Several Applications of Generalized Solutions in some Examples of Waves Propagation

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Abstract

This paper deals with some examples of physically meaningful models where the use of Generalized Functions allows to solve some situations where the usual ways are not always suitable, since a product of non regular functions may accur. We first describe the modelization of very simple cases as the model of the velocity field corresponding to a temporary stop and restart (in road traffic, in biology for chrysalis,..). Then we show how to build an acceptable shock condition, even when the model is not a conservative one, for an application in elastodynamics, then in a general class of nonhomogeneous hyperbolic systems (including the SaintVenant equations and many of the usual models in Hydrodynamics, Waves Equations...). Next we present some interesting behaviours of the solution which can be profiled as the succession of many short shocks and regular waves, as in the Roll waves phenomenon, approximating a regular curve which never corresponds to a solution by itself, and may be an example of Generalized Solutions.

1 What is a wave?

We propose to define a wave as a profile $\varphi(x,t)$ travelling along a trajectory $x = x(t)$ and which can be continuously modified during this travel. This can be modelized as follows

$$\varphi(x(t+\Delta t), t+\Delta t) \;=\; \varphi(x,t)\,(1 \;+\; A\,\Delta t) \;+\; B\,\Delta t\;, \tag{1.1}$$

for a time step Δt and some given coefficients A ($internal\ effects$) and B ($external\ effects$) which may depend on the profile φ, the space and the time variables. Using a Taylor expansion one gets from (1.1), writing $c = x'(t)$ as the celerity,

$$\varphi_t \;+\; c\,\varphi_x \;=\; A\,\varphi \;+\; B\,. \tag{1.2}$$

This corresponds to a scalar first order equation. We shall extend this model to the cases where A or B may present some singularities as a Dirac impulse, or a discontinuity.

As a first example we propose the following savings account model. The amount of money on the account at time t when the interest rate is $x(t)$ corresponds to $\varphi(x(t),t)$ in (1.1) with $A = x(t)$. The parameter B will correspond to the deposits ($B > 0$) or the withdrawals ($0 > B \geq -\varphi$) and is shaped as a Dirac comb since these operations are done at discrete times. In practice, the rate is a piecewise constant parameter, and is modified also at some discrete times. If t_0 corresponds to such a time, the profile $\varphi(x(t),t)$ will present a kink for $B = 0$ at $t = t_0$ and even a jump for $B \neq 0$

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at $t = t_0$. That way, the product $c\varphi_x$ may be interpreted as a "Dirac$\times$Heavyside" product in the first case, and even to "Dirac square" term in the second case. However the solution stay easy to compute by working along the characteristics, that is the trajectories $x = x(t)$.

A more difficult example appears when the rate $x(t)$ is a function of the amount φ, that is $x(t) = f(\varphi)$ (where $f' > 0$ is expected). We get a quasi linear equation of the form

$$\varphi_t \; + \; (\varphi \, f(\varphi) \; + \; B) \, f'(\varphi) \, \varphi_x \; = \; \varphi \, f(\varphi) \; + \; B \; ,$$

and when a shock curve occurs the product $B \; f'(\varphi)\varphi_x$ may have the form "Dirac$\times$Heaviside$\times$Dirac".

A second example corresponds corresponds to the propagation of an elastic wave in a medium made of two materials of different soundspeeds c_1 and c_2 separated by an interface. In the one dimension case, this corresponds to two half intervals $]-\infty, x_0[, \;]x_0, +\infty[$ where x_0 is the position of the interface. The profile φ will correspond to the amplitude of the wave, the term A in (1.2) has the form $A = -\alpha \; \delta(x - x_0)$, with $0 \leq \alpha \leq 1$ corresponding to the reflected part of the wave and δ denoting the Dirac pulse. We have $B = 0$ here. We get at the interface

$$\varphi(x_0 + 0) \; = \; \varphi(x_0 - 0) \, (1 - \alpha) \; ,$$

which makes φ discontinuous at $x = x_0$, and corresponds to the different interpretations of two different "Dirac$\times$Heavyside" products, one for the product $c\varphi_x$ and the second for the product $A\varphi_x$.

A third example corresponds to the "stop and go" problem consisting into writing the good formulation of the velocity field of a mobile having a constant speed v_0 everywhere but at one point where this speed is set to zero for a finite time T, before restarting with the same velocity c_0. For a given $\epsilon > 0$, this velocity field can be approached by

$$c_\epsilon(x) \; = \; \begin{cases} c_0 & for \; x < x_0 \; or \; x > x_0 + \epsilon \\ \frac{\epsilon}{T} & for \; x_0 < x < x_0 + \epsilon \; . \end{cases} \tag{1.3}$$

It is clear that the parameter T will disappear when passing to the limit on c_ϵ as ϵ goes to zero. Nevertheless, by using the slowness, which is the invert of the velocity (that is $l = \frac{1}{c}$) one gets

$$l_\epsilon(x) \; \equiv \; \frac{1}{c_\epsilon(x)} \; = \; \begin{cases} l_0 = \frac{1}{c_0} & for \; x < x_0 \; or \; x > x_0 + \epsilon, \\ \frac{T}{\epsilon} & for \; x_0 < x < x_0 + \epsilon \; , \end{cases} \tag{1.4}$$

which converges as a distribution towards $l(x) = l_0 + T\delta(x - x_0)$. This means that the expected formulation of our velocity field should be

$$c(x) \; = \; \frac{c_0}{1 \; + \; c_0 \, T \; \delta(x - x_0)} \; , \tag{1.5}$$

involving a division by a Dirac pulse... Such a situation occurs when for example one crosses a colleague in the corridor of the institute and stop to chat for a while, or in a traffic flow at a stop or a light, or also to represent the chrysalis step in the life of a butterfly. In this last case, in (1.2), c corresponds to the physiological ageing velocity and the variable x corresponds to the age, $A < 0$ is the death rate and $B = 0$. We may notice that another possible formulation of (1.2) is here

$$l(x) \, \varphi_t \; + \; \varphi_x \; = \; A \, l(x) \, \varphi \;\; , \tag{1.6}$$

which corresponds to handle the slowness as a porosity term. We avoid that way the division by the Dirac pulse, but refind two multiplications of distributions, one of the Dirac square type, the second of the "Dirac×Heavyside" type.

All these algebraic operations find a sense in the set of Genaralized Functions proposed by J.F.Colombeau (see [1]) which is bigger than the set of Distributions, and may be interpreted by breaking each distribution into several (many) classes for which the algebraic operations are suitable. For example by denoting by H the Heavyside distribution, the quantities H^m, $m \in \mathbb{N}$ corresponds to the same distribution than H but to different classes in the Colombeau Algebra.

2 A pressure velocity model

We describe the state at a given point of a medium of fixed density ρ_0 by its pressure p and its velocity v. We restrict the study to the one dimension case. These variables are linked by the dynamical equation (first principle)

$$\rho_0 \, (\, v_t \; + \; v \; v_x \,) \; + \; p_x \; = \; 0 \; . \tag{2.1}$$

and by a state law (similar to the Hooke law) which can be written under the following dynamical form

$$p_t \; + \; v \; p_x \; + \; \rho_0 \, c^2(p) \; u_x \; = \; 0 \; , \tag{2.2}$$

where $c = c(p) \; > 0$ corresponds to the soundspeed when the pressure has the value p. We propose here a way to derive this function $c(p)$ from experimental data obtained from velocity measures of physical shock waves, even though the system (2.1), (2.2) is not under a conservative form.

The first drawback in this case is that no Rankine Hugoniot condition can be directly derived from (2.1), (2.2) using a simple computation of distributions. Nevertheless, from physical measures using experimental impacts (or shocks) with other materials and different velocities, it is possible to obtain point by point a state law of the following form, called shock polar curves,

$$u \; = \; g(p) \; , \tag{2.3}$$

which describe the only possible states (u, p) on one side on a shock wave when the material is still at rest (state $(0, 0)$) on the other side. More details on this experiment are to be found in [2]. In practice g is an increasing concave function, and the usual formulation has the form

$$u \; = \; \alpha_0 \left(\sqrt{1 \; + \; \beta_0 \, p} \; - \; 1 \right) , \tag{2.4}$$

where α_0, β_0 are constants linked to the material. As a matter of fact, if c_0 is the sounspeed (at rest) we always have the relation

$$\alpha_0 \, \beta_0 \, \rho_0 \, c_0 \; = \; 2, \tag{2.5}$$

in M.K.S. units. The state (u, p) is a constant state here.

We shall use this polar shock curve to derive a relation between $c(p)$ and $g(p)$ and also a general Rankine Hugoniot compatibility condition. To do that, we need an asumption on the relative behaviour of the pressure and the velocity on a shock wave. We assume that they are shaped by the same Generalized Function in the equivalence class of the Heavyside distribution. This fact was already established for the fluids dynamics (see [3]) since it allows to refind the usual Rankine Hugoniot for the pressure velocity model in that case, very similar to the one studied here.

Denoting by H the Heavyside Generalized Function representant the pressure jump and also the velocity jump, we have

$$v \ = \ u \ (\ 1 \ - \ \ H(x - x(t)))$$

on the edge of the shock, where the pressure reads $p\Big(1 - H(x - x(t)\Big)$. We put that in (2.1) to get

$$\rho_0 \ \Big(u \ H'(x - x(t))x'(t) \ - \ u^2(1 - H(x - xt))H'(x - x(t))\Big) \ = \ p \ H'(x - x(t) \ ,$$

where

$$(1 - H) \ H' \ = \ \frac{1}{2} \ H' \ , \tag{2.6}$$

since the Heaviside representant is the same. This leads to

$$\left(\ \rho_0 \, u \ x'(t) \ - \ \rho_0 \ \frac{u^2}{2} \ - \ p \ \right) \ H'(x - x(t)) \ \ = \ \ 0 \ ,$$

and to a first Rankine Hugoniot condition

$$x'(t) \ = \ \frac{u}{2} \ + \ \frac{p}{\rho_0 \ u} \ . \tag{2.7}$$

Since u and p are constant, we get that $x'(t)$ is a constant (denoted k) and we get $x(t) = kt$ if the jump starts from the point $x(0) = 0$.

From the second equation (2.2), we get by the same way

$$p \ H' \ - \ u \ (1 - H) \ p \ H' \ = \ \rho_0 \, u \ c^2(p(1 - H)) \ H'$$

that we integrate with respect to x, for t frozen, and using the function

$$\Phi(q) \ = \ \int_0^q \ c^2(\xi) \ d\xi \quad (\textit{note that } \Phi(0) = 0), \tag{2.8}$$

to get

$$kp \ H(x - kt) \ - \ \frac{up}{2} \ H(x - kt) \ + \ \rho_0 \frac{u}{p} \ \Phi(p(1 - H(x - kt)) \ = \ Constant \tag{2.9}$$

Now for $x < kt$ we get

$$Constant \ = \ \rho_0 \ \frac{u}{p} \ \Phi(p) \ ,$$

and for $x > kt$

$$Constant \ = \ kp \ - \ \frac{up}{2} \ .$$

We have got, together with (2.7)

$$kp \ = \ \frac{up}{2} \ + \ \rho_0 \ \frac{u}{p} \ \Phi(p) \ = \ \frac{up}{2} \ + \ \frac{p^2}{\rho_0 u} \ , \tag{2.10}$$

which gives

$$\Phi(p) \ = \ \frac{p^3}{\rho_0 \ u^2} \ = \ \frac{p^3}{\rho_0 \ g(p)^2} \ , \tag{2.11}$$

which is the expected relation between $\Phi(p)$ and $g(p)$. We get

$$\rho_0 \, c^2(p) \;=\; \frac{d}{dp}\left(\frac{p^3}{g(p)^2}\right) \;=\; \frac{3\,p^2\,g(p)\;-\;2\,p^3\,g'(p)}{g(p)^3}\;. \tag{2.12}$$

The equations (2.10) are similar to the Rankine Hugoniot conditions. Using the proposed form (2.4), where $\beta_0 p$ is small in practice (we have $\beta_0 < 10^{-10}$ in M.K.S. units, and $\beta_0 p \sim 10^{-2}$ corresponds to $p \sim 10^8 = 1kbar$, which is very high pressure as at 1000m under the level of the sea for example). We set $X = \sqrt{1+\beta_0 p} - 1$ and get from (2.11)

$$c^2(p) \;=\; \frac{1}{\rho_0^2\,\alpha_0^2\,\beta_0^2}\,\frac{(2+X)^2\,(1+2X)}{1+X}\;, \tag{2.13}$$

which gives after a Taylor expansion

$$c^2(p) \;\approx\; \frac{4}{\rho_0^2\,\alpha_0^2\,\beta_0^2}\,(1\;+\;\beta_0 p\;-\;\frac{7}{16}\,\beta_0^2 p^2)\;.$$

Now, using (2.5), we have the approximation

$$c(p) \;\approx\; c_0\,\sqrt{1+\beta_0 p}\;. \tag{2.14}$$

Another method to be able to handle the shocks is to give a conservative form to (2.2). This can be done by the introduction of a new parameter $q = q(p)$ satisfying (for example) the following transport equation

$$q_t \;+\; (qv)_x \;=\; 0\;. \tag{2.15}$$

We derive from (2.15) that p satisfies

$$q'(p)\,(p_t \;+\; v\,p_x) \;+\; q(p)\,v_x \;=\; 0\;,$$

to be identified to (2.2), which makes that $q(p)$ will be such that

$$\frac{q'(p)}{q(p)} \;=\; \frac{1}{\rho_0\,c^2(p)}\;. \tag{2.16}$$

The Rankine Hugoniot condition derived from the conservative system (2.1) (2.15) reads

$$\rho_0\,\overline{q}\,\Delta v^2 \;=\; \Delta p\;\Delta q\;. \tag{2.17}$$

For a shock wave corresponding to (2.3), we get here

$$\rho_0\,u^2 \;=\; 2\,(q(p) - q_0)$$

with $q_0 = q(0)$, that is

$$q(p) \;=\; q_0\,\frac{2p\;+\;\rho_0\,g(p)^2}{2p\;-\;\rho_0\,g(p)^2}\;, \tag{2.18}$$

and from (2.16) and (2.18) we derive now

$$\rho_0\, c^2(p) \;=\; \frac{4\; p^2 \;-\; \rho_0^2\, g(p)^4}{4\; \rho_0\, (4pg(p)g'(p) \;-\; g(p)^2)}\;. \tag{2.19}$$

The two formulations (2.12) and (2.19) give different formulations of the sounspeed. However, by using (2.4) and $X = \sqrt{1+\beta_0 p} - 1$ as before, we get, since $\rho_0 c_0 \alpha_0 \beta_0 = 2$,

$$c^2(p) \;=\; c_0^2\,(1+X)\,\left(1 + X + \frac{X^2}{4}\left(1 - \frac{\alpha_0^2}{c_0^2}\right)\right)\,, \tag{2.20}$$

which gives the approximation

$$c^2(p) \;=\; c_0^2\left(1 \;+\; \beta_0 p \;+\; \frac{\beta_0^3 p^3}{8} \;+\; ...\right)\,,$$

and we get the same approximation (2.14) as before, with a higher order here.

Our conclusion is that the two methods are suitable to handle the shock waves, but they give different expressions of the soundspeed, which are very close in practice, when the pressure is not too large.

3 Some source term interactions

We consider the following class of second order nonlinear strictly hyperbolic systems whose first equation reads

$$q_t \;+\; m_x \;=\; 0\;, \tag{3.1}$$

and involving coefficients which are only dependant on the components q and m of the solution. Hence the second equation has the form

$$m_t \;+\; 2u\; m_x \;+\; (c^2 - u^2)\; q_x \;+\; h(q,m) \;=\; 0\;, \tag{3.2}$$

where $u = u(q,m)$, $c = c(q,m) \geq 0$ and h is called the source term. We shall restrict our study to the case $c = c(q)$ and when the set

$$H_0 \;=\; \{\; (q,m) \in \mathbb{R}^2 \mid h(q,m) \;=\; 0\; \} \tag{3.3}$$

corresponds to a single regular curve in the phase plane (q,m).

In this case, the system (3.1) (3.2) can be written under a conservative form only when u satisfies

$$\frac{\partial u}{\partial q} \;+\; u\,\frac{\partial u}{\partial m} \;=\; 0\;,$$

which is the well known Burgers equation in the phase variables. And well known solution are (at least)

$$u \;=\; \frac{m}{q} \quad or \quad u \;=\; constant\;.$$

In the first case the function u corresponds to a velocity and we recognize the usual models fo hydrodynamics, hydraulics, elastodynamics, etc... In the second case which can be reduced to

$u = 0$ by a simple transformation, we recognize a nonlinear wave equation. In the two cases Equation 2 becomes

$$m_t + (F(q,m))_x + h(q,m) = 0 . \qquad (3.4)$$

The function F is a flux, whose gradient is $(c^2 - u^2, 2u)$.

By denoting with the symbol Δ the jump, a shock wave will satisfy the Hugoniot condition

$$(\Delta m)^2 = \Delta F \, \Delta q , \qquad (3.5)$$

and we can obtain some intermediate values denoted $\overline{u}$ and $\overline{q}$ such that

$$\Delta F = 2\,\overline{u}\Delta m + \left((c(\overline{q}))^2 - (\overline{u})^2\right) \Delta q ,$$

and get

$$(\Delta m - \overline{u}\Delta q)^2 = c(\overline{q})^2 \, (\Delta q)^2 ,$$

that is

$$\frac{\Delta m}{\Delta q} = \overline{u} \pm c(\overline{q}) . \qquad (3.6)$$

We can generalize the Hugoniot condition (3.6) to the non conservative case by fitting some intermediate values $\overline{u}$ and $\overline{q}$ according to some (external) physical argument. This can be also obtained by a direct computation of a shock solution of the nonconservative equation (3.6) using the Colombeau Algebra as in Section 2 above. We have to notice that these intermediate values $\overline{u}$ and $\overline{q}$ seldom correspond to the usual arithmetic average of the values on both sides of the shock.

Let us now consider a regular wave. Hyperbolicity means that a linkage between the variables does exist along this wave. This can be easily seen from the Riemann analysis, which allows to build the Riemann invariants in the homogeneous case. In the general case we propose to write this linkage by using the form $m = m(q)$ that we introduce into (3.1) (3.2) and derive the linear system

$$\begin{pmatrix} 1 & m'(q) \\ m'(q) & 2um'(q) + c^2 - u^2 \end{pmatrix} \begin{pmatrix} q_t \\ q_x \end{pmatrix} = \begin{pmatrix} 0 \\ -h(q, m(q)) \end{pmatrix} . \qquad (3.7)$$

If the matrix in (3.7) is a singular one, then

$$(m'(q) - u)^2 = c^2$$

that is

$$m'(q) = u \pm c , \qquad (3.8)$$

which is a differential equation whose solutions are the Riemann invariants (as a matter of fact the constants obtained by integrating correspond exactly to the Riemann invariants). By the way, introducing (3.8) into (3.7) leads to the two equations

$$q_t + m'(q)\, q_x = 0 \ , \ m'(q)\, (q_t + m'(q)\, q_x) = -h(q, m(q)),$$

which are compatible only when $h(q, m(q)) \equiv 0$. This means that the Riemann analysis is no longer suitable in the non homogeneous case, and also that in this case the matrix in (3.7) must be a regular one. Hence solving (3.7) is possible and gives

$$\frac{(m'(q) - u)^2 - c^2}{h(q, m(q))} q_x = 1 . \tag{3.9}$$

which has the form

$$\psi'(q) q_x = 1 , \tag{3.10}$$

and by integrating with respect to x, one gets

$$\psi(q) = x - K(t) ,$$

where $K(t)$ is some function of t to be specified. Next we derive with respect to t and get

$$K'(t) = -\psi'(q) q_t = m'(q) gy'(q) q_x = m'(q) , \tag{3.11}$$

by using the first equation in (3.7) and (3.10).

Now we derive the equation $K'(t) = m'(q)$ with respect to x and get

$$m''(q) = 0 . \tag{3.12}$$

Otherwise $q_x = 0$, then $q_t = 0$ and $h = 0$ which is not expected.
We get from (3.12) the formulae

$$m(q) = A q - B , \quad K(t) = At - C , \tag{3.13}$$

where A, B and C are real constants. This result is somewhere a surprising one for it indicates that the linkage in the non homogeneous case is a linear one, which is far simpler than in the homogeneous case where the Riemann invariants are obtained. The constant A corresponds to the wave velocity, and in one of the conservative cases (that is for $m = qu$ here) we have

$$A = u + \frac{B}{q} , \tag{3.14}$$

which means that $\frac{B}{q}$ corresponds to the soundspeed c for a characteristic wave.

From (3.10) we also get

$$\psi'(q) = (x - A t) + C ,$$

which indicates that the wave is a traveling wave of velocity A, the profile of which is given by $q(x,t) = \left(\psi'\right)^{-1}(x - At + C)$ when we can compute the primitive ψ of ψ'. This corresponds to an hypothesis of the form

$$\psi' \in L^1_{loc}(\mathcal{V}(q)) , \tag{3.15}$$

where $\mathcal{V}(q)$ is a neighborough of a value q which is concerned. In many applications involving a friction term for example, this hypothesis is not respected, which may develop some singularities in the numerical simulations.

We have got that a general regular wave will be such that (3.9) reads

$$\frac{(A \,-\, u(q, Aq - B))^2 \,-\, c(q)^2}{h(q, Aq - B)}\, q_x \,=\, 1\,, \tag{3.16}$$

which reduces in the conservative cases to

$$\frac{B^2 \,-\, c^2\, q^2}{q^2\, h(q, Aq - B)}\, q_x \,=\, 1 \;\; for\; u = \frac{m}{q}\,, \qquad \frac{A^2 \,-\, c^2}{h(q, Aq - B)}\, q_x \,=\, 1 \;\; for\; u = 0\,. \tag{3.17}$$

Let us consider now a state $M_0 = (q_0, m_0)$ in the phase plane and a straight line passing through M_0, which may be written

$$A \,=\, \frac{m \,-\, m_0}{q \,-\, q_0} \qquad or \;\; B \,=\, \frac{q_0\, m \,-\, m_0\, q}{q \,-\, q_0}\,.$$

Let us denote by $T(M_0)$ the set of states $M = (q, m)$ such that the numerator of (3.16) (or (3.17) in the conservative case) is zero. This set is reduced to a non linear curve made of two branches of equations

$$T_{+}(M_0): \qquad m \,=\, m_0 \,+\, (u(q, m) + c(q))\,(q - q_0)\,, \tag{3.18}$$

$$T_{-}(M_0): \qquad m \,=\, m_0 \,+\, (u(q, m) - c(q))\,(q - q_0)\,. \tag{3.19}$$

In the conservative cases, these equations are explicit and read

$$T_{+}(M_0): \;\; m \,=\, \frac{q}{q_0}\, m_0 + \frac{q}{q_0}\,(q - q_0)\, c(q)\; for\; u = \frac{m}{q}\,,\;\; m \,=\, m_0 + (q - q_0) c(q)\; for\; u = 0, \tag{3.20}$$

$$T_{-}(M_0): \;\; m \,=\, \frac{q}{q_0}\, m_0 - \frac{q}{q_0}\,(q - q_0)\, c(q)\; for\; u = \frac{m}{q}\,,\;\; m \,=\, m_0 - (q - q_0) c(q)\; for\; u = 0. \tag{3.21}$$

We define now the characteristic lines by

$$K_{+}(M_0): \qquad m \,=\, m_0 \,+\, (u(q_0, m_0) + c(q_0))\,(q - q_0)\,, \tag{3.22}$$

$$K_{-}(M_0): \qquad m \,=\, m_0 \,+\, (u(q_0, m_0) - c(q_0))\,(q - q_0)\,. \tag{3.23}$$

We have the following result:

Proposition 3.1 *The characteristic line $K_{+}(M_0)$ is tangent to the curve $T_{+}(M_0)$ at $M = M_0$, the characteristic line $K_{-}(M_0)$ is tangent to the curve $T_{-}(M_0)$ at $M = M_0$, and*

$$M \,\in\, T_{+}(M_0) \;\Leftrightarrow\; M_0 \,\in\, K_{+}(M)\;\;, \quad M \,\in\, T_{-}(M_0) \;\Leftrightarrow\; M_0 \,\in\, K_{-}(M)\,. \tag{3.24}$$

The proof is obvious and based on different interpretations of the formulae (3.18) and (3.19) ([4]). We propose the following definition.

Definition 3.2 *A state $M_0 = (q_0, m_0)$ is called a regular state when*

$$c(q_0)\,\frac{\partial u}{\partial m}(q_0, m_0) \;+\; c'(q_0) \;\geq\; \left|\frac{\partial u}{\partial q}(q_0, m_0) + u(q_0, m_0)\frac{\partial u}{\partial m}(q_0, m_0)\right| . \tag{3.25}$$

We can notice that any state (q_0, m_0) with $q_0 > 0$ is a regular state in the conservative case since $c = c(q)$ is a positive increasing function in this case.

Let us consider now a regular state $M_0 = (q_0, m_0)$ such that $h(q_0, m_0) > 0$ and the two characteristic lines passing through M_0 in the phase plane. We get using (3.25) and a Taylor expansion in the neighborough of M_0, following the characteristic lines

$$(q - q_0)\left((u \;\pm\; c_0 \;-\; u(q_0, m_0))^2 \;-\; c^2(q)\right) \;\leq\; 0\,,$$

which means, using (3.25) and $h(q_{\grave{a}}, m_0) > 0$,

$$q_x \;\leq\; 0 \;\; for\; q > q_0, \quad q_x \;\leq\; 0\; for\; q < q_0$$

or that the state $M_0 = (q_0, m_0)$ is an attractive state along the two characteristic lines when $h(q_0, m_0) > 0$. Similarly for $h(q_0, m_0) < 0$ the state $M_0 = (q_0, m_0)$ is a repulsive state along the two characteristic lines. When $M_0 \in H_0$, that is $h(q_0, m_0) = 0$, then q_x is of constant sign along the two characteristic lines. We have the following result, where $H_+ = \{(q, m) | h(q, m) > 0\}$ and $H_- = \{(q, m) | h(q, m) < 0\}$.

Proposition 3.3 *For a regular solution the set H_0 can be crossed at a point S only using a characteristic line $K_+(S)$ or $K_-(S)$, in only one direction, from the set H_- towards the set H_+.*

Let $S = (q_S, m_S) \in H_0$ such that the characteristic line $K_+(S)$ is not tangent to H_0. The slope of $K_+(S)$ is denoted $A_S = u_S + c_S$, with $u_S = u(q_S, m_S)$, $c_S = c(q_S)$. We take now two regular states $D = (q_D, m_D) \in K_+(S)$, $E = (q_E, m_E) \in K_+(S)$ such that $D \in H_-$, $E \in H_+$ and D and E are linkable by a shock wave, that is satisfying (3.6) where necessarily $\overline{u} + c(\overline{q}) = A_S$ (we have also to require an entropy condition to be satisfied, for example $u(q_E, m_E) > u(q_D, m_D)$ which works in many cases). That way, one can join D to S then S to E following $K_+(S)$, according to Proposition 3.3, which corresponds to a regular wave, and then E to D by a shock wave. We have built a periodic entropy solution of (3.1), (3.2) which has a constant velocity A_S. Such a solution corresponds for example to the Roll Waves in hydraulics, where $h(q, m)$ involves the slope of the topography and a friction term. Now we can make D to go to S, and so does E on the other side, and get at the limit that S corresponds to a state which is a constant solution of (3.1), (3.2). We come to the following conclusion, which looks as a paradoxal one: " *When the set H_0 does not correspond to a Riemann invariant, then it corresponds to a set of particular solutions".*

We have the same result by using the other characteristic line K$_-$(S). In [4], we conjecture that H_0 corresponds to so called "creeping waves" obtained by the limit of a succession of regular and shock waves which are not of constant velocity, using a sequence D_k in H_- and a sequence E_k in H_+, which makes a sequence $S_k \in H_0$, with q_{S_k} monotone, such that a shock wave links E_k to D_{k+1}. At the limit these shock waves vanish and we get that the values of the solution correspond to a subset of H_0, which is not reduced to one point.

This example may be an example of a generalized solution in the sense of the Generalized Functions for a Partial Differential System. It is not a solution in the classical sense although it may be regular enough to perform the derivations, but it is the limit of weak solutions.

4 Conclusion

We have proposed several examples where the notion of Generalized Solutions is applicable in a suitable way, and enable to solve some problems which really occur in the physical appications. The examples given in Section 2 and Section 3 allow to build some numerical solvers to be used in numerical methods to compute the fluxes through the interfaces of the mesh. The condition (3.15) may be not satisfied for some orientations of the interfaces when for example a gravity term and a friction terms are present. This can develop some unstabilities whose origin is not easy to find. The analysis of the behaviour of a solution close to a well balanced state often needs to use some of the tools discussed here. Among these cases the dynamical models in Environmental Sciences are particularly concerned.

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