Traffic Flow on a Road Network

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Abstract

This paper is concerned with a fluidodynamic model for traffic flow. More precisely, we consider a single conservation law, deduced from conservation of the number of cars, defined on a road network that is a collection of roads with junctions. The evolution problem is underdetermined at junctions, hence we choose to have some fixed rules for the distribution of traffic plus an optimization criteria for the flux. We prove existence, uniqueness and stability of solutions to the Cauchy problem.

Our method is based on wave front tracking approach, see [6], and works also for boundary data and time dependent coefficients of traffic distribution at junctions, so including traffic lights.

Key Words: scalar conservation laws, traffic flow.

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1 Introduction

This paper deals with a fluidodynamic model of heavy traffic on a road network. More precisely, we consider the conservation law formulation proposed by Lighthill and Whitham [12] and Richards [13]. This nonlinear framework is based simply on the conservation of cars and is described by the equation:

$$\rho_t + f(\rho)_x = 0, \tag{1.1}$$

where $\rho = \rho(t,x) \in [0, \rho_{max}]$, $(t,x) \in I\!R^2$, is the *density* of cars, $v(t,x)$ is the velocity and $f(\rho) = v\,\rho$ is the flux. This model is appropriate to reveal shocks formation as it is natural for conservation laws, whose solutions may develop discontinuities in finite time even for smooth initial data, [6]. In most cases one assumes that v is a function of ρ only and that the corresponding flux is a concave function. We make the same assumption, moreover we let f have a unique maximum $\sigma \in]0, \rho_{max}[$ and for notational simplicity assume $\rho_{max} = 1$.

Here we deal with a network of roads, as in [11]. This means that we have a finite number of roads modeled by intervals $[a_i, b_i]$ (with one of the two endpoints possibly infinite) that meet at some junctions. For endpoints that do not touch a junction (and are not infinite), we assume to have a given boundary data and solve the corresponding boundary problem, as in [1, 2, 3, 5]. The key role is played by junctions at which the system is underdetermined even after prescribing the conservation of cars, that can be written as the Rankine-Hugoniot relation:

$$\sum_{i=1}^{n} f(\rho_i(t,b_i)) = \sum_{j=n+1}^{n+m} f(\rho_j(t,a_j)), \tag{1.2}$$

where ρ_i, $i = 1,\ldots,n$, are the car densities on incoming roads, while ρ_j, $j = n+1,\ldots,n+m$, are the car densities on outgoing roads. In [11], the Riemann problem, that is the problem with constant initial data on each road, is solved maximizing a concave function of the fluxes and it is proved existence of weak solutions for Cauchy problems with suitable initial data of bounded variation. In this paper we assume that:

(A) there are some prescribed preferences of drivers, that is the traffic from incoming roads is distributed on outgoing roads according to fixed coefficients;

(B) respecting (A), drivers choose so as to maximize fluxes.

To deal with rule (A), we fix a matrix

$$A \doteq \{\alpha_{ji}\}_{j=n+1,\ldots n+m,\, i=1,\ldots,n} \in I\!R^{m\times n},$$

such that

$$\alpha_{ji} \neq \alpha_{ji'}, \qquad 0 < \alpha_{jn} < 1, \qquad \sum_{j=n+1}^{n+m} \alpha_{ji} = 1, \tag{1.3}$$

for each $i' \neq i = 1, ..., n$ and $j = n+1, ..., n+m$, where α_{ji} is the percentage of drivers arriving from the $i-$th incoming road that take the $j-$th outgoing road. Notice that with only the rule (A) Riemann problems are still underdetermined. This choice represents a situation in which drivers have a final destination, hence distribute on outgoing roads according to a fixed law, but maximize the flux whenever possible. We are able to solve uniquely Riemann problems and, in case of simple junctions with two incoming and outgoing roads, to generate a Lipschitz semigroup, defined on L^1, whose trajectories are weak solutions and respect rules (A) and (B) in case of bounded variation (the same conditions are not meaningful if the solution is only L^1). Our main technique is the use of a front tracking algorithm and suitable approximations and functionals to control the total variation. We refer the reader to [6] for the general theory of conservation laws and for a discussion of wave front tracking algorithms.

