Weak solutions for a hyperbolic system with unilateral constraint and mass loss

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Abstract

We consider isentropic gas dynamics equations with unilateral constraint on the density and mass loss. The γ and pressureless pressure laws are considered. We propose an entropy weak formulation of the system that incorporates the constraint and Lagrange multiplier, for which we prove weak stability and existence of solutions. The nonzero pressure model is approximated by a kinetic BGK relaxation model, while the pressureless model is approximated by a sticky-blocks dynamics with mass loss.

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Contents

1 Introduction and models **2**
- 1.1 Models . . . 2
- 1.2 Main results . . . 5

2 Stability of the entropy weak product **6**

3 Isentropic model with nonzero pressure **7**
- 3.1 BGK model . . . 7
- 3.2 Properties of the kinetic entropy . . . 9
- 3.3 Existence for the BGK model . . . 11
- 3.4 Kinetic invariant domains . . . 12
- 3.5 Relaxation limit via compensated compactness . . . 13
- 3.6 Relaxation limit for $\gamma = 3$ via averaging lemma . . . 15

4 Pressureless model **16**
- 4.1 Sticky blocks dynamics . . . 17
- 4.2 Properties of sticky blocks . . . 18
- 4.3 Existence of a solution . . . 20
- 4.4 Loss of the strong exclusion relation . . . 20

1 Introduction and models

1.1 Models

The aim of this paper is to introduce a weak formulation and establish weak stability and existence for solutions to some one-dimensional systems of conservations laws with unilateral constraint. Such system arises for example in the modeling of two-phase flows, see [8], as

$$\left\{ \begin{array}{l} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p(\rho) + \pi) = 0, \end{array} \right. \tag{1.1}$$

with constraint and pressure Lagrange multiplier

$$0 \leq \rho \leq 1, \qquad \pi \geq 0, \tag{1.2}$$

and exclusion relation

$$(1-\rho)\pi = 0. \tag{1.3}$$

This system was studied in [20] with viscosity. Existence and weak stability of suitable weak solutions is obtained in [2] in the pressureless case $p(\rho) = 0$, but however, the general nonzero pressure case remains open. We refer to [22], [1], [18], [19], [16], [20] and [2] for other hyperbolic problems with constraints. Some general formulations can be found in [13].

Here we are going to consider a slightly different model with mass loss, that can be written as

$$\left\{ \begin{array}{l} \partial_t \rho + \partial_x(\rho u) = Q, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p(\rho)) = Qu, \end{array} \right. \tag{1.4}$$

with constraint and mass loss rate Lagrange multiplier

$$0 \leq \rho \leq 1, \qquad Q \leq 0, \tag{1.5}$$

and exclusion relation

$$(1-\rho)Q = 0. \tag{1.6}$$

This model is based on the physical idea to remove from the density ρ what overflows with 1 as rain could make overflow a reservoir or a river. The term Qu on the right-hand side of the momentum equation expresses that the overflowing matter travels at velocity u. This interpretation is especially relevant for the Saint-Venant equations when $p(\rho) = \kappa\rho^2$. We are going to consider here pressure laws of the form

$$p(\rho) = \kappa\rho^\gamma, \qquad 1 < \gamma \leq 3,\ \kappa \geq 0. \tag{1.7}$$

The pressureless case $\kappa = 0$ is very particular. Existence and properties for the system of pressureless gas without constraint have been studied in [17], [9], [12], [15] and [10].

The main difficulty in the model is to give a suitable sense to the products Qu in (1.4) and ρQ in (1.6), because Q is only a measure, and ρ, u can be discontinuous. Indeed, since Q can be nonzero only where $\rho = 1$, we have formal formulas obtained by frozing $\rho = 1$ in (1.4),

$$Q = \mathbb{1}_{\rho=1}\partial_x u, \qquad Qu = \mathbb{1}_{\rho=1}(\partial_t u + \partial_x u^2), \tag{1.8}$$

but of course this is again meaningless. Therefore, we provide the following entropy weak formulation of the problem, that involves what we call entropy weak products. The idea is to introduce a different velocity $v(t,x)$ for the lost matter, and write the system

$$\left\{ \begin{array}{l} \partial_t\rho + \partial_x(\rho u) = Q, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p(\rho)) = Qv, \end{array} \right. \tag{1.9}$$

with as before

$$0 \leq \rho \leq 1, \qquad Q \leq 0. \tag{1.10}$$

We take $v \in L^\infty(Q)$, so that the product Qv is well-defined as a measure. We need then to formulate in a weak sense that $Qv = Qu$, and that $Q\rho = Q$. In order to do so, we require the family of entropy weak product inequalities

$$\partial_t\, \eta_S(\rho,u) + \partial_x\, G_S(\rho,u) \leq Q\, \eta_S'(1,v)\cdot(1,v), \tag{1.11}$$

for any convex entropy η_S in a suitable family parametrized by a convex function S, where G_S is its entropy flux, and η_S' is its derivative with respect to $(\rho,\rho u)$. Since v, by definition, is defined Q a.e., the term on the right-hand side of (1.11) is well-defined. In order to see that (1.9)-(1.11) is a weak formulation of (1.4)-(1.6), we observe first that any suitable solution to (1.4)-(1.6) also solves (1.9)-(1.11) with $v = u$. Conversely, if we have a sufficiently smooth solution to (1.9)-(1.11), then multiplying (1.9) by $\eta_S'(\rho,u)$ and comparing with (1.11), we get

$$Q\, \eta_S'(\rho,u)\cdot(1,v) \leq Q\, \eta_S'(1,v)\cdot(1,v). \tag{1.12}$$

If we take for the entropy the physical energy

$$\eta(\rho, u) = \rho u^2/2 + \frac{\kappa}{\gamma - 1}\rho^\gamma, \tag{1.13}$$

we have $\eta'(\rho, u) = (\gamma\kappa\rho^{\gamma-1}/(\gamma - 1) - u^2/2, u)$, and (1.12) gives

$$Q\left[\frac{\gamma\kappa}{\gamma - 1}(\rho^{\gamma-1} - 1) - (v - u)^2/2\right] \leq 0. \tag{1.14}$$

Together with the constraints (1.10), we deduce that $Qv = Qu$ and $Q\rho = Q$, except in the pressureless case $\kappa = 0$, in which we can only conclude $Qv = Qu$. We shall see in Subsection 4.4 that in this case the formulation really fails to give $Q\rho = Q$ in the strong sense.

Our main result is that the entropy formulation of the system (1.9)-(1.11) is weakly stable. We are able to prove a priori estimates and compactness of suitable approximations, that lead to existence for the Cauchy problem. For the nonzero pressure model, the approximate solutions are obtained by a kinetic BGK equation with additional projection to enforce the constraint $\rho \leq 1$. For the pressureless model, the approximation is based on the notion of sticky blocks that has been introduced in [8] and used in [2], but here with a different dynamics based on mass loss.

We look for solutions with regularities

$$\rho \in L_t^\infty(0, \infty; L_x^\infty(\mathbb{R}) \cap L_x^1(\mathbb{R})), \tag{1.15}$$

$$u \in L_t^\infty(0, \infty; L_x^\infty(\mathbb{R})), \tag{1.16}$$

$$Q \in \mathcal{M}([0, \infty[\times\mathbb{R}), \quad v \in L^\infty(Q). \tag{1.17}$$

The density ρ and the momentum density ρu are a priori not continuous with respect to time, because Q could contain Dirac distributions in time. However, (1.8) suggests that it should not be the case, but it is an open question to decide whether or not it is the case.

Thus, we consider weak solutions in the sense that for all $\varphi \in \mathcal{D}([0, \infty[\times\mathbb{R})$,

$$\int_0^\infty \int_{\mathbb{R}} [\rho\partial_t\varphi + \rho u\partial_x\varphi]\, dtdx + \int_{\mathbb{R}} \rho^0(x)\varphi(0, x)\, dx = -\int_{[0,\infty[}\int_{\mathbb{R}} \varphi Q, \tag{1.18}$$

$$\int_0^\infty \int_{\mathbb{R}} [\rho u\partial_t\varphi + (\rho u^2 + p(\rho))\partial_x\varphi]\, dtdx + \int_{\mathbb{R}} \rho^0(x)u^0(x)\varphi(0, x)\, dx = -\int_{[0,\infty[}\int_{\mathbb{R}} \varphi Qv. \tag{1.19}$$

It includes the initial data

$$\rho(0, x) = \rho^0(x), \quad \rho(0, x)u(0, x) = \rho^0(x)u^0(x). \tag{1.20}$$

We assume that $\rho^0 \in L^1(\mathbb{R})$, so that we can bound a priori the mass loss, $\int\int -Q\, dtdx \leq \int \rho^0\, dx$.

1.2 Main results

The following compactness result is valid for the two possible pressure laws (isentropic model with nonzero pressure or pressureless model).

