big\{\big\|u_x(s)\big\|_{\mathbf{L}^1}\big\|DA\big\|_{\mathbf{L}^\infty}\big\|z_x(s)\big\|_{\mathbf{L}^1} + \big\|z(s)\big\|_{\mathbf{L}^\infty}\big\|DA\big\|_{\mathbf{L}^\infty}\big\|u_x(s)\big\|_{\mathbf{L}^1}\Big\}\,ds\\
&\leq 2\delta_0\kappa\,\big\|DA\big\|_{\mathbf{L}^\infty} \cdot \int_0^t \frac{1}{\sqrt{t-s}}\,\big\|z_x(s)\big\|_{\mathbf{L}^1}\,ds\,.
\end{aligned}$$

Consider first the case of smooth initial data. We shall argue by contradiction. Assume that there exists a first time $\tau < \hat{t}$ such that the equality in (2.8) holds. Then, observing that

$$\int_0^t \frac{1}{\sqrt{s(t-s)}}\,ds = \int_0^1 \frac{1}{\sqrt{\sigma(1-\sigma)}}\,d\sigma = \pi < 4$$

we compute

$$\begin{aligned}\big\|z_x(\tau)\big\|_{\mathbf{L}^1} &\leq \frac{\kappa}{\sqrt{\tau}}\delta_0 + 2\kappa\delta_0\,\big\|DA\big\|_{\mathbf{L}^\infty} \cdot \int_0^\tau \frac{1}{\sqrt{\tau-s}}\,\frac{2\delta_0\kappa}{\sqrt{s}}\,ds \\ &< \frac{\kappa\delta_0}{\sqrt{\tau}} + 16\kappa^2\kappa_A\delta_0^2 \;\leq\; \frac{2\kappa\delta_0}{\sqrt{\tau}}\,,\end{aligned}$$

reaching a contradiction. Hence, (2.8) is satisfied as a strict inequality for all $t \in [0,\hat{t}]$. Observing that this estimate depends only on the $\mathbf{L}^1$ norms of u_x and z, by an approximation argument we obtain the same bound for general initial data, not necessarily smooth. Since $z \doteq u_x$ is a particular solution of (2.3), the bounds (2.8) certainly apply also to $z_x = u_{xx}$.

A similar technique is used to establish (2.9). Indeed, we can write

$$z_{xx}(t) = G_x^*(t/2) * z_x(t/2) - \int_{t/2}^t G_x^*(t-s) * \Big\{\big(z \bullet A(u)\big)u_x(s) + \big(A(u)-A^*\big)z_x(s)\Big\}_x\,ds\,. \qquad (2.12)$$

We will prove (2.9) first in the case $z_{xx} = u_{xxx}$, then in the general case. If (2.9) is satisfied as an equality at a first time $\tau < \hat{t}$, using (2.12) and recalling the definitions (2.7) we compute

$$\begin{aligned}\big\|z_{xx}(\tau)\big\|_{\mathbf{L}^1} \leq{}& \frac{\kappa}{\sqrt{\tau/2}} \cdot \frac{2\kappa\delta_0}{\sqrt{\tau/2}} + \int_{\tau/2}^\tau \frac{\kappa}{\sqrt{\tau-s}} \cdot \Big\{\big\|z_x \bullet A(u)u_x(s)\big\|_{\mathbf{L}^1} + \big\|z \bullet \big(u_x \bullet A(u)\big)u_x(s)\big\|_{\mathbf{L}^1} \\ &+ \big\|z \bullet A(u)u_{xx}(s)\big\|_{\mathbf{L}^1} + \big\|u_x \bullet A(u)z_x(s)\big\|_{\mathbf{L}^1} + \Big\|\big(A(u)-A^*\big)z_{xx}(s)\Big\|_{\mathbf{L}^1}\Big\}\,ds \\ \leq{}& \frac{2\kappa^2\delta_0}{\tau/2} + \int_{\tau/2}^\tau \frac{\kappa}{\sqrt{\tau-s}} \cdot \Big\{\delta_0\|DA\|_{\mathbf{L}^\infty}\big\|z_{xx}(s)\big\|_{\mathbf{L}^1} + \delta_0\|D^2A\|_{\mathbf{L}^\infty}\big\|u_{xx}(s)\big\|_{\mathbf{L}^1}^2 \\ &+ \delta_0\|DA\|_{\mathbf{L}^\infty}\big\|u_{xxx}(s)\big\|_{\mathbf{L}^1} + \delta_0\|DA\|_{\mathbf{L}^\infty}\big\|z_{xx}(s)\big\|_{\mathbf{L}^1} + \delta_0\|DA\|_{\mathbf{L}^\infty}\big\|z_{xx}(s)\big\|_{\mathbf{L}^1}\Big\}ds \\ \leq{}& \frac{4\kappa^2\delta_0}{\tau} + \kappa\delta_0\big(4\kappa^2\delta_0^2\|D^2A\|_{\mathbf{L}^\infty} + 20\kappa^2\delta_0\|DA\|_{\mathbf{L}^\infty}\big)\int_{\tau/2}^\tau \frac{1}{s\sqrt{\tau-s}}\,ds\,, \\ <{}& \frac{4\kappa^2\delta_0}{\tau} + 20\kappa^3\kappa_A\delta_0^2 \cdot \frac{4}{\sqrt{\tau/2}} \;<\; \frac{5\kappa^2\delta_0}{\tau}\,,\end{aligned}$$

reaching a contradiction.

Finally, using (2.12) and (2.8)-(2.9), the bounds in (2.10) are proved by the estimate

$$\begin{aligned}\big\|z_{xx}(\tau)\big\|_{\mathbf{L}^\infty} \leq{}& \frac{\kappa}{\sqrt{\tau/2}} \cdot \frac{5\kappa^2\delta_0}{\tau/2} + \int_{\tau/2}^\tau \frac{\kappa}{\sqrt{\tau-s}} \cdot \Big\{\big\|z_x \bullet A(u)u_x(s)\big\|_{\mathbf{L}^\infty} + \big\|z \bullet (u_x \bullet A(u))u_x(s)\big\|_{\mathbf{L}^\infty} \\ &+ \big\|z \bullet A(u)u_{xx}(s)\big\|_{\mathbf{L}^\infty} + \big\|u_x \bullet A(u)z_x(s)\big\|_{\mathbf{L}^\infty} + \big\|(A(u)-A^*)z_{xx}(s)\big\|_{\mathbf{L}^\infty}\Big\}\,ds \\ \leq{}& \frac{10\sqrt{2}\kappa^3\delta_0}{\tau\sqrt{\tau}} + \big(8\kappa^4\delta_0^3\|D^2A\| + 46\kappa^4\delta_0^2\|DA\|\big)\int_{\tau/2}^\tau \frac{1}{s^{3/2}\sqrt{\tau-s}}\,ds \\ \leq{}& \frac{15\kappa^3\delta_0}{\tau\sqrt{\tau}} + 46\kappa^4\kappa_A\delta_0^2 \cdot \frac{4}{\tau/2} \;<\; \frac{16\kappa^3\delta_0}{\tau\sqrt{\tau}}\,.\end{aligned}$$

□

Corollary 2.2. *In the same setting as Proposition 2.1, assume that the bounds (2.6) hold on a larger interval* $[0,T]$. *Then for all* $t\in[\hat t,\,T]$ *there holds*

$$\big\|u_{xx}(t)\big\|_{\mathbf{L}^1}\,,\ \big\|u_x(t)\big\|_{\mathbf{L}^\infty}\,,\ \big\|z_x(t)\big\|_{\mathbf{L}^1}\ =\ \mathcal{O}(1)\cdot\delta_0^2\,, \tag{2.13}$$

$$\big\|u_{xxx}(t)\big\|_{\mathbf{L}^1}\,,\ \big\|u_{xx}(t)\big\|_{\mathbf{L}^\infty}\,,\ \big\|z_{xx}(t)\big\|_{\mathbf{L}^1}\ =\ \mathcal{O}(1)\cdot\delta_0^3\,, \tag{2.14}$$

$$\big\|u_{xxx}(t)\big\|_{\mathbf{L}^\infty}\,,\ \big\|z_{xx}(t)\big\|_{\mathbf{L}^\infty}\ =\ \mathcal{O}(1)\cdot\delta_0^4\,. \tag{2.15}$$

Proof. It suffices to apply Proposition 2.1 on the interval $[t-\hat t,\ t]$. □

Proposition 2.3. *Let* $u=u(t,x)$, $z=z(t,x)$ *be solutions of (2.2), (2.3) respectively, such that*

$$Tot.Var.\big\{u(0,\cdot)\big\}\leq\frac{\delta_0}{4\kappa}\,,\qquad\qquad \big\|z(0)\big\|_{\mathbf{L}^1}\leq\frac{\delta_0}{4\kappa}\,. \tag{2.16}$$

Then u,z *are well defined on the whole interval* $[0,\hat t]$ *in (2.7), and satisfy*

$$\big\|u_x(t)\big\|_{\mathbf{L}^1}\leq\frac{\delta_0}{2}\,,\qquad\qquad \big\|z(t)\big\|_{\mathbf{L}^1}\leq\frac{\delta_0}{2}\qquad\qquad t\in[0,\,\hat t]\,. \tag{2.17}$$

Proof. We have the identity

$$z(t)=G^*(t)*z(0)+\int_0^t G^*(t-s)*\Big\{\big(A^*-A(u)\big)z_x(s)-\big(z\bullet A(u)\big)u_x(s)\Big\}\,ds\,. \tag{2.18}$$

As before, we first establish the result for $z=u_x$, then for a general solution z of (2.3). Assume that there exists a first time $\tau<\hat t$ where the bound in (2.17) is satisfied as an equality. Estimating the right hand side of (2.18) by means of (2.5) and (2.8), we obtain

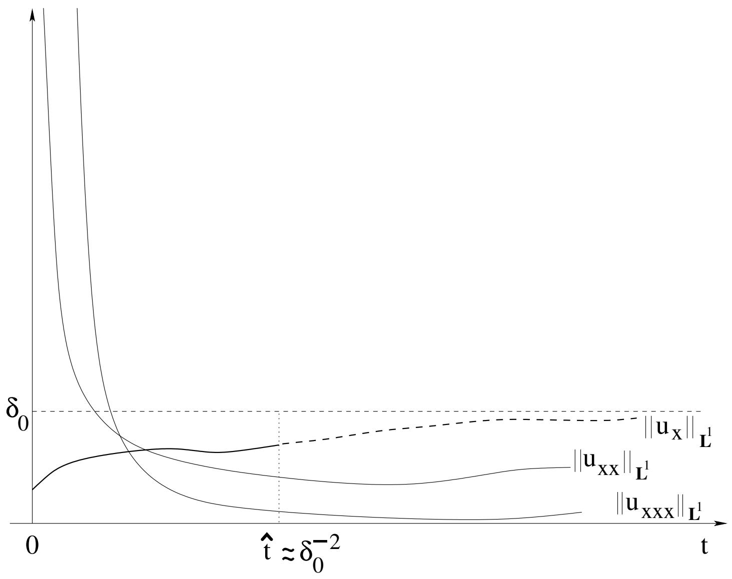
$$\begin{aligned}\big\|z(\tau)\big\|_{\mathbf{L}^1}&\leq\frac{\kappa\delta_0}{4\kappa}+\int_0^\tau\frac{2\kappa\delta_0^2}{\sqrt{s}}\,\|DA\|_{\mathbf{L}^\infty}\,ds\\&\leq\frac{\delta_0}{4}+4\kappa\,\kappa_A\delta_0^2\sqrt{\tau}\ <\ \frac{\delta_0}{2}\,,\end{aligned}$$

reaching a contradiction. □

To simplify the proofs, in all previous results we used the same hypotheses on the functions u_x and z. However, observing that z solves a linear homogeneous equation, similar estimates can be immediately derived without any restriction on the initial size $\big\|z(0)\big\|_{\mathbf{L}^1}$. In particular, from Proposition 2.3 it follows

Corollary 2.4. *Let* $u=u(t,x)$, $z=z(t,x)$ *be solutions of (2.2), (2.3) respectively, such that* $\big\|u_x(0)\big\|_{\mathbf{L}^1}\leq\delta_0/4\kappa$. *Then* u,z *are well defined on the whole interval* $[0,\hat t]$ *in (2.7), and satisfy*

$$\big\|u_x(t)\big\|_{\mathbf{L}^1}\leq2\kappa\big\|u_x(0)\big\|_{\mathbf{L}^1}\,,\qquad\qquad \big\|z(t)\big\|_{\mathbf{L}^1}\leq2\kappa\big\|z(0)\big\|_{\mathbf{L}^1}\,,\qquad\qquad t\in[0,\,\hat t]\,. \tag{2.19}$$



31.eps figure 1

A summary of the main estimates is illustrated in fig. 1. On the initial interval $t \in [0, \hat{t}]$, with $\hat{t} \approx 1/\delta_0^2$ we have

$$\big\|u_x(t)\big\|_{\mathbf{L}^1} \leq \delta_0, \tag{2.20}$$

while the norms of the higher derivatives decay:

$$\big\|u_{xx}\big\|_{\mathbf{L}^1} = \mathcal{O}(1) \cdot \delta_0/\sqrt{t}\,, \qquad \big\|u_{xxx}\big\|_{\mathbf{L}^1} = \mathcal{O}(1) \cdot \delta_0/t\,.$$

On the other hand, for $t \geq \hat{t}$, as long as (2.20) remains valid we also have

$$\big\|u_{xx}\big\|_{\mathbf{L}^1} = \mathcal{O}(1) \cdot \delta_0^2\,, \qquad \big\|u_{xxx}\big\|_{\mathbf{L}^1} = \mathcal{O}(1) \cdot \delta_0^3\,.$$

These bounds (the solid lines in fig. 1) were obtained in the present section by standard parabolic-type estimates. The most difficult part of the proof is to obtain the estimate (2.20) for large times $t \in [\hat{t}, \infty[$ (the broken line in fig. 1). This will require hyperbolic-type estimates, based on the local decomposition of the gradient u_x as a sum of travelling waves, and on a careful analysis of all interaction terms.

3 - Outline of the BV estimates

It is our aim to derive global a priori bounds on the total variation of solutions of

$$u_t + A(u)u_x = u_{xx} \tag{3.1}$$

for small initial data. We always assume that the system is strictly hyperbolic, so that each matrix $A(u)$ has real distinct eigenvalues $\lambda_i(u)$ as in (1.3), and dual bases of right and left eigenvectors

$r_i(u)$, $l_i(u)$ normalized as in (1.4). The directional derivative of a function $\phi = \phi(u)$ in the direction of the vector $\mathbf{v}$ is written

$$\mathbf{v} \bullet \phi(u) \doteq D\phi \cdot \mathbf{v} = \lim_{\epsilon \to 0} \frac{\phi(u + \epsilon \mathbf{v}) - \phi(u)}{\epsilon}, \tag{3.2}$$

while

$$[r_j, r_k] \doteq r_j \bullet r_k - r_k \bullet r_j$$

denotes a Lie bracket. In order to obtain uniform bounds on Tot.Var.$\{u(t,\cdot)\}$ for all $t > 0$, our basic strategy is as follows. We choose $\delta_0 > 0$ sufficiently small and consider an initial data $u(0,\cdot) = \bar{u}$ satisfying the first inequality in (2.16). By Proposition 2.3, the corresponding solution is well defined on the initial time interval $[0, \hat{t}]$ and its total variation remains bounded, according to (2.17). The main task is to establish BV estimates on the remaining interval $[\hat{t}, \infty[$. For this purpose, we decompose the gradient u_x along a suitable basis of unit vectors $\tilde{r}_1, \ldots, \tilde{r}_n$, say

$$u_x = \sum_{i=1}^{n} v_i \tilde{r}_i \,. \tag{3.3}$$

Differentiating (3.1), we obtain a system of n evolution equations for these scalar components

$$v_{i,t} + (\tilde{\lambda}_i v_i)_x - v_{i,xx} = \phi_i \qquad\qquad i = 1, \ldots, n\,. \tag{3.4}$$

Since the left hand side is in conservation form, (3.4) implies

$$\big\| v_i(t,\cdot)\big\|_{\mathbf{L}^1} \le \big\| v_i(\hat{t},\cdot)\big\|_{\mathbf{L}^1} + \int_{\hat{t}}^{\infty} \int \big|\phi_i(t,x)\big|\, dx dt \tag{3.5}$$

for all $t \ge \hat{t}$. By (3.3) it follows

$$\text{Tot.Var.}\big\{u(t,\cdot)\big\} = \big\|u_x(t,\cdot)\big\|_{\mathbf{L}^1} \le \sum_i \big\|v_i(t,\cdot)\big\|_{\mathbf{L}^1} \,. \tag{3.6}$$

In order to obtain a uniform bound on the total variation, the key step is thus to construct the basis of unit vectors $\{\tilde{r}_1, \ldots, \tilde{r}_n\}$ in (3.3) in a clever way, so that the functions ϕ_i on the right hand side of (3.4) become integrable on the half plane $\{t > \hat{t},\ x \in I\!R\}$.

As a preliminary, we observe that the choice $\tilde{r}_i \doteq r_i(u)$, the i-th eigenvector of the matrix $A(u)$, seems quite natural. This choice was indeed adopted in [BiB1], where the authors proved Theorem 1 restricted to the special class of systems where all Rankine-Hugoniot curves are straight lines. Unfortunately, for general $n \times n$ hyperbolic systems it does not work. To understand why, let us write

$$u_x^i \doteq l_i(u) \cdot u_x \tag{3.7}$$

for the i-th component of u_x in this basis of eigenvectors. As shown in [BiB1], these components satisfy the system of evolution equations

$$\begin{aligned}(u_x^i)_t + &(\lambda_i u_x^i)_x - (u_x^i)_{xx} \\ &= l_i \cdot \left\{ \sum_{j \ne k} \lambda_j [r_j, r_k] u_x^j u_x^k + 2 \sum_{j,k} (r_k \bullet r_j)(u_x^j)_x u_x^k + \sum_{j,k,\ell} [r_\ell,\ r_k \bullet r_j] u_x^j u_x^k u_x^\ell \right\} \\ &\doteq \phi_i \,.\end{aligned} \tag{3.8}$$

Assume that the i-th characteristic field is genuinely nonlinear, with shock and rarefaction curves not coinciding, and consider a travelling wave solution $u(t,x) = U(x - \lambda t)$, representing a viscous i-shock. It is then easy to see that the right hand side of (3.8) is not identically zero. Since it corresponds to a travelling wave, the integral

$$\int \big|\phi_i(t,x)\big|\, dx \neq 0$$

is constant in time. Hence ϕ_i is certainly not integrable over the half plane $\{t > \hat{t},\ x \in \mathbb{R}\}$.

The previous example clearly points out a basic requirement for our decomposition (3.3). Namely, in connection with a viscous travelling wave, the source terms ϕ_i in (3.4) should vanish identically. To achieve this goal, we shall seek a decomposition of u_x not along eigenvectors of the matrix $A(u)$, but as a **sum of gradients of viscous travelling waves.** More precisely, consider a smooth function $u : \mathbb{R} \mapsto \mathbb{R}^n$. At each point x, depending on the second order jet (u, u_x, u_{xx}), we shall uniquely determine n travelling waves $U_1, \ldots, U_n$ passing through $u(x)$. We then write u_x in the form (3.3), as the sum of the gradients of these waves. As a guideline, we shall try to achieve the following relations:

$$U_i(x) = u(x) \qquad\qquad i = 1, \ldots, n, \tag{3.9}$$

$$\sum_i U_i'(x) = u_x(x)\,, \qquad\qquad \sum_i U_i''(x) = u_{xx}(x)\,. \tag{3.10}$$

Details of this construction will be worked out in the next two sections.

4 - A center manifold of viscous travelling waves

To carry out our program, we must first select certain families of travelling waves, depending on the correct number of parameters to fit the data. Given a state $u \in \mathbb{R}^n$, a second order jet $(u_x,\, u_{xx})$ determines $2n$ scalar parameters. In order to uniquely satisfy the equations (3.10), we thus need to construct n families of travelling wave profiles through u, each depending on two scalar parameters. This will be achieved by an application of the center manifold theorem.

Travelling waves for the viscous hyperbolic system (3.1) correspond to (possibly unbounded) solutions of

$$\big(A(U) - \sigma\big)U' = U''. \tag{4.1}$$

We write (4.1) as a first order system on the space $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$:

$$\begin{cases} \dot{u} = v\,, \\ \dot{v} = \big(A(u) - \sigma\big)v\,, \\ \dot{\sigma} = 0\,. \end{cases} \tag{4.2}$$

Let a state u^* be given and fix an index $i \in \{1, \ldots, n\}$. Linearizing (4.2) at the equilibrium point $P^* \doteq \big(u^*,\, 0,\, \lambda_i(u^*)\big)$ we obtain the linear system

$$\begin{cases} \dot{u} = v\,, \\ \dot{v} = \big(A(u^*) - \lambda_i(u^*)\big)v\,, \\ \dot{\sigma} = 0\,. \end{cases} \tag{4.3}$$

Let $\{r_1^*, \ldots, r_n^*\}$ and $\{l_1^*, \ldots, l_n^*\}$ be dual bases of right and left eigenvectors of $A(u^*)$ normalized as in (1.4). We call $(V_1, \ldots, V_n)$ the coordinates of a vector $v \in I\!R^n$ w.r.t. this basis, so that

$$|r_i^*| = 1, \qquad v = \sum_j V_j r_j^*, \qquad V_j \doteq l_j^* \cdot v\,.$$

The center subspace $\mathcal{N}$ for (4.3) consists of all vectors $(u, v, \sigma) \in I\!R^n \times I\!R^n \times I\!R$ such that

$$V_j = 0 \qquad \text{for all } \ j \neq i, \tag{4.4}$$

and therefore has dimension $n+2$. By the center manifold theorem [V], there exists a smooth manifold $\mathcal{M} \subset I\!R^{n+n+1}$, tangent to $\mathcal{N}$ at the stationary point P^*, which is locally invariant under the flow of (4.2). This manifold has dimension $n+2$ and can be locally defined by the $n-1$ equations

$$V_j = \varphi_j(u, V_i, \sigma) \qquad j \neq i. \tag{4.5}$$

We can assume that the $n-1$ smooth scalar functions φ_j are defined on the domain

$$\mathcal{D} \doteq \Big\{ |u - u^*| < \epsilon, \qquad |V_i| < \epsilon, \qquad \big|\sigma - \lambda_i(u^*)\big| < \epsilon \Big\}.$$

Moreover, the tangency condition implies

$$\varphi_j(u, V_i, \sigma) = \mathcal{O}(1) \cdot \Big(|u - u^*|^2 + |V_i|^2 + \big|\sigma - \lambda_i(u^*)\big|^2 \Big). \tag{4.6}$$

We now take a closer look at the flow on this center manifold. By construction, every trajectory

$$t \mapsto P(t) \doteq \big(u(t),\, v(t),\, \sigma(t)\big)$$

of (4.2), which remains within a small neighborhood of the point $P^* \doteq \big(u^*, 0, \lambda_i(u^*)\big)$ for all $t \in I\!R$, must lie entirely on the manifold $\mathcal{M}$. In particular, $\mathcal{M}$ contains all viscous i-shock profiles joining a pair of states u^-, u^+ sufficiently close to u^*. Moreover, all equilibrium points $(u, 0, \sigma)$ with $|u - u^*| < \epsilon$ and $\big|\sigma - \lambda_i(u^*)\big| < \epsilon$ must lie on $\mathcal{M}$. Hence

$$\varphi_j(u, 0, \sigma) = 0 \qquad \text{for all } \ j \neq i. \tag{4.7}$$

By (4.7) and the smoothness of the functions φ_j, we can "factor out" the component V_i and write

$$\varphi_j(u, V_i, \sigma) = \psi_j(u, V_i, \sigma) \cdot V_i,$$

for suitable smooth functions ψ_j. From (4.6) it follows

$$\psi_j \to 0 \qquad \text{as} \qquad (u, V_i, \sigma) \to \big(u^*, 0, \lambda_i(u^*)\big). \tag{4.8}$$

On the manifold $\mathcal{M}$ we thus have

$$v = \sum_k V_k r_k^* = V_i \cdot \left(r_i^* + \sum_{j \neq i} \psi_j(u, V_i, \sigma)\, r_j^* \right) \doteq V_i\, r_i^\sharp(u, V_i, \sigma). \tag{4.9}$$

By (4.8), the function $r^\sharp$ defined by the last equality in (4.9) satisfies

$$r_i^\sharp(u, V_i, \sigma) \to r_i^* \qquad \text{as} \qquad (u, V_i, \sigma) \to (u^*, 0, \lambda_i(u^*)). \tag{4.10}$$

Remark 4.1. Trajectories on the center manifold correspond to the profiles of viscous travelling i-waves. We thus expect that the derivative $\dot{u} = v$ should be a vector "almost parallel" to the eigenvector $r_i^* \doteq r_i(u^*)$. This is indeed confirmed by (4.10).

We can now define the new variable

$$v_i = v_i(u, V_i, \sigma) \doteq V_i \cdot \big| r_i^\sharp(u, V_i, \sigma) \big|. \tag{4.11}$$

As (u, V_i, σ) range in a small neighborhood of $(u^*, 0, \lambda_i(u^*))$, by (4.10) the vector $r_i^\sharp$ remains close to the eigenvector r_i^*. In particular, its norm remains uniformly positive. Therefore, the transformation $V_i \longleftrightarrow v_i$ is invertible and smooth. We can thus reparametrize the center manifold $\mathcal{M}$ in terms of the variables $(u, v_i, \sigma) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$. Moreover, we define the unit vector

$$\tilde{r}_i(u, v_i, \sigma) \doteq \frac{r_i^\sharp}{|r_i^\sharp|}. \tag{4.12}$$

Observe that $\tilde{r}_i$ is also a smooth function of its arguments. With the above definitions, instead of (4.5) we can write the manifold $\mathcal{M}$ in terms of the equation

$$v = v_i \tilde{r}_i. \tag{4.13}$$

The above construction of a center manifold can be repeated for every $i = 1, \ldots, n$. We thus obtain n center manifolds $\mathcal{M}_i \subset \mathbb{R}^{2n+1}$ and vector functions $\tilde{r}_i = \tilde{r}_i(u, v_i, \sigma_i)$ such that

$$|\tilde{r}_i| \equiv 1, \tag{4.14}$$

$$\mathcal{M}_i = \big\{ (u, v, \sigma_i) \; ; \quad v = v_i \, \tilde{r}_i(u, v_i, \sigma_i) \big\}, \tag{4.15}$$

as $(u, v_i, \sigma_i) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ ranges in a neighborhood of $\big(u^*, \, 0, \, \lambda_i(u^*)\big)$.

We derive here some useful identities, for later use. The partial derivatives of $\tilde{r}_i = \tilde{r}_i(u, v_i, \sigma_i)$ w.r.t. its arguments will be written as

$$\tilde{r}_{i,u} \doteq \frac{\partial}{\partial u} \tilde{r}_i, \qquad \tilde{r}_{i,v} \doteq \frac{\partial}{\partial v_i} \tilde{r}_i, \qquad \tilde{r}_{i,\sigma} \doteq \frac{\partial}{\partial \sigma_i} \tilde{r}_i.$$

Clearly, $\tilde{r}_{i,u}$ is an $n \times n$ matrix, while $\tilde{r}_{i,v}$, $\tilde{r}_{i,\sigma}$ are n-vectors. Higher order derivatives are denoted as $\tilde{r}_{i,u\sigma}$, $\tilde{r}_{i,\sigma\sigma} \ldots$ We claim that

$$\tilde{r}_i(u, 0, \sigma_i) = r_i(u) \qquad \text{for all} \;\; u, \sigma_i. \tag{4.16}$$

Indeed, consider again the equations for a viscous travelling i-wave:

$$u_{xx} = \big(A(u) - \sigma_i\big) u_x. \tag{4.17}$$

For a solution contained in the center manifold, taking the derivative w.r.t. x of

$$u_x = v = v_i \tilde r_i(u, v_i, \sigma_i) \tag{4.18}$$

and using (4.17) we obtain

$$v_{i,x}\tilde r_i + v_i \tilde r_{i,x} = \big(A(u) - \sigma_i\big) v_i \tilde r_i. \tag{4.19}$$

Since $|\tilde r_i| \equiv 1$, the vector $\tilde r_i$ is perpendicular to its derivative $\tilde r_{i,x}$. Taking the inner product of (4.19) with $\tilde r_i$ we thus obtain

$$v_{i,x} = (\tilde\lambda_i - \sigma_i) v_i\,, \tag{4.20}$$

where we defined the speed $\tilde\lambda_i = \tilde\lambda_i(u, v_i, \sigma_i)$ as the inner product

$$\tilde\lambda_i \doteq \big\langle \tilde r_i\,,\ A(u)\tilde r_i\big\rangle\,. \tag{4.21}$$

Using (4.20) in (4.19) and dividing by v_i we finally obtain

$$(\tilde\lambda_i - \sigma_i) v_i \tilde r_i + v_i\big(\tilde r_{i,u}\tilde r_i v_i + \tilde r_{i,v}(\tilde\lambda_i - \sigma_i) v_i\big) = \big(A(u) - \sigma_i\big) v_i \tilde r_i\,, \tag{4.22}$$

$$v_i\big(\tilde r_{i,u}\tilde r_i + \tilde r_{i,v}(\tilde\lambda_i - \sigma_i)\big) = \big(A(u) - \tilde\lambda_i\big)\tilde r_i\,. \tag{4.23}$$

By (4.23), as $v_i \to 0$, the unit vector $\tilde r_i(u, v_i, \sigma_i)$ approaches an eigenvector of the matrix $A(u)$, while $\tilde\lambda_i$ approaches the corresponding eigenvalue. By continuity, this establishes (4.16).

In turn, by the smoothness of the vector field $\tilde r_i$ we also have

$$\begin{aligned} \tilde r_i(u, v_i, \sigma_i) - r_i(u) &= \mathcal{O}(1)\cdot v_i\,, & \tilde r_{i,\sigma} &= \mathcal{O}(1)\cdot v_i\,,\\ \tilde r_{i,u\sigma} &= \mathcal{O}(1)\cdot v_i\,, & \tilde r_{i,\sigma\sigma} &= \mathcal{O}(1)\cdot v_i\,, \end{aligned} \tag{4.24}$$

Using (4.24), from (4.21) one obtains

$$\big|\tilde\lambda_i(u, v_i, \sigma_i) - \lambda_i(u)\big| = \mathcal{O}(1)\cdot v_i\,, \qquad \tilde\lambda_{i,\sigma} = \mathcal{O}(1)\cdot v_i\,. \tag{4.25}$$

A further identity will be of use. Differentiating (4.19) one finds

$$v_{i,xx}\tilde r_i + 2 v_{i,x}\tilde r_{i,x} + v_i \tilde r_{i,xx} = \big(A(u) v_i \tilde r_i\big)_x - \sigma_i v_{i,x}\tilde r_i - \sigma_i v_i \tilde r_{i,x}\,. \tag{4.26}$$

From the identities

$$\big\langle \tilde r_i,\, \tilde r_{i,x}\big\rangle = 0\,, \qquad \big\langle \tilde r_i,\, \tilde r_{i,xx}\big\rangle = -\big\langle \tilde r_{i,x},\, \tilde r_{i,x}\big\rangle,$$

taking the inner product of (4.19) with $\tilde r_{i,x}$ we obtain

$$\big\langle \tilde r_i,\, \tilde r_{i,xx}\big\rangle v_i = -\big\langle \tilde r_{i,x},\, A(u)\tilde r_i\big\rangle v_i\,. \tag{4.27}$$

Taking now the inner product of (4.26) with $\tilde r_i$ we find

$$v_{i,xx} + \big\langle \tilde r_i, \tilde r_{i,xx}\big\rangle v_i = \big\langle \tilde r_i,\ (A(u)\tilde r_i v_i)_x\big\rangle - \sigma_i v_{i,x}\,.$$

Since $v_{i,t} + \sigma_i v_{i,x} = 0$, using the identity (4.27) we conclude

$$v_{i,t} + (\tilde\lambda_i v_i)_x - v_{i,xx} = 0, \tag{4.28}$$

where $\tilde\lambda_i$ is the speed at (4.21).

Remark 4.2. It is important to appreciate the difference between the identities

$$\big(A(u) - \lambda_i\big) r_i = 0\,, \qquad \big(A(u) - \tilde\lambda_i\big)\tilde r_i = v_i\big(\tilde r_{i,u}\tilde r_i + \tilde r_{i,v}(\tilde\lambda_i - \sigma_i)\big), \tag{4.29}$$

satisfied respectively by an eigenvector r_i and by a unit vector $\tilde r_i$ parallel to the gradient of a travelling wave. Decomposing u_x along the eigenvectors r_i one obtains the evolution equations (3.8), with non-integrable source terms on the right hand side. When a similar computation is performed in connection with the vectors $\tilde r_i$, thanks to the presence of the additional terms on the right hand side in (4.29) a crucial cancellation is achieved. In this case, we will show that the source terms ϕ_i in (3.4) are integrable over the half plane $x \in I\!R$, $t > \hat t$.

5 - Gradient decomposition

Let $u : \mathbb{R} \mapsto \mathbb{R}^n$ be a smooth function with small total variation. At each point x, we seek a decomposition of the gradient u_x in the form (3.3), where $\tilde r_i = \tilde r_i(u, v_i, \sigma_i)$ are the vectors defining the center manifold in (4.15). To uniquely determine the $\tilde r_i$, we should first define the wave strengths v_i and speeds σ_i in terms of u, u_x, u_{xx}.

Consider first the special case where u is precisely the profile of a viscous travelling wave of the j-th family (contained in the center manifold $\mathcal{M}_j$). In this case, our decomposition should clearly contain one single component:

$$u_x = v_j \tilde r_j(u, v_j, \sigma_j)\,. \tag{5.1}$$

It is easy to guess what v_j, σ_j in (5.1) should be. Indeed, since by construction $|\tilde r_j| = 1$, the quantity

$$v_j = \pm |u_x|$$

is the signed strength of the wave. Notice also that for a travelling wave the vectors u_x and u_t are always parallel, since $u_t = -\sigma_j u_x$ where σ_j is the speed of the wave. We can thus write

$$u_t = u_{xx} - A(u) u_x = \omega_j \tilde r_j(u, v_j, \sigma_j) \tag{5.2}$$

for some scalar ω_j. The speed of the wave is now obtained as $\sigma_j = -\omega_j / v_j$.

Motivated by the previous analysis, as a first attempt we define

$$u_t = u_{xx} - A(u) u_x \tag{5.3}$$

and try to find scalar quantities v_i, ω_i such that

$$\begin{cases} u_x = \sum_i v_i \,\tilde r_i(u, v_i, \sigma_i), \\ u_t = \sum_i \omega_i \,\tilde r_i(u, v_i, \sigma_i), \end{cases} \qquad \sigma_i = -\frac{\omega_i}{v_i}\,. \tag{5.4}$$

The trouble with (5.4) is that the vectors $\tilde r_i$ are defined only for speeds σ_i close to the i-th characteristic speed $\lambda_i^* \doteq \lambda_i(u^*)$. However, when $u_x \approx 0$ one has $v_i \approx 0$ and the ratio ω_i / v_i may become arbitrarily large.

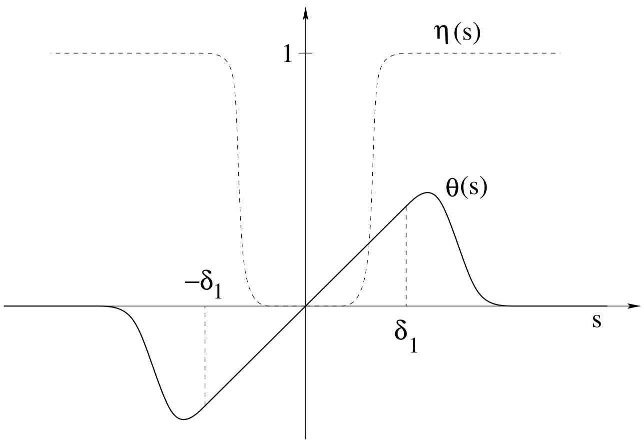


figure 2

To overcome this problem, we introduce a cutoff function (fig. 2). Fix $\delta_1 \in]0,\, 1/3]$ sufficiently small. Define a smooth odd function $\theta : I\!R \mapsto [-2\delta_1,\, 2\delta_1]$ such that

$$\theta(s) = \begin{cases} s & \text{if} \quad |s| \leq \delta_1 \\ 0 & \text{if} \quad |s| \geq 3\delta_1 \end{cases} \qquad |\theta'| \leq 1, \qquad |\theta''| \leq 4/\delta_1 \,. \tag{5.5}$$

We now rewrite (5.4) in terms of the new variable w_i, related to ω_i by $\omega_i \doteq w_i - \lambda_i^* v_i$. We require that σ_i coincides with $-\omega_i/v_i$ only when this ratio is sufficiently close to $\lambda_i^* \doteq \lambda_i(u^*)$. Our basic equations thus take the form

$$\begin{cases} u_x = \displaystyle\sum_i v_i\, \tilde r_i(u, v_i, \sigma_i), \\ u_t = \displaystyle\sum_i (w_i - \lambda_i^* v_i)\, \tilde r_i(u, v_i, \sigma_i), \end{cases} \tag{5.6}$$

where

$$u_t = u_{xx} - A(u)u_x \,, \qquad \sigma_i = \lambda_i^* - \theta\left(\frac{w_i}{v_i}\right) \,. \tag{5.7}$$

Notice that σ_i is not well defined when $v_i = w_i = 0$. However, recalling (4.16), in this case we have $\tilde r_i = r_i(u)$, regardless of σ_i. Hence the two equations in (5.6) are still meaningful.

Remark 5.1. The decomposition (5.6) corresponds to viscous travelling waves U_i such that

$$U_i(x) = u(x), \qquad U_i'(x) = v_i \tilde r_i \,, \qquad U_i'' = \big(A(u) - \sigma_i\big)U_i' \,.$$

From the first equation in (5.6) it follows

$$u_x(x) = \sum_i U_i'(x) \,.$$

If $\sigma_i = \lambda_i^* - w_i/v_i$ for all $i = 1, \ldots, n$, i.e. if none of the cutoff functions is active, then

$$\begin{aligned} u_{xx}(x) &= u_t + A(u)u_x \\ &= \sum_i (w_i - \lambda_i^* v_i)\tilde r_i + A(u) \sum_i v_i \tilde r_i \\ &= \sum_i \big(A(u) - \sigma_i\big) v_i \tilde r_i \\ &= \sum_i U_i''(x) \,. \end{aligned}$$

In this case, both of the equalities in (3.10) hold. Notice however that the second equality in (3.10) may fail if $|w_i/v_i| > \delta_1$ for some i.

Lemma 5.2. *For $|u - u^*|$, $|u_x|$ and $|u_{xx}|$ sufficiently small, the system of $2n$ equations (5.6) has a unique solution $(v, w) = (v_1, \ldots, v_n,\, w_1, \ldots, w_n)$. The map $(u, u_x, u_{xx}) \mapsto (v, w)$ is smooth outside the n manifolds $\mathcal{N}_i \doteq \{v_i = w_i = 0\}$; moreover it is $\mathcal{C}^{1,1}$, i.e. continuously differentiable with Lipschitz continuous derivatives on a whole neighborhood of the point $(u^*, 0, 0)$.*

Proof. Given (v,w) in a neighborhood of $(0,0)\in I\!R^{2n}$, the vectors u_x, u_t are uniquely determined. Hence the solution of (5.6)-(5.7) is certainly unique. To prove its existence, consider the mapping $\Lambda : I\!R^n\times I\!R^n\times I\!R^n\mapsto I\!R^{2n}$ defined by

$$\Lambda(u,v,w)\doteq\sum_{i=1}^n \Lambda_i(u,v_i,w_i), \tag{5.8}$$

$$\Lambda_i(u,v_i,w_i)\doteq\begin{pmatrix} v_i\,\tilde r_i\big(u,\ v_i,\ \lambda_i^*-\theta(w_i/v_i)\big)\\ (w_i-\lambda_i^*v_i)\,\tilde r_i\big(u,\ v_i,\ \lambda_i^*-\theta(w_i/v_i)\big)\end{pmatrix}. \tag{5.9}$$

This map is well defined and continuous also when $v_i=0$, because in this case (4.16) implies $\tilde r_i=r_i(u)$. Computing the Jacobian matrix of partial derivatives w.r.t. (v_i,w_i) we find

$$\begin{aligned}\frac{\partial\Lambda_i}{\partial(v_i,w_i)}&=\begin{pmatrix}\tilde r_i & 0\\ -\lambda_i^*\tilde r_i & \tilde r_i\end{pmatrix}\\ &+\begin{pmatrix} v_i\tilde r_{i,v}+(w_i/v_i)\theta_i'\tilde r_{i,\sigma} & -\theta_i'\tilde r_{i,\sigma}\\ w_i\tilde r_{i,v}-\lambda_i^*v_i\tilde r_{i,v}-\lambda_i^*(w_i/v_i)\theta_i'\tilde r_{i,\sigma}+(w_i/v_i)^2\theta_i'\tilde r_{i,\sigma} & \lambda_i^*\theta_i'\tilde r_{i,\sigma}-(w_i/v_i)\theta_i'\tilde r_{i,\sigma}\end{pmatrix}.\end{aligned} \tag{5.10}$$

Here and throughout the following, by θ_i, θ_i' we denote the function θ and its derivative, evaluated at the point $s=w_i/v_i$. By (5.10) we can write

$$\frac{\partial\Lambda}{\partial(v,w)}=B_0(u,v,w)+B_1(u,v,w), \tag{5.11}$$

Because of (4.24), the matrix functions B_0, B_1 are well defined and continuous also when $v_i=0$. Moreover, for (v,w) small, B_0 has a uniformly bounded inverse and $B_1\to 0$ as $(v,w)\to 0$. Since $\Lambda(u,0,0)=(0,0)\in I\!R^{2n}$, we conclude that the map $(v,w)\mapsto\Lambda(u;\,v,w)$ is $\mathcal{C}^1$ and invertible in a neighborhood of the origin. Therefore, given (u,u_x,u_{xx}), there exist unique values of (v,w) such that

$$\Lambda(u,v,w)=\big(u_x,\ u_{xx}-A(u)u_x\big). \tag{5.12}$$

The inverse of the map Λ w.r.t. the variables v,w will be denoted by $\Lambda^{-1}(u;\,p,q)$. In other words,

$$\Lambda^{-1}(u;\,p,q)=(v,w)\qquad\text{iff}\qquad\Lambda(u;\,v,w)=(p,q)\,.$$

Since $\tilde r_i(u,0,\sigma_i)=r_i(u)$, we have

$$\Lambda(u,0,w)=\Big(\,0\,,\ \sum_i w_ir_i(u)\Big).$$

Therefore,

$$\Lambda^{-1}(u,0,q)=(0,w)\qquad\text{where}\qquad w_i=l_i(u)\cdot q\,.$$

In particular, $\Lambda^{-1}(u,0,0)=(0,0)\in I\!R^{2n}$. Concerning first derivatives (which we regard here as linear operators), we have

$$\frac{\partial\Lambda(u;0,w)}{\partial(v,w)}\cdot(\hat v,\hat w)=B_0(u;0,w)\cdot(\hat v,\hat w)=\left(\sum_i\hat v_ir_i(u)\ ,\ \sum_i\big(\hat w_i-\lambda_i^*\hat v_i\big)r_i(u)\right), \tag{5.13}$$

$$\frac{\partial \Lambda^{-1}(u;0,q)}{\partial(p,q)} \cdot (\hat p, \hat q) = (\hat v, \hat w) \qquad \text{where} \qquad \hat v_i = l_i(u)\cdot \hat p\,, \qquad \hat w_i = l_i(u)\cdot \hat q + \lambda_i^* \hat v_i\,. \tag{5.14}$$

We shall not compute the second derivatives explicitly. However, one easily checks that

$$\frac{\partial^2 \Lambda}{\partial v_i \partial v_j} = \frac{\partial^2 \Lambda}{\partial v_i \partial w_j} = \frac{\partial^2 \Lambda}{\partial w_i \partial w_j} = 0 \qquad \text{if} \quad i \neq j\,. \tag{5.15}$$

Moreover, recalling (4.24) and (5.5), we have the estimate

$$\frac{\partial^2 \Lambda}{\partial v_i^2}\,,\ \frac{\partial^2 \Lambda}{\partial v_i \partial w_i}\,,\ \frac{\partial^2 \Lambda}{\partial w_i^2} \ =\ \mathcal{O}(1)\cdot \frac{1}{\delta_1}\,. \tag{5.16}$$

Since the cutoff function θ vanishes for $|s| \geq 3\delta_1$, it is clear that each Λ_i is smooth outside the manifold $\mathcal{N}_i \doteq \big\{(v,w)\,;\ v_i = w_i = 0\big\}$, having codimension 2. Since all second derivatives are uniformly bounded outside the n manifolds $\mathcal{N}_i$, we conclude that Λ is continuously differentiable with Lipschitz continuous first derivatives on a whole neighborhood of the point $(u^*, 0, 0)$. Hence the same holds for Λ^{-1}. □

Remark 5.3. By possibly performing a linear transformation of variables, we can assume that the matrix $A(u^*)$ is diagonal, hence its eigenvectors $r_1^*, \ldots r_n^*$ form an orthonormal basis:

$$\big\langle r_i^*,\, r_j^*\big\rangle = \delta_{ij}\,. \tag{5.17}$$

Observing that

$$\big|\tilde r_i(u, v_i, \sigma_i) - r_i^*\big| = \mathcal{O}(1)\cdot\big(|u-u^*| + |v_i|\big), \tag{5.18}$$

from (4.16) and the above assumption we deduce

$$\begin{aligned}\big\langle \tilde r_i(u, v_i, \sigma_i),\, \tilde r_j(u, v_j, \sigma_j)\big\rangle &= \delta_{ij} + \mathcal{O}(1)\cdot\big(|u-u^*| + |v_i| + |v_j|\big)\\ &= \delta_{ij} + \mathcal{O}(1)\cdot\delta_0\,,\end{aligned} \tag{5.19}$$

$$\big\langle \tilde r_i,\, \tilde r_j\big\rangle = \mathcal{O}(1)\cdot\delta_0\,, \qquad \big\langle \tilde r_i,\, A(u)\tilde r_j\big\rangle = \mathcal{O}(1)\cdot\delta_0 \qquad \text{for} \quad j \neq i\,. \tag{5.20}$$

Another useful consequence of (5.17)-(5.18) is the following. Choosing $\delta_0 > 0$ small enough, the decomposition (5.6) will satisfy

$$|u_x| \leq \sum_i |v_i| \leq 2\sqrt{n}\,|u_x|\,. \tag{5.21}$$

We conclude this section by deriving estimates corresponding to (2.13)-(2.15), valid for the components v_i, w_i. In the following, given a solution $u = u(t,x)$ of (3.1) with small total variation, we consider the decomposition (5.6) of u_x in terms of gradients of travelling waves. It is understood that the vectors $\tilde r_i$ are constructed as in Section 4, taking $P_i^* \doteq \big(u^*, 0, \lambda_i(u^*)\big)$ as basic points in the construction of the center manifolds $\mathcal{M}_i$. Here $u^* \doteq u(t, -\infty)$ is the constant state in (2.1).

