Remarks on Weak Solutions to the Navier-Stokes Equations for 2-D Compressible Isothermal Fluids with Spherically Symmetric Initial Data *

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Abstract

We prove the existence of global weak solutions to the Cauchy problem for the Navier-Stokes equations of 2-dimensional compressible isothermal fluids when ρ_0 and $\boldsymbol{m}_0$ are spherically symmetric, $\rho_0 \in L^1 \cap L_M$ and $\boldsymbol{m}_0/\sqrt{\rho_0} \in L^2$, where ρ_0 and $\boldsymbol{m}_0$ are the initial density and momentum respectively, L_M is the Orlicz space over $\mathbb{R}^2$ with $M = M(s) = (1+s)\ln(1+s) - s$. The proof is based on a compactness lemma which gives a compactness result concerning $H^{n/2}$ and L_M in $\mathbb{R}^n$.

Mathematics Subject Classification (2000): 76N10, 35M10, 35Q30
Keywords: 2-D compressible Navier-Stokes equations, isothermal fluids, spherically symmetric weak solutions, global existence, Orlicz spaces, weak convergence

1 Introduction

We prove the global existence of spherically symmetric weak solutions to the Cauchy problem for the Navier-Stokes equations of compressible isothermal fluids in $\mathbb{R}^2$ with initial density in $L^1(\mathbb{R}^2) \cap L_M(\mathbb{R}^2)$ and initial momentum divided by the square root of the initial density in $L^2(\mathbb{R}^2)$:

$$\partial_t \rho + \mathrm{div}(\rho \boldsymbol{u}) = 0, \tag{1.1}$$

$$\partial_t(\rho \boldsymbol{u}) + \mathrm{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) - \mu \Delta \boldsymbol{u} - (\lambda + \mu)\nabla \mathrm{div}\, \boldsymbol{u} + \nabla P(\rho) = 0, \tag{1.2}$$

*supported by the Special Funds for Major State Basic Research Projects (No. G1999032801), the NNSF and the CAEP.

with initial data

$$\rho(0,x) = \rho_0(x), \quad (\rho \boldsymbol{u})(0,x) = \boldsymbol{m}_0(x), \quad x \in \mathbb{R}^2 \tag{1.3}$$

and the pressure $P(\rho)$ satisfying

$$P(\rho) = a\rho. \tag{1.4}$$

Here $\rho(t,x)$ is the fluid density, $\boldsymbol{u}(t,x) = (u_1(t,x), u_2(t,x))$ is the fluid velocity, λ and μ are the constant viscosity coefficients satisfying $\mu > 0$, $3\lambda + 2\mu \geq 0$, and a is a positive constant. In this paper we study the spherically symmetric case and assume hence that

$$\rho_0 = \rho_0(|x|), \quad \boldsymbol{m}_0 = \frac{x}{|x|} m_0(|x|), \quad x \in \mathbb{R}^2. \tag{1.5}$$

The Cauchy problem and initial boundary value problems for the system (1.1), (1.2) in $\mathbb{R}^n$ ($n \geq 2$) have been studied by a number of mathematicians in the last decades and significant progress has been made on the existence of global solutions for large initial data. For isentropic fluids, i.e. $P(\rho) = a\rho^\gamma$ with $\gamma > 1$, Lions [14] (also Feireisl, Novotný and Petzeltová [8, 7]) used the method of weak convergence and proved the global existence of weak solutions for any $\gamma > n/2$, see also [6, 5] on time-periodic solutions and the existence for self-gravitating fluids. Similar global existence for the Cauchy problem was obtained for any $\gamma > 1$ provided that initial data are spherically symmetric or axisymmetric [10, 11]. For isothermal fluids, i.e. $P(\rho) = a\rho$, to our best knowledge the global existence is proved by Hoff only in the case of spherically symmetric initial data. In [9] Hoff obtained the global existence of spherically symmetric weak solutions for initial density in the BV-space, and the BV initial density is required in order to get compactness of the approximate density.

The aim of this paper is to prove the global existence of a weak solution to (1.1)–(1.5) for ρ_0 in the Orlicz space $L_M(\mathbb{R}^2)$ and $\boldsymbol{m}_0/\sqrt{\rho_0}$ in $L^2(\mathbb{R}^2)$, and therefore this paper extends the result in [9]. Our proof here, different from that in [9], follows easily from a compactness lemma (i.e. Lemma 3.1), which gives a compactness result concerning $H^{n/2}(\mathbb{R}^n)$ and $L_M(\mathbb{R}^n)$. It should be pointed out that many estimates in this paper remain valid also for non-symmetric solutions, and the spherical symmetry of solutions is needed only in the derivation of convergence of $\rho^\epsilon \boldsymbol{u}^\epsilon \otimes \boldsymbol{u}^\epsilon$ (cf. Remark 2.2). We also mention that for general initial data, the existence of global solutions still remains open for n-dimensional isentropic fluids with $1 \leq \lambda \leq n/2$. Concerning the global existence of spherically symmetric and axisymmetric weak solutions to the Euler equations in the domain exterior to a ball, we refer to the papers of Chen and Glimm [3, 4].

Before presenting the main result, we introduce some notation used throughout this paper. Let Ω be a domain in $\mathbb{R}^2$. Let m be an integer and let $1 \leq p \leq \infty$. By $W^{m,p}(\Omega)$ we denote the usual Sobolev space defined over Ω. $W^{m,2}(\Omega) \equiv H^m(\Omega)$, $W^{0,p}(\Omega) \equiv L^p(\Omega)$. $L_M(\Omega)$ and $L_N(\Omega)$ denote the Orlicz spaces defined over Ω with $M = M(s) := (1+s)\ln(1+s) - s$ and $N = N(s) := e^s - s - 1$. The space $E_M(\Omega)$ is defined as the closure of the set of all bounded measurable functions on Ω with compact support with respect to the Orlicz norm $\|\cdot\|_{L_M(\Omega)}$ (see e.g. [1, 12]). $L^p(I,B)$ denotes the space of all strongly measurable, p^{th}-power integrable (essentially bounded if $p=\infty$) functions from I to B, $I \subset \mathbb{R}$ an interval, B a Banach space. $C^0(I, B-w)$ is the space of all functions which are in $L^\infty(I,B)$ and continuous in t with values in B endowed with the weak topology.

Now, let us recall the definition of weak solutions to (1.1)–(1.3).

Definition 1.1 *We call $(\rho(t,x), \boldsymbol{u}(t,x))$ a global weak solution of (1.1)–(1.3), if*
1) $\rho \geq 0$ a.e., and for any $T>0$,

$$\begin{aligned} &\rho \in L^\infty([0,T], L^1(\mathbb{R}^2)), \quad \rho|\boldsymbol{u}|^2 \in L^\infty([0,T], L^1(\mathbb{R}^2)), \\ &\boldsymbol{u} \in L^2([0,T], H^1_{\mathrm{loc}}(\mathbb{R}^2)), \quad \rho, \rho\boldsymbol{u} \in C^0([0,T], L^1_{\mathrm{loc}}(\mathbb{R}^2)-w), \\ &(\rho, \rho u)(0,\cdot) = (\rho_0, \boldsymbol{m}_0) \ \textit{weakly in} \ L^1_{\mathrm{loc}}(\mathbb{R}^2)\times L^1_{\mathrm{loc}}(\mathbb{R}^2). \end{aligned} \tag{1.6}$$

2) For any $t_2 > t_1 \geq 0$ and any $\phi \in C^1_0(\mathbb{R}\times\mathbb{R}^2)$, the following holds:

$$\int_{\mathbb{R}^2} \rho\phi\, dx|_{t_1}^{t_2} - \int_{t_1}^{t_2}\int_{\mathbb{R}^2} (\rho\partial_t\phi + \rho\boldsymbol{u}\cdot\nabla\phi)\, dxdt = 0, \tag{1.7}$$

$$\begin{aligned} \int_{\mathbb{R}^2} \rho u_i\phi\, dx|_{t_1}^{t_2} &- \int_{t_1}^{t_2}\int_{\mathbb{R}^2} \{\rho u_i\partial_t\phi + \rho u_i\boldsymbol{u}\cdot\nabla\phi + P(\rho)\partial_{x_i}\phi\}\, dxdt \\ &+ \int_{t_1}^{t_2}\int_{\mathbb{R}^2} \{\mu\nabla u_i\cdot\nabla\phi + (\lambda+\mu)\mathrm{div}\boldsymbol{u}\,\partial_{x_i}\phi\}\, dxdt = 0. \end{aligned} \tag{1.8}$$

Thus the main result of this paper reads:

Theorem 1.1 *Assume that $\rho_0 \in L^1(\mathbb{R}^2)\cap L_M(\mathbb{R}^2)$, $\boldsymbol{m}_0/\sqrt{\rho_0} \in L^2(\mathbb{R}^2)$ and that (1.5) holds. Then there exists a global weak solution $(\rho, \boldsymbol{u})$ of (1.1)–(1.4), such that for any $T>0$, $\rho \in L^\infty([0,T], L_M(\mathbb{R}^2))$, $\nabla\boldsymbol{u} \in L^2([0,T], L^2(\mathbb{R}^2))$, $\rho = \rho(t,|x|)$ and $\boldsymbol{u} = u(t,|x|)x/|x|$ for some u, and the following estimate holds.*

$$E(t) + \int_0^t D(s)ds \leq E(0)e^{ta^2/(\lambda+2\mu)}, \quad t\geq 0, \tag{1.9}$$

where

$$\begin{aligned} E(t) &:= \frac{1}{2}\int_{\mathbb{R}^2}\{\rho|\boldsymbol{u}|^2 + 2(1+\rho)\ln(1+\rho)\}dx, \\ D(t) &:= \int_{\mathbb{R}^2}\{\mu|\nabla\boldsymbol{u}|^2 + (\lambda+\mu)|\mathrm{div}\boldsymbol{u}|^2\}dx. \end{aligned}$$

The proof of Theorem 1.1 will be given in the next section.

2 Proof of Theorem 1.1

In this section we first consider an approximate problem to (1.1)–(1.5) in a ball and derive the a priori estimates uniformly with respect to the radius of the ball, then we let the radius of the ball go to infinity and obtain a global spherically symmetric weak solution by applying the key Lemma 3.1 in Appendix, which gives a compactness result referring to $H^{n/2}(\mathbb{R}^n)$ and $L_M(\mathbb{R}^n)$. We shall also point out in the proof at which point the spherical symmetry of the initial data is needed (cf. Remark 2.1).

Throughout this and the next sections, the same letter C (sometimes used as $C(X,\cdots)$ to emphasize the dependence of C on $X,\cdots$) will denote various positive constants which do not depend on ϵ.

To construct the approximate problem, we first mollify the initial data. Let $0<\epsilon<1/2$ and

$$\rho_0^\epsilon(x):=(\rho_0*j_\epsilon)(x)+\epsilon^3,\quad \boldsymbol{m}_0^\epsilon(x):=\frac{\sqrt{\rho_0^\epsilon(x)}}{\sqrt{\rho_0(x)}}\boldsymbol{m}_0(x)\chi_\epsilon(|x|),\tag{2.1}$$

where j_ϵ is the symmetric Fridriches mollifier and $\chi_\epsilon\in C^\infty(\mathbb{R})$ satisfying $\chi_\epsilon(r)=0$ for $r\ge\epsilon^{-1}$ and $\chi_\epsilon(r)=1$ for $r\le\epsilon^{-1}-1$. Notice that due to the spherical symmetry of the mollifier j_ϵ, ρ_0^ϵ and $\boldsymbol{m}_0^\epsilon$ are spherically symmetric, i.e.

$$\rho_0^\epsilon(x)=\rho_0^\epsilon(|x|),\quad \boldsymbol{m}_0^\epsilon(x)=\frac{x}{|x|}\frac{\rho_0^\epsilon(|x|)}{\rho_0(|x|)}m_0(|x|)\chi_\epsilon(|x|).$$

Recalling that $\rho_0\in L^1(\mathbb{R}^2)$, $\boldsymbol{m}_0/\sqrt{\rho_0}\in L^2(\mathbb{R}^2)$, it follows from Cauchy-Schwarz's inequality that $\boldsymbol{m}_0\in L^1(\mathbb{R}^2)$. Thus, by Theorem 8.21 in [1] and a straightforward calculation, we easily find that as $\epsilon\to 0$,

$$\|\rho_0^\epsilon-\epsilon^3-\rho_0\|_{L_M(\mathbb{R}^2)\cap L^1(\mathbb{R}^2)}+\left\|\frac{\boldsymbol{m}_0^\epsilon}{\sqrt{\rho_0^\epsilon}}-\frac{\boldsymbol{m}_0}{\sqrt{\rho_0}}\right\|_{L^2(\mathbb{R}^2)}+\|\boldsymbol{m}_0^\epsilon-\boldsymbol{m}_0\|_{L^1_{\rm loc}(\mathbb{R}^2)}\to 0,\tag{2.2}$$

which gives

$$\begin{aligned}
&\int_{|x|\le\epsilon^{-1}}|(1+\rho_0^\epsilon)\ln(1+\rho_0^\epsilon)-(1+\rho_0)\ln(1+\rho_0)|\,dx\\
&=\int_{|x|\le\epsilon^{-1}}\Big|(1+\rho_0^\epsilon-\epsilon^3)\ln(1+\rho_0^\epsilon-\epsilon^3)-(1+\rho_0)\ln(1+\rho_0)\\
&\quad+\epsilon^3\ln(1+\rho_0^\epsilon)+(1+\rho_0^\epsilon-\epsilon^3)\ln\left(1+\frac{\epsilon^3}{1+\rho_0^\epsilon-\epsilon^3}\right)\Big|\,dx\\
&\le\int_{|x|\le\epsilon^{-1}}\big|(1+\rho_0^\epsilon-\epsilon^3)\ln(1+\rho_0^\epsilon-\epsilon^3)-(1+\rho_0)\ln(1+\rho_0)\big|\,dx\\
&\quad+\epsilon^3\int_{|x|\le\epsilon^{-1}}(\rho_0^\epsilon+1)dx\to 0,\qquad\text{as }\epsilon\to 0.
\end{aligned}\tag{2.3}$$

For simplicity, denote $\Omega_\epsilon:=\{x\in\mathbb{R}^2\mid |x|<\epsilon^{-1}\}$. We now consider the the following problem in the domain $(0,\infty)\times\Omega_\epsilon$:

$$\partial_t\rho^\epsilon+{\rm div}(\rho^\epsilon\boldsymbol{u}^\epsilon)=0,\tag{2.4}$$

$$\partial_t(\rho^\epsilon\boldsymbol{u}^\epsilon)+{\rm div}(\rho^\epsilon\boldsymbol{u}^\epsilon\otimes\boldsymbol{u}^\epsilon)-\mu\Delta\boldsymbol{u}^\epsilon-(\lambda+\mu)\nabla{\rm div}\boldsymbol{u}^\epsilon+a\nabla\rho^\epsilon=0,\tag{2.5}$$

$$\boldsymbol{u}^\epsilon|_{\partial\Omega_\epsilon}=0,\tag{2.6}$$

$$\rho^\epsilon(0,x)=\rho_0^\epsilon(x),\ \ (\rho^\epsilon\boldsymbol{u}^\epsilon)(0,x)=\boldsymbol{m}_0^\epsilon(x),\qquad x\in\Omega_\epsilon.\tag{2.7}$$

By virtue of the definition (2.1) of ρ_0^ϵ, $\rho_0^\epsilon \in BV(\mathbb{R}^2)$ and $\epsilon^3 \le \rho_0^\epsilon(x) \le \epsilon^{-2}\|\rho_0\|_{L^1(\mathbb{R}^2)} + \epsilon^3$ for all $x \in \mathbb{R}^2$. So, recalling (2.2), we can apply the Theorem in the introduction of [9] (also cf. Theorems 5.6 and 5.7) and obtain hence that there is a global weak solution $(\rho^\epsilon(t,x), \boldsymbol{u}^\epsilon(t,x))$ of (2.4)–(2.7), such that

$$\begin{aligned}
&\rho^\epsilon,\ \rho^\epsilon \boldsymbol{u}^\epsilon \in C^0([0,\infty), L^1(\Omega_\epsilon)),\\
&(\rho^\epsilon(0,\cdot), (\rho^\epsilon \boldsymbol{u}^\epsilon)(0,\cdot)) = (\rho_0^\epsilon, \boldsymbol{u}_0^\epsilon) \ \text{ in } L^1(K) \text{ for any } K \subset \Omega_\epsilon,\\
&\sqrt{\rho^\epsilon}\boldsymbol{u}^\epsilon \in L^\infty([0,\infty), L^2(\Omega_\epsilon)),\ \ \nabla \boldsymbol{u}^\epsilon \in L^2([0,\infty), L^2(\Omega_\epsilon)),\\
&\rho^\epsilon \ge 0 \ \text{ a.e.,} \qquad \rho^\epsilon = \rho^\epsilon(t,|x|),\ \boldsymbol{u}^\epsilon = \frac{x}{|x|}u^\epsilon(t,|x|) \text{ for some } u^\epsilon.
\end{aligned} \tag{2.8}$$

Moreover, the following a priori estimates hold, the proof of which can be found in [15, p. 9]:

$$E_\epsilon(t) + \int_0^t D_\epsilon(s)ds \le E^\epsilon(0)e^{ta^2/(\lambda+2\mu)}, \qquad t \ge 0, \tag{2.9}$$

$$\int_{\Omega_\epsilon} \rho^\epsilon(t,x)dx = \int_{\Omega_\epsilon} \rho_0^\epsilon(x)dx \le \int_{\mathbb{R}^2} \rho_0 dx + C\epsilon, \qquad t \ge 0, \tag{2.10}$$

where

$$\begin{aligned}
E_\epsilon(t) &:= \int_{\Omega_\epsilon} \left\{ \frac{\rho^\epsilon|\boldsymbol{u}^\epsilon|^2}{2} + (1+\rho^\epsilon)\ln(1+\rho^\epsilon) \right\}(t,x)dx,\\
D_\epsilon(t) &:= \int_0^t \int_{\Omega_\epsilon} \left\{ \mu|\nabla \boldsymbol{u}^\epsilon|^2 + (\lambda+\mu)|\mathrm{div}\,\boldsymbol{u}^\epsilon|^2 \right\} dx d\tau.
\end{aligned}$$

Remark 2.1 *In fact, the existence theorems in [9] are for the Cauchy problem. However, the same arguments work for the Dirichlet problem (2.4)–(2.7), and a similar existence theorem can be obtained.*

On the other hand, for any $K \subset\subset \mathbb{R}^2$, let ϵ be small enough such that $K \subset \Omega_\epsilon$, then we get from $(2.8)_3$ and (2.9) that

$$\begin{aligned}
\int_0^t \int_K |\boldsymbol{u}^\epsilon|^2 dx d\tau &\le C(K) \int_0^t \int_K \frac{|\boldsymbol{u}^\epsilon|^2}{|x|^2} dx d\tau\\
&\le C(K) \int_0^t \int_0^{\epsilon^{-1}} \frac{|u^\epsilon(\tau,r)|^2}{r^2} r dr d\tau\\
&\le C(K) \int_0^t \int_{\Omega_\epsilon} |\nabla \boldsymbol{u}^\epsilon|^2 dx d\tau\\
&\le C(K) E^\epsilon(0) e^{ta^2/(\lambda+2\mu)}, \qquad t \ge 0.
\end{aligned} \tag{2.11}$$

Now, we extend both $\rho^\epsilon(t,x)$ and $\boldsymbol{u}^\epsilon(t,x)$ to be zero for $x \in \mathbb{R}^2 \backslash \Omega_\epsilon$. For simplicity, we still denote by $(\rho^\epsilon(t,x), \boldsymbol{u}^\epsilon(t,x))$ this extension. By virtue of (2.2), (2.3), $(2.8)_3$ and

(2.9)–(2.11), we can extract a subsequence of $(\rho^\epsilon, \boldsymbol{u}^\epsilon)$, still denoted by $(\rho^\epsilon, \boldsymbol{u}^\epsilon)$, such that (also cf. iii) in Appendix)

$$\begin{aligned} &\rho^\epsilon \rightharpoonup \rho = \rho(t,|x|) \quad \text{weak-}* \text{ in } L^\infty([0,T], L_M(\mathbb{R}^2)) \cap L^\infty([0,T], L^1_{\rm loc}(\mathbb{R}^2)), \\ &\boldsymbol{u}^\epsilon \rightharpoonup \boldsymbol{u} = \frac{x}{|x|}u(t,|x|) \text{ for some } u, \quad \text{weakly in } L^2([0,T], H^1_{\rm loc}(\mathbb{R}^2)), \end{aligned} \tag{2.12}$$

where $T>0$ be an arbitrary but fixed constant. Moreover, by the lower semicontinuity of weak convergence, $\rho \in L^\infty([0,T], L^1(\mathbb{R}^2))$ and $\nabla \boldsymbol{u} \in L^2([0,T], L^2(\mathbb{R}^2))$.

In the sequel, we show that $(\rho, \boldsymbol{u})$ obtained in (2.12) is indeed a weak solution of (1.1)–(1.5). For any $K \subset\subset \mathbb{R}^2$, the estimates (2.9) and (2.11) imply that $\boldsymbol{u}^\epsilon$ is bounded from above in $L^2([0,T], H^1(K))$ uniformly in ϵ provided ϵ is sufficiently small, while by Eq. (2.4) and (2.9) we find that $\partial_t \rho^\epsilon$ is uniformly bounded in $L^\infty([0,T], W^{-1,1}(K))$. Thus, applying Lemma 3.1, we obtain

$$\rho^\epsilon \boldsymbol{u}^\epsilon \rightharpoonup \rho \boldsymbol{u} \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^+ \times K). \tag{2.13}$$

To prove the weak convergence of $\rho^\epsilon \boldsymbol{u}^\epsilon \otimes \boldsymbol{u}^\epsilon$ to $\rho \boldsymbol{u} \otimes \boldsymbol{u}$, we utilize the spherical symmetry of $\boldsymbol{u}^\epsilon$ $(= u^\epsilon x/|x|)$. Recalling $K \subset \Omega_\epsilon$, we deduce that

$$\begin{aligned} |\boldsymbol{u}^\epsilon(t,x)|^2 = |u^\epsilon(t,|x|)|^2 &\le 2\int_{|x|}^{\epsilon^{-1}} |u^\epsilon(t,r)\partial_r u^\epsilon(t,r)|dr \\ &\le \int_0^{\epsilon^{-1}} \left\{ |\partial_r u^\epsilon(t,r)|^2 + \frac{|u^\epsilon(t,r)|^2}{r^2} \right\} rdr \\ &\le C\int_{\Omega_\epsilon} |\nabla \boldsymbol{u}^\epsilon(t,x)|^2 dx, \end{aligned}$$

which, by (2.9),

$$\int_0^T \|\boldsymbol{u}^\epsilon(\tau,\cdot)\|^2_{L^\infty(K)} d\tau \le C(T,K). \tag{2.14}$$

Hence, by virtue of (2.9) and (2.10), $\{\rho^\epsilon \boldsymbol{u}^\epsilon\}$ is bounded in $L^2([0,T], L_M(K))$ uniformly with respect to ϵ. On the other hand, by Eq. (2.5) and (2.9) we see that $\partial_t(\rho^\epsilon \boldsymbol{u}^\epsilon)$ is uniformly bounded in $L^2([0,T], W^{-1,1}(K))$. Therefore we can apply Lemma 3.1 and (2.13) to get

$$\rho^\epsilon \boldsymbol{u}^\epsilon \otimes \boldsymbol{u}^\epsilon \rightharpoonup \rho \boldsymbol{u} \otimes \boldsymbol{u} \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^+ \times K). \tag{2.15}$$

Remark 2.2 *The spherical symmetry of the solutions is needed only in the derivation of (2.14), which gives* $L^\infty([0,T], L_M(K)) \cdot L^2([0,T], H^1(K)) \hookrightarrow L^2([0,T], L_M(K))$ *such that one can apply Lemma 3.1. In general, however, this imbedding is not true. In the general case we have only* $L^\infty([0,T], L_M(K)) \cdot L^2([0,T], H^1(K)) \hookrightarrow L^2([0,T], L_{M_{1/2}}(K))$ *with* $M_{1/2}$ *being the complementary Young function to* $e^{s^2}-1$.

We proceed to prove Theorem 1.1. Using $(2.12)_1$ and (2.13), and following the same procedure as in the derivation of (3.6) in Appendix, we can prove that for any $\phi \in$

$C_0^\infty(\mathbb{R}\times\mathbb{R}^2)$ and any $t\geq 0$,

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{\mathbb{R}^2}(\rho^\epsilon\phi)(t,x)dx &= \int_{\mathbb{R}^2}(\rho\phi)(t,x)dx,\\ \lim_{\epsilon\to 0}\int_{\mathbb{R}^2}(\rho^\epsilon\boldsymbol{u}^\epsilon\phi)(t,x)dx &= \int_{\mathbb{R}^2}(\rho\boldsymbol{u}\phi)(t,x)dx.\end{aligned}\tag{2.16}$$

Hence, letting ϵ be appropriately small such that $\operatorname{supp}\phi\subset[0,T]\times\Omega_\epsilon$ for some T, we multiply (2.4) and (2.5) by ϕ (test functions) in $L^2((t_1,t_2)\times\mathbb{R}^2)$ with $t_2>t_1\geq 0$ and then integrate by parts. Letting $\epsilon\to 0$, and using (2.2), (2.12), and (2.13), (2.15) and (2.16), we see that $(\rho,\boldsymbol{u})$ satisfies (1.7) and (1.8) for any $\phi\in C_0^\infty(\mathbb{R}\times\mathbb{R}^2)$. Moreover, a density argument shows that (1.7), (1.8) still hold for any $\phi\in C_0^1(\mathbb{R}\times\mathbb{R}^2)$.

It follows from (2.12), (1.7), (1.8) and Lemma C.1 of [13, Appendix C] that $\rho,\rho\boldsymbol{u}\in C^0([0,T],L^1_{\rm loc}(\mathbb{R}^2)-w)$. Moreover, recalling $\rho^\epsilon\boldsymbol{u}^\epsilon\in L^2([0,T],L_M(K))$ for any $K\subset\subset\mathbb{R}^2$, we find by $(2.12)_1$, (2.13), (2.8) and (2.9) that (1.6) is satisfied. Finally, the estimate (1.9) follows from (2.2), (2.9), (2.11), and (2.12), (2.15) and the lower semicontinuity of weak convergence. This completes the proof. $\square$

3 Appendix

We first recall some basic facts on the Orlicz spaces (see e.g. [1, 12, 15]), then using these facts, we prove a compactness lemma which is crucial in the proof of Theorem 1.1.

i) $M=M(s):=(1+s)\ln(1+s)-s$ and $N=N(s):=e^s-s-1$ are mutually complementary Young functions, and M satisfies the Δ_2 condition, that is, for large values of s, there holds

$$M(2s)\leq aM(s),\tag{3.1}$$

for some $a>0$;

ii) Let Ω be a domain in $\mathbb{R}^n$, then the conjugate space, $(L_M(\Omega))^*$, of $L_M(\Omega)$ is $L_N(\Omega)$;

iii) The Orlicz space $L_M(\Omega)$ is E_N-weakly compact with N being the complementary N-function to M, i.e. for any sequence $\{v_n\}\in L_M(\Omega)$ uniformly bounded, there are a subsequence of $\{v_n\}$, still denoted by $\{v_n\}$ and a $v\in L_M(\Omega)$, such that $\int_\Omega v_n\phi dx\to\int_\Omega v\phi dx$ for any $\phi\in E_N(\Omega)$. In fact, this can be proved by using the convexity of the function $(1+x)\ln(1+x)-x$, the absolute continuity of the Radon measure associated with $(1+|v_n|)\ln(1+|v_n|)-|v_n|$ and the denseness of C_0^∞-functions in $E_N(\Omega)$.

The following is the key compactness lemma which gives some compactness concerning $H^{n/2}$ and L_M:

Lemma 3.1 *Let U be a bounded domain in $\mathbb{R}^n$, and p,q be conjugate numbers. Assume that $\{g^\epsilon(t,x)\}$ and $\{v^\epsilon(t,x)\}$ are bounded uniformly in ϵ in $L^p([0,T],L_M(U))$ and $L^q([0,T],H^{n/2}(U))$ respectively, and that as $\epsilon\downarrow 0$, g^ϵ and v^ϵ converge weakly to g and v in $L^p([0,T],L_M(U))$ and $L^q([0,T],H^{n/2}(U))$, respectively. Moreover, assume that $\{\partial_t g^\epsilon\}$ is uniformly bounded in $L^\gamma([0,T],W^{-m,1}(U))$ for some $\gamma>1$ and $m>0$. Then $g^\epsilon v^\epsilon$ converges weakly to gv in the sense of $\mathcal{D}'((0,T)\times U)$ as $\epsilon\downarrow 0$.*

Proof. Without loss of generality, we may assume that there is a number $l > n/2$, such that $\{\partial_t g^\epsilon\}$ is uniformly bounded in $L^\gamma([0,T], H^{-l}(U))$. In fact, by the Sobolev imbedding theorem, we have $W^{-m,1}(U) \hookrightarrow H^{-l}(U)$ for some l appropriately large. For the sake of clear presentation, we break up the proof into the three steps.

STEP 1. We claim that

$$L_M(U) \hookrightarrow\hookrightarrow H^{-n/2}(U). \tag{3.2}$$

Let $\{w^\epsilon\}$ be a bounded sequence in $L_M(U)$. Then by virtue of iii) in Appendix, there is a subsequence of $\{w^\epsilon\}$, still denoted by $\{w^\epsilon\}$ for simplicity, such that w^ϵ converges E_N-weakly to $w \in L_M(U)$ as $\epsilon \to 0$. Now denote by B the closed unit ball in $H_0^{n/2}(U)$. Then, noticing $H^{n/2}(U) \hookrightarrow L_N(U) = L_M(U)^*$ (cf. [1]), we have by a contradiction argument that

$$\lim_{\epsilon \to 0} \sup_{\phi \in B} \left| \int_U \phi(w^\epsilon - w)\, dx \right| = 0, \tag{3.3}$$

which proves (3.2). In fact, if (3.3) does not hold, then there exist a positive $\delta > 0$, a sequence of $\{\phi_\epsilon\} \subset B$ and a subsequence of $\{w^\epsilon\}$, still denoted by $\{w^\epsilon\}$, such that

$$\left| \int_U \phi_\epsilon(w^\epsilon - w)\, dx \right| \geq \delta, \qquad \text{as } \epsilon \to 0. \tag{3.4}$$

On the other hand, since $H^{n/2}(U) \hookrightarrow\hookrightarrow L_N(U)$ (see Prop. 4.3 of [16]) and the space $E_N(U)$ is the closure of the set of all bounded measurable functions on U with bounded support with respect to the Orlicz norm $\|\cdot\|_{L_N}$, B is compact in $E_N(U)$. Therefore there exists a $\phi \in E_N(U)$ such that as $\epsilon \to 0$,

$$\|\phi_\epsilon - \phi\|_{L_N(U)} = 0. \tag{3.5}$$

Combining (3.4) with (3.5), we get

$$\lim_{\epsilon \to 0} \left| \int_U \phi(w^\epsilon - w)\, dx \right| \geq \delta,$$

which contradicts to the fact that $w^\epsilon \rightharpoonup w$ E_N-weakly in $L_M(U)$.

STEP 2. As $\{\partial_t g^\epsilon(t,x)\}$ is uniformly bounded in $L^\gamma([0,T], H^{-l}(U))$, we see that $\{g^\epsilon(t,x)\}$ is uniformly bounded in $C([0,T], H^{-l}(U))$. Moreover,

$$\|g^\epsilon(t_1, \cdot) - g^\epsilon(t_2, \cdot)\|_{H^{-l}(U)} \leq C|t_1 - t_2|^{\frac{\gamma-1}{\gamma}},$$

and by Rellich's Selection Theorem, we have

$$\text{for any fixed } t, \quad g^\epsilon(t,x) \text{ lies in a compact subset of } H^{-l'}(U)$$

for all $l' > l$. Therefore Ascoli-Arzelà's Theorem (see [2]) implies that there is a function $g \in C([0,T], H^{-l'}(U))$ auch that

$$\max_{0 \leq t \leq T} \|g^\epsilon(t,\cdot) - g(t,\cdot)\|_{H^{-l'}(U)} \to 0, \quad \text{as } \epsilon \to 0. \tag{3.6}$$

Thus by virtue of (3.2), the fact $H^{-n/2}(U) \hookrightarrow H^{-l'}(U)$ and Lemma 2.1 in [17, Chapter 3], for any $\delta > 0$, there exists a positive constant $C(\delta)$ such that

$$\|g^\epsilon(t,\cdot) - g(t,\cdot)\|_{H^{-n/2}(U)} \le \delta \|g^\epsilon(t,\cdot) - g(t,\cdot)\|_{L_M(U)} + C(\delta)\|g^\epsilon(t,\cdot) - g(t,\cdot)\|_{H^{-l'}(U)},$$

which implies that

$$\begin{aligned} \|g^\epsilon - g\|_{L^p([0,T],H^{-n/2}(U))} \le\; & \delta \|g^\epsilon - g\|_{L^p([0,T],L_M(U))} \\ & + C(\delta) T^{1/p} \max_{0\le t\le T} \|g^\epsilon(,\cdot) - g(t,\cdot)\|_{H^{-l'}(U)}. \end{aligned} \tag{3.7}$$

From (3.7) and (3.6) we conclude that

$$g^\epsilon(t,x) \to g(t,x) \quad \text{in} \quad L^p([0,T], H^{-n/2}(U)). \tag{3.8}$$

Step 3. Note that $L^p([0,T], H^{-n/2}(U)) = \Big(L^q([0,T], H_0^{n/2}(U))\Big)^*$, which together with (3.8) implies that as $\epsilon \to 0$,

$$g^\epsilon v^\epsilon \rightharpoonup gv \quad \text{in} \quad \mathcal{D}'((0,T)\times U).$$

This completes the proof of the lemma. □

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