KINETIC APPROXIMATION OF A BOUNDARY VALUE PROBLEM FOR CONSERVATION LAWS

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Abstract

We design numerical schemes for systems of conservation laws with boundary conditions. These schemes are based on relaxation approximations taking the form of discrete BGK models with kinetic boundary conditions. The resulting schemes are Riemann solver free and easily extendable to higher order in time or in space. For scalar equations convergence is proved. We show numerical examples, including solutions of Euler equations.

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1 Introduction

In this paper, we study a new class of numerical approximations of the initial-boundary value problem for the one-dimensionnal conservation law:

$$\partial_t u + \partial_x F(u) = 0, \tag{1}$$

where $u(t,x) \in \mathbb{R}^p$ $(p \geq 1)$ for $x \geq 0$, $t \geq 0$, and F is a Lipschitz continuous function.

We denote u^0 the initial data:

$$u(0,x) = u^0(x), \quad x \geq 0, \tag{2}$$

and u_b the boundary data:

$$u(t,0) = u_b(t), \quad t \geq 0. \tag{3}$$

It is well-known that condition (3) can be imposed only in regions where it is compatible with the initial data and the flux F. In the scalar case $p = 1$, Bardos, Le Roux and Nédélec have proved existence and uniqueness of the entropy solution when the boundary condition is written as [2]:

$$\max\{\mathrm{sgn}(u(t,0) - u_b(t))(F(u(t,0)) - F(k)) : k \in I(u_b(t), u(t,0))\} = 0. \tag{4}$$

For systems, such a general result is not available, see for example [3], [12], and references therein.

The convergence of some numerical schemes such as Gocunov, Lax-Friedrichs, Glimm schemes, has been studied in the scalar case by A.-Y. Le Roux in [18]. In the framework of finite volumes, the reader can refer to the article of S. Benharbit, A. Chalabi and J.-P. Vila [4].

Our schemes are based on a kinetic approximation of the problem (1). Consider the BGK-like system:

$$\partial_t f_k^\epsilon + \lambda_k \partial_x f_k^\epsilon = \frac{1}{\epsilon}(M_k(Pf^\epsilon) - f_k^\epsilon), \quad k \in \{1, \ldots, N\} \tag{5}$$

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where the λ_k are fixed velocities, P is defined by $Pf = \sum_k f_k$, and ϵ is a positive parameter. Each f_k^ϵ takes values in $\mathbb{R}^p$. In addition, we impose the corresponding initial and boundary data:

$$f_k^\epsilon(0,x) = M_k(u^0(x)) \tag{6}$$

and

$$f_k^\epsilon(t,0) = M_k(u_b(t)) \ \text{ if } \ \lambda_k > 0. \tag{7}$$

Functions M_k are known as Maxwellian functions depending on the macroscopic quantities $u^\epsilon = Pf^\epsilon$, the flux F and the velocities λ_k. The following compatibility conditions link systems (1) and (5):

$$\sum_{k=1}^{N} M_k(w) = w, \quad \sum_{k=1}^{N} \lambda_k M_k(w) = F(w) \tag{8}$$

for all $w \in I$, a domain of $\mathbb{R}^p$ containing all the values of the solution of the problem (1), (2), (3). Denoting $v^\epsilon = \sum_{k=1}^{N} \lambda_k f_k^\epsilon$, system (5) gives:

$$\left\{ \begin{array}{lcl} \partial_t u^\epsilon + \partial_x v^\epsilon & = & 0, \\ \partial_t v^\epsilon + \displaystyle\sum_{k=1}^{N} \lambda_k^2 \partial_x f_k^\epsilon & = & \dfrac{1}{\epsilon}\left(F(u^\epsilon) - v^\epsilon\right), \end{array} \right. \tag{9}$$

so that if f^ϵ converges (in a strong enough sense) to f when ϵ tends to zero, then u^ϵ converges to a solution of system (1).

Rigorous convergence results exist for the scalar case $p = 1$. In [30], W.C. Wang and Z. Xin studied the relaxation model with boundary condition

$$\left\{ \begin{array}{l} \partial_t u^\epsilon + \partial_x v^\epsilon = 0, \\ \\ \partial_t v^\epsilon + \lambda^2 \partial_x u^\epsilon = \frac{1}{\epsilon}(F(u^\epsilon) - v^\epsilon), \end{array} \right. \tag{10}$$

with

$$\left\{ \begin{array}{l} u^\epsilon(t,0) = u_b(t), \\ \\ u^\epsilon(0,x) = u^0(x), \\ \\ v^\epsilon(0,x) = F(u^0(x)). \end{array} \right. \tag{11}$$

The system (10) can be put as a 2×2 kinetic system of form (5) with velocities $-\lambda, \lambda$ and was introduced for numerical purposes by S. Jin and Z. Xin [16]. W.C. Wang and Z. Xin showed the existence and uniqueness of the solution under strong restrictions on the data, namely that the initial boundary condition must be a small perturbation of a constant data u^*, with a non transonic condition on said constant state (i.e. $F'(u^*) \neq 0$).

Working with the same 2×2 system, but with a (7) -like boundary condition , R. Natalini and A. Terracina, (see [24]) succeeded in proving the same kind of result, but without any restrictions neither on the data (as in [30]), nor on flux F (as in [26]). It is obvious that the number of conditions to impose is equal to the number of velocities entering the domain (in our case those velocities are positive). As we have a scalar boundary condition a degree of freedom remains. For example in the relaxation case (10) , we can explain this with the following expression where α is a real number varying in $]-1:1]$:

$$f_+^\epsilon(t,0) = M_+(u_b(t)) - \alpha(f_-^\epsilon(t,0) - M_-(u_b(t))).$$

If $\alpha = 1$ one has the condition (11) , while $\alpha = 0$ is the boundary condition studied in [24]. In [33], $\alpha \neq -1$ was shown to be a necessary condition of well-possedness as far as $F' \neq 0$.

The result of R. Natalini and A. Terracina has been extended to the case of any set of velocities by V. Milisic [21].

The case of systems is an open problem. The recent work of F. Berthelin and F. Bouchut [5] concerning a kinetic approximation for isentropic gas dynamics is to be mentionned.

A necessary and crucial point for the kinetic approximations to be stable, as well as to be compatible with entropy conditions, is the monotonicity of the Maxwellian function in the following sense:

Definition 1.1 *The function M is Monotone Non Decreasing (MND) on I if the eigenvalues of $M_k'(u)$ are non negative for all k and all $u \in I$.*

Under some hypothesis which are trivial in the scalar case, it is easy to show that if M is MND on I, then the following condition is satisfied for all $m \in \{1, \ldots, p\}$ and all $u \in I$:

$$\min_k \lambda_k \leq a_m(u) \leq \max_k \lambda_k \tag{12}$$

where the $a_m(u)$ are the eigenvalues of $F'(u)$. Such subcharacteristic conditions are well-known in the general setting of relaxation problems [31], [20]. In the scalar case the fact that M is MND is a sufficient condition of convergence, for the initial boundary value problem, as well as for the Cauchy problem (see [23]). For some systems, D. Serre [29] shows convergence under condition (12), while monotonicity is involved in F. Bouchut's work about compatibility with entropy conditions, [6].

From the numerical viewpoint, we discretize the problem (5), (6), (7) and making ϵ tend to zero, we obtain a numerical scheme for the initial boundary value problem for the conservation law (1). Such kinetic schemes are known for the Cauchy problem. They were first introduced in the framework of the Boltzmann approach of hydrodynamic problems, see for example [28], [27]. For general conservation laws, S. Jin and Z. Xin introduced a relaxation approximation in 1995 [16] and constructed related numerical schemes for the Cauchy problem. In one space dimension, Jin and Xin's system (10) owns a kinetic form (5). We also mention the important paper of A. Harten, P.D. Lax, and B. van Leer [15] where a kinetic interpretation of flux splitting schemes is given. Numerical schemes based on the system (5) have been constructed, and their convergence was studied in [1] for the Cauchy problem. The main tool is the splitting between the two present operators: transport and collision. The first is easily solved thanks to its linearity and diagonality, while some great properties of the macroscopic variables during the collision allow us an explicit formula to solve the second step. This technique gives a Riemann-solver free scheme, for a scalar equation ($u \in \mathbb{R}$), as for systems. Higher order precision in space as well as in time can be reached easily.

Here, our goal is to extend these techniques to the semi-bounded domain case. The numerical schemes are constructed for a general system of conservation laws, while stability and convergence are studied for the scalar case only. The main difficulties lie in the influence of the boundary condition in all estimates, particularly the ones on the total variation, so that the involved tools are new.

The paper is organized as follows: in section 2, we explain the method used to produce kinetic schemes, afterwards, in section 3, we establish some standard stability results necessary to prove compactness properties of the approximate solution of (1) in the scalar case, then we prove that this solution tends to a weak solution of (1) . The convergence having been established, we show in section 4, that kinetic schemes give the entropy solution in the case of first order approximation in space, when the discretization step goes to 0, in the sense of [2]. Using results of the first part, we show in section 5 how to construct second order schemes specific to the boundary problem. It is not a difficult task to extend the ideas of this paper, at least formally, to higher space dimensions, considering the multidimensional kinetic models constructed in [1]. In section 6, we present some numerical experiments for both scalar and system cases, in one and two space dimensions.

2 The kinetic schemes for initial-boundary condition

2.1 Notations

Before giving the explicit form of the kinetic schemes, we introduce some useful notations. In this work we restrict ourselves to the study on uniform grid, that is

. in space

$$\mathbb{R}^+ = \bigcup_{i \in \mathbb{N}} [i\Delta x, (i+1)\Delta x]$$

. in time

$$(0,T) = \bigcup_{0 \leq n \leq L-1} [n\Delta t, (n+1)\Delta t]$$

The coefficients Δt, Δx represent time and space steps, and we call :

$$x_{i-\frac{1}{2}} = i\Delta x, \quad x_i = (i+\tfrac{1}{2})\Delta x, \quad t^n = n\Delta t.$$

We are looking for a discrete solution of the form

$$u_\Delta(t,x) = \sum_{n=0}^{L-1} \sum_{i\in\mathbb{N}} u_i^n \mathbb{I}_{[x_{i-\frac{1}{2}},x_{i+\frac{1}{2}}]\times[t^n,t^{n+1}[}(t,x)$$

where u_i^n is an approximation of

$$\frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(t^n,x)\,dx$$

We define in the same way,

$$u_\Delta^n(x) = \sum_{i\geq 0} u_i^n \mathbb{I}_{[x_{i-\frac{1}{2}},x_{i+\frac{1}{2}}]}(x).$$

The notations for f are similar. The data of the problem (1) are approximated by

$$\begin{aligned} u_b^n &= u_{-1}^n = \tfrac{1}{\Delta t}\textstyle\int_{t^n}^{t^{n+1}} u_b(t)dt, & 0\leq n\leq L-1,\\ u_i^0 &= \tfrac{1}{\Delta x}\textstyle\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u^0(x)dx, & i\in\mathbb{N}, \end{aligned}$$

and we take for all k:

$$\begin{aligned} f_{b,k}^n &= f_{-1,k}^n = M_k(u_b^n), & 0\leq n\leq L-1,\\ f_{i,k}^0 &= M_k(u_i^0), & i\in\mathbb{N}. \end{aligned}$$

2.2 First order in time discretizations

To discretize the system (5) we carry out an operator split, that is we first solve the transport part, and afterwards the collision one. In each case, we explain precisely what is done on the boundary.