Lemma 5.4. *In the same setting as Proposition 2.1, assume that the bounds (2.6) hold on a larger interval $[0,T]$. Then for all $t \in [\hat t, T]$, the decomposition (5.6) is well defined. The components v_i, w_i satisfy the estimates*

$$\big\|v_i(t)\big\|_{\mathbf{L}^1},\ \big\|w_i(t)\big\|_{\mathbf{L}^1} \ =\ \mathcal{O}(1)\cdot\delta_0\,, \tag{5.22}$$

$$\big\|v_i(t)\big\|_{\mathbf{L}^\infty},\ \big\|w_i(t)\big\|_{\mathbf{L}^\infty},\big\|v_{i,x}(t)\big\|_{\mathbf{L}^1},\ \big\|w_{i,x}(t)\big\|_{\mathbf{L}^1} \ =\ \mathcal{O}(1)\cdot\delta_0^2\,, \tag{5.23}$$

$$\big\|v_{i,x}(t)\big\|_{\mathbf{L}^\infty},\ \big\|w_{i,x}(t)\big\|_{\mathbf{L}^\infty} \ =\ \mathcal{O}(1)\cdot\delta_0^3\,. \tag{5.24}$$

Proof. By Lemma 5.2, in a neighborhood of the origin the map $(v,w) \mapsto \Lambda(u,v,w)$ in (5.8) is well defined, locally invertible, and continuously differentiable with Lipschitz continuous derivatives. Hence, for $\delta_0 > 0$ suitably small, the $\mathbf{L}^\infty$ bounds in (2.13) and (2.14) guarantee that the decomposition (5.6) is well defined. From the identity (5.12) it now follows

$$v_i, w_i = \mathcal{O}(1) \cdot \big(|u_x| + |u_{xx}|\big).$$

By (2.6) and (2.13)-(2.14) this yields the $\mathbf{L}^1$ bounds in (5.22) and the $\mathbf{L}^\infty$ bounds in (5.23). Differentiating (5.12) w.r.t. x we obtain

$$\frac{\partial \Lambda}{\partial u} u_x + \frac{\partial \Lambda}{\partial(v,w)}(v_x, w_x) = \Big(u_{xx},\ u_{xxx} - A(u)u_{xx} - \big(u_x \bullet A(u)\big)u_x\Big). \tag{5.25}$$

Using the estimate

$$\frac{\partial \Lambda}{\partial u} = \mathcal{O}(1) \cdot \big(|v| + |w|\big),$$

since the derivative $\partial\Lambda/\partial(v,w)$ has bounded inverse, from (5.25) we deduce

$$(v_x, w_x) = \mathcal{O}(1) \cdot \Big(|u_{xx}| + |u_{xxx}| + |u_x|^2 + |u_x|\big(|v| + |w|\big)\Big).$$

This yields the remaining estimates in (5.23) and (5.24). □

6 - Bounds on the source terms

We now consider a smooth solution $u = u(t,x)$ of (3.1) and let v_i, w_i, be the corresponding components in the decomposition (5.6), which are well defined in view of Lemma 5.2. The equations governing the evolution of these $2n$ components can be written in the form

$$\begin{cases} v_{i,t} + (\tilde\lambda_i v_i)_x - v_{i,xx} = \phi_i\,, \\ w_{i,t} + (\tilde\lambda_i w_i)_x - w_{i,xx} = \psi_i\,. \end{cases} \tag{6.1}$$

As in (4.21), we define here the speed $\tilde\lambda_i \doteq \langle \tilde r_i\,,\ A(u)\tilde r_i\rangle$. The source terms ϕ_i, ψ_i can be computed by differentiating (3.1) and using the implicit relations (5.6). However, it is not necessary to carry out in detail all these computations. Indeed, we are interested not in the exact form of these terms, but only in an upper bound for the norms $\|\phi_i\|_{\mathbf{L}^1}$ and $\|\psi_i\|_{\mathbf{L}^1}$.

Before giving these estimates, we provide an intuitive explanation of how the source terms arise. Consider first the special case where u is precisely one of the travelling wave profiles on the center manifold (fig. 3a), say $u(t,x) = U_j(x - \sigma_j t)$. We then have

$$u_x = v_j \tilde r_j\,, \qquad u_t = (w_j - \lambda_j^* v_j)\tilde r_j\,, \qquad v_i = w_i = 0 \quad \text{for } i \neq j\,,$$

and therefore

$$\begin{cases} v_{i,t} + (\tilde\lambda_i v_i)_x - v_{i,xx} = 0\,, \\ w_{i,t} + (\tilde\lambda_i w_i)_x - w_{i,xx} = 0\,. \end{cases} \tag{6.2}$$

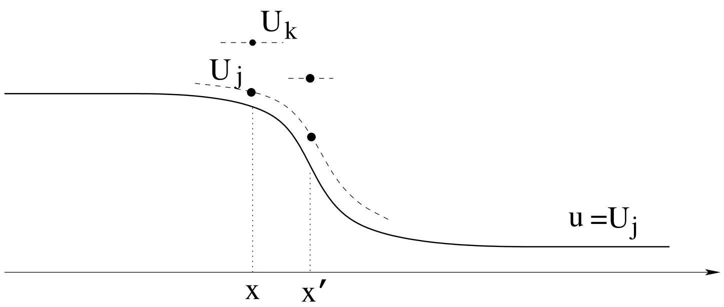


figure 3a

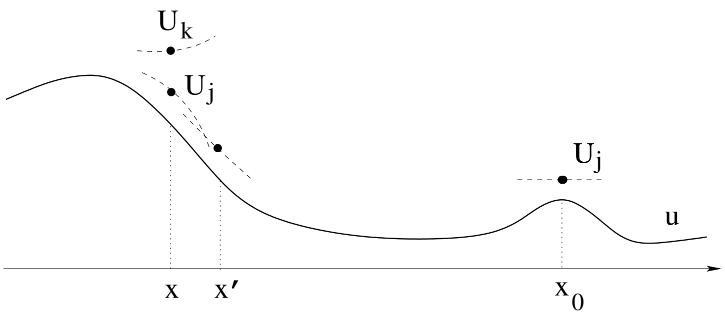


figure 3b

Indeed, this is obvious when $i \neq j$. The identity $\phi_j = 0$ follows from (4.28), while the relation $w_j = (\lambda_j^* - \sigma_j)v_j$ implies $\psi_j = 0$.

Next, consider the case of a general solution $u = u(t, x)$. The sources on the right hand sides of (6.1) arise for three different reasons (fig. 3b).

1. The ratio $|w_j/v_j|$ is large and hence the cutoff function θ in (5.7) is active. Typically, this will happen near a point x_0 where $u_x = 0$ but $u_t = u_{xx} \neq 0$. In this case the identity (4.28) fails because of a "wrong" choice of the speed: $\sigma_j \neq \lambda_j^* - (w_j/v_j)$.

2. Waves of two different families $j \neq k$ are present at a given point x. These will produce quadratic source terms, due to transversal interactions.

3. Since the decomposition (3.10) is defined pointwise, it may well happen that the travelling j-wave profile U_j at a point x is not the same as the profile U_j at a nearby point x'. Indeed, these two travelling waves may have slightly different speeds. It is the rate of change in this speed, i.e. $\sigma_{j,x}$, that determines the infinitesimal interaction between nearby waves of the same family. A detailed analysis will show that the corresponding source terms can only be linear or quadratic w.r.t. $\sigma_{j,x}$, with the square of the strength of the wave always appearing as a factor. These terms can thus be estimated as $\mathcal{O}(1) \cdot v_j^2 \sigma_{j,x} + \mathcal{O}(1) \cdot v_j^2 \sigma_{j,x}^2$.

Lemma 6.1. *The source terms in (6.1) satisfy the bounds*

$$
\begin{aligned}
\phi_i,\ \psi_i = & \,\mathcal{O}(1)\cdot\sum_j \big(|v_{j,x}|+|w_{j,x}|\big)\cdot|w_j-\theta_j v_j| && \textbf{(wrong speed)}\\
&+\mathcal{O}(1)\cdot\sum_j |v_{j,x}w_j - v_j w_{j,x}| && \textbf{(change in speed, linear)}\\
&+\mathcal{O}(1)\cdot\sum_j \left|v_j\left(\frac{w_j}{v_j}\right)_x\right|^2\cdot\chi_{\{|w_j/v_j|<3\delta_1\}} && \textbf{(change in speed, quadratic)}\\
&+\mathcal{O}(1)\cdot\sum_{j\neq k}\big(|v_jv_k|+|v_{j,x}v_k|+|v_jw_k|+|v_{j,x}w_k|+|v_jw_{k,x}|+|w_jw_k|\big)\\
& && \textbf{(interaction of waves of different families)}
\end{aligned}
$$

From a direct inspection of the equations (6.1), it will be clear that the source terms depend only on the third order jet $(u,\,u_x,\,u_{xx},\,u_{xxx})$. Since all functions $\phi_i,\ \psi_i$ vanish in the case of a travelling wave, for a general solution u their size can be estimated in terms of the distance between the third order jet of u and the (nearest) jet of some travelling wave. This is indeed the strategy adopted in the following proof. An alternative proof, based on more direct calculations, will be given in Appendix A.

Proof of Lemma 6.1. The conclusion will be reached in several steps.

1. The vector $(u_x,u_t)=\Lambda(u,v,w)$ satisfies the evolution equation

$$
\begin{pmatrix}u_x\\u_t\end{pmatrix}_t+\left(\begin{bmatrix}A(u)&0\\0&A(u)\end{bmatrix}\begin{pmatrix}u_x\\u_t\end{pmatrix}\right)_x-\begin{pmatrix}u_x\\u_t\end{pmatrix}_{xx}=\begin{pmatrix}0\\ \big(u_x\bullet A(u)\big)u_t-\big(u_t\bullet A(u)\big)u_x\end{pmatrix}. \tag{6.3}
$$

Observe that, in the conservative case $A(u)=Df(u)$, the right hand side vanishes because

$$
\big(u_x\bullet A(u)\big)u_t=\big(u_t\bullet A(u)\big)u_x=D^2f(u)\,(u_x\otimes u_t)\,.
$$

In the general case, recalling (5.6) we deduce

$$
\big(u_x\bullet A(u)\big)u_t-\big(u_t\bullet A(u)\big)u_x=\mathcal{O}(1)\cdot\sum_{j\neq k}\big(|v_jv_k|+|v_jw_k|\big). \tag{6.4}
$$

2. For notational convenience, we introduce the variable $z\doteq(v,w)$ and write $\tilde\lambda$ for the $2n\times 2n$ diagonal matrix with entries $\tilde\lambda_i$ defined at (4.21):

$$
\tilde\lambda\doteq\begin{pmatrix}\mathrm{diag}(\tilde\lambda_i)&0\\0&\mathrm{diag}(\tilde\lambda_i)\end{pmatrix}.
$$

From (6.3) it now follows

$$
\begin{aligned}
&\frac{\partial\Lambda}{\partial u}u_t+\frac{\partial\Lambda}{\partial z}\begin{pmatrix}v\\w\end{pmatrix}_t+\left(\begin{bmatrix}A(u)&0\\0&A(u)\end{bmatrix}\Lambda\right)_x-\frac{\partial\Lambda}{\partial z}\begin{pmatrix}v\\w\end{pmatrix}_{xx}-\frac{\partial\Lambda}{\partial u}u_{xx}\\
&\qquad-\frac{\partial^2\Lambda}{\partial u^{[2]}}(u_x\otimes u_x)-\frac{\partial^2\Lambda}{\partial z^{[2]}}\cdot\begin{pmatrix}v_x\\w_x\end{pmatrix}\otimes\begin{pmatrix}v_x\\w_x\end{pmatrix}-2\frac{\partial^2\Lambda}{\partial u\,\partial z}\,u_x\otimes\begin{pmatrix}v_x\\w_x\end{pmatrix}\\
&\qquad=\begin{pmatrix}0\\ \big(u_x\bullet A(u)\big)u_t-\big(u_t\bullet A(u)\big)u_x\end{pmatrix}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{\partial\Lambda}{\partial z}\left[\begin{pmatrix} v\\ w\end{pmatrix}_t+\left(\tilde\lambda\begin{pmatrix} v\\ w\end{pmatrix}\right)_x-\begin{pmatrix} v\\ w\end{pmatrix}_{xx}\right]\\
&\quad=\frac{\partial\Lambda}{\partial z}\left(\tilde\lambda\begin{pmatrix} v\\ w\end{pmatrix}\right)_x-\left(\begin{bmatrix} A(u) & 0\\ 0 & A(u)\end{bmatrix}\Lambda\right)_x+\frac{\partial\Lambda}{\partial u}A(u)u_x+\begin{pmatrix} 0\\ (u_x\bullet A(u))u_t-(u_t\bullet A(u))u_x\end{pmatrix}\\
&\qquad+\frac{\partial^2\Lambda}{\partial u^{[2]}}(u_x\otimes u_x)+\frac{\partial^2\Lambda}{\partial z^{[2]}}\cdot\begin{pmatrix} v_x\\ w_x\end{pmatrix}\otimes\begin{pmatrix} v_x\\ w_x\end{pmatrix}+2\frac{\partial^2\Lambda}{\partial u\,\partial z}u_x\otimes\begin{pmatrix} v_x\\ w_x\end{pmatrix}\\
&\quad\doteq E
\end{aligned}
\tag{6.5}
$$

Since the differential $\partial\Lambda/\partial z$ has uniformly bounded inverse, the right hand sides in (6.1) clearly satisfy the bounds

$$
\phi_i=\mathcal{O}(1)\cdot E\,,\qquad\qquad \psi_i=\mathcal{O}(1)\cdot E,\qquad\qquad i=1,\ldots,n\,. \tag{6.6}
$$

3. To estimate the quantity E in (6.5), it is convenient to introduce the function

$$
\Lambda_i(u,v_i,w_i,\sigma_i)\doteq\begin{pmatrix} v_i\tilde r_i(u,v_i,\sigma_i)\\ (w_i-\lambda_i^*v_i)\tilde r_i(u,v_i,\sigma_i)\end{pmatrix}, \tag{6.7}
$$

so that $\Lambda=\sum\Lambda_i$ and $E=\sum E_i$, where

$$
\begin{aligned}
E_i=&\frac{\partial\Lambda_i}{\partial z_i}\begin{pmatrix} \tilde\lambda_i v_i\\ \tilde\lambda_i w_i\end{pmatrix}_x-\left(\begin{bmatrix} A(u) & 0\\ 0 & A(u)\end{bmatrix}\Lambda_i\right)_x+\frac{\partial\Lambda_i}{\partial\sigma_i}\tilde\lambda_i\,\sigma_{i,x}+\frac{\partial\Lambda_i}{\partial u}\sum_j A(u)v_j\tilde r_j\\
&+\begin{pmatrix} 0\\ \pi_i\big((u_x\bullet A(u))u_t-(u_t\bullet A(u))u_x\big)\end{pmatrix}+\frac{\partial^2\Lambda_i}{\partial u^{[2]}}u_x\otimes u_x+\frac{\partial^2\Lambda_i}{\partial z_i^{[2]}}\cdot\begin{pmatrix} v_{i,x}\\ w_{i,x}\end{pmatrix}\otimes\begin{pmatrix} v_{i,x}\\ w_{i,x}\end{pmatrix}\\
&+\frac{\partial^2\Lambda_i}{\partial\sigma_i^2}\sigma_{i,x}^2+2\frac{\partial^2\Lambda_i}{\partial\sigma_i\,\partial z_i}\begin{pmatrix} \sigma_{i,x}v_{i,x}\\ \sigma_{i,x}w_{i,x}\end{pmatrix}+2\frac{\partial^2\Lambda_i}{\partial u\,\partial z_i}\begin{pmatrix} v_{i,x}\\ w_{i,x}\end{pmatrix}\otimes u_x+2\frac{\partial^2\Lambda_i}{\partial u\,\partial\sigma_i}\sigma_{i,x}u_x\\
&+\frac{\partial\Lambda_i}{\partial\sigma_i}\left(\frac{\partial^2\sigma_i}{\partial v_i^2}v_{i,x}^2+2\frac{\partial^2\sigma_i}{\partial v_i\partial w_i}v_{i,x}w_{i,x}+\frac{\partial^2\sigma_i}{\partial w_i^2}w_{i,x}^2\right).
\end{aligned}
\tag{6.8}
$$

Notice that in (6.5) we regarded Λ as a function of the three independent variables (u,v,w), while in (6.8) we think Λ as a function of the four independent variables (u,v,w,σ). Regarding the σ_i as independent variables, one has the advantage that the maps $\Lambda_i=\Lambda_i(u,v_i,w_i,\sigma_i)$ are now smooth, while $\Lambda_i=\Lambda_i(u,v,w)$ in (5.8) was only $\mathcal{C}^{1,1}$, because of the singularities of the map $(u,v_i,w_i)\mapsto\sigma_i$ in (5.7). The last term in (6.8) is due to the nonlinear dependence of σ_i w.r.t. v_i,w_i. By $\pi_i(\mathbf{v})$ we denoted the i-th component of a vector $\mathbf{v}$ w.r.t. the basis $\{r_1^*,\ldots,r_n^*\}$. Also notice that in the previous computation we used the identity

$$
\begin{aligned}
&\frac{\partial\Lambda_i}{\partial\sigma_i}\frac{\partial\sigma_i}{\partial v_i}\tilde\lambda_{i,x}v_i+\frac{\partial\Lambda_i}{\partial\sigma_i}\frac{\partial\sigma_i}{\partial w_i}\tilde\lambda_{i,x}w_i\\
&\quad=\tilde\lambda_{i,x}\begin{pmatrix} w_i/v_i\cdot\theta_i'\tilde r_{i,\sigma} & -\theta_i'\tilde r_{i,\sigma}\\ w_i/v_i(w_i/v_i-\lambda_i^*)\theta_i'\tilde r_{i,\sigma} & -(w_i/v_i-\lambda_i^*)\theta_i'\tilde r_{i,\sigma}\end{pmatrix}\begin{pmatrix} v_i\\ w_i\end{pmatrix}=\begin{pmatrix} 0\\ 0\end{pmatrix}.
\end{aligned}
$$

4. By Lemma 5.2, the inverse map Λ^{-1} sets a one to one correspondence

$$
(u,u_x,u_{xx})\mapsto(u,v,w)
$$

between two neighborhoods of the point $(u^*, 0, 0) \in I\!R^{3n}$. This map is $\mathcal{C}^1$ with Lipschitz continuous derivative. It can be prolonged to a map

$$(u, u_x, u_{xx}, u_{xxx}) \mapsto (u, v, w, \sigma, v_x, w_x, \sigma_x) \tag{6.9}$$

which is one-to-one, but of course not onto. Indeed, (5.6) and the identity $u_t + A(u)u_x = u_{xx}$ together imply

$$\begin{aligned} &\sum_i w_i \tilde r_i + \sum_i \big(A(u) - \lambda_i^*\big) v_i \tilde r_i \\ &\qquad = \sum_i v_{i,x} \tilde r_i + \sum_{ij} v_i \tilde r_{i,u}\, v_j \tilde r_j + \sum_i v_i \tilde r_{i,v} v_{i,x} + \sum_i v_i \tilde r_{i,\sigma} \sigma_{i,x}\,. \end{aligned} \tag{6.10}$$

A vector $(u, v, w, \sigma, v_x, w_x, \sigma_x) \in I\!R^{7n}$ corresponds to some third order jet $(u, u_x, u_{xx}, u_{xxx})$ provided that it satisfies the vector equation (6.10), together with

$$\sigma_i = \lambda_i^* - \theta\left(\frac{w_i}{v_i}\right), \qquad \sigma_{i,x} = \frac{w_i v_{i,x} - w_{i,x} v_i}{v_i^2}\, \theta'\left(\frac{w_i}{v_i}\right) \qquad i = 1, \ldots, n\,. \tag{6.11}$$

5. By the analysis at (6.2), $E_i(u, v^\diamond, w^\diamond, \sigma^\diamond, v_x^\diamond, w_x^\diamond, \sigma_x^\diamond) = 0$ whenever the argument corresponds to the third order jet of a viscous travelling i-wave. This is the case if

$$v_j^\diamond = w_j^\diamond = v_{j,x}^\diamond = w_{j,x}^\diamond = 0, \qquad \sigma_{j,x}^\diamond = 0, \qquad \text{for all } \; j \neq i, \tag{6.12}$$

$$\begin{cases} v_{i,x}^\diamond = (\tilde\lambda_i - \sigma_i^\diamond) v_i^\diamond\,, \\ w_{i,x}^\diamond = (\tilde\lambda_i - \sigma_i^\diamond) w_i^\diamond\,, \end{cases} \qquad \left|\frac{w_i^\diamond}{v_i^\diamond}\right| < 3\delta_1\,, \qquad \sigma_i^\diamond = \lambda_i^* - \frac{w_i^\diamond}{v_i^\diamond}\,, \qquad \sigma_{i,x}^\diamond = 0\,. \tag{6.13}$$

In order to estimate $E_i(u, v, w, \sigma, v_x, w_x, \sigma_x)$ we proceed as follows. We introduce a new vector $(u, v^\diamond, w^\diamond, \sigma^\diamond, v_x^\diamond, w_x^\diamond, \sigma_x^\diamond)$ corresponding to the jet of a travelling i-wave, by setting

$$v_i^\diamond = v_i\,, \qquad w_i^\diamond = \theta\left(\frac{w_i}{v_i}\right) v_i, \qquad \sigma_i^\diamond = \sigma_i = \lambda_i^* - \frac{w_i^\diamond}{v_i^\diamond}\,. \tag{6.14}$$

The quantities $v_{i,x}^\diamond, w_{i,x}^\diamond, \sigma_{i,x}^\diamond$ are then defined according to (6.13), while the components $j \neq i$ are as in (6.12). The above construction implies $E_i^\diamond \doteq E_i(u, v^\diamond, w^\diamond, \sigma^\diamond, v_x^\diamond, w_x^\diamond, \sigma_x^\diamond) = 0$. Hence $E_i = E_i - E_i^\diamond$.

6. Taking the inner product of (6.10) with $\tilde r_i$, recalling that $\tilde r_i$ has unit norm and is thus orthogonal to its derivatives, we obtain

$$w_i + (\tilde\lambda_i - \lambda_i^*) v_i = v_{i,x} + \Theta_i\,, \tag{6.15}$$

where

$$\begin{aligned} \Theta_i &= \sum_{j\neq i} \Big\langle \tilde r_i,\ \big(\lambda_j^* - A(u)\big) \tilde r_j \Big\rangle v_j + \sum_{j\neq i} \sum_k \langle \tilde r_i,\ \tilde r_{j,u} \tilde r_k \rangle v_j v_k \\ &\quad + \sum_{j\neq i} \langle r_i,\, r_{j,v} \rangle v_j v_{j,x} + \sum_{j\neq i} \langle r_i,\, r_{j,\sigma} \rangle v_j \sigma_{j,x} - \sum_{j\neq i} \langle r_i,\, r_j \rangle (w_j - v_{j,x}) \\ &= \mathcal{O}(1) \cdot \delta_0 \sum_{j\neq i} \big(|v_j| + |w_j - v_{j,x}|\big)\,. \end{aligned} \tag{6.16}$$

The above estimate on Θ_i is obtained using (5.20) together with the $\mathbf{L}^\infty$ bounds in (5.23)-(5.24) and the bound on $\tilde r_{j,\sigma}$ in (4.24). Summing (6.15) over $i = 1, \ldots, n$ and recalling that

$$\delta_0 << 1\,, \qquad |\tilde\lambda_i - \lambda_i^*| = \mathcal{O}(1)\cdot\delta_0\,, \qquad |v_i|,\, |w_i| = \mathcal{O}(1)\cdot\delta_0^2\,,$$

from (6.16) we deduce

$$\sum_i |w_i - v_{i,x}| = \mathcal{O}(1)\cdot\delta_0\,\sum_j |v_j|\,. \tag{6.17}$$

We can now write

$$\begin{aligned} v_{i,x} &= w_i + (\tilde\lambda_i - \lambda_i^*)v_i + \mathcal{O}(1)\cdot\delta_0 \sum_{j\neq i}\big(|v_j| + |w_j - v_{j,x}|\big)\\ &= (\tilde\lambda_i - \sigma_i)v_i + (w_i - \theta_i v_i) + \mathcal{O}(1)\cdot\delta_0 \sum_{j\neq i}\big(|v_j| + |w_j - v_{j,x}|\big)\,. \end{aligned} \tag{6.18}$$

We recall that $\theta_i \doteq \theta(w_i/v_i)$. The first equality in (6.18) yields the implications

$$|w_i| < 3\delta_1|v_i| \qquad\Longrightarrow\qquad v_{i,x} = \mathcal{O}(1)\cdot v_i + \mathcal{O}(1)\cdot\delta_0\sum_{j\neq i}|v_j|\,. \tag{6.19}$$

$$|w_i| > \delta_1|v_i| \qquad\Longrightarrow\qquad v_i = \mathcal{O}(1)\cdot v_{i,x} + \mathcal{O}(1)\cdot\delta_0\sum_{j\neq i}|v_j|\,. \tag{6.20}$$

Moreover, using both equalities in (6.18) we deduce

$$\begin{aligned} (\tilde\lambda_i - \sigma_i)w_i &= (\tilde\lambda_i - \sigma_i)\big[v_{i,x} - (\tilde\lambda_i - \lambda_i^*)v_i\big] + \mathcal{O}(1)\cdot\delta_0\sum_{j\neq i}\big(|v_j| + |w_j - v_{j,x}|\big)\\ &= (\tilde\lambda_i - \sigma_i)v_{i,x} - (\tilde\lambda_i - \lambda_i^*)\big(v_{i,x} - (w_i - \theta_i v_i)\big) + \mathcal{O}(1)\cdot\delta_0\sum_{j\neq i}\big(|v_j| + |w_j - v_{j,x}|\big)\\ &= \frac{w_i}{v_i}v_{i,x} - \left(\frac{w_i}{v_i} - \theta_i\right)v_{i,x} + (\tilde\lambda_i - \lambda_i^*)(w_i - \theta_i v_i) + \mathcal{O}(1)\cdot\delta_0\sum_{j\neq i}\big(|v_j| + |w_j - v_{j,x}|\big)\,, \end{aligned}$$

and hence, by (6.20),

$$w_{i,x} - (\tilde\lambda_i - \sigma_i)w_i = \frac{w_{i,x}v_i - w_i v_{i,x}}{v_i} + \mathcal{O}(1)\cdot\left|\frac{v_{i,x}}{v_i}\right|\,|w_i - \theta_i v_i| + \mathcal{O}(1)\cdot\delta_0\sum_{j\neq i}\big(|v_j| + |w_j - v_{j,x}|\big)\,.$$

From the definitions (6.13)-(6.14), using the above estimates we obtain

$$\begin{aligned} |w_i - w_i^\diamondsuit| &= |w_i - \theta_i v_i|\,,\\ |v_{i,x} - v_{i,x}^\diamondsuit| &= \mathcal{O}(1)\cdot|w_i - \theta_i v_i| + \mathcal{O}(1)\cdot\delta_0\sum_{j\neq i}\big(|v_j| + |w_j - v_{j,x}|\big)\,,\\ |w_{i,x} - w_{i,x}^\diamondsuit| &= \left|\frac{w_{i,x}v_i - w_i v_{i,x}}{v_i}\right| + \mathcal{O}(1)\cdot\left|\frac{v_{i,x}}{v_i}\right|\,|w_i - \theta_i v_i| + \mathcal{O}(1)\cdot\delta_0\sum_{j\neq i}\big(|v_j| + |w_j - v_{j,x}|\big)\,. \end{aligned} \tag{6.21}$$

7. We now compute

$$
\begin{aligned}
E_i &= E_i - E_i^\diamondsuit \\
&= \frac{\partial\Lambda_i}{\partial z_i}\begin{pmatrix}\tilde\lambda_i v_i\\ \tilde\lambda_i w_i\end{pmatrix}_x - \left(\begin{bmatrix}A(u) & 0\\ 0 & A(u)\end{bmatrix}\Lambda_i\right)_x - \frac{\partial\Lambda_i^\diamondsuit}{\partial z_i}\begin{pmatrix}\tilde\lambda_i^\diamondsuit v_i^\diamondsuit\\ \tilde\lambda_i^\diamondsuit w_i^\diamondsuit\end{pmatrix}_x - \left(\begin{bmatrix}A(u) & 0\\ 0 & A(u)\end{bmatrix}\Lambda_i^\diamondsuit\right)_x \\
&\quad + \frac{\partial\Lambda_i}{\partial\sigma_i}\tilde\lambda_i\,\sigma_{i,x} + \frac{\partial\Lambda_i}{\partial u}\sum_{j\neq i} A(u)v_j\tilde r_j \\
&\quad + \begin{pmatrix}0\\ \pi_i\big((u_x\bullet A(u))u_t - (u_t\bullet A(u))u_x\big)\end{pmatrix} + \left(\frac{\partial^2\Lambda_i}{\partial u^{[2]}}u_x\otimes u_x + \frac{\partial^2\Lambda_i^\diamondsuit}{\partial u^{[2]}}\,v_i\tilde r_i^\diamondsuit\otimes v_i\tilde r_i^\diamondsuit\right) \\
&\quad + \frac{\partial^2\Lambda_i}{\partial z_i^{[2]}}\cdot\begin{pmatrix}v_{i,x}\\ w_{i,x}\end{pmatrix}\otimes\begin{pmatrix}v_{i,x}\\ w_{i,x}\end{pmatrix} - \frac{\partial^2\Lambda_i^\diamondsuit}{\partial z_i^{[2]}}\cdot\begin{pmatrix}v_{i,x}^\diamondsuit\\ w_{i,x}^\diamondsuit\end{pmatrix}\otimes\begin{pmatrix}v_{i,x}^\diamondsuit\\ w_{i,x}^\diamondsuit\end{pmatrix} \\
&\quad + \frac{\partial^2\Lambda_i}{\partial\sigma_i^2}\sigma_{i,x}^2 + 2\frac{\partial^2\Lambda_i}{\partial u\partial z_i}\begin{pmatrix}v_{i,x}\\ w_{i,x}\end{pmatrix}\otimes u_x - 2\frac{\partial^2\Lambda_i^\diamondsuit}{\partial u\partial z_i}\begin{pmatrix}v_{i,x}^\diamondsuit\\ w_{i,x}^\diamondsuit\end{pmatrix}\otimes v_i\tilde r_i^\diamondsuit + 2\frac{\partial^2\Lambda_i}{\partial u\,\partial\sigma_i}\sigma_{i,x}u_x \\
&\quad + 2\frac{\partial^2\Lambda_i}{\partial z_i\partial\sigma_i}\begin{pmatrix}\sigma_{i,x}v_{i,x}\\ \sigma_{i,x}w_{i,x}\end{pmatrix} + \frac{\partial\Lambda_i}{\partial\sigma_i}\left(\frac{\partial^2\sigma_i}{\partial^2 v_i}v_{i,x}^2 + 2\frac{\partial^2\sigma_i}{\partial v_i\partial w_i}v_{i,x}w_{i,x} + \frac{\partial^2\sigma_i}{\partial^2 w_i}w_{i,x}^2\right).
\end{aligned}
\tag{6.22}
$$

Observe that the quantities $u, v_i, \sigma_i, \tilde r_i, \tilde\lambda_i$ remain the same in the computations of E_i and $E_i^\diamondsuit$. Moreover, all the terms involving derivatives w.r.t. σ_i vanish when we compute $E_i^\diamondsuit$.

In the remaining steps, we will examine the various terms on the right hand side of (6.22) and show that they can all be bounded according to the lemma. As a preliminary, we observe that by (6.7) and (4.24) the derivatives of the smooth function $\Lambda_i = \Lambda_i(u, z_i, \sigma_i)$ satisfy

$$
\frac{\partial\Lambda_i}{\partial u},\quad \frac{\partial^2\Lambda_i}{\partial u^{[2]}},\quad \frac{\partial^2\Lambda_i}{\partial z_i\partial\sigma_i} = \mathcal{O}(1)\cdot\big(|v_i| + |w_i|\big),
\tag{6.23}
$$

$$
\frac{\partial\Lambda_i}{\partial\sigma_i},\quad \frac{\partial^2\Lambda_i}{\partial u\partial\sigma_i},\quad \frac{\partial^2\Lambda_i}{\partial\sigma_i^2} = \mathcal{O}(1)\cdot\big(|v_i^2| + |v_i w_i|\big)\,.
\tag{6.24}
$$

8. We start by collecting some transversal terms. Using (6.4) and (6.23)-(6.24) we obtain

$$
\begin{aligned}
&\frac{\partial\Lambda_i}{\partial u}\sum_{j\neq i}A(u)v_j\tilde r_j + \begin{pmatrix}0\\ \pi_i\big((u_x\bullet A(u))u_t - (u_t\bullet A(u))u_x\big)\end{pmatrix} \\
&\qquad + \frac{\partial^2\Lambda_i}{\partial u^{[2]}}\left(\sum_{j,k}v_j v_k\tilde r_j\otimes\tilde r_k - v_i^2\tilde r_i\otimes\tilde r_i\right) + 2\frac{\partial^2\Lambda_i}{\partial u\partial z_i}\sum_{j\neq i}\begin{pmatrix}v_{i,x}\\ w_{i,x}\end{pmatrix}\otimes v_j\tilde r_j \\
&= \mathcal{O}(1)\cdot\sum_{j\neq k}\big(|v_j v_k| + |w_j v_k|\big) + \mathcal{O}(1)\cdot\sum_{j\neq i}\big(|w_j w_i| + |v_j v_{i,x}| + |v_j w_{i,x}|\big).
\end{aligned}
\tag{6.25}
$$

Here and in the following, by "transversal terms" we mean terms whose size is bounded by products of distinct components $j\neq k$, as in (6.25).

9. We now look at terms involving derivatives w.r.t. σ_i. One should here keep in mind that, if

$\sigma_{i,x} \neq 0$, then both sides of the implication (6.19) hold true. Using (6.23) we obtain

$$\begin{aligned}
\frac{\partial^2 \Lambda_i}{\partial z_i \partial \sigma_i} \begin{pmatrix} \sigma_{i,x} v_{i,x} \\ \sigma_{i,x} w_{i,x} \end{pmatrix} &= \mathcal{O}(1) \cdot v_i \big(|v_{i,x}| + |w_{i,x}|\big) \sigma_{i,x} \\
&= \mathcal{O}(1) \cdot v_i^2 \sigma_{i,x} + \mathcal{O}(1) \cdot (w_i v_{i,x} - w_{i,x} v_i) \sigma_{i,x} + \text{transversal terms} \\
&= \mathcal{O}(1) \cdot |w_i v_{i,x} - w_{i,x} v_i| + \mathcal{O}(1) \cdot \left| v_i \cdot \left(\frac{w_i}{v_i}\right)_x \right|^2 \cdot \chi_{\{|w_i/v_i| < 3\delta_1\}} + \text{transversal terms}.
\end{aligned} \tag{6.26}$$

An application of (6.24) yields

$$\begin{aligned}
\frac{\partial \Lambda_i}{\partial \sigma_i} \tilde{\lambda}_i \sigma_{i,x} + \frac{\partial^2 \Lambda_i}{\partial \sigma_i^2} \sigma_{i,x}^2 + 2 \frac{\partial^2 \Lambda_i}{\partial u \partial \sigma_i} \sigma_{i,x} u_x &= \mathcal{O}(1) \cdot v_i^2 \big(|\sigma_{i,x}| + |\sigma_{i,x}^2|\big) + \text{transversal terms} \\
&= \mathcal{O}(1) \cdot |v_i w_{i,x} - w_i v_{i,x}| + \mathcal{O}(1) \cdot \left| v_i \cdot \left(\frac{w_i}{v_i}\right)_x \right|^2 \cdot \chi_{\{|w_i/v_i| < 3\delta_1\}} + \text{transversal terms}.
\end{aligned} \tag{6.27}$$

Next, we observe that the quantity

$$\frac{\partial^2 \sigma_i}{\partial^2 v_i} v_{i,x}^2 + 2 \frac{\partial^2 \sigma_i}{\partial v_i \partial w_i} v_{i,x} w_{i,x} + \frac{\partial^2 \sigma_i}{\partial^2 w_i} w_{i,x}^2$$

vanishes in the special case where $w_{i,x} = (w_i/v_i) v_{i,x}$. In general, using (6.19) and (6.24) one obtains

$$\begin{aligned}
&\frac{\partial \Lambda_i}{\partial \sigma_i} \left(\frac{\partial^2 \sigma_i}{\partial^2 v_i} v_{i,x}^2 + 2 \frac{\partial^2 \sigma_i}{\partial v_i \partial w_i} v_{i,x} w_{i,x} + \frac{\partial^2 \sigma_i}{\partial^2 w_i} w_{i,x}^2 \right) \\
&\quad = \frac{\partial \Lambda_i}{\partial \sigma_i} \left\{ \left(-\theta_i'' \frac{w_i^2}{v_i^4} - 2\theta_i' \frac{w_i}{v_i^3} \right) v_{i,x}^2 + 2 \left(\theta_i'' \frac{w_i}{v_i^3} + \theta_i' \frac{1}{v_i^2} \right) v_{i,x} w_{i,x} - \theta_i'' \frac{w_{i,x}^2}{v_i^2} \right\} \\
&\quad = \frac{\partial \Lambda_i}{\partial \sigma_i} \left\{ 2\theta_i' (v_i w_{i,x} - w_i v_{i,x}) \frac{v_{i,x}}{v_i} - \theta_i'' \Big(\frac{v_i w_{i,x} - w_i v_{i,x}}{v_i^2} \Big)^2 \right\} \\
&\quad = \mathcal{O}(1) \cdot |v_i w_{i,x} - w_i v_{i,x}| + \mathcal{O}(1) \cdot \left| v_i \left(\frac{w_i}{v_i}\right)_x \right|^2 \cdot \chi_{\{|w_j/v_j| < 3\delta_1\}} + \text{transversal terms}.
\end{aligned} \tag{6.28}$$

10. We now complete the analysis of the remaining terms. As a preliminary, we observe that the only difference between $\Lambda_i^\diamondsuit$ and Λ_i is due to the fact that one may have $w_i^\diamondsuit \neq w_i$. The first equality in (6.21) thus implies

$$\big|\Lambda_i^\diamondsuit - \Lambda_i\big|, \quad \big|D\Lambda_i^\diamondsuit - D\Lambda_i\big|, \quad \big|D^2\Lambda_i^\diamondsuit - D^2\Lambda_i\big| \;=\; \mathcal{O}(1) \cdot |w_i - \theta_i v_i| + \mathcal{O}(1) \cdot \delta_0 \sum_{j \neq i} |v_j|. \tag{6.29}$$

By (6.20) and (6.29), if we compute Λ_i or its partial derivatives at the point (u, v_i, w_i, σ_i) instead of $(u, v_i^\diamondsuit, w_i^\diamondsuit, \sigma_i^\diamondsuit) = (u, v_i, w_i^\diamondsuit, \sigma_i)$, the difference in each of the corresponding terms in (6.22) will have magnitude

$$\mathcal{O}(1) \cdot |w_i - \theta_i v_i| \cdot \big(|v_{i,x}| + |w_{i,x}|\big) + \text{transversal terms}.$$

For example,

$$\begin{aligned}
\left(\frac{\partial^2 \Lambda_i}{\partial u^{[2]}} - \frac{\partial^2 \Lambda_i^\diamondsuit}{\partial u^{[2]}} \right) (v_i \tilde{r}_i \otimes v_i \tilde{r}_i) &= \mathcal{O}(1) \cdot |w_i - \theta_i v_i|\, v_i^2 + \text{transversal terms} \\
&= \mathcal{O}(1) \cdot |w_i - \theta_i v_i|\, v_{i,x}^2 + \text{transversal terms}.
\end{aligned} \tag{6.30}$$

Indeed, if $w_i \neq \theta_i v_i$, then both sides of the implication (6.20) hold true.

Observing that $\partial^2\Lambda_i/\partial w_i^2 = 0$ and using again (6.18), we have

$$\begin{aligned}
&\frac{\partial^2\Lambda_i}{\partial z_i^{[2]}}\left[\begin{pmatrix} v_{i,x} \\ w_{i,x}\end{pmatrix}\otimes\begin{pmatrix} v_{i,x} \\ w_{i,x}\end{pmatrix} - \begin{pmatrix} v_{i,x}^{\diamondsuit} \\ w_{i,x}^{\diamondsuit}\end{pmatrix}\otimes\begin{pmatrix} v_{i,x}^{\diamondsuit} \\ w_{i,x}^{\diamondsuit}\end{pmatrix}\right] \\
&\qquad = \mathcal{O}(1)\cdot\big(v_{i,x}^2 - (v_{i,x}^{\diamondsuit})^2\big) + \mathcal{O}(1)\cdot\big(v_{i,x}w_{i,x} - v_{i,x}^{\diamondsuit}w_{i,x}^{\diamondsuit}\big) \\
&\qquad = \mathcal{O}(1)\cdot v_{i,x}|w_i - \theta_i v_i| + \mathcal{O}(1)\cdot w_{i,x}|w_i - \theta_i v_i| + \mathcal{O}(1)|w_i v_{i,x} - v_i w_{i,x}| + \text{transversal terms}\,.
\end{aligned} \tag{6.31}$$

In a similar way, using (6.20) and (6.21) one derives the estimate

$$\begin{aligned}
\frac{\partial^2\Lambda_i}{\partial u\partial z_i}\begin{pmatrix} v_{i,x} - v_{i,x}^{\diamondsuit} \\ w_{i,x} - w_{i,x}^{\diamondsuit}\end{pmatrix}\otimes v_i\tilde r_i &= \mathcal{O}(1)\cdot v_{i,x}|w_i - \theta_i v_i| + \mathcal{O}(1)\cdot|w_i v_{i,x} - v_i w_{i,x}| \\
&\qquad + \text{transversal terms}\,.
\end{aligned} \tag{6.32}$$

Using the identity (4.23), we now compute

$$\begin{aligned}
&\frac{\partial\Lambda_i}{\partial z_i}\begin{pmatrix}\tilde\lambda_i v_i \\ \tilde\lambda_i w_i\end{pmatrix}_x - \left(\begin{bmatrix} A(u) & 0 \\ 0 & A(u)\end{bmatrix}\Lambda_i\right)_x = \begin{pmatrix} v_i\tilde r_{i,v}(\tilde\lambda_i v_i)_x \\ (w_i - \lambda_i^* v_i)\tilde r_{i,v}(\tilde\lambda_i v_i)_x\end{pmatrix} + \begin{pmatrix} v_i A(u)\tilde r_{i,v} v_{i,x} \\ (w_i - \lambda_i^* v_i)A(u)\tilde r_{i,v} v_{i,x}\end{pmatrix} \\
&\quad + \begin{bmatrix}\tilde r_i & 0 \\ -\lambda_i^*\tilde r_i & \tilde r_i\end{bmatrix}\begin{pmatrix}\tilde\lambda_{i,x} v_i \\ \tilde\lambda_{i,x} w_i\end{pmatrix} - v_i\begin{bmatrix}\tilde r_{i,u}\tilde r_i + (\tilde\lambda_i - \sigma_i)\tilde r_{i,v} & 0 \\ -\lambda_i^*(\tilde r_{i,u}\tilde r_i + (\tilde\lambda_i - \sigma_i)\tilde r_{i,v}) & \tilde r_{i,u}\tilde r_i + (\tilde\lambda_i - \sigma_i)\tilde r_{i,v}\end{bmatrix}\begin{pmatrix} v_{i,x} \\ w_{i,x}\end{pmatrix} \\
&\quad - \begin{bmatrix} A(u) & 0 \\ 0 & A(u)\end{bmatrix}\frac{\partial\Lambda_i}{\partial\sigma_i}\sigma_{i,x} - \sum_j\begin{bmatrix} DA(u)\tilde r_j & 0 \\ 0 & DA(u)\tilde r_j\end{bmatrix} v_j\Lambda_i - \sum_j\begin{bmatrix} A(u) & 0 \\ 0 & A(u)\end{bmatrix}\frac{\partial\Lambda_i}{\partial u} v_j\tilde r_j\,.
\end{aligned}$$

With similar arguments as above, we obtain

$$\begin{aligned}
&\begin{pmatrix} (\tilde\lambda_i v_i)_x v_i\tilde r_{i,v} \\ (\tilde\lambda_i v_i)_x v_i(w_i - \lambda_i^* v_i)\tilde r_{i,v}\end{pmatrix} - \begin{pmatrix} (\tilde\lambda_i^{\diamondsuit} v_i^{\diamondsuit})_x v_i^{\diamondsuit}\tilde r_{i,v}^{\diamondsuit} \\ (\tilde\lambda_i^{\diamondsuit} v_i^{\diamondsuit})_x v_i^{\diamondsuit}(w_i^{\diamondsuit} - \lambda_i^* v_i^{\diamondsuit})\tilde r_{i,v}^{\diamondsuit}\end{pmatrix} \\
&\qquad = \mathcal{O}(1)\cdot|v_i w_{i,x} - v_{i,x} w_i|\cdot\chi_{\{|w_i/v_i|\le 3\delta_1\}} + \mathcal{O}(1)\cdot v_{i,x}|w_i - \theta_i v_i| + \text{transversal terms}\,,
\end{aligned}$$

$$\begin{pmatrix}\tilde\lambda_{i,x} v_i \\ \tilde\lambda_{i,x} w_i\end{pmatrix} - \begin{pmatrix}\tilde\lambda_{i,x}^{\diamondsuit} v_i^{\diamondsuit} \\ \tilde\lambda_{i,x}^{\diamondsuit} w_i^{\diamondsuit}\end{pmatrix} = \mathcal{O}(1)\cdot|v_i w_{i,x} - v_{i,x} w_i| + \mathcal{O}(1)\cdot v_{i,x}|w_i - \theta_i v_i| + \text{transversal terms}\,,$$

$$\begin{pmatrix} v_i v_{i,x} \\ v_i w_{i,x}\end{pmatrix} - \begin{pmatrix} v_i^{\diamondsuit} v_{i,x}^{\diamondsuit} \\ v_i^{\diamondsuit} w_{i,x}^{\diamondsuit}\end{pmatrix} = \mathcal{O}(1)\cdot|v_i w_{i,x} - v_{i,x} w_i| + \mathcal{O}(1)\cdot v_{i,x}|w_i - \theta_i v_i| + \text{transversal terms}\,.$$

The above estimates together imply

$$\begin{aligned}
&\frac{\partial\Lambda_i}{\partial z_i}\begin{pmatrix}\tilde\lambda_i v_i \\ \tilde\lambda_i w_i\end{pmatrix}_x - \left(\begin{bmatrix} A(u) & 0 \\ 0 & A(u)\end{bmatrix}\Lambda_i\right)_x - \frac{\partial\Lambda_i^{\diamondsuit}}{\partial z_i}\begin{pmatrix}\tilde\lambda_i^{\diamondsuit} v_i^{\diamondsuit} \\ \tilde\lambda_i^{\diamondsuit} w_i^{\diamondsuit}\end{pmatrix}_x - \left(\begin{bmatrix} A(u) & 0 \\ 0 & A(u)\end{bmatrix}\Lambda_i^{\diamondsuit}\right)_x \\
&\qquad = \mathcal{O}(1)\cdot|w_i - \theta_i v_i|\big(|v_{i,x}| + |w_{i,x}|\big) + \mathcal{O}(1)\cdot|w_i v_{i,x} - v_i w_{i,x}| + \text{transversal terms}\,.
\end{aligned} \tag{6.33}$$

This completes the proof of Lemma 6.1. $\square$

7 - Transversal wave interactions

The goal of this section is to establish an a priori bound on the total amount of interactions between waves of different families. More precisely, let $u = u(t,x)$ be a solution of the parabolic system (3.1) and assume that

$$\big\|u_x(t)\big\|_{\mathbf{L}^1} \leq \delta_0 \qquad\qquad t \in [0,T]\,. \tag{7.1}$$

In this case, for $t \geq \hat t$, by Corollary 2.2 all higher derivatives will be suitably small and we can thus define the components v_i, w_i according to (5.6)-(5.7). These will satisfy the linear evolution equation (6.1), with source terms ϕ_i, ψ_i described in Lemma 6.1. Assuming that

$$\int_{\hat t}^T \int \big|\phi_i(t,x)\big| + \big|\psi_i(t,x)\big|\,dxdt \leq \delta_0\,, \qquad\qquad i = 1,\ldots,n, \tag{7.2}$$

and relying on the bounds (5.22)–(5.24), we shall prove the estimate

$$\int_{\hat t}^T \int \sum_{j\neq k} \big(|v_j v_k| + |v_{j,x} v_k| + |v_j w_k| + |v_{j,x} w_k| + |v_j w_{k,x}| + |w_j w_k|\big)\,dxdt = \mathcal{O}(1)\cdot\delta_0^2\,. \tag{7.3}$$

As a preliminary, we establish a more general estimate on solutions of two independent linear parabolic equations, with strictly different drifts.