The main difficulty in solving systems of conservation laws is the control of the total variation, see [6]. It is easy to see that for a single conservation law the total variation is decreasing, however in our case it may increase due to interaction of waves with junctions. The problem is quite delicate, as shown in Appendix B, where an example is given of a single wave of arbitrarily small strength (variation) that, interacting with a junction, generates waves whose strengths are bounded away from zero. Hence we can not expect any bound on the total variation of the solution in term of the total variation of the initial data, as it is the case for systems. This arbitrarily large magnification of total variation is possible only if waves crossing the value σ interact with junctions at which the boundary data of the roads are *bad*, that is in $[0, \sigma]$ for incoming roads and in $[\sigma, 1]$ for outgoing roads. We thus have first to deal with special data of bounded variation that have a finite number of crossing of the value σ. The sum of the number of these crossing plus the number of *bad* boundary data is proved to be decreasing along front tracking approximate solutions.

However, this is not enough since the variation can still increase due to interactions with junctions (and there is no bound on the number of interactions). The conserved quantity is the total variation of the flux. We prove this fact for junctions with only two incoming roads and two outgoing ones, and show in Appendix A that the conclusion does not hold for for junctions with three (or more) incoming and outgoing roads. Unfortunately the total variation of the flux is not equivalent to the total variation of ρ, since $f'(\sigma) = 0$. We thus have to approximate the flux with one having never vanishing derivative and a corner at σ, and then pass to the limit.

Our techniques are quite flexible, so we can deal with time dependent coefficients for the rule (A). In particular we can model traffic lights and also in this case the control of total variation is extremely delicate. An arbitrarily small change in the coefficients can produce waves whose strength is bounded away from zero. Still it is possible to consider periodic coefficients, a case of particular interest for applications. We can also deal with roads with different fluxes: this can be treated in the same way with the necessary notational modifications.

There is an interesting ongoing discussion on hydrodynamic modelization for heavy traffic flow. In particular some models using systems of two conservation laws have been proposed, see [4, 9, 10]. We do no treat this aspect.

The paper is organized as follows. in Section 2 we give the definition of weak entrtopic solution and following (A) and (B) we introduce an admissibility condition. In Section 3 we prove the existence and uniqueness of admissible solutions for the Riemann Problem in a junction, then using this we describe the construction of the approximants for the Cauchy Problem (see Section 4). In section 5 we prove the monotonicity of the number of big waves for piecewise constant solutions. Assuming that f' is bounded away from 0 and that there are at most two incoming and outcoming roads in each junction we prove the monotonicity of the total variation of the flux (see Section 6) and existence, uniqueness and stability of admissible solutions for the Cauchy Problem with suitable BV initial data. Using these results we show the existence of a unique Lipschitz semigroup of solutions defined on L^1 (see Section 8) also in the case in which f is smooth. In Section 9 we describe what happens when there are traffic lights and time dependent coefficients. In Appendix A we show with an example that the total variation of the flux can increase when there are three incoming and three outcoming roads in a junction. Finally, in Appendix B we show that the interaction of a small wave with a junction can produce a uniformly big wave.

2 Basic Definitions

We consider a network of roads, that is a modeled by a finite collection of intervals $I_i = [a_i, b_i] \subset \mathbb{R}$, $i = 1, \ldots, N$, possibly with either $a_i = -\infty$ or $b_i = +\infty$, on which we consider the equation (1.1). Hence the datum is given by a finite collection of functions ρ_i defined on $[0, +\infty[\times I_i$.

On each road I_i we want ρ_i to be a weak entropic solution, that is for $\varphi : I_i \to \mathbb{R}$ smooth with compact support on $]0, +\infty[\times]a_i, b_i[$

$$\int_0^{+\infty} \int_{a_i}^{b_i} \Big(\rho_i \frac{\partial \varphi}{\partial t} + f(\rho_i) \frac{\partial \varphi}{\partial x}\Big) dx dt = 0, \tag{2.4}$$

and for every $k \in \mathbb{R}$ and $\tilde{\varphi} : I_i \to \mathbb{R}$ smooth, positive with compact support on $]0, +\infty[\times]a_i, b_i[$

$$\int_0^{+\infty} \int_{a_i}^{b_i} \Big(|\rho_i - k| \frac{\partial \tilde{\varphi}}{\partial t} + \mathrm{sgn}\ (\rho_i - k)(f(\rho_i) - f(k)) \frac{\partial \tilde{\varphi}}{\partial x}\Big) dx dt \geq 0, \tag{2.5}$$

It is well known that for any initial data in L^∞, defined on the whole $\mathbb{R}$, there exists a unique weak entropic solution depending in a Lipschitz continuous way from the initial data in L^1_{loc}.