Theorem 1.1 *Let us consider a sequence of solutions (ρ_n, u_n, Q_n, v_n) with regularities (1.15)-(1.17) with uniform bounds in their respective spaces of (1.15)-(1.17), satisfying (1.18)-(1.19) and (1.10)-(1.11). Initial data ρ_n^0, u_n^0 are supposed to satisfy*

$$0 \leq \rho_n^0 \leq 1, \qquad (\rho_n^0)_{n\geq 0} \text{ is bounded in } L^1(\mathbb{R}), \tag{1.21}$$

$$(u_n^0)_{n\geq 0} \text{ is bounded in } L^\infty(\mathbb{R}). \tag{1.22}$$

In the pressureless case, we also assume that the Oleinik inequality (1.26) holds, and that Q_n is bounded in $L^\infty_{loc}(]0,\infty[,\mathcal{M}_{loc}(\mathbb{R}))$. Then, up to a subsequence, as $n\to\infty$, $(\rho_n, u_n, Q_n, v_n) \rightharpoonup (\rho, u, Q, v)$ satisfying (1.15)-(1.17), in the following sense

$$\rho_n \rightharpoonup \rho, \qquad u_n \rightharpoonup u \qquad \text{in } L^\infty_{w*}(]0,\infty[\times\mathbb{R}), \tag{1.23}$$

$$Q_n \rightharpoonup Q, \qquad Q_n v_n \rightharpoonup Qv \qquad \text{in } \mathcal{M}([0,\infty[\times\mathbb{R})w*, \tag{1.24}$$

where (ρ, u, Q, v) is a solution to (1.18)-(1.19) and (1.10)-(1.11) with initial data ρ^0, u^0 defined by

$$\rho_n^0 \rightharpoonup \rho^0 \text{ in } L^\infty_{w*}(\mathbb{R}), \quad \text{and} \quad \rho_n^0 u_n^0 \rightharpoonup \rho^0 u^0 \text{ in } L^\infty_{w*}(\mathbb{R}). \tag{1.25}$$

In the pressureless case, we also get (1.26). In the nonzero pressure case, we have the convergence a.e. $\rho_n \to \rho$, $\rho_n u_n \to \rho u$.

We turn now to existence. The theorem is again the same for both pressure laws.

Theorem 1.2 *Let $\rho^0 \in L^1(\mathbb{R})$ such that $0 \leq \rho^0 \leq 1$ and $u^0 \in L^\infty(\mathbb{R})$. Then there exists (ρ, u, Q, v) with regularities (1.15)-(1.17) satisfying (1.18)-(1.19) and (1.10)-(1.11).*

In the pressureless context, we have more precise results.

Theorem 1.3 *In addition, in the pressureless case $\kappa = 0$, the solution of Theorem 1.2 satisfies*

$$\partial_x u(t,x) \leq \tfrac{1}{t}, \tag{1.26}$$

$$TV_{[a,b]}(u(t,.)) \leq \tfrac{2(b-a)}{t} + 2\|u^0\|_{L^\infty} \quad \forall a < b, \tag{1.27}$$

$$\operatorname{essinf} u^0(x) \leq u(t,x) \leq \operatorname{esssup} u^0(x), \tag{1.28}$$

$$Q \in L^\infty_{loc}(]0,\infty[,\mathcal{M}_{loc}(\mathbb{R})), \tag{1.29}$$

$$\rho, \rho u \in C(]0,\infty[, L^\infty w*), \tag{1.30}$$

$$\partial_t(\rho S(u)) + \partial_x(\rho u S(u)) = Q_S \text{ in }]0,\infty[\times\mathbb{R} \tag{1.31}$$

for every $S \in C(\mathbb{R})$, where $Q_S \in \mathcal{M}([0,\infty[\times\mathbb{R})$ satisfies

$$|Q_S| \leq \|S\|_{L^\infty} |Q|. \tag{1.32}$$

The remainder of the paper is organized as follows. In Section 2, we prove the stability of the entropy weak product formulation. In Section 3, we study the non-zero pressure case. We prove the existence of solutions for a BGK model with relaxation of the type of those introduced in [6]. We obtain kinetic entropy inequalities and the existence of kinetic invariant domains, following [24], [4]. Using compensated compactness, we prove the convergence, as $\varepsilon \to 0$, towards the nonzero pressure gas dynamics model with mass loss. Finally, we provide an alternate analysis of the BGK model in the particular case $\gamma = 3$ using averaging lemma. In Section 4, we study the pressureless model. We introduce the sticky blocks dynamics, and specify in this case how entropy weak product inequalities arise. Finally, we prove that the strong exclusion relation is lost in weak limits for the pressureless model.

2 Stability of the entropy weak product

The weak stability of the formulation (1.11) becomes clear with the two following lemmas.

Lemma 2.1 *Let Q_n be nonpositive measures and $v_n \in L^\infty(Q_n)$. If $(Q_n)_{n\geq 0}$ is a sequence bounded in $\mathcal{M}_{loc}([0,\infty[\times\mathbb{R})$ and $(\|v_n\|_{L^\infty(Q_n)})_{n\geq 0}$ is bounded, then there exists a measure Q and a function $v \in L^\infty(Q)$ such that after extraction of a subsequence, $Q_n \rightharpoonup Q$, $Q_n v_n \rightharpoonup Qv$ and $Q_n\varphi(v_n) \rightharpoonup Q^\varphi \leq Q\varphi(v)$ for any convex function φ.*

Proof. The measures Q_n are bounded in $\mathcal{M}_{loc}$, thus for a subsequence $Q_n \rightharpoonup Q$ in $\mathcal{M}_{loc}w*$. By diagonal extraction, there exists a subsequence such that

$$Q_n\varphi(v_n) \rightharpoonup Q^\varphi$$

for every φ continuous. We have $|Q_n v_n| \leq C|Q_n|$, thus at the limit $|Q^{Id}| \leq -CQ$ and therefore there exists $v \in L^\infty(Q)$ such that $Q^{Id} = Qv$.
We compare now Q^φ and $Q\varphi(v)$ for φ convex. A convex function φ can be written as

$$\varphi(v) = \sup\{av + b\ ;\ a, b \text{ such that } \varphi \geq aId + b\}.$$

Let φ be a convex function and let $a, b \in \mathbb{R}$ such that $\varphi \geq aId + b$. The measure Q_n is nonpositive thus $Q_n\varphi(v_n) \leq Q_n(av_n + b)$, which gives at the limit $Q^\varphi \leq Q(av + b)$. Since this is true for any a, b such that $\varphi \geq aId + b$, we conclude that $Q^\varphi \leq Q\varphi(v)$. $\square$

Lemma 2.2 *The function $v \mapsto \eta_S'(1, v) \cdot (1, v)$ is convex for $S : \mathbb{R} \to \mathbb{R}$ convex and C^1. Furthermore, it is a nonnegative function as soon as $S \geq 0$.*

Proof. We have first to specify what are the entropies η_S. We take the so called weak entropies, that are defined as

$$\eta_S(\rho, u) = \int_{\mathbb{R}} \chi(\rho, \xi - u) S(\xi)\, d\xi, \quad S \text{ convex}, \tag{2.1}$$

where χ is defined by (3.7)-(3.9) in the case $\kappa > 0$, and by $\chi(\rho, \xi) = \rho\delta(\xi)$ if $\kappa = 0$ (in other words $\eta_S(\rho, u) = \rho S(u)$). Recalling that prime denotes differentiation with respect to $(\rho, \rho u)$, we can express the desired quantity and get, for $1 < \gamma < 3$,

$$\eta_S'(1, v) \cdot (1, v) = \frac{1-\theta}{J_\lambda} \int_{-1}^{1} (1 - z^2)^{\lambda - 1} S(v + a_\gamma z)\, dz, \tag{2.2}$$

for $\gamma = 3$,

$$\eta_S'(1, v) \cdot (1, v) = \Big(S(v + \sqrt{3\kappa}) + S(v - \sqrt{3\kappa}) \Big)/2, \tag{2.3}$$

and for $\kappa = 0$,

$$\eta_S'(1, v) \cdot (1, v) = S(v). \tag{2.4}$$

The result follows obviously. □

Remark 2.1 The sign assertion in Lemma 2.2 is helpful because the right-hand side in (1.11) becomes itself nonpositive when $S \geq 0$ and we deduce the decrease of $\int \eta_S \, dx$.

3 Isentropic model with nonzero pressure

In this section, we introduce a kinetic BGK relaxation model that approximates the problem with nonzero pressure $p(\rho) = \kappa \rho^\gamma$ with $\kappa > 0$, $1 < \gamma \leq 3$. We first prove existence of solutions for the BGK model and establish kinetic invariant domains leading to uniform bounds. Then we let the relaxation parameter ε tend to 0, and get an entropy solution to (1.9)-(1.11) via compensated compactness. For the special case $\gamma = 3$, an alternate proof via averaging lemma is provided.

3.1 BGK model

We consider the following kinetic BGK relaxation model, which is obtained from the one of [3], [4] by including an overall projection onto the constraint $\rho \leq 1$,

$$\partial_t f + \xi \partial_x f = \frac{M^*[f] - f}{\varepsilon} \quad \text{in }]0, \infty[\times \mathbb{R} \times \mathbb{R}, \tag{3.1}$$

where $f = f(t, x, \xi) = (f_0(t, x, \xi), f_1(t, x, \xi)) \in \mathbb{R}^2$, $t > 0$, $x \in \mathbb{R}$, $\xi \in \mathbb{R}$,

$$f(t, x, \xi) \in D_\xi, \tag{3.2}$$

$$\rho(t, x) = \int_{\mathbb{R}} f_0(t, x, \xi)\, d\xi, \;\; \rho(t, x) u(t, x) = \int_{\mathbb{R}} f_1(t, x, \xi)\, d\xi, \tag{3.3}$$

$$M^*[f](t, x, \xi) = M^* \left(\rho(t, x), u(t, x), \xi \right), \tag{3.4}$$

$$M^*(\rho, u, \xi) = M(\min(1, \rho), u, \xi), \tag{3.5}$$

and M is the Maxwellian defined by

$$M(\rho, u, \xi) = \Big(\chi(\rho, \xi - u), ((1 - \theta) u + \theta \xi) \chi(\rho, \xi - u) \Big), \tag{3.6}$$

$$\chi(\rho, \xi) = c_{\gamma, \kappa} \left(a_\gamma^2 \rho^{\gamma - 1} - \xi^2 \right)_+^\lambda, \tag{3.7}$$

$$\theta = \frac{\gamma-1}{2}, \quad \lambda = \frac{1}{\gamma-1} - \frac{1}{2}, \quad c_{\gamma,\kappa} = a_\gamma^{-2/(\gamma-1)}/J_\lambda, \tag{3.8}$$

$$J_\lambda = \int_{-1}^{1} (1-z^2)^\lambda \, dz = \sqrt{\pi}\Gamma(\lambda+1)/\Gamma(\lambda+3/2), \quad a_\gamma = \frac{2\sqrt{\gamma\kappa}}{\gamma-1}. \tag{3.9}$$

We complete (3.1) by initial data

$$f(0,x,\xi) = f^0(x,\xi), \tag{3.10}$$

satisfying energy bounds. For $1 < \gamma < 3$, we take

$$D_\xi = D = \{(f_0, f_1) \in \mathbb{R}^2, \; f_0 > 0 \text{ or } f_1 = f_0 = 0\}, \tag{3.11}$$

while if $\gamma = 3$,

$$D_\xi = \{(f_0, f_1) \in \mathbb{R}^2, \; f_1 = \xi f_0 \text{ and } 0 \leq f_0 \leq 1/2\sqrt{3\kappa}\}, \tag{3.12}$$

and the Maxwellian simplifies in

$$M(\rho,u,\xi) = \frac{1}{2\sqrt{3\kappa}} \mathbb{1}_{u-\sqrt{3\kappa}\rho < \xi < u+\sqrt{3\kappa}\rho} K(\xi), \qquad K(\xi) = (1,\xi). \tag{3.13}$$

In this case a single scalar equation on f_0 can be written, the second one on f_1 being proportional to it.

The entropies involved in the inequality (1.11) are the so called weak entropies, defined by

$$\eta_S(\rho,u) = \int_{\mathbb{R}} \chi(\rho,\xi-u) S(\xi) \, d\xi, \quad S \text{ convex}, \tag{3.14}$$

and their entropy fluxes are defined by

$$G_S(\rho,u) = \int_{\mathbb{R}} \Big((1-\theta)u + \theta\xi \Big) \chi(\rho,\xi-u) S(\xi) \, d\xi. \tag{3.15}$$

To each entropy η_S we can associate a kinetic entropy, in the case $\gamma < 3$ they can be defined from a kernel by

$$H_S(f,\xi) = \int_{\mathbb{R}} \Phi(\rho(f,\xi), u(f,\xi), \xi, v) S(v) \, dv, \text{ for } f \neq 0, \quad H_S(0,\xi) = 0, \tag{3.16}$$

where

$$u(f,\xi) = \frac{f_1/f_0 - \theta\xi}{1-\theta}, \quad \rho(f,\xi) = a_\gamma^{-\frac{2}{\gamma-1}} \left(\left(\frac{f_1/f_0 - \xi}{1-\theta} \right)^2 + (f_0/c_{\gamma,\kappa})^{1/\lambda} \right)^{\frac{1}{\gamma-1}} \tag{3.17}$$

is the inverse relation for $f = M(\rho,u,\xi)$. The kernel Φ is symmetric in ξ, v and satisfies in particular $\Phi \geq 0$ and $\int_{\mathbb{R}} (1,v)\Phi(\rho,u,\xi,v) \, dv = M(\rho,u,\xi)$. For more details about this model, we refer to [4].

For $\gamma = 3$, the kinetic entropy is simply given for $f = (f_0, \xi f_0) \in D_\xi$, by

$$H_S(f, \xi) = S(\xi) f_0, \tag{3.18}$$

so that finally for any $1 \le \gamma \le 3$,

$$\eta_S(\rho, u) = \int_{\mathbb{R}} H_S(M(\rho, u, \xi), \xi)\, d\xi, \qquad G_S(\rho, u) = \int_{\mathbb{R}} \xi\, H_S(M(\rho, u, \xi), \xi)\, d\xi. \tag{3.19}$$

Let us finally introduce the kinetic invariant domains. The macroscopic invariant domains are defined as follows for any $\omega_{min} < \omega_{max}$,

$$\tilde{D} = \{(\rho, u) \in \mathbb{R}^+ \times \mathbb{R};\ \rho = 0 \ \mathit{or}\ \omega_{min} \le \omega_1 \le \omega_2 \le \omega_{max}\}, \tag{3.20}$$

where the Riemann invariants ω_1, ω_2 are given by

$$\omega_1 = u - a_\gamma \rho^{\frac{\gamma-1}{2}},\ \omega_2 = u + a_\gamma \rho^{\frac{\gamma-1}{2}}. \tag{3.21}$$

Their corresponding kinetic invariant domains are defined in the case $1 < \gamma < 3$ by

$$\tilde{D}_\xi = \{f \in D;\ f = 0 \ \mathit{or}\ \omega_{min} \le \omega_1(f, \xi) \le \omega_2(f, \xi) \le \omega_{max}\}, \tag{3.22}$$

where

$$\omega_1(f, \xi) = u(f, \xi) - a_\gamma \rho(f, \xi)^{\frac{\gamma-1}{2}},\ \omega_2(f, \xi) = u(f, \xi) + a_\gamma \rho(f, \xi)^{\frac{\gamma-1}{2}}, \tag{3.23}$$

and in the case $\gamma = 3$

$$\tilde{D}_\xi = \{f \in D_\xi;\ 0 \le f_0 \le \frac{1}{2\sqrt{3\kappa}} \mathbb{1}_{\omega_{min} < \xi < \omega_{max}}\}. \tag{3.24}$$

3.2 Properties of the kinetic entropy

We recall the value of the moments of M,

$$\int_{\mathbb{R}} M(\rho, u, \xi)\, d\xi = (\rho, \rho u), \qquad \int_{\mathbb{R}} \xi M(\rho, u, \xi)\, d\xi = (\rho u, \rho u^2 + \kappa \rho^\gamma),$$

and

$$\int_{\mathbb{R}} \frac{1}{2} \xi^2 M_0(\rho, u, \xi)\, d\xi = \frac{1}{2} \rho u^2 + \frac{\kappa}{\gamma - 1} \rho^\gamma \equiv \eta(\rho, u),$$

for every $\rho \ge 0$ and $u \in \mathbb{R}$. It results from the definition of a convex function that

Lemma 3.1 *For every $S : \mathbb{R} \to \mathbb{R}$ convex, we have*

$$\begin{aligned} &(\omega_2 - \omega_1) S(\xi) + (\omega_1 - \xi) S(\omega_2) + (\xi - \omega_2) S(\omega_1) \\ &\le 0 \ \mathit{if}\ \omega_1 \le \xi \le \omega_2, \\ &\ge 0 \ \mathit{if}\ \xi \le \omega_1 \le \omega_2 \ \mathit{or}\ \omega_1 \le \omega_2 \le \xi. \end{aligned} \tag{3.25}$$

As in [3], [4], we need a subdifferential inequality in order to prove boundedness of the entropy, namely

Proposition 3.2 (Subdifferential inequality) *If $S : \mathbb{R} \to \mathbb{R}$ is convex, of class C^1, then for every $\rho \geq 0$, $u, \xi \in \mathbb{R}$ and $f \in D_\xi$, we have*

$$H_S(f,\xi) \geq H_S(M(\rho,u,\xi),\xi) + T_S(\rho,u)\cdot(f - M(\rho,u,\xi)), \tag{3.26}$$

with

$$T_S(\rho,u) = \frac{1}{J_\lambda}\int_{-1}^{1}(1-z^2)^\lambda \begin{pmatrix} S(u+a_\gamma\rho^\theta z) + (\theta a_\gamma \rho^\theta z - u)S'(u+a_\gamma\rho^\theta z) \\ S'(u+a_\gamma\rho^\theta z) \end{pmatrix} dz, \tag{3.27}$$

which coincides with $\eta_S'(\rho,u)$ when $\rho > 0$, where prime denotes differentiation with respect to the conservative variables $(\rho, q \equiv \rho u)$. We also have if $f \neq 0$

$$(H_S'(f,\xi) - T_S(\rho,u))\cdot(M(\rho,u,\xi) - f) \leq 0, \tag{3.28}$$

where if $\gamma = 3$, $H_S'(f,\xi) = S(\xi)(1,0)$ by convention.

Proof. The case $\gamma < 3$ was treated in [4], thus let us assume that $\gamma = 3$. We set $\omega_1 = u - \sqrt{3\kappa}\rho$, $\omega_2 = u + \sqrt{3\kappa}\rho$, thus $\omega_2 - \omega_1 = 2\sqrt{3\kappa}\rho \geq 0$. We have for $\rho > 0$

$$\begin{aligned} T_S(\rho,u) &= \frac{1}{2\sqrt{3\kappa}\rho}\begin{pmatrix} (\sqrt{3\kappa}\rho - u)S(u+\sqrt{3\kappa}\rho) + (\sqrt{3\kappa}\rho + u)S(u-\sqrt{3\kappa}\rho) \\ S(u+\sqrt{3\kappa}\rho) - S(u-\sqrt{3\kappa}\rho) \end{pmatrix} \\ &= \frac{1}{\omega_2-\omega_1}\begin{pmatrix} -\omega_1 S(\omega_2) + \omega_2 S(\omega_1) \\ S(\omega_2) - S(\omega_1) \end{pmatrix}, \end{aligned} \tag{3.29}$$

and $T_S(0,u) = (S(u) - uS'(u), S'(u))$. We compute

$$\begin{aligned} A &= H_S(f,\xi) - H_S(M(\rho,u,\xi),\xi) - T_S(\rho,u)\cdot(f - M(\rho,u,\xi)) \\ &= [f_0 - M_0(\rho,u,\xi)][S(\xi) - T_S(\rho,u)\cdot(1,\xi)]. \end{aligned} \tag{3.30}$$

Then, if $\rho > 0$,

$$S(\xi) - T_S(\rho,u)\cdot(1,\xi) = [(\omega_2-\omega_1)S(\xi) + (\omega_1-\xi)S(\omega_2) + (\xi-\omega_2)S(\omega_1)]/(\omega_2-\omega_1), \tag{3.31}$$

and if $\rho = 0$,

$$S(\xi) - T_S(\rho,u)\cdot(1,\xi) = S(\xi) - S(u) - S'(u)(\xi - u) \geq 0. \tag{3.32}$$

If $\xi \leq \omega_1$ or $\xi \geq \omega_2$, then $M_0(\rho,u,\xi) = 0$, and $S(\xi) - T_S(\rho,u)\cdot(1,\xi) \geq 0$ thanks to Lemma 3.1, thus $A \geq 0$. If $\omega_1 < \xi < \omega_2$, then $M_0(\rho,u,\xi) = \frac{1}{2\sqrt{3\kappa}} \geq f_0$ and $S(\xi) - T_S(\rho,u)\cdot(1,\xi) \leq 0$, thus $A \geq 0$ also, which proves (3.26) and (3.28). $\square$

The kinetic entropy associated to the physical energy is denoted by $H = H_{v^2/2}$. We recall that $H \geq 0$. Applying the previous result to $\min(1,\rho)$ and integrating in ξ, we get a minimization principle.

Corollary 3.3 (Entropy minimization principle) *Assume that $S : \mathbb{R} \to \mathbb{R}$ is convex of class C^1 and such that $|S(v)| \leq B(1+v^2)$ for some $B \geq 0$. Consider $f \in L^1(\mathbb{R}_\xi)$ such that $f \in D_\xi$ a.e. and*

$$\int_{\mathbb{R}} H(f(\xi), \xi)\, d\xi < \infty. \tag{3.33}$$

Then $H_S(f(\xi), \xi)$ and $H_S(M^*[f](\xi), \xi)$ lie in $L^1(\mathbb{R}_\xi)$, and setting $(\rho, \rho u) = \int_{\mathbb{R}} f\, d\xi$ we have

$$\int_{\mathbb{R}} H_S(M^*[f](\xi), \xi)\, d\xi + T_S(\min(1, \rho), u) \cdot (1, u)(\rho - 1)_+ \leq \int_{\mathbb{R}} H_S(f(\xi), \xi)\, d\xi. \tag{3.34}$$

For further reference, we also provide the following obvious estimate.

Lemma 3.4 *For every $\rho \geq 0$ and $u, \xi \in \mathbb{R}$, we have $0 \leq M_0^*(\rho, u, \xi) \leq M_0(\rho, u, \xi)$.*

3.3 Existence for the BGK model

We proceed as in [3] in order to apply Schauder's theorem and to get the existence of global solutions. We notice that the proofs simplify for $\gamma = 3$ because $H(f, \xi) = f_0\, \xi^2/2$ is linear and because the kinetic system becomes in fact a rank-one model, and as a consequence we mainly only study the convergence of the first component which is non-negative. In any case we get the following result.

Theorem 3.5 *Assume that $f^0 \in L^1(\mathbb{R}_x \times \mathbb{R}_\xi)$ satisfies $f^0(x, \xi) \in D_\xi$ a.e. in $\mathbb{R} \times \mathbb{R}$ and*

$$\iint_{\mathbb{R}\times\mathbb{R}} H(f^0(x, \xi), \xi)\, dx d\xi = C_H^0 < \infty. \tag{3.35}$$

Then there exists a solution f to (3.1)-(3.12) satisfying

$$f \in (C \cap L^\infty)_t([0, \infty[, L^1(\mathbb{R}_x \times \mathbb{R}_\xi)), \tag{3.36}$$

$$\forall t \geq 0,\ f(t, x, \xi) \in D_\xi \text{ a.e. in } \mathbb{R} \times \mathbb{R}, \tag{3.37}$$

$$\forall t \geq 0,\ \iint_{\mathbb{R}\times\mathbb{R}} f_0(t, x, \xi)\, dx d\xi \leq \iint_{\mathbb{R}\times\mathbb{R}} f_0^0(x, \xi)\, dx d\xi, \tag{3.38}$$

$$\forall t \geq 0,\ \iint_{\mathbb{R}\times\mathbb{R}} H(f(t, x, \xi), \xi)\, dx d\xi \leq C_H^0, \tag{3.39}$$

$$\partial_t\Big(\int_{\mathbb{R}} f_0\, d\xi\Big) + \partial_x\Big(\int_{\mathbb{R}} \xi f_0\, d\xi\Big) = -\frac{(\rho - 1)_+}{\varepsilon} \leq 0, \tag{3.40}$$

$$
\begin{aligned}
&\frac{d}{dt}\iint\limits_{\mathbb{R}\times\mathbb{R}} H(f(t,x,\xi),\xi)\,dxd\xi \\
= \;& -\frac{1}{\varepsilon}\iint\limits_{\mathbb{R}\times\mathbb{R}} \Big(H'(f,\xi) - (\frac{\gamma\kappa}{\gamma-1}(\rho^*)^{\gamma-1} - \frac{u^2}{2}, u)\Big)\cdot(f - M^*[f])\,dxd\xi \\
& -\frac{1}{\varepsilon}\Big(\frac{\gamma\kappa}{\gamma-1}(\rho^*)^{\gamma-1} + \frac{u^2}{2}\Big)(\rho-1)_+ \qquad (3.41)\\
\le \;& 0,
\end{aligned}
$$

where $(\rho,\rho u) = \int_{\mathbb{R}} f\,d\xi$ *and* $\rho^* = \min(1,\rho)$.

We notice that we have a representation of the solution f,

$$
f(t,x,\xi) = f^0(x - t\xi,\xi)e^{-t/\varepsilon} + \frac{1}{\varepsilon}\int_0^t e^{-s/\varepsilon}M^*[f](t-s, x-s\xi,\xi)\,ds. \qquad (3.42)
$$

3.4 Kinetic invariant domains

By using Corollary 3.3 and Lemma 2.2 in a computation similar to (3.41), we get

Proposition 3.6 *Assume that* $S:\mathbb{R}\to\mathbb{R}$ *is convex of class* C^1 *and such that* $0\le S(v)\le B(1+v^2)$ *for some* $B\ge 0$*. Then, with* f *the solution of Theorem 3.5,*

$$
\iint\limits_{\mathbb{R}\times\mathbb{R}} H_S(f(t,x,\xi),\xi)\,dxd\xi \le \iint\limits_{\mathbb{R}\times\mathbb{R}} H_S(f^0(x,\xi),\xi)\,dxd\xi. \qquad (3.43)
$$

Noticing that the invariant domain (3.20) is stable by the projection $(\rho,u)\mapsto(\min(1,\rho),u)$, this allows to obtain invariant domains and bounds for ρ, u, f and $M^*[f]$.

Theorem 3.7 *For any* $\omega_{min} < \omega_{max}$*, the system (3.1) has the property that* $\tilde{D}_\xi$ *is a family of convex kinetic invariant domains. Moreover, the set* $\tilde{D}_\xi$ *is associated with the invariant domain* $\tilde{D}$ *in the sense that*

$$
\forall(\rho,u)\in\tilde{D},\; M(\rho,u,\xi)\in\tilde{D}_\xi \text{ a.e.}\xi, \qquad (3.44)
$$

and

$$
\begin{aligned}
&\text{for any } f(\xi)\in L^1(\mathbb{R}_\xi) \text{ such that } f(\xi)\in\tilde{D}_\xi \text{ a.e. } \xi,\\
&(\rho,u)\in\tilde{D} \text{ with } (\rho,\rho u) = \int_{\mathbb{R}} f(\xi)\,d\xi. \qquad (3.45)
\end{aligned}
$$

In particular, if the initial data of Theorem 3.5 satisfies $f^0(x,\xi)\in\tilde{D}_\xi$ *for a.e.* x, ξ*, then, denoting by* f *the solution obtained in Theorem 3.5,* (ρ,u) *defined by (3.3) verify* $\forall t\ge 0,\; (\rho(t,x),u(t,x))\in\tilde{D}$ *for a.e.* x.
Besides $f(t,x,\xi)\in\tilde{D}_\xi$ $\forall t\ge 0$ *and consequently* f *has compact support with respect to* ξ, $supp_\xi f\subset[\omega_{min},\omega_{max}]$.
Furthermore ρ, u, f, $M[f]$ *and* $M^*[f]$ *are uniformly bounded in* L^∞.

Proof. Using the functions $S(v) = (v-\omega_{max})_+^2$, $S(v) = (\omega_{min}-v)_+^2$ in Proposition 3.6, and the fact that $\mathbb{1}_{(\xi-u)^2<3\kappa\rho^2}\le\mathbb{1}_{\omega_{min}<\xi<\omega_{max}}\Leftrightarrow]u-\sqrt{3\kappa}\rho, u+\sqrt{3\kappa}\rho[\subset]\omega_{min},\omega_{max}[$, we obtain the result by adapting the proof of [4]. □

3.5 Relaxation limit via compensated compactness

In this section, we prove the stability Theorem 1.1 and the existence Theorem 1.2 in the case of nonzero pressure.

Proof of Theorem 1.1 for nonzero pressure.
Let (ρ_n, u_n, Q_n, v_n) satisfy the assumptions of Theorem 1.1. Then

$$\partial_t \eta_S(\rho_n, u_n) + \partial_x G_S(\rho_n, u_n) \leq Q_n \eta_S'(1, v_n) \cdot (1, v_n), \tag{3.46}$$

and since the right-hand side is bounded in $\mathcal{M}_{loc}$, we can apply the compensated compactness result of [21] and it gives that, up to a subsequence, $(\rho_n, \rho_n u_n)$ converge a.e. in $]0,\infty[\times\mathbb{R}$ when $n \to \infty$ to some functions $(\rho, \rho u)$. Using Lemmas 2.1 and 2.2, we can pass to the limit in (3.46), while the limit in (1.18)-(1.19) is obvious. $\square$

We turn now to the existence result (Theorem 1.2) and prove the relaxation of (3.1) to (1.9)-(1.11).

Theorem 3.8 *Let us denote by f_ε the solution of Theorem 3.5 with the same initial data $f^0(x,\xi) \in L^1(\mathbb{R}\times\mathbb{R})$ that satisfies $f^0(x,\xi) \in \tilde{D}_\xi$ a.e. for some $\omega_{min} < \omega_{max}$, and the energy bound (3.35). Then $(\rho_\varepsilon, u_\varepsilon)$ defined by (3.3) are uniformly bounded in L^∞, and passing if necessary to subsequences, $(\rho_\varepsilon, \rho_\varepsilon u_\varepsilon)$ converge a.e. in $]0,\infty[\times\mathbb{R}$ when $\varepsilon \to 0$ to $(\rho, \rho u)$, $-(\rho_\varepsilon - 1)_+/\varepsilon \rightharpoonup Q$, $-(\rho_\varepsilon - 1)_+ u_\varepsilon/\varepsilon \rightharpoonup Qv$, where (ρ, u, Q, v) have the regularities (1.15)-(1.17) and satisfy (1.18)-(1.19) and (1.10)-(1.11), with initial data $(\rho^0, \rho^0 u^0) = \int f^0 d\xi$.*

Proof. The bounds of Theorem 3.7 give that ρ_ε, u_ε f_ε and the support in ξ of f_ε are uniformly bounded. Then, the renormalization result for a transport equation of [7] gives for any convex C^1 function S

$$\partial_t H_S(f_\varepsilon, \xi) + \xi \partial_x H_S(f_\varepsilon, \xi) = H_S'(f_\varepsilon, \xi) \cdot (M^*[f_\varepsilon] - f_\varepsilon)/\varepsilon.$$

By integration in ξ, it yields

$$\begin{aligned} &\partial_t \int_{\mathbb{R}} H_S(f_\varepsilon, \xi)\, d\xi + \partial_x \int_{\mathbb{R}} \xi H_S(f_\varepsilon, \xi)\, d\xi \\ = &\frac{1}{\varepsilon} \int_{\mathbb{R}} (H_S'(f_\varepsilon, \xi) - T_S(\rho_\varepsilon^*, u_\varepsilon)) \cdot (M^*[f_\varepsilon] - f_\varepsilon)\, d\xi \\ &+ T_S(\rho_\varepsilon^*, u_\varepsilon) \cdot \int_{\mathbb{R}} \frac{M^*[f_\varepsilon] - f_\varepsilon}{\varepsilon}\, d\xi, \end{aligned} \tag{3.47}$$

with $\rho_\varepsilon^* = \min(1, \rho_\varepsilon)$. Define

$$Q_\varepsilon = -\frac{(\rho_\varepsilon - 1)_+}{\varepsilon} \leq 0. \tag{3.48}$$

Then

$$\int_{\mathbb{R}} \frac{M^*[f_\varepsilon] - f_\varepsilon}{\varepsilon}\, d\xi = Q_\varepsilon\ (1, u_\varepsilon), \tag{3.49}$$

and

$$Q_\varepsilon\ T_S(\rho_\varepsilon^*, u_\varepsilon) = Q_\varepsilon\ T_S(1, u_\varepsilon) = Q_\varepsilon\ \eta_S'(1, u_\varepsilon), \tag{3.50}$$

thus by (3.28)

$$\begin{aligned}
& \partial_t \eta_S(\rho_\varepsilon^*, u_\varepsilon) + \partial_x G_S(\rho_\varepsilon^*, u_\varepsilon) \\
\leq \; & \partial_t \int_{\mathbb{R}} \Big(H_S(M^*[f_\varepsilon], \xi) - H_S(f_\varepsilon, \xi) \Big)\, d\xi \\
& + \partial_x \int_{\mathbb{R}} \xi \Big(H_S(M^*[f_\varepsilon], \xi) - H_S(f_\varepsilon, \xi) \Big)\, d\xi \\
& + Q_\varepsilon \eta_S'(1, u_\varepsilon) \cdot (1, u_\varepsilon).
\end{aligned} \tag{3.51}$$

Next, we observe that Q_ε is bounded in $L^1(]0,\infty[\times\mathbb{R})$ since

$$\iint -Q_\varepsilon dx dt \leq \iint_{\mathbb{R}\times\mathbb{R}} f_0^0(x,\xi)\, dx d\xi, \tag{3.52}$$

as a consequence of (3.40). We deduce that $(\rho_\varepsilon - 1)_+$ tends to 0 in L^1, and after extraction of a subsequence, $\rho_\varepsilon - \rho_\varepsilon^* \to 0$ a.e. Provided that $f_\varepsilon - M^*[f_\varepsilon] \to 0$ a.e. t, x, ξ, using (3.51), we can then apply the compensated compactness result of [21] which gives that up to a subsequence, $(\rho_\varepsilon^*, \rho_\varepsilon^* u_\varepsilon)$ (and also $(\rho_\varepsilon, \rho_\varepsilon u_\varepsilon)$) converge a.e. in $]0,\infty[\times\mathbb{R}$ when $\varepsilon \to 0$ to some $(\rho, \rho u)$, with $(\rho, u) \in \tilde{D}$, $0 \leq \rho \leq 1$. By applying Lemma 2.1, we get $Q_\varepsilon \rightharpoonup Q$, $Q_\varepsilon u_\varepsilon \rightharpoonup Qv$ with $Q \in \mathcal{M}([0,\infty[\times\mathbb{R})$, $v \in L^\infty(Q)$. Using again $f_\varepsilon - M^*[f_\varepsilon] \to 0$ a.e. t, x, ξ and with Lemma 2.2, the limit in (3.51) gives (1.11). A direct integration of (3.1) also gives (1.18)-(1.19) at the limit. Thus it only remains to prove that $f_\varepsilon - M^*[f_\varepsilon] \to 0$ a.e. t, x, ξ. This can be justified as follows. From (3.41), the integral

$$\iiint_{]0,T[\times\mathbb{R}\times\mathbb{R}} \Big(H'(f_\varepsilon, \xi) - T_{\frac{v^2}{2}}(\rho_\varepsilon^*, u_\varepsilon) \Big) \cdot \frac{M^*[f_\varepsilon] - f_\varepsilon}{\varepsilon}\, dt dx d\xi \tag{3.53}$$

is bounded uniformly in ε. Thus, if $\gamma < 3$, we can adapt the dissipation result of [4] and get that $f_\varepsilon - M^*[f_\varepsilon] \to 0$ a.e. t, x, ξ, replacing the use of the identity $\int_{\mathbb{R}} (f_\varepsilon)_0\, d\xi = \int_{\mathbb{R}} (M[f_\varepsilon])_0\, d\xi$ by

$$0 \leq \iiint_{(t,x)\in B} \Big((f_\varepsilon)_0 - (M^*[f_\varepsilon])_0 \Big)\, dt dx d\xi \leq K\varepsilon,$$

for any bounded set B. For $\gamma = 3$, the study is a little bit different and we refer to the next section for the precise analysis, that leads to the same result $f_\varepsilon - M^*[f_\varepsilon] \to 0$ a.e. t, x, ξ. □

Proof of Theorem 1.2. Let ρ^0, u^0 satisfy $\rho^0 \in L^1(\mathbb{R})$, $0 \leq \rho^0 \leq 1$ and $u^0 \in L^\infty(\mathbb{R})$. Then there exists ω_{min}, ω_{max} such that $(\rho^0, u^0) \in \tilde{D}$ a.e. We take $f^0(x,\xi) = M(\rho^0(x), u^0(x), \xi) \in \tilde{D}_\xi$. Since

$$\iint_{\mathbb{R}\times\mathbb{R}} H(M(\rho^0, u^0, \xi), \xi) dx d\xi = \int_{\mathbb{R}} \eta(\rho^0, u^0) dx < \infty,$$

we can apply Theorem 3.8 and we get the result. □

3.6 Relaxation limit for $\gamma = 3$ via averaging lemma

In this section, we complete the proof of Theorem 3.8 in the case $\gamma = 3$ by proving that $f_\varepsilon - M^*[f_\varepsilon] \to 0$ a.e. t, x, ξ also in this case, and we give an alternate compactness argument via averaging lemma instead of compensated compactness, following the ideas of [14], [23], [11], [25] and [26]. Let f_ε be the solution of Theorem 3.5 with the same initial data $f^0(x,\xi)$ and $(\rho_\varepsilon, u_\varepsilon)$ be the approximate solutions to (1.9) defined by (3.3). In order to prove the compactness of ρ_ε and $\rho_\varepsilon u_\varepsilon$, we use the compactness averaging lemma of [14] in the following form.

Proposition 3.9 *Let $g_\varepsilon \in L^\infty(]0,\infty[\times\mathbb{R}\times\mathbb{R})$ satisfy*

$$\partial_t g_\varepsilon + \xi \partial_x g_\varepsilon = -\lambda_\varepsilon - \partial^2_{\xi\xi}\mu_\varepsilon, \tag{3.54}$$

for some nonnegative measures λ_ε, μ_ε locally bounded uniformly in ε. If g_ε is bounded in L^∞ uniformly in ε, then $\int_{\mathbb{R}} g_\varepsilon(t,x,\xi)\psi(\xi)\,d\xi$ belongs to a compact set of $L^p_{loc}(]0,\infty[\times\mathbb{R})$, $1 < p < \infty$, for any $\psi \in C^\infty_c(\mathbb{R})$.

Proposition 3.10 *The solution f_ε of Theorem 3.5 for $\gamma = 3$ satisfies*

$$\partial_t (f_\varepsilon)_0 + \xi \partial_x (f_\varepsilon)_0 = -\lambda_\varepsilon - \partial^2_{\xi\xi}\mu_\varepsilon, \tag{3.55}$$

where $(\lambda_\varepsilon)_{\varepsilon>0}$ and $(\mu_\varepsilon)_{\varepsilon>0}$ are nonnegative measures bounded uniformly in ε. Therefore, by Proposition 3.9, ρ_ε and $\rho_\varepsilon u_\varepsilon$ are locally compact.

Proof. We set $\lambda_\varepsilon = (M_0[f_\varepsilon] - M_0^*[f_\varepsilon])/\varepsilon$ and $h_\varepsilon = ((f_\varepsilon)_0 - M_0[f_\varepsilon])/\varepsilon$. We have $\lambda_\varepsilon \geq 0$ thanks to Lemma 3.4. Since $\int_{\mathbb{R}} h_\varepsilon\, d\xi = 0$, $\int_{\mathbb{R}} \xi h_\varepsilon\, d\xi = 0$ and h_ε has compact support in ξ, there exists a distribution μ_ε with compact support in ξ such that $h_\varepsilon = \partial^2_{\xi\xi}\mu_\varepsilon$. Thus we have (3.55). We integrate this equality and get

$$\iiint_{]0,T[\times\mathbb{R}\times\mathbb{R}} \lambda_\varepsilon \;\leq\; \iint_{\mathbb{R}\times\mathbb{R}} f_0^0(x,\xi)\,dxd\xi.$$

Take now a test function $\varphi(t,x,\xi) = \varphi_1(t,x)\varphi_2(\xi)$, with φ_1, φ_2 nonnegative and of class C^∞_c, and define $\phi \in C^\infty$ by $\varphi_2 = \partial^2_{\xi\xi}\phi$. We have that ϕ is convex, and by the entropy minimization principle

$$\langle \mu_\varepsilon, \varphi\rangle = \langle \partial^2_{\xi\xi}\mu_\varepsilon, \varphi_1\phi\rangle = \iiint_{]0,T[\times\mathbb{R}\times\mathbb{R}} \varphi_1\phi h_\varepsilon \;\geq 0,$$

thus $\mu_\varepsilon \geq 0$. We integrate now (3.55) against $\xi^2/2$, and we get

$$\iiint_{]0,T[\times\mathbb{R}\times\mathbb{R}} \mu_\varepsilon \;\leq\; \iint_{\mathbb{R}\times\mathbb{R}} \frac{\xi^2}{2} f_0^0(x,\xi)\,dxd\xi,$$

which concludes the proof. $\square$

Proof of Theorem 3.8 when $\gamma = 3$.
The beginning of the proof is the same, we can replace compensated compactness by Proposition 3.10, but it remains to get that $f_\varepsilon - M^*[f_\varepsilon] \to 0$ a.e. t, x, ξ. For a subsequence, we have

$$\rho_\varepsilon \to \rho, \qquad \rho_\varepsilon u_\varepsilon \to \rho u \qquad \text{a.e. } t, x, \tag{3.56}$$

with $(\rho, u) \in \tilde{D}$. The bound (3.53) implies that for a subsequence,

$$\frac{1}{2}\Big[(\xi - u_\varepsilon)^2 - 3\kappa(\rho_\varepsilon^*)^2\Big]\Big[(f_\varepsilon)_0 - M_0(\rho_\varepsilon^*, u_\varepsilon, \xi)\Big] \to 0 \qquad \text{a.e. } t, x, \xi. \tag{3.57}$$

Since $\rho_\varepsilon - \rho_\varepsilon^* \to 0$ a.e., it gives

$$\Big[(\xi - u_\varepsilon)^2 - 3\kappa\rho_\varepsilon^2\Big]\Big[(f_\varepsilon)_0 - M_0(\rho_\varepsilon, u_\varepsilon, \xi)\Big] \to 0 \qquad \text{a.e. } t, x, \xi, \tag{3.58}$$

and we recall that this quantity is nonnegative.

We set $E = \{(t,x) \in]0,T[\times\mathbb{R};\ \rho(t,x) > 0\}$. On E, we have $u_\varepsilon \to u$ a.e., and from (3.58), $(f_\varepsilon)_0 \to M_0(\rho, u, \xi)$ a.e. ξ since the set of ξ such that $(\xi - u(t,x))^2 = 3\kappa\rho^2(t,x)$ has measure zero. Then, for any bounded domain B in (t,x), by passing to the limit as $\varepsilon \to 0$ in

$$\begin{aligned}&\iint_{(t,x)\in B\cap E}\int_\xi (f_\varepsilon)_0\, dtdxd\xi + \iint_{\substack{(t,x)\in B\\(t,x)\notin E}}\int_\xi (f_\varepsilon)_0\, dtdxd\xi\\ = &\iint_{(t,x)\in B\cap E}\int_\xi M_0[f_\varepsilon]\, dtdxd\xi + \iint_{\substack{(t,x)\in B\\(t,x)\notin E}}\int_\xi M_0[f_\varepsilon]\, dtdxd\xi,\end{aligned}$$

we get that

$$\iint_{\substack{(t,x)\in B\\(t,x)\notin E}}\int_\xi (f_\varepsilon)_0(t,x,\xi)\, dtdxd\xi \to \iint_{\substack{(t,x)\in B\\(t,x)\notin E}}\int_\xi M_0(\rho,u,\xi)\, dtdxd\xi = 0.$$

Finally we get that, up to an extraction,

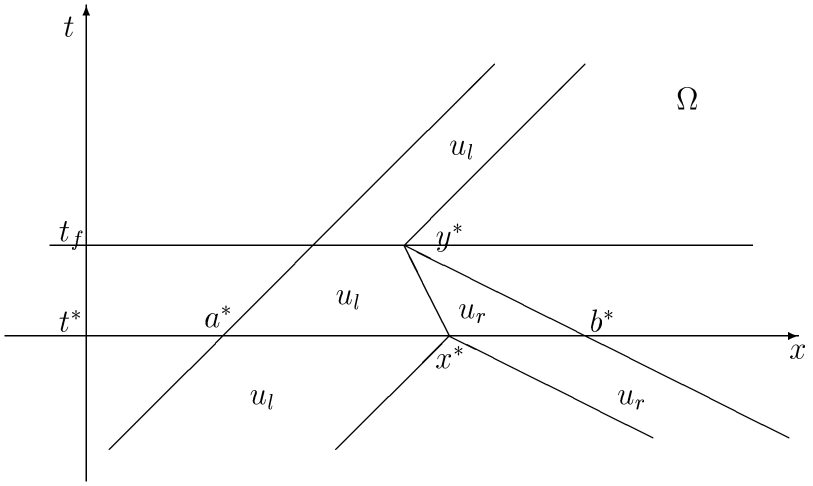
$$(f_\varepsilon)_0 \to M_0(\rho, u, \xi) \qquad \text{a.e. } t, x, \xi. \tag{3.59}$$

Since $(f_\varepsilon)_1 = \xi(f_\varepsilon)_0$, we also get $(f_\varepsilon)_1 \to \xi M_0(\rho,u,\xi) = M_1(\rho,u,\xi)$ a.e., and the result follows. □

4 Pressureless model

This section is devoted to the proof of Theorems 1.1-1.3 when $p = 0$. We build a sticky blocks dynamics with mass loss that solves the system for particular data, that is used to approximate arbitrary initial data.

Figure 1:



The analysis is similar to that of [2], and differs from the one for the system of pressureless gases without constraint, that gives Dirac distributions on ρ in finite time (see [5], [9], [12]). The entropies and entropy fluxes are defined by

$$\eta_S(\rho,u) = \rho S(u), \quad G_S(\rho,u) = \rho u S(u), \tag{4.1}$$

for any $S : \mathbb{R} \to \mathbb{R}$ convex. They satisfy

$$\eta'_S(1,v)\cdot(1,v) = S(v), \tag{4.2}$$

thus the entropy inequalities (1.11) write

$$\partial_t(\rho S(u)) + \partial_x(\rho u S(u)) \le Q S(v). \tag{4.3}$$

4.1 Sticky blocks dynamics

Let us consider a volume fraction $\rho(t,x)$ and a momentum density $\rho(t,x)u(t,x)$ given by

$$\rho(t,x) = \sum_{i=1}^{n} \mathbb{1}_{a_i(t)<x<b_i(t)}, \quad \rho(t,x)u(t,x) = \sum_{i=1}^{n} u_i(t)\mathbb{1}_{a_i(t)<x<b_i(t)}, \tag{4.4}$$

with $a_1(t) < b_1(t) \le a_2(t) < b_2(t) \le \cdots \le a_n(t) < b_n(t)$. The time evolution is defined as follows. The number of blocks n indeed depends on t, but is piecewise constant. As long as the blocks do not meet, they move at constant velocity $u_i(t)$. When two blocks collide at a time t^*, the dynamics is exhibited in Figure 1, and is defined as follows. The volume fraction ρ is given locally by

$$\rho(t,x) = \begin{cases} \mathbb{1}_{a_l(t)<x<b_l(t)} + \mathbb{1}_{a_r(t)<x<b_r(t)} & \text{if } t < t^*, \\ \mathbb{1}_{a_l(t)<x<b_r(t)} & \text{if } t^* \le t < t_f, \\ \mathbb{1}_{a(t)<x<b(t)} & \text{if } t \ge t_f. \end{cases} \tag{4.5}$$

and the momentum density ρu by

$$\rho u(t,x) = \begin{cases} u_l \mathbb{1}_{a_l(t)<x<b_l(t)} + u_r \mathbb{1}_{a_r(t)<x<b_r(t)} & \text{if } t < t^*, \\ u_l \mathbb{1}_{a_l(t)<x<c(t)} + u_r \mathbb{1}_{c(t)<x<b_r(t)} & \text{if } t^* \leq t < t_f, \\ u^* \mathbb{1}_{a(t)<x<b(t)} & \text{if } t \geq t_f. \end{cases} \tag{4.6}$$

We have $a_l(t) = a^* + u_l(t-t^*)$, $b_l(t) = x^* + u_l(t-t^*)$, $a_r(t) = x^* + u_r(t-t^*)$, $b_r(t) = b^* + u_r(t-t^*)$, $c(t) = x^* + u_{loss}(t-t^*)$, $a(t) = a_l(t_f) + u^*(t-t_f)$ and $b(t) = b_r(t_f) + u^*(t-t_f)$, with $u_r \leq u_{loss} \leq u_l$,

$$t_f = \min\Big(t^* + (b^* - x^*)/(u_{loss} - u_r), t^* + (x^* - a^*)/(u_l - u_{loss})\Big), \tag{4.7}$$

$$u^* = \begin{cases} u_l & \text{if } (x^* - a^*)/(u_l - u_{loss}) > (b^* - x^*)/(u_{loss} - u_r), \\ u_r & \text{if } (x^* - a^*)/(u_l - u_{loss}) < (b^* - x^*)/(u_{loss} - u_r). \end{cases} \tag{4.8}$$

One of the right or left block disappears, and in case of equality in (4.8), $a(t) = b(t)$ and all the mass disappears.

When more than two blocks collide at the same time, or if a block collides at a time t_1 with two blocks that are colliding since a time t^* with $t^* < t_1 < t_f$, we perform the collisions locally at each interface between two blocks. We notice that with this construction, ρ and ρu are both continuous in time with values in $L^1(\mathbb{R})$.

It only remains to define the interface velocity u_{loss}, and we can indeed take any relation

$$u_{loss} = \Phi(u_l, u_r), \tag{4.9}$$

where Φ is defined for $u_l > u_r$ and satisfies

$$u_r \leq \Phi(u_l, u_r) \leq u_l. \tag{4.10}$$

4.2 Properties of sticky blocks

We have the following consistency result.

Theorem 4.1 *There exists a nonpositive measure $Q(t,x) \in \mathcal{M}(]0,\infty[\times\mathbb{R})$ with concentrations only in x, such that with $\rho(t,x)$ and $u(t,x)$ defined by (4.4), and with the above defined dynamics, we get a solution to the weak formulation (1.18)-(1.19), (4.3), with $v = u_l + u_r - u_{loss}$.*

Proof. As long as there is no collision, each block moves at the constant velocity u_i, and (ρ, u) solves the pressureless Euler system. In a neighborhood of initial data, the proof is given in [2]. Let us now look at the case of a collision of two blocks at a time t^*. Let $\varphi(t,x)$ be a smooth function with support in Ω of Figure 1 such that there is no other contact in Ω. We have

$$\begin{aligned} &\int_{t^*}^{t_f} \int_{a_l(t)}^{c(t)} [S(u_l)\partial_t \varphi(t,x) + S(u_l) u_l \partial_x \varphi(t,x)] \, dt dx \\ = \quad & S(u_l) \left\{ \int_{t^*}^{t_f} \left[\frac{d}{dt} \int_{a_l(t)}^{c(t)} \varphi(t,x) \, dx - u_{loss} \varphi(t, c(t)) + u_l \varphi(t, a_l(t)) \right] dt \right. \end{aligned}$$

$$
\begin{aligned}
& + \int_{t^*}^{t_f} u_l\Big(\varphi(t,c(t)) - \varphi(t,a_l(t))\Big)\,dt\Big\} \\
= \;& S(u_l)\left\{\int_{a_l(t_f)}^{c(t_f)} \varphi(t_f,x)\,dx - \int_{a^*}^{x^*} \varphi(t^*,x)\,dx + \int_{t^*}^{t_f} (u_l - u_{loss})\varphi(t,c(t))\,dt\right\},
\end{aligned}
$$

and similarly

$$
\begin{aligned}
& \int_{t^*}^{t_f}\int_{c(t)}^{b_r(t)} [S(u_r)\partial_t\varphi(t,x) + S(u_r)u_r\partial_x\varphi(t,x)]\ dtdx \\
= \;& S(u_r)\left\{\int_{c(t_f)}^{b_r(t_f)} \varphi(t_f,x)\,dx - \int_{x^*}^{b^*} \varphi(t^*,x)\,dx + \int_{t^*}^{t_f} (u_{loss} - u_r)\varphi(t,c(t))\,dt\right\}.
\end{aligned}
$$

Thus by addition

$$
\begin{aligned}
& \int_{t^*}^{t_f}\int_{\mathbb{R}} [\rho S(u)\partial_t\varphi + \rho S(u)u\partial_x\varphi]\ dtdx \\
= \;& \int_{\mathbb{R}}\Big(\rho S(u)\varphi\Big)(t_f,x)\,dx - \int_{\mathbb{R}}\Big(\rho S(u)\varphi\Big)(t^*,x)\,dx \\
& + \int_{t^*}^{t_f}\Big[(u_l - u_{loss})S(u_l) + (u_{loss} - u_r)S(u_r)\Big]\varphi(t,c(t))\,dt,
\end{aligned}
$$

and finally

$$
\begin{aligned}
& \langle \partial_t(\rho S(u)) + \partial_x(\rho u S(u)), \varphi\rangle \\
= \;& -\langle \rho S(u), \partial_t\varphi\rangle - \langle \rho u S(u), \partial_x\varphi\rangle \\
= \;& -\int_{t^*}^{t_f}\Big[(u_l - u_{loss})S(u_l) + (u_{loss} - u_r)S(u_r)\Big]\varphi(t,c(t))\,dt.
\end{aligned}
$$

Therefore we have (1.31) with

$$
Q_S = -\Big[(u_l - u_{loss})S(u_l) + (u_{loss} - u_r)S(u_r)\Big]\mathbb{1}_{t^*<t<t_f}\delta(x - c(t)). \tag{4.11}
$$

For $S \equiv 1$, we get

$$
Q(t,x) = -(u_l - u_r)\mathbb{1}_{t^*<t<t_f}\delta(x - c(t)), \tag{4.12}
$$

and for $S = \mathrm{Id}$, we get

$$
(Qv)(t,x) = -(u_l - u_r)(u_l + u_r - u_{loss})\mathbb{1}_{t^*<t<t_f}\delta(x - c(t)). \tag{4.13}
$$

Since $u_l > u_r$ is necessary to have the collision, we have $Q \leq 0$, and (4.13) gives $v = u_l + u_r - u_{loss}$. We prove now the entropy weak product inequality. Since

$$
QS(v) = -(u_l - u_r)S(u_l + u_r - u_{loss})\mathbb{1}_{t^*<t<t_f}\delta(x - c(t)), \tag{4.14}
$$

we obtain, by using that $u_r \leq u_{loss} \leq u_l$, which imply that $u_r \leq v \leq u_l$, that for S convex,

$$\partial_t(\rho S(u)) + \partial_x(\rho u S(u)) = Q_S \leq QS(v), \tag{4.15}$$

which concludes the proof. □

Similarly to the model with pressure Lagrange multiplier of [8], we have the discrete Oleinik entropy inequality

$$u_i(t) - u_{i-1}(t) \leq \frac{a_i(t) - b_{i-1}(t)}{t} \quad \text{for } 2 \leq i \leq n. \tag{4.16}$$

Extending the value of $u(t,x)$ to all x by linear interpolation between two blocks and by putting a constant at infinity, we get that the sticky blocks dynamics satisfies Oleinik's condition (1.26). Then, we observe that according to (4.12)-(4.13), the formulas (1.8) hold, and in particular we have

$$|Q| \leq |\partial_x u|, \tag{4.17}$$

thus $Q \in L^\infty_{loc}(]0,\infty[, \mathcal{M}_{loc}(\mathbb{R}))$. We also have the maximum principle (1.28) and the entropy equality (1.31) with

$$Q_S(t,x) = \left[\frac{u_l - u_{loss}}{u_l - u_r} S(u_l) + \frac{u_{loss} - u_r}{u_l - u_r} S(u_r)\right] Q(t,x). \tag{4.18}$$

Thus, we get (1.32) for every S continuous.

4.3 Existence of a solution

We do not detail the proof of Theorem 1.1, since it is very close to that of [2]. The same argument is used to prove that for any S continuous, $\rho_n S(u_n) \rightharpoonup \rho S(u)$. Only Lemma 2.1 is new and gives directly the entropy weak product inequality.

Remark 4.1 In Theorem 1.1, we also have the existence of $Q_S \in \mathcal{M}([0,\infty[\times\mathbb{R})$ such that (1.31) and (1.32) are satisfied.

The proof of Theorems 1.2 and 1.3 is straightforward. As in [2], we approximate the initial data by blocks. For this data, we build the solution by the above sticky blocks dynamics. Uniform bounds follow from the analysis of Subsection 4.2. Then, we use the stability to get the solution.

4.4 Loss of the strong exclusion relation

This section is devoted to a counterexample that shows that in the pressureless case, the exclusion relation in the strong sense (1.6) is not included in the entropy weak product inequality (1.11). We construct a sequence of sticky blocks solutions to which we can apply the stability theorem, but for which the limit does not satisfy the strong exclusion relation $(1-\rho)Q = 0$.

We consider initially two blocks of height $1/2$,

$$\rho^0(x) = \frac{1}{2}(\mathbb{1}_{0<x<1} + \mathbb{1}_{2<x<3}), \qquad \rho^0(x)u^0(x) = \frac{1}{2}(\mathbb{1}_{0<x<1} - \mathbb{1}_{2<x<3}). \tag{4.19}$$

We use the following approximation of these initial data,

$$\rho_n^0(x) = \sum_{k=1}^{n} \mathbb{1}_{\frac{2k-1}{2n}<x<\frac{2k}{2n}} + \sum_{k=1}^{n} \mathbb{1}_{2+\frac{2(k-1)}{2n}<x<2+\frac{2k-1}{2n}}, \tag{4.20}$$

$$\rho_n^0(x)u_n^0(x) = \sum_{k=1}^{n} \mathbb{1}_{\frac{2k-1}{2n}<x<\frac{2k}{2n}} - \sum_{k=1}^{n} \mathbb{1}_{2+\frac{2(k-1)}{2n}<x<2+\frac{2k-1}{2n}}. \tag{4.21}$$

Using the dynamics of Section 4.1 with $\Phi(u_l, u_r) = (u_l + u_r)/2$, we get a solution (ρ_n, u_n, Q_n, v_n) to (1.18)-(1.19), (1.10)-(1.11) with regularities (1.15)-(1.17), which is indeed given by

$$\rho_n = \sum_{k=1}^{n} \mathbb{1}_{t+\frac{2k-1}{2n}<x<t+\frac{2k}{2n}} \mathbb{1}_{x<3/2} + \sum_{k=1}^{n} \mathbb{1}_{2-t+\frac{2(k-1)}{2n}<x<2-t+\frac{2k-1}{2n}} \mathbb{1}_{x>3/2}, \tag{4.22}$$

$$\rho_n u_n = \sum_{k=1}^{n} \mathbb{1}_{t+\frac{2k-1}{2n}<x<t+\frac{2k}{2n}} \mathbb{1}_{x<3/2} - \sum_{k=1}^{n} \mathbb{1}_{2-t+\frac{2(k-1)}{2n}<x<2-t+\frac{2k-1}{2n}} \mathbb{1}_{x>3/2}, \tag{4.23}$$

$$Q_n = -2\sum_{k=1}^{n} \mathbb{1}_{\frac{1}{2}+\frac{2(k-1)}{2n}<t<\frac{1}{2}+\frac{2k-1}{2n}} \delta(x - \frac{3}{2}), \qquad v_n = 0. \tag{4.24}$$

Now, as $n \to \infty$, one can check easily that

$$Q_n \rightharpoonup -\mathbb{1}_{\frac{1}{2}<t<\frac{3}{2}} \delta(x - \frac{3}{2}), \tag{4.25}$$

$$\rho_n \rightharpoonup \rho = \frac{1}{2}\mathbb{1}_{t<x<t+1}\mathbb{1}_{x<\frac{3}{2}} + \frac{1}{2}\mathbb{1}_{2-t<x<3-t}\mathbb{1}_{x>\frac{3}{2}}, \tag{4.26}$$

and

$$\rho_n u_n \rightharpoonup \rho u = \frac{1}{2}\mathbb{1}_{t<x<t+1}\mathbb{1}_{x<\frac{3}{2}} - \frac{1}{2}\mathbb{1}_{2-t<x<3-t}\mathbb{1}_{x>\frac{3}{2}}. \tag{4.27}$$

We use the stability result and we get that $(\rho, u, Q, 0)$ is a solution to (1.18)-(1.19), (1.10)-(1.11) with regularities (1.15)-(1.17) for the initial data ρ^0 and $\rho^0 u^0$. This solution is half of a usual sticky block solution, as can be easily checked in the pressureless case, we can multiply any weak solution by a factor between 0 and 1, it is still a solution. It can be also interpreted as the sticky block solution with constraint $\rho \le 1/2$. The two blocks with height 1/2 loose their mass when they collide, all the mass disappears though ρ is staying less than 1/2. The exclusion relation is lost because here Q does not vanish even if $\rho < 1$ everywhere. We get instead $(\frac{1}{2} - \rho)Q = 0$. However, it seems that for the initial data (4.19), there is no solution that satisfies the strong exclusion relation. Therefore, this weak solution should be considered as the right one, and one could say that anyway it satisfies the exclusion relation if we define ρ to take the value 1 on the line $x = 3/2$, $1/2 < t < 3/2$.

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