Lemma 7.1. *Let $z, z^\sharp$ be solutions of the two independent scalar equations*

$$\begin{aligned} z_t + \big(\lambda(t,x)z\big)_x - z_{xx} &= \varphi(t,x)\,,\\ z^\sharp_t + \big(\lambda^\sharp(t,x)z^\sharp\big)_x - z^\sharp_{xx} &= \varphi^\sharp(t,x)\,, \end{aligned} \tag{7.4}$$

defined for $t \in [0,T]$. Assume that

$$\inf_{t,x} \lambda^\sharp(t,x) \;-\; \sup_{t,x}\lambda(t,x) \;\geq\; c \;>\; 0\,. \tag{7.5}$$

Then

$$\begin{aligned} &\int_0^T \int \big|z(t,x)\big|\,\big|z^\sharp(t,x)\big|\,dxdt\\ &\qquad \leq \frac{1}{c}\left(\int \big|z(0,x)\big|\,dx + \int_0^T\int \big|\varphi(t,x)\big|\,dxdt\right)\left(\int \big|z^\sharp(0,x)\big|\,dx + \int_0^T\int \big|\varphi^\sharp(t,x)\big|\,dxdt\right). \end{aligned} \tag{7.6}$$

Proof. We consider first the homogeneous case, where $\varphi = \varphi^\sharp = 0$. Define the interaction potential

$$Q(z, z^\sharp) \doteq \iint K(x-y)\,\big|z(x)\big|\,\big|z^\sharp(y)\big|\,dxdy\,, \tag{7.7}$$

with

$$K(s) \doteq \begin{cases} 1/c & \text{if} \quad s \geq 0,\\ 1/c\cdot e^{cs/2} & \text{if} \quad s < 0. \end{cases} \tag{7.8}$$

Computing the distributional derivatives of the kernel K we find that $cK' - 2K''$ is precisely the Dirac distribution, i.e. a unit mass at the origin. A direct computations now yields

$$
\begin{aligned}
&\frac{d}{dt}Q\big(z(t), z^\sharp(t)\big) \\
&\quad = \frac{d}{dt}\iint K(x-y)\big|z(x)\big|\,\big|z^\sharp(y)\big|\,dxdy \\
&\quad = \iint K(x-y)\Big\{\big(z_{xx} - (\lambda z)_x\big)\,\mathrm{sign}z(x)\big|z^\sharp(y)\big| + \big|z(x)\big|\big(z^\sharp_{yy} - (\lambda^\sharp z^\sharp)_y\big)\,\mathrm{sign}z^\sharp(y)\Big\}\,dxdy \\
&\quad = \iint K'(x-y)\Big\{\lambda\big|z(x)\big|\,\big|z^\sharp(y)\big| - \lambda^\sharp\big|z(x)\big|\,\big|z^\sharp(y)\big|\Big\}\,dxdy \\
&\qquad + \iint K''(x-y)\Big\{\big|z(x)\big|\,\big|z^\sharp(y)\big| + \big|z(x)\big|\,\big|z^\sharp(y)\big|\Big\}\,dxdy \\
&\quad \le -\iint \big(cK' - 2K''\big)\big|z(x)\big|\,\big|z^\sharp(y)\big|\,dxdy \\
&\quad = -\int \big|z(x)\big|\,\big|z^\sharp(x)\big|\,dx\,.
\end{aligned}
$$

Therefore

$$
\int_0^T\!\!\int \big|z(t,x)\big|\,\big|z^\sharp(t,x)\big|\,dxdt \le Q\big(z(0),\,z^\sharp(0)\big) \le \frac{1}{c}\,\big\|z(0)\big\|_{\mathbf{L}^1}\big\|z^\sharp(0)\big\|_{\mathbf{L}^1}\,. \tag{7.9}
$$

proving the lemma in the homogeneous case.

To handle the general case, call $\Gamma, \Gamma^\sharp$ the Green functions for the corresponding linear homogenous systems. The general solution of (7.4) can thus be written in the form

$$
\begin{aligned}
z(t,x) &= \int \Gamma(t,x,0,y)z(0,y)dy + \int_0^t\!\!\int \Gamma(t,x,s,y)\varphi(s,y)dyds, \\
z^\sharp(t,x) &= \int \Gamma^\sharp(t,x,0,y)z^\sharp(0,y)dy + \int_0^t\!\!\int \Gamma^\sharp(t,x,s,y)\varphi^\sharp(s,y)dyds.
\end{aligned} \tag{7.10}
$$

From (7.9) it follows

$$
\int_{\max\{s,s'\}}^T \int \Gamma(t,x,s,y)\cdot\Gamma^\sharp(t,x,s',y')\,dxdt \le \frac{1}{c} \tag{7.11}
$$

for every couple of initial points (s,y) and (s',y'). The estimate (7.6) now follows from (7.11) and the representation formula (7.10). □

Remark 7.2. Exactly the same estimate (7.6) would be true also for a system without viscosity. In particular, if

$$
z_t + \big(\lambda(t,x)z\big)_x = 0, \qquad\qquad z^\sharp_t + \big(\lambda^\sharp(t,x)z^\sharp\big)_x = 0,
$$

and if the speeds satisfy the gap condition (7.5), then

$$
\frac{d}{dt}\left[\frac{1}{c}\iint_{x<y}\big|z^\sharp(t,x)z(t,y)\big|\,dxdy\right] \le -\int \big|z(t,x)\big|\,\big|z^\sharp(t,x)\big|\,dx\,.
$$

In the case where viscosity is present, our definition (7.7)-(7.8) thus provides a natural counterpart to the Glimm interaction potential between waves of different families, introduced in [G] for strictly hyperbolic systems.

Lemma 7.1 allows us to estimate the integral of the terms $|v_i v_k|$, $|v_j w_k|$ and $|w_j w_k|$ in (7.3). We now work toward an estimate of the remaining terms $|v_{j,x} v_k|$, $|v_{j,x} w_k|$ and $|v_j w_{k,x}|$, containing one derivative w.r.t. x.

Lemma 7.3. *Let $z, z^\sharp$ be solutions of (7.4) and assume that (7.5) holds, together with the estimates*

$$\int_0^T \int \big|\varphi(t,x)\big|\, dxdt \leq \delta_0, \qquad \int_0^T \int \big|\varphi^\sharp(t,x)\big|\, dxdt \leq \delta_0\,, \tag{7.12}$$

$$\big\|z(t)\big\|_{\mathbf{L}^1}\,,\ \big\|z^\sharp(t)\big\|_{\mathbf{L}^1} \leq \delta_0\,, \qquad \big\|z_x(t)\big\|_{\mathbf{L}^1}\,,\ \big\|z^\sharp(t)\big\|_{\mathbf{L}^\infty} \leq C^*\delta_0^2\,, \tag{7.13}$$

$$\big\|\lambda_x(t)\big\|_{\mathbf{L}^\infty}\,,\ \big\|\lambda_x(t)\big\|_{\mathbf{L}^1} \leq C^*\delta_0\,, \qquad \lim_{x\to-\infty} \lambda(t,x) = 0 \tag{7.14}$$

for all $t \in [0,T]$. Then one has the bound

$$\int_0^T \int \big|z_x(t,x)\big|\,\big|z^\sharp(t,x)\big|\, dxdt = \mathcal{O}(1)\cdot\delta_0^2. \tag{7.15}$$

Proof. The left hand side of (7.15) is clearly bounded by the quantity

$$\mathcal{I}(T) \doteq \sup_{(\tau,\xi)\in[0,T]\times I\!R} \int_0^{T-\tau} \int \big|z_x(t,x) z^\sharp(t+\tau, x+\xi)\big|\, dxdt \leq (C^*\delta_0^2)^2\cdot T\,,$$

the last inequality being a consequence of (7.13). For $t > 1$ we can write z_x in the form

$$z_x(t,x) = \int G_x(1,y) z(t-1,\, x-y)\, dy + \int_0^1 \int G_x(s,y)\big[\varphi - (\lambda z)_x\big](t-s,\, x-y)\, dyds\,,$$

where $G(t,x) \doteq \exp\{-x^2/4t\}/2\sqrt{\pi t}$ is the standard heat kernel. Using (7.6) we obtain

$$
\begin{aligned}
&\int_1^{T-\tau}\int \big|z_x(t,x)\,z^\sharp(t+\tau,x+\xi)\big|\,dxdt\\
&\le \int_1^{T-\tau}\iint \Big|G_x(1,y)\,z(t-1,\,x-y)\,z^\sharp(t+\tau,x+\xi)\Big|\,dydxdt\\
&\quad+\int_1^{T-\tau}\int\int_0^1\int \|\lambda_x\|_{\mathbf{L}^\infty}\Big|G_x(s,y)\,z(t-s,\,x-y)\,z^\sharp(t+\tau,x+\xi)\Big|\,dydsdxdt\\
&\quad+\int_1^{T-\tau}\int\int_0^1\int \|\lambda\|_{\mathbf{L}^\infty}\Big|G_x(s,y)\,z_x(t-s,\,x-y)\,z^\sharp(t+\tau,x+\xi)\Big|\,dydsdxdt\\
&\quad+\int_1^{T-\tau}\int\int_{t-1}^{t}\int \Big|G_x(t-s,\,x-y)\,\varphi(s,y)\,z^\sharp(t+\tau,x+\xi)\Big|\,dydsdxdt\\
&\le \left(\int \big|G_x(1,y)\big|\,dy+\|\lambda_x\|_{\mathbf{L}^\infty}\int_0^1\int \big|G_x(s,y)\big|\,dyds\right)\\
&\qquad\cdot\sup_{s,y,\tau,\xi}\left(\int_1^{T-\tau}\int \big|z(t-s,\,x-y)\big|\,\big|z^\sharp(t+\tau,x+\xi)\big|\,dxdt\right)\\
&\quad+\left(\|\lambda\|_{\mathbf{L}^\infty}\cdot\int_0^1\int \big|G_x(s,y)\big|\,dyds\right)\cdot\left(\sup_{s,y,\tau,\xi}\int_1^{T-\tau}\int \big|z_x(t-s,\,x-y)\big|\,\big|z^\sharp(t+\tau,x+\xi)\big|\,dxdt\right)\\
&\quad+\|z^\sharp\|_{\mathbf{L}^\infty}\cdot\int_0^1\int \big|G_x(s,y)\big|\,dsdy\cdot\int_0^T\int \big|\varphi(t,x)\big|\,dxdt\\
&\le \left(\frac{1}{\sqrt{\pi}}+\|\lambda_x\|_{\mathbf{L}^\infty}\frac{2}{\sqrt{\pi}}\right)\frac{4\delta_0^2}{c}+\|\lambda\|_{\mathbf{L}^\infty}\frac{2}{\sqrt{\pi}}\mathcal{I}(T)+C^*\delta_0^2\frac{2}{\sqrt{\pi}}\delta_0\,.
\end{aligned}
\tag{7.16}
$$

On the initial time interval $[0,1]$, by (7.13) one has

$$
\int_0^1\int \big|z_x(t,x)\,z^\sharp(t+\tau,\,x+\xi)\big|\,dxdt \le \int_0^1 \big\|z_x(t)\big\|_{\mathbf{L}^1}\,\big\|z^\sharp(t+\tau)\big\|_{\mathbf{L}^\infty}\,dt \le (C^*\delta_0^2)^2\,. \tag{7.17}
$$

Moreover, (7.14) implies

$$
\|\lambda\|_{\mathbf{L}^\infty} \le \|\lambda_x\|_{\mathbf{L}^1} \le C^*\delta_0 << 1\,.
$$

From (7.16) and (7.17) it thus follows

$$
\mathcal{I}(T) \le (C^*\delta_0^2)^2+\frac{4\delta_0^2}{c}+\frac{1}{2}\mathcal{I}(T)+C^*\delta_0^3\,.
$$

For δ_0 sufficiently small, this implies $\mathcal{I}(T)\le 9\delta_0^2/c$, proving the lemma. □

Using the two previous lemmas we now prove the estimate (7.3). Setting $z \doteq v_j$, $z^\sharp \doteq v_k$, $\lambda \doteq \tilde\lambda_j$, $\lambda^\sharp \doteq \tilde\lambda_k$, an application of Lemma 7.1 yields the desired bound on the integral of $|v_j v_k|$. Moreover, Lemma 7.3 allows us to estimate the integral of $|v_{j,x}v_k|$. Notice that the assumptions (7.13)-(7.14) are a consequence of (5.22)-(5.23). The simplifying condition $\lambda(t,-\infty)=0$ in (7.14) can be easily achieved, using a new space coordinate $x' \doteq x-\lambda_j^* t$.

The other terms $|v_j w_k|$, $|w_j w_k|$, $|v_{j,x}w_k|$ and $|v_j w_{k,x}|$ are handled similarly.

8 - Functionals related to shortening curves

We now study the interaction of viscous waves of the same family. As in the previous section, let $u = u(t,x)$ be a solution of the parabolic system (3.1) whose total variation remains bounded according to (7.1). Assume that the components v_i, w_i satisfy the evolution equation (6.1), with source terms ϕ_i, ψ_i bounded as in (7.2). Relying on the bounds (5.22)–(5.24), for each $i = 1, \ldots, n$ we shall prove the estimates

$$\int_{\hat t}^{T} \int |w_{i,x} v_i - w_i v_{i,x}| \, dxdt = \mathcal{O}(1) \cdot \delta_0^2, \tag{8.1}$$

$$\int_{\hat t}^{T} \int_{|w_i/v_i|<3\delta_1} |v_i|^2 \left| \left(\frac{w_i}{v_i} \right)_x \right|^2 dxdt = \mathcal{O}(1) \cdot \delta_0^3. \tag{8.2}$$

The above integrals will be controlled in terms of two functionals, related to shortening curves. Consider a parametrized curve in the plane $\gamma : I\!R \mapsto I\!R^2$. Assuming that γ is sufficiently smooth, its **length** is computed by

$$\mathcal{L}(\gamma) \doteq \int \big|\gamma_x(x)\big| \, dx \,. \tag{8.3}$$

Following [BiB2], we also define the **area** functional as the integral of a wedge product:

$$\mathcal{A}(\gamma) \doteq \frac{1}{2} \iint_{x<y} \big|\gamma_x(x) \wedge \gamma_x(y)\big| \, dxdy \,. \tag{8.4}$$

To understand its geometrical meaning, observe that if γ is a closed curve, the integral

$$\frac{1}{2} \int \gamma(y) \wedge \gamma_x(y) \, dy = \frac{1}{2} \iint_{x<y} \gamma_x(x) \wedge \gamma_x(y) \, dx \, dy$$

yields the sum of the areas of the regions enclosed by the curve γ, multiplied by the corresponding winding number. In general, the quantity $\mathcal{A}(\gamma)$ provides an upper bound for the area of the convex hull of γ.

Let now $\gamma = \gamma(t,x)$ be a planar curve which evolves in time, according to the vector equation

$$\gamma_t + \lambda \gamma_x = \gamma_{xx} \,. \tag{8.5}$$

Here $\lambda = \lambda(t,x)$ is a sufficiently smooth scalar function. It is then clear that the length $\mathcal{L}\big(\gamma(t)\big)$ of the curve is a decreasing function of time. It was shown in [BiB2] that also the area functional $\mathcal{A}\big(\gamma(t)\big)$ is monotonically decreasing. Moreover, the amount of decrease dominates the area swept by the curve during its motion. An intutive way to see this is the following. In the special case where γ is a polygonal line, with vertices at the points $P_0, \ldots, P_m$, the integral in (8.4) reduces to a sum:

$$\mathcal{A}(\gamma) = \frac{1}{2} \sum_{i<j} \big|\mathbf{v}_i \wedge \mathbf{v}_j\big| \,, \qquad\qquad \mathbf{v}_i \doteq P_i - P_{i-1} \,.$$

If we now replace γ by a new curve γ' obtained by replacing two consecutive edges $\mathbf{v}_h$, $\mathbf{v}_k$ by one single edge (fig. 4b), the area between γ and γ' is precisely $|\mathbf{v}_h \wedge \mathbf{v}_k|/2$, while an easy computation yields

$$\mathcal{A}(\gamma') \leq \mathcal{A}(\gamma) - \frac{1}{2} |\mathbf{v}_h \wedge \mathbf{v}_k| \,.$$

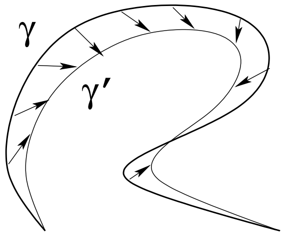


figure 4a

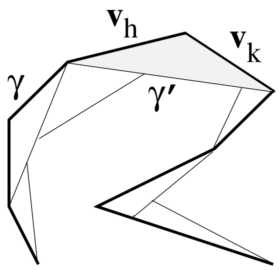


figure 4b

The estimate on the area swept by a smooth curve (fig. 4a) is now obtained by approximating a shortening curve γ by a sequence of polygonals, each obtained from the previous one by replacing two consecutive edges by a single segment.

We shall apply the previous geometric considerations toward a proof of the estimates of (8.1)-(8.2). Let v, w be two scalar functions, satisfying

$$\begin{aligned} v_t + (\lambda v)_x - v_{xx} &= \phi\,, \\ w_t + (\lambda w)_x - w_{xx} &= \psi\,. \end{aligned} \tag{8.6}$$

Define the planar curve γ by setting

$$\gamma(t,x) = \left(\int_{-\infty}^{x} v(t,y)dy,\ \int_{-\infty}^{x} w(t,y)dy \right). \tag{8.7}$$

Integrating (8.6) w.r.t. x, one finds the corresponding evolution equation for γ :

$$\gamma_t + \lambda\gamma_x - \gamma_{xx} = \Phi(t,x) \doteq \left(\int_{-\infty}^{x} \phi(t,y)dy,\ \int_{-\infty}^{x} \psi(t,y)dy \right). \tag{8.8}$$

In particular, if no sources were present, the motion of the curve would reduce to (8.5). At each fixed time t, we now define the **Length Functional** as

$$\mathcal{L}(t) = \mathcal{L}\big(\gamma(t)\big) = \int \sqrt{v^2(t,x) + w^2(t,x)}\ dx \tag{8.9}$$

and the **Area Functional** as

$$\mathcal{A}(t) = \mathcal{A}\big(\gamma(t)\big) = \frac{1}{2} \int\!\!\int_{x<y} \big|v(t,x)w(t,y) - v(t,y)w(t,x)\big|\, dxdy\,. \tag{8.10}$$

We now estimate the time derivative of the above functionals, in the general case when sources are present.

Lemma 8.1 *Let v, w be solutions of (8.6), defined for $t \in [0,T]$. For each t, assume that the maps $x \mapsto v(t,x)$ $x \mapsto w(t,x)$ and $x \mapsto \lambda(t,x)$ are $\mathcal{C}^{1,1}$, i.e. continuously differentiable with Lipschitz derivative. Then the corresponding area functional (8.10) satisfies*

$$\frac{d}{dt}\mathcal{A}(t) \leq -\int \Big|v_x(t,x)w(t,x) - v(t,x)w_x(t,x)\Big|dx + \big\|v(t)\big\|_{\mathbf{L}^1}\big\|\psi(t)\big\|_{\mathbf{L}^1} + \big\|w(t)\big\|_{\mathbf{L}^1}\big\|\phi(t)\big\|_{\mathbf{L}^1}\,. \tag{8.11}$$

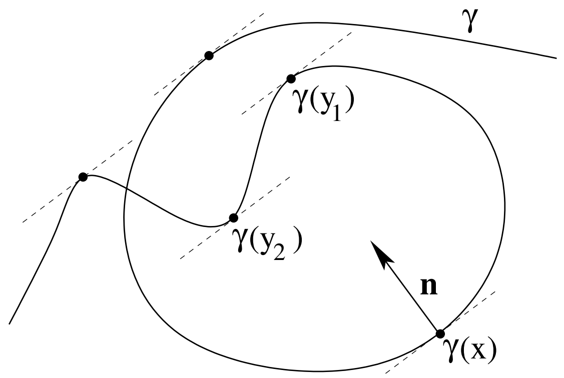


figure 5

Proof. In the following, given a curve γ, at each point x where $\gamma_x \neq 0$ we define the unit normal $\mathbf{n} = \mathbf{n}(x)$ (see fig. 5), oriented so that $\gamma_x(x) \wedge \mathbf{n} = \big|\gamma_x(x)\big| > 0$. For every vector $\mathbf{v} \in I\!R^2$ this implies

$$\gamma_x(x) \wedge \mathbf{v} = \big|\gamma_x(x)\big| \, \big\langle \mathbf{n}\,,\ \mathbf{v} \big\rangle\,.$$

Given a unit vector $\mathbf{n}$, we shall also consider the projection of γ along $\mathbf{n}$, namely

$$y \mapsto \chi^{\mathbf{n}}(y) \doteq \big\langle \mathbf{n}\,,\ \gamma(y) \big\rangle\,.$$

If $\gamma = \gamma(t,x)$ is any smooth curve evolving in time, the time derivative of the area functional in (8.4) can be computed as

$$\begin{aligned}
\frac{d\mathcal{A}}{dt} &= \frac{1}{2} \int\!\!\int_{x<y} \mathrm{sign}\big(\gamma_x(x) \wedge \gamma_x(y)\big) \Big\{ \gamma_{xt}(x) \wedge \gamma_x(y) + \gamma_x(x) \wedge \gamma_{xt}(y) \Big\}\, dxdy \\
&= \frac{1}{2} \int\!\!\int \mathrm{sign}\big(\gamma_x(x) \wedge \gamma_x(y)\big) \Big\{ \gamma_x(x) \wedge \gamma_{xt}(y) \Big\}\, dydx \\
&= \frac{1}{2} \int \big|\gamma_x(x)\big| \bigg(\int \mathrm{sign} \big\langle \mathbf{n}\,,\ \gamma_x(y) \big\rangle \cdot \big\langle \mathbf{n}\,,\ \gamma_{xt}(y) \big\rangle \, dy \bigg)\, dx \\
&= \frac{1}{2} \int \big|\gamma_x(x)\big| \, \frac{d}{dt} \Big(\mathrm{Tot.Var.}\{\chi^{\mathbf{n}}\} \Big)\, dx
\end{aligned}$$

For each x, we are here choosing the unit normal $\mathbf{n} = \mathbf{n}(x)$ to the curve $\gamma(t,\cdot)$ at the point x. We emphasize that, in the last expression, the derivative w.r.t. time of the total variation is taken regarding $\mathbf{n}$ as a constant. To compute this derivative, assume that the function $y \mapsto \chi^{\mathbf{n}(x)}(y)$ has a finite number of local maxima and minima, say attained at the points (fig. 5)

$$y_{-p} < \cdots < y_{-1} < y_0 = x < y_1 < \cdots < y_q\,.$$

Assume, in addition, that its derivative $d\chi^{\mathbf{n}}/dy$ changes sign across every such point. Then

$$\frac{d}{dt} \Big(\mathrm{Tot.Var.}\{\chi^{\mathbf{n}}\} \Big) = -\mathrm{sign} \big\langle \mathbf{n}\,,\ \gamma_{xx}(x) \big\rangle \cdot 2 \sum_{-p \le \alpha \le q} (-1)^{\alpha} \big\langle \mathbf{n}\,,\ \gamma_t(y_\alpha) \big\rangle\,. \tag{8.13}$$

Notice the sign factor in front of (8.13). If the inner product $\big\langle \mathbf{n}\,,\ \gamma_{xx}(x) \big\rangle$ is positive, the even indices α correspond to local minima and the odd indices to local maxima. The opposite is true if the inner product is negative. We now apply (8.13) the curve γ considered at (8.7)-(8.8). Observing that

$$\big\langle \mathbf{n}\,,\ \gamma_x(x)\big\rangle = 0\,, \qquad\qquad \mathrm{sign}\,\big\langle \mathbf{n}\,,\ \gamma_{xx}(y_\alpha)\big\rangle = (-1)^\alpha\cdot \mathrm{sign}\,\big\langle \mathbf{n}\,,\ \gamma_{xx}(x)\big\rangle\,,$$

one obtains

$$\begin{aligned}
\frac{d\mathcal{A}}{dt} &= -\int \big|\gamma_x(x)\big|\mathrm{sign}\,\big\langle \mathbf{n}\,,\ \gamma_{xx}(x)\big\rangle\cdot\left(\sum_{-p\le\alpha\le q}(-1)^\alpha\big\langle \mathbf{n}\,,\ \gamma_t(y_\alpha)\big\rangle\right)\,dx\\
&\le -\int\sum_\alpha \big|\gamma_x(x)\wedge\gamma_{xx}(y_\alpha)\big|\,dx + \int\big|\gamma_x(x)\big|\cdot\left|\sum_\alpha(-1)^\alpha\big\langle \mathbf{n}\,,\ \Phi(y_\alpha)\big\rangle\right|\,dx\\
&\le -\int\big|\gamma_x(x)\wedge\gamma_{xx}(x)\big|\,dx + \iint\big|\gamma_x(x)\wedge\Phi_x(y)\big|\,dydx\\
&= -\int\left|\begin{pmatrix}v(x)\\ w(x)\end{pmatrix}\wedge\begin{pmatrix}v_x(x)\\ w_x(x)\end{pmatrix}\right|\,dx + \iint\left|\begin{pmatrix}v(x)\\ w(x)\end{pmatrix}\wedge\begin{pmatrix}\phi(y)\\ \psi(y)\end{pmatrix}\right|\,dydx\,.
\end{aligned} \tag{8.14}$$

From this estimate, (8.11) clearly follows. Notice that, by an approximation argument, we can assume that the functions $\chi^{\mathbf{n}(x)}$ have the required regularity for almost every $(t,x)\in I\!R^2$. $\square$

Lemma 8.2. *Together with the hypotheses of Lemma 8.1, at a fixed time t assume that $\gamma_x(t,x)\neq 0$ for every x. Then*

$$\frac{d}{dt}\mathcal{L}(t)\le -\frac{1}{(1+9\delta_1^2)^{3/2}}\int_{|w/v|\le 3\delta_1}\big|v(t)\big|\,\left|\left(\frac{w(t)}{v(t)}\right)_x\right|^2\,dx + \big\|\phi(t)\big\|_{\mathbf{L}^1}+\big\|\psi(t)\big\|_{\mathbf{L}^1}\,. \tag{8.15}$$

Proof. As a preliminary, recalling that

$$\gamma_x = (v,w)\,, \qquad\qquad \gamma_{xt}+(\lambda\gamma_x)_x-\gamma_{xxx} = (\phi,\psi)\,,$$

we derive the identities

$$\begin{aligned}
|\gamma_{xx}|^2\,|\gamma_x|^2 - \big\langle\gamma_x,\,\gamma_{xx}\big\rangle^2 &= \big(v_x^2+w_x^2\big)(v^2+w^2) - \big(vv_x+ww_x\big)^2\\
&= \big(vw_x - v_xw\big)^2 = v^4\big|(w/v)_x\big|^2,
\end{aligned}$$

$$\frac{|v|^3}{|\gamma_x|^3} = \frac{1}{\big(1+(w/v)^2\big)^{3/2}}\,.$$

Thanks to the assumption that γ_x never vanishes, we can now integrate by parts and obtain

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) &= \int\frac{\big\langle\gamma_x,\,\gamma_{xt}\big\rangle}{\sqrt{\big\langle\gamma_x,\,\gamma_x\big\rangle}}\,dx = \int\left\{\frac{\big\langle\gamma_x,\,\gamma_{xxx}\big\rangle}{|\gamma_x|} - \frac{\big\langle\gamma_x,\,(\lambda\gamma_x)_x\big\rangle}{|\gamma_x|} + \frac{\big\langle\gamma_x,\,(\phi,\psi)\big\rangle}{|\gamma_x|}\right\}\,dx\\
&= \int\left\{|\gamma_x|_{xx} - \big(\lambda\,|\gamma_x|\big)_x - \frac{|\gamma_{xx}|^2 - \big\langle\gamma_x/|\gamma_x|,\,\gamma_{xx}\big\rangle^2}{|\gamma_x|}\right\}\,dx + \int\frac{\big\langle\gamma_x,\,(\phi,\psi)\big\rangle}{|\gamma_x|}\,dx\\
&\le -\int\frac{|v|\,\big|(w/v)_x\big|^2}{\big(1+(w/v)^2\big)^{3/2}}\,dx + \big\|\phi(t)\big\|_{\mathbf{L}^1}+\big\|\psi(t)\big\|_{\mathbf{L}^1}\,.
\end{aligned}$$

Since the integrand is non-negative, the last inequality clearly implies (8.15). $\square$

Remark 8.3. Let $u = u(t,x)$ be a solution to a scalar, viscous conservation law

$$u_t + f(u)_x - u_{xx} = 0\,,$$

and consider the planar curve $\gamma \doteq \big(u,\ f(u) - u_x\big)$ whose components are respectively the conserved quantity and the flux (fig. 6). If $\lambda \doteq f'$, the components $v \doteq u_x$ and $w \doteq -u_t$ evolve according to (8.6), with $\phi = \psi = 0$, hence $\gamma_t + \lambda\gamma_x - \gamma_{xx} = 0$. Defining the speed $s(x) \doteq -u_t(x)/u_x(x)$, the area functional $\mathcal{A}(\gamma)$ in (8.4) can now be written as

$$\begin{aligned}
\mathcal{A}(\gamma) &= \frac{1}{2}\iint_{x<y} \big|u_x(x)u_t(y) - u_t(x)u_x(y)\big|\,dxdy \\
&= \frac{1}{2}\iint_{x<y} \big|u_x(x)\,dx\big|\cdot\big|u_x(y)\,dy\big|\cdot\big|s(x)-s(y)\big| \\
&= \frac{1}{2}\iint_{x<y} [\text{wave at } x]\times[\text{wave at } y]\times[\text{difference in speeds}]\,.
\end{aligned}$$

It now becomes clear that the area functional can be regarded as an interaction potential between waves of the same family. In the case where viscosity is present, this provides a counterpart to the interaction functional introduced in [L4] in connection with strictly hyperbolic systems.

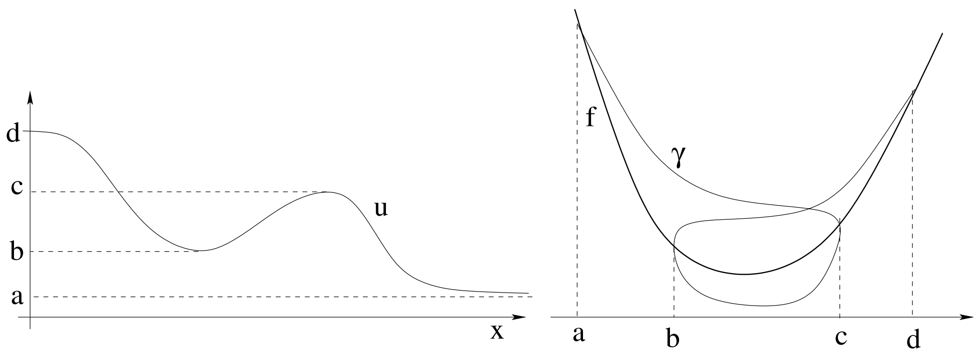


figure 6

Recalling that the components v_i, w_i satisfy the equations (6.1), we can apply the previous lemmas with $v \doteq v_i$, $w \doteq w_i$, $\lambda \doteq \tilde\lambda_i$, $\phi \doteq \phi_i$, $\psi \doteq \psi_i$, calling $\mathcal{L}_i$ and $\mathcal{A}_i$ the corresponding length and area functionals. For $t \in [\hat t,\, T]$, the bounds (5.22)–(5.23) yield

$$\mathcal{A}_i(t) \le \big\|v_i(t)\big\|_{\mathbf{L}^\infty}\cdot\big\|w_i(t)\big\|_{\mathbf{L}^1} = \mathcal{O}(1)\cdot\delta_0^3, \tag{8.16}$$

$$\mathcal{L}_i(t) \le \big\|v_i(t)\big\|_{\mathbf{L}^1} + \big\|w_i(t)\big\|_{\mathbf{L}^1} = \mathcal{O}(1)\cdot\delta_0. \tag{8.17}$$

Using (8.11) we now obtain

$$\begin{aligned}
&\int_{\hat t}^T\int \big|w_{i,x}v_i - w_i v_{i,x}\big|\,dxdt \\
&\qquad\le \int_{\hat t}^T\left[-\frac{d}{dt}\mathcal{A}_i(t)\right]dt + \int_{\hat t}^T\Big(\big\|v_i(t)\big\|_{\mathbf{L}^1}\big\|\psi_i(t)\big\|_{\mathbf{L}^1} + \big\|w_i(t)\big\|_{\mathbf{L}^1}\big\|\phi_i(t)\big\|_{\mathbf{L}^1}\Big)\,dt \\
&\qquad\le \mathcal{A}_i(\hat t) + \sup_{t\in[\hat t,T]}\Big(\big\|v_i(t)\big\|_{\mathbf{L}^1} + \big\|w_i(t)\big\|_{\mathbf{L}^1}\Big)\cdot\int_{\hat t}^T\int\Big(\big|\phi_i(t,x)\big| + \big|\psi_i(t,x)\big|\Big)\,dxdt \\
&\qquad= \mathcal{O}(1)\cdot\delta_0^2\,,
\end{aligned} \tag{8.18}$$

proving (8.1). To establish (8.2), we first observe that, by an approximation argument, it is not restrictive to assume that the set of points in the t-x plane where $v_{i,x}(t,x) = w_{i,x}(t,x) = 0$ is at most countable. In this case, for almost every $t \in [\hat{t}, T]$ the inequality (8.15) is valid. Moreover, our choice $\delta_1 \leq 1/3$ implies $(1 + 9\delta_1^2)^{3/2} < 4$. Therefore

$$\begin{aligned}\int_{\hat{t}}^{T} \int_{|w_i/v_i| < 3\delta_1} |v_i| \left| \left(\frac{w_i}{v_i} \right)_x \right|^2 dx dt &\leq 4 \int_{\hat{t}}^{T} \left[-\frac{d}{dt} \mathcal{L}_i(t) \right] dt + \int_{\hat{t}}^{T} \Big(\big\| \phi_i(t) \big\|_{\mathbf{L}^1} + \big\| \psi_i(t) \big\|_{\mathbf{L}^1} \Big) dt \\ &\leq 4 \mathcal{L}_i(\hat{t}) + \int_{\hat{t}}^{T} \int \Big(\big| \phi_i(t,x) \big| + \big| \psi_i(t,x) \big| \Big) dx dt \\ &= \mathcal{O}(1) \cdot \delta_0 .\end{aligned} \tag{8.19}$$

Using the bound (5.23) on $\|v_i\|_{\mathbf{L}^\infty}$, from (8.19) we deduce (8.2).

9 - Energy estimates

In the same setting as the two previous sections, we shall now prove the estimate

$$\int_{\hat{t}}^{T} \int \big(|v_{i,x}| + |w_{i,x}| \big) \, |w_i - \theta_i v_i| \, dx dt = \mathcal{O}(1) \cdot \delta_0^2 . \tag{9.1}$$

We recall that $\theta_i \doteq \theta(w_i/v_i)$, where θ is the cut-off function introduced in (5.5). Notice that the integrand can be $\neq 0$ only when $|w_i/v_i| > \delta_1$.

Consider another cut-off function $\eta : \mathbb{R} \mapsto [0,1]$ such that (fig. 2)

$$\eta(s) = \begin{cases} 0 & \text{if} \quad |s| \leq 3\delta_1/5 , \\ 1 & \text{if} \quad |s| \geq 4\delta_1/5 . \end{cases} \tag{9.2}$$

We can assume that η is a smooth even function, such that

$$|\eta'| \leq 21/\delta_1 , \qquad\qquad |\eta''| \leq 101/\delta_1^2 .$$

A third cut-off function

$$\bar{\eta}(s) \doteq \eta\big(|s| - \delta_1/5 \big) \leq \eta(s)$$

will also be used. For convenience, we shall write $\eta_i \doteq \eta(w_i/v_i)$, $\bar{\eta}_i \doteq \bar{\eta}(w_i/v_i)$. As a preliminary, we prove some simple estimates relating the sizes of v_i, w_i and $v_{i,x}$. It is here useful to keep in mind the bounds

$$\tilde{\lambda}_i - \lambda_i^* = \mathcal{O}(1) \cdot |\tilde{r}_i - r_i^*| = \mathcal{O}(1) \cdot \delta_0 , \qquad\qquad |v_i|, \, |w_i| = \mathcal{O}(1) \cdot \delta_0^2 , \tag{9.3}$$

valid for $t \geq \hat{t}$ and $i = 1, \ldots, n$. Recall also our choice of the constants

$$0 < \delta_0 << \delta_1 \leq \frac{1}{3} . \tag{9.4}$$

Lemma 9.1. *If* $|w_i/v_i| \geq 3\delta_1/5$, *then*

$$|w_i| \leq 2|v_{i,x}| + \mathcal{O}(1) \cdot \delta_0 \sum_{j \neq i} |v_j| , \qquad\qquad |v_i| \leq \frac{5}{2\delta_1} |v_{i,x}| + \mathcal{O}(1) \cdot \delta_0 \sum_{j \neq i} |v_j| . \tag{9.5}$$

On the other hand, if $|w_i/v_i| \leq \delta_1$, then
$$|v_{i,x}| \leq 2\delta_1 |v_i| + \mathcal{O}(1)\cdot\delta_0 \sum_{j\neq i} |v_j|\,. \tag{9.6}$$

Proof. We recall the first estimate in (6.18):
$$\begin{aligned} v_{i,x} &= w_i + (\tilde\lambda_i - \lambda_i^*)v_i + \Theta_i \\ &= w_i + (\tilde\lambda_i - \lambda_i^*)v_i + \mathcal{O}(1)\cdot\delta_0\sum_{j\neq i}\big(|v_j| + |w_j - v_{j,x}|\big)\,, \end{aligned} \tag{9.7}$$
with Θ_i defined as in (6.16). By (9.3)-(9.4), from the condition $|w_i/v_i| \geq 3\delta_1/5$ two cases can arise. On one hand, if
$$|\Theta_i| \leq \frac{\delta_1}{10}|v_i|\,, \tag{9.8}$$
then
$$|v_{i,x}| \geq \frac{3\delta_1}{5}|v_i| - \mathcal{O}(1)\cdot\delta_0|v_i| - \frac{\delta_1}{10}|v_i| \geq \frac{2\delta_1}{5}|v_i|\,,$$
and hence
$$|v_i| \leq \frac{5}{2\delta_1}|v_{i,x}|, \qquad\qquad |w_i| \leq |v_{i,x}| + \frac{\delta_1}{5}|v_i| \leq 2|v_{i,x}|\,. \tag{9.9}$$
On the other hand, if (9.8) fails, then by (9.3) and (6.17) we conclude
$$\big|\tilde\lambda_i - \lambda_i^*\big|\,|v_i| = \mathcal{O}(1)\cdot\delta_0^2\sum_{j\neq i}|v_j|\,. \tag{9.10}$$
In both cases, the estimates in (9.5) hold.

Next, assume that $|w_i/v_i| \leq \delta_1$. Observing that $|\tilde\lambda_i - \lambda_i^*| = \mathcal{O}(1)\cdot\delta_0 < \delta_1/2$ and using (6.16)-(6.17), from (9.7) we deduce
$$|v_{i,x}| \leq \delta_1|v_i| + \frac{\delta_1}{2}|v_i| + \mathcal{O}(1)\cdot\delta_0\sum_j|v_j|\,, \tag{9.11}$$
proving (9.6). □

Toward a proof of the estimate (9.1), we first reduce the integrand to a more tractable expression. Since the term $|w_i - \theta_i v_i|$ vanishes when $|w_i/v_i| \leq \delta_1$, and is $\leq |w_i|$ otherwise, by (9.5) we always have the bound
$$|w_i - \theta_i v_i| \leq |\bar\eta_i w_i| \leq \bar\eta_i\left(2|v_{i,x}| + \mathcal{O}(1)\cdot\delta_0\sum_{j\neq i}|v_j|\right).$$
Therefore
$$\begin{aligned} \big(|v_{i,x}| + |w_{i,x}|\big)\cdot|w_i - \theta_i v_i| &\leq \big(|v_{i,x}| + |w_{i,x}|\big)\bar\eta_i\left(2|v_{i,x}| + \mathcal{O}(1)\cdot\delta_0\sum_{j\neq i}|v_j|\right) \\ &\leq 2\eta_i v_{i,x}^2 + 2\bar\eta_i|v_{i,x}w_{i,x}| + \sum_{j\neq i}\big(|v_j v_{i,x}| + |v_j w_{i,x}|\big) \\ &\leq 3\eta_i v_{i,x}^2 + \bar\eta_i w_{i,x}^2 + \sum_{j\neq i}\big(|v_j v_{i,x}| + |v_j w_{i,x}|\big) \end{aligned} \tag{9.12}$$

.

Notice that here we can assume $\mathcal{O}(1)\cdot\delta_0<1$. Since we already proved the bounds (7.3) on the integrals of transversal terms, to prove (9.1) we only need to consider the integrals of $v_{i,x}^2$ and $w_{i,x}^2$, in the region where $\eta_i\neq 0$. In both cases, energy type estimates will be used.

We start with $v_{i,x}^2$. Multiplying the first equation in (6.1) by $\eta_i v_i$ and integrating by parts, we obtain

$$\begin{aligned}
\int \eta_i v_i\phi_i\,dx &= \int\Big\{\eta_i v_i v_{i,t}+\eta_i v_i(\tilde\lambda_i v_i)_x-\eta_i v_i v_{i,xx}\Big\}\,dx\\
&=\int\Big\{\eta_i(v_i^2/2)_t-\eta_i\tilde\lambda_i v_i v_{i,x}-\eta_{i,x}\tilde\lambda_i v_i^2+\eta_i v_{i,x}^2+\eta_{i,x}v_{i,x}v_i\Big\}\,dx\\
&=\int\Big\{\big(\eta_i v_i^2/2\big)_t+(\tilde\lambda_i\eta_i)_x(v_i^2/2)-\big(\eta_{i,t}+2\tilde\lambda_i\eta_{i,x}-\eta_{i,xx}\big)(v_i^2/2)+\eta_i v_{i,x}^2+2\eta_{i,x}v_i v_{i,x}\Big\}\,dx\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
\int \eta_i v_{i,x}^2\,dx = -\frac{d}{dt}\left[\int \eta_i v_i^2/2\,dx\right]+\int\big(\eta_{i,t}+\tilde\lambda_i\eta_{i,x}-\eta_{i,xx}\big)(v_i^2/2)\,dx\\
-\int\tilde\lambda_{i,x}\eta_i(v_i^2/2)\,dx-2\int\eta_{i,x}v_i v_{i,x}\,dx+\int\eta_i v_i\phi_i\,dx\,.
\end{aligned}\tag{9.13}$$

A direct computation yields

$$\begin{aligned}
\eta_{i,t}+\tilde\lambda_i\eta_{i,x}-\eta_{i,xx} &= \eta_i'\left(\frac{w_{i,t}}{v_i}-\frac{v_{i,t}w_i}{v_i^2}\right)+\tilde\lambda_i\eta_i'\left(\frac{w_{i,x}}{v_i}-\frac{v_{i,x}w_i}{v_i^2}\right)\\
&\quad-\eta_i''\left(\frac{w_i}{v_i}\right)_x^2-\eta_i'\left(\frac{w_{i,xx}}{v_i}-\frac{v_{i,xx}w_i}{v_i^2}-2\frac{v_{i,x}w_{i,x}}{v_i^2}+2\frac{v_{i,x}^2 w_i}{v_i^3}\right)\\
&=\Big[\eta_i'\big(w_{i,t}+(\tilde\lambda_i w_i)_x-w_{i,xx}\big)/v_i-\eta_i' w_i\big(v_{i,t}+(\tilde\lambda_i v_i)_x-v_{i,xx}\big)/v_i^2\Big]\\
&\quad+2v_{i,x}\eta_i'/v_i\cdot(w_i/v_i)_x-\eta_i''(w_i/v_i)_x^2\\
&=\eta_i'\left(\frac{\psi_i}{v_i}-\frac{w_i}{v_i}\frac{\phi_i}{v_i}\right)+2\eta_i'\frac{v_{i,x}}{v_i}\left(\frac{w_i}{v_i}\right)_x-\eta_i''\left(\frac{w_i}{v_i}\right)_x^2.
\end{aligned}\tag{9.14}$$

Since $\tilde\lambda_{i,x}=(\tilde\lambda_i-\lambda_i^*)_x$, integrating by parts and using the second estimate in (9.5) one obtains

$$\begin{aligned}
\left|\int\tilde\lambda_{i,x}\eta_i(v_i^2/2)\,dx\right| &= \left|\int(\tilde\lambda_i-\lambda_i^*)\big(\eta_{i,x}v_i^2/2+\eta_i v_i v_{i,x}\big)\,dx\right|\\
&\le\|\tilde\lambda_i-\lambda_i^*\|_{\mathbf{L}^\infty}\cdot\left\{\frac12\int|\eta_i'|\,|w_{i,x}v_i-v_{i,x}w_i|\,dx+\frac{5}{2\delta_1}\int\eta_i v_{i,x}^2\,dx+\mathcal{O}(1)\cdot\delta_0\int\sum_{j\neq i}|v_{i,x}v_j|\,dx\right\}\\
&\le\int|w_{i,x}v_i-v_{i,x}w_i|\,dx+\frac12\int\eta_i v_{i,x}^2\,dx+\delta_0\int\sum_{j\neq i}|v_{i,x}v_j|\,dx\,.
\end{aligned}\tag{9.15}$$

Indeed, by (9.3)-(9.4), $|\tilde\lambda_i-\lambda_i^*|=\mathcal{O}(1)\cdot\delta_0<<\delta_1$. Using (9.14)-(9.15) in (9.13) we now obtain

$$\begin{aligned}\frac12\int\eta_i\,v_{i,x}^2\,dx\le&-\frac{d}{dt}\left[\int\frac{\eta_iv_i^2}{2}\,dx\right]+\frac12\int|\eta_i'|\big(|v_i\psi_i|+|w_i\phi_i|\big)\,dx+\int\left|\eta_i'v_iv_{i,x}\left(\frac{w_i}{v_i}\right)_x\right|\,dx\\&+\frac12\int\left|\eta_i''v_i^2\left(\frac{w_i}{v_i}\right)_x^2\right|\,dx+\int\big|w_{i,x}v_i-w_iv_{i,x}\big|\,dx\\&+\delta_0\int\sum_{j\neq i}|v_{i,x}v_j|\,dx+2\int|\eta_{i,x}v_iv_{i,x}|\,dx+\int|v_i\phi_i|\,dx\,.\end{aligned}\tag{9.16}$$

Recalling the definition of η_i, on regions where $\eta_i'\neq0$ one has $|w_i/v_i|<\delta_1$, hence the bounds (9.6) hold. In turn, they imply

$$\begin{aligned}\big|\eta_{i,x}v_iv_{i,x}\big|&=\left|\eta_i'v_iv_{i,x}\left(\frac{w_i}{v_i}\right)_x\right|\\&\le2\left|\delta_1\eta_i'v_i^2\left(\frac{w_i}{v_i}\right)_x\right|+\mathcal{O}(1)\cdot\delta_0\sum_{j\neq i}\left|\eta_i'v_iv_j\left(\frac{w_i}{v_i}\right)_x\right|\\&\le2\big|\delta_1\eta_i'\big|\,|w_{i,x}v_i-w_iv_{i,x}|+\mathcal{O}(1)\cdot\delta_0|\eta_i'|\sum_{j\neq i}\left(|v_jw_{i,x}|+|v_jv_{i,x}|\left|\frac{w_i}{v_i}\right|\right)\,.\end{aligned}\tag{9.17}$$

Using the bounds (5.22)-(5.23). (7.2)-(7.3) and (8.1)-(8.2), from (9.16) we conclude

$$\begin{aligned}\int_{\hat t}^T\int\eta_i\,v_{i,x}^2\,dxdt\le&\int\eta_iv_i^2(\hat t,x)\,dx+\mathcal{O}(1)\cdot\int_{\hat t}^T\int\big(|v_i\psi_i|+|w_i\phi_i|\big)\,dxdt\\&+\mathcal{O}(1)\cdot\int_{\hat t}^T\int|w_{i,x}v_i-w_iv_{i,x}|\,dxdt+\mathcal{O}(1)\cdot\delta_0\int_{\hat t}^T\int\sum_{j\neq i}\big(|v_jw_{i,x}|+|v_jv_{i,x}|\big)\,dxdt\\&+\mathcal{O}(1)\cdot\int_{\hat t}^T\int_{|w_i/v_i|<\delta_1}\big|v_i(w_i/v_i)_x\big|^2\,dxdt+2\delta_0\int_{\hat t}^T\int\sum_{j\neq i}|v_{i,x}v_j|\,dxdt+2\int_{\hat t}^T\int|v_i\phi_i|\,dxdt\\=&\,\mathcal{O}(1)\cdot\delta_0^2\,.\end{aligned}\tag{9.18}$$