We assume that the roads are connected by some junctions. Each junction J is given by a finite number of incoming roads and a finite number of outgoing roads, thus we identify J with $((i_1,\ldots,i_n),(j_1,\ldots,j_m))$ where the first n–tuple indicates the set of incoming roads and the second m–tuple indicates the set of outgoing roads. We assume that each road can be incoming road at most for one junction and outgoing at most for one junction.

Hence the complete model is given by a couple $(\mathcal{I},\mathcal{J})$, where $\mathcal{I}=\{I_i : i=1,\ldots,N\}$ is the collection of roads and $\mathcal{J}$ the collection of junctions.

Fix a junction J with incoming roads, say $I_1,\ldots,I_n$, and outgoing roads, say $I_{n+1},\ldots,I_{n+m}$. A weak solution at the junction J is a collection of functions $\rho_i : [0,+\infty[\times I_i \to \mathbb{R}$, $i=1,\ldots,n+m$, such that

$$\sum_{l=0}^{n+m}\left(\int_0^{+\infty}\int_{a_l}^{b_l}\left(\rho_l\frac{\partial\varphi_l}{\partial t}+f(\rho_l)\frac{\partial\varphi_l}{\partial x}\right)dxdt\right)=0 \tag{2.6}$$

for each $\varphi_1,...,\varphi_{n+m}$ smooth having compact support in $]0,+\infty[\times\mathbb{R}$, that are also smooth across the junction,i.e.

$$\varphi_i(\cdot,b_i)=\varphi_j(\cdot,a_j),\qquad \frac{\partial\varphi_i}{\partial x}(\cdot,b_i)=\frac{\partial\varphi_j}{\partial x}(\cdot,a_j),\qquad i=1,...,n,\ j=n+1,...,n+m.$$

Remark 2.1 Let $\rho=(\rho_1,\ldots,\rho_{n+m})$ be a solution of (1.1) such that each $x\to\rho_i(t,x)$ has bounded variation. We can deduce that it satisfies the *Rankine-Hugoniot Condition* in the junction J, namely

$$\sum_{i=1}^{n} f(\rho_i(t,b_i))=\sum_{j=n+1}^{n+m} f(\rho_j(t,a_j)), \tag{2.7}$$

for each $t>0$.

Remark 2.2 The assumption $\alpha_{ji}\neq\alpha_{ji'}$ in (1.3)is needed in order to have uniqueness of solutions to Riemann problems, see Section 3. However, this condition can be relaxed requiring, for example, that if $\alpha_{ji}=\alpha_{ji'}$ then the fluxes from road i and i' coincide.

The rules (A) and (B) can be given explicitly only for solutions with bounded variation at each time as in next definition.

Definition 2.1 *Let $\rho=(\rho_1,\ldots,\rho_{n+m})$ be such that $\rho_i(t,\cdot)$ is of bounded variation. Then ρ is an admissible weak solution of (1.1) related to the matrix A satisfying (1.3) at the junction J if and only if following properties hold:*

(i) ρ *is a weak solution at the junction;*

(ii) $f(\rho_j(\cdot, a_j+)) = \sum_{i=1}^{n} \alpha_{ji} f(\rho_i(\cdot, b_i-)),$ *for each* $j = n+1, ..., n+m;$

(iii) $\sum_{i=1}^{n} f(\rho_i(\cdot, b_i-)) + \sum_{j=n+1}^{n+m} f(\rho_j(\cdot, a_j+))$ *is maximum subject to (ii).*

For every road $I_i = [a_i, b_i]$, if $a_i > -\infty$ and I_i is not the outgoing of any junction, or $b_i < +\infty$ and I_i is not the incoming road of any function, then a boundary data $\psi_i : [0, +\infty[\to I\!R$ is given. In this case we ask ρ_i to satisfy $\rho_i(t, a_i) = \psi_i(t)$ (or $\rho_i(t, b_i) = \psi_i(t)$) in the sense of [5].