Transport

We discretize on $[t^n,t^{n+1}]$ the set of transport equations:

$$\left\{\begin{array}{ll} \partial_t f_k + \lambda_k\partial_x f_k = 0, \quad k=1,\ldots,N\,, & \\ f(t^n,x) = f_\Delta^n(x), & \\ f_k(t,0) = f_{b,k}^n & if\ \ \lambda_k>0. \end{array}\right. \tag{13}$$

We write here the scheme in the formulation of Harten [14] as a convex combination, that will include both first and second order space approximation of the transport part :

$$i\geq 0, \quad \left\{\begin{array}{ll} \lambda_k>0 & f_{i,k}^{n+\frac{1}{2}} = f_{i,k}^n(1-D_{i-\frac{1}{2},k}^n) + D_{i-\frac{1}{2},k}^n f_{i-1,k}^n,\\ \lambda_k\leq 0 & f_{i,k}^{n+\frac{1}{2}} = f_{i,k}^n(1-D_{i+\frac{1}{2},k}^n) + D_{i+\frac{1}{2},k}^n f_{i+1,k}^n. \end{array}\right. \tag{14}$$

Remark that the boundary value $f_{b,k}^n = f_{-1,k}^n$ is needed only for positive velocities.

The time step restriction is:

$$\max_{1\leq k\leq N} |\lambda_k|\frac{\Delta t}{\Delta x}\leq 1. \tag{15}$$

Collision

For the whole grid the compatibility relations (8) allow us to solve :

$$\left\{\begin{array}{ll} \partial_t f_k(t) = \frac{1}{\epsilon}(M_k(u)-f_k(t)) & \forall k\in\{1,\ldots,N\},\\ f_k(t^n) = f_k^{n+\frac{1}{2}}, & \\ u = Pf = \sum_k f_k & \end{array}\right.$$

exactly on $[t^n,t^{n+1}]$, giving :

$$u_i^{n+1} = u_i^{n+\frac{1}{2}} \tag{16}$$

and

$$f_{i,k}(t^{n+1}) = (1 - e^{-\frac{\Delta t}{\epsilon}})M_k(u_i^{n+\frac{1}{2}}) + e^{-\frac{\Delta t}{\epsilon}} f_{i,k}^{n+\frac{1}{2}}, \quad i \geq 0. \tag{17}$$

for fixed ϵ. Thus we may take the limit $\epsilon = 0$ and we have

$$f_{i,k}^{n+1} = M_k(u_i^{n+\frac{1}{2}}) = M_k(u_i^{n+1}), \ \ i \geq 0. \tag{18}$$

We underline that the collision step is solved at the border cell in the same way as it is inside the grid. Thus, we have defined a complete step in time, for the whole grid, so that we can write the complete scheme in macroscopic variables.

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}(g_{i+\frac{1}{2}}^n - g_{i-\frac{1}{2}}^n), \quad i \geq 0 \tag{19}$$

where

$$\frac{\Delta t}{\Delta x} g_{i+\frac{1}{2}}^n = \sum_{\lambda_k > 0} D_{i-\frac{1}{2},k}^n M_k(u_i^n) - \sum_{\lambda_k \leq 0} D_{i+\frac{1}{2},k}^n M_k(u_{i+1}^n), \quad i \geq 0.$$

When transport is approximated by the upwind first order scheme, the previous fluxes can be written as

$$g_{i+\frac{1}{2}}^n = \sum_{\lambda_k > 0} \lambda_k M_k(u_i^n) + \sum_{\lambda_k < 0} \lambda_k M_k(u_{i+1}^n),$$

while the flux on the boundary interface is

$$g_{-\frac{1}{2}}^n = \sum_{\lambda_k > 0} \lambda_k M_k(u_b^n) + \sum_{\lambda_k < 0} \lambda_k M_k(u_0^n).$$

Note that there is no ghost cell, and that on the whole grid we apply the same scheme. The time step restriction is the one given by the transport part of the scheme, namely condition (15) . As we always suppose that the Maxwellian functions are MND, the subcharacteristic condition (12) is satisfied. Hence, we have also the usual condition:

$$\max_{1 \leq m \leq p} |a_m(u)| \frac{\Delta t}{\Delta x} \leq 1.$$

2.3 Higher order in time

To construct higher order in time approximations we remark that the we have obtained numerical flux functions g such that formula (19) may be viewed as a first order discretization of the differential equation:

$$u_i' = u_i - \frac{\overline{g_{i+\frac{1}{2}}} - \overline{g_{i-\frac{1}{2}}}}{\Delta x} \ . \tag{20}$$

Here $\overline{g_{i+\frac{1}{2}}}$ is the limit of $g_{i+\frac{1}{2}}$ when $\Delta t \to 0$ and Δx is fixed. This limit exists if one uses for (13) an upwind first order scheme or a higher order MUSCL reconstruction-transport-projection scheme, see the examples below. Therefore, one may discretize (20) with any high order, for example Runge-Kutta method. The stability condition remains identical. As usual the order in time has to be related with order in space.

3 Stability results in the scalar case

For sections 3 and 4, we restrict ourselves to the convergence of kinetic numerical schemes towards the unique entropy solution of (1) where u is a scalar, and F a flux in $C^1(\mathbb{R}; \mathbb{R})$. Following are some preliminary results useful for the rest of this work.

3.1 Preliminaries

To recall certain tools of [2] useful to include boundary effects in the entropy inequality, we present the following definition.

Definition 3.1 *A function $u \in BV(\mathbb{R}^+ \times]0,T[)$ is a solution of problem (1) (2) (3) if for all $c \in \mathbb{R}$ and all non negative test function $\phi \in C^2(\mathbb{R}^+ \times]0,T[)$ with a compact support in $\mathbb{R}^+ \times]0,T[$, it satisfies the following inequality :*

$$
\begin{aligned}
&\int_{\mathbb{R}^+}\int_0^T |u-c|\partial_t\phi + \operatorname{sgn}(u-c)(F(u)-F(c))\partial_x\phi \, dt\, dx \\
&\qquad \int_{\mathbb{R}^+} |u^0(x)-c|\phi(0,x)\, dx + \int_0^T \operatorname{sgn}(u_b(t)-c)(F(u(t,0))-F(c))\phi(t,0)\, dt \geq 0
\end{aligned}
\tag{21}
$$

This entropy inequality is used in [2] to obtain uniqueness in L^1, using Kružkov techniques.

Now we present a very useful tool, specific to kinetic approximation, see [6] for more general formulation. The following theorem links the entropy functions of (1) and the existence of kinetic entropy functions associated to (5) . Note that all results use only an invariant domain I in the phase space ($u \in I$), this is coherent with the boundary problem.

Theorem 3.1 *For a given entropy pair (S,G) of (1) , suppose that $u \in I \subset \mathbb{R}$ and that $f_k \in I_k$ then the two following results are equivalent :*

*i) The Maxwellians are monotone non decreasing (*MND*) functions of u,*

ii) there exists S_k such that :

(E1) $\forall k$ S_k is a convex function of f_k

(E2) $\sum_k S_k(M_k(u)) = S(u) + C$ for $u \in I$ and C is a constant.

(E3) if $u_f = \sum_k f_k \in I$ then the Maxwellians minimize the kinetic entropy sum :

$$\sum_k S_k(M_k(u_f)) \leq \sum_k S_k(f_k).$$

3.2 Assumptions

Our hypothesis are the following :

* On the data, we suppose that

$$(H1)\begin{cases} u^0(x) \in L^\infty(\mathbb{R}_+) \cap BV(\mathbb{R}_+) \cap L^1(\mathbb{R}_+) \\ u_b(t) \in L^\infty(0,T) \cap BV(0,T) \cap L^1(0,T) \end{cases}$$

* On the Maxwellians, we need monotonicity (MND), that is

$$\text{(H2)} \quad M_k'(u) \geq 0 \ \ \forall u \in I, \quad I = [-\mu_\infty, \mu_\infty]$$

with

$$\mu_\infty = \max(||u_b||_{L^\infty(\mathbb{R}_t^+)}, ||u^0||_{L^\infty(\mathbb{R}_x^+)})$$

* On the transport part of the scheme we impose that

$$\text{(H3)} \quad \forall i,n,k \ \ D^n_{i-\frac{1}{2},k} \in [0,1]$$

3.3 L^∞ bounds

Lemma 3.1 *Under the assumptions (H1)-(H3), the kinetic schemes (19) are L^∞ stable: for all n and all $i \geq 0$:*

$$u_i^n \in I.$$

Proof : Note that by (H2), the boundary condition, for every n can be estimated as

$$M_k(-\mu_\infty) \leq f^n_{b,k} \leq M_k(\mu_\infty).$$

Results are shown by induction. Suppose that for all $i \geq 0$:

$$-\mu_\infty \leq u_i^n \leq \mu_\infty.$$

Then

$$M_k(-\mu_\infty) \leq f_{i,k}^n \leq M_k(\mu_\infty).$$

Thus, by transport's convex formulation

$$\begin{aligned}(1-D^n_{i-\frac{1}{2},k})M_k(-\mu_\infty)+D^n_{i-\frac{1}{2},k}M_k(-\mu_\infty) &\leq (1-D^n_{i-\frac{1}{2},k})f^n_{i,k}+D^n_{i-\frac{1}{2},k}f^n_{i-1,k} \leq \\ &\leq (1-D^n_{i-\frac{1}{2},k})M_k(\mu_\infty)+D^n_{i-\frac{1}{2},k}M_k(\mu_\infty).\end{aligned}$$

for the positive velocity components, for example, which gives

$$M_k(-\mu_\infty) \leq f_{i,k}^{n+\frac{1}{2}} \leq M_k(\mu_\infty).$$

The first remark enables us to claim the same property on the first cell as well. Because $u_i^{n+\frac{1}{2}} = \sum_k f_{i,k}^{n+\frac{1}{2}}$ we have

$$-\mu_\infty \leq u_i^{n+\frac{1}{2}} = u_i^{n+1} \leq \mu_\infty$$

which ends the proof. $\diamondsuit$

3.4 BV bounds

This part is the most difficult to establish, because it is specific to the boundary problem. In fact, in the Cauchy problem when applying the Green-Gauss formula there is no border term, while in the initial-boundary value problem there is, and need to be controlled. For the our scheme, the estimates do not involve the same techniques as in the continuous case (developed in [21]). In fact in the discrete case, the BV bounds are obtained thanks to an "entropy-like" relationship, as in the continuous case one uses the kinetic equations.

In this section we first show that the estimates of total variation in space do not need the total variation in time on the boundary, as they do in [24], but only for convergence towards the entropy solution (see section 4.1 below).

Proposition 3.1 *Under the same hypotheses (H1)-(H3), we have*

$$\begin{aligned}TV(u_\Delta^{n+1}) = TV(f_\Delta^{n+1}) &\leq TV(u_\Delta^0) + TV(u_{b,\Delta}) + |u_0^0 - u_b^0| \\ &\leq TV(u^0) + TV(u_b) + 2\mu_\infty\end{aligned}$$

where $TV(f_\Delta^{n+1}) = \sum_k TV(f_{k,\Delta}^{n+1})$.