We now perform a similar computation for $w_{i,x}^2$. Multiplying the second equation in (6.1) by $\bar\eta_iw_i$ and integrating by parts, we obtain

$$\int\bar\eta_iw_i\psi_i\,dx=\int\Big\{\big(\bar\eta_iw_i^2/2\big)_t+(\tilde\lambda_i\bar\eta_i)_x(w_i^2/2)-\big(\bar\eta_{i,t}+2\tilde\lambda_i\bar\eta_{i,x}-\bar\eta_{i,xx}\big)(w_i^2/2)+\bar\eta_iw_{i,x}^2+2\bar\eta_{i,x}w_iw_{i,x}\Big\}\,dx\,.$$

Therefore, the identity (9.13) still holds, with v_i,ϕ_i replaced by w_i,ψ_i, respectively:

$$\begin{aligned}\int\bar\eta_iw_{i,x}^2\,dx=&-\frac{d}{dt}\left[\int\bar\eta_iw_i^2/2\,dx\right]+\int\big(\bar\eta_{i,t}+\tilde\lambda_i\bar\eta_{i,x}-\bar\eta_{i,xx}\big)(w_i^2/2)\,dx\\&-\int\tilde\lambda_{i,x}\bar\eta_i(w_i^2/2)\,dx-2\int\bar\eta_{i,x}w_iw_{i,x}\,dx+\int\bar\eta_iw_i\psi_i\,dx\,.\end{aligned}\tag{9.19}$$

The equality (9.14) can again be used, with η_i replaced by $\bar\eta_i$. To obtain a suitable replacement for (9.15), we observe that, if $\bar\eta_i\neq0$ then (9.5) implies

$$|w_iw_{i,x}|\le2|v_{i,x}w_{i,x}|+\mathcal{O}(1)\cdot\delta_0\sum_{j\neq i}|v_jw_{i,x}|\le v_{i,x}^2+w_{i,x}^2+\mathcal{O}(1)\cdot\delta_0\sum_{j\neq i}|v_jw_{i,x}|\,.$$

Integrating by parts we thus obtain

$$
\begin{aligned}
\left|\int \tilde\lambda_{i,x}\bar\eta_i (w_i^2/2)\,dx\right| &= \left|\int (\tilde\lambda_i-\lambda_i^*)\big(\bar\eta_{i,x}w_i^2/2+\bar\eta_i w_i w_{i,x}\big)\,dx\right|\\
&\le \|\tilde\lambda_i-\lambda_i^*\|_{\mathbf{L}^\infty}\cdot\left\{\int |\bar\eta_i'|\,|w_{i,x}v_i-v_{i,x}w_i|\,\left|\frac{w_i^2}{v_i^2}\right|\,dx+\int \eta_i v_{i,x}^2\,dx\right.\\
&\qquad\qquad \left.+\int \bar\eta_i w_{i,x}^2\,dx+\mathcal{O}(1)\cdot\delta_0\int\sum_{j\neq i}|v_j w_{i,x}|\,dx\right\}\\
&\le \int |w_{i,x}v_i-v_{i,x}w_i|\,dx+\frac12\int \eta_i v_{i,x}^2\,dx+\frac12\int\bar\eta_i w_{i,x}^2\,dx+\mathcal{O}(1)\cdot\delta_0\int\sum_{j\neq i}|v_jw_{i,x}|\,dx\,.
\end{aligned}
\tag{9.20}
$$

Using (9.14) and (9.20) in (9.19) and observing that $|w_i^2/v_i^2|\le\delta_1^2$ on the region where $\bar\eta_i'\neq 0$, we now obtain an estimate similar to (9.16):

$$
\begin{aligned}
\frac12\int\bar\eta_i\,w_{i,x}^2\,dx &\le -\frac{d}{dt}\left[\int\frac{\bar\eta_i w_i^2}{2}\,dx\right]+\frac{\delta_1^2}{2}\int|\bar\eta_i'|\big(|v_i\psi_i|+|w_i\phi_i|\big)\,dx+\delta_1^2\int\left|\bar\eta_i' v_i v_{i,x}\left(\frac{w_i}{v_i}\right)_x\right|\,dx\\
&\quad+\frac{\delta_1^2}{2}\int\left|\bar\eta_i'' v_i^2\left(\frac{w_i}{v_i}\right)_x^2\right|\,dx+\int\big|w_{i,x}v_i-w_iv_{i,x}\big|\,dx+\frac12\int\eta_i v_{i,x}^2\,dx\\
&\quad+\delta_0\int\sum_{j\neq i}|v_jw_{i,x}|\,dx+2\int|\bar\eta_{i,x}w_iw_{i,x}|\,dx+\int|w_i\psi_i|\,dx\,.
\end{aligned}
\tag{9.21}
$$

We now observe that $\bar\eta_i'\neq 0$ only when $4\delta_1/5<|w_i/v_i|<\delta_1$. In this case one has $\eta_i=1$ and moreover, recalling our choice $\delta_1<1/3$,

$$
\left|v_i\left(\frac{w_i}{v_i}\right)_x\right|^2 \;\ge\; w_{i,x}^2-2\left|\frac{w_i}{v_i}\right||w_{i,x}v_{i,x}|-\left|\frac{w_i}{v_i}\right|^2 v_{i,x}^2\;\ge\;\frac12 w_{i,x}^2-\frac12 v_{i,x}^2\,.
$$

Hence

$$
\big|\bar\eta_{i,x}w_iw_{i,x}\big|=\mathcal{O}(1)\cdot\big|v_i(w_i/v_i)_x\big|^2\cdot\chi_{\{|w_i/v_i|<\delta_1\}}+\mathcal{O}(1)\cdot\eta_i v_{i,x}^2\,.
\tag{9.22}
$$

Using the bounds (5.22)-(5.23), (7.2)-(7.3), (8.1)-(8.2), (9.17)-(9.18) and (9.22), from (9.21) we conclude

$$
\begin{aligned}
\int_{\hat t}^T\int\bar\eta_i\,w_{i,x}^2\,dxdt &\le \int\bar\eta_i w_i^2(\hat t,x)\,dx+\mathcal{O}(1)\cdot\int_{\hat t}^T\int\big(|v_i\psi_i|+|w_i\phi_i|\big)\,dxdt\\
&\quad+\mathcal{O}(1)\cdot\int_{\hat t}^T\int|w_{i,x}v_i-w_iv_{i,x}|\,dxdt+\mathcal{O}(1)\cdot\delta_0\int_{\hat t}^T\int\sum_{j\neq i}\big(|v_jw_{i,x}|+|v_jv_{i,x}|\big)\,dxdt\\
&\quad+\mathcal{O}(1)\cdot\int_{\hat t}^T\int_{|w_i/v_i|<\delta_1}\big|v_i(w_i/v_i)_x\big|^2\,dxdt+\mathcal{O}(1)\cdot\int_{\hat t}^T\int\eta_i\,v_{i,x}^2\,dxdt\\
&\quad+\delta_0\int_{\hat t}^T\int\sum_{j\neq i}|w_{i,x}v_j|\,dxdt+2\int_{\hat t}^T\int|w_i\psi_i|\,dxdt\\
&=\mathcal{O}(1)\cdot\delta_0^2\,.
\end{aligned}
\tag{9.23}
$$

Using (9.18) and (9.23) in (9.12), we obtain the desired estimate (9.1).

10 - Proof of the BV estimates

In this section we conclude the proof of the uniform BV bounds. Consider any initial data $\bar u : I\!R \mapsto I\!R^n$, with

$$\text{Tot.Var.}\{\bar u\} \leq \frac{\delta_0}{8\sqrt{n}\,\kappa}, \qquad \lim_{x\to-\infty} \bar u(x) = u^* \in K. \tag{10.1}$$

We recall that κ is the constant defined at (2.5), related to the Green kernel G^* of the linearized equation (2.4). This constant actually depends on the matrix $A(u^*)$, but it is clear that it remains uniformly bounded when u^* varies in a compact set $K \subset I\!R^n$.

An application of Corollary 2.4 yields the existence of the solution to the Cauchy problem (1.10), (1.2) on an initial interval $[0, \hat t]$, satisfying the bound

$$\big\| u_x(\hat t)\big\|_{\mathbf{L}^1} \leq \frac{\delta_0}{4\sqrt{n}}. \tag{10.2}$$

This solution can be prolonged in time as long as its total variation remains small. Define the time

$$T \doteq \sup\left\{\tau\ ;\quad \sum_i \int_{\hat t}^{\tau}\!\!\int \big|\phi_i(t,x)\big| + \big|\psi_i(t,x)\big|\,dxdt \leq \frac{\delta_0}{2}\right\}. \tag{10.3}$$

If $T < \infty$, a contradiction is obtained as follows. By (5.21) and (10.2), for all $t \in [\hat t,\, T]$ one has

$$\begin{aligned}
\big\|u_x(t)\big\|_{\mathbf{L}^1} &\leq \sum_i \big\|v_i(t)\big\|_{\mathbf{L}^1}\\
&\leq \sum_i \left(\big\|v_i(\hat t)\big\|_{\mathbf{L}^1} + \int_{\hat t}^{T}\!\!\int \big|\phi_i(t,x)\big|\,dxdt\right)\\
&\leq 2\sqrt{n}\,\big\|u_x(\hat t)\big\|_{\mathbf{L}^1} + \frac{\delta_0}{2} \ \leq\ \delta_0\,.
\end{aligned} \tag{10.4}$$

Using Lemma 6.1 and the bounds (7.3), (8.1), (8.2) and (9.1) we now obtain

$$\sum_i \int_{\hat t}^{T}\!\!\int \big|\phi_i(t,x)\big| + \big|\psi_i(t,x)\big|\,dxdt = \mathcal{O}(1)\cdot\delta_0^2 \ <\ \frac{\delta_0}{2}, \tag{10.5}$$

provided that δ_0 was chosen suitably small. Therefore T cannot be a supremum. This contradiction with (10.3) shows that the total variation remains $< \delta_0$ for all $t \in [\hat t,\, \infty[$. In particular, the solution u is globally defined.

Remark 10.1. The estimates (8.1) and (9.1) were obtained under the assumption (7.2) on the source terms. A posteriori, by (10.5) the integral of the source terms is quadratic w.r.t. δ_0. Using (10.5) instead of (7.2) in the inequalities (8.18) and (9.18), (9.23), we now see that the quantities in (8.1) and (9.1) are both $= \mathcal{O}(1)\cdot\delta_0^3$. Recalling that δ_0 is the order of magnitude of the total variation, we see here another analogy with the the purely hyperbolic case [G]. Namely, the total amount of interactions between waves of different families is of quadratic order w.r.t. the total variation, while the interaction between waves of the same family is cubic.

Remark 10.2. Within the previous proof, we constructed wave speeds $\sigma_i \doteq \lambda_i^* - \theta(w_i/v_i)$ for which the following holds. Decomposing the gradients u_x, u_t according to

$$\begin{cases} u_x = \sum_i v_i\, \tilde r_i(u, v_i, \sigma_i), \\ u_t = \sum_i (w_i - \lambda_i^* v_i)\, \tilde r_i(u, v_i, \sigma_i), \end{cases} \tag{10.6}$$

the components v_i, w_i then satisfy

$$\begin{cases} v_{i,t} + (\tilde\lambda_i v_i)_x - v_{i,xx} = \phi_i(t,x)\,, \\ w_{i,t} + (\tilde\lambda_i w_i)_x - w_{i,xx} = \psi_i(t,x)\,, \end{cases} \tag{10.7}$$

where all source terms ϕ_i,ψ_i are integrable:

$$\int_0^\infty \int \big|\phi_i(t,x)\big|\, dxdt < \delta_0\,, \qquad\qquad \int_0^\infty \int \big|\psi_i(t,x)\big|\, dxdt < \delta_0\,. \tag{10.8}$$

In general, the speeds σ_i defined at (5.7) are not even continuous, as functions of t, x. However, by a suitable mollification we can find slightly different speed functions $\sigma_i(t,x)$ which are smooth and such that the corresponding decomposition (10.6) is achieved in terms of (smooth) functions v_i, w_i satisfying a system of the form (10.7), with source terms again bounded as in (10.8).

We conclude this section by studying the continuous dependence w.r.t. time of the solution $t \mapsto u(t,\cdot)$. By (10.4) we have

$$\text{Tot.Var.}\big\{u(t)\big\} = \big\|u_x(t)\big\|_{\mathbf{L}^1} \le \delta_0 \qquad \text{for all } \ t > 0. \tag{10.9}$$

By the estimate (2.8) in Proposition 2.1, the second derivative satisfies

$$\big\|u_{xx}(t)\big\|_{\mathbf{L}^1} \le \begin{cases} 2\kappa\delta_0/\sqrt{t} & \text{if } \ t < \hat t, \\ 2\kappa\delta_0/\sqrt{\hat t} & \text{if } \ t \ge \hat t. \end{cases} \tag{10.10}$$

Therefore, from (1.10) it easily follows

$$\big\|u_t(t)\big\|_{\mathbf{L}^1} \le L'\Big(1 + \frac{1}{2\sqrt{t}}\Big),$$

for some constant L'. For any $t > s \ge 0$ we now have

$$\begin{aligned} \big\|u(t) - u(s)\big\|_{\mathbf{L}^1} &\le \int_s^t \big\|u_t(\tau)\big\|_{\mathbf{L}^1}\, d\tau \\ &\le L'\Big(|t-s| + \big|\sqrt{t} - \sqrt{s}\,\big|\Big). \end{aligned} \tag{10.11}$$

Remark 10.3. A more careful analysis shows that in (10.11) one can actually take $L' = \mathcal{O}(1)\cdot$ Tot.Var.$\{\bar u\}$. However, this sharper estimate will not be needed in the sequel.

11 - Stability estimates

Let $u = u(t,x)$ be any solution of (3.1) with small total variation. The evolution of a first order perturbation $z = z(t,x)$ is then governed by the linear equation

$$z_t + \big(A(u)z\big)_x - z_{xx} = \big(u_x \bullet A(u)\big)z - \big(z \bullet A(u)\big)u_x\,. \tag{11.1}$$

As usual, by "$\bullet$" we denote a directional derivative. The primary goal of our analysis is to establish the bound

$$\big\|z(t,\cdot)\big\|_{\mathbf{L}^1} \leq L\,\big\|z(0,\cdot)\big\|_{\mathbf{L}^1} \qquad \text{for all } \ t \geq 0\,, \tag{11.2}$$

for some constant L. By a standard homotopy argument [B1], [BiB1], this implies the uniform stability of solutions, w.r.t. the $\mathbf{L}^1$ distance. Indeed, consider two initial data $\bar u, \bar v$ with suitably small total variation. We can assume that $u^* \doteq \bar u(-\infty) = \bar v(-\infty)$, otherwise $\|\bar u - \bar v\|_{\mathbf{L}^1} = \infty$ and there is nothing to prove. Consider the smooth path of initial data

$$\theta \mapsto \bar u^\theta \doteq \theta \bar u + (1-\theta)\bar v, \qquad\qquad \theta \in [0,1],$$

and call $t \mapsto u^\theta(t,\cdot)$ the solution of (3.1) with initial data $\bar u^\theta$. The tangent vector

$$z^\theta(t,x) \doteq \frac{du^\theta}{d\theta}(t,x)$$

is then a solution of the linearized Cauchy problem

$$z_t^\theta + \big[DA(u^\theta)\cdot z^\theta\big]u_x^\theta + A(u^\theta)z_x^\theta = z_{xx}^\theta,$$

$$z^\theta(0,x) = \bar z^\theta(x) = \bar u(x) - \bar v(x),$$

hence it satisfies (11.2) for every θ. For every $t \geq 0$ we now have

$$\begin{aligned}\big\|u(t) - v(t)\big\|_{\mathbf{L}^1} &\leq \int_0^1 \left\|\frac{du^\theta(t)}{d\theta}\right\|_{\mathbf{L}^1} d\theta \\ &\leq L\cdot \int_0^1 \left\|\frac{du^\theta(0)}{d\theta}\right\|_{\mathbf{L}^1} d\theta \\ &= L\cdot \|\bar u - \bar v\|_{\mathbf{L}^1}\,.\end{aligned}$$

This proves the Lipschitz continuous dependence of solutions of (3.1) w.r.t. the initial data, with a Lipschitz constant independent of time. In particular, it shows that all solutions with small total variation are uniformly stable.

Remark 11.1. In the hyperbolic case, a priori estimates on first order tangent vectors for solutions with shocks were first derived in [B2]. However, even with the aid of these estimates, controlling the $\mathbf{L}^1$ distance between any two solutions remains a difficult task. Indeed, a straightforward use of the homotopy argument fails, due to lack of regularity. These difficulties were eventually overcome in [BC1] and [BCP], at the price of heavy technicalities. On the other hand, in the present case with viscosity, all solutions are smooth and the homotopy argument goes through without any effort.

Throughout the following, we consider a reference solution $u = u(t,x)$ of (3.1) with small total variation. According to Remark 10.2, we can assume that there exist smooth functions v_i, w_i, σ_i for which the decomposition (10.6) holds, together with (10.7) and (10.8).

The techniques that we shall use to prove (11.2) are similar to those used to control the total variation. By (2.19) we already know that the desired estimate holds on the initial time interval $[0, \hat{t}]$. To obtain a uniform estimate valid for all $t > 0$, we decompose the vector z along a basis of unit vectors and derive an evolution equation for these scalar components. At first sight, it looks promising to write

$$z = \sum_i z_i \tilde{r}_i(u, v_i, \sigma_i),$$

where $\tilde{r}_1, \ldots, \tilde{r}_n$ are the same vectors used in the decomposition of u_x at (5.6). Unfortunately, this choice would lead to non-integrable source terms. Instead, we shall use a different basis of unit vectors $\hat{r}_1, \ldots, \hat{r}_n$, depending not only on the reference solution u but also on the perturbation z.

Toward this decomposition, we introduce the variable

$$\Upsilon \doteq z_x - A(u)z\,,$$

related to the flux of z. By (11.1), this quantity evolves according to the equation

$$\begin{aligned}\Upsilon_t + \big(A(u)\Upsilon\big)_x - \Upsilon_{xx} = &\Big[\big(u_x \bullet A(u)\big)z - \big(z \bullet A(u)\big)u_x\Big]_x - A(u)\Big[\big(u_x \bullet A(u)\big)z - \big(z \bullet A(u)\big)u_x\Big] \\ &+ \big(u_x \bullet A(u)\big)\Upsilon - \big(u_t \bullet A(u)\big)z\,.\end{aligned} \tag{11.3}$$

We now decompose z, Υ according to

$$\begin{cases} z = \sum_i h_i \tilde{r}_i\big(u, v_i, \lambda_i^* - \theta(g_i/h_i)\big)\,, \\ \Upsilon = \sum_i (g_i - \lambda_i^* h_i)\tilde{r}_i\big(u, v_i, \lambda_i^* - \theta(g_i/h_i)\big)\,, \end{cases} \tag{11.4}$$

where θ is the cutoff function introduced at (5.5). In the following we shall write

$$\hat{r}_i \doteq \tilde{r}_i\big(u, v_i,\, \lambda_i^* - \theta(g_i/h_i)\big),$$

to distinguish these unit vectors from the vectors $\tilde{r}_i\big(u, v_i,\, \lambda_i^* - \theta(w_i/v_i)\big)$ previously used in the decomposition (5.6) of u_x. Moreover we introduce the speed

$$\hat{\lambda}_i \doteq \big\langle \hat{r}_i,\, A(u)\hat{r}_i \big\rangle\,, \tag{11.5}$$

and denote by

$$\hat{\theta}_i \doteq \theta(g_i/h_i) \tag{11.6}$$

the correction in the speed for the perturbation. The next result, similar to Lemma 5.2, provides the existence and regularity of the decomposition (11.4).

Lemma 11.2 *Let $|u - u^*|$ and $|v|$ be sufficiently small. Then for all $z, \Upsilon \in I\!R^n$ the system of $2n$ equations (11.4) has a unique solution $(h_1, \ldots, h_n, g_1, \ldots, g_n)$. The map $(z, \Upsilon) \mapsto (h, g)$ is Lipschitz continuous. Moreover, it is smooth outside the n manifolds $\widehat{\mathcal{N}}_i \doteq \{h_i = g_i = 0\}$.*

Proof. The uniqueness of the decomposition is clear. To prove the existence, consider the mapping $\widehat{\Lambda} : I\!R^{2n} \mapsto I\!R^{2n}$ defined by

$$\widehat{\Lambda}(h, g) \doteq \sum_{i=1}^n \widehat{\Lambda}_i(h_i, g_i), \tag{11.7}$$

$$\widehat{\Lambda}_i(h_i,g_i) \doteq \begin{pmatrix} h_i\,\tilde r_i\big(u,\,v_i,\,\lambda_i^*-\theta(g_i/h_i)\big) \\ (g_i-\lambda_i^* h_i)\,\tilde r_i\big(u,\,v_i,\,\lambda_i^*-\theta(g_i/h_i)\big) \end{pmatrix}. \tag{11.8}$$

Computing the Jacobian matrix of partial derivatives we find

$$\frac{\partial \widehat{\Lambda}_i}{\partial(h_i,g_i)} = \begin{pmatrix} \hat r_i + (g_i/h_i)\hat\theta_i'\hat r_{i,\sigma} & -\hat\theta_i'\hat r_{i,\sigma} \\ -\lambda_i^*\hat r_i - \lambda_i^*(g_i/h_i)\hat\theta_i'\hat r_{i,\sigma} + (g_i/h_i)^2\hat\theta_i'\hat r_{i,\sigma} & \hat r_i + \lambda_i^*\hat\theta_i'\hat r_{i,\sigma} - (g_i/h_i)\hat\theta_i'\hat r_{i,\sigma} \end{pmatrix}. \tag{11.9}$$

By (4.24), $\hat r_{i,\sigma} = \mathcal{O}(1)\cdot v_i$. Hence, for v_i small enough, the differential $D\widehat{\Lambda}$ is invertible. By the implicit function theorem, $\widehat{\Lambda}$ is a one-to-one map whose range covers a whole neighborhood of the origin. Observing that $\widehat{\Lambda}$ is positively homogeneous of degree 1, we conclude that the decomposition is well defined and Lipschitz continuous on the whole space $\mathbb{R}^{2n}$. Outside the manifolds $\widehat{\mathcal{N}}_i$, $i=1,\ldots,n$, the smoothness of the decomposition is clear. $\square$

Writing the identity $\Upsilon = z_x - A(u)z$ in terms of the decomposition (11.4) we obtain

$$\sum_i (g_i-\lambda_i^*h_i)\hat r_i = \sum_i h_{i,x}\hat r_i - \sum_i A(u)h_i\hat r_i + \sum_{ij} h_i\hat r_{i,u}\,v_j\tilde r_j + \sum_i h_i\hat r_{i,v}v_{i,x} - \sum_i h_i\hat r_{i,\sigma}\hat\theta_{i,x}\,. \tag{11.10}$$

Taking the inner product with $\hat r_i$ and observing that $\hat r_i$ is a unit vector and hence is perpendicular to its derivatives, we obtain

$$g_i = h_{i,x} - (\hat\lambda_i - \lambda_i^*)h_i + \widehat{\Theta}_i$$

with

$$\widehat{\Theta}_i = -\sum_{j\neq i}\big\langle \hat r_i\,,\,A(u)\hat r_j\big\rangle h_j + \sum_{j\neq i}\sum_k \big\langle \hat r_i\,,\,\hat r_{j,u}\tilde r_k\big\rangle h_j v_k + \sum_{j\neq i}\big\langle \hat r_i\,,\,\hat r_{j,v}\big\rangle h_j v_{j,x} - \sum_{j\neq i}\big\langle \hat r_i\,,\,\hat r_{j,\sigma}\big\rangle h_j\hat\theta_{j,x}\,. \tag{11.11}$$

Hence, by (5.20) and (4.24),

$$g_i = h_{i,x} - (\hat\lambda_i - \lambda_i^*)h_i + \mathcal{O}(1)\cdot\delta_0\sum_{j\neq i}\big(|h_j|+|v_j|\big)\,. \tag{11.12}$$

A straightforward consequence of (11.12) is the following analogue of Lemma 9.1.

Corollary 11.3. *If $|g_i/h_i|\geq 3\delta_1/5$, then*

$$|g_i| \leq 2|h_{i,x}| + \mathcal{O}(1)\cdot\delta_0\sum_{j\neq i}\big(|v_j|+|h_j|\big), \qquad |h_i| \leq \frac{5}{2\delta_1}|h_{i,x}| + \mathcal{O}(1)\cdot\delta_0\sum_{j\neq i}\big(|v_j|+|h_j|\big)\,. \tag{11.13}$$

On the other hand, if $|g_i/h_i|\leq 4\delta_1/5$, then

$$|h_{i,x}| \leq \delta_1|h_i| + \mathcal{O}(1)\cdot\delta_0\sum_{j\neq i}\big(|v_j|+|h_j|\big)\,. \tag{11.14}$$

Our eventual goal is to show that the components h_i, g_i satisfy a system of evolution equations of the form

$$\begin{cases} h_{i,t} + (\tilde\lambda_i h_i)_x - h_{i,xx} = \hat\phi_i\,, \\ g_{i,t} + (\tilde\lambda_i g_i)_x - g_{i,xx} = \hat\psi_i\,, \end{cases} \tag{11.15}$$

where the source terms on the right hand sides are integrable on $[\hat{t}, \infty[\times I\!R$. Before embarking in calculations, we must first dispose of a technical difficulty due to the lack of regularity of the equations (11.4).

Since our equations (3.1) and (11.1) are uniformly parabolic, it is clear that for $t > 0$ all solutions are smooth. Moreover, by Remark 10.2, we can slightly modify the speeds σ_i occurring in the decomposition of u_x, so that (10.6)–(10.8) hold and the corresponding functions v_i are now smooth. On the other hand, the map $\widehat{\Lambda}$ in (11.7) is only Lipschitz continuous, hence the same is true in general for the functions $h_i = h_i(t,x)$ and $g_i = g_i(t,x)$. Indeed, at points where $h_i = g_i = 0$ for some index i, the derivatives $h_{i,x}$ or $g_{i,x}$ may well be discontinuous. In this case, the equations (11.15) would make no sense. To avoid this unpleasant situation, we observe that each manifold $\widehat{\mathcal{N}}_i$ has codimension 2. Given the smooth functions z, Υ and $\epsilon > 0$, by an arbitrarily small perturbation we can construct new functions $z^\sharp, \Upsilon^\sharp$ satisfying

$$\|z^\sharp - z\|_{\mathcal{C}^2} + \|\Upsilon^\sharp - \Upsilon\|_{\mathcal{C}^2} < \epsilon$$

and such that the corresponding decomposition (11.4) is $\mathcal{C}^\infty$ outside a countable set of isolated points $(t_m, x_m)_{m\geq 1}$. A further implementation of this technique yields

Lemma 11.4. *Let z, Υ be solutions of (11.1) and (11.3) respectively. Then for any $\epsilon > 0$ there exists smooth functions $z^\sharp$, $\Upsilon^\sharp$ such that the corresponding coefficients in the decomposition (11.4) are smooth except at countably many isolated points (t_m, x_m), $m \geq 1$. Moreover, these perturbed functions solve the system of equations*

$$\begin{aligned}
z^\sharp_t + \big(A(u)z^\sharp\big)_x - z^\sharp_{xx} &= \big(u_x \bullet A(u)\big)z^\sharp - \big(z^\sharp \bullet A(u)\big)u_x + e_1(t,x)\,,\\
\Upsilon^\sharp_t + \big(A(u)\Upsilon^\sharp\big)_x - \Upsilon^\sharp_{xx} &= \Big[\big(u_x \bullet A(u)\big)z^\sharp - \big(z^\sharp \bullet A(u)\big)u_x\Big]_x - A(u)\Big[\big(u_x \bullet A(u)\big)z^\sharp - \big(z^\sharp \bullet A(u)\big)u_x\Big]\\
&\quad + \big(u_x \bullet A(u)\big)\Upsilon^\sharp - \big(u_t \bullet A(u)\big)z^\sharp + e_2(t,x)\,,
\end{aligned}$$

for some perturbations e_1, e_2 such that

$$\int_{\hat{t}}^\infty \int \big|e_1(t,x)\big| + \big|e_2(t,x)\big|\,dxdt < \epsilon\,.$$

Thanks to this lemma, we can study the time evolution of the components h_i, g_i by means of a second order parabolic system, at the price of an arbitrarily small perturbation on the right hand side. In the remainder of the paper, for simplicity we derive all the estimates in the case $e_1 = e_2 = 0$. The general case easily follows by an approximation argument.

In Section 6 we showed that the source terms in the equations (6.1) could be reduced to four basic types. The following result is an analogue of Lemma 6.1, providing an estimate for the source terms in the equations (11.15). The proof, involving lengthy calculations, will be given in Appendix B.

Lemma 11.5. *The source terms in the equations (11.15) satisfy the estimates*

$$\begin{aligned}
\hat\phi_i(t,x),\ \hat\psi_i(t,x) &= \mathcal{O}(1)\cdot\sum_j \Big(|h_{j,x}|+|h_jv_j|+|g_jv_j|+|g_{j,x}|\Big)\,|w_j-\theta_jv_j| \\
&+\mathcal{O}(1)\cdot\sum_j \Big(|v_jh_{j,x}-h_jv_{j,x}|+|v_{j,x}g_j-g_{j,x}v_j|+|h_jw_{j,x}-w_jh_{j,x}|+|g_jw_{j,x}-g_{j,x}w_j|\Big) \\
&+\mathcal{O}(1)\cdot\sum_j \big(|v_j|+|h_j|\big)\left|h_j\left(\frac{g_j}{h_j}\right)_x^2\right|\cdot\chi_{\{|g_j/h_j|<3\delta_1\}} \\
&+\mathcal{O}(1)\cdot\sum_{j\neq k} \Big(|h_jv_k|+|h_{j,x}v_k|+|h_jv_{k,x}|+|h_jw_k|+|g_jv_k|+|g_{j,x}v_k|+|g_jv_{k,x}|+|h_jh_k|+|h_jg_k|\Big) \\
&+\mathcal{O}(1)\cdot\sum_j \Big(|h_j\phi_j|+|h_j\psi_j|+|g_j\phi_j|+|g_j\psi_j|\Big).
\end{aligned} \tag{11.16}$$

The key step in establishing the bound (11.2) is to prove

Lemma 11.6. *Consider a solution z of (11.1), satisfying*

$$\big\|z(t)\big\|_{\mathbf{L}^1}\leq\delta_0 \qquad \textit{for all } t\in[0,T], \tag{11.17}$$

and assume that the source terms in (11.4) satisfy

$$\int_{\hat t}^T\int\big|\hat\phi_i(t,x)\big|+\big|\hat\psi_i(t,x)\big|\,dxdt\leq\delta_0 \qquad\qquad i=1,\ldots,n. \tag{11.18}$$

Then for each $i=1,\ldots,n$ one has the estimates

$$\int_{\hat t}^T\int\big|\hat\phi_i(t,x)\big|\,dxdt=\mathcal{O}(1)\cdot\delta_0^2\,, \qquad\qquad \int_{\hat t}^T\int\big|\hat\psi_i(t,x)\big|\,dxdt=\mathcal{O}(1)\cdot\delta_0^2\,. \tag{11.19}$$

Assuming the validity of this lemma, we can easily recover the estimate (11.2). Indeed, since the equations (11.1) are linear, it suffices to prove the estimate in the case where

$$\big\|z(0)\big\|_{\mathbf{L}^1}=\frac{\delta_0}{8\sqrt{n}\,\kappa}\,. \tag{11.20}$$

We recall that κ is the constant defined at (3.5). By Corollary 2.4, on the initial interval $[0,\hat t]$ we have

$$\big\|z(t)\big\|_{\mathbf{L}^1}\leq 2\kappa\big\|z(0)\big\|_{\mathbf{L}^1}=\frac{\delta_0}{4\sqrt{n}} \qquad\qquad t\in[0,\,\hat t\,]\,. \tag{11.21}$$

Define the time

$$T\doteq\sup\left\{\tau\ ;\quad \sum_i\int_{\hat t}^\tau\int\big|\hat\phi_i(t,x)\big|+\big|\hat\psi_i(t,x)\big|\,dxdt\leq\frac{\delta_0}{2}\right\}. \tag{11.22}$$

If $T < \infty$, a contradiction is obtained as follows. First, we observe that the inequalities in (5.21) remain valid for the decomposition of z, namely

$$|z| \leq \sum_i |h_i| \leq 2\sqrt{n}\,|z|\,. \tag{11.23}$$

For every $\tau \in [\hat t,\, T]$, by (11.22) and (11.23) one has

$$\begin{aligned}
\big\|z(\tau)\big\|_{\mathbf{L}^1} &\leq \sum_i \big\|h_i(\tau)\big\|_{\mathbf{L}^1} \\
&\leq \sum_i \left(\big\|h_i(\hat t)\big\|_{\mathbf{L}^1} + \int_{\hat t}^{\tau} \int \big|\hat\phi_i(t,x)\big|\, dxdt \right) \\
&\leq 2\sqrt{n}\,\big\|z(\hat t)\big\|_{\mathbf{L}^1} + \frac{\delta_0}{2} \;\leq\; \delta_0\,.
\end{aligned} \tag{11.24}$$

We can thus use Lemma 11.5 and conclude

$$\sum_i \int_{\hat t}^{T} \int \big|\hat\phi_i(t,x)\big| + \big|\hat\psi_i(t,x)\big|\, dxdt = \mathcal{O}(1)\cdot \delta_0^2 \;<\; \frac{\delta_0}{2}\,, \tag{11.25}$$

provided that δ_0 was chosen suitably small. Therefore T cannot be a supremum. This contradiction shows that the bound (11.2) holds for all $t \geq 0$ and $z \in \mathbf{L}^1$, with $L \doteq 8\kappa\sqrt{n}$. The remainder of this section is aimed at establishing the estimates (11.19).

Proof of Lemma 11.6. By Corollary 2.2, for $t \in [\hat t, T]$, as long as $\big\|z(t)\big\|_{\mathbf{L}^1} \leq \delta_0$ we also have the bounds

$$\big\|z_x(t)\big\|_{\mathbf{L}^1} = \mathcal{O}(1)\cdot\delta_0^2\,, \qquad \big\|z_{xx}(t)\big\|_{\mathbf{L}^1} = \mathcal{O}(1)\cdot\delta_0^3\,, \qquad \big\|z_{xx}(t)\big\|_{\mathbf{L}^\infty} = \mathcal{O}(1)\cdot\delta_0^4\,.$$

By Lemma 11.2, the map $(z, \Upsilon) \mapsto (h, g)$ is uniformly Lipschitz continuous. From the previous bounds, for every $t \in [\hat t, T]$ and all $j = 1, \ldots, n$ it thus follows

$$\big\|h_{j,x}(t)\big\|_{\mathbf{L}^1}\,,\quad \big\|g_{j,x}(t)\big\|_{\mathbf{L}^1}\,,\quad \big\|h_j(t)\big\|_{\mathbf{L}^\infty}\,,\quad \big\|g_j(t)\big\|_{\mathbf{L}^\infty} \;=\; \mathcal{O}(1)\cdot\delta_0^2\,, \tag{11.26}$$

$$\big\|h_{j,x}(t)\big\|_{\mathbf{L}^\infty}\,,\quad \big\|g_{j,x}(t)\big\|_{\mathbf{L}^\infty} \;=\; \mathcal{O}(1)\cdot\delta_0^3\,. \tag{11.27}$$

Recalling that v_i, w_i, h_i, g_i satisfy the systems of equations (10.7) and (11.15) with source terms bounded by (10.8) and (11.18), we now provide an estimate on the integrals of all terms on the right hand side of (11.16).

The same techniques used in Section 7 yield an estimate on all transversal terms, with $j \neq k$:

$$\begin{aligned}
\int_{\hat t}^{T}\int \Big(&|h_j v_k| + |h_{j,x} v_k| + |h_j v_{k,x}| + |h_j w_k| + |g_j v_k| + |g_{j,x} v_k| + |g_j v_{k,x}| + |h_j h_k| + |h_j g_k| \Big)\, dxdt \\
&= \mathcal{O}(1)\cdot\delta_0^2\,.
\end{aligned} \tag{11.28}$$

From (10.8) and (11.26) one easily obtains

$$\begin{aligned}\int_{\hat t}^T\int\Big(|h_j\phi_j|+|h_j\psi_j|+|g_j\phi_j|+|g_j\psi_j|\Big)\,dxdt\ &\leq\ \int_{\hat t}^T\int\Big(\|h_j\|_{\mathbf{L}^\infty}+\|g_j\|_{\mathbf{L}^\infty}\Big)\cdot\big(|\phi_j|+|\psi_j|\big)\,dxdt\\ &=\ \mathcal{O}(1)\cdot\delta_0^3\,.\end{aligned}\tag{11.29}$$

A further set of terms will now be bounded using functionals related to shortening curves, as in Section 8. At each fixed time $t\in[\hat t,\,T]$, for $i=1,\ldots,n$ consider the curves

$$\gamma_i^{(v,h)}(x)\doteq\left(\int_{-\infty}^x v_i(t,y)dy,\ \ \int_{-\infty}^x h_i(t,y)dy\right).$$

By obvious meaning of notations, we also consider the curves $\gamma_i^{(v,g)}$, $\gamma_i^{(w,h)}$, $\gamma_i^{(w,g)}$, $\gamma_i^{(h,g)}$. By (6.1) and (11.15), the evolution of these curves is governed by vector equations similar to (8.8). For example,

$$\gamma_{i,t}^{(v,h)}+\tilde\lambda\gamma_{i,x}^{(v,h)}-\gamma_{i,xx}^{(v,h)}=\left(\int_{-\infty}^x\phi_i(t,y)dy,\ \ \int_{-\infty}^x\hat\phi_i(t,y)dy\right).$$

As in (8.9)-(8.10), we introduce the corresponding *Length* and *Area Functionals*, by setting

$$\mathcal{L}_i^{(v,h)}(t)=\mathcal{L}\big(\gamma_i^{(v,h)}(t)\big)=\int\sqrt{v_i^2(t,x)+h_i^2(t,x)}\ dx\,,$$

$$\mathcal{A}_i^{(v,h)}(t)=\mathcal{A}\big(\gamma_i^{(v,h)}(t)\big)=\frac12\int\!\!\int_{x<y}\big|v_i(t,x)h_i(t,y)-v_i(t,y)h_i(t,x)\big|\,dxdy\,.$$

Similarly we define $\mathcal{L}_i^{(v,g)}(t)$, $\mathcal{A}_i^{(v,g)}(t)$, etc... A computation entirely analogous to (8.18) now yields the bounds

$$\begin{aligned}\int_{\hat t}^T\int\Big(&|v_ih_{i,x}-h_iv_{i,x}|\ +\ |v_{i,x}g_i-g_{i,x}v_i|\ +\ |h_iw_{i,x}-w_ih_{i,x}|\\ &+\ |g_iw_{i,x}-g_{i,x}w_i|\ +\ |g_ih_{i,x}-h_ig_{i,x}|\Big)dxdt=\mathcal{O}(1)\cdot\delta_0^2\,.\end{aligned}\tag{11.30}$$

Moreover, repeating the argument in (8.19) we obtain

$$\int_{\hat t}^T\int_{|w_i/v_i|<3\delta_1}\left|h_i\left(\frac{g_i}{h_i}\right)_x^2\right|\,dxdt=\mathcal{O}(1)\cdot\delta_0\,.\tag{11.31}$$

Using the bounds (5.23) on $\|v_i\|_{\mathbf{L}^\infty}$ and (11.26) on $\|h_i\|_{\mathbf{L}^\infty}$, from (11.31) we deduce

$$\int_{\hat t}^T\int_{|w_i/v_i|<3\delta_1}\big(|v_i|+|h_i|\big)\left|h_i\left(\frac{g_i}{h_i}\right)_x^2\right|\,dxdt=\mathcal{O}(1)\cdot\delta_0^3\,.\tag{11.32}$$

The integrals of the remaining terms in (11.16) will be bounded by means of energy estimates. For convenience, we write $\hat\eta_i\doteq\eta(g_i/h_i)$, where η is the cutoff function introduced at (9.2). In Appendix C we will prove the estimates

$$\int_0^T\int\hat\eta_i\,h_{i,x}^2\,dx=\mathcal{O}(1)\cdot\delta_0^2\,,\tag{11.33}$$

$$\int_0^T \int \hat\eta_i \, g_{i,x}^2 \, dx = \mathcal{O}(1) \cdot \delta_0^2 \,. \tag{11.34}$$

Using (11.33)-(11.34) we now bound the terms containing the "wrong speed" $|w_i - \theta_i v_i|$. All these terms can be $\neq 0$ only when $|w_i/v_i| > \delta_1$. Hence by (6.20) we can write

$$\big(|h_i v_i| + |g_i v_i|\big)\, |w_i - \theta_i v_i| = \mathcal{O}(1) \cdot \big(|h_i| + |g_i|\big) \left(\Big| v_{i,x} \, (w_i - \theta_i v_i) \Big| + \sum_{j \neq i} \Big| v_j \, (w_i - \theta_i v_i) \Big| \right).$$

By (7.3), (9.1) and (11.26) it thus follows

$$\int_{\hat t}^T \int \big(|h_i v_i| + |g_i v_i|\big)\, |w_i - \theta_i v_i| dx dt = \mathcal{O}(1) \cdot \delta_0^4 \,. \tag{11.35}$$

To estimate the remaining terms, we split the domain according to the size of $|g_i/h_i|$.

CASE 1: $|g_i/h_i| > 4\delta_1/5$, $|w_i/v_i| > \delta_1$. Recalling (9.5) we then have

$$\begin{aligned}
\big(|h_{i,x}| + |g_{i,x}|\big)\, |w_i - \theta_i v_i| &\leq \big(|h_{i,x}| + |g_{i,x}|\big)\, |w_i| \\
&= \big(|h_{i,x}| + |g_{i,x}|\big) \Big(2|v_{i,x}| + \mathcal{O}(1) \cdot \sum_{j \neq i} |v_j| \Big) \\
&\leq \big(h_{i,x}^2 + g_{i,x}^2 + 2 v_{i,x}^2\big) + \mathcal{O}(1) \cdot \sum_{j \neq i} |h_{i,x} v_j| + \mathcal{O}(1) \cdot \sum_{j \neq i} |g_{i,x} v_j| \,.
\end{aligned}$$

Using (11.33)-(11.34), (9.18) and (11.28), we conclude

$$\begin{aligned}
&\int_{\hat t}^T \int_{|g_i/h_i| > 4\delta_1/5} \big(|h_{i,x}| + |g_{i,x}|\big)\, |w_i - \theta_i v_i| \, dx dt \\
&\quad \leq \int_{\hat t}^T \int \big(\hat\eta_i \, h_{i,x}^2 + \hat\eta_i \, g_{i,x}^2 + 2\eta_i \, v_{i,x}^2\big) \, dx dt + \mathcal{O}(1) \cdot \int_{\hat t}^T \int \sum_{j \neq i} \big(|h_{i,x} v_j| + |g_{i,x} v_j|\big) \, dx dt \\
&\quad = \mathcal{O}(1) \cdot \delta_0^2 \,.
\end{aligned} \tag{11.36}$$

CASE 2: $|g_i/h_i| \leq 4\delta_1/5$, $|w_i/v_i| > \delta_1$. In this case we have

$$|g_i v_i| \leq \frac{4\delta_1 |h_i|}{5} \, \frac{|w_i|}{\delta_1} = \frac{4}{5} |h_i w_i| \,. \tag{11.37}$$

Using (11.14) we can write

$$\begin{aligned}
|h_{i,x}| \, |w_i - \theta_i v_i| &\leq |h_{i,x} w_i| \\
&= \left| g_i - (\hat\lambda_i - \lambda_i^*) h_i + \mathcal{O}(1) \cdot \delta_0 \sum_{j \neq i} \big(|h_j| + |v_j|\big) \right| \, |w_i| \\
&\leq \delta_1 |h_i w_i| + \mathcal{O}(1) \cdot \delta_0 \sum_{j \neq i} \big(|h_j| + |v_j|\big) |w_i| \,.
\end{aligned} \tag{11.38}$$

By (11.28), the integral of the last terms on the right hand side of (11.38) is $\mathcal{O}(1)\cdot\delta_0^3$. Concerning the first term, using (11.36) and then (6.18) and (11.14), we can write

$$\begin{aligned}\frac{1}{5}|h_iw_i| &\le |h_iw_i-g_iv_i|\\ &\le |h_iv_{i,x}-h_{i,x}v_i|+|h_i|\,\big|w_i-v_{i,x}\big|+|v_i|\,\big|g_i-h_{i,x}\big|\\ &= |h_iv_{i,x}-h_{i,x}v_i|+\mathcal{O}(1)\cdot\delta_0|h_i|\Big(|w_i|+\sum_{j\neq i}|v_j|\Big)+\mathcal{O}(1)\cdot\delta_0|w_i|\Big(|h_i|+\sum_{j\neq i}\big(|h_j|+|v_j|\big)\Big).\end{aligned}$$

Hence, for δ_0 small, one has

$$|h_iw_i|\le 6|h_iv_{i,x}-h_{i,x}v_i|+\mathcal{O}(1)\cdot\sum_{j\neq i}\big(|h_iv_j|+|w_ih_j|+|w_iv_j|\big)\,.$$

By (11.28) and (11.30) it follows

$$\begin{aligned}\int_{\hat t}^T\int_{|g_i/h_i|\le 4\delta_1/5}|h_{i,x}|\,\big|w_i-\theta_iv_i\big|\,dxdt &\le \int_{\hat t}^T\int_{|g_i/h_i|\le 4\delta_1/5,\ |w_i/v_i|>\delta_1}|h_{i,x}w_i|\,dxdt\\ &= \mathcal{O}(1)\cdot\delta_0^2\,.\end{aligned}\tag{11.39}$$

Recalling that $\delta_1\le 1$, the last remaining term can now be bounded as

$$\begin{aligned}|g_{i,x}|\,\big|w_i-\theta_iv_i\big| &\le |g_{i,x}w_i|\\ &\le |g_{i,x}w_i-g_iw_{i,x}|+|g_iw_{i,x}|\\ &\le |g_{i,x}w_i-g_iw_{i,x}|+\frac{4\delta_1}{5}|h_iw_{i,x}|\\ &\le |g_{i,x}w_i-g_iw_{i,x}|+|h_iw_{i,x}-h_{i,x}w_i|+|h_{i,x}w_i|\,.\end{aligned}$$

By (11.30) and (11.39) we conclude

$$\int_{\hat t}^T\int_{|g_i/h_i|\le 4\delta_1/5}|g_{i,x}|\,\big|w_i-\theta_iv_i\big|\,dxdt=\mathcal{O}(1)\cdot\delta_0^2\,.\tag{11.40}$$

This completes the proof of Lemma 11.6. □

12 - Propagation speed

Consider two solutions u,v of the same viscous system (1.10), whose initial data coincide outside a bounded interval $[a,b]$. Since the system is parabolic, at a given time $t>0$ one may well have $u(t,x)\neq v(t,x)$ for all $x\in\mathbb{R}$. Yet, we want to show that the bulk of the difference $|u-v|$ remains confined within a bounded interval $[a-\beta t,\ b+\beta t]$. This result will be useful in the final section of the paper, because it implies the finite propagation speed of vanishing viscosity limits.