Our aim is to solve the Cauchy problem on $[0, +\infty[$ for a given initial and boundary data as in next definition.

Definition 2.2 *Given* $\bar{\rho}_i : I_i \to I\!R$ *and possibly* $\psi_i : [0, +\infty[\to I\!R$*, functions of* L^∞*, a collection of functions* $\rho = (\rho_1, \ldots, \rho_N)$ *with* $\rho_i : [0, +\infty[\times I_i \to I\!R$ *continuous as functions from* $[0, +\infty[$ *into* L^1_{loc}*, is an admissible solution if* ρ_i *is a weak entropic solution to (1.1) on* I_i*,* $\rho_i(0, x) = \bar{\rho}_i(x)$ *a.e.,* $\rho_i(t, b_i) = \psi_i(t)$ *in the sense of [5], finally such that at each junction* ρ *is a weak solution and is an admissible weak solution in case of bounded variation.*

The treatment of boundary data in the sense of [5] can be done in the same way as in [1, 2, 3], thus we treat the case without boundary data. All the stated results hold also for the case with boundary data with the obvious modifications.

On the flux f we make the following assumption

$(\mathcal{F})$ $f : [0, 1] \to I\!R$ is smooth, strictly concave (i.e. $f'' \leq -c < 0$ for some $c > 0$), $f(0) = f(1) = 0$, $|f'(x)| \leq C < +\infty$ and there exists $\sigma \in]0, 1[$ such that $f'(\sigma) = 0$ (that is σ is a strict maximum).

3 The Riemann Problem

In this section we study Riemann problems. For a scalar conservation law a Riemann problem is a Cauchy problem for an initial data of Heaviside type, that is piecewise constant with only one discontinuity. The solutions of these problems are the building blocks to construct solutions to the Cauchy problem via wave front tracking. These solutions are formed by continuous waves called rarefactions and traveling discontinuities called shocks. The speed of waves are related to the values of $f'(\rho)$.

Analogously, we call Riemann problem for the road network the Cauchy problem corresponding to an initial data that is piecewise constant on each road. The solutions on each road I_i can be constructed in the same way as for the scalar conservation law, hence it suffices to describe the solution at junctions. Because of finite propagation speed, it is enough to study the Riemann Problem for a single junction.

As explained in the Introduction, we first have to treat the case of fluxes with nonvanishing derivative, hence we assume that

($\mathcal{F}1$) $f : [0,1] \to I\!R$ is continuous, strictly concave, $f(0) = f(1) = 0$ and there exists $\sigma \in]0,1[$ such that f is smooth on $[0,\sigma[$ and on $]\sigma,1]$ and

$$0 < c \leq |f'(x)| \leq C < +\infty, \tag{3.8}$$

for each $x \in [0,\sigma[\cup]\sigma,1]$.

Consider a junction J in which there are n roads with incoming traffic and m roads with outgoing traffic. For simplicity we indicate by

$$(t,x) \in I\!R_+ \times I_i \mapsto \rho_i(t,x) \in [0,1], \quad i = 1,...,n \tag{3.9}$$

the densities of the cars on the road with incoming traffic and

$$(t,x) \in I\!R_+ \times I_j \mapsto \rho_j(t,x) \in [0,1], \quad j = n+1,...,n+m \tag{3.10}$$

those on the roads with outgoing traffic.

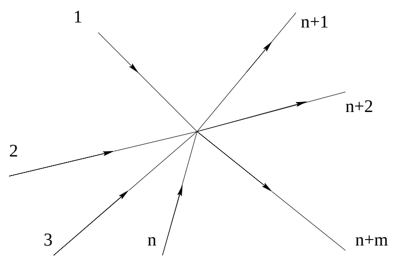

figure 1

We need some more notation:

Definition 3.1 *Let $\tau : [0,1] \to [0,1]$, $\tau(\sigma) = \sigma$, be the map satisfying the following*

$$\tau(\rho) \neq \rho, \qquad f(\tau(\rho)) = f(\rho),$$

for each $\rho \neq \sigma$.