Proof :

1 Collision

Thank to (H2), it is easy to show that

$$TV(u_\Delta^{n+1}) = TV(f_\Delta^{n+1}) \leq TV(f_\Delta^{n+\frac{1}{2}})$$

2 Transport

The boundary condition is introduced in this part of the scheme so that we must care on the different directions of transport.

. $\lambda_k > 0$

For the first cell we have :

$$|f_{1,k}^{n+\frac{1}{2}} - f_{0,k}^{n+\frac{1}{2}}| \leq (1-D^n_{-\frac{1}{2},k})|f^n_{1,k} - f^n_{0,k}| + D^n_{-\frac{1}{2},k}|f^n_{0,k} - M_k(u_b^n)|$$

For the whole grid, we then have

$$\begin{aligned}\sum_{i=1}^{\infty}|f_{i,k}^{n+\frac{1}{2}}-f_{i-1,k}^{n+\frac{1}{2}}| &\le \sum_{i=1}^{\infty}(1-D_{i-\frac{1}{2},k}^{n})|f_{i,k}^{n}-f_{i-1,k}^{n}| \\ &+\sum_{i=1}^{\infty}D_{i-\frac{1}{2},k}^{n}|f_{i,k}^{n}-f_{i-1,k}^{n}|+D_{-\frac{1}{2},k}^{n}|f_{0,k}^{n}-M_k(u_b^n)| \\ &\le \sum_{i=1}^{\infty}|f_{i,k}^{n}-f_{i-1,k}^{n}|+D_{-\frac{1}{2},k}^{n}|f_{0,k}^{n}-M_k(u_b^n)|\end{aligned}$$

. $\lambda_k < 0$

In the same way

$$\begin{aligned}\sum_{i=1}^{\infty}|f_{i,k}^{n+\frac{1}{2}}-f_{i-1,k}^{n+\frac{1}{2}}| &\le \sum_{i=1}^{\infty}(1-D_{-\frac{1}{2},k}^{n})|f_{i,k}^{n}-f_{i-1,k}^{n}| \\ &+\sum_{i=2}^{\infty}D_{-\frac{1}{2},k}^{n}|f_{i,k}^{n}-f_{i-1,k}^{n}| \\ &\le \sum_{i=1}^{\infty}|f_{i,k}^{n}-f_{i-1,k}^{n}|-D_{\frac{1}{2},k}^{n}|f_{1,k}^{n}-f_{0,k}^{n}|\end{aligned}$$

. $\lambda_k = 0$

$$\sum_{i=1}^{\infty}|f_{i,k}^{n+\frac{1}{2}}-f_{i-1,k}^{n+\frac{1}{2}}|=\sum_{i=1}^{\infty}|f_{i,k}^{n}-f_{i-1,k}^{n}|$$

The total variation of the transport part can thus be written as

$$\sum_k TV(f_k^{n+\frac{1}{2}}) \le \sum_k TV(f_k^n)+\sum_{\lambda_k>0}D_{-\frac{1}{2},k}^{n}|f_{0,k}^{n}-M_k(u_b^n)|-\sum_{\lambda_k<0}D_{\frac{1}{2},k}^{n}|f_{1,k}^{n}-f_{0,k}^{n}|$$

For the whole step we have,

$$\sum_k TV(f_k^{n+1}) \le \sum_k TV(f_k^n)+\sum_{\lambda_k>0}D_{-\frac{1}{2},k}^{n}|f_{0,k}^{n}-M_k(u_b^n)|-\sum_{\lambda_k<0}D_{\frac{1}{2},k}^{n}|f_{1,k}^{n}-f_{0,k}^{n}| \tag{22}$$

To control the two last terms in the previous estimate, we need the following key argument

Lemma 3.2 *Under the same hypotheses, we have*

$$\begin{aligned}\sum_{\lambda_k>0}D_{-\frac{1}{2},k}^{n}|f_{0,k}^{n}-M_k(u_b^n)| & -\sum_{\lambda_k<0}D_{\frac{1}{2},k}^{n}|f_{1,k}^{n}-f_{0,k}^{n}| \le \\ & \sum_k|f_{0,k}^{n}-M_k(u_b^{n-1})|-\sum_k|f_{0,k}^{n+1}-M_k(u_b^n)|+|u_b^n-u_b^{n-1}|\end{aligned}$$

Proof : In a standard way, we have

$$\sum_k|f_{0,k}^{n+1}-M_k(c)|=\sum_k|M_k(u_0^{n+\frac{1}{2}})-M_k(c)|=|u_0^{n+\frac{1}{2}}-c|\le\sum_k|f_{0,k}^{n+\frac{1}{2}}-M_k(c)|$$

Now taking $c=u_b^n$, we have that

$$\sum_k|f_{0,k}^{n+1}-M_k(u_b^n)|\le\sum_k|f_{0,k}^{n+\frac{1}{2}}-M_k(u_b^n)|$$

As for the transport part, we have

$$
\begin{aligned}
\sum_k |f_{0,k}^{n+\frac12} - M_k(u_b^n)| &\le \sum_{\lambda_k>0} |(1-D^n_{-\frac12,k})f_{0,k}^n + D^n_{-\frac12,k}M_k(u_b^n) - M_k(u_b^n)| \\
&+ \sum_{\lambda_k\le 0} |(1-D^n_{\frac12,k})f_{0,k}^n + D^n_{\frac12,k}f_{1,k}^n - M_k(u_b^n)| \\
&\le \sum_{\lambda_k>0}(1-D^n_{-\frac12,k})|f_{0,k}^n - M_k(u_b^n)| \\
&+ \sum_{\lambda_k\le 0}\left[D_{\frac12,k}|f_{0,k}^n - f_{1,k}^n| + |f_{0,k}^n - M_k(u_b^n)|\right] \\
&\le \sum_k |f_{0,k}^n - M_k(u_b^n)| \\
&+ \sum_{\lambda_k<0} D_{\frac12,k}|f_{0,k}^n - f_{1,k}^n| - \sum_{\lambda_k>0} D_{-\frac12,k}|f_{0,k}^n - M_k(u_b^n)| \\
&\le \sum_k |f_{0,k}^n - M_k(u_b^{n-1})| + \sum_k |M_k(u_b^n) - M_k(u_b^{n-1})| \\
&+ \sum_{\lambda_k<0} D_{\frac12,k}|f_{0,k}^n - f_{1,k}^n| - \sum_{\lambda_k>0} D_{-\frac12,k}|f_{0,k}^n - M_k(u_b^n)|,
\end{aligned}
$$

which ends the proof. ♢
Thanks to the last result, (22) becomes

$$TV(f_\Delta^{n+1}) \le TV(f_\Delta^n) + \sum_k |f_{0,k}^n - M_k(u_b^{n-1})| - \sum_k |f_{0,k}^{n+1} - M_k(u_b^n)| + |u_b^n - u_b^{n-1}|$$

which by induction writes

$$
\begin{aligned}
TV(f_\Delta^{n+1}) &\le TV(f_\Delta^0) + \sum_k |f_{0,k}^0 - M_k(u_b^0)| + TV(u_{b,\Delta}). \\
&\le TV(u_\Delta^0) + |u_{0,\Delta}^0 - u_{b,\Delta}^0| + TV(u_{b,\Delta})
\end{aligned}
$$

We projected the initial condition on the equilibrium manifold, which gives the desired result. ♢

3.5 $L^1(\mathbb{R}^+)$ bounds

We show now how to use the TV stability to establish the L^1 estimates.

Lemma 3.3 *Under the assumptions (H1)-(H3), the kinetic schemes (19) are L^1 stable.*

$$||u_\Delta^{n+1}||_{L^1(\mathbb{R}^+)} = ||f_\Delta^{n+1}||_{L^1(\mathbb{R}^+)} \le ||u^0||_{L^1(\mathbb{R}^+)} + ||u_b||_{L^1(0,T)} + T\ CK$$

where $||f_\Delta^{n+1}||_{L^1(\mathbb{R}^+)} = \sum_k ||f_{k,\Delta}^{n+1}||_{L^1(\mathbb{R}^+)}$, *$C$ is proportional to* $\sup_k |\lambda_k|$, *and*

$$K = TV(u^0, \mathbb{R}^+) + TV(u_b, (0,T) + 2\mu_\infty. \tag{23}$$

Proof :

1 Collision

By (H2) we have

$$||u_\Delta^{n+1}||_{L^1(\mathbb{R}^+)} = ||f_\Delta^{n+1}||_{L^1(\mathbb{R}^+)} \le ||f_\Delta^{n+\frac12}||_{L^1(\mathbb{R}^+)}.$$

2 Transport part

For positive velocity components, we can see that

$$|f_{i,k}^{n+\frac12}| \le |f_{i,k}^n| + D^n_{i-\frac12,k}|f_{i,k}^n - f_{i-1,k}^n|$$

for all $i \ge 1$, while on the edge, we have

$$|f_{0,k}^{n+\frac12}| \le (1 - D^n_{-\frac12,k})|f_{i,k}^n| + D_{-\frac12,k}|M_k(u_b^n)|.$$

So that after summing over i, one has

$$\sum_{i\geq 0}|f_{i,k}^{n+\frac{1}{2}}| \leq \sum_{i\geq 0}|f_{i,k}^n| + TV(f_{i,k}^n) + |M_k(u_b^n)|$$

For the non entering characteristics, we have no boundary terms, so that for every $i \geq 0$ we have

$$|f_{i,k}^{n+\frac{1}{2}}| \leq |f_{i,k}^n| + D_{i+\frac{1}{2},k}^n|f_{i,k}^n - f_{i+1,k}^n|,$$

which after summing over the whole grid gives

$$\sum_{i\geq 0}|f_{i,k}^{n+\frac{1}{2}}| \leq \sum_{i\geq 0}|f_{i,k}^n| + TV(f_{k,\Delta}^n).$$

For all components we can write

$$\sum_k ||f_{k,\Delta}^{n+\frac{1}{2}}||_{L^1(\mathbb{R}^+)} \leq \sum_k ||f_{k,\Delta}^n||_{L^1(\mathbb{R}^+)} + \Delta x \sum_k TV(f_{k,\Delta}^n) + \Delta x \sum_{\lambda_k>0} |M_k(u_b^n)|$$

which gives, on u

$$\begin{aligned} ||u_\Delta^{n+1}||_{L^1(\mathbb{R}^+)} &\leq ||u_\Delta^n||_{L^1(\mathbb{R}^+)} + \Delta x TV(u_\Delta^n, \mathbb{R}^+) + \Delta x \sum_{\lambda_k>0} |M_k(u_b^n)| \\ &\leq ||u^0||_{L^1(\mathbb{R}^+)} + n\Delta x K + ||u_b||_{L^1(0,T)} \end{aligned}$$

where K is defined in (23) . We conclude by noting that Δt and Δx satisfy the CFL condition (15) and their ratio is fixed.

$\diamondsuit$

3.6 Convergence results

Lemma 3.4 *Let u_Δ be the numerical solution given by (19) . If the assumptions (H1)-(H3) are satisfied, then u_Δ is equicontinuous in time i.e.:*

$$||u_\Delta^{n+1} - u_\Delta^n||_{L^1(\mathbb{R}_+)} \leq CK\Delta x$$

where K is defined in (23) .

Proof : For the transport part, we take for example the positive velocity components, for $i \geq 1$:

$$\begin{aligned} |f_{i,k}^{n+\frac{1}{2}} - f_{i,k}^n| &\leq D_{i-\frac{1}{2}}^n|f_{i,k}^n - f_{i-1,k}^n| \\ &\leq |f_{i,k}^n - f_{i-1,k}^n| \end{aligned}$$

as for the boundary cell one has :

$$|f_{0,k}^{n+\frac{1}{2}} - f_{0,k}^n| \leq |f_{0,k}^n - M_k(u_b^n)| \leq 2M_k(\mu_\infty).$$

and summing over $i \geq 1$ we have that

$$\sum_{i\geq 0}|f_{i,k}^{n+\frac{1}{2}} - f_{i,k}^n| \leq TV(f_k^n, \mathbb{R}^+) + 2M_k(\mu_\infty)$$

Doing the same on other components, we have that

$$\begin{aligned} |f_{i,k}^{n+\frac{1}{2}} - f_{i,k}^n| &\leq D_{i+\frac{1}{2}}^n|f_{i,k}^n - f_{i+1,k}^n| \\ &\leq |f_{i,k}^n - f_{i+1,k}^n| \end{aligned}$$

which gives finally,

$$\begin{aligned} ||u_\Delta^{n+1} - u_\Delta^n||_{L^1(\mathbb{R}^+)} &= \Delta x \sum_i |u^{n+1} - u_i^n| = \Delta x \sum_i |u^{n+\frac{1}{2}} - u_i^n| \\ &\leq \Delta x (\sum_k TV(f_k^n, \mathbb{R}^+) + 2 \sum_{\lambda_k > 0} M_k(\mu_\infty)) \\ &\leq \Delta x (TV(u_\Delta^n, \mathbb{R}^+) + 2\mu_\infty). \end{aligned}$$

$\diamondsuit$

As a corollary, we have the equicontinuity in time :

Lemma 3.5 *Under the same hypotheses as before,*

$$\forall t, t' \in [0,T], \quad ||u_\Delta(t) - u_\Delta(t')||_{L^1(\mathbb{R}_+)} \leq CK(\Delta t + |t - t'|) \tag{24}$$

where K is defined in (23) .

Theorem 3.2 *Let $T > 0$, suppose that hypothesis (H1)-(H3) hold, and let $\frac{\Delta t}{\Delta x}$ be constant. As $\Delta x \to 0$ the sequence u_Δ converges in $\mathrm{L}^\infty(0,T;\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}))$ to a solution u of (1)(2), and $u \in C^0([0,T], \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})) \cap \mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{R}))$.*

Proof : We follow exactly the method of [10] and just give a sketch of the proof: for all $t \geq 0$, $\{u_\Delta(t), \quad \Delta t > 0\}$ is bounded in $\mathrm{L}^1 \cap \mathrm{L}^\infty$. Moreover, by proposition 3.1, we can apply Fréchet-Kolmogorov theorem and obtain a relatively compact set of ${\mathrm{L}^1}_{\mathrm{loc}}$. Using the equicontinuity property (24) as in the proof of Ascoli-Arzela's theorem we obtain convergence in $\mathrm{L}^\infty(0,T;\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}))$ and a.e. the limit is a solution of the problem by Lax-Wendroff's theorem.
$\diamondsuit$

4 First order approximation (FO) and entropy solution

In this section, we restrict ourselves to the first order in space approximation, that is

$$D^n_{i-\frac{1}{2},k} = \xi_k = |\lambda_k| \frac{\Delta t}{\Delta x}.$$

In that case, assumption (H3) is equivalent to the time step restriction (15) . Thanks to the previous section we can claim the following theorem.

Theorem 4.1 *Suppose that hypothesis (H1)-(H3) are satisfied. Then, when Δx tends to zero and $\frac{\Delta t}{\Delta x}$ is constant,*

$$u_\Delta \to u \text{ in } L^\infty([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^+)).$$

Here u is a weak solution of (1)(2), and $u \in C^0([0,T], L^1_{\mathrm{loc}}(\mathbb{R}^+))$.

To prove that the numerical approximation calculated with our first order scheme tends to the entropy solution, we apply the Lax-Wendroff type arguments. As we established the convergence we can now show the convergence to the unique entropy solution when $\Delta x \to 0$. This is necessary because the boundary introduces new terms that must satisfy certain properties of stability.

To show the consistency of the trace with the boundary condition (see [2]), we need the following estimates on the trace of the solution given by the FO scheme.

Lemma 4.1 *Under the assumptions (H1)-(H3), one has*

$$\begin{aligned} \sum_{p=1}^n \sum_{\lambda_k < 0} \xi_k |M_k(u_0^p) - M_k(u_0^{p-1})| &\leq TV(u_{b,\Delta}, (0,T)) + TV(u^0_\Delta, \mathbb{R}^+) + |u_0^0 - u_b^0|. \\ &\leq TV(u_b, (0,T)) + TV(u^0, \mathbb{R}^+) + 2\mu_\infty \end{aligned}$$

Proof : As before, we use the convexity of the absolute value on the linear transport, to obtain the estimates on the trace.

. $\lambda_k > 0$

$$|f_{i,k}^{n+\frac{1}{2}} - f_{i,k}^{n-\frac{1}{2}}| \leq (1-\xi_k)|f_{i,k}^n - f_{i,k}^{n-1}| + \xi_k |f_{i-1,k}^n - f_{i-1,k}^{n-1}|$$

which gives after summing over all cells of the grid:

$$\sum_{i=0}^{\infty} |f_{i,k}^{n+\frac{1}{2}} - f_{i,k}^{n-\frac{1}{2}}| \leq \sum_{i=0}^{\infty} |f_{i,k}^n - f_{i,k}^{n-1}| + \xi_k |M_k(u_b^n) - M_k(u_b^{n-1})|$$

. $\lambda_k < 0$

$$|f_{i,k}^{n+\frac{1}{2}} - f_{i,k}^{n-\frac{1}{2}}| \leq (1-\xi_k)|f_{i,k}^n - f_{i,k}^{n-1}| + \xi_k |f_{i+1,k}^n - f_{i+1,k}^{n-1}|$$

Applying the same ideas on the negative velocity components.

$$\sum_{i=0}^{\infty} |f_{i,k}^{n+\frac{1}{2}} - f_{i,k}^{n-\frac{1}{2}}| \leq \sum_{i=0}^{\infty} |f_{i,k}^n - f_{i,k}^{n-1}| - \xi_k |M_k(u_0^n) - M_k(u_0^{n-1})|$$

. $\lambda_k = 0$

$$|f_{i,k}^{n+\frac{1}{2}} - f_{i,k}^{n-\frac{1}{2}}| = |f_{i,k}^n - f_{i,k}^{n-1}|$$

The whole transport step yields :

$$\begin{aligned} \sum_{i,k} |f_{i,k}^{n+\frac{1}{2}} - f_{i,k}^{n-\frac{1}{2}}| \quad &\leq \sum_{i,k} |f_{i,k}^n - f_{i,k}^{n-1}| \\ &+ \sum_{\lambda_k>0} \xi_k |M_k(u_b^n) - M_k(u_b^{n-1})| - \sum_{\lambda_k<0} \xi_k |M_k(u_0^n) - M_k(u_0^{n-1})| \end{aligned}$$

For the projection step,

$$\sum_k |f_{i,k}^{n+1} - f_{i,k}^n| \leq \sum_k |f_{i,k}^{n+\frac{1}{2}} - f_{i,k}^{n-\frac{1}{2}}| \quad \forall i \geq 0$$

that gives

$$\begin{aligned} \sum_{\lambda_k<0} \xi_k |M_k(u_0^n) - M_k(u_0^{n-1})| \quad &\leq |u_b^n - u_b^{n-1}| \\ &+ \sum_{i,k} |f_{i,k}^n - f_{i,k}^{n-1}| - \sum_k |f_{i,k}^{n+1} - f_{i,k}^n|. \end{aligned}$$

By summing one gets :

$$\sum_{p=1}^{n} \sum_{\lambda_k<0} \xi_k |M_k(u_0^p) - M_k(u_0^{p-1})| \leq TV(u_b,(0,T)) + \sum_{i,k} |f_{i,k}^1 - f_{i,k}^0|$$

or

$$\sum_{i,k} |f_{i,k}^1 - f_{i,k}^0| = \sum_{i,k} |M_k(u_i^{\frac{1}{2}}) - M_k(u_i^0)| = \sum_i |u_i^{\frac{1}{2}} - u_i^0|$$

and this can be estimated by

$$\sum_i |u_i^{\frac{1}{2}} - u_i^0| \leq TV(u^0, \mathbb{R}^+) + |u_0^0 - u_b^0|$$

$\diamond$

Let pose

$$\omega_{k,\Delta} = (\omega_k^n)_{1\leq n\leq N} = (M_k(u_0^n))_{1\leq n\leq N} \quad \forall k \text{ s.t. } \lambda_k > 0$$

Corollary 4.1 *Under the previous assumptions (H1)-(H3), $\{\omega_{k,\Delta}, \Delta x > 0\}$ is bounded in $BV(0,T)$ and we can extract an $L^1(0,T)$ convergent subsequence.*

4.1 Discrete entropy inequalities

To apply usual tools of the Lax-Wendroff theorem, we do not need to specify how to treat the boundary, so that we use the standard discrete entropy inequalities already established for the Cauchy problem.

Lemma 4.2 *Under the hypotheses (H1)-(H3), and for $i \geq 0$, we have :*

$$S(u_i^{n+1}) - S(u_i^n) + \frac{\Delta t}{\Delta x}[G^n_{i+\frac{1}{2}} - G^n_{i-\frac{1}{2}}] \leq 0$$

where

$$S(u) = \textstyle\sum_k S_k(M_k(u))$$

and

$$G^n_{i+\frac{1}{2}} = G(u^n_{i+1}, u^n_i) = \{\textstyle\sum_{\lambda_k>0} \lambda_k S_k(M_k(u_i^n)) + \sum_{\lambda_k<0} \lambda_k S_k(M_k(u^n_{i+1}))\}$$

while the boundary flux reads

$$G^n_{-\frac{1}{2}} = \{\sum_{\lambda_k>0} \lambda_k S_k(M_k(u_b^n)) + \sum_{\lambda_k<0} \lambda_k S_k(M_k(u_0^n))\}$$

Proof : We use the transport-projection formulation as in (14) and in (18) . Then it is enough to use the transport's convex form and to apply a convex entropy.

$$\begin{aligned} \lambda_k > 0 \quad & S_k(f_{i,k}^{n+\frac{1}{2}}) \leq S_k(f^n_{i,k})(1-\xi_k) + \xi_k S_k(f^n_{i-1,k}) \\ \lambda_k \leq 0 \quad & S_k(f_{i,k}^{n+\frac{1}{2}}) \leq S_k(f^n_{i,k})(1-\xi_k) + \xi_k S_k(f^n_{i+1,k}) \end{aligned} \tag{25}$$

Summing up over k :

$$\sum_k S_k(f_i^{n+\frac{1}{2}}) - \sum_k S_k(f_i^n) \leq \sum_{\lambda_k>0} \xi_k(S_k(f^n_{i,k}) - S_k(f^n_{i-1,k})) + \sum_{\lambda_k<0} (S_k(f^n_{i,k}) - S_k(f^n_{i+1,k})) \tag{26}$$

And because $S_k(f_i^n) = S_k(M_k(u_i^n))$,

$$S(u_i^n) = \sum_k S_k(f_i^n).$$

and

$$\sum_{\lambda_k>0} \xi_k(S_k(f^n_{i,k}) - S_k(f^n_{i-1,k})) + \sum_{\lambda_k<0} (S_k(f^n_{i,k}) - S_k(f^n_{i+1,k})) = -\frac{\Delta t}{\Delta x}[G^n_{i+\frac{1}{2}} - G^n_{i-\frac{1}{2}}].$$

Because the Maxwellian functions are MND, by theorem (3.1) , they minimize the microscopic entropy functions, for a given state :

$$S(u_i^{n+1}) = \sum_k S_k(f_{i,k}^{n+1}) \leq \sum_k S_k(f_{i,k}^{n+\frac{1}{2}}) \tag{27}$$

using this estimate in (26) , we obtain the desired discrete entropy inequality. $\diamondsuit$

4.2 Passing to the limit towards a continuous inequality

Theorem 4.2 *The solution given by the first order scheme (19) tends, when the discretization step Δx goes to 0 and $\frac{\Delta t}{\Delta x}$ is constant, towards the entropy solution satisfying*

$$\int_{\mathbb{R}_+\times(0,T)} |u-c|\partial_t\phi \quad +\mathrm{sgn}(u-c)(f(u)-f(c))\partial_x\phi dxdt + \int_{\mathbb{R}_+} |u(0,x)-c|\phi(0,x)dx$$

$$+\int_0^T \mathrm{sgn}(u_b(t)-c)(f(u(t,0))-f(c))\phi(t,0)dt \geq 0$$

for every positive test function $\phi \in C^1_0([0,T[\times\mathbb{R}_+)$.

Using both the estimates established previously, we are able to express the weak inequality, but for discrete variables. And when the discretization step Δx tends to zero, we find the same result as in [24].

Lemma 4.3 *When Δx tends to zero and $\frac{\Delta t}{\Delta x}$ is constant, the limit of relaxed scheme (19) satisfies to the following entropy inequality :*

$$\int_{(0,T)\times\mathbb{R}_+} S(u)\partial_t\phi + G(u)\partial_x\phi \; dt\; dx + \sum_{\lambda_k>0}\lambda_k \int_{(0,T)} S_k(M_k(u_b(t)))\phi(t,0)\phi \; dt$$

$$+\sum_{\lambda_k<0}\lambda_k \int_{(0,T)} S_k(\omega_k)\phi(t,0)\; dt + \int_{\mathbb{R}_+} S(u^0(x))\phi(0,x)\; dx \geq 0$$

where ω_k is the limit obtained on the boundary terms from outgoing variables, and ϕ is a test function $\phi \in C_0^1([0,T[\times\mathbb{R}_+)$.

Proof : We call ϕ_i^n the piecewise constant approximation of a positive test function of $C_0^1([0,T[\times\mathbb{R}_+)$.

$$\phi_i^n = \frac{1}{\Delta x\Delta t}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\int_{t^n}^{t^{n+1}} \phi(t,x)\; dt\; dx \quad \text{with} \quad \phi \in C_0^1(\mathbb{R}_+\times[0,T))$$

We multiply inequalities obtained above by ϕ_i^n and we sum all them over the space-time domain $\mathbb{R}_+\times(0,T)$.

$$\sum_{i=0}^{\infty}\sum_{n=0}^{L}(S(u_i^{n+1})-S(u_i^n))\phi_i^n + \frac{\Delta t}{\Delta x}\sum_{i=0}^{\infty}\sum_{n=1}^{L}(G_{i+\frac{1}{2}}^n - G_{i-\frac{1}{2}}^n)\phi_i^n \leq 0$$

Integration by parts gives in the first term the following quantities :

$$-\Delta x\sum_{i=0}^{\infty} S(u_i^{L+1})\phi_i^L + \Delta x\sum_{i=0}^{\infty}\sum_{n=1}^{L} S(u_i^n)(\phi_i^n - \phi_i^{n-1}) + \Delta x\sum_{i=0}^{\infty} S(u_i^0)\phi_i^0$$

$$+\Delta t\sum_{n=0}^{L}\sum_{i=0}^{\infty} G_{i+\frac{1}{2}}(\phi_{i+1}^n - \phi_i^n) + \Delta t\sum_{n=0}^{L} G_{-\frac{1}{2}}^n\phi_0^n \geq 0$$

The first term vanishes because of the support of the test function, so we can neglect it, and the previous expression becomes :

$$\int_{\mathbb{R}+}\int_0^T S(u_\Delta)\frac{\phi_\Delta(t,x)-\phi_\Delta(t-\Delta t,x)}{\Delta t}\; dt\; dx + \int_{\mathbb{R}+} S(u_\Delta(0,x))\phi_\Delta(0,x)\; dx$$

$$+\int_0^T\int_{\mathbb{R}+} G_\Delta(t,x)\frac{(\phi_\Delta(t,x+\Delta x)-\phi_\Delta(t,x))}{\Delta x}dx\; dt + \int_0^T G_\Delta(t,\tfrac{\Delta x}{2})\phi_\Delta(t,\tfrac{\Delta x}{2})\; dt \geq 0$$

where

$$G_\Delta(t,x) = \sum_{\lambda_k>0}\lambda_k S_k(M_k(u_\Delta(t,x))) + \sum_{\lambda_k<0}\lambda_k S_k(M_k(u_\Delta(t,x+\Delta x)))$$

Because of stability arguments established before, and regularity of the test functions and their approximations the previous expression tends to :

$$\int_0^\infty\int_0^T S(u)\partial_t\phi\; dt\; dx \quad +\int_0^\infty S(u(0,x))\phi(0,x)\; dx$$

$$+\int_0^T\int_0^\infty G(t,x)\partial_x\phi dx\; dt + \int_0^T G_b(t,0)\phi(t,0)\; dt \geq 0$$

where

$$G(t,x) = \sum_k \lambda_k S_k(M_k(u(t,x)))$$

and

$$G_b(t,0) = \sum_{\lambda_k>0}\lambda_k S_k(M_k(u_b(t))) + \sum_{\lambda_k<0}\lambda_k S_k(\omega_k)$$

And the last term is obtained as extracted subsequence of $\omega_{\Delta,k}$ thanks to lemma 4.1, because estimates are uniform in Δx. $\diamondsuit$

Here we recall the lemma used in [21]

Lemma 4.4 *Suppose that the Maxwellian functions are* MND. *If we call* $S_k(f_k) = |f_k - M_k(c)|$, *the microscopic entropy functions associated to the Kružkov entropy for (1) we have*

$$\begin{aligned} &\textstyle\sum_{\lambda_k>0} \lambda_k |M_k(u_b) - M_k(c)| + \sum_{\lambda_k<0} \lambda_k |f_k - M_k(c)| \\ &\le \operatorname{sgn}(u_b - c)[\textstyle\sum_{\lambda_k>0} \lambda_k M_k(u_b) + \sum_{\lambda_k<0} \lambda_k f_k - F(c)] \end{aligned} \tag{28}$$

Proof of theorem 4.2 : Now we apply the lemma 4.3 to the Kružkov entropy functions and using (28) , we have the following estimate

$$\begin{aligned} &\textstyle\int_{(0,T)\times\mathbb{R}_+} \sum_k |u - c|\partial_t \phi + \operatorname{sgn}(u - c)(F(u) - F(c))\partial_x \phi \; dt\; dx + \int_{\mathbb{R}_+} |u^0(x) - c|\phi \; dx \\ &+ \textstyle\int_{(0,T)} \operatorname{sgn}(u_b - c)[\sum_{\lambda_k>0} \lambda_k M_k(u_b) + \sum_{\lambda_k<0} \lambda_k \omega_k - F(c)]\phi \; dt \ge 0 \end{aligned} \tag{29}$$

Taking a special test function ϕ, such that

$$\phi(t, x) = \rho(t) \max\{0, 1 - \frac{x}{\eta}\}, \qquad \eta \in \mathbb{R}^+ ,$$

where ρ is a positive test function in $C_0^\infty(0, T)$, one gets :

$$\begin{aligned} &\textstyle\int_0^T \operatorname{sgn}(u(t, 0) - c)(F(u(t, 0)) - F(c))\rho(t)dt \\ &\le \textstyle\int_0^T \operatorname{sgn}(u_b(t) - c)[\sum_{\lambda_k>0} \lambda_k M_k(u_b) + \sum_{\lambda_k<0} \lambda_k \omega_k(t) - F(c)]\rho(t)dt, \end{aligned}$$

letting η tend to zero. As this is true for every positive test function $\rho(t)$, it is true almost everywhere so that one has :

$$\operatorname{sgn}(u(t, 0) - c)(F(u(t, 0)) - F(c)) \le \operatorname{sgn}(u_b(t) - c)[\sum_{\lambda_k>0} \lambda_k M_k(u_b) + \sum_{\lambda_k<0} \lambda_k \omega_k(t) - F(c)].$$

Taking $c > \sup(u(t, 0), u_b(t))$ and $c < \inf(u(t, 0), u_b(t))$ it yields :

$$\sum_{\lambda_k<0} \lambda_k \omega_k(t) = F(u(t, 0)) - \sum_{\lambda_k>0} \lambda_k M_k(u_b(t)),$$

we replace $\sum_{\lambda_k<0} \lambda_k \omega_k(t)$ by its value in (29) and we get the desired result $\diamondsuit$

5 Stability of second order schemes

In this part, we focus our attention on second order discretizations constructed by MUSCL methods on the transport part. In general, such schemes do not satisfy discrete entropy inequalities, so that there is no hope to converge to the entropy solution of the problem. Nevertheless, we can obtain convergence to a weak solution of (1)(2) by satisfying the requirements of section 3, namely the assumption (H3). A particular interest is given to what can be done on the boundary coefficients.

5.1 Second order convection

To approximate the transport part (13) by a second order precision scheme, we use a reconstruction-transport-projection method. As usual, given the $f_{i,k}^n$, we construct a piecewise linear function

$$\overline{f_{i,k}^n}(x) = f_{i,k}^n + (x - x_i)\sigma_{i,k}^n, \quad x \in]x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}[.$$

Then we solve exactly the transport equations on $[t^n, t^{n+1}]$ and we project the result on the set of piecewise constant functions on the cells. We have the following formulation:

$$f_{i,k}^{n+\frac{1}{2}} = (1 - \xi_k)f_{i,k}^n + \xi_k f_{i-1,k}^n - \frac{\xi_k(1 - \xi_k)}{2}(\sigma_{i,k}^n - \sigma_{i-1,k}^n),$$

if the velocity is positive, while for $\lambda_k < 0$:

$$f_{i,k}^{n+\frac{1}{2}} = (1 - \xi_k)f_{i,k}^n + \xi_k f_{i+1,k}^n + \frac{\xi_k(1 - \xi_k)}{2}(\sigma_{i,k}^n - \sigma_{i+1,k}^n),$$

which give an explicit formulation to $D^n_{i+\frac{1}{2},k}$ that reads,

$$\begin{cases} D^n_{i-\frac{1}{2},k} & = \xi_k(1 + \frac{(1-\xi_k)}{2}\frac{(\sigma^n_{i,k}-\sigma^n_{i-1,k})}{\Delta f^n_{i-\frac{1}{2},k}}) \quad \lambda_k > 0 \\ \\ D^n_{i+\frac{1}{2},k} & = \xi_k(1 - \frac{(1-\xi_k)}{2}\frac{(\sigma^n_{i,k}-\sigma^n_{i+1,})}{\Delta f^n_{i+\frac{1}{2},k}}) \quad \lambda_k < 0. \end{cases}$$

The $\sigma^n_{i,k}$ are limited slopes:

$$\sigma^n_{i,k} = \min\bmod\,(X_{1,k,i}\Delta f^n_{i+\frac{1}{2},k}, X_{2,k,i}f^n_{i-\frac{1}{2},k})$$

with

$$\min\bmod\,(a,b) = \min(|a|,|b|) * \frac{\operatorname{sgn}(a)+\operatorname{sgn}(b)}{2}$$

and

$$\Delta f^n_{i+\frac{1}{2},k} = f^n_{i+1,k} - f^n_{i,k}.$$

The two coefficients $X_{1,k,i}, X_{2,k,i}$ amplifying the gradients inside the minmod can be chosen to make the upwind scheme closer to the central scheme, without loss of stability, see [11].

5.2 Focus on the boundary SO approximation

We establish all stability criteria that allow us to apply the Lax-Wendroff theorem so that the macroscopic solution u_Δ given by the scheme tends to a weak solution of problem (1) . This is independent of slopes that we decide to apply for the boundary cells. For precision sakes at fixed $\Delta x, \Delta t$, we propose some ideas.

For the entering characteristics

On a positive velocity transport equation with boundary imposed at the interface $x_{-\frac{1}{2}}$ we have the following equation:

$$f^{n+\frac{1}{2}}_{0,k} = \frac{1}{\Delta x}\int_0^{\Delta x} \overline{f^n_{0,k}}(x)dx - \xi_k\frac{1}{\Delta t}\int_0^{\Delta t}(f_{\frac{1}{2}}(t) - f_{-\frac{1}{2}}(t))dt$$

but while at the cell interface $x_{\frac{1}{2}}$ one can write the solution as a transported quantity, this can no more be done on the boundary. Therefore, even if the boundary condition is approximated by a piecewise linear function, the slope will not pass through the integral.

Following this argument we can write the corresponding slope for the boundary

$$\sigma^n_{-1,k} = 0.$$

Note that this does not reduce the boundary cell to first order approximation because still on the other interface the slope limiter σ^n_0 is active.

In fact for this slope, we have the imposed boundary condition at the edge of the cell, in the center, the value of $f^n_{0,k}$, so that we can consider that the slope σ^n_0 can be defined as

$$\sigma^n_0 = \sigma(X_{1,k,0}(f^n_{1,k} - f^n_{0,k}), 2X_{2,k,0}(f^n_{0,k} - M_k(u^n_b))).$$

For outgoing characteristics

For the slope of the cell $i = 0$ we dispose only of the right values, that is $f^n_{1,k} - f^n_{0,k}$, so that we propose the following slope

$$\sigma^n_{0,k} = X_{1,k,0}(f^n_{1,k} - f^n_{0,k})$$

Proposition 5.1 *If for* $\lambda_k > 0$

$$0 \le X_{1,k,i} \le \frac{2}{1-\xi_k}\ , \qquad 0 \le X_{2,k,i} \le \frac{2}{\xi_k}\ ,$$

for every $i \ge 1$*, while*

$$X_{1,k,0} \le \frac{2}{1-\xi_k}\ , \qquad X_{2,k,0} \le \frac{1}{\xi_k}\ ,$$

and if for $\lambda_k < 0$*, and* $i \ge 0$

$$X_{1,k,i} \le \frac{2}{\xi_k}\ , \qquad X_{2,k,i} \le \frac{2}{1-\xi_k}\ ,$$

then for any $i \ge 0$*, and every* $k \in \{1,..,N\}$

$$D^n_{i-\frac{1}{2},k} \in [0,1].$$

Proof : By classical arguments for all $i \geq 1$, (see [11]), the result is valid. We must especially take care on the boundary cell ($i = 0$).

1) $\lambda_k < 0$

In this case, the Harten's convex coefficients are :

$$D^n_{\frac{1}{2},k} = \xi_k(1 - \frac{(1-\xi_k)}{2}(\sigma(\frac{X_{1,k,1}}{r^n_1}, X_{2,k,1}) - X_{1,k,0}))$$

If $X_{1,k,0} \leq \frac{2}{\xi_k}$ we have $D^n_{\frac{1}{2},k} \leq 1$, as if $X_{2,k,1} \leq \frac{2}{(1-\xi_k)}$, $D^n_{\frac{1}{2},k}$ is greater or equal to 0.

2) $\lambda_k > 0$

As $\sigma^n_{-1} = 0$ and $\sigma^n_0 = \sigma(X_{1,k,0}(f^n_{1,k} - f^n_{0,k}), 2X_{2,k,0}(f^n_{0,k} - M_k(u_b)))$ then the incremental form yields :

$$D^n_{-\frac{1}{2},k} = \xi_k(1 + \frac{(1-\xi_k)}{2}(\sigma(\frac{X_{1,k,0}(f^n_{1,k} - f^n_{0,k})}{f^n_0 - M_k(u^n_b)}, 2X_{2,k,0})))$$

and obviously $D^n_{-\frac{1}{2},k}$ is positive, while $D^n_{-\frac{1}{2},k} \leq 1$ if $X_{2,k,0} \leq \frac{1}{\xi_k}$.

$\diamondsuit$

The previous bound on the convex form of the SO transport scheme, allows us to claim, as for the FO approximation:

Theorem 5.1 *If the Maxwellian functions are* MND, *and the slope limiters verify conditions of proposition 5.1 then*

$$u_\Delta \to u \text{ in } L^\infty([0,T]; L^1_{\text{loc}}(\mathbb{R}^+))$$

where u is a weak solution of (1) such that

$$u \in C^0([0,T]; L^1_{\text{loc}}(\mathbb{R}^+))$$

6 Numerical experiments

6.1 The scalar case

First we study the equation (1) in the scalar case with

$$F(u) = \frac{u^3}{3}$$

which admits sonic points.

To experiment first and second order schemes we take an oscillating boundary condition, while the initial condition is a constant:

$$\begin{cases} u_b(t) = \sin(6t) & t \in \mathbb{R}^+ \\ u^0(x) = 0.0 & x \in \mathbb{R}^+ \end{cases}$$

Actually, the exact solution can not be obtained solving a Riemann problem at the positive side of the $\mathbb{R}$ axes, because the boundary condition depends on time. We can also remark that the sonic point is frequently crossed.

6.1.1 Kinetic approximations

- **Two velocities model**

  We approximate (1) by the relaxation system (10) which reads

$$\partial_t f_\pm \pm \lambda \partial_x f_\pm = \frac{1}{\epsilon}(M_\pm(u) - f_\pm)$$

with

$$M_\pm(u) = \frac{1}{2}(u \pm \frac{F(u)}{\lambda}).$$

This model discretized at the first order gives the wellknown Lax Friedrichs scheme. The maxwellian functions are MND if and only if the subcharacteristic condition is satisfied:

$$\lambda > \sup_{u\in I} |F'(u)|. \tag{30}$$

In fact, we actualize the value of λ at each time step, by taking the maximum of $|F'(u^n_\Delta)|$ on the whole grid.

As explained in the introduction, there is a freedom degree while imposing the boundary condition that reads

$$f^n_{b,+} = M_+(u^n_b) - \alpha(f^n_{0,-} - M_-(u^n_b)).$$

We show tests for two particular values of α. For $\alpha = 0$ we have the equilibrium on the positive characteristic f_+ that was studied above (called NT on the figures). The boundary condition studied in [30, 8] is obtained for $\alpha = 1$ (called XW).

All numerical tests show that schemes for those two values provide different approximation for a fixed discretization step Δx. However, they both tend to the same solution as Δx goes to 0.

- **Three velocities model**

  In order to improve the approximation, we now use the following kinetic system, already proposed in [1]:

$$\begin{cases} \partial_t f^\epsilon_+ + \lambda\partial_x f^\epsilon_+ & = \frac{1}{\epsilon}(M_+(u) - f^\epsilon_+) \\ \partial_t f^\epsilon_0 & = \frac{1}{\epsilon}(M_0(u) - f^\epsilon_0) \\ \partial_t f^\epsilon_- - \lambda\partial_x f^\epsilon_- & = \frac{1}{\epsilon}(M_-(u) - f^\epsilon_-) \end{cases}$$

  with $\lambda > 0$ and

$$\begin{cases} M_+(u) = (\int_0^u [F'(s)]_+ \, ds)/\lambda, \\ M_-(u) = -(\int_0^u [F'(s)]_- \, ds)/\lambda, \\ M_0(u) = u - (\int_0^u |F'(s)| ds)/\lambda, \end{cases}$$

  where $[\cdot]_\pm$ is the positive (resp. negative) part. We impose at the boundary only :

$$f_+(t, 0) = M_+(u_b(t)).$$

  Here, M is MND if and only if the condition (30) is satisfied, as for the relaxation model.

  At first order, this model gives the wellknown Enquist-Osher scheme. On the graphics this scheme is called $3V_eq$. Note that as $F'(u) = u^2$, $M_-(u) = 0$ and the scheme is purely upwind.

6.1.2 FO schemes and results

Here we present some first order results for different Δx, the CFL is equal to 0.7, we can see that as expected the three velocities model gives much better results.

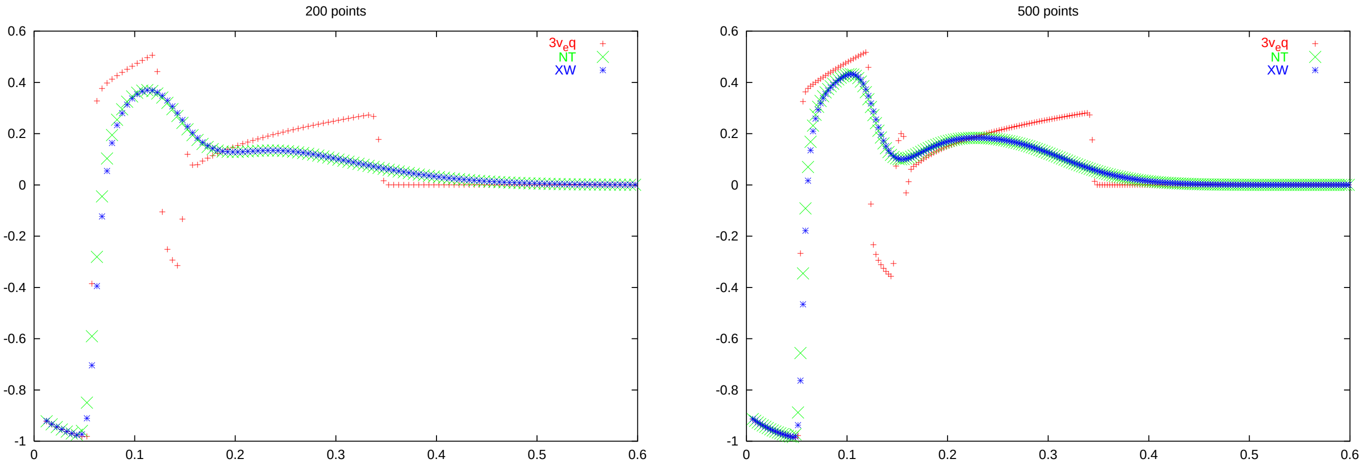


Figure 1: FO schemes for the cubic flux: 200 & 500 points

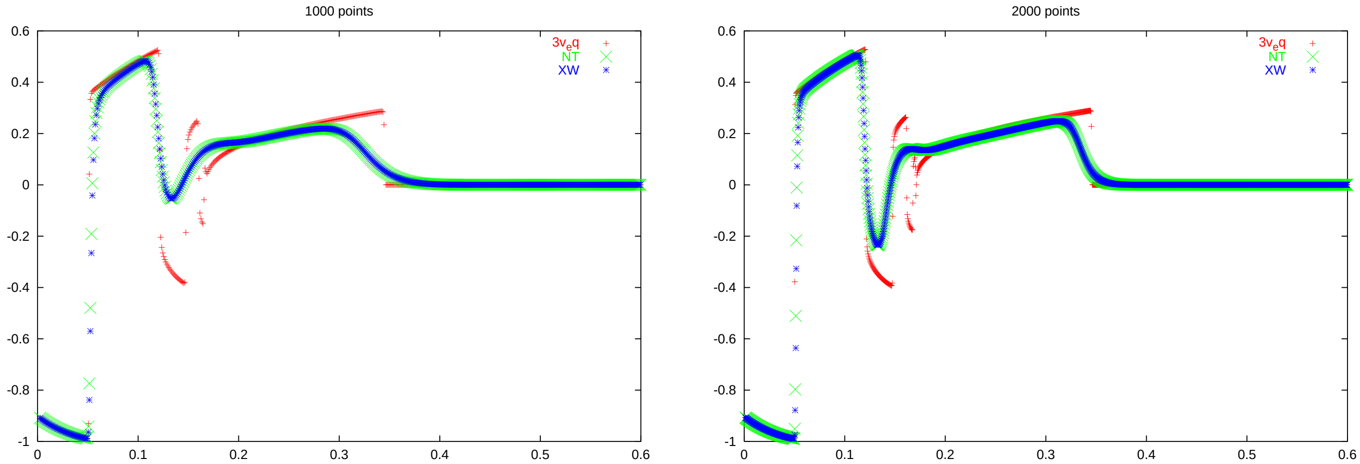


Figure 2: FO schemes: 1000 & 2000 points

6.1.3 Second order schemes and slope limiters

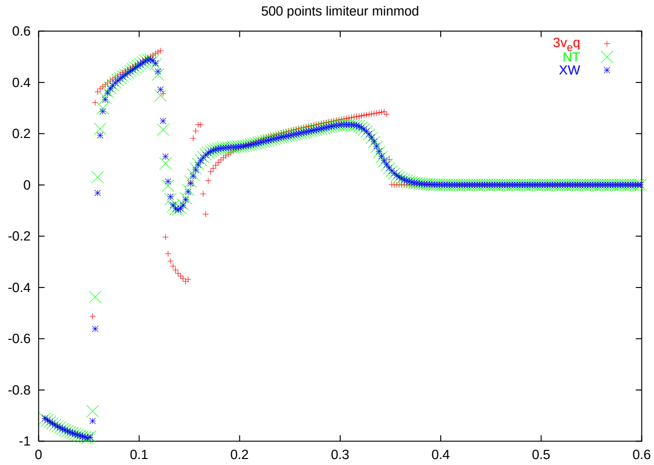
For the same data, we present numerical results for the second order schemes. We use two types of slope limiters. The first one is the well-known $\min\mathrm{mod}$ limiter that reads

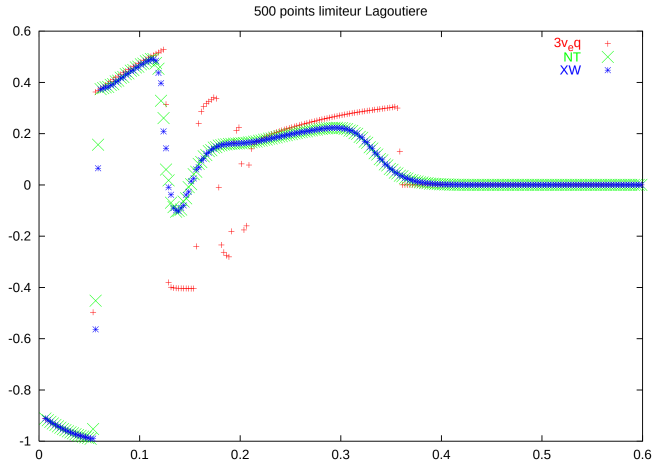
$$\sigma(a,b) = \min\mathrm{mod}\,(a,b) = \frac{\mathrm{sgn}(a)+\mathrm{sgn}(b)}{2}\,\min(|a|,|b|)$$

while the second is optimized to reduce the numerical dissipation of the transport scheme, and can be written :

	$\sigma(a,b)$
$\lambda_k > 0$	$2\ \min\mathrm{mod}\,(\frac{a}{(1-\xi)}, \frac{b}{\xi})$
$\lambda_k < 0$	$2\ \min\mathrm{mod}\,(\frac{a}{\xi}, \frac{b}{(1-\xi)})$

where $\xi = \frac{\lambda \Delta t}{\Delta x}$. This limiter was introduced by F. Lagoutière and B. Després in [12], and gives great precision as we can see on pictures below.





We can notice that a fast oscillating structure appears for $x \in [0.1, 0.2]$, it is a compression effect of slow waves by waves of greater amplitude. We can also remark that the structure becomes more and more complicated as Δx goes to 0.

6.2 Euler's equations

We approximate now a weak solution u of Euler's system of gas dynamics, in one and two dimensions. This system has the abstract form

$$\begin{cases} \partial_t u + \sum_{d=1}^{D} \partial_{x_d} F_d(u) = 0 & \forall (t,x) \in [0,T] \times \Omega \\ u(0,x) = u^0(x) & \forall x \in \Omega \end{cases}$$

and we would like to impose the following boundary condition where it is possible :

$$u(t,x) = u_b(t,x) \ \ \forall x \in \partial\Omega, t \geq 0$$

u is a time-space depending vector of $\mathbb{R}^p$. The system is hyperbolic (symetrisable) and the fluxes F_d are Lipschitz continuous from $\mathbb{R}^p$ to $\mathbb{R}^p$. In [1], the kinetic models are presented in a multidimensional version to which we add here a boundary condition:

$$\begin{cases} \partial_t f_k + \overrightarrow{\lambda_k} \cdot \overrightarrow{\nabla} f_k = \frac{1}{\epsilon}(M_k(u) - f_k) & \forall (t,x) \in [0,T] \times \Omega; \ \ k \in \{1, \ldots, N\} \\ f_k(0,x) = f_k^0(x) & \forall x \in \Omega \\ f_k(t,x) = M_k(u_b(t,x)) & \forall x \in \partial\Omega, t \geq 0 \quad \forall k \text{ s.t. } \overrightarrow{\nu} \cdot \overrightarrow{\lambda_k} < 0 \end{cases}$$

Here $\overrightarrow{\nu}$ is the outgoing normal vector, defined for every (t,x) in $[0,T] \times \partial\Omega$, while $\overrightarrow{\lambda_k}$ are vectors of $\mathbb{R}^D$, and M_k are Lipschitz continuous functions defined on $\mathbb{R}^p$ with values on $\mathbb{R}^p$. The compatibility (8) are extended as: for $u \in I$, some fixed rectangle of $\mathbb{R}^p$:

$$\begin{cases} \sum_k M_k(u) = u \\ \sum_k (\lambda_k)_d M_k(u) = F_d(u) \quad d = \{1, \ldots, D\} \end{cases}$$

As ine the one-dimensional case, we impose that M is MND in the sense of definition 1.1.

6.2.1 Euler's equations in one space dimension

We consider the one dimensional Euler system, on the bounded domain $[0,1]$:

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 & \forall x \in [0,1], \\ \partial_t(\rho v) + \partial_x(\rho v^2 + p) = 0, & \\ \partial_t E + \partial_x((E+p)v) = 0 & \end{cases} \tag{31}$$

where ρ, v, p and E are respectively the density, velocity, pressure and total energy of a perfect gas:

$$p = (\gamma - 1)(E - \frac{1}{2}\rho v^2) \ . \tag{32}$$

with $\gamma = 1.4$.

$$u = \begin{pmatrix} \rho \\ \rho v \\ E \end{pmatrix} \text{ and } F(u) = \begin{pmatrix} \rho v \\ \rho v^2 + p \\ (E+p)v \end{pmatrix}.$$

We use the relaxation model (10) . The maxwellian functions are MND if

$$\sigma(M'_{\pm}) > 0$$

wich reads on λ

$$\lambda > \max_{u \in I} |\sigma(F'(u))|$$

where I is a fixed rectangle of $\mathbb{R}^p$.

We have tested our schemes with the two interacting blast waves benchmark. The initial data is built with three constant states of a gamma-law gas. The domain is bounded by two walls separated by a distance of unity. The density is uniform, while the pressure is :

$$p = \begin{cases} 1000, & x < \frac{1}{10}, \\ 0.01, & x \in [\frac{1}{10}, \frac{9}{10}], \\ 100, & x > \frac{9}{10} \ . \end{cases}$$

To explain how we modelize reflecting walls with the kinetic model, let u_0^n be the vector value in the first cell of the grid. On the entering characteristics we apply at $x = 0$ for example

$$f_{+,b}^n = M_+ \begin{pmatrix} \rho_0^n \\ -\rho_0^n v_0^n \\ E_0^n \end{pmatrix}.$$

as at $x = 1$, we impose

$$f_{-,b}^n = M_- \begin{pmatrix} \rho_J^n \\ -\rho_J^n v_J^n \\ E_J^n \end{pmatrix}.$$

where J is the last cell of the grid. In fact, we inject a reflected state inside the domain.

The grid is uniform, and contains 400 points. We use the 2 velocities model with a fifth order approximation in space and of third order in time. Two schemes were implemented, the first is based on a local subcharacteristic condition, that is, the condition yields only locally inside the dependence domain, for the second one, the same condition is evaluated on the whole grid before each time step.

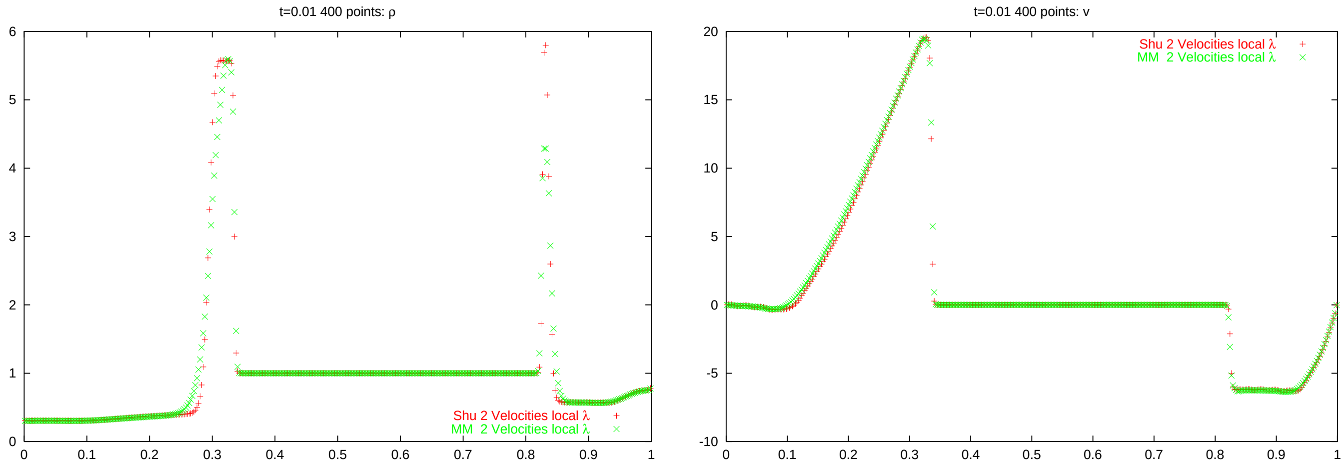


Figure 3: Density ρ and velocity u at t=0.01

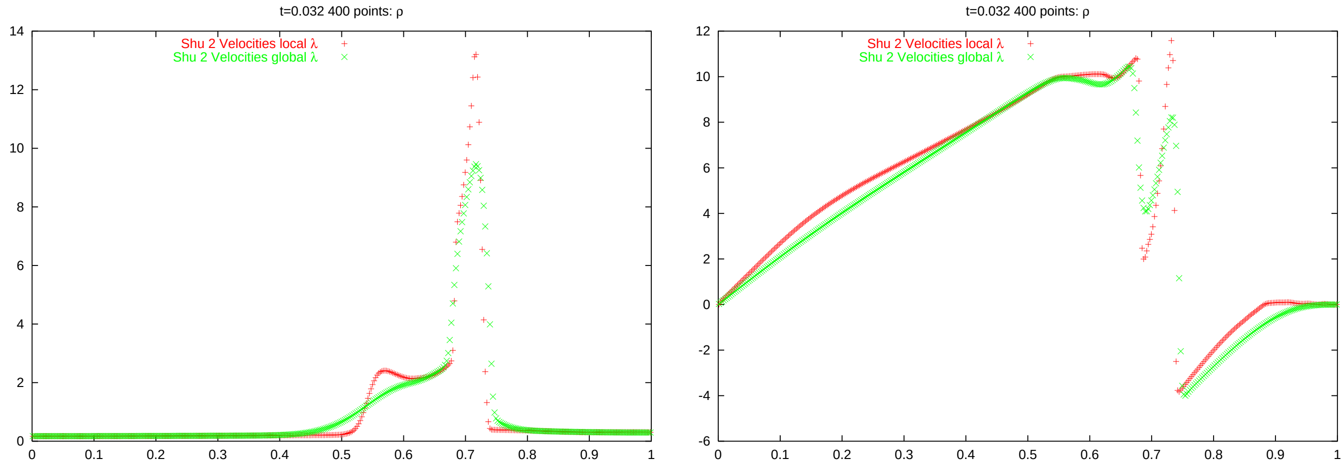


Figure 4: ρ and u at t=0.032

We can see that the local scheme is less diffusive than the global one, specially for longer time (see figure 4). But for both cases, the results are relevant.

6.3 A Mach 3 Wint Tunel With a Step

This benchmark used in several papers (see [32]), begins with uniform Mach 3 flow in a wind tunnel containing a step. The wind tunnel is 1 length units wide and 3 lengths units long. The step is 0.2 length units high and is located 0.6 length units long. At the left, we imposed a supersonic inflow boundary condition, while at the right side we imposed an outgoing condition. Initially the wind tunnel is filled with a gamma-law gas, with $\gamma = 1.4$, which everywhere has

$$\begin{cases} \rho = 1.4 \\ u = 3.0 \\ v = 0.0 \\ E = \frac{1}{2}\rho(u^2 + v^2) + \frac{p}{\gamma - 1} \qquad \text{with } p = 1.0 \end{cases}$$

On the boundary we impose reflecting boundary condition , that is

$$\overrightarrow{\nu} \cdot \overrightarrow{u} = 0$$

where $\overrightarrow{\nu}$ is the outgoing normal vector for points of the boundary. We let evolve the other components, as in the 1D case. To solve the 2D Euler equations we use the following 4 velocities model

- As velocity vecors we use

$$\overrightarrow{\lambda_1} = \lambda_x \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\overrightarrow{\lambda_2} = \lambda_y \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\overrightarrow{\lambda_3} = \lambda_x \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\overrightarrow{\lambda_4} = \lambda_x \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- while the corresponding Maxwellians are

$$M_1(u) = \frac{1}{4}[u + \frac{2F}{\lambda_x}]$$

$$M_2(u) = \frac{1}{4}[u - \frac{2G}{\lambda_y}]$$

$$M_3(u) = \frac{1}{4}[u - \frac{2F}{\lambda_x}]$$

$$M_4(u) = \frac{1}{4}[u + \frac{2G}{\lambda_y}]$$

F and G are the x (resp. y) fluxes given by the Euler system.

The stability condition reads

$$\lambda_x > \sup_{u \in I} |\sigma(F'(u))|$$

$$\lambda_y > \sup_{u \in I} |\sigma(G'(u))|$$

where $\sigma(\cdot)$ is the spectrum of the matrices.

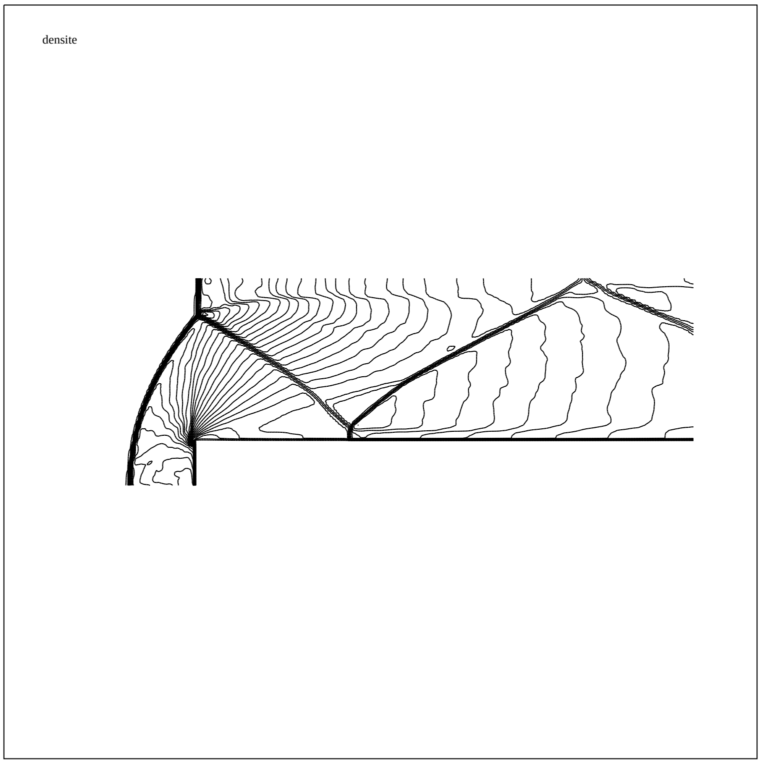


Figure 5: calculations with Despres & Lagoutiere limiters for CFL=0.2, at $t = 3.5$

The results presented are obtained with a second order approximation in space and first order in time. We can see that a strong shock appears on the bottom boundary, but it's effects on the solution are rather small. The approximation of the strong shock reflecting on the walls is quite efficient.

6.4 Double Mach Reflection of a Strong Shock

This test problem describes the reflection of a planar Mach shock in air, hitting a wedge. The setup is a Mach 10 shock which initially makes 60^0 angle with the reflecting wall. Ahead of the shock the undisturbed air has density 1.4 and pressure 1.0. The computational domain is $[0, 4] \times [0, 1]$ and the reflecting wall lies at the bottom of the domain starting at $x = 1/6$. See [32] for a more detailed description of the setup.

The kinetic model is a generalization of the previous one:

$$\lambda_{jd} = \delta_{jd}\lambda_{md}, \quad \lambda_{2+j,d} = \delta_{jd}\lambda_{pd}, \quad j = 1, 2, \quad d = 1, 2,$$

where δ_{jd} is the Kroenecker symbol and $\lambda_{md} < \lambda_{pd}$. The Maxwellian functions are:

$$\left\{\begin{array}{lcl} M_d(u) & = & \dfrac{1}{\lambda_{pd}-\lambda_{md}}\left(\dfrac{\lambda_{pd}}{2}u - F_d(u)\right), \\ & & \\ M_{2+d}(u) & = & \dfrac{1}{\lambda_{pd}-\lambda_{md}}\left(-\dfrac{\lambda_{md}}{2}u + F_d(u)\right). \end{array}\right. \qquad d = 1,2.$$

If $\lambda_{md} = -\lambda_{pd}$ we recover the previous model. In general the function M is monotone if, for all $u \in I$,

$$\lambda_{md} \leq a_{ld}(u) \leq \lambda_{pd}, \quad d = 1,2, \quad l = 1,4$$

where the $a_{ld}(u)$ are the eigenvalues of $F'_d(u)$.

Figure 6 shows the density at time $t = 0.2$, for a $cfl = 0.3$ and for $\Delta x = \Delta y = 1/120$. The solution is displayed with 30 isolines, and is obtained with a second order scheme in time as in space. This scheme uses Van Leer's limiters for the space second order approximation.

The double Mach reflection and the jet produced by it are clearly discernible on the figure. The weak shock generated at the kink in the main reflected shock and the contact discontinuity emerging from the three-shock interaction are fairly broad.

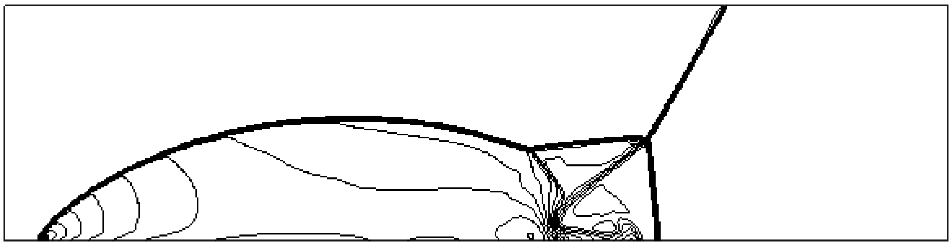


Figure 6: Density at $t = 0.2$

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