Lemma 12.1 *For some constants $\alpha,\beta>0$ the following holds. Let u,v be solutions of (1.10) with small total variation, whose initial data satisfy*

$$u(0,x)=v(0,x)\qquad\qquad x\notin[a,b]\,.\tag{12.1}$$

Then for all $x \in I\!R$, $t > 0$ one has

$$\big|u(t,x) - v(t,x)\big| \leq \big\|u(0) - v(0)\big\|_{\mathbf{L}^\infty} \cdot \min\Big\{\alpha e^{\beta t - (x-b)},\ \alpha e^{\beta t + (x-a)}\Big\}. \tag{12.2}$$

On the other hand, assuming that

$$u(0,x) = v(0,x) \qquad\qquad x \in [a,b], \tag{12.3}$$

one has

$$\big|u(t,x) - v(t,x)\big| \leq \big\|u(0) - v(0)\big\|_{\mathbf{L}^\infty} \cdot \Big(\alpha e^{\beta t - (x-a)} + \alpha e^{\beta t + (x-b)}\Big). \tag{12.4}$$

Proof. **1.** As a first step, we consider a solution z of the linearized system

$$z_t + \big[A(u)z\big]_x + \big[DA(u)\cdot z\big]u_x - \big[DA(u)\cdot u_x\big]z = z_{xx} \tag{12.5}$$

with initial data satisfying

$$\begin{cases} \big|z(0,x)\big| \leq 1 & \text{if} \quad x \leq 0, \\ \ z(0,x) = 0 & \text{if} \quad x > 0. \end{cases}$$

We will show that $z(t,x)$ becomes exponentially small on a domain of the form $\{x > \beta t\}$. More precisely, let $B(t)$ be a continuous increasing function such that

$$B(t) \geq 1 + 2\|A\|_\infty \int_0^t \left(\frac{1}{\sqrt{t-s}} + \sqrt{\pi}\right) B(s)\,ds\,, \qquad\qquad B(0) = 1.$$

One can show that such a function exists, satisfying the additional inequality $B(t) \leq 2e^{Ct}$, for some constant C large enough and for all $t \geq 0$. We claim that

$$\big|z(t,x)\big| \leq E(t,x) \doteq B(t)\ \exp\left\{4\|DA\|_{\mathbf{L}^\infty}\int_0^t \big\|u_x(s)\big\|_{\mathbf{L}^\infty} ds + t - x\right\} \tag{12.6}$$

for all $x \in I\!R$ and $t \geq 0$. Indeed, any solution of (12.5) admits the integral representation

$$\begin{aligned} z(t) = G(t) * z(0) &- \int_0^t G_x(t-s) * \big[A(u)z\big](s)ds \\ &+ \int_0^t G(t-s) * \Big[\big(u_x \bullet A(u)\big)z(s) - \big(z \bullet A(u)\big)u_x(s)\Big]ds\,, \end{aligned}$$

in terms of convolutions with the standard heat kernel $G(t,x) \doteq e^{-x^2/4t}/2\sqrt{\pi t}$. Therefore

$$\begin{aligned} \big|z(t,x)\big| \leq \int G(t,x-y)\big|z(0,y)\big|dy &+ \|A\|_{\mathbf{L}^\infty}\int_0^t\!\!\int \Big|G_x(t-s,x-y)\Big|\big|z(s,y)\big|dyds \\ &+ 2\|DA\|_{\mathbf{L}^\infty}\int_0^t\!\!\int \big\|u_x(s)\big\|_{\mathbf{L}^\infty} G(t-s,x-y)\big|z(s,y)\big|dyds\,. \end{aligned} \tag{12.7}$$

For every $t > 0$ the following estimates hold (see Appendix D for details):

$$\int G(t,\,x-y)\,\big|z(0,y)\big|\,dy < \int \frac{e^{-(x-y)^2/4t}}{2\sqrt{\pi t}}\,e^{-y}dy = e^{t-x}\,, \tag{12.8}$$

$$\|A\|_{\mathbf{L}^\infty} \int_0^t \int \big|G_x(t-s, x-y)\big| E(s,y)\,dy\,ds \le \frac{1}{2}E(t,x) - \frac{1}{2}e^{t-x}, \tag{12.9}$$

$$2\|DA\|_{\mathbf{L}^\infty} \int_0^t \big\|u_x(s)\big\|_{\mathbf{L}^\infty} \left(\int G(t-s, x-y)E(s,y)\,dy \right) ds \le \frac{1}{2}E(t,x) - \frac{1}{2}e^{t-x}. \tag{12.10}$$

The bounds (12.7)–(12.10) show that, if (12.6) is satisfied for all $t \in [0, \tau[$, then at time $t = \tau$ one always has a strict inequality: $\big|z(\tau,x)\big| < E(\tau,x)$. A simple argument now yields the validity of (12.6) for all $t > 0$ and $x \in I\!R$.

2. Recalling (10.10) we have

$$\big\|u_x(s)\big\|_{\mathbf{L}^\infty} \le \max\left\{ \frac{2\kappa\delta_0}{\sqrt{s}}\ \frac{2\kappa\delta_0}{\sqrt{\hat{t}}} \right\}.$$

From the definition of E at (12.6), for some constants $\alpha, \beta > 0$ we now obtain

$$\big|z(t,x)\big| \le E(t,x) \le 2e^{Ct} \exp\left\{ 4\|DA\|_{\mathbf{L}^\infty} \cdot 2\kappa\delta_0 \big(2\sqrt{t} + t/\sqrt{\hat{t}}\,\big) \right\} e^{t-x} \le \alpha e^{\beta t - x}. \tag{12.11}$$

3. More generally, let now z be a solution of (12.5) whose initial data satisfies

$$\begin{cases} \big|z(0,x)\big| \le \rho & \text{if} \quad x \le b\,, \\ z(0,x) = 0 & \text{if} \quad x > b\,. \end{cases}$$

By the linearity of the equations (11.1) and translation invariance, a straightforward extension of the above arguments yields

$$\big|z(t,x)\big| \le \rho \cdot \alpha e^{\beta t - (x-b)}.$$

On the other hand, if

$$\begin{cases} \big|z(0,x)\big| \le \rho & \text{if} \quad x \ge a\,, \\ z(0,x) = 0 & \text{if} \quad x < a\,, \end{cases}$$

then

$$\big|z(t,x)\big| \le \rho \cdot \alpha e^{\beta t + (x-a)}.$$

4. Having established the corresponding bounds on first order tangent vectors, the estimates (12.2) and (12.4) can now be recovered by a simple homotopy argument. For each $\theta \in [0,1]$, let u^θ be the solution of (1.10) with initial data

$$u^\theta(0) = \theta u(0) + (1-\theta)v(0).$$

Moreover, call z^θ the solution of the linearized Cauchy problem

$$z^\theta_t + \big[DA(u^\theta) \cdot z^\theta\big] u^\theta_x + A(u^\theta) z^\theta_x = z^\theta_{xx},$$

$$z^\theta(0,x) = u(0,x) - v(0,x).$$

If (12.1) holds, then by the previous analysis all functions z^θ satisfy the two inequalities

$$\big|z^\theta(t,x)\big| \le \big\|u(0) - v(0)\big\|_{\mathbf{L}^\infty} \cdot \alpha e^{\beta t - (x-b)},$$
$$\big|z^\theta(t,x)\big| \le \big\|u(0) - v(0)\big\|_{\mathbf{L}^\infty} \cdot \alpha e^{\beta t + (x-a)}.$$

Therefore

$$\begin{aligned}\big|u(t,x)-v(t,x)\big| &\leq \int_0^1 \left|\frac{du^\theta(t,x)}{d\theta}\right| d\theta = \int_0^1 \big|z^\theta(t,x)\big|\, d\theta \\ &\leq \big\|u(0)-v(0)\big\|_{\mathbf{L}^\infty}\cdot \min\Big\{\alpha e^{\beta t-(x-b)},\ \alpha e^{\beta t-(a-x)}\Big\}.\end{aligned}$$

This proves (12.2). On the other hand, if (12.3) holds, we consider a third solution w of (1.10), with initial data

$$w(0,x)=\begin{cases} u(0,x) & \text{if}\quad x\leq b,\\ v(0,x) & \text{if}\quad x\geq a.\end{cases}$$

For every $x\in I\!\!R$ and $t>0$, the previous arguments now yield

$$\begin{aligned}\big|u(t,x)-w(t,x)\big| &\leq \big\|u(0)-w(0)\big\|_{\mathbf{L}^\infty}\cdot \alpha e^{\beta t+(x-b)},\\ \big|v(t,x)-w(t,x)\big| &\leq \big\|w(0)-v(0)\big\|_{\mathbf{L}^\infty}\cdot \alpha e^{\beta t-(x-a)}.\end{aligned}$$

Combining these two inequalities we obtain (12.4). □

13 - The vanishing viscosity limit

Up to now, all the analysis has been concerned with solutions of the parabolic system (1.10), with unit viscosity. Our results, however, can be immediatly applied to the Cauchy problem

$$u^\varepsilon_t + A(u^\varepsilon)u^\varepsilon_x = \varepsilon\, u^\varepsilon_{xx}\,,\qquad\qquad u^\varepsilon(0,x)=\bar u(x) \tag{13.1}$$

for any $\varepsilon>0$. Indeed, as remarked in the Introduction, a function u^ε is a solution of (13.1) if and only if

$$u^\varepsilon(t,x)=u(t/\varepsilon,\ x/\varepsilon), \tag{13.2}$$

where u is the solution of the Cauchy problem

$$u_t + A(u)u_x = u_{xx}\,,\qquad\qquad u(0,x)=\bar u(\varepsilon x)\,. \tag{13.3}$$

Since the rescaling (13.2) does not change the total variation, from our earlier analysis we easily obtain the first part of Theorem 1. Namely, for every initial data $\bar u$ with sufficiently small total variation, the corresponding solution $u^\varepsilon(t)\doteq S^\varepsilon_t\bar u$ is well defined for all times $t\geq 0$. The bounds (1.15)–(1.17) follow from

$$\text{Tot.Var.}\big\{u^\varepsilon(t)\big\} = \text{Tot.Var.}\big\{u(t/\varepsilon)\big\}\leq C\,\text{Tot.Var.}\{\bar u\}, \tag{13.4}$$

$$\big\|u^\varepsilon(t)-v^\varepsilon(t)\big\|_{\mathbf{L}^1} = \varepsilon\big\|u(t)-v(t)\big\|_{\mathbf{L}^1}\leq \varepsilon L\,\big\|u(0)-v(0)\big\|_{\mathbf{L}^1} = \varepsilon L\,\frac{1}{\varepsilon}\|\bar u-\bar v\|_{\mathbf{L}^1}\,, \tag{13.5}$$

$$\big\|u^\varepsilon(t)-u^\varepsilon(s)\big\|_{\mathbf{L}^1}\leq \varepsilon\big\|u(t/\varepsilon)-u(s/\varepsilon)\big\|_{\mathbf{L}^1}\leq \varepsilon L'\left(\left|\frac{t}{\varepsilon}-\frac{s}{\varepsilon}\right| + \left|\sqrt{\frac{t}{\varepsilon}}-\sqrt{\frac{s}{\varepsilon}}\right|\right). \tag{13.6}$$

Moreover, if $\bar u(x)=\bar v(x)$ for $x\in[a,b]$, then (12.4) implies

$$\big|u^\varepsilon(t,x)-v^\varepsilon(t,x)\big|\leq \|\bar u-\bar v\|_{\mathbf{L}^\infty}\cdot\left\{\alpha\exp\Big(\frac{\beta t-(x-a)}{\varepsilon}\Big)+\alpha\exp\Big(\frac{\beta t+(x-b)}{\varepsilon}\Big)\right\}. \tag{13.7}$$

We now consider the vanishing viscosity limit. Call $\mathcal{U} \subset \mathbf{L}^1_{loc}$ the set of all functions $\bar{u} : I\!R \mapsto I\!R^n$ with small total variation, satisfying (1.14). For each $t \geq 0$ and every initial condition $\bar{u} \in \mathcal{U}$, call $S^\varepsilon_t \bar{u} \doteq u^\varepsilon(t, \cdot)$ the corresponding solution of (13.1). Thanks to the uniform BV bounds (13.4), we can apply Helly's compactness theorem and obtain a sequence $\varepsilon_\nu \to 0$ such that

$$\lim_{\nu \to \infty} u^{\varepsilon_\nu}(t, \cdot) = u(t, \cdot) \qquad \text{in} \quad \mathbf{L}^1_{loc}\,. \tag{13.8}$$

holds for some BV function $u(t, \cdot)$. By extracting further subsequences and then using a standard diagonalization procedure, we can assume that the limit in (13.8) exists for all rational times t and all solutions u^ε with initial data in a countable dense set $\mathcal{U}^* \subset \mathcal{U}$. Adopting a semigroup notation, we thus define

$$S_t \bar{u} \doteq \lim_{m \to \infty} S^{\varepsilon_m}_t \bar{u} \qquad \text{in } \mathbf{L}^1_{loc}\,, \tag{13.9}$$

for some particular subsequence $\varepsilon_m \to 0$. By the uniform continuity of the maps $(t, \bar{u}) \mapsto u^\varepsilon(t, \cdot) \doteq S^\varepsilon_t \bar{u}$, stated in (13.5)-(13.6), the set of couples $(t, \bar{u})$ for which the limit (13.9) exists must be closed in $I\!R_+ \times \mathcal{U}$. Therefore, this limit is well defined for all $\bar{u} \in \mathcal{U}$ and $t \geq 0$.

Remark 13.1. The function $u(t, \cdot) = S_t \bar{u}$ is here defined as a limit in $\mathbf{L}^1_{loc}$. Since it has bounded variation, we can remove any ambiguity concerning its pointwise values by choosing, say, a right continuous representative:

$$u(t, x) = \lim_{y \to x+} u(t, y).$$

With this choice, the function u is certainly jointly measurable w.r.t. t, x (see [B5], p.16).

To complete the proof of Theorem 1, we need to show that the map S defined at (13.9) is a semigroup, satisfies the continuity properties (1.18) and does not depend on the choice of the subsequence $\{\varepsilon_m\}$. These results will be achieved in several steps.

1. (Continuous dependence) Let S be the map defined by (13.9). Then

$$\big\|S_t \bar{u} - S_t \bar{v}\big\|_{\mathbf{L}^1} = \sup_{r>0} \int_{-r}^{r} \Big| \big(S_t \bar{u}\big)(x) - \big(S_t \bar{v}\big)(x) \Big|\, dx\,.$$

For every $r > 0$, the convergence in $\mathbf{L}^1_{loc}$ implies

$$\int_{-r}^{r} \Big| \big(S_t \bar{u}\big)(x) - \big(S_t \bar{v}\big)(x) \Big|\, dx = \lim_{m \to \infty} \int_{-r}^{r} \Big| \big(S^{\varepsilon_m}_t \bar{u}\big)(x) - \big(S^{\varepsilon_m}_t \bar{v}\big)(x) \Big|\, dx \leq L\, \big\|\bar{u} - \bar{v}\big\|_{\mathbf{L}^1}\,.$$

because of (13.5). This yields the Lipschitz continuous dependence w.r.t. the initial data:

$$\big\|S_t \bar{u} - S_t \bar{v}\big\|_{\mathbf{L}^1} \leq L\, \big\|\bar{u} - \bar{v}\big\|_{\mathbf{L}^1}\,. \tag{13.10}$$

The continuous dependence w.r.t. time is proved in a similar way. By (13.6), for every $r > 0$ we have

$$\begin{aligned}
\int_{-r}^{r} \Big| \big(S_t \bar{v}\big)(x) - \big(S_s \bar{v}\big)(x) \Big|\, dx &= \lim_{m \to \infty} \int_{-r}^{r} \Big| \big(S^{\varepsilon_m}_t \bar{v}\big)(x) - \big(S^{\varepsilon_m}_s \bar{v}\big)(x) \Big|\, dx \\
&\leq \lim_{m \to \infty} \varepsilon_m L' \left(\left| \frac{t}{\varepsilon_m} - \frac{s}{\varepsilon_m} \right| + \left| \sqrt{\frac{t}{\varepsilon_m}} - \sqrt{\frac{s}{\varepsilon_m}} \right| \right) \\
&= L' |t - s|\,.
\end{aligned}$$

Hence

$$\left\|S_t\bar v - S_s\bar v\right\|_{\mathbf{L}^1} \le L'|t-s|\,. \tag{13.11}$$

Together, (13.10) and (13.11) yield (1.18).

2. (Finite propagation speed) Consider any interval $[a,b]$ and two initial data $\bar u, \bar v$, with $\bar u(x) = \bar v(x)$ for $x \in [a,b]$. By (13.7), for every $t \ge 0$ and $x \in\,]a+\beta t,\ b-\beta t[$ one has

$$\begin{aligned}\Big|\big(S_t\bar u\big)(x) - \big(S_t\bar v\big)(x)\Big| &\le \limsup_{m\to\infty}\Big|\big(S_t^{\varepsilon_m}\bar u\big)(x) - \big(S_t^{\varepsilon_m}\bar v\big)(x)\Big| \\ &\le \lim_{m\to\infty}\|\bar u - \bar v\|_{\mathbf{L}^\infty}\cdot\left\{\alpha\exp\Big(\frac{\beta t-(x-a)}{\varepsilon_m}\Big) + \alpha\exp\Big(\frac{\beta t+(x-b)}{\varepsilon_m}\Big)\right\} \\ &= 0\,. \end{aligned} \tag{13.12}$$

In other words, the restriction of the function $S_t\bar u \in \mathbf{L}^1_{loc}$ to a given interval $[a',b']$ depends only on the values of the initial data $\bar u$ on the interval $[a'-\beta t,\ b'+\beta t]$. Using (13.12), we now prove a sharper version of the continuous dependence estimate (13.10):

$$\int_a^b \Big|\big(S_t\bar u\big)(x) - \big(S_t\bar v\big)(x)\Big|\,dx \le L\cdot\int_{a-\beta t}^{b+\beta t}\big|\bar u(x) - \bar v(x)\big|\,dx\,. \tag{13.13}$$

valid for every $\bar u, \bar v$ and $t \ge 0$. Indeed, define the auxiliary function

$$\bar w(x) = \begin{cases}\bar u(x) & \text{if } \ x \in [a-\beta t,\ b+\beta t]\,,\\ \bar v(x) & \text{if } \ x \notin [a-\beta t,\ b+\beta t]\,.\end{cases}$$

Using the finite propagation speed, we now have

$$\begin{aligned}\int_a^b \Big|\big(S_t\bar u\big)(x) - \big(S_t\bar v\big)(x)\Big|\,dx &= \int_a^b \Big|\big(S_t\bar w\big)(x) - \big(S_t\bar v\big)(x)\Big|\,dx \\ &\le L\,\|\bar w - \bar v\|_{\mathbf{L}^1} = L\cdot\int_{a-\beta t}^{b+\beta t}\big|\bar u(x) - \bar v(x)\big|\,dx\,.\end{aligned}$$

3. (Semigroup property) We now show that the map $(t,\bar u)\mapsto S_t\bar u$ is a semigroup, i.e.

$$S_0\bar u = \bar u\,, \qquad\qquad S_sS_t\bar u = S_{s+t}\bar u\,. \tag{13.14}$$

Since every S^ε is a semigroup, the first equality in (13.14) is a trivial consequence of the definition (13.9). To prove the second equality, we observe that

$$S_{s+t}\bar u = \lim_{m\to\infty} S_s^{\varepsilon_m}S_t^{\varepsilon_m}\bar u\,, \qquad\qquad S_sS_t\bar u = \lim_{m\to\infty} S_s^{\varepsilon_m}S_t\bar u\,. \tag{13.15}$$

We can assume $s > 0$. Fix any $r > 0$ and consider the function

$$\tilde u_m(x) \doteq \begin{cases}\big(S_t\bar u\big)(x) & \text{if } \ |x| > r+2\beta s\,,\\ \big(S_t^{\varepsilon_m}\bar u\big)(x) & \text{if } \ |x| < r+2\beta s\,.\end{cases}$$

Observing that $S_t^{\varepsilon_m}\bar u \to S_t\bar u$ in $\mathbf{L}^1_{loc}$ and hence $\tilde u_m \to S_t\bar u$ in $\mathbf{L}^1$, we can use (13.7) and (13.5) and obtain

$$\begin{aligned}
\limsup_{m\to\infty} \int_{-r}^{r} & \left|\big(S_s^{\varepsilon_m}S_t^{\varepsilon_m}\bar u\big)(x) - \big(S_s^{\varepsilon_m}S_t\bar u\big)(x)\right| dx \\
&\le \lim_{m\to\infty} 2r\cdot \sup_{|x|<r}\left|\big(S_s^{\varepsilon_m}S_t^{\varepsilon_m}\bar u\big)(x) - \big(S_s^{\varepsilon_m}\tilde u_m\big)(x)\right| + \lim_{m\to\infty}\left\|S_s^{\varepsilon_m}\tilde u_m - S_s^{\varepsilon_m}S_t\bar u\right\|_{\mathbf{L}^1} \\
&\le \lim_{m\to\infty} 2r\left\|S_t^{\varepsilon_m}\bar u - \tilde u_m\right\|_{\mathbf{L}^\infty}\cdot 2\alpha e^{-\beta s/\varepsilon_m} + \lim_{m\to\infty} L\cdot\left\|\tilde u_m - S_t\bar u\right\|_{\mathbf{L}^1} \\
&= 0\,.
\end{aligned}$$

By (13.15), this proves the second identity in (13.14).

4. (Tame Oscillation) We now exhibit a regularity property which is shared by all semigroup trajectories. This property, introduced in [BG], plays a key role in the proof of uniqueness. We begin by recalling the main definitions. Given $a<b$ and $\tau\ge 0$, we denote by Tot.Var.$\big\{u(\tau)\,;\ ]a,b[\,\big\}$ the total variation of $u(\tau,\cdot)$ over the open interval $]a,b[$. Moreover, consider the triangle

$$\Delta^\tau_{a,b} \doteq \big\{(t,x)\,;\quad t>\tau,\ a+\beta(t-\tau)<x<b-\beta(t-\tau)\big\}\,.$$

The oscillation of u over $\Delta^\tau_{a,b}$ will be denoted by

$$\text{Osc.}\big\{u\,;\ \Delta^\tau_{a,b}\big\} \doteq \sup\Big\{\big|u(t,x)-u(t',x')\big|\,;\quad (t,x),(t',x')\in\Delta^\tau_{a,b}\Big\}.$$

We claim that each function $u(t,x)=\big(S_t\bar u\big)(x)$ satisfies the *tame oscillation property*: there exists a constant C' such that, for every $a<b$ and $\tau\ge 0$, one has

$$\text{Osc.}\big\{u\,;\ \Delta^\tau_{a,b}\big\} \le C'\cdot \text{Tot.Var.}\big\{u(\tau)\,;\ ]a,b[\,\big\}\,. \tag{13.16}$$

Indeed, let a,b,τ be given, together with an initial data $\bar u$. By the semigroup property, it is not restrictive to assume $\tau=0$. Consider the auxiliary initial condition

$$\bar v(x) \doteq \begin{cases} \bar u(x) & \text{if}\quad a<x<b,\\ \bar u(a+) & \text{if}\quad x\le a,\\ \bar u(b-) & \text{if}\quad x\ge b,\end{cases} \tag{13.17}$$

and call $v(t,x)\doteq\big(S_t\bar v\big)(x)$ the corresponding trajectory. Observe that

$$\lim_{x\to-\infty} v(t,x) = \bar u(a+)$$

for every $t\ge 0$. Using (1.15) and the finite propagation speed, we can thus write

$$\begin{aligned}
\text{Osc.}\big\{u\,;\ \Delta^\tau_{a,b}\big\} = \text{Osc.}\big\{v\,;\ \Delta^\tau_{a,b}\big\} &\le 2\sup_t\Big(\text{Tot.Var.}\big\{S_t\bar v\big\}\Big)\\
&\le 2C\cdot\text{Tot.Var.}\{\bar v\} = 2C\cdot\text{Tot.Var.}\big\{u(\tau)\,;\ ]a,b[\,\big\}\,,
\end{aligned}$$

proving (13.16) with $C'=2C$.

5. (Conservation equations) Assume that the system (13.1) is in conservation form, i.e. $A(u)=Df(u)$ for some flux function f. In this special case, we claim that every vanishing viscosity limit is a weak solution of the system of conservation laws (1.1). Indeed, with the usual notations, if ϕ is a $\mathcal{C}^2$ function with compact support contained in the half plane $\{x \in I\!R,\ t > 0\}$, one can repeatedly integrate by parts and obtain

$$\begin{aligned}\iint \big[u\,\phi_t + f(u)\phi_x\big]\,dxdt \;&=\; \lim_{m\to\infty} \iint \big[u^{\varepsilon_m}\,\phi_t + f(u^{\varepsilon_m})\phi_x\big]\,dxdt\\ = -\lim_{m\to\infty} \iint \big[u_t^{\varepsilon_m}\,\phi + f(u^{\varepsilon_m})_x\phi\big]\,dxdt \;&=\; -\lim_{m\to\infty} \iint \varepsilon_m\, u_{xx}^{\varepsilon_m}\,\phi\,dxdt\\ = -\lim_{m\to\infty} \iint \varepsilon_m\, u^{\varepsilon_m}\phi_{xx}\,dxdt \;&=\; 0\,.\end{aligned}$$

An easy approximation argument shows that the identity (1.5) holds more generally, assuming only $\phi \in \mathcal{C}_c^1$.

6. (Approximate jumps) From the uniform bound on the total variation and the Lipschitz continuity w.r.t. time, it follows that each function $u(t,x) = \big(S_t\bar u\big)(x)$ is a BV function, jointly w.r.t. the two variables t, x. In particular, an application of Theorem 2.6 in [B5] yields the existence of a set of times $\mathcal{N} \subset I\!R_+$ of measure zero such that, for every $(\tau,\xi) \in I\!R_+ \times I\!R$ with $\tau \notin \mathcal{N}$, the following holds. Calling

$$u^- \doteq \lim_{x\to\xi-} u(\tau,x), \qquad\qquad u^+ \doteq \lim_{x\to\xi+} u(\tau,x), \tag{13.18}$$

there exists a finite speed λ such that the function

$$U(t,x) \doteq \begin{cases} u^- & \text{if} \quad x < \lambda t,\\ u^+ & \text{if} \quad x > \lambda t,\end{cases} \tag{13.19}$$

for every constant $\kappa > 0$ satisfies

$$\lim_{r\to 0+} \frac{1}{r^2} \int_{-r}^{r}\int_{-\kappa r}^{\kappa r} \big|u(\tau+t,\ \xi+x) - U(t,x)\big|\,dxdt = 0, \tag{13.20}$$

$$\lim_{r\to 0+} \frac{1}{r} \int_{-\kappa r}^{\kappa r} \big|u(\tau+r,\ \xi+x) - U(r,x)\big|\,dx = 0. \tag{13.21}$$

In the case where $u^- \neq u^+$, we say that (τ,ξ) is a point of *approximate jump* for the function u. On the other hand, if $u^- = u^+$ (and hence λ can be chosen arbitrarily), we say that u is *approximately continuous* at (τ,ξ). The above result can thus be restated as follows: with the exception of a null set $\mathcal{N}$ of "interaction times", the solution u is either approximately continuous or has an approximate jump discontinuity at each point (τ,ξ).

7. (Shock conditions) Assume again that the system is in conservation form. Consider a semigroup trajectory $u(t,\cdot) = S_t\bar u$ and a point (τ,ξ) where u has an approximate jump. Since u is a weak solution, the states u^-, u^+ and the speed λ in (13.19) must satisfy the Rankine-Hugoniot equations

$$\lambda\,(u^+ - u^-) = f(u^+) - f(u^-). \tag{13.22}$$

For a proof, see Theorem 4.1 in [B5].

If u is a limit of vanishing viscosity approximations, the same is true of the solution U in (13.19). In particular (see [MP] or [D]), the *Liu shock conditions* must hold. More precisely, call

$s \mapsto S_i(s)$ the parametrized shock curve through u^- and let $\lambda_i(s)$ be the speed of the corresponding shock. If $u^+ = S_i(s)$ for some s, then

$$\lambda_i(s') \geq \lambda_i(s) \qquad \text{for all } s' \in [0, s]\,. \tag{13.23}$$

Under the additional assumption that each characterictic field is either linearly degenerate or genuinely nonlinear, it is well known that the Liu conditions imply the Lax shock conditions:

$$\lambda_i(u^+) \leq \lambda \leq \lambda_i(u^-). \tag{13.24}$$

8. (Uniqueness in a special case) Assume that the system is in conservation form and that each characterictic field is either linearly degenerate or genuinely nonlinear. By the previous steps, the semigroup trajectory $u(t,\cdot) = S_t\bar{u}$ provides a weak solution to the Cauchy problem (1.1)-(1.2) which satisfies the Tame Oscillation and the Lax shock conditions. By a well known uniqueness theorem in [BG], [B5], such a weak solution is unique and coincides with the limit of front tracking approximations. In particular, it does not depend on the choice of the subsequence $\{\varepsilon_m\}$:

$$S_t\bar{u} = \lim_{\varepsilon\to 0+} S_t^\varepsilon \bar{u}\,,$$

i.e. the same limit actually holds over all real values of ε.

The above results already yield a proof of Theorem 1 in the special case where the system is in conservation form and satisfies the standard assumptions (H). To handle the general (non-conservative) case, we shall need to understand first the solution of the Riemann problem.

14 - The non-conservative Riemann problem

Aim of this section is to characterize the vanishing viscosity limit for solutions u^ε of (13.1), in the case of Riemann data

$$\bar{u}(x) = \begin{cases} u^- & \text{if } \ x < 0, \\ u^+ & \text{if } \ x > 0. \end{cases} \tag{14.1}$$

More precisely, we will show that, as $\varepsilon \to 0+$, the solutions u^ε converge to a self-similar limit $\omega(t,x) = \tilde{\omega}(x/t)$. We first describe a method for constructing this solution ω.

As a first step, given a left state u^- and $i \in \{1,\ldots,n\}$, we seek a one-parameter curve of right states $u^+ = \Psi_i(s)$ such that the non-conservative Riemann problem

$$\omega_t + A(\omega)\omega_x = 0, \qquad\qquad \omega(0,x) = \begin{cases} u^- & \text{if } \ x < 0 \\ u^+ & \text{if } \ x > 0 \end{cases} \tag{14.2}$$

admits a vanishing viscosity solution consisting only of i-waves. In the case where the system is in conservation form and the i-th field is genuinely nonlinear, it is well known [Lx] that one should take

$$\Psi_i(s) = \begin{cases} R_i(s) & \text{if } \ s \geq 0, \\ S_i(s) & \text{if } \ s < 0. \end{cases}$$

Here R_i and S_i are the i-th rarefaction and shock curves through u^-, respectively. We now describe a method for constructing such curve Ψ_i in the general case.

Fix $\epsilon, s > 0$. Consider the family $\Gamma \subset \mathcal{C}^0\big([0,s]\,;\ I\!\!R^n \times I\!\!R \times I\!\!R\big)$ of all continuous curves

$$\tau \mapsto \gamma(\tau) = \big(u(\tau),\, v_i(\tau),\, \sigma_i(\tau)\big), \qquad\qquad \tau \in [0,s]\,,$$

with

$$u(0) = u^-, \qquad \big|u(\tau) - u^-\big| \leq \epsilon, \qquad \big|v_i(\tau)\big| \leq \epsilon, \qquad \big|\sigma_i(\tau) - \lambda_i(u^-)\big| \leq \epsilon.$$

In connection with a given curve $\gamma \in \Gamma$, define the scalar flux function

$$f_i(\gamma,\tau) \doteq \int_0^\tau \tilde\lambda_i\big(u(\varsigma),\, v_i(\varsigma),\, \sigma_i(\varsigma)\big)\, d\varsigma \qquad\qquad \tau \in [0,\, s]\,, \tag{14.3}$$

where $\tilde\lambda_i$ is the speed in (4.21). Moreover, consider the lower convex envelope

$$\mathrm{conv}\, f_i(\gamma,\tau) \doteq \inf\Big\{\theta f_i(\gamma,\tau') + (1-\theta) f_i(\gamma,\tau'')\,;\quad \theta \in [0,1]\,,\ \ \tau',\tau'' \in [0,s]\,,\ \ \tau = \theta\tau' + (1-\theta)\tau''\Big\}\,.$$

We now define a continuous mapping $\mathcal{T}_{i,s} : \Gamma \mapsto \Gamma$ by setting $\mathcal{T}_{i,s}\gamma = \hat\gamma = (\hat u, \hat v_i, \hat\sigma_i)$, where

$$\begin{cases} \hat u(\tau) \doteq u^- + \displaystyle\int_0^\tau \tilde r_i\big(u(\varsigma),\, v_i(\varsigma),\, \sigma_i(\varsigma)\big)\, d\varsigma\,, \\ \hat v_i(\tau) \doteq f_i(\gamma,\tau) - \mathrm{conv}\, f_i(\gamma,\tau)\,, \\ \hat\sigma_i(\tau) \doteq \dfrac{d}{d\tau}\,\mathrm{conv}\, f_i(\gamma,\tau)\,. \end{cases} \tag{14.4}$$

We recall that $\tilde r_i$ are the unit vectors that define the center manifold in (4.13). Because of the bounds

$$\big|\hat u(\tau) - u^-\big| \leq \tau \leq s\,,$$

$$\big|\hat\sigma_i(\tau) - \lambda_i(u^-)\big| = \mathcal{O}(1)\cdot \sup_{\varsigma\in[0,s]} \Big|\tilde\lambda_i\big(u(\varsigma),\, v_i(\varsigma),\, \sigma_i(\varsigma)\big) - \lambda_i(u^-)\Big| = \mathcal{O}(1)\cdot s\,,$$

$$\big|\hat v_i(\tau)\big| = \mathcal{O}(1)\cdot s \sup_{\varsigma\in[0,s]} \Big|\tilde\lambda_i\big(u(\varsigma),\, v_i(\varsigma),\, \sigma_i(\varsigma)\big) - \lambda_i(u^-)\Big| = \mathcal{O}(1)\cdot s^2\,,$$

it is clear that for $0 \leq s << \epsilon << 1$ the transformation $\mathcal{T}_{i,s}$ maps Γ into itself. We now show that, in this same range of parameters, $\mathcal{T}_{i,s}$ is a contraction with respect to the weighted norm

$$\big\|(u, v_i, \sigma_i)\big\|_\dagger \doteq \|u\|_{\mathbf{L}^\infty} + \|v_i\|_{\mathbf{L}^\infty} + \epsilon\,\|\sigma_i\|_{\mathbf{L}^\infty}\,.$$

Indeed, consider two curves $\gamma, \gamma' \in \Gamma$. For each $\tau \in [0,s]$ one has

$$\begin{aligned} \|\hat u - \hat u'\|_{\mathbf{L}^\infty} &\leq \int_0^s \Big|\tilde r_i(u, v_i, \sigma_i) - \tilde r_i(u', v_i', \sigma_i')\Big|\, d\varsigma \\ &= \mathcal{O}(1)\cdot s\,\Big(\big\|u - u'\big\|_{\mathbf{L}^\infty} + \|v_i - v_i'\|_{\mathbf{L}^\infty} + \|v_i\|_{\mathbf{L}^\infty}\|\sigma_i - \sigma_i'\|_{\mathbf{L}^\infty}\Big), \end{aligned}$$

$$\begin{aligned} \|\hat v_i - \hat v_i'\|_{\mathbf{L}^\infty} &\leq 2\big\|f_i(\gamma) - f_i(\gamma')\big\|_{\mathbf{L}^\infty} \\ &\leq 2\int_0^s \Big|\tilde\lambda_i(u, v_i, \sigma_i) - \tilde\lambda_i(u', v_i', \sigma_i')\Big|\, d\varsigma \\ &= \mathcal{O}(1)\cdot\, s\,\Big(\big\|u - u'\big\|_{\mathbf{L}^\infty} + \|v_i - v_i'\|_{\mathbf{L}^\infty} + \|v_i\|_{\mathbf{L}^\infty}\|\sigma_i - \sigma_i'\|_{\mathbf{L}^\infty}\Big)\,, \end{aligned}$$

$$
\|\hat\sigma_i(\tau)-\hat\sigma_i'\|_{\mathbf{L}^\infty} \le \sup_{\tau\in[0,s]}\left|\frac{d}{d\tau}\mathrm{conv}\, f_i(\gamma,\tau)-\frac{d}{d\tau}\mathrm{conv}\, f_i'(\gamma,\tau)\right| \le \sup_{\tau\in[0,s]}\left|\frac{d}{d\tau}\, f_i(\gamma,\tau)-\frac{d}{d\tau}\, f_i'(\gamma,\tau)\right|
$$

$$
\le \big\|\tilde\lambda_i-\tilde\lambda_i'\big\|_{\mathbf{L}^\infty} = \mathcal{O}(1)\cdot\Big(\big\|u-u'\big\|_{\mathbf{L}^\infty}+\|v_i-v_i'\|_{\mathbf{L}^\infty}+\|v_i\|_{\mathbf{L}^\infty}\|\sigma_i-\sigma_i'\|_{\mathbf{L}^\infty}\Big)\,.
$$

For some constant C_0, the previous estimates imply

$$
\|\hat\gamma-\hat\gamma'\|_\dagger \le C_0\,\epsilon\,\|\gamma-\gamma'\|_\dagger \le \frac{1}{2}\|\gamma-\gamma'\|_\dagger\,, \tag{14.5}
$$

provided that ϵ is sufficiently small. Therefore, by the contraction mapping principle, the map $\mathcal{T}_{i,s}$ admits a unique fixed point, i.e. a continuous curve $\gamma=(u,v_i,\sigma_i)$ such that

$$
\begin{cases} u(\tau)=u^-+\displaystyle\int_0^\tau \tilde r_i\big(u(\varsigma),\, v_i(\varsigma),\, \sigma_i(\varsigma)\big)\,d\varsigma\,,\\ v_i(\tau)=f_i(\gamma,\tau)-\mathrm{conv}\, f_i(\gamma,\tau)\,,\\ \sigma_i(\tau)=\dfrac{d}{d\tau}\mathrm{conv}\, f_i(\gamma,\tau)\,. \end{cases} \tag{14.6}
$$

Recalling the definition (14.3), from the continuity of u,v_i,σ_i it follows that the maps $\tau\mapsto u(\tau)$, $\tau\mapsto v_i(\tau)$ and $\tau\mapsto f_i(\gamma,\tau)$ are continuously diferentiable. We now show that, taking $u^+=u(s)$ corresponding to the endpoint of this curve γ, the Riemann problem (14.2) admits a self-similar solution containing only i-waves.

Lemma 14.1 *In the previous setting, let $\gamma:\tau\mapsto\big(u(\tau),\, v_i(\tau),\, \sigma_i(\tau)\big)$ be the fixed point of the transformation $\mathcal{T}_{i,s}$. Define the right state $u^+\doteq u(s)$. Then the unique vanishing viscosity solution of the Riemann problem (14.2) is the function*

$$
\omega(t,x)\doteq\begin{cases} u^- & \text{if}\quad x/t\le\sigma_i(0)\,,\\ u(\tau) & \text{if}\quad x/t=\sigma_i(\tau)\,,\\ u^+ & \text{if}\quad x/t\ge\sigma_i(s)\,. \end{cases} \tag{14.7}
$$

Proof. With the semigroup notations introduced in Theorem 1, we will show that, for every $t\ge0$,

$$
\lim_{\varepsilon\to0+}\big\|\omega(t)-S_t^\varepsilon\omega(0)\big\|_{\mathbf{L}^1}=0\,. \tag{14.8}
$$

The proof will be given in several steps.

1. Assume that we can construct a family v^ε of solutions to

$$
v_t+A(v)v_x=\varepsilon\, v_{xx}\,, \tag*{$(14.9)_\varepsilon$}
$$

with

$$
\lim_{\varepsilon\to0+}\big\|v^\varepsilon(t)-\omega(t)\big\|_{\mathbf{L}^1}=0 \tag{14.10}
$$

for all $t\in[0,1]$. Then (14.8) follows. Indeed, by a simple rescaling we immediately have a family of solutions v^ε such that $(14.9)_\varepsilon$-(14.10) hold on any fixed interval $[0,T]$. For every $t\in[0,T]$, since by assumption $v^\varepsilon(t)=S_t^\varepsilon v^\varepsilon(0)$, using (1.16) we obtain

$$
\lim_{\varepsilon\to0+}\big\|\omega(t)-S_t^\varepsilon\omega(0)\big\|_{\mathbf{L}^1}\le\lim_{\varepsilon\to0+}\big\|\omega(t)-v^\varepsilon(t)\big\|_{\mathbf{L}^1}+\lim_{\varepsilon\to0+}\big\|v^\varepsilon(t)-S_t^\varepsilon\omega(0)\big\|_{\mathbf{L}^1}
$$

$$
\le 0+L\cdot\lim_{\varepsilon\to0+}\big\|v^\varepsilon(0)-\omega(0)\big\|_{\mathbf{L}^1}\;=\;0\,.
$$

2. For notational convenience, call VVL the set of all vanishing viscosity limits, i.e. all functions $v : [0,1] \times I\!\!R \mapsto I\!\!R^n$ such that

$$\lim_{\varepsilon \to 0+} \big\| v^\varepsilon(t) - v(t) \big\|_{\mathbf{L}^1} = 0 \qquad\qquad t \in [0,1] \tag{14.11}$$

for some family of solutions v^ε of $(14.9)_\varepsilon$. By Step 1, it suffices to show that the function ω at (14.7) lies in VVL.

Let us make some preliminary considerations. Consider a piecewise smooth function $v = v(t,x)$ which provides a classical solution to the quasilinear system

$$v_t + A(v)v_x = 0 \qquad\qquad t \in [0,1]\,,$$

outside a finite number of straight lines, say $x = x_j(t)$, $j = 1, \ldots, N$. Assume that there exists $\delta > 0$ and constant states ω_j^-, ω_j^+ such that

$$v(t,x) = \begin{cases} \omega_j^- & \text{if} \quad x_j(t) - \delta \leq x < x_j(t)\,, \\ \omega_j^+ & \text{if} \quad x_j(t) < x \leq x_j(t) + \delta\,, \end{cases}$$

Moreover, assume that each pair of states ω_j^-, ω_j^+ can be connected by a viscous travelling wave having speed $\dot x_j$. Finally, let v be constant on each of the two regions where $x > r_0$ or $x < -r_0$, for some r_0 sufficiently large. Under all of the above hypotheses, it is then clear that $v \in VVL$. Indeed, a family of viscous approximations v^ε can be constructed by a simplified version of the singular perturbation technique used in [GX].

As a second observation, notice that if we have a sequence of functions $v_m \in VVL$ with

$$\lim_{m \to \infty} \big\| v_m(t) - v(t) \big\|_{\mathbf{L}^1} = 0 \qquad\qquad t \in [0,1]\,,$$

then also $v \in VVL$.

3. Consider first the (generic) case where the set of points in which f_i is disjoint from its convex envelope is a finite union of open intervals (fig. 7), say

$$\Big\{ \tau \in [0,s]\,; \quad f_i(\gamma,\tau) > \operatorname{conv} f_i(\gamma,\tau) \Big\} = \bigcup_{j=1}^{N} \,]a_j,\, b_j[\,. \tag{14.12}$$

Our strategy is to prove that $\omega \in VVL$ first in this special case. Later we shall deal with the general case, by an approximation argument.

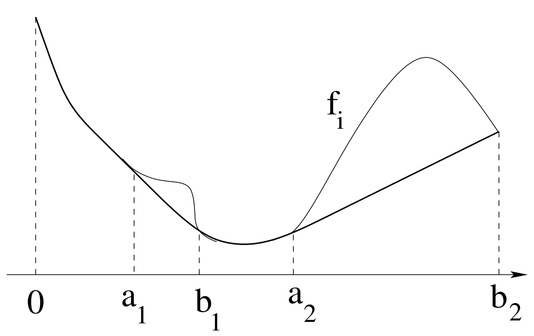


figure 7

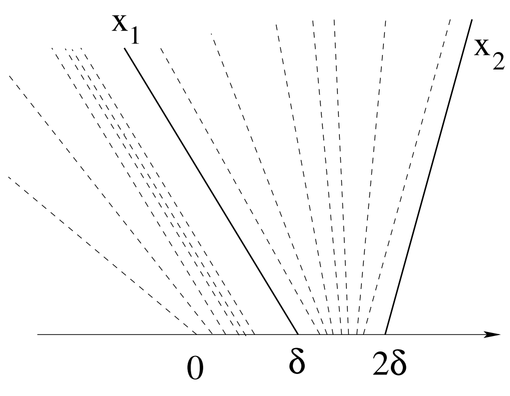


figure 8

If (14.12) holds, we can make the two following observations.

(i) For each $j = 1, \ldots, N$, we claim that the left and right states $u(a_j)$, $u(b_j)$ are connected by a viscous travelling profile U such that

$$U'' = \big(A(U) - \sigma_{ij}\big)U', \qquad U(-\infty) = u(a_j), \quad U(\infty) = u(b_j). \tag{14.13}$$

Here σ_{ij} is the constant speed

$$\sigma_{ij} \doteq \sigma_i(\tau) = \frac{d}{d\tau} \operatorname{conv} f_i(\gamma, \tau) \qquad \tau \in [a_j, b_j]\,.$$

To construct the function U, consider the variable transformation $]a_j,\, b_j[\, \mapsto I\!R$, say $\tau \mapsto x(\tau)$, defined by

$$x\Big(\frac{a_j + b_j}{2}\Big) = 0\,, \qquad \frac{dx(\tau)}{d\tau} = \frac{1}{v_i(\tau)}\,.$$

Let $\tau = \tau(x)$ be its inverse. Then the function $U(x) \doteq u\big(\tau(x)\big)$ is the required travelling wave profile. Indeed, U obviously takes the correct limits at $\pm\infty$. Moreover,

$$U' = \frac{du}{d\tau}\,\frac{d\tau}{dx} = v_i \tilde r_i,$$

$$\begin{aligned} U'' &= v_{i,x}\tilde r_i + v_i \tilde r_{i,x} \\ &= v_i\big(v_{i,\tau}\tilde r_i + v_i \tilde r_{i,\tau}\big) \\ &= v_i(\tilde\lambda_i - \sigma_{ij})\tilde r_i + v_i^2\big(\tilde r_{i,u}\tilde r_i + \tilde r_{i,v}(\tilde\lambda_i - \sigma_{ij})\big). \end{aligned}$$

Recalling the identity (4.22), we see that U also satisfies the differental equation in (14.13), thus proving our claim.

(ii) On the intervals where $f_i(\gamma, \tau) = \operatorname{conv} f_i(\gamma, \tau)$ we have $v_i(\tau) = 0$. Hence, by the first equation in (14.6) and by (4.16), $u_\tau = \tilde r_i = r_i(u)$ is an i-eigenvector of the matrix $A(u)$.

4. In general, even if the condition (14.12) is satisfied, we do not expect that the function ω has the regularity specified in Step 2. However, we now show that it can be approximated in $\mathbf{L}^1$ by functions ω_δ satisfying all the required assumptions. To fix the ideas, let

$$0 \doteq b_0 \le a_1 < b_1 \le a_2 < b_2 \le \cdots \le a_N < b_N \le a_{N+1} \doteq s\,.$$

A piecewise smooth viscosity solution can be defined as follows (fig. 8). Fix a small $\delta > 0$. For each $k = 0, \ldots, N$, consider a smooth non-decreasing map

$$\tau_k : \big[k\delta,\, (k+1)\delta\big] \mapsto [b_k,\, a_{k+1}]$$

such that

$$\tau_k(x) = \begin{cases} b_k & \text{if } \ x \le k\delta + \delta/3, \\ a_{k+1} & \text{if } \ x \ge k\delta + 2\delta/3. \end{cases}$$

We then define the initial condition

$$\omega_\delta(0, x) \doteq \begin{cases} u(0) & \text{if } \ x < 0\,, \\ u\big(\tau_k(x)\big) & \text{if } \ k\delta < x < (k+1)\delta\,, \\ u(s) & \text{if } \ x > (N+1)\delta\,. \end{cases}$$

A corresponding solution of the Cauchy problem can then be constructed by the method of characteristics:

$$\omega_\delta(t,x) \doteq \begin{cases} u(0) = u^- & \text{if} \quad x < \sigma_i(0)\,t\,, \\ u(0) & \text{if} \quad x = y + \sigma_i\big(\tau_k(y)\big)t \ \text{ for some } \ y \in \,\big]k\delta,\,(k+1)\delta\big[\,, \\ u(s) = u^+ & \text{if} \quad x > (N+1)\delta + \sigma_i(s)t\,. \end{cases}$$

It is clear that the above function ω_δ satisfies all of the assumptions considered in Step 2. Hence $\omega_\delta \in VVL$. Letting $\delta \to 0$ we have $\big\|\omega_\delta(t) - \omega(t)\big\|_{\mathbf{L}^1} \to 0$ for every t. Therefore, by the last observation in Step 2 we conclude that also $\omega \in VVL$.

5. To prove the Lemma in the general case, where the set in (14.12) may by the union of infinitely many open intervals, we use an approximation argument. For each $\delta > 0$, by slightly perturbing the values of A, we can construct a second matrix valued function A' with

$$\sup_u \big|A'(u) - A(u)\big| \leq \delta\,, \tag{14.14}$$

such that the following properties hold. For some right state $\tilde u^+$ with $|\tilde u^+ - u^+| \leq \delta$, the non-conservative Riemann problem

$$u_t + A'(u)u_x = 0, \qquad\qquad u(0,x) = \begin{cases} u^- & \text{if} \quad x < 0, \\ \tilde u^+ & \text{if} \quad x > 0. \end{cases} \tag{14.15}$$

admits a self-similar solution ω' which is limit of vanishing viscosity approximations and satisfies

$$\int_{-3\beta}^{3\beta} \big|\omega'(t,x) - \omega(t,x)\big|\,dx \leq \delta \qquad\qquad t \in [0,1]\,. \tag{14.16}$$

Clearly, the fact that ω' is a limit of vanishing viscosity approximations can be achieved by choosing A' so that a corresponding transformation $\mathcal{T}'_{s'}$ will admit as fixed point some curve $\gamma' : \tau \mapsto \big(u'(\tau)\,, v'_i(\tau)\,, \sigma'_i(\tau)\big)$ for which $u'(0) = u^-$, $u'(s') \approx u^+$ and with $f_i(\gamma', \tau)$ differing from its convex envelope on a finite number of open intervals.

Call ω^ε the solution of the viscous Riemann problem (13.1) with initial data (14.1). Using (13.7) with $v^\varepsilon \equiv u^+$, $a = 0$, $b = \infty$, for all $t \in [0,1]$ we obtain

$$\lim_{\varepsilon \to 0} \int_\beta^\infty \big|\omega^\varepsilon(t,x) - \omega(t,x)\big|\,dx \leq \lim_{\varepsilon \to 0} \int_\beta^\infty |u^- - u^+| \cdot \alpha e^{(\beta t - x)/\varepsilon}\,dx = 0\,. \tag{14.17}$$

Similarly,

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{-\beta} \big|\omega^\varepsilon(t,x) - \omega(t,x)\big|\,dx = 0\,. \tag{14.18}$$

To establish the convergence also on the interval $[-\beta, \beta]$, call v^ε the solution of the Cauchy problem

$$v^\varepsilon_t + A'(v^\varepsilon)v^\varepsilon_x = \varepsilon v^\varepsilon_{xx}\,, \qquad\qquad v^\varepsilon(0,x) = \begin{cases} u^- & \text{if} \quad x < 0\,, \\ \tilde u^+ & \text{if} \quad 0 < x < 3\beta\,, \\ u^+ & \text{if} \quad x > 3\beta\,. \end{cases}$$

Clearly,

$$\lim_{\varepsilon \to 0} \int_{-\beta}^{\beta} \big|v^\varepsilon(t,x) - \omega'(t,x)\big|\,dx = 0\,. \tag{14.19}$$

because ω' is a vanishing viscosity limit and because of the finite propagation speed. Using the triangle inequality we can write

$$\begin{aligned}\limsup_{\varepsilon\to 0}\int_{-\beta}^{\beta}\big|\omega^\varepsilon(t,x)-\omega(t,x)\big|\,dx &\le \limsup_{\varepsilon\to 0}\int_{-\beta}^{\beta}\big|\omega^\varepsilon(t,x)-v^\varepsilon(t,x)\big|\,dx\\ &+\lim_{\varepsilon\to 0}\int_{-\beta}^{\beta}\big|v^\varepsilon(t,x)-\omega'(t,x)\big|\,dx+\int_{-\beta}^{\beta}\big|\omega'(t,x)-\omega(t,x)\big|\,dx\,.\end{aligned}\tag{14.20}$$

Since $t\mapsto\omega^\varepsilon(t)=S_t^\varepsilon\omega(0)$ is a trajectory of the Lipschitz semigroup S^ε, recalling (14.14) we have the estimate

$$\begin{aligned}\big\|v^\varepsilon(t)-\omega^\varepsilon(t)\big\|_{\mathbf{L}^1} &\le L\,\big\|v^\varepsilon(0)-\omega^\varepsilon(0)\big\|_{\mathbf{L}^1}+L\cdot\int_0^t\left\{\lim_{h\to 0+}\frac{\big\|v^\varepsilon(s+h)-S_h^\varepsilon v^\varepsilon(s)\big\|_{\mathbf{L}^1}}{h}\right\}ds\\ &\le 3\beta L\,|\tilde u^+-u^+|+L\cdot\int_0^t\left\{\int\Big|A(v^\varepsilon(s,x))-A'(v^\varepsilon(s,x))\Big|\,\big|v_x^\varepsilon(s,x)\big|\,dx\right\}ds\\ &\le 3\beta L\delta+L\delta\,C\cdot\text{Tot.Var.}\big\{v^\varepsilon(0)\big\}\\ &\le C''\delta\,,\end{aligned}\tag{14.21}$$

for some constant C''. Estimating the right hand side of (14.20) by means of (14.21), (14.19) and (14.16), we obtain

$$\limsup_{\varepsilon\to 0}\int_{-\beta}^{\beta}\big|\omega^\varepsilon(t,x)-\omega(t,x)\big|\,dx\le C''\delta+0+\delta.$$

Since $\delta>0$ can be arbitrarily small, together with (14.17)-(14.18) this yields

$$\lim_{\varepsilon\to 0}\big\|\omega^\varepsilon(t)-\omega(t)\big\|_{\mathbf{L}^1}=0\qquad\qquad\text{for all } t\in[0,1]\,,$$

completing the proof. □

Remark 14.2. The transformation $\mathcal{T}_{i,s}$ defined at (14.4) depends on the vectors $\tilde r_i$, and hence on center manifold (which is not unique). However, the curve γ that we obtain as fixed point of $\mathcal{T}_{i,s}$ involves only a concatenation of bounded travelling profiles or stationary solutions. These are bounded solutions of (4.2), and will certainly be included in every center manifold. For this reason, the curve γ (and hence the solution of the Riemann problem) is independent of our choice of the center manifold.

For negative values of the parameter s, a right state $u^+=\Psi_i(s)$ can be constructed exactly in the same way as before, except that one now takes the upper concave envelope of f_i :

$$\operatorname{conc} f_i(\gamma,\tau)\doteq\sup\Big\{\theta f_i(\gamma,\tau')+(1-\theta)f_i(\gamma,\tau'')\,;\quad\theta\in[0,1]\,,\ \ \tau',\tau''\in[0,s]\,,\ \ \tau=\theta\tau'+(1-\theta)\tau''\Big\}\,,$$

instead of the lower convex envelope.

Our next step is to study the regularity of the curve of right states $u^+=\Psi_i(s)$.

Lemma 14.3. *Given a left state u^- and $i\in\{1,\dots,n\}$, the curve of right states $s\mapsto\Psi_i(s)$ is Lipschitz continuous and satisfies*

$$\lim_{s\to 0}\frac{d\Psi_i(s)}{ds}=r_i(u^-).\tag{14.22}$$

Proof. We assume $s > 0$, the other case being entirely similar. For sake of clarity, let us introduce some notations. For fixed i and $s > 0$, let $\gamma^{i,s} = (u^{i,s}, v_i^s, \sigma_i^s)$ be the fixed point of the transformation $\mathcal{T}_{i,s} : \Gamma \mapsto \Gamma$ in (14.4). Notice that, as soon as s is fixed, we can choose $\epsilon = \mathcal{O}(1) \cdot s$ in the definition of the domain Γ. By definition we now have

$$\Psi_i(s) \doteq u^{i,s}(s)\,.$$

For $0 < s' < s$, let $\gamma' \doteq (u', v_i', \sigma_i')$ be the restriction of $\gamma^{i,s}$ to the subinterval $[0, s']$. Since $\mathcal{T}_{i,s'}$ is a strict contraction, the distance of γ' from the fixed point of $\mathcal{T}_{i,s'}$ is estimated as

$$\big\|\gamma' - \gamma^{i,s'}\big\|_\dagger = \mathcal{O}(1) \cdot \big\|\gamma' - \mathcal{T}_{i,s'}\gamma'\big\|_\dagger = \mathcal{O}(1) \cdot (s - s')\, s\,.$$

In particular,

$$\big|u^{i,s'}(s') - u^{i,s}(s')\big| = \mathcal{O}(1) \cdot (s - s')\, s\,.$$

Observing that

$$u^{i,s}(s) - u^{i,s}(s') = \int_{s'}^{s} \tilde r_i\big(u^{i,s}(\varsigma),\, v_i^s(\varsigma),\, \sigma_i^s(\varsigma)\big)\, d\varsigma = (s - s') \cdot r_i(u^-) + \mathcal{O}(1) \cdot (s - s')s\,,$$

we conclude

$$\big|\Psi_i(s) - \Psi_i(s') - (s - s')r_i(u^-)\big| = \mathcal{O}(1) \cdot (s - s')\, s\,. \tag{14.23}$$

By (14.23), the map $s \mapsto \Psi_i(s)$ is Lipschitz continuous, hence differentiable almost everywhere, by Rademacher's theorem. The limit in (14.22) is again a consequence of (14.23). □

Thanks to the previous analysis, the solution of the general Riemann problem (14.2) can now be constructed following a standard procedure. Given a left state u^-, call $s \mapsto \Psi_i(s)(u^-)$ the curve of right states that can be connected to u^- by i-waves. Consider the composite mapping

$$\Psi : (s_1, \ldots, s_n) \mapsto \Psi_n(s_n) \circ \cdots \circ \Psi_1(s_1)(u^-)\,.$$

By Lemma 14.3 and a version of the implicit function theorem valid for Lipschitz continuous maps (see [Cl], p.253), Ψ is a one-to-one mapping from a neighborhood of the origin in $I\!R^n$ onto a neighborhood of u^-. Hence, for all u^+ sufficiently close to u^-, one can find unique values $s_1, \ldots, s_n$ such that $\Psi(s_1, \ldots, s_n) = u^+$. In turn, this yields intermediate states $u_0 = u^-, u_1, \ldots, u_n = u^+$ such that each Riemann problem with data u_{i-1}, u_i admits a vanishing viscosity solution $\omega_i = \omega_i(t, x)$ consisting only of i-waves. By strict hyperbolicity, we can now choose intermediate speeds

$$-\infty \doteq \lambda_0' < \lambda_1' < \lambda_2' < \cdots < \lambda_{n-1}' < \lambda_n' = \infty$$

such that all i-waves in the solution ω_i have speeds contained inside the interval $[\lambda_{i-1}', \lambda_i']$. The general solution of the general Riemann problem (14.2) is then given by

$$\omega(t, x) = \omega_i(t, x) \qquad\qquad \text{for } \lambda_{i-1}' < \frac{x}{t} < \lambda_i'\,. \tag{14.24}$$

Because of Lemma 14.1, it is clear that the function ω is the unique limit of viscous approximations:

$$\lim_{\varepsilon \to 0+} \big\|\omega(t) - S_t^\varepsilon \omega(0)\big\|_{\mathbf{L}^1} = 0 \qquad\qquad \text{for every } t \geq 0\,. \tag{14.25}$$

15 - Viscosity solutions and uniqueness of the semigroup

In [B3], one of the authors introduced a definition of *viscosity solution* for a system of conservation laws, based on local integral estimates. Assuming the existence of a Lipschitz semigroup of entropy weak solutions, it was proved that such a semigroup is necessarily unique and every viscosity solution coincides with a semigroup trajectory. We shall follow here exactly the same approach, in order to prove the uniqueness of the Lipschitz semigroup constructed in (13.9) as limit of vanishing viscosity approximations.

Toward the definition of a *viscosity solution* for the general hyperbolic system

$$u_t + A(u)u_x = 0, \tag{15.1}$$

we first introduce some notations. Given a function $u = u(t,x)$ and a point (τ,ξ), we denote by $U^\sharp_{(u;\tau,\xi)}$ the solution of the Riemann problem (14.1) with initial data

$$u^- = \lim_{x\to\xi-} u(\tau,x), \qquad\qquad u^+ = \lim_{x\to\xi+} u(\tau,x). \tag{15.2}$$

Of course, we refer here to the vanishing viscosity solution constructed in Section 14. In addition, we define $U^\flat_{(u;\tau,\xi)}$ as the solution of a linear hyperbolic Cauchy problem with constant coefficients:

$$w_t + \widehat{A}w_x = 0, \qquad\qquad w(0,x) = u(\tau,x). \tag{15.3}$$

Here $\widehat{A} \doteq A\big(u(\tau,\xi)\big)$. Observe that (15.3) is obtained from the quasilinear system (15.1) by "freezing" the coefficients of the matrix $A(u)$ at the point (τ,ξ) and choosing $u(\tau)$ as initial data.

As in [B3], the notion of *viscosity solution* is now defined by locally comparing a function u with the self-similar solution of a Riemann problem and with the solution of a linear hyperbolic system with constant coefficients.

Definition 15.1. A function $u = u(t,x)$ is a **viscosity solution** of the system (15.1) if $t \mapsto u(t,\cdot)$ is continuous as a map with values into $\mathbf{L}^1_{loc}$, and moreover the following integral estimates hold.

(i) At every point (τ,ξ), for every $\beta' > 0$ one has

$$\lim_{h\to 0+} \frac{1}{h}\int_{\xi-\beta' h}^{\xi+\beta' h} \Big|u(\tau+h,\ x) - U^\sharp_{(u;\tau,\xi)}(h,\ x-\xi)\Big|\,dx \;=\; 0. \tag{15.4}$$

(ii) There exist constants $C,\beta > 0$ such that, for every $\tau \geq 0$ and $a < \xi < b$, one has

$$\limsup_{h\to 0+} \frac{1}{h}\int_{a+\beta h}^{b-\beta h} \Big|u(\tau+h,\ x) - U^\flat_{(u;\tau,\xi)}(h,x)\Big|\,dx \leq C\cdot\Big(\text{Tot.Var.}\big\{u(\tau);\ ]a,\,b[\ \big\}\Big)^2. \tag{15.5}$$

The main result of this section shows that the above viscosity solutions coincide precisely with the limits of vanishing viscosity approximations.

Lemma 15.2. *Let $S : \mathcal{D}\times[0,\infty[\,\mapsto \mathcal{D}$ be a semigroup of vanishing viscosity solutions, constructed as limit of a sequence S^{ε_m} as in (13.9) and defined on a domain $\mathcal{D}\subset \mathbf{L}^1_{loc}$ of functions with small total variation. A map $u : [0,T]\mapsto\mathcal{D}$ satisfies*

$$u(t) = S_t u(0) \qquad\qquad \textit{for all } t\in[0,T] \tag{15.6}$$

if and only if u is a viscosity solution of (15.1).

Proof. **Necessity:** Assume that (15.6) holds. By (13.11), the map $t \mapsto u(t)$ is continuous. Let any $\beta' > 0$ be given and let L, β be the constants in (13.13). Then, for any (τ, ξ), an application of (13.13) yields

$$\begin{aligned}
&\int_{\xi-\beta' h}^{\xi+\beta' h} \Big| u(\tau+h,\ x) - U^\sharp_{(u;\tau,\xi)}(h,\ x-\xi) \Big|\, dx \\
&\leq L \cdot \left\{ \int_{\xi-(\beta+\beta')h}^{\xi} \Big| u(\tau,x) - u(\tau,\xi-) \Big|\, dx + \int_{\xi}^{\xi+(\beta+\beta')h} \Big| u(\tau,x) - u(\tau,\xi+) \Big|\, dx \right\} \\
&\leq L(\beta+\beta')h \left\{ \sup_{\xi-(\beta+\beta')h\, <\, x\, <\, \xi} \Big| u(\tau,x) - u(\tau,\xi-) \Big| \ + \sup_{\xi\, <\, x\, <\, \xi+(\beta+\beta')h} \Big| u(\tau,x) - u(\tau,\xi+) \Big| \right\}.
\end{aligned}$$

Hence (15.4) is clear.

To prove the second estimate, fix τ and $a < \xi < b$. Define the function

$$\bar v(x) \doteq \begin{cases} u(\tau, a+) & \text{if} \quad x \leq a\,, \\ u(\tau, x) & \text{if} \quad a < x < b\,, \\ u(\tau, b-) & \text{if} \quad x \geq b\,. \end{cases}$$

Call $v^\varepsilon, w^\varepsilon$ respectively the solutions of the viscous systems

$$v^\varepsilon_t + A(v^\varepsilon) v^\varepsilon_x = \varepsilon\, v^\varepsilon_{xx}\,, \qquad\qquad w^\varepsilon_t + \widehat A w^\varepsilon_x = \varepsilon\, w^\varepsilon_{xx}\,, \tag{15.7}$$

with the same initial data $v^\varepsilon(0,x) = w^\varepsilon(0,x) = \bar v(x)$.

Recalling that S^ε is a semigroup with Lipschitz constant L, as in [B3], [B5] we can use the error formula

$$\begin{aligned}
\big\| w^\varepsilon(h) - v^\varepsilon(h) \big\|_{\mathbf{L}^1} &= \big\| w^\varepsilon(h) - S^\varepsilon_h \bar v \big\|_{\mathbf{L}^1} \\
&\leq L \cdot \int_0^h \left\{ \liminf_{r \to 0+} \frac{\big\| w^\varepsilon(t+r) - S^\varepsilon_r w^\varepsilon(t) \big\|_{\mathbf{L}^1}}{r} \right\} dt \\
&\leq L \cdot \int_0^h \int \Big| \widehat A - A\big(w^\varepsilon(t,x)\big) \Big|\, \big| w^\varepsilon_x(t,x) \big|\, dx\, dt \\
&\leq L\, h \left(\sup_{t,x} \Big| A\big(w^\varepsilon(0,\xi)\big) - A\big(w^\varepsilon(t,x)\big) \Big| \right) \cdot \sup_t \big\| w^\varepsilon_x(t) \big\|_{\mathbf{L}^1} \\
&\leq C\, h \Big(\text{Tot.Var.}\{\bar v\} \Big)^2,
\end{aligned}$$

for some constant C. Letting $\varepsilon \to 0$ and using the estimate (13.13) on the finite speed of propagation, we obtain

$$\begin{aligned}
\frac{1}{h} \int_{a+\beta h}^{b-\beta h} \Big| u(\tau+h,\ x) - U^\flat_{(u;\tau,\xi)}(h,\ x) \Big|\, dx &\leq \frac{1}{h} \lim_{\varepsilon \to 0} \int_{a+\beta h}^{b-\beta h} \Big| v^\varepsilon(h,x) - w^\varepsilon(h,x) \Big|\, dx \\
\leq C \Big(\text{Tot.Var.}\{\bar v\} \Big)^2 \ &= \ C \Big(\text{Tot.Var.}\big\{\bar u\,;\ ]a,b[\,\big\} \Big)^2.
\end{aligned}$$

This proves (15.5), with β the constant in (13.13).

Sufficiency: Let $u = u(t,x)$ be a viscosity solution of (15.1). By assumption, the map $t \mapsto u(t)$ is continuous with values in a domain $\mathcal{D} \subset \mathbf{L}^1_{loc}$ of functions with small total variation. From (15.5) and this uniform bound on the total variation it follows that this map is actually Lipschitz continuous:

$$\big\|u(t) - u(s)\big\|_{\mathbf{L}^1} \leq L'' \,|t-s|\,, \tag{15.8}$$

for some constant L'' and all $s,t \in [0,T]$. Let L be the Lipschitz constant of the semigroup S, as in (13.13). Given any interval $[a,b]$, thanks to (15.8) one has the error estimate

$$\begin{aligned}\int_{a+t\beta}^{b-t\beta} & \Big|u(t,x) - \big(S_t u(0)\big)(x)\Big|\,dx \\ & \leq L\cdot \int_0^t \left\{\liminf_{h\to 0+}\frac{1}{h}\int_{a+(\tau+h)\beta}^{b-(\tau+h)\beta}\Big|u(\tau+h,x) - \big(S_h u(\tau)\big)(x)\Big|\,dx\right\} d\tau\,.\end{aligned} \tag{15.9}$$

To prove the identity (15.6) it thus suffices to show that the integrand on the right hand side of (15.9) vanishes for all $\tau \in [0,T]$.

Fix any time $\tau \in [0,T]$ and let $\epsilon > 0$ be given. Since the total variation of $u(\tau,\cdot)$ is finite, we can choose finitely many points

$$a + \tau\beta = x_0 < x_1 < \cdots < x_N = b - \tau\beta$$

such that, for every $j = 1,\ldots,N$,

$$\text{Tot.Var.}\big\{u(\tau,\cdot)\,;\ ]x_{j-1},\,x_j[\,\big\} < \epsilon\,.$$

By the necessity part of the theorem, which has been already proved, the function $w(t,\cdot) \doteq S_{t-\tau}u(\tau)$ is itself a viscosity solution and hence it also satisfies the estimates (15.4)-(15.5). We now consider the midpoints $y_j \doteq (x_{j-1}+x_j)/2$. Using the estimate (15.4) at each of the points $\xi = x_j$ and the estimate (15.5) with $\xi \doteq y_j$ on each of the intervals $]x_{j-1},\ x_j[$, taking $\beta > 0$ sufficiently large we now compute

$$\begin{aligned}\limsup_{h\to 0+}\frac{1}{h}&\int_{a+(\tau+h)\beta}^{b-(\tau+h)\beta}\Big|u(\tau+h,x) - \big(S_h u(\tau)\big)(x)\Big|\,dx \\ \leq \sum_{j=1}^{N-1}&\limsup_{h\to 0+}\frac{1}{h}\int_{x_j-h\beta}^{x_j+h\beta}\bigg(\Big|u(\tau+h,x) - U^\sharp_{(u;\tau,x_j)}(\tau+h,\ x)\Big| \\ &\qquad\qquad + \Big|U^\sharp_{(u;\tau,x_j)}(\tau+h,\ x) - \big(S_h u(\tau)\big)(x)\Big|\bigg)\,dx \\ & + \sum_{j=1}^{N}\limsup_{h\to 0+}\frac{1}{h}\int_{x_{j-1}+h\beta}^{x_j-h\beta}\bigg(\Big|u(\tau+h,x) - U^\flat_{(u;\tau,y_j)}(h,\ x)\Big| \\ &\qquad\qquad + \Big|U^\flat_{(u;\tau,y_j)}(h,\ x) - \big(S_h u(\tau)\big)(x)\Big|\bigg)\,dx \\ \leq 0 + &\sum_{j=1}^{N} C\left(\text{Tot.Var.}\big\{u(\tau);\ ]x_{j-1},\ x_j[\big\}\right)^2 \\ \leq C\,\epsilon &\cdot \text{Tot.Var.}\big\{u(\tau);\ ]a+\tau\beta\,,\ b-\tau\beta[\big\}\,.\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, the integrand on the right hand side of (15.9) must vanish at time τ. This completes the proof of the lemma. $\square$

Remark 15.3. From the proof of the sufficiency part, it is clear that the identity (15.6) still holds if we require that the integral estimates (15.4) hold only for τ outside a set of times $\mathcal{N} \subset [0,T]$ of measure zero. By a well known result in the theory of BV functions [EG], any BV function of two variables $u = u(t,x)$ is either approximately continuous or has an approximate jump discontinuity at every point (τ, ξ), with τ outside a set $\mathcal{N}$ having zero measure. To decide whether a function u is a viscosity solution, it thus suffices to check (15.4) only at points of approximate jump, where the Riemann problem is solved in terms of a single shock.

Using Lemma 15.2, we now obtain at one stroke the uniqueness of viscosity solutions and of vanishing viscosity limits:

Completion of the proof of Theorem 1. What remains to be proved is that the whole family of viscous approximations converges to a unique limit, i.e.

$$\lim_{\varepsilon\to 0+} S_t^\varepsilon \bar u = S_t \bar u\,, \tag{15.10}$$

where the limit holds over all real values of ε and not only along a particular sequence $\{\varepsilon_m\}$. If (15.10) fails, we can find $\bar v$, τ and two different sequences $\varepsilon_m, \varepsilon'_m \to 0$ such that

$$\lim_{m\to\infty} S_\tau^{\varepsilon_m} \bar v \neq \lim_{m\to\infty} S_\tau^{\varepsilon'_m} \bar v\,. \tag{15.11}$$

By extracting further subsequences, we can assume that the limits

$$\lim_{m\to\infty} S_t^{\varepsilon_m} \bar u = S_t \bar u\,, \qquad\qquad \lim_{m\to\infty} S_t^{\varepsilon'_m} \bar u = S'_t \bar u\,, \tag{15.12}$$

exist in $\mathbf{L}^1_{loc}$, for all $t \geq 0$ and $\bar u \in \mathcal{U}$. By the analysis in Section 13, both S and S' are semigroups of vanishing viscosity solutions. In particular, the necessity part of Lemma 14.2 implies that the map $t \mapsto v(t) \doteq S_t \bar v$ is a viscosity solution of (15.1), while the sufficiency part implies $v(t) = S'_t v(0)$ for all $t \geq 0$. But this is in contradiction with (15.11), hence the unique limit (15.10) is well defined. □

Remark 15.4. The above uniqueness result is obtained within the family of vanishing viscosity limits of the form $(1.13)_\varepsilon$, with unit viscosity matrix. In the more general case $(1.21)_\varepsilon$, if the system is not in conservation form, we expect that the limit of solutions as $\varepsilon \to 0$ will depend on the form of the viscosity matrices $B(u)$. Indeed, by choosing different matrices $B(u)$, one will likely alter the vanishing viscosity solutions of the Riemann problems (14.2). In turn, this affects the definition of viscosity solution at (15.4).

16 - Dependence on parameters and large time asymptotics

We wish to derive here a simple estimate on how the viscosity solution changes, depending on hyperbolic matrices $A(u)$.

Corollary 16.1. *Assume that the two hyperbolic systems*

$$u_t + A(u)u_x = 0,$$
$$u_t + \widehat{A}(u)u_x = 0\,,$$

both satisfy the hypotheses of Theorem 1. Call $S, \widehat{S}$ the corresponding semigroups of viscosity solutions. Then, for every initial data $\bar u$ with small total variation, one has the estimate

$$\big\|S_t\bar u - \widehat{S}_t\bar u\big\|_{\mathbf{L}^1} = \mathcal{O}(1)\cdot t\Big(\sup_u \big|\widehat{A}(u) - A(u)\big|\Big)\cdot \text{Tot.Var.}\{\bar u\}\,. \tag{16.1}$$

Proof. Call $S^\varepsilon, \widehat{S}^\varepsilon$ the semigroups of solutions to the corresponding viscous problems

$$u_t + A(u)u_x = \varepsilon\, u_{xx}, \qquad\qquad u_t + \widehat{A}(u)u_x = \varepsilon\, u_{xx}\,.$$

Let L be the Lipschitz constant in (1.16) and call $w^\varepsilon(t) \doteq \widehat{S}^\varepsilon_t\bar u$. For every $t \geq 0$ we have the error estimate

$$\begin{aligned}
\big\|\widehat{S}^\varepsilon_t\bar u - S^\varepsilon_t\bar u\big\|_{\mathbf{L}^1} &= \big\|w^\varepsilon(t) - S^\varepsilon_t\bar u\big\|_{\mathbf{L}^1}\\
&\leq L\cdot\int_0^t \left\{\liminf_{h\to 0+}\frac{\big\|w^\varepsilon(s+h) - S^\varepsilon_h w^\varepsilon(s)\big\|_{\mathbf{L}^1}}{h}\right\} ds\\
&\leq L\cdot\int_0^t\int \Big|\widehat{A}\big(w^\varepsilon(s,x)\big) - A\big(w^\varepsilon(s,x)\big)\Big|\,\big|w^\varepsilon_x(s,x)\big|\,dxds\\
&\leq L\left(\sup_u\big|\widehat{A}(u) - A(u)\big|\right)\int_0^t \big\|w^\varepsilon_x(s)\big\|_{\mathbf{L}^1}\,ds\\
&= \mathcal{O}(1)\cdot t\left(\sup_u\big|\widehat{A}(u) - A(u)\big|\right)\text{Tot.Var.}\{\bar u\}.
\end{aligned}$$

□

Next, we show that some semigroup trajectories are asymptotically self-similar.

Corollary 16.2. *Under the same assumption of Theorem 1, consider an initial data $\bar u$ with small total variation, such that*

$$\int_{-\infty}^0 \big|\bar u(x) - u^-\big|\,dx + \int_0^\infty \big|\bar u(x) - u^+\big|\,dx < \infty\,, \tag{16.2}$$

for some states u^-, u^+. Call $\omega(t,x) = \tilde\omega(x/t)$ the self-similar solution of the corresponding Riemann problem (14.2). Then the solution of the viscous Cauchy problem

$$u_t + A(u)u_x = u_{xx}\,, \qquad\qquad u(0,x) = \bar u(x) \tag{16.3}$$

satisfies

$$\lim_{\tau\to\infty}\int \big|u(\tau,\,\tau y) - \tilde\omega(y)\big|\,dy = 0\,. \tag{16.4}$$

Proof. The assumption on ω implies that the limit (14.25) holds. For fixed τ, call $\varepsilon \doteq 1/\tau$ and consider the function $v^\varepsilon(t,x) \doteq u(\tau x,\,\tau t)$. Clearly, v^ε satisfies the equation

$$v^\varepsilon_t + A(v^\varepsilon)v^\varepsilon_x = \varepsilon\, v^\varepsilon_{xx}\,, \qquad\qquad v^\varepsilon(0,x) = \bar u(x/\varepsilon).$$

Therefore,

$$\begin{aligned}
\int \big|u(\tau,\,\tau y) - \tilde\omega(y)\big|\,dy &= \int \big|v^\varepsilon(1,x) - \omega(1,x)\big|\,dx\\
&\leq \big\|S^\varepsilon_1 v^\varepsilon(0) - S^\varepsilon_1\omega(0)\big\|_{\mathbf{L}^1} + \big\|S^\varepsilon_1\omega(0) - \omega(1)\big\|_{\mathbf{L}^1}\,.
\end{aligned} \tag{16.5}$$

Observing that

$$\begin{aligned}\left\|S_1^\varepsilon v^\varepsilon(0) - S_1^\varepsilon \omega(0)\right\|_{\mathbf{L}^1} &\leq L \cdot \left\|S_1^\varepsilon v^\varepsilon(0) - S_1^\varepsilon \omega(0)\right\|_{\mathbf{L}^1} \\ &= L\varepsilon \left(\int_{-\infty}^0 \left|\bar{u}(x) - u^-\right| dx + \int_0^\infty \left|\bar{u}(x) - u^+\right| dx\right),\end{aligned}$$

and using (14.25), from (16.5) we obtain (16.4). □

Appendix A

We derive here the explicit form of the evolution equations (6.1), for the variables v_i and w_i defined by the decomposition

$$u_x = \sum_i v_i \tilde{r}_i\big(u, v_i, \lambda_i^* - \theta(w_i/v_i)\big), \qquad u_t = \sum_i \big(w_i - \lambda_i^* v_i\big)\tilde{r}_i\big(u, v_i, \lambda_i^* - \theta(w_i/v_i)\big). \tag{A.1}$$

By checking one by one all source terms, we then provide an alternative proof of Lemma 6.1. The computations are lengthy but straightforward: one has to rewrite the evolution equations for u_x and u_t :

$$\begin{cases} (u_x)_t + \big(A(u)u_x\big)_x - (u_x)_{xx} = 0\,, \\ (u_t)_t + \big(A(u)u_t\big)_x - (u_t)_{xx} = \big(u_x \bullet A(u)\big)u_t - \big(u_t \bullet A(u)\big)u_x\,, \end{cases} \tag{A.2}$$

in terms of v_i, w_i. For convenience, we set $\theta_i \doteq \theta(w_i/v_i)$. The fundamental relation (4.23) can be written as

$$v_i \tilde{r}_{i,u}\tilde{r}_i - A(u)\tilde{r}_i = -\tilde{\lambda}_i \tilde{r}_i + \big(-\tilde{\lambda}_i + \lambda_i^* - \theta_i\big)v_i \tilde{r}_{i,v}\,. \tag{A.3}$$

Differentiating (A.1) w.r.t. x and using (A.3) we obtain

$$\begin{aligned} u_{xx} - A(u)u_x &= \sum_i v_{i,x}\tilde{r}_i + \sum_i v_i \tilde{r}_{i,x} - \sum_i A(u)v_i\tilde{r}_i \\ &= \sum_i v_{i,x}\tilde{r}_i + \sum_i v_i\Big[v_i\tilde{r}_{i,u}\tilde{r}_i - A(u)\tilde{r}_i\Big] \\ &\quad + \sum_i v_i\Big[v_{i,x}\tilde{r}_{i,v} - \theta_i'\big((v_iw_{i,x} - w_iv_{i,x})/v_i^2\big)\tilde{r}_{i,\sigma}\Big] + \sum_{i\neq j} v_iv_j\tilde{r}_{i,u}\tilde{r}_j \\ &= \sum_i (v_{i,x} - \tilde{\lambda}_iv_i)\tilde{r}_i + \sum_i(-\tilde{\lambda}_i + \lambda_i^* - \theta_i)v_i^2\tilde{r}_{i,v} \\ &\quad + \sum_i v_i\Big[v_{i,x}\tilde{r}_{i,v} - \theta_i'\big((v_iw_{i,x} - w_iv_{i,x})/v_i^2\big)\tilde{r}_{i,\sigma}\Big] + \sum_{i\neq j} v_iv_j\tilde{r}_{i,u}\tilde{r}_j \\ &= \sum_i \big(v_{i,x} - \tilde{\lambda}_iv_i\big)\Big[\tilde{r}_i + v_i\tilde{r}_{i,v} + \theta_i'\big(w_i/v_i\big)\tilde{r}_{i,\sigma}\Big] + \sum_i \big(w_{i,x} - \tilde{\lambda}_iw_i\big)\Big[-\theta_i'\tilde{r}_{i,\sigma}\Big] \\ &\quad + \sum_i v_i^2\big(\lambda_i^* - \theta_i\big)\tilde{r}_{i,v} + \sum_{i\neq j} v_iv_j\tilde{r}_{i,u}\tilde{r}_j\,, \end{aligned} \tag{A.4}$$

$$
\begin{aligned}
u_{tx} - A(u)u_t &= \sum_i (w_{i,x} - \lambda_i^* v_{i,x})\tilde r_i + \sum_i (w_i - \lambda_i^* v_i)\tilde r_{i,x} - \sum_i (w_i - \lambda_i^* v_i) A(u)\tilde r_i \\
&= \sum_i (w_{i,x} - \lambda_i^* v_{i,x})\tilde r_i + \sum_i (w_i - \lambda_i^* v_i)\big[v_i \tilde r_{i,u}\tilde r_i - A(u)\tilde r_i\big] \\
&\quad + \sum_i (w_i - \lambda_i^* v_i)\Big[v_{i,x}\tilde r_{i,v} - \theta_i'\big((v_i w_{i,x} - w_i v_{i,x})/v_i^2\big)\tilde r_{i,\sigma}\Big] + \sum_{i\neq j}\big(w_i - \lambda_i^* v_i\big) v_j \tilde r_{i,u}\tilde r_j \\
&= \sum_i (w_{i,x} - \tilde\lambda_i w_i)\tilde r_i - \sum_i \lambda_i^* (v_{i,x} - \tilde\lambda_i v_i)\tilde r_i + \sum_i (w_i - \lambda_i^* v_i)\big(-\tilde\lambda_i + \lambda_i^* - \theta_i\big) v_i \tilde r_{i,v} \\
&\quad + \sum_i (w_i - \lambda_i^* v_i)\Big[v_{i,x}\tilde r_{i,v} - \theta_i'\big((v_i w_{i,x} - w_i v_{i,x})/v_i^2\big)\tilde r_{i,\sigma}\Big] + \sum_{i\neq j}\big(w_i - \lambda_i^* v_i\big) v_j \tilde r_{i,u}\tilde r_j \\
&= \sum_i \big(v_{i,x} - \tilde\lambda_i v_i\big)\Big[w_i \tilde r_{i,v} + \theta_i'\big(w_i/v_i\big)^2 \tilde r_{i,\sigma}\Big] + \sum_i \big(w_{i,x} - \tilde\lambda_i w_i\big)\Big[\tilde r_i - \big(\theta_i' w_i/v_i\big)\tilde r_{i,\sigma}\Big] \\
&\quad + \sum_i v_i w_i \big(\lambda_i^* - \theta_i\big)\tilde r_{i,v} + \sum_{i\neq j} w_i v_j \tilde r_{i,u}\tilde r_j \\
&\quad - \sum_i \lambda_i^* \bigg\{\Big(v_{i,x} - \tilde\lambda_i v_i\Big)\Big[\tilde r_i + v_i \tilde r_{i,v} + \big(\theta_i' w_i/v_i\big)\tilde r_{i,\sigma}\Big] + \Big(w_{i,x} - \tilde\lambda_i w_i\Big)\Big[-\theta_i'\tilde r_{i,\sigma}\Big] \\
&\qquad\qquad + v_i^2\big(\lambda_i^* - \theta_i\big)\tilde r_{i,v} + \sum_{j\neq i} v_i v_j \tilde r_{i,u}\tilde r_j\bigg\}.
\end{aligned}
\tag{A.5}
$$

Differentiating (A.1) w.r.t. t one obtains

$$
\begin{aligned}
u_{xt} &= \sum_i v_{i,t}\tilde r_i + \sum_i v_i \tilde r_{i,t} \\
&= \sum_i v_{i,t}\tilde r_i + \sum_i v_i\Big[v_{i,t}\tilde r_{i,v} - \theta_i'\big((w_{i,t}v_i - w_i v_{i,t})/v_i^2\big)\tilde r_{i\sigma}\Big] + \sum_{i,j} v_i\big(w_j - \lambda_j^* v_j\big)\tilde r_{i,u}\tilde r_j \\
&= \sum_i v_{i,t}\Big[\tilde r_i + v_i \tilde r_{i,v} + \big(\theta_i' w_i/v_i\big)\tilde r_{i,\sigma}\Big] + \sum_i w_{i,t}\Big[-\theta_i'\tilde r_{i,\sigma}\Big] + \sum_{i,j} v_i\big(w_j - \lambda_j^* v_j\big)\tilde r_{i,u}\tilde r_j\,,
\end{aligned}
\tag{A.6}
$$

$$
\begin{aligned}
u_{tt} &= \sum_i \big(w_{i,t} - \lambda_i^* v_{i,t}\big)\tilde r_i + \sum_i \big(w_i - \lambda_i^* v_i\big)\tilde r_{i,t} \\
&= \sum_i \big(w_{i,t} - \lambda_i^* v_{i,t}\big)\tilde r_i + \sum_i \big(w_i - \lambda_i^* v_i\big)\Big[v_{i,t}\tilde r_{i,v} - \theta_i'\big((w_{i,t}v_i - w_i v_{i,t})/v_i^2\big)\tilde r_{i\sigma}\Big] \\
&\quad + \sum_{i,j}\big(w_i - \lambda_i^* v_i\big)\big(w_j - \lambda_j^* v_j\big)\tilde r_{i,u}\tilde r_j \\
&= \sum_i v_{i,t}\Big[w_i \tilde r_{i,v} + \theta_i'\big(w_i/v_i\big)^2\tilde r_{i,\sigma}\Big] + \sum_i w_{i,t}\Big[\tilde r_i - \theta_i'\big(w_i/v_i\big)\tilde r_{i,\sigma}\Big] + \sum_{i,j} w_i\big(w_j - \lambda_j^* v_j\big)\tilde r_{i,u}\tilde r_j \\
&\quad - \sum_i \lambda_i^*\bigg\{v_{i,t}\Big[\tilde r_i + v_i\tilde r_{i,v} + \big(\theta_i' w_i/v_i\big)\tilde r_{i,\sigma}\Big] - w_{i,t}\theta_i'\tilde r_{i,\sigma} + \sum_j v_i\big(w_j - \lambda_j^* v_j\big)\tilde r_{i,u}\tilde r_j\bigg\}.
\end{aligned}
\tag{A.7}
$$

Differentiating again $u_{xx} - A(u)u_x$ and $u_{tx} - A(u)u_t$ w.r.t. x, from (A.4) and (A.5) one finds

$$
\begin{aligned}
u_{tx} &= \big(u_x\big)_{xx} - \big(A(u)u_x\big)_x \\
&= \sum_i \big(v_{i,xx} - (\tilde\lambda_i v_i)_x\big)\Big[\tilde r_i + v_i \tilde r_{i,v} + \theta_i'(w_i/v_i)\tilde r_{i,\sigma}\Big] + \sum_i \big(w_{i,xx} - (\tilde\lambda_i w_i)_x\big)\Big[-\theta_i' \tilde r_{i,\sigma}\Big] \\
&\quad + \sum_i (v_{i,x} - \tilde\lambda_i v_i)\Big[\sum_j v_j \tilde r_{i,u}\tilde r_j + 2v_{i,x}\tilde r_{i,v} + \Big(-\theta_{i,x} + (\theta_i' w_i/v_i)_x\Big)\tilde r_{i,\sigma} + \sum_j v_j v_i \tilde r_{i,vu}\tilde r_j \\
&\qquad + v_i v_{i,x}\tilde r_{i,vv} + \big(-v_i\theta_{i,x} + \theta_i' v_{i,x} w_i/v_i\big)\tilde r_{i,v\sigma} + \sum_j v_j \theta_i'(w_i/v_i)\tilde r_{i,\sigma u}\tilde r_j - \theta_{i,x}\theta_i'(w_i/v_i)\tilde r_{i,\sigma\sigma}\Big] \\
&\quad + \sum_i (w_{i,x} - \tilde\lambda_i w_i)\Big[-\theta_{i,x}'\tilde r_{i,\sigma} - \sum_j \theta_i' v_j \tilde r_{i,\sigma u}\tilde r_j - v_{i,x}\theta_i'\tilde r_{i,\sigma v} + \theta_i'\theta_{i,x}\tilde r_{i,\sigma\sigma}\Big] \\
&\quad + \sum_i \big(v_i^2(\lambda_i^* - \theta_i)\big)_x \tilde r_{i,v} + \sum_i v_i^2(\lambda_i^* - \theta_i)\Big[\sum_j v_j \tilde r_{i,vu}\tilde r_j + v_{i,x}\tilde r_{i,vv} - \theta_{i,x}\tilde r_{i,v\sigma}\Big] \\
&\quad + \sum_{i\neq j} \big(v_i v_j\big)_x \tilde r_{i,u}\tilde r_j + \sum_{i\neq j} v_i v_j \Big[\sum_k v_k\big(\tilde r_{i,uu}(\tilde r_j\otimes\tilde r_k) + \tilde r_{i,u}\tilde r_{j,u}\tilde r_k\big) \\
&\qquad + v_{j,x}\tilde r_{i,u}\tilde r_{j,v} + v_{i,x}\tilde r_{i,uv}\tilde r_j - \theta_{j,x}\tilde r_{i,u}\tilde r_{j,\sigma} - \theta_{i,x}\tilde r_{i,u\sigma}\tilde r_j\Big],
\end{aligned}
\tag{A.8}
$$

$$
\begin{aligned}
&(u_t)_{xx} - \big(A(u)u_t\big)_x \\
&= \sum_i \big(v_{i,xx} - (\tilde\lambda_i v_i)_x\big)\Big[w_i \tilde r_{i,v} + \theta_i'\big(w_i/v_i\big)^2 \tilde r_{i,\sigma}\Big] + \sum_i \big(w_{i,xx} - (\tilde\lambda_i w_i)_x\big)\Big[\tilde r_i - \theta_i'(w_i/v_i)\tilde r_{i,\sigma}\Big] \\
&\quad + \sum_i (v_{i,x} - \tilde\lambda_i v_i)\Big[w_{i,x}\tilde r_{i,v} + \sum_j w_i v_j\, \tilde r_{i,vu}\tilde r_j + w_i v_{i,x}\, \tilde r_{i,vv} + \Big(-w_i\theta_{i,x} + \theta_i'(w_i/v_i)^2 v_{i,x}\Big)\tilde r_{i,v\sigma} \\
&\qquad + \big(\theta_i'(w_i/v_i)^2\big)_x \tilde r_{i,\sigma} + \sum_j v_j\theta_i'(w_i/v_i)^2 \tilde r_{i,\sigma u}\tilde r_j - \theta_i'(w_i/v_i)^2\theta_{i,x}\tilde r_{i,\sigma\sigma}\Big] \\
&\quad + \sum_i (w_{i,x} - \tilde\lambda_i w_i)\Big[\sum_j v_j \tilde r_j \tilde r_{i,u} + v_{i,x}\tilde r_{i,v} - \big(\theta_{i,x} + (\theta_i' w_i/v_i)_x\big)\tilde r_{i,\sigma} - \sum_j v_j\theta_i'(w_i/v_i)\tilde r_{i,\sigma u}\tilde r_j \\
&\qquad - v_{i,x}\theta_i'(w_i/v_i)\tilde r_{i,\sigma v} + \theta_{i,x}\theta_i'(w_i/v_i)\tilde r_{i,\sigma\sigma}\Big] + \sum_i \big(w_i v_i(\lambda_i^* - \theta_i)\big)_x \tilde r_{i,v} \\
&\quad + \sum_i w_i v_i\big(\lambda_i^* - \theta_i\big)\Big[\sum_j v_j \tilde r_{i,vu}\tilde r_j + v_{i,x}\tilde r_{i,vv} - \theta_{i,x}\tilde r_{i,v\sigma}\Big] + \sum_{i\neq j}(w_i v_j)_x \tilde r_{i,u}\tilde r_j \\
&\quad + \sum_{i\neq j} w_i v_j \Big[\sum_k v_k\big(\tilde r_{i,uu}(\tilde r_j\otimes\tilde r_k) + \tilde r_{i,u}\tilde r_{j,u}\tilde r_k\big) + v_{j,x}\tilde r_{i,u}\tilde r_{j,v} + v_{i,x}\tilde r_{i,vu}\tilde r_j - \theta_{j,x}\tilde r_{i,u}\tilde r_{j,\sigma} - \theta_{i,x}\tilde r_{i,u\sigma}\tilde r_j\Big] \\
&\quad - \sum_i \lambda_i^* \Big\{(v_{i,x} - \tilde\lambda_i v_i)\Big[\tilde r_i + v_i\tilde r_{i,v} + \theta_i'(w_i/v_i)\tilde r_{i,\sigma}\Big] + \big(w_{i,x} - \tilde\lambda_i w_i\big)\Big[-\theta_i'\tilde r_{i,\sigma}\Big] \\
&\qquad + v_i^2\big(\lambda_i^* - \theta_i\big)\tilde r_{i,v} + \sum_{j\neq i} v_i v_j \tilde r_{i,u}\tilde r_j\Big\}_x .
\end{aligned}
\tag{A.9}
$$

Substituting the expressions (A.6)–(A.9) inside (A.2) and observing that

$$\big(u_x \bullet A(u)\big)u_t - \big(u_t \bullet A(u)\big)u_x = \sum_{j\neq i}(w_i - \lambda_i^* v_i)v_j\Big[\big(\tilde r_j \bullet A(u)\big)\tilde r_i - \big(\tilde r_i \bullet A(u)\big)\tilde r_j\Big],$$

we finally obtain an implicit system of $2n$ scalar equations, describing the evolution of the components v_i, w_i:

$$\begin{aligned}
&\sum_i \Big(v_{i,t} + \big(\tilde\lambda_i v_i\big)_x - v_{i,xx}\Big)\Big[\tilde r_i + v_i \tilde r_{i,v} + \theta_i'(w_i/v_i)\tilde r_{i,\sigma}\Big] + \sum_i \Big(w_{i,t} + \big(\tilde\lambda_i w_i\big)_x - w_{i,xx}\Big)\Big[-\theta_i'\tilde r_{i,\sigma}\Big]\\
&\quad = \sum_i \tilde r_{i,u}\tilde r_i\Big[v_i\big(v_{i,x} - \tilde\lambda_i v_i\big) - v_i\big(w_i - \lambda_i^* v_i\big)\Big]\\
&\qquad + \sum_{i\neq j}\tilde r_{i,u}\tilde r_j\Big[\big(v_{i,x} - \tilde\lambda_i v_i\big)v_j + \big(v_i v_j\big)_x - v_i\big(w_j - \lambda_j^* v_j\big)\Big]\\
&\qquad + \sum_i \tilde r_{i,v}\Big[2v_{i,x}\big(v_{i,x} - \tilde\lambda_i v_i\big) + \big(v_i^2(\lambda_i^* - \theta_i)\big)_x\Big]\\
&\qquad + \sum_i \tilde r_{i,\sigma}\Big[\big(v_{i,x} - \tilde\lambda_i v_i\big)\big(- \theta_{i,x} + (\theta_i' w_i/v_i)_x\big) - \big(w_{i,x} - \tilde\lambda_i w_i\big)\theta_{i,x}'\Big]\\
&\qquad + \sum_i \tilde r_{i,vu}\tilde r_i\Big[v_i^2\big(v_{i,x} - \tilde\lambda_i v_i\big) + v_i^3(\lambda_i^* - \theta_i)\Big]\\
&\qquad + \sum_{i\neq j}\tilde r_{i,vu}\tilde r_j\Big[v_i v_j\big(v_{i,x} - \tilde\lambda_i v_i\big) + v_j v_i^2(\lambda_i^* - \theta_i)\Big]\\
&\qquad + \sum_i \tilde r_{i,vv}\Big[v_i v_{i,x}\big(v_{i,x} - \tilde\lambda_i v_i\big) + v_{i,x}v_i^2(\lambda_i^* - \theta_i)\Big]\\
&\qquad + \sum_i \tilde r_{i,v\sigma}\Big[\big(v_{i,x} - \tilde\lambda_i v_i\big)\Big(-v_i\theta_{i,x} + \theta_i' v_{i,x} w_i/v_i\Big) - \big(w_{i,x} - \tilde\lambda_i w_i\big)v_{i,x}\theta_i' - v_i^2(\lambda_i^* - \theta_i)\theta_{i,x}\Big]\\
&\qquad + \sum_i \tilde r_{i,\sigma u}\tilde r_i\Big[\big(v_{i,x} - \tilde\lambda_i v_i\big)\theta_i' w_i - \big(w_{i,x} - \tilde\lambda_i w_i\big)v_i\theta_i'\Big]\\
&\qquad + \sum_{i\neq j}\tilde r_{i,\sigma u}\tilde r_j\Big[\big(v_{i,x} - \tilde\lambda_i v_i\big)v_j\theta_i' w_i/v_i - \big(w_{i,x} - \tilde\lambda_i w_i\big)v_j\theta_i'\Big]\\
&\qquad + \sum_i \tilde r_{i,\sigma\sigma}\Big[-\big(v_{i,x} - \tilde\lambda_i v_i\big)\theta_{i,x}\theta_i' w_i/v_i + \big(w_{i,x} - \tilde\lambda_i w_i\big)\theta_i'\theta_{i,x}\Big]\\
&\qquad + \sum_{i\neq j} v_i v_j\Bigg[\sum_k v_k\big(\tilde r_{i,uu}(\tilde r_j \otimes \tilde r_k) + \tilde r_{i,u}\tilde r_{j,u}\tilde r_k\big) + v_{j,x}\tilde r_{i,u}\tilde r_{j,v} + v_{i,x}\tilde r_{i,uv}\tilde r_j - \theta_{j,x}\tilde r_{i,u}\tilde r_{j,\sigma} - \theta_{i,x}\tilde r_{i,u\sigma}\tilde r_j\Bigg]\\
&\quad \doteq \sum_i a_i(t,x),
\end{aligned}\tag{A.10}$$

$$
\begin{aligned}
&\sum_i\Big(v_{i,t}+(\lambda_i v_i)_x-v_{i,xx}\Big)\Big[w_i\tilde r_{i,v}+\theta_i'(w_i/v_i)^2\tilde r_{i,\sigma}\Big]+\sum_i\Big(w_{i,t}+(\lambda_i w_i)_x-w_{i,xx}\Big)\Big[\tilde r_i-\theta_i'(w_i/v_i)\tilde r_{i,\sigma}\Big]\\
&\qquad-\sum_i\lambda_i^*\bigg\{\Big(v_{i,t}+\big(\tilde\lambda_i v_i\big)_x-v_{i,xx}\Big)\Big[\tilde r_i+v_i\tilde r_{i,v}+\theta_i'(w_i/v_i)\tilde r_{i,\sigma}\Big]-\Big(w_{i,t}+\big(\tilde\lambda_i w_i\big)_x-w_{i,xx}\Big)\Big[\theta_i'\tilde r_{i,\sigma}\Big]\bigg\}\\
&\quad=\sum_i\tilde r_{i,u}\tilde r_i\Big[(w_{i,x}-\tilde\lambda_i w_i)v_i-w_i\big(w_i-\lambda_i^* v_i\big)\Big]\\
&\qquad+\sum_{i\neq j}\tilde r_{i,u}\tilde r_j\Big[(w_{i,x}-\tilde\lambda_i w_i)v_j-w_i\big(w_j-\lambda_j^* v_j\big)+\big(w_i v_j\big)_x\Big]\\
&\qquad+\sum_i\tilde r_{i,v}\Big[(v_{i,x}-\lambda_i v_i)w_{i,x}+\big(w_{i,x}-\tilde\lambda_i w_i\big)v_{i,x}+\big(w_i v_i(\lambda_i^*-\theta_i)\big)_x\Big]\\
&\qquad+\sum_i\tilde r_{i,\sigma}\Big[(v_{i,x}-\tilde\lambda_i v_i)\big(\theta_i'(w_i/v_i)^2\big)_x-(w_{i,x}-\tilde\lambda_i w_i)\big(\theta_{i,x}+(\theta_i' w_i/v_i)_x\big)\Big]\\
&\qquad+\sum_i\tilde r_{i,vu}\tilde r_i\Big[\big(v_{i,x}-\tilde\lambda_i v_i\big)w_i v_i+w_i v_i^2(\lambda_i^*-\theta_i)\Big]\\
&\qquad+\sum_i\tilde r_{i,vv}\Big[\big(v_{i,x}-\tilde\lambda_i v_i\big)w_i v_{i,x}+w_i v_i v_{i,x}(\lambda_i^*-\theta_i)\Big]\\
&\qquad+\sum_{i\neq j}\tilde r_{i,vu}\tilde r_j\Big[\big(v_{i,x}-\tilde\lambda_i v_i\big)w_i v_j+w_i v_i v_j(\lambda_i^*-\theta_i)\Big]\\
&\qquad+\sum_i\tilde r_{i,v\sigma}\Big[(v_{i,x}-\tilde\lambda_i v_i)\big(-w_i\theta_{i,x}+\theta_i'(w_i/v_i)^2 v_{i,x}\big)-(w_{i,x}-\tilde\lambda_i w_i)\theta_i' v_{i,x}w_i/v_i-w_i v_i(\lambda_i^*-\theta_i)\theta_{i,x}\Big]\\
&\qquad+\sum_i\tilde r_{i,\sigma u}\tilde r_i\Big[\big(v_{i,x}-\tilde\lambda_i v_i\big)\theta_i' w_i^2/v_i-\big(w_{i,x}-\tilde\lambda_i w_i\big)\theta_i' w_i\Big]\\
&\qquad+\sum_{i\neq j}\tilde r_{i,\sigma u}\tilde r_j\Big[(v_{i,x}-\tilde\lambda_i v_i)v_j\theta_i'(w_i/v_i)^2-(w_{i,x}-\tilde\lambda_i w_i)v_j\theta_i' w_i/v_i\Big]\\
&\qquad+\sum_i\tilde r_{i,\sigma\sigma}\Big[-\big(v_{i,x}-\tilde\lambda_i v_i\big)\theta_i'(w_i/v_i)^2\theta_{i,x}+\big(w_{i,x}-\tilde\lambda_i w_i\big)\theta'\theta_{i,x}w_i/v_i\Big]\\
&\qquad+\sum_{i\neq j}w_i v_j\Big[\sum_k v_k\big(\tilde r_{i,uu}(\tilde r_j\otimes\tilde r_k)+\tilde r_{i,u}\tilde r_{j,u}\tilde r_k\big)+v_{j,x}\tilde r_{i,u}\tilde r_{j,v}+v_{i,x}\tilde r_{i,uv}\tilde r_j-\theta_{j,x}\tilde r_{i,u}\tilde r_{j,\sigma}-\theta_{i,x}\tilde r_{i,u\sigma}\tilde r_j\Big]\\
&\qquad+\sum_{i\neq j}(w_i-\lambda_i^* v_i)v_j\Big[\big(\tilde r_j\bullet A(u)\big)\tilde r_i-\big(\tilde r_i\bullet A(u)\big)\tilde r_j\Big]\\
&\qquad-\sum_i\lambda_i^* a_i(t,x)\\
&\quad\doteq\ \sum_i b_i(t,x)-\sum_i\lambda_i^* a_i(t,x).
\end{aligned}
\tag{A.11}
$$

Recalling the expression (5.10) for the differential $\partial\Lambda/\partial(v,w)$, we recognize that the equations (A.10)-(A.11) provide the explicit form of the system (6.5). The uniform invertibility of the differential of Λ implies the estimates

$$\phi_j,\ \psi_j=\mathcal{O}(1)\cdot\sum_i\big(|a_i|+|b_i|\big).$$

To prove Lemma 6.1, it thus suffices to show that all the terms in the summations defining a_i, b_i have the correct order of magnitude.

First of all, one checks that all those terms which involve a product of distinct components $i \neq j$ can be bounded as

$$\mathcal{O}(1) \cdot \sum_{j \neq k} \big(|v_j v_k| + |v_{j,x} v_k| + |v_j w_k| + |v_{j,x} w_k| + |v_j w_{k,x}| + |w_j w_k| \big). \tag{A.12}$$

In most cases, this estimate is straightforward. For the terms containing the factor $\theta_{i,x}$ or $\theta_{j,x}$ this is proved as follows. Recalling the bounds (4.24) we have, for example,

$$\theta_{j,x} \tilde r_{j,\sigma} = \mathcal{O}(1) \cdot v_j \, \theta_j' \frac{w_{j,x} v_j - w_j v_{j,x}}{v_j^2} = \mathcal{O}(1) \cdot \big(|w_{j,x}| + |v_{j,x}| \big) = \mathcal{O}(1) \cdot \delta_0^3 \,,$$

because of (5.24). Hence

$$v_i v_j \theta_{j,x} \tilde r_{i,u} \tilde r_{j,\sigma} = \mathcal{O}(1) \cdot \delta_0^3 |v_i v_j| \,.$$

Next, we look at each one of the remaining terms on the right hand side of (A.10) and (A.11) and show that its size can be bounded as claimed by Lemma 6.1. To appreciate the following computations, one should keep in mind that:

1. By (6.18) there holds

$$v_{i,x} - (\tilde\lambda_i - \lambda_i^*) v_i - w_i = \mathcal{O}(1) \cdot \delta_0 \sum_{j \neq i} \big(|v_j| + |w_j - v_{j,x}| \big) \,.$$

2. By (5.5) the cutoff functions satisfy $\theta_i' = \theta_i'' = 0$ whenever $|w_i/v_i| \geq 3\delta_1$.

3. By (4.24) we have $\tilde r_{i,\sigma}/v_i \,,\ \tilde r_{i,\sigma\sigma}/v_i \,,\ \tilde r_{i,\sigma u}/v_i = \mathcal{O}(1)$.

4. One can have $|w_i - \theta_i v_i| \neq 0$ only when $|w_i| > \delta_1 |v_i|$. In this case, (6.20) yields

$$v_i = \mathcal{O}(1) \cdot v_{i,x} + \mathcal{O}(1) \cdot \delta_0 \sum_{j \neq i} |v_j| \,.$$

What follows is a list of the various terms, first those appearing in a_i, then the ones in b_i.

Coefficients of $\tilde r_{i,u} \tilde r_i$:

$$v_i (v_{i,x} - \tilde\lambda_i v_i) - v_i (w_i - \lambda_i^* v_i) = v_i \big[v_{i,x} - (\tilde\lambda_i - \lambda_i^*) v_i - w_i \big],$$

$$v_i (w_{i,x} - \tilde\lambda_i w_i) - w_i (w_i - \lambda_i^* v_i) = \big[v_i w_{i,x} - v_{i,x} w_i \big] + w_i \big[v_{i,x} - (\tilde\lambda_i - \lambda_i^*) v_i - w_i \big].$$

Coefficients of $\tilde r_{i,v}$:

$$\begin{aligned} 2v_{i,x}(v_{i,x} - \tilde\lambda_i v_i) + \big(v_i^2 (\lambda_i^* - \theta_i) \big)_x &= 2v_{i,x} \big[v_{i,x} - (\tilde\lambda_i - \lambda_i^*) v_i - \theta_i v_i \big] + \theta_i' \big[v_{i,x} w_i - v_i w_{i,x} \big] \\ &= 2v_{i,x} \Big[v_{i,x} - (\tilde\lambda_i - \lambda_i^*) v_i - w_i \Big] + 2v_{i,x} \big[w_i - \theta_i v_i \big] + \theta_i' \big[v_{i,x} w_i - v_i w_{i,x} \big], \end{aligned}$$

$$
\begin{aligned}
&w_{i,x}(v_{i,x}-\tilde\lambda_i v_i)+v_{i,x}(w_{i,x}-\tilde\lambda_i w_i)+\big(w_iv_i(\lambda_i^*-\theta_i)\big)_x\\
&\qquad =2w_{i,x}\big[v_{i,x}-(\tilde\lambda_i-\lambda_i^*)v_i-w_i\big]+2w_{i,x}\big[w_i-\theta_iv_i\big]+\big(\lambda_i^*-\theta_i-\tilde\lambda_i+\theta_i'w_i/v_i\big)\big[v_{i,x}w_i-v_iw_{i,x}\big].
\end{aligned}
$$

Coefficients of $\tilde r_{i,\sigma}/v_i$:

$$
\begin{aligned}
v_i\big(v_{i,x}-\tilde\lambda_iv_i\big)\Big(-\theta_{i,x}+(\theta_i'w_i/v_i)_x\Big)-v_i\big(w_{i,x}-\tilde\lambda_iw_i\big)\theta_{i,x}' &= -\big(v_iw_{i,x}-w_iv_{i,x}\big)\theta_i''\big(w_i/v_i\big)_x\\
&= -\theta_i''\Big[v_i\,\big(w_i/v_i\big)_x\Big]^2,
\end{aligned}
$$

$$
v_i\big(v_{i,x}-\tilde\lambda_iv_i\big)\Big(\theta_i'(w_i/v_i)^2\Big)_x-v_i\big(w_{i,x}-\tilde\lambda_iw_i\big)\Big(\theta_{i,x}+(\theta_i'w_i/v_i)_x\Big)=-\Big(\theta_i''(w_i/v_i)+2\theta_i'\Big)\Big[v_i\,\big(w_i/v_i\big)_x\Big]^2.
$$

Coefficients of $\tilde r_{i,vu}\tilde r_i$:

$$
v_i^2\big(v_{i,x}-\tilde\lambda_iv_i\big)+v_i^3(\lambda_i^*-\theta_i)=v_i^2\Big[v_{i,x}-(\tilde\lambda_i-\lambda_i^*)v_i-w_i\Big]+v_i^2\big[w_i-\theta_iv_i\big],
$$

$$
v_iw_i\big(v_{i,x}-\tilde\lambda_iv_i\big)+v_i^2w_i(\lambda_i^*-\theta_i)=v_iw_i\Big[v_{i,x}-(\tilde\lambda_i-\lambda_i^*)v_i-w_i\Big]+v_iw_i\big[w_i-\theta_iv_i\big].
$$

Coefficients of $\tilde r_{i,vv}$:

$$
v_iv_{i,x}(v_{i,x}-\tilde\lambda_iv_i)+v_{i,x}v_i^2(\lambda_i^*-\theta_i)=v_iv_{i,x}\big[v_{i,x}-(\tilde\lambda_i-\lambda_i^*)v_i-w_i\big]+v_iv_{i,x}\big[w_i-\theta_iv_i\big],
$$

$$
w_iv_{i,x}\big(v_{i,x}-\tilde\lambda_iv_i\big)+v_{i,x}v_iw_i(\lambda_i^*-\theta_i)=w_iv_{i,x}\Big[v_{i,x}-\big(\tilde\lambda_i-\lambda_i^*\big)v_i-w_i\Big]+w_iv_{i,x}\big[w_i-\theta_iv_i\big].
$$

Coefficients of $\tilde r_{i,v\sigma}$:

$$
\begin{aligned}
&(v_{i,x}-\tilde\lambda_iv_i)\big(-v_i\theta_{i,x}+\theta_i'v_{i,x}w_i/v_i\big)-(w_{i,x}-\tilde\lambda_iw_i)\theta_i'v_{i,x}-v_i^2(\lambda_i^*-\theta_i)\theta_{i,x}\\
&\quad =v_{i,x}\theta_i'\big(v_{i,x}w_i-v_iw_{i,x}\big)/v_i-\theta_{i,x}v_i\Big(v_{i,x}-(\tilde\lambda_i-\lambda_i^*+\theta_i)v_i\Big)\\
&\quad =2\theta_i'\left(v_{i,x}\frac{w_i}{v_i}-w_{i,x}\right)\Big\{\big[v_{i,x}-(\tilde\lambda_i-\lambda_i^*)v_i-w_i\big]+\big[w_i-\theta_iv_i\big]\Big\}-\big(\tilde\lambda_i-\lambda_i^*+\theta_i\big)\theta_i'\big[w_{i,x}v_i-w_iv_{i,x}\big],
\end{aligned}
$$

$$
\begin{aligned}
\big(v_{i,x}-\tilde\lambda_iv_i\big)&\big(-w_i\theta_{i,x}+\theta_i'(w_i/v_i)^2v_{i,x}\big)-\big(w_{i,x}-\tilde\lambda_iw_i\big)\theta_i'v_{i,x}w_i/v_i-v_iw_i(\lambda_i^*-\theta_i)\theta_{i,x}\\
&=2\theta_i'\frac{w_i}{v_i}\left(v_{i,x}\frac{w_i}{v_i}-w_{i,x}\right)\Big\{\big[v_{i,x}-(\tilde\lambda_i-\lambda_i^*)v_i-w_i\big]+\big[w_i-\theta_iv_i\big]\Big\}\\
&\qquad-\big(\tilde\lambda_i-\lambda_i^*+\theta_i\big)\theta_i'\big[w_{i,x}v_i-w_iv_{i,x}\big]w_i/v_i\,.
\end{aligned}
$$

Coefficients of $\tilde r_{i,\sigma u}\tilde r_i/v_i$:

$$
(v_{i,x}-\tilde\lambda_iv_i)w_iv_i\theta_i'-(w_{i,x}-\tilde\lambda_iw_i)v_i^2\theta_i'=\theta_i'v_i\big[v_{i,x}w_i-v_iw_{i,x}\big],
$$

$$
(v_{i,x}-\tilde\lambda_iv_i)w_i^2\theta_i'-(w_{i,x}-\tilde\lambda_iw_i)w_iv_i\theta_i'=\theta_i'w_i\big[v_{i,x}w_i-v_iw_{i,x}\big].
$$

Coefficients of $\tilde r_{i,\sigma\sigma}/v_i$:

$$
-(v_{i,x}-\tilde\lambda_iv_i)w_i\theta_i'\theta_{i,x}+(w_{i,x}-\tilde\lambda_iw_i)v_i\theta_i'\theta_{i,x}=(\theta_i')^2\Big[v_i\,\big(w_i/v_i\big)_x\Big]^2,
$$

$$
-(v_{i,x}-\tilde\lambda_iv_i)w_i\big(w_i/v_i\big)\theta_i'\theta_{i,x}+\big(w_{i,x}-\tilde\lambda_iw_i\big)w_i\theta_i'\theta_{i,x}=-(\theta_i')^2\frac{w_i}{v_i}\Big[v_i\,\big(w_i/v_i\big)_x\Big]^2.
$$

This completes our analysis, showing that all terms in the summations that define a_i, b_i have the correct order of magnitude, as claimed by Lemma 6.1. □

Appendix B

We compute here the source terms $\hat{\phi}_i, \hat{\psi}_i$ in the equations (11.15) for the components of a first order perturbation, and prove Lemma 11.4. We recall that

$$z = \sum_i h_i \tilde{r}_i\big(u, v_i, \lambda_i^* - \theta(g_i/h_i)\big), \qquad \Upsilon = \sum_i \big(g_i - \lambda_i^* h_i\big)\tilde{r}_i\big(u, v_i, \lambda_i^* - \theta(g_i/h_i)\big), \tag{B.1}$$

$$\hat{\theta}_i \doteq \theta\left(\frac{g_i}{h_i}\right), \qquad \hat{r}_i \doteq \tilde{r}_i(u, v_i,\, \lambda_i^* - \hat{\theta}_i), \qquad \hat{\lambda}_i \doteq \big\langle \hat{r}_i,\, A(u)\hat{r}_i\big\rangle.$$

As in (A.4)–(A.11), the computations are lengthy but straightforward: one has to rewrite the evolution equations for z and Υ :

$$\begin{cases} z_t + \big(A(u)z\big)_x - z_{xx} = \big(u_x \bullet A(u)\big)z - \big(z \bullet A(u)\big)u_x\,, \\ \Upsilon_t + \big(A(u)\Upsilon\big)_x - \Upsilon_{xx} = \Big[\big(u_x \bullet A(u)\big)z - \big(z \bullet A(u)\big)u_x\Big]_x - A(u)\Big[\big(u_x \bullet A(u)\big)z - \big(z \bullet A(u)\big)u_x\Big] \\ \qquad\qquad + \big(u_x \bullet A(u)\big)\Upsilon - \big(u_t \bullet A(u)\big)z\,, \end{cases} \tag{B.2}$$

in terms of h_i, g_i. The fundamental relation (4.23) implies

$$A(u)\hat{r}_i = \hat{\lambda}_i \hat{r}_i + v_i\big(\hat{r}_{i,u}\hat{r}_i + (\hat{\lambda}_i - \lambda_i^* + \hat{\theta}_i)\hat{r}_{i,v}\big)\,. \tag{B.3}$$

Differentiating (B.1) w.r.t. x and using (B.3) we obtain

$$\begin{aligned} z_x - A(u)z &= \sum_i h_{i,x}\hat{r}_i + \sum_i h_i\Big[v_i \hat{r}_{i,u}\tilde{r}_i - A(u)\hat{r}_i\Big] + \sum_i h_i\Big[v_{i,x}\hat{r}_{i,v} - \big(\hat{\theta}_i'(h_i g_{i,x} - g_i h_{i,x})/h_i^2\big)\hat{r}_{i,\sigma}\Big] \\ &\quad + \sum_{i\neq j} h_i v_j \hat{r}_{i,u}\tilde{r}_j \\ &= \sum_i \big(h_{i,x} - \hat{\lambda}_i h_i\big)\hat{r}_i + \sum_i h_i v_i r_{i,u}\big(\tilde{r}_i - \hat{r}_i\big) + \sum_i h_i\Big[v_{i,x} - (\hat{\lambda}_i - \lambda_i^* + \hat{\theta}_i)v_i\Big]\hat{r}_{i,v} \\ &\quad - \sum_i \hat{\theta}_i'\left[\big(g_{i,x} - \hat{\lambda}_i g_i\big) - \frac{g_i}{h_i}\big(h_{i,x} - \hat{\lambda} h_i\big)\right]\hat{r}_{i,\sigma} + \sum_{i\neq j} h_i v_j \hat{r}_{i,u}\tilde{r}_j, \\ &= \sum_i \big(h_{i,x} - \hat{\lambda}_i h_i\big)\Big[\hat{r}_i + \hat{\theta}_i'(g_i/h_i)\hat{r}_{i,\sigma}\Big] - \sum_i \big(g_{i,x} - \hat{\lambda}_i g_i\big)\hat{\theta}_i'\hat{r}_{i,\sigma} \\ &\quad + \sum_i h_i v_i \tilde{r}_{i,u}\big(\tilde{r}_i - \hat{r}_i\big) + \sum_i h_i\Big[v_{i,x} - (\hat{\lambda}_i - \lambda_i^* + \hat{\theta}_i)v_i\Big]\hat{r}_{i,v} + \sum_{i\neq j} h_i v_j r_{i,u}\tilde{r}_j\,, \end{aligned} \tag{B.4}$$

$$
\begin{aligned}
\Upsilon_x - A(u)\Upsilon &= \sum_i \big(g_{i,x} - \lambda_i^* h_{i,x}\big)\hat r_i + \sum_i \big(g_i - \lambda_i^* h_i\big)\Big[v_i \hat r_{i,u}\tilde r_i - A(u)\hat r_i\Big] \\
&\quad + \sum_i \big(g_i - \lambda_i^* h_i\big)\Big[v_{i,x}\hat r_{i,v} - \big(\hat\theta_i'(h_i g_{i,x} - g_i h_{i,x})/h_i^2\big)\hat r_{i,\sigma}\Big] + \sum_{i\neq j}\big(g_i - \lambda_i^* h_i\big)v_j \hat r_{i,u}\tilde r_j \\
&= \sum_i \big(g_{i,x} - \hat\lambda_i g_i\big)\hat r_i - \sum_i \lambda_i^*\big(h_{i,x} - \hat\lambda_i h_i\big)\hat r_i + \sum_i \big(g_i - \lambda_i^* h_i\big)v_i \hat r_{i,u}(\tilde r_i - \hat r_i) \\
&\quad + \sum_i \big(g_i - \lambda_i^* h_i\big)\Big[v_{i,x} - (\hat\lambda_i - \lambda_i^* + \hat\theta_i)v_i\Big]\hat r_{i,v} \\
&\quad - \sum_i \hat\theta_i'\left(\frac{g_i}{h_i} - \lambda_i^*\right)\left[\big(g_{i,x} - \hat\lambda_i g_i\big) - \frac{g_i}{h_i}\big(h_{i,x} - \hat\lambda_i h_i\big)\right]\hat r_{i,\sigma} + \sum_{i\neq j}\big(g_i - \lambda_i^* h_i\big)v_j \hat r_{i,u}\tilde r_j \\
&= \sum_i \big(h_{i,x} - \hat\lambda_i h_i\big)\Big[\hat\theta_i'\big(g_i/h_i\big)^2 \hat r_{i,\sigma}\Big] + \sum_i \big(g_{i,x} - \hat\lambda_i g_i\big)\Big[\hat r_i - \hat\theta_i'(g_i/h_i)\hat r_{i,\sigma}\Big] \\
&\quad + \sum_i g_i\Big[v_{i,x} - (\hat\lambda - \lambda_i^* + \hat\theta_i)v_i\Big]\hat r_{i,v} + \sum_i g_i v_i \hat r_{i,u}\big(\tilde r_i - \hat r_i\big) + \sum_{i\neq j} g_i v_j \hat r_{i,u}\tilde r_j \\
&\quad - \sum_i \lambda_i^*\Big\{\big(h_{i,x} - \hat\lambda_i h_i\big)\Big[\hat r_i + \hat\theta_i'(g_i/h_i)\hat r_{i,\sigma}\Big] - \big(g_{i,x} - \hat\lambda_i g_i\big)\hat\theta_i'\hat r_{i,\sigma} \\
&\qquad + h_i\Big[v_{i,x} - (\hat\lambda_i - \lambda_i^* + \hat\theta_i)v_i\Big]\hat r_{i,v} + h_i v_i \hat r_{i,u}\big(\tilde r_i - \hat r_i\big) + \sum_{j\neq i} h_i v_j \hat r_{i,u}\tilde r_j\Big\}.
\end{aligned}
\tag{B.5}
$$

Next, differentiating (B.1) w.r.t. t we obtain

$$
\begin{aligned}
z_t &= \sum_i h_{i,t}\hat r_i + \sum_i h_i\big(\hat r_{i,u}u_t + v_{i,t}\hat r_{i,v} - \hat\theta_{i,t}\hat r_{i,\sigma}\big) \\
&= \sum_i h_{i,t}\hat r_i + \sum_i h_i\Big[v_{i,t}\hat r_{i,v} - \big(\hat\theta_i'(g_{i,t}h_i - g_i h_{i,t})/h_i^2\big)\hat r_{i,\sigma}\Big] + \sum_{i,j} h_i\big(w_j - \lambda_j^* v_j\big)\hat r_{i,u}\tilde r_j \\
&= \sum_i h_{i,t}\Big[\hat r_i + \hat\theta_i'(g_i/h_i)\hat r_{i,\sigma}\Big] - \sum_i \hat\theta_i' g_{i,t}\hat r_{i,\sigma} + \sum_i h_i v_{i,t}\hat r_{i,v} + \sum_{i,j} h_i\big(w_j - \lambda_j^* v_j\big)\hat r_{i,u}\tilde r_j\,,
\end{aligned}
\tag{B.6}
$$

$$
\begin{aligned}
\Upsilon_t &= \sum_i \big(g_{i,t} - \lambda_i^* h_{i,t}\big)\hat r_i + \sum_i \big(g_i - \lambda_i^* h_i\big)\Big(\hat r_{i,u}u_t + v_{i,t}\hat r_{i,v} - \hat\theta_{i,t}\hat r_{i,\sigma}\Big) \\
&= \sum_i \big(g_{i,t} - \lambda_i^* h_{i,t}\big)\hat r_i + \sum_i \big(g_i - \lambda_i^* h_i\big)\Big[v_{i,t}\hat r_{i,v} - \big(\hat\theta_i'(g_{i,t}h_i - g_i h_{i,t})/h_i^2\big)\hat r_{i,\sigma}\Big] \\
&\quad + \sum_{i,j}\big(g_i - \lambda_i^* h_i\big)\big(w_j - \lambda_j^* v_j\big)\hat r_{i,u}\tilde r_j \\
&= \sum_i h_{i,t}\Big[\hat\theta_i'(g_i/h_i)^2\hat r_{i,\sigma}\Big] + \sum_i g_{i,t}\Big[\hat r_i - \hat\theta_i'(g_i/h_i)\hat r_{i,\sigma}\Big] + \sum_i g_i v_{i,t}\hat r_{i,v} + \sum_{i,j} g_i(w_j - \lambda_j^* v_j)\hat r_{i,u}\tilde r_j \\
&\quad - \sum_i \lambda_i^*\Big\{h_{i,t}\big[\hat r_i + \hat\theta_i'(g_i/h_i)\hat r_{i,\sigma}\big] - \hat\theta_i' g_{i,t}\hat r_{i,\sigma} + h_i v_{i,t}\hat r_{i,v} + \sum_j h_i\big(w_j - \lambda_j^* v_j\big)\hat r_{i,u}\tilde r_j\Big\}.
\end{aligned}
\tag{B.7}
$$

Differentiating again $z_x - A(u)z$ and $\Upsilon_x - A(u)\Upsilon$ w.r.t. x, from (B.4) and (B.5) one finds

$$
\begin{aligned}
z_{xx} - \big(A(u)z\big)_x = {} & \sum_i \big(h_{i,xx} - (\hat\lambda_i h_i)_x\big)\Big[\hat r_i + \hat\theta_i'(g_i/h_i)\hat r_{i,\sigma}\Big] - \sum_i \big(g_{i,xx} - (\hat\lambda_i g_i)_x\big)\hat\theta_i'\hat r_{i,\sigma} \\
& + \sum_i \big(h_{i,x} - \hat\lambda_i h_i\big)\Big[\sum_j v_j \hat r_{i,u}\tilde r_j + v_{i,x}\hat r_{i,v} + \big(-\hat\theta_{i,x} + (\hat\theta_i' g_i/h_i)_x\big)\hat r_{i,\sigma} \\
& \qquad + \hat\theta_i' v_{i,x}(g_i/h_i)\hat r_{i,\sigma v} + \sum_j v_j \hat\theta_i'(g_i/h_i)\hat r_{i,u\sigma}\tilde r_j - \hat\theta_{i,x}\hat\theta_i'(g_i/h_i)\hat r_{i,\sigma\sigma}\Big] \\
& + \sum_i \big(g_{i,x} - \hat\lambda_i g_i\big)\Big[-\hat\theta_i''(g_i/h_i)_x \hat r_{i,\sigma} - \sum_j \hat\theta_i' v_j \hat r_{i,u\sigma}\tilde r_j - v_{i,x}\hat\theta_i'\hat r_{i,v\sigma} + \hat\theta_i'\hat\theta_{i,x}\hat r_{i,\sigma\sigma}\Big] \\
& + \sum_i \big(h_i v_i \hat r_{i,u}(\tilde r_i - \hat r_i)\big)_x + \sum_i \Big(h_i\big(v_{i,x} - (\hat\lambda_i - \lambda_i^* + \hat\theta_i)v_i\big)\Big)_x \hat r_{i,v} \\
& + \sum_i h_i\big(v_{i,x} - (\hat\lambda_i - \lambda_i^* + \hat\theta_i)v_i\big)\Big[\sum_j v_j \hat r_{i,vu}\tilde r_j + v_{i,x}\hat r_{i,vv} - \hat\theta_{i,x}\hat r_{i,v\sigma}\Big] \\
& + \sum_{i\neq j}\big(h_i v_j\big)_x \hat r_{i,u}\tilde r_j + \sum_{i\neq j} h_i v_j \Big[\sum_k v_k\big(\hat r_{i,u}\tilde r_{j,u}\tilde r_k + \tilde r_{i,uu}(\tilde r_j\otimes\tilde r_k)\big) \\
& \qquad + v_{j,x}\hat r_{i,u}\tilde r_{j,v} + v_{i,x}\hat r_{i,vu}\tilde r_j - \theta_{j,x}\tilde r_{i,u}\tilde r_{j,\sigma} - \hat\theta_{i,x}\hat r_{i,\sigma u}\tilde r_j\Big], \qquad (B.8)
\end{aligned}
$$

$$
\begin{aligned}
\Upsilon_{xx} - \big(A(u)\Upsilon\big)_x = {} & \sum_i \big(h_{i,xx} - (\hat\lambda_i h_i)_x\big)\Big[\hat\theta_i'(g_i/h_i)^2\hat r_{i,\sigma}\Big] + \sum_i \big(g_{i,xx} - (\hat\lambda_i g_i)_x\big)\Big[\hat r_i - \hat\theta_i'(g_i/h_i)\hat r_{i,\sigma}\Big] \\
& + \sum_i \big(h_{i,x} - \hat\lambda_i h_i\big)\Big[\hat\theta_i'(w_i/v_i)^2 v_{i,x}\hat r_{i,v\sigma} + \big(\hat\theta_i'(g_i/h_i)^2\big)_x \hat r_{i,\sigma} \\
& \qquad + \sum_j v_j\hat\theta_i'\big(g_i/h_i\big)^2\hat r_{i,\sigma u}\hat r_j - \hat\theta_i'(g_i/h_i)^2\hat\theta_{i,x}\hat r_{i,\sigma\sigma}\Big] \\
& + \sum_i \big(g_{i,x} - \tilde\lambda_i g_i\big)\Big[\sum_j v_j\hat r_{i,u}\tilde r_j + v_{i,x}\hat r_{i,v} - \big(\hat\theta_{i,x} + (\hat\theta_i' g_i/h_i)_x\big)\hat r_{i,\sigma} \\
& \qquad - \sum_j v_j\hat\theta_i'(g_i/h_i)\hat r_{i,\sigma u}\tilde r_j - v_{i,x}\hat\theta_i'(g_i/h_i)\hat r_{i,v\sigma} + \hat\theta_{i,x}\hat\theta_i'(g_i/h_i)\hat r_{i,\sigma\sigma}\Big] \\
& + \sum_i \Big[g_i\big(v_{i,x} - (\hat\lambda_i - \lambda_i^* + \hat\theta_i)v_i\big)\Big]_x \hat r_{i,v} + \sum_i \Big(g_i v_i \hat r_{i,u}(\tilde r_i - \hat r_i)\Big)_x \\
& + \sum_i g_i\big(v_{i,x} - (\hat\lambda_i - \lambda_i^* + \hat\theta_i)v_i\big)\Big[\sum_j v_j\hat r_{i,v}\tilde r_j + v_{i,x}\hat r_{i,vv} - \hat\theta_{i,x}\hat r_{i,v\sigma}\Big] + \sum_{i\neq j}(g_i v_j)_x \hat r_{i,u}\tilde r_j \\
& + \sum_{i\neq j} g_i v_j\Big[\sum_k v_k\big(\hat r_{i,u}\tilde r_{j,u}\tilde r_k + \hat r_{i,uu}(\tilde r_j\otimes\tilde r_k)\big) + v_{j,x}\hat r_{i,u}\tilde r_{j,v} + v_{i,x}\hat r_{i,vu}\tilde r_j - \theta_{j,x}\hat r_{i,u}\tilde r_{j,\sigma} - \hat\theta_{i,x}\hat r_{i,\sigma u}\tilde r_j\Big] \\
& - \sum_i \lambda_i^*\Big\{\big(h_{i,x} - \hat\lambda_i h_i\big)\Big[\hat r_i + \hat\theta_i'(g_i/h_i)\hat r_{i,\sigma}\Big] - \big(g_{i,x} - \hat\lambda_i g_i\big)\hat\theta_i'\hat r_{i,\sigma} \\
& \qquad + h_i\big(v_{i,x} - (\hat\lambda_i - \lambda_i^* + \hat\theta_i)v_i\big)\hat r_{i,v} + h_i v_i\hat r_{i,u}\big(\tilde r_i - \hat r_i\big) + \sum_{j\neq i} v_i v_j \hat r_{i,u}\tilde r_j\Big\}_x . \qquad (B.9)
\end{aligned}
$$

Substituting the expressions (B.6)–(B.9) inside (B.2) we obtain an implicit system of $2n$ scalar equations governing the evolution of the components h_i, g_i:

$$
\begin{aligned}
&\sum_i \Big(h_{i,t} + (\hat\lambda_i h_i)_x - h_{i,xx}\Big)\Big[\hat r_i + \hat\theta_i'(g_i/h_i)\hat r_{i,\sigma}\Big] + \sum_i \Big(g_{i,t} + \big(\hat\lambda_i g_i\big)_x - g_{i,xx}\Big)\Big[-\hat\theta_i'\hat r_{i,\sigma}\Big] \\
&= \sum_i \hat r_{i,u}\tilde r_i \Big[v_i(h_{i,x} - \hat\lambda_i h_i) - h_i(w_i - \lambda_i^*)\Big] \\
&\quad + \sum_{i\neq j} \hat r_{i,u}\tilde r_j \Big[(h_{i,x} - \hat\lambda_i h_i)v_j + \big(h_i v_j\big)_x - h_i(w_j - \lambda_j^* v_j)\Big] \\
&\quad + \sum_i \hat r_{i,v}\Big[(h_{i,x} - \hat\lambda_i h_i)v_{i,x} + \Big(h_i\big(v_{i,x} - (\hat\lambda_i - \lambda_i^* + \hat\theta_i)v_i\big)\Big)_x - h_i v_{i,t}\Big] \\
&\quad + \sum_i \hat r_{i,\sigma}\Big[(h_{i,x} - \hat\lambda_i h_i)\big(-\hat\theta_{i,x} + (\hat\theta_i' g_i/h_i)_x\big) - (g_{i,x} - \hat\lambda_i g_i)\hat\theta_{i,x}'\Big] \\
&\quad + \sum_i \hat r_{i,vu}\tilde r_i\Big[h_i v_i\big(v_{i,x} - (\hat\lambda_i - \lambda_i^* + \hat\theta_i)v_i\big)\Big] \\
&\quad + \sum_{i\neq j} \hat r_{i,vu}\tilde r_j\Big[h_i v_j\big(v_{i,x} - (\hat\lambda_i - \lambda_i^* + \hat\theta_i)v_i\big)\Big] \\
&\quad + \sum_i \hat r_{i,vv}\Big[h_i v_{i,x}\big(v_{i,x} - (\hat\lambda_i - \lambda_i^* + \hat\theta_i)v_i\big)\Big] \\
&\quad + \sum_i \hat r_{i,v\sigma}\Big[\big(h_{i,x} - \hat\lambda_i h_i\big)\hat\theta_i' v_{i,x} g_i/h_i - \big(g_{i,x} - \hat\lambda_i g_i\big)v_{i,x}\hat\theta_i' - h_i\big(v_{i,x} - (\hat\lambda_i - \lambda_i^* + \hat\theta_i)v_i\big)\hat\theta_{i,x}\Big] \\
&\quad + \sum_i \hat r_{i,\sigma u}\tilde r_i\Big[\big(h_{i,x} - \hat\lambda_i h_i\big)v_i\hat\theta_i' g_i/h_i - \big(g_{i,x} - \hat\lambda_i g_i\big)v_i\hat\theta_i'\Big] \\
&\quad + \sum_{i\neq j} \hat r_{i,\sigma u}\tilde r_j\Big[\big(h_{i,x} - \hat\lambda_i h_i\big)v_j\hat\theta_i' g_i/h_i - \big(g_{i,x} - \hat\lambda_i g_i\big)v_j\hat\theta_i'\Big] \\
&\quad + \sum_i \hat r_{i,\sigma\sigma}\Big[-\big(h_{i,x} - \hat\lambda_i h_i\big)\hat\theta_i'\hat\theta_{i,x} g_i/h_i + \big(g_{i,x} - \hat\lambda_i g_i\big)\hat\theta_i'\hat\theta_{i,x}\Big] + \sum_i \Big(h_i v_i \hat r_{i,u}(\tilde r_i - \hat r_i)\Big)_x \\
&\quad + \sum_{i\neq j} h_i v_j\Big[\sum_k v_k\big(\hat r_{i,u}\tilde r_{j,u}\tilde r_k + \hat r_{i,uu}(\tilde r_j \otimes \tilde r_k)\big) + v_{j,x}\hat r_{i,u}\tilde r_{j,v} + v_{i,x}\hat r_{vu}\tilde r_j - \theta_{j,x}\hat r_{i,u}\tilde r_{j,\sigma} - \hat\theta_{i,x}\hat r_{\sigma u}\tilde r_j\Big] \\
&\quad + \sum_{i,j} h_i v_j\Big[\big(\tilde r_j \bullet A(u)\big)\hat r_i - \big(\hat r_i \bullet A(u)\big)\tilde r_j\Big] \\
&\doteq \sum_i \hat a_i(t,x),
\end{aligned}
\tag{B.10}
$$

$$
\begin{aligned}
& \sum_i \Big(h_{i,t} + \big(\hat\lambda_i h_i\big)_x - h_{i,xx}\Big)\Big[\hat\theta_i'(g_i/h_i)^2 \hat r_{i,\sigma}\Big] + \sum_i \Big(g_{i,t} + \big(\hat\lambda_i g_i\big)_x - g_{i,xx}\Big)\Big[\hat r_i - \hat\theta_i'(g_i/h_i)\hat r_{i,\sigma}\Big] \\
& - \sum_i \lambda_i^* \Big\{ \Big(h_{i,t} + (\hat\lambda_i h_i)_x - h_{i,xx}\Big)\Big[\hat r_i + \hat\theta_i'(g_i/h_i)\hat r_{i,\sigma}\Big] + \sum_i \Big(g_{i,t} + (\hat\lambda_i g_i)_x - g_{i,xx}\Big)\Big[-\hat\theta_i' \hat r_{i,\sigma}\Big]\Big\} \\
= & \sum_i \hat r_{i,u}\tilde r_i \Big[\big(g_{i,x} - \hat\lambda_i g_i\big) v_i - g_i\big(w_i - \lambda_i^* v_i\big)\Big] \\
& + \sum_{i\neq j} \hat r_{i,u}\tilde r_j \Big[(g_{i,x} - \hat\lambda_i g_i) v_j - g_i(w_j - \lambda_j^* v_j) + (g_i v_j)_x\Big] \\
& + \sum_i \hat r_{i,v}\Big[\Big(g_i\big(v_{i,x} - (\hat\lambda_i - \lambda_i^* + \hat\theta_i)v_i\big)\Big)_x + \big(g_{i,x} - \hat\lambda_i g_i\big) v_{i,x} - g_i v_{i,t}\Big] \\
& + \sum_i \hat r_{i,\sigma}\Big[\big(h_{i,x} - \hat\lambda_i h_i\big)\big(\hat\theta_i'(g_i/h_i)^2\big)_x - \big(g_{i,x} - \hat\lambda_i g_i\big)\big(\hat\theta_{i,x} + (\hat\theta_i' g_i/h_i)_x\big)\Big] \\
& + \sum_i \hat r_{i,vu}\tilde r_i \Big[\big(v_{i,x} - (\hat\lambda_i - \lambda_i^* + \hat\theta_i)v_i\big) g_i v_i\Big] \\
& + \sum_i \hat r_{i,vv}\Big[\big(v_{i,x} - (\hat\lambda_i - \lambda_i^* + \hat\theta_i)v_i\big) g_i v_{i,x}\Big] \\
& + \sum_{i\neq j} \hat r_{i,vu}\tilde r_j \Big[\big(v_{i,x} - (\tilde\lambda_i - \lambda_i^* + \hat\theta_i)v_i\big) g_i v_j\Big] \\
& + \sum_i \hat r_{i,v\sigma}\Big[\big(h_{i,x} - \hat\lambda_i h_i\big)\hat\theta_i'(g_i/h_i)^2 v_{i,x} - \big(g_{i,x} - \hat\lambda_i g_i\big)\hat\theta_i' v_{i,x} g_i/h_i - g_i\big(v_{i,x} - (\hat\lambda_i - \lambda_i^* + \hat\theta_i)v_i\big)\hat\theta_{i,x}\Big] \\
& + \sum_i \hat r_{i,\sigma u}\tilde r_i \Big[\big(h_{i,x} - \hat\lambda_i h_i\big)\hat\theta_i'(g_i/h_i)^2 v_i - \big(g_{i,x} - \hat\lambda_i g_i\big)\hat\theta_i' v_i g_i/h_i\Big] \\
& + \sum_{i\neq j} \hat r_{i,\sigma u}\tilde r_j \Big[\big(h_{i,x} - \hat\lambda_i h_i\big) v_j \hat\theta_i'(g_i/h_i)^2 - \big(g_{i,x} - \hat\lambda_i g_i\big) v_j \hat\theta_i' g_i/h_i\Big] \\
& + \sum_i \hat r_{i,\sigma\sigma}\Big[-\big(h_{i,x} - \hat\lambda_i h_i\big)\hat\theta_i'(g_i/h_i)^2\hat\theta_{i,x} + \big(g_{i,x} - \hat\lambda_i g_i\big)\hat\theta'\hat\theta_{i,x} g_i/h_i\Big] + \sum_i \Big(g_i v_i \hat r_{i,u}(\tilde r_i - \hat r_i)\Big)_x \\
& + \sum_{i\neq j} g_i v_j \Big[\sum_k v_k\big(\hat r_{i,u}\tilde r_{j,u}\tilde r_k + \hat r_{i,uu}(\tilde r_j \otimes \tilde r_k)\big) + v_{j,x}\hat r_{i,u}\tilde r_{j,v} + v_{i,x}\hat r_{i,vu}\tilde r_j - \theta_{j,x}\hat r_{i,u}\tilde r_{j,\sigma} - \hat\theta_{i,x}\hat r_{i,\sigma u}\tilde r_j\Big] \\
& + \sum_{i,j}\big(w_i h_j - v_i g_j\big)\big(\tilde r_i \bullet A(u)\big)\hat r_j + \sum_{i,j}\Big[v_i h_j\Big(\big(\tilde r_i \bullet A(u)\big)\hat r_j - \big(\hat r_j \bullet A(u)\big)\tilde r_i\Big)\Big]_x \\
& + \sum_{i\neq j}(\lambda_j^* - \lambda_i^*) v_i h_j \,\big(\tilde r_i \bullet A(u)\big)\hat r_j + \sum_{i,j} v_i h_j A(u)\Big[\big(\tilde r_i \bullet A(u)\big)\hat r_j - \big(\hat r_j \bullet A(u)\big)\tilde r_i\Big] \\
& - \sum_i \lambda_i^* \hat a_i(t,x) \\
\doteq & \sum_i \hat b_i(t,x) - \sum_i \lambda_i^* \hat a_i(t,x).
\end{aligned}
\tag{B.11}
$$

Recalling the expression (11.11) for the differential $\partial\widehat{\Lambda}/\partial(h,g)$, we can write (B.10)-(B.11) in the more compact form

$$\frac{\partial\widehat{\Lambda}}{\partial(h,g)}\cdot\begin{pmatrix}[h_{i,t}+(\hat\lambda_i h_i)_x-h_{i,xx}]\\ [g_{i,t}+\big(\hat\lambda_i g_i\big)_x-g_{i,xx}]\end{pmatrix}=\sum_i\begin{pmatrix}\hat a_i\\ \hat b_i-\lambda_i^*\hat a_i\end{pmatrix}.$$

By the uniform invertibility of the differential of $\widehat{\Lambda}$, to prove the estimates stated in Lemma 11.4, it suffices to show that, for every $i=1,\ldots,n$, the four quantities

$$\hat a_i\,,\qquad \hat b_i\,,\qquad \big((\tilde\lambda_i-\hat\lambda_i)h_i\big)_x\,,\qquad \big((\tilde\lambda_i-\hat\lambda_i)g_i\big)_x\,,$$

can all be bounded according to the right hand side of (11.16).

We start by looking at all the terms in the expressions (B.10)-(B.11) for $\hat a_i$ and $\hat b_i$. First of all, one checks that all those terms which involve a product of distinct components $i\neq j$ can be bounded as

$$\mathcal{O}(1)\cdot\sum_{j\neq k}\Big(|h_jh_k|+|h_jv_k|+|h_{j,x}v_k|+|h_jv_{k,x}|+|h_jw_k|+|g_jv_k|+|g_{x,j}v_k|+|g_jv_{k,x}|+|g_jw_k|\Big).\quad (B.12)$$

For convenience, quantities whose size is bounded as in (B.12) will be called “transversal terms”. More generally, quantities whose size is bounded according to the right hand side of (11.16) will be called “admissible terms”. We denote by $\mathcal{A}$ the family of all admissible terms. We now exhibit various additional terms which are admissible.

1. By (6.18) it follows

$$\begin{aligned}&\big(|h_i|+|g_i|+|h_{i,x}|+|g_{i,x}|\big)\Big|v_{i,x}-(\tilde\lambda_i-\lambda_i^*)v_i-w_i\Big|\\ &\qquad=\mathcal{O}(1)\cdot\delta_0\sum_{j\neq i}\big(|h_iv_j|+|g_iv_j|+|h_{i,x}v_j|+|g_{i,x}v_j|\big)\ \in\ \mathcal{A}\,.\end{aligned}\qquad (B.13)$$

2. Two other other admissible terms are

$$\begin{aligned}h_i[w_{i,x}v_i-w_iv_{i,x}]&=[h_iw_{i,x}-w_ih_{i,x}]v_i+w_i[h_{i,x}v_i-h_iv_{i,x}]\in\mathcal{A}\,,\\ g_i[w_{i,x}v_i-w_iv_{i,x}]&=[g_iw_{i,x}-w_ig_{i,x}]v_i+w_i[g_{i,x}v_i-g_iv_{i,x}]\in\mathcal{A}\,.\end{aligned}\qquad (B.14)$$

3. We now consider terms that involve the difference between the speeds: $\hat\theta_i-\theta_i$. We claim that the following four quantities are admissible:

$$h_iv_i(\hat\theta_i-\theta_i),\quad g_iv_i(\hat\theta_i-\theta_i),\quad h_{i,x}v_i(\hat\theta_i-\theta_i),\quad g_{i,x}v_i(\hat\theta_i-\theta_i)\in\mathcal{A}\,.\qquad (B.15)$$

Indeed, from the definitions and the bounds (4.24) it follows

$$|\hat\lambda_i-\tilde\lambda_i|=\mathcal{O}(1)\cdot|\hat r_i-\tilde r_i|=\mathcal{O}(1)\cdot v_i\,|\hat\theta_i-\theta_i|=\mathcal{O}(1)\cdot\delta_0\,|\hat\theta_i-\theta_i|.\qquad (B.16)$$

Since $|\theta'|\leq 1$, one has

$$|\hat\theta_i-\theta_i|\leq\big|(g_i/h_i)-(w_i/v_i)\big|.$$

Using (6.18) and (11.12) we now obtain

$$\begin{aligned}
|h_i v_i|\big|\hat\theta_i-\theta_i\big| &\le |g_i v_i - w_i h_i| \\
&= \Bigg|\Big(h_{i,x}+(\hat\lambda_i-\lambda_i^*)h_i+\mathcal{O}(1)\cdot\delta_0\sum_{j\neq i}\big(|h_j|+|v_j|\big)\Big)v_i \\
&\qquad -\Big(v_{i,x}+(\tilde\lambda_i-\lambda_i^*)v_i+\mathcal{O}(1)\cdot\delta_0\sum_{j\neq i}|v_j|\Big)h_i\Bigg| \\
&= \Bigg|(h_{i,x}v_i-v_{i,x}h_i)+(\hat\lambda_i-\tilde\lambda_i)v_ih_i+\mathcal{O}(1)\cdot\delta_0\sum_{j\neq i}\big(|v_jv_i|+|h_jv_i|\big)\Bigg| \\
&\le \big|h_{i,x}v_i-v_{i,x}h_i\big|+\mathcal{O}(1)\cdot\delta_0\big|\hat\theta_i-\theta_i\big||h_iv_i|+\mathcal{O}(1)\cdot\delta_0\sum_{j\neq k}\big(|v_jv_k|+|h_jv_k|\big)\,.
\end{aligned}$$

Hence

$$|g_iv_i-w_ih_i|\le 2\big|h_{i,x}v_i-v_{i,x}h_i\big|+\mathcal{O}(1)\cdot\delta_0\sum_{j\neq k}\big(|v_jv_k|+|h_jv_k|\big)\in\mathcal{A}\,, \tag{B.17}$$

showing that the quantity $h_iv_i(\hat\theta_i-\theta_i)$ is admissible.

Observing that $\hat\theta_i-\theta_i\neq 0$ only if either $|g_i/h_i|\le 6\delta_1$ and $|w_i/v_i|\le 3\delta_1$, or else $|g_i/h_i|\ge 6\delta_1$ and $|w_i/v_i|\le 3\delta_1$, we can write

$$\begin{aligned}
\Big|g_iv_i(\hat\theta_i-\theta_i)\Big| &\le |g_i/h_i|\,\big|h_iv_i(\hat\theta_i-\theta_i)\big|\cdot\chi_{\big\{|g_i/h_i|\le 6\delta_1\big\}}+2\delta_1|g_iv_i|\cdot\chi_{\big\{|g_i/h_i|\ge 6\delta_1,\ |w_i/v_i|\le 3\delta_1\big\}} \\
&\le 6\delta_1|g_i/h_i|\,\big|h_iv_i(\hat\theta_i-\theta_i)\big|+4\delta_1\big|g_iv_i-w_ih_i\big|.
\end{aligned}$$

Hence $g_iv_i(\hat\theta_i-\theta_i)\in\mathcal{A}$. In turn, using (11.12) we obtain

$$h_{i,x}v_i(\hat\theta_i-\theta_i)=g_iv_i\big(\hat\theta_i-\theta_i\big)+(\hat\lambda_i-\lambda_i^*)h_iv_i(\hat\theta_i-\theta_i)+\mathcal{O}(1)\cdot\delta_0\sum_{j\neq i}\big(|v_iv_j|+|v_ih_j|\big),$$

showing that the term $h_{i,x}v_i(\hat\theta_i-\theta_i)$ is also admissible. Finally, using (6.18) one can write

$$\begin{aligned}
g_{i,x}v_i(\hat\theta_i-\theta_i) &= (\hat\theta_i-\theta_i)\big[g_{i,x}v_i-v_{i,x}g_i\big]+g_iv_{i,x}(\hat\theta_i-\theta_i) \\
&= (\hat\theta_i-\theta_i)\big[g_{i,x}v_i-v_{i,x}g_i\big]+g_iw_i(\hat\theta_i-\theta_i) \\
&\qquad +(\tilde\lambda_i-\lambda_i^*)g_iv_i(\hat\theta_i-\theta_i)+\mathcal{O}(1)\cdot\delta_0\sum_{j\neq i}|v_iv_j|\,.
\end{aligned}$$

To estimate the term $g_iw_i(\hat\theta_i-\theta_i)$, we observe that $\hat\theta_i-\theta_i=0$ if $|w_i/v_i|$ and $|g_i/h_i|$ are both $\ge 3\delta_1$. Hence, using again (6.18), we can write

$$\begin{aligned}
\big|g_iw_i\big(\hat\theta_i-\theta_i\big)\big| &= 3\delta_1\big|g_iv_i\big(\hat\theta_i-\theta_i\big)\big|\cdot\chi_{\big\{|w_i/v_i|<3\delta_1\big\}}+3\delta_1\big|h_iw_i(\hat\theta_i-\theta_i)\big|\cdot\chi_{\big\{|g_i/h_i|<3\delta_1\big\}} \\
&= 3\delta_1|g_iv_i|\big|\hat\theta_i-\theta_i\big|+\Big|h_i\big(v_{i,x}-(\tilde\lambda_i-\lambda_i^*)v_i\big)\Big|\big|\hat\theta_i-\theta_i\big|+\mathcal{O}(1)\cdot\sum_{j\neq i}|h_iv_j|\,.
\end{aligned}$$

By the previous estimates, this shows that $g_{i,x}v_i(\hat\theta_i-\theta_i)\in\mathcal{A}$, completing the proof of (B.15). By (B.16), the following terms are also admissible:

$$h_i(\tilde\lambda_i-\hat\lambda_i),\quad g_i(\tilde\lambda_i-\hat\lambda_i),\quad h_{i,x}(\tilde\lambda_i-\hat\lambda_i),\quad g_{i,x}(\tilde\lambda_i-\hat\lambda_i)\in\mathcal{A}. \tag{B.18}$$

4. Next, we claim that

$$h_i(\tilde r_i-\hat r_i)_x\,,\quad g_i(\tilde r_i-\hat r_i)_x\,,\quad h_i(\tilde\lambda_i-\hat\lambda_i)_x\,,\quad g_i(\tilde\lambda_i-\hat\lambda_i)_x\in\mathcal{A}\,. \tag{B.19}$$

Indeed, one can write

$$\begin{aligned}h_i(\tilde r_i-\hat r_i)_x=h_iv_i(\hat\theta_i-\theta_i)&\left\{\sum_j v_j\frac{(\tilde r_{i,u}-\hat r_{i,u})\tilde r_j}{v_i(\hat\theta_i-\theta_i)}+v_{i,x}\frac{\tilde r_{i,v}-\hat r_{i,v}}{v_i(\hat\theta_i-\theta_i)}\right\}\\&+\hat\theta_i'(g_i/h_i)\big[v_{i,x}h_i-h_{i,x}v_i\big](\hat r_{i,\sigma}/v_i)+\hat\theta_i'\big[v_ig_{i,x}-g_iv_{i,x}\big](\hat r_{i,\sigma}/v_i)\\&+(w_i/v_i)\theta_i'\big[v_{i,x}h_i-v_ih_{i,x}\big](\tilde r_{i,\sigma}/v_i)+\theta_i'\big[h_{i,x}w_i-h_iw_{i,x}\big](\tilde r_{i,\sigma}/v_i),\end{aligned}$$

$$\begin{aligned}g_i(\tilde r_i-\hat r_i)_x=g_iv_i(\hat\theta_i-\theta_i)&\left\{\sum_j v_j\frac{(\tilde r_{i,u}-\hat r_{i,u})\tilde r_j}{v_i(\hat\theta_i-\theta_i)}+v_{i,x}\frac{\tilde r_{i,v}-\hat r_{i,v}}{v_i(\hat\theta_i-\theta_i)}\right\}\\&+\hat\theta_i'(g_i/h_i)^2\big[v_{i,x}h_i-h_{i,x}v_i\big](\hat r_{i,\sigma}/v_i)+\hat\theta_i'(g_i/h_i)\big[v_ig_{i,x}-g_iv_{i,x}\big](\hat r_{i,\sigma}/v_i)\\&+(w_i/v_i)\theta_i'\big[v_{i,x}g_i-v_ig_{i,x}\big](\tilde r_{i,\sigma}/v_i)+\theta_i'\big[g_{i,x}w_i-g_iw_{i,x}\big](\tilde r_{i,\sigma}/v_i).\end{aligned}$$

By (4.24), the above expressions within braces are uniformly bounded. Hence the first two quantities in (B.19) are admissible. To prove the admissibility of the last two terms it suffices to repeat the above computation, with $\tilde r_i$ and $\hat r_i$ replaced by by $\tilde\lambda_i$ and $\hat\lambda_i$.

In a similar way as in Appendix A, we are now ready to check one by one all the (non-tranversal) terms in the expressions of $\hat a_i,\hat b_i$ in (B.10)-(B.11), showing that all of them are admissible.

Coefficients of $\hat r_{i,u}\tilde r_i$:

$$\begin{aligned}v_i(h_{i,x}-&\hat\lambda_ih_i)-h_i(w_i-\lambda_i^*v_i)\\&=\big[v_ih_{i,x}-h_iv_{i,x}\big]+\big[v_ih_i(\tilde\lambda_i-\hat\lambda_i)\big]+\Big[h_i\big(v_{i,x}-(\tilde\lambda_i-\lambda_i^*)v_i-w_i\big)\Big],\end{aligned}$$

$$\begin{aligned}v_i(g_{i,x}-&\hat\lambda_ig_i)-g_i(w_i-\lambda_i^*v_i)\\&=\big[g_{i,x}v_i-v_{i,x}g_i\big]+\big[v_ig_i(\tilde\lambda_i-\hat\lambda_i)\big]+\Big[g_i\big(v_{i,x}-(\tilde\lambda_i-\lambda_i^*)v_i-w_i\big)\Big].\end{aligned}$$

Coefficients of $\hat r_{i,v}$:

$$\begin{aligned}v_{i,x}(h_{i,x}-\hat\lambda_ih_i)+&\Big(h_i\big(v_{i,x}-(\hat\lambda_i-\lambda_i^*+\hat\theta_i)v_i\big)\Big)_x-h_iv_{i,t}\\&=2\Big[h_{i,x}\big(v_{i,x}-(\tilde\lambda_i-\lambda_i^*)v_i-w_i\big)\Big]+2\big[h_{i,x}(w_i-\theta_iv_i)\big]\\&\quad+\big[h_iv_{i,x}-h_{i,x}v_i\big]\big(\lambda_i^*-\hat\theta_i-\hat\lambda_i+\hat\theta_i'g_i/h_i\big)+\hat\theta_i'\big[v_ig_{i,x}-v_{i,x}g_i\big]\\&\quad+2\big[h_{i,x}v_i(\hat\lambda_i+\hat\theta_i-\tilde\lambda_i-\theta_i)\big]+\Big[h_i\big((\tilde\lambda_i-\hat\lambda_i)v_i\big)_x\Big]-h_i\phi_i\,,\end{aligned}$$

$$
\begin{aligned}
\Big(g_i\big(v_{i,x} - (\hat\lambda_i - \lambda_i^* + \hat\theta_i)v_i\big)\Big)_x &+ v_{i,x}(g_{i,x} - \hat\lambda_i g_i) - g_i v_{i,t} \\
&= 2\Big[g_{i,x}\big(v_{i,x} - \big(\tilde\lambda_i - \lambda_i^*\big)v_i - w_i\big)\Big] + 2\Big[g_{i,x}\big(w_i - \theta_i v_i\big)\Big] \\
&\quad + \big(\lambda_i^* - \hat\theta_i - \hat\lambda_i + \hat\theta_i' g_i/h_i\big)\left[v_{i,x}g_i - v_i g_{i,x}\right] + \hat\theta_i'(g_i/h_i)^2\left[v_i h_{i,x} - v_{i,x}h_i\right] \\
&\quad + 2\big[g_{i,x}v_i(\tilde\lambda_i + \theta_i - \hat\lambda_i - \hat\theta_i)\big] + \Big[g_i\big((\tilde\lambda_i - \hat\lambda_i)v_i\big)_x\Big] - g_i\phi_i\,.
\end{aligned}
$$

Coefficients of $\hat r_{i,\sigma}/v_i$:

$$
v_i(h_{i,x} - \hat\lambda_i h_i)\big(-\hat\theta_{i,x} + (\hat\theta_i' g_i/h_i)_x\big) - v_i(g_{i,x} - \hat\lambda_i g_i)\hat\theta_{i,x}' = -v_i\left[\hat\theta_i'' h_i\left(\frac{g_i}{h_i}\right)^2_x\right],
$$

$$
v_i\big(h_{i,x} - \hat\lambda_i h_i\big)\left(\hat\theta_i'\big(g_i/h_i\big)^2\right)_x + v_i\big(g_{i,x} - \hat\lambda_i g_i\big)\left(\hat\theta_{i,x}' - \big(\hat\theta_i' g_i/h_i\big)_x\right) = -v_i\left[\left(\hat\theta_i''\frac{g_i}{h_i} + 2\hat\theta_i'\right)h_i\left(\frac{g_i}{h_i}\right)^2_x\right].
$$

Coefficients of $\hat r_{i,vu}\tilde r_i$:

$$
\begin{aligned}
v_i h_i\big(v_{i,x} - (\hat\lambda_i - \lambda_i^* + \hat\theta_i)v_i\big) &= \Big[v_i h_i\big(v_{i,x} - (\tilde\lambda_i - \lambda_i^*)v_i - w_i\big)\Big] + \Big[v_i h_i\big(w_i - \theta_i v_i\big)\Big] \\
&\quad + \Big[v_i^2 h_i(\tilde\lambda_i + \theta_i - \hat\lambda_i - \hat\theta_i)\Big],
\end{aligned}
$$

$$
\begin{aligned}
v_i g_i\big(v_{i,x} - (\hat\lambda_i - \lambda_i^* + \hat\theta_i)v_i\big) &= \Big[v_i g_i\big(v_{i,x} - (\tilde\lambda_i - \lambda_i^*)v_i - w_i\big)\Big] + \Big[v_i g_i(w_i - \theta_i v_i)\Big] \\
&\quad + \Big[v_i^2 g_i(\tilde\lambda_i + \theta_i - \hat\lambda_i - \hat\theta_i)\Big].
\end{aligned}
$$

Coefficients of $\hat r_{i,vv}$:

$$
\begin{aligned}
h_i v_{i,x}\big(v_{i,x} - (\hat\lambda_i - \lambda_i^* + \hat\theta_i)v_i\big) &= \Big[h_i v_{i,x}\big(v_{i,x} - (\tilde\lambda_i - \lambda_i^* + \theta_i)v_i\big)\Big] + \Big[h_i v_i v_{i,x}(\tilde\lambda_i + \theta_i - \hat\lambda_i - \hat\theta_i)\Big] \\
&= \Big[v_i h_{i,x}\big(v_{i,x} - (\tilde\lambda_i - \lambda_i^*)v_i - w_i\big)\Big] + \Big[h_i v_i v_{i,x}(\tilde\lambda_i + \theta_i - \hat\lambda_i - \hat\theta_i)\Big] \\
&\quad + v_i\left[h_{i,x}(w_i - \theta_i v_i)\right] + \big(v_{i,x} + (\tilde\lambda_i - \lambda_i^*)v_i - \theta_i v_i\big)\left[h_i v_{i,x} - v_i h_{i,x}\right],
\end{aligned}
$$

$$
\begin{aligned}
g_i v_{i,x}\big(v_{i,x} - (\hat\lambda_i - \lambda_i^*)v_i + \hat\theta_i v_i\big) &= v_i\Big[g_{i,x}\big(v_{i,x} - (\tilde\lambda_i - \lambda_i^*)v_i - w_i\big)\Big] + \left[g_i v_i v_{i,x}(\tilde\lambda_i + \theta_i - \hat\lambda_i - \hat\theta_i)\right] \\
&\quad + v_i\left[g_{i,x}(w_i - \theta_i v_i)\right] + \big(v_{i,x} - (\tilde\lambda_i - \lambda_i^*)v_i - \theta_i v_i\big)\left[g_i v_{i,x} - v_i g_x\right].
\end{aligned}
$$

Coefficients of $\hat r_{i,v\sigma}$:

$$
\begin{aligned}
\big(h_{i,x} - \hat\lambda_i h_i\big)&\hat\theta_i'(g_i/h_i)v_{i,x} - (g_{i,x} - \hat\lambda_i g_i)\hat\theta_i' v_{i,x} - h_i\big(v_{i,x} - (\hat\lambda_i - \lambda_i^* + \hat\theta_i)v_i\big)\hat\theta_{i,x} \\
&= v_{i,x}\hat\theta_i'(h_{i,x}g_i - h_i g_{i,x})/h_i - \hat\theta_{i,x}h_i\big(v_{i,x} - (\hat\lambda_i - \lambda_i^* + \hat\theta_i)v_i\big) \\
&= 2\hat\theta_i'\big(h_{i,x}(g_i/h_i) - g_{i,x}\big)\Big[\big(v_{i,x} - (\tilde\lambda_i - \lambda_i^*)v_i - w_i\big) + (w_i - \theta_i v_i)\Big] \\
&\quad + \Big(2(\tilde\lambda_i - \lambda_i^* + \theta_i) - (\hat\lambda_i - \lambda_i^* + \hat\theta_i)\Big)\Big\{\hat\theta_i'\left[v_{i,x}g_i - v_i g_{i,x}\right] - \hat\theta_i'(g_i/h_i)\left[v_{i,x}h_i - v_i h_{i,x}\right]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
&\big(h_{i,x}-\hat\lambda_i h_i\big)\hat\theta_i'(g_i/h_i)^2 v_{i,x}-\big(g_{i,x}-\hat\lambda_i g_i\big)\hat\theta_i' v_{i,x}g_i/h_i-g_i\big(v_{i,x}-(\hat\lambda_i-\lambda_i^*+\hat\theta_i)v_i\big)\hat\theta_{i,x}\\
&\quad=2\theta_i'\frac{g_i}{h_i}\left(h_{i,x}\frac{g_i}{h_i}-g_{i,x}\right)\Big[\big(v_{i,x}-(\tilde\lambda_i-\lambda_i^*)v_i-w_i\big)+(w_i-\theta v_i)\Big]\\
&\qquad+\Big(2(\tilde\lambda_i-\lambda_i^*+\theta_i)-(\hat\lambda_i-\lambda_i^*+\hat\theta_i)\Big)\Big\{\hat\theta_i'(g_i/h_i)\Big[v_{i,x}g_i-v_ig_{i,x}\Big]-\hat\theta_i'(g_i/h_i)^2\Big[v_{i,x}h_i-v_ih_{i,x}\Big]\Big\}.
\end{aligned}
$$

Coefficients of $\hat r_{i,\sigma u}\tilde r_i/v_i$:

$$
(h_{i,x}-\hat\lambda h_i)\hat\theta_i' v_i^2 g_i/h_i-(g_{i,x}-\hat\lambda_i g_i)v_i^2\hat\theta_i'=\hat\theta_i' v_i\big[v_{i,x}g_i-v_ig_{i,x}\big]+\hat\theta_i' v_i(g_i/h_i)\big[v_ih_{i,x}-h_iv_{i,x}\big],
$$

$$
\begin{aligned}
&(h_{i,x}-\hat\lambda h_i)\hat\theta_i' v_i^2(g_i/h_i)^2-(g_{i,x}-\hat\lambda_i g_i)\hat\theta_i' v_i^2(g_i/h_i)\\
&\qquad=\hat\theta_i' v_i(g_i/h_i)\Big\{\big[v_{i,x}g_i-v_ig_{i,x}\big]+(g_i/h_i)\big[v_ih_{i,x}-h_iv_{i,x}\big]\Big\}.
\end{aligned}
$$

Coefficients of $\hat r_{i,\sigma\sigma}/v_i$:

$$
-\big(h_{i,x}-\hat\lambda_i h_i\big)v_i(g_i/h_i)\hat\theta_i'\hat\theta_{i,x}+\big(g_{i,x}-\hat\lambda_i g_i\big)v_i\hat\theta_i'\hat\theta_{i,x}=(\hat\theta_i')^2v_ih_i\big[(g_i/h_i)_x\big]^2,
$$

$$
-\big(h_{i,x}-\hat\lambda_i h_i\big)v_i(g_i/h_i)^2\hat\theta_i'\hat\theta_{i,x}+\big(g_{i,x}-\hat\lambda_i g_i\big)\hat\theta_i'\hat\theta_{i,x}v_i(g_i/h_i)=(\hat\theta_i')^2v_ig_i\big[(g_i/h_i)_x\big]^2.
$$

There are a few remaining terms in (B.10)-(B.11) which we now examine. Recalling (B.14) we have

$$
\begin{aligned}
\Big(h_iv_i\hat r_{i,u}(\tilde r_i-\hat r_i)\Big)_x=&\mathcal{O}(1)\cdot h_{i,x}v_i^2(\theta_i-\hat\theta_i)+\mathcal{O}(1)\cdot h_iv_iv_{i,x}(\theta_i-\hat\theta_i)\\
&+\mathcal{O}(1)\cdot h_iv_i^2(\theta_i-\hat\theta_i)+\mathcal{O}(1)\cdot h_iv_i(\tilde r_i-\hat r_i)_x\,,
\end{aligned}
$$

$$
\begin{aligned}
\Big(g_iv_i\hat r_{i,u}(\tilde r_i-\hat r_i)\Big)_x=&\mathcal{O}(1)\cdot g_{i,x}v_i^2(\theta_i-\hat\theta_i)+\mathcal{O}(1)\cdot g_iv_iv_{i,x}(\theta_i-\hat\theta_i)\\
&+\mathcal{O}(1)\cdot g_iv_i^2(\theta_i-\hat\theta_i)+\mathcal{O}(1)\cdot g_iv_i(\tilde r_i-\hat r_i)_x\,,
\end{aligned}
$$

$$
\begin{aligned}
h_iv_i\Big[\big(\tilde r_i\bullet A(u)\big)\hat r_i-\big(\hat r_i\bullet A(u)\big)\tilde r_i\Big]&=\mathcal{O}(1)\cdot h_iv_i(\tilde r_i-\hat r_i)\,,\\
h_iv_iA(u)\Big[\big(\tilde r_i\bullet A(u)\big)\hat r_i-\big(\hat r_i\bullet A(u)\big)\tilde r_i\Big]&=\mathcal{O}(1)\cdot h_iv_i(\tilde r_i-\hat r_i)\,.\\
(w_ih_i-v_ig_i)\big(\tilde r_i\bullet A(u)\big)\hat r_i&=\mathcal{O}(1)\cdot|w_ih_i-g_iv_i|\,,
\end{aligned}
$$

$$
\Big[v_ih_i\Big(\big(\tilde r_i\bullet A(u)\big)\hat r_i-\big(\hat r_i\bullet A(u)\big)\tilde r_i\Big)\Big]_x=\mathcal{O}(1)\cdot\big(|v_{i,x}h_i|+|v_ih_{i,x}|\big)|\tilde r_i-\hat r_i|+\mathcal{O}(1)\cdot v_ih_i(\tilde r_i-\hat r_i)_x\,,
$$

These terms are all admissible because of (B.15)–(B.19).

We have thus completed the analysis of all terms in (B.10)-(B.11), showing that the quantities $\hat a_i$, $\hat b_i$ are admissible. The admissibility of the terms $\big((\tilde\lambda_i-\hat\lambda_i)h_i\big)_x$ and $\big((\tilde\lambda_i-\hat\lambda_i)g_i\big)_x$ follows immediately from (B.18) and (B.19). This completes the proof of Lemma 11.4. □

Appendix C

Aim of this section is to derive energy estimates for the components h_i, g_i and prove the bounds (11.33)-(11.34). We write the evolution equations (11.15) for the components h_i, g_i in the form

$$\begin{cases} h_{i,t} + (\tilde\lambda_i h_i)_x - h_{i,xx} = \hat\phi_i\,, \\ g_{i,t} + (\tilde\lambda_i g_i)_x - g_{i,xx} = \hat\psi_i\,. \end{cases} \tag{C.1}$$

For convenience, we define $\hat\eta_i \doteq \eta(g_i/h_i)$. Multiplying the first equation in (C.1) by $h_i\hat\eta_i$ and integrating by parts, we obtain

$$\begin{aligned}\int \hat\eta_i h_i \hat\phi_i\,dx &= \int \Big\{\hat\eta_i h_i h_{i,t} + \hat\eta_i h_i(\tilde\lambda_i h_i)_x - \hat\eta_i h_i h_{i,xx}\Big\}\,dx \\ &= \int \Big\{\hat\eta_i (h_i^2/2)_t - \hat\eta_i\tilde\lambda_i h_i h_{i,x} - \hat\eta_{i,x}\tilde\lambda_i h_i^2 + \hat\eta_i h_{i,x}^2 + \hat\eta_{i,x} h_{i,x} h_i\Big\}\,dx \\ &= \int \Big\{\big(\hat\eta_i h_i^2/2\big)_t + (\tilde\lambda_i\hat\eta_i)_x (h_i^2/2) - \big(\hat\eta_{i,t} + 2\tilde\lambda_i\hat\eta_{i,x} - \hat\eta_{i,xx}\big)(h_i^2/2) + \hat\eta_i h_{i,x}^2 - 2\hat\eta_{i,x} h_i h_{i,x}\Big\}\,dx\,.\end{aligned}$$

Therefore

$$\begin{aligned}\int \hat\eta_i h_{i,x}^2\,dx = -\frac{d}{dt}\left[\int \hat\eta_i h_i^2/2\,dx\right] + \int \big(\hat\eta_{i,t} + \tilde\lambda_i\hat\eta_{i,x} - \hat\eta_{i,xx}\big)(h_i^2/2)\,dx \\ - \int \tilde\lambda_{i,x}\hat\eta_i(h_i^2/2)\,dx + \int \hat\eta_i h_i\hat\phi_i\,dx + 2\int \hat\eta_{i,x} h_i h_{i,x}\,dx\,.\end{aligned} \tag{C.2}$$

As in (9.14), a direct computation yields

$$\hat\eta_{i,t} + \tilde\lambda_i\hat\eta_{i,x} - \hat\eta_{i,xx} = \hat\eta_i'\left(\frac{\hat\psi_i}{h_i} - \frac{g_i}{h_i}\frac{\hat\phi_i}{h_i}\right) + 2\hat\eta_i'\frac{h_{i,x}}{h_i}\left(\frac{g_i}{h_i}\right)_x - \hat\eta_i''\left(\frac{g_i}{h_i}\right)_x^2. \tag{C.3}$$

Since $\tilde\lambda_{i,x} = (\tilde\lambda_i - \lambda_i^*)_x$, integrating by parts and using the second estimate in (11.13) one obtains

$$\begin{aligned}\left|\int \tilde\lambda_{i,x}\hat\eta_i(h_i^2/2)\,dx\right| &= \left|\int (\tilde\lambda_i - \lambda_i^*)\big(\hat\eta_{i,x}h_i^2/2 + \hat\eta_i h_i h_{i,x}\big)\,dx\right| \\ &\leq \|\tilde\lambda_i - \lambda_i^*\|_{\mathbf{L}^\infty}\cdot\left\{\frac{1}{2}\int |\hat\eta_i'|\,|g_{i,x}h_i - g_i h_{i,x}|\,dx + \frac{5}{2\delta_1}\int \hat\eta_i h_{i,x}^2\,dx \right. \\ &\qquad\left. + \mathcal{O}(1)\cdot\delta_0\sum_{j\neq i}\int \big(|h_i v_j| + |h_i h_j|\big)\,dx\right\} \\ &\leq \int |g_{i,x}h_i - h_{i,x}g_i|\,dx + \frac{1}{2}\int \hat\eta_i h_{i,x}^2\,dx + \delta_0\sum_{j\neq i}\int \big(|h_i v_j| + |h_i h_j|\big)\,dx\,,\end{aligned} \tag{C.4}$$

because

$$|\tilde\lambda_i - \lambda_i^*| = \mathcal{O}(1)\cdot\delta_0 << \delta_1 \leq 1\,.$$

Using (C.3) and (C.4) in (C.2), we now obtain

$$\begin{aligned}\frac{1}{2}\int \hat\eta_i\,h_{i,x}^2\,dx \leq -\frac{d}{dt}\left[\int \frac{\hat\eta_i h_i^2}{2}\,dx\right] + \frac{1}{2}\int |\hat\eta_i'|\big(|h_i\hat\psi_i| + |g_i\hat\phi_i|\big)\,dx + \int \left|\hat\eta_i' h_i h_{i,x}\left(\frac{g_i}{h_i}\right)_x\right|\,dx \\ + \frac{1}{2}\int \left|\hat\eta_i'' h_i^2\left(\frac{g_i}{h_i}\right)_x^2\right|\,dx + \int |g_{i,x}h_i - g_i h_{i,x}|\,dx \\ + \delta_0\sum_{j\neq i}\int \big(|h_i v_j| + |h_i h_j|\big)\,dx + \int |h_i\hat\phi_i|\,dx + 2\int \big|\hat\eta_{i,x} h_i h_{i,x}\big|\,dx\,.\end{aligned} \tag{C.5}$$

Recalling the definition of $\hat\eta_i$, on regions where $\hat\eta_i' \neq 0$ one has $|g_i/h_i| \leq 4\delta_1/5$, hence the bounds (11.14) hold. In turn, they imply

$$
\begin{aligned}
\big|\hat\eta_{i,x} h_i h_{i,x}\big| &= \left|\hat\eta_i' h_i h_{i,x} \left(\frac{g_i}{h_i}\right)_x\right| \\
&\leq \frac{5}{2\delta_1}\left|\hat\eta_i' h_i^2 \left(\frac{g_i}{h_i}\right)_x\right| + \mathcal{O}(1)\cdot\delta_0 \sum_{j\neq i} \left|\hat\eta_i' h_i \left(\frac{g_i}{h_i}\right)_x\right| \big(|v_j| + |h_j|\big) \\
&= \mathcal{O}(1)\cdot |g_{i,x}h_i - g_i h_{i,x}| + \mathcal{O}(1)\cdot\delta_0 \sum_{j\neq i}\big(|v_j g_{i,x}| + |v_j h_{i,x}| + |h_j g_{i,x}| + |h_j h_{i,x}|\big)\,.
\end{aligned}
\tag{C.6}
$$

Using (C.6) and then the bounds (11.18), (11.26), (11.28), (11.30) and (11.31), from (C.5) we conclude

$$
\begin{aligned}
\int_{\hat t}^T \int \hat\eta_i\, h_{i,x}^2\,dxdt &\leq \int \hat\eta_i h_i^2(\hat t,x)\,dx + \mathcal{O}(1)\cdot \int_{\hat t}^T\int \big(|h_i\hat\psi_i| + |g_i\hat\phi_i|\big)\,dxdt \\
&\qquad + \mathcal{O}(1)\cdot \int_{\hat t}^T \int |g_{i,x}h_i - g_i h_{i,x}|\,dxdt + \mathcal{O}(1)\cdot \int_{\hat t}^T \int_{g_i/h_i|<\delta_1} \big|h_i (g_i/h_i)_x\big|^2\,dxdt \\
&\qquad + \mathcal{O}(1)\cdot\delta_0 \sum_{j\neq i}\int_{\hat t}^T\int \big(|v_j g_{i,x}| + |v_j h_{i,x}| + |h_j g_{i,x}| + |h_j h_{i,x}|\big)\,dxdt \\
&\qquad + \delta_0 \sum_{j\neq i}\int_{\hat t}^T\int \big(|h_i v_j| + |h_i h_j|\big)\,dxdt + 2\int_{\hat t}^T\int |h_i\hat\phi_i|\,dxdt \\
&= \mathcal{O}(1)\cdot\delta_0^2\,,
\end{aligned}
\tag{C.7}
$$

proving the estimate (11.33)

We now perform a similar computation for $g_{i,x}^2$. Define $\check\eta_i \doteq \bar\eta(h_i/g_i)$, where $\bar\eta(s) = \eta\big(|s| - \delta_1/5\big)$. Multiplying the second equation in (C.1) by $\check\eta_i g_i$ and integrating by parts, one obtains

$$
\int \check\eta_i g_i \hat\psi_i\,dx = \int \Big\{ \big(\check\eta_i g_i^2/2\big)_t + (\tilde\lambda_i\check\eta_i)_x (g_i^2/2) - \big(\check\eta_{i,t} + 2\tilde\lambda_i \check\eta_{i,x} - \check\eta_{i,xx}\big)(g_i^2/2) + \check\eta_i g_{i,x}^2 - 2\check\eta_{i,x} g_i g_{i,x}\Big\}\,dx\,.
$$

Therefore, the identity (C.2) still holds, with $h_i, \hat\phi_i, \hat\eta_i$ replaced by $g_i, \hat\psi_i, \check\eta_i$, respectively:

$$
\begin{aligned}
\int \check\eta_i g_{i,x}^2\,dx = -\frac{d}{dt}\left[\int \check\eta_i g_i^2/2\,dx\right] + \int \big(\check\eta_{i,t} + \tilde\lambda_i\check\eta_{i,x} - \check\eta_{i,xx}\big)(g_i^2/2)\,dx \\
- \int \tilde\lambda_{i,x}\check\eta_i (g_i^2/2)\,dx + \int \check\eta_i g_i\hat\psi_i\,dx + 2\int \check\eta_{i,x} g_i g_{i,x}\,dx\,.
\end{aligned}
\tag{C.8}
$$

The equality (C.3) can again be used, with $\hat\eta_i$ replaced by $\check\eta_i$. To obtain a suitable replacement for (C.4) we observe that, if $\check\eta_i \neq 0$, then (11.13) implies

$$
|g_i g_{i,x}| \leq 2|h_{i,x} g_{i,x}| + \mathcal{O}(1)\cdot\delta_0 \sum_{j\neq i}\big(|v_j g_{i,x}| + |h_j g_{i,x}|\big)
$$

and hence

$$
|g_i g_{i,x}| \leq h_{i,x}^2 + g_{i,x}^2 + \mathcal{O}(1)\cdot\delta_0\sum_{j\neq i}\big(|v_j g_i| + |h_j g_i|\big)\,.
$$

Integrating by parts we thus obtain

$$
\begin{aligned}
\left|\int \tilde\lambda_{i,x}\check\eta_i(g_i^2/2)\,dx\right| &= \left|\int (\tilde\lambda_i-\lambda_i^*)\big(\check\eta_{i,x}g_i^2/2+\check\eta_i g_i g_{i,x}\big)\,dx\right| \\
&\le \|\tilde\lambda_i-\lambda_i^*\|_{\mathbf{L}^\infty}\cdot\left\{\int |\check\eta_i'|\,|g_{i,x}h_i-g_ih_{i,x}|\,\left|\frac{g_i^2}{h_i^2}\right|\,dx+\int \check\eta_i h_{i,x}^2\,dx\right. \\
&\qquad \left. +\int \check\eta_i g_{i,x}^2\,dx+\mathcal{O}(1)\cdot\delta_0\sum_{j\neq i}\int \big(|v_jg_i|+|h_jg_i|\big)\,dx\right\} \\
&\le \int |g_{i,x}h_i-h_{i,x}g_i|\,dx+\frac12\int \check\eta_i h_{i,x}^2\,dx+\frac12\int \check\eta_i g_{i,x}^2\,dx+\delta_0\sum_{j\neq i}\int \big(|v_jg_i|+|h_jg_i|\big)\,dx\,.
\end{aligned}
\tag{C.9}
$$

Using (C.3) and (C.9) in (C.8) and observing that $|g_i^2/h_i^2|\le \delta_1^2$ on the region where $\check\eta_i'\neq 0$, we now obtain an estimate similar to (C.5):

$$
\begin{aligned}
\frac12\int \check\eta_i\,g_{i,x}^2\,dx &\le -\frac{d}{dt}\left[\int \frac{\check\eta_i g_i^2}{2}\,dx\right]+\frac{\delta_1^2}{2}\int |\check\eta_i'|\big(|h_i\hat\psi_i|+|g_i\hat\phi_i|\big)\,dx+\delta_1^2\int\left|\check\eta_i'h_ih_{i,x}\left(\frac{g_i}{h_i}\right)_x\right|\,dx \\
&\quad +\frac{\delta_1^2}{2}\int\left|\check\eta_i''h_i^2\left(\frac{g_i}{h_i}\right)_x^2\right|\,dx+\int |g_{i,x}h_i-g_ih_{i,x}|\,dx+\frac12\int \check\eta_ih_{i,x}^2\,dx \\
&\quad +\delta_0\sum_{j\neq i}\int\big(|v_jg_i|+|h_jg_i|\big)\,dx+\int |g_i\hat\psi_i|\,dx+2\int\big|\check\eta_{i,x}g_ig_{i,x}\big|\,dx\,.
\end{aligned}
\tag{C.10}
$$

We now observe that $\check\eta_i'\neq 0$ only when $4\delta_1/5<|g_i/h_i|<\delta_1$. In this case one has $\hat\eta_i=1$ and moreover, recalling our choice $\delta_1<1/3$,

$$
\left|h_i\left(\frac{g_i}{h_i}\right)_x\right|^2 \;\ge\; g_{i,x}^2-2\left|\frac{g_i}{h_i}\right|\,|g_{i,x}h_{i,x}|-\left|\frac{g_i}{h_i}\right|^2h_{i,x}^2\;\ge\;\frac12 g_{i,x}^2-\frac12 h_{i,x}^2\,.
$$

Hence

$$
(\hat\eta_i-\check\eta_i)g_{i,x}^2+\big|\check\eta_{i,x}g_ig_{i,x}\big| = \mathcal{O}(1)\cdot\big|h_i(g_i/h_i)_x\big|^2\cdot\chi_{\{|h_i/g_i|<\delta_1\}}+\mathcal{O}(1)\cdot\hat\eta_ih_{i,x}^2\,. \tag{C.11}
$$

Using (C.6) and then the bounds (C.7), (11.18), (11.26), (11.28), (11.30)-(11.31) and (C.11), from (C.10) we conclude

$$
\begin{aligned}
\int_{\hat t}^T\int \hat\eta_i\,g_{i,x}^2\,dxdt &\le \int \hat\eta_ig_i^2(\hat t,x)\,dx+\mathcal{O}(1)\cdot\int_{\hat t}^T\int\big(|h_i\hat\psi_i|+|g_i\hat\phi_i|\big)\,dxdt \\
&\quad +\mathcal{O}(1)\cdot\int_{\hat t}^T\int |g_{i,x}h_i-g_ih_{i,x}|\,dxdt+\mathcal{O}(1)\cdot\int_{\hat t}^T\int_{|g_i/h_i|<\delta_1}\big|h_i(g_i/h_i)_x\big|^2\,dxdt \\
&\quad +\mathcal{O}(1)\cdot\delta_0\sum_{j\neq i}\int_{\hat t}^T\int\big(|v_jg_{i,x}|+|v_jh_{i,x}|+|h_jg_{i,x}|+|h_jh_{i,x}|\big)\,dxdt \\
&\quad +\mathcal{O}(1)\cdot\int_{\hat t}^T\int \hat\eta_i\,h_{i,x}^2\,dxdt+\delta_0\sum_{j\neq i}\int\big(|v_jg_i|+|h_jg_i|\big)\,dx+2\int_{\hat t}^T\int |h_i\hat\phi_i|\,dxdt \\
&= \mathcal{O}(1)\cdot\delta_0^2\,,
\end{aligned}
\tag{C.12}
$$

proving the estimate (11.34).

Appendix D

We derive here the two estimates (12.9)-(12.10), used in the proof of Lemma 12.1.

$$
\begin{aligned}
&\|A\|_{\mathbf{L}^\infty} \int_0^t \int \big|G_x(t-s,x-y)\big| E(s,y)\,dy\,ds \\
&= \|A\|_{\mathbf{L}^\infty} \int_0^t \int \frac{|x-y|}{4(t-s)\sqrt{\pi(t-s)}} B(s) \\
&\qquad\qquad \cdot \exp\Big\{ -\frac{(x-y)^2}{4(t-s)} + 4\|DA\|_{\mathbf{L}^\infty} \int_0^s \big\|u_x(\sigma)\big\|_{\mathbf{L}^\infty} d\sigma + s - y \Big\} dy\,ds \\
&\le \|A\|_{\mathbf{L}^\infty} \exp\left\{ 4\|DA\|_{\mathbf{L}^\infty} \int_0^t \big\|u_x(\sigma)\big\|_{\mathbf{L}^\infty} d\sigma + t - x \right\} \\
&\qquad\qquad \cdot \int_0^t \frac{B(s)}{4(t-s)\sqrt{\pi(t-s)}} \left(\int |x-y| \exp\left\{ -\frac{(y+2(t-s)-x)^2}{4(t-s)} \right\} dy \right) ds \\
&= \exp\left\{ 4\|DA\|_{\mathbf{L}^\infty} \int_0^t \big\|u_x(\sigma)\big\|_{\mathbf{L}^\infty} d\sigma + t - x \right\} \int_0^t \frac{\|A\|_{\mathbf{L}^\infty} B(s)}{\sqrt{\pi(t-s)}} \left(\int \big|\zeta - \sqrt{t-s}\big| e^{-\zeta^2} d\zeta \right) ds \\
&\le \exp\left\{ 4\|DA\|_{\mathbf{L}^\infty} \int_0^t \big\|u_x(\sigma)\big\|_{\mathbf{L}^\infty} d\sigma + t - x \right\} \int_0^t \|A\|_{\mathbf{L}^\infty} \left(\frac{1}{\sqrt{t-s}} + \sqrt{\pi} \right) B(s)\,ds \\
&\le \exp\left\{ 4\|DA\|_{\mathbf{L}^\infty} \int_0^t \big\|u_x(\sigma)\big\|_{\mathbf{L}^\infty} d\sigma + t - x \right\} \left(\frac{B(t)}{2} - \frac{1}{2} \right) \\
&= \frac{1}{2} E(t,x) - \frac{1}{2} \exp\left\{ 4\|DA\|_{\mathbf{L}^\infty} \int_0^t \big\|u_x(\sigma)\big\|_{\mathbf{L}^\infty} d\sigma + t - x \right\} \\
&\le \frac{1}{2} E(t,x) - \frac{1}{2} e^{t-x},
\end{aligned}
$$

$$
\begin{aligned}
&2\|DA\|_{\mathbf{L}^\infty} \int_0^t \big\|u_x(s)\big\|_{\mathbf{L}^\infty} \left(\int G(t-s,x-y) E(s,y)\,dy \right) ds \\
&\qquad = 2\|DA\|_{\mathbf{L}^\infty} \int_0^t B(s) \cdot \exp\left\{ 4\|DA\|_{\mathbf{L}^\infty} \int_0^s \big\|u_x(\sigma)\big\|_{\mathbf{L}^\infty} d\sigma + s \right\} \\
&\qquad\qquad\qquad \cdot \frac{\big\|u_x(s)\big\|_{\mathbf{L}^\infty}}{2\sqrt{\pi(t-s)}} \left(\int \exp\left\{ -\frac{(x-y)^2}{4(t-s)} - y \right\} dy \right) ds \\
&\qquad \le B(t) e^{t-x} \int_0^t 2\|DA\|_{\mathbf{L}^\infty} \big\|u_x(s)\big\|_{\mathbf{L}^\infty} \exp\left\{ 4\|DA\|_{\mathbf{L}^\infty} \int_0^s \big\|u_x(\sigma)\big\|_{\mathbf{L}^\infty} d\sigma \right\} ds \\
&\qquad = B(t) e^{t-x} \left[\frac{1}{2} \exp\left\{ 4\|DA\|_{\mathbf{L}^\infty} \int_0^t \big\|u_x(\sigma)\big\|_{\mathbf{L}^\infty} d\sigma \right\} - \frac{1}{2} \right] \\
&\qquad \le \frac{1}{2} E(t,x) - \frac{1}{2} e^{t-x}.
\end{aligned}
$$

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