Clearly τ is well defined and satisfies

$$0 \leq \rho \leq \sigma \Longleftrightarrow \sigma \leq \tau(\rho) \leq 1, \qquad \sigma \leq \rho \leq 1 \Longleftrightarrow 0 \leq \tau(\rho) \leq \sigma.$$

The main result of this section is the following Theorem.

Theorem 3.1 *Let $f:[0,1]\to I\!R$ satisfy $(\mathcal{F}1)$ and $\rho_{1,0},...,\rho_{n+m,0}\in[0,1]$ be constants. There exists an unique admissible weak solution, in the sense of Definition 2.1, $\rho=(\rho_1,...,\rho_{n+m})$ of (1.1) at the junction J such that*

$$\rho_1(0,\cdot)\equiv\rho_{1,0},......,\rho_{n+m}(0,\cdot)\equiv\rho_{n+m,0}.$$

Moreover, there exist a unique $(n+m)-$tuple $(\hat\rho_1,...,\hat\rho_{n+m})\in[0,1]^{n+m}$ such that

$$\hat\rho_i\in\begin{cases}\{\rho_{i,0}\}\cup]\tau(\rho_{i,0})1] & \text{if } 0\le\rho_{i,0}\le\sigma,\\ [\sigma,1] & \text{if } \sigma\le\rho_{i,0}\le1,\end{cases}\qquad i=1,...,n \tag{3.11}$$

and

$$\hat\rho_j\in\begin{cases}[0,\sigma] & \text{if } 0\le\rho_{j,0}\le\sigma,\\ \{\rho_{j,0}\}\cup[0,\tau(\rho_{j,0})[& \text{if } \sigma\le\rho_{j,0}\le1,\end{cases}\qquad j=n+1,...,n+m. \tag{3.12}$$

Fixed $i\in\{1,...,n\}$, if $\rho_{i,0}\le\hat\rho_i$ there results

$$\rho_i(t,x)=\begin{cases}\rho_{i,0} & \text{if } x\le\frac{f(\hat\rho_i)-f(\rho_{i,0})}{\hat\rho_i-\rho_{i,0}}t,\\ \hat\rho_i & \text{otherwise},\end{cases} \tag{3.13}$$

and if $\hat\rho_i<\rho_{i,0}$

$$\rho_i(t,x)=\begin{cases}\rho_{i,0} & \text{if } x\le f'(\rho_{i,0})t,\\ (f')^{-1}\left(\frac{x}{t}\right) & \text{if } f'(\rho_{i,0})t\le x\le f'(\hat\rho_i)t,\\ \hat\rho_i & \text{if } x>f'(\hat\rho_i)t.\end{cases} \tag{3.14}$$

Otherwise, fixed $j\in\{n+1,...,n+m\}$, if $\rho_{j,0}\le\hat\rho_j$ there results

$$\rho_j(t,x)=\begin{cases}\hat\rho_j & \text{if } x\le f'(\hat\rho_j)t,\\ (f')^{-1}\left(\frac{x}{t}\right) & \text{if } f'(\hat\rho_j)t\le x\le f'(\rho_{j,0})t,\\ \rho_{j,0} & \text{if } x>f'(\rho_{j,0})t,\end{cases} \tag{3.15}$$

and if $\hat\rho_j<\rho_{j,0}$

$$\rho_j(t,x)=\begin{cases}\hat\rho_j & \text{if } x\le\frac{f(\rho_{j,0})-f(\hat\rho_j)}{\hat\rho_i-\rho_{i,0}}t,\\ \rho_{j,0} & \text{otherwise}.\end{cases} \tag{3.16}$$

PROOF. Define the map

$$E:(\gamma_1,...,\gamma_n)\in I\!R^n\longmapsto\sum_{i=1}^n\gamma_i$$

and the sets

$$\Omega_i\doteq\begin{cases}[0,f(\rho_{i,0})] & \text{if } 0\le\rho_{i,0}\le\sigma,\\ [0,f(\sigma)] & \text{if } \sigma\le\rho_{i,0}\le1,\end{cases}\qquad i=1,...,n,$$

$$\Omega_j \doteq \begin{cases} [0, f(\sigma)] & \text{if } 0 \le \rho_{j,0} \le \sigma, \\ [0, f(\rho_{j,0})] & \text{if } \sigma \le \rho_{j,0} \le 1, \end{cases} \qquad j = n+1, ..., n+m,$$

$$\Omega \doteq \Big\{ (\gamma_1, ..., \gamma_n) \in \Omega_1 \times .... \times \Omega_n \big| A \cdot (\gamma_1, ..., \gamma_n)^T \in \Omega_{n+1} \times .... \times \Omega_{n+m} \Big\}.$$

Since E is linear, the set Ω is closed, convex and not empty. By (1.3) there exists a unique vector $(\hat{\gamma}_1, ..., \hat{\gamma}_n) \in \Omega$ such that

$$E(\hat{\gamma}_1, ..., \hat{\gamma}_n) = \max_{(\gamma_1, ..., \gamma_n) \in \Omega} E(\gamma_1, ..., \gamma_n).$$

Fix $i \in \{1, ..., n\}$, let $\hat{\rho}_i \in [0,1]$ be such that

$$f(\hat{\rho}_i) = \hat{\gamma}_i, \quad \hat{\rho}_i \in \begin{cases} \{\rho_{i,0}\} \cup]\tau(\rho_{i,0}) 1] & \text{if } 0 \le \rho_{i,0} \le \sigma, \\ [\sigma, 1] & \text{if } \sigma \le \rho_{i,0} \le 1. \end{cases}$$

By $(\mathcal{F}1)$, $\hat{\rho}_i$ exists and is unique. Let

$$\hat{\gamma}_j \doteq \sum_{i=1}^{n} \alpha_{ji} \hat{\gamma}_i, \qquad j = n+1, ..., n+m$$

and $\hat{\rho}_j \in [0,1]$ be such that

$$f(\hat{\rho}_j) = \hat{\gamma}_j, \quad \hat{\rho}_j \in \begin{cases} [0, \sigma] & \text{if } 0 \le \rho_{j,0} \le \sigma, \\ \{\rho_{j,0}\} \cup [0, \tau(\rho_{j,0})[& \text{if } \sigma \le \rho_{j,0} \le 1. \end{cases}$$

Since $(\hat{\gamma}_1, ..., \hat{\gamma}_n) \in \Omega$, $\hat{\rho}_j$ exists and is unique. Solving the Riemann Problem (see [6, Chapter 6]) on each road, the thesis is proved. □

4 The Wave Front Tracking Algorithm

Once the solution to a Riemann problem is provided, we are able to construct piecewise constant approximations via wave-front tracking. The construction is very similar to that for scalar conservation law, see [6], hence we only briefly describe it.

Let ρ_0 be a piecewise constant map defined on the road network. We want to construct a solution of (1.1) with initial condition $\rho(0, \cdot) \equiv \rho_0$. We begin by solving the Riemann Problems on each road in correspondence of the jumps of ρ_0 and the Riemann Problems at junctions determined by the values of ρ_0 (see Theorem 3.1). We split each rarefaction wave into a rarefaction fan formed by rarefaction shocks, that are discontinuities traveling with the Rankine-Hugoniot speed. We always split rarefaction waves inserting the value σ (if it is in the range of the rarefaction), in order to control the number of big waves defined in next Section.

When a wave interacts with another one we simply solve the new Riemann Problem, if otherwise it reaches a junction then we solve the Riemann Problem at the junction. Since the wave speed is bounded there are finitely many waves on the network at each time $t \geq 0$. We call the obtained function *a wave front tracking approximate solution.* Given a general initial data, we approximate it by a sequence of piecewise constant functions and construct the corresponding approximate solutions. If they converge in L^1_{loc}, then the limit is a weak entropic solution on each road, see [6] for a proof.

5 Estimates on the Number of Big Waves

In this Section we consider big waves, that are the waves crossing the value σ. For these waves the variation of $f(\rho)$ is not comparable to the variation in ρ, more precisely the former can vanish while the second is different from zero. Since only the variation of $f(\rho)$ happens to be conserved we need to control the number of big waves.

Let ρ a piecewise constant map defined on the network and J a junction with n roads with incoming traffic and m roads with outgoing traffic as in Section 3. Define the set

$$\Phi_J(\rho) \doteq \big\{ i \in \{1,...,n\} \big| \rho_i(b_i-) \in [0,\sigma] \big\} \cup \big\{ j \in \{n+1,...,n+m\} \big| \rho_j(a_j+) \in [\sigma,1] \big\}.$$

For each road I_i we denote by $\{x_\alpha \in I_i : \alpha \in A_i\}$ and by $\{(\rho(x_\alpha-),\rho(x_\alpha+)) : \alpha \in A_i\}$ the set of discontinuity points and the set of discontinuities, respectively, of the map ρ_i on the road I_i. We define

$$G^i(\rho) \doteq \big\{ (\rho(x_\alpha-),\rho(x_\alpha+)) \big| \ \alpha \in A_i, \ \mathrm{sgn}\ \big(\rho_k(x_\alpha-)-\sigma\big) \cdot \mathrm{sgn}\ \big(\rho_k(x_\alpha+)-\sigma\big) \leq 0 \big\},$$

with the agreement sgn $(0) = 0$, and the functional

$$N(\rho) \doteq \sum_{J\in\mathcal{J}} \#\big(\Phi_J(\rho)\big) + \sum_{i=1}^{N} \#\big(G^i(\rho)\big),$$

where # indicates the cardinality of a set.

The main result of this section is the following.

Lemma 5.1 *Let $f : [0,1] \to I\!\!R$ satisfy $(\mathcal{F}1)$ and ρ be a piecewise constant wave front tracking approximate solution of (1.1) on the net. Then the map*

$$t > 0 \longmapsto N\big(\rho(t,\cdot)\big)$$

does not increase.

PROOF. We begin considering a single junction J with n roads with incoming traffic and m roads with outgoing traffic and an equilibrium configuration $(\rho_{1,0},...,\rho_{n+m,0}) \in [0,1]^{n+m}$, namely the solution of the Riemann Problem in the junction with that initial data is constant. Suppose that a wave on one road arrives to the junction at time $\bar{t}$ and there is no other wave on the roads, then we claim that $N\big(\rho(\bar{t}-,\cdot)\big) = N\big(\rho(\bar{t}+,\cdot)\big)$. Assume that the wave is on an incoming road, for example the first one and let $(\rho_1, \rho_{1,0})$ be the values on the left and right side of the wave respectively. Since the wave is approaching the junction, its speed is positive and so $0 \le \rho_1 \le \sigma$, moreover since it is the unique wave

$$G^i\big(\rho(\bar{t}-,\cdot)\big) = \emptyset, \qquad i = 2,...,n+m.$$

Let $(\hat{\rho}_1,...,\hat{\rho}_{n+m})$ be the solution to the Riemann Problem with initial data $(\rho_1, \rho_{2,0}..., \rho_{n+m,0})$ (see Theorem 3.1), there results

$$\hat{\rho}_1 \in \{\rho_1\} \cup [\sigma, 1]$$

$$\hat{\rho}_i \in \{\rho_{i,0}\} \cup [\sigma,1], \qquad i = 2,...,n$$

$$\hat{\rho}_j \in \{\rho_{j,0}\} \cup [0,\sigma], \qquad j = n+1,...,n+m.$$

In the following we study the change of the functional N due to the presence of new waves. If a new rarefaction is produced then it can not cross the value σ, otherwise there would be rarefaction shocks with positive and negative velocity at the same time. Hence each functional G^i can not be bigger than one after the interaction. By abuse of notation, we indicate the whole rarefaction fan as a single wave for notational simplicity.

So, fixed $i \in \{2,...,n\}$, if $i \notin \Phi_J\big(\rho(\bar{t}-,\cdot)\big)$, then

$$i \notin \Phi_J\big(\rho(\bar{t}+,\cdot)\big), \qquad (\rho_{i,0}, \hat{\rho}_i) \notin G^i\big(\rho(\bar{t}+,\cdot)\big),$$

and if $i \in \Phi_J\big(\rho(\bar{t}-,\cdot)\big)$ we have

$$\hat{\rho}_i = \rho_{i,0} \Longrightarrow i \in \Phi_J\big(\rho(\bar{t}+,\cdot)\big),\ (\rho_{i,0}, \hat{\rho}_i) \notin G^i\big(\rho(\bar{t}+,\cdot)\big),$$

$$\sigma \le \hat{\rho}_i \le 1 \Longrightarrow i \notin \Phi_J\big(\rho(\bar{t}+,\cdot)\big),\ (\rho_{i,0}, \hat{\rho}_i) \in G^i\big(\rho(\bar{t}+,\cdot)\big).$$

On the other hand, fixed $j \in \{n+1,...,n+m\}$, if $j \notin \Phi_J\big(\rho(\bar{t}-,\cdot)\big)$, then

$$j \notin \Phi_J\big(\rho(\bar{t}+,\cdot)\big), \qquad (\hat{\rho}_j, \rho_{j,0}) \notin G^j\big(\rho(\bar{t}+,\cdot)\big),$$

and if $j \in \Phi_J\big(\rho(\bar{t}-,\cdot)\big)$ we have

$$\hat{\rho}_j = \rho_{j,0} \Longrightarrow j \in \Phi_J\big(\rho(\bar{t}+,\cdot)\big),\ (\hat{\rho}_j, \rho_{j,0}) \notin G^j\big(\rho(\bar{t}+,\cdot)\big),$$

$$0 \le \hat{\rho}_j \le \sigma \Longrightarrow j \notin \Phi_J\big(\rho(\bar{t}+,\cdot)\big),\ (\hat{\rho}_j, \rho_{j,0}) \in G^j\big(\rho(\bar{t}+,\cdot)\big).$$

Hence the contribution to N due to roads I_i, $i = 2, \ldots, n+m$, does not increase. Let us now treat the waves on the first road.

Notice that if $\rho_{1,0} = \sigma$ then $\rho_1 \neq \sigma$ hence

$$1 \in \Phi_J\big(\rho(\bar{t}-, \cdot)\big), \qquad (\rho_1, \rho_{1,0}) \in G^1\big(\rho(\bar{t}-, \cdot)\big)$$

and N can not increase. The same conclusion holds if $0 \leq \rho_{1,0} < \sigma$ and $\rho_1 = \sigma$.

Now, if $0 \leq \rho_{1,0} < \sigma$ and $\rho_1 \neq \sigma$, then there results

$$1 \in \Phi_J\big(\rho(\bar{t}-, \cdot)\big), \qquad (\rho_1, \rho_{1,0}) \notin G^1\big(\rho(\bar{t}-, \cdot)\big)$$

and

$$\hat{\rho}_1 = \rho_1 \Longrightarrow 1 \in \Phi_J\big(\rho(\bar{t}+, \cdot)\big),\ (\rho_1, \hat{\rho}_1) \notin G^1\big(\rho(\bar{t}+, \cdot)\big),$$

$$\hat{\rho}_1 \neq \rho_1 \Longrightarrow \sigma < \hat{\rho}_1 \leq 1 \Longrightarrow 1 \notin \Phi_J\big(\rho(\bar{t}+, \cdot)\big),\ (\rho_1, \hat{\rho}_1) \in G^1\big(\rho(\bar{t}+, \cdot)\big).$$

If $\sigma < \rho_{1,0} \leq 1$ and $\rho_1 = \sigma$ we have

$$1 \notin \Phi_J\big(\rho(\bar{t}-, \cdot)\big), \qquad (\rho_1, \rho_{1,0}) \in G^1\big(\rho(\bar{t}-, \cdot)\big)$$

and

$$\hat{\rho}_1 = \rho_1 = \sigma \Longrightarrow 1 \in \Phi_J\big(\rho(\bar{t}+, \cdot)\big),\ (\rho_1, \hat{\rho}_1) \notin G^1\big(\rho(\bar{t}+, \cdot)\big),$$

$$\sigma < \hat{\rho}_1 \leq 1 \Longrightarrow 1 \notin \Phi_J\big(\rho(\bar{t}+, \cdot)\big),\ (\rho_1, \hat{\rho}_1) \in G^1\big(\rho(\bar{t}+, \cdot)\big).$$

Finally, if $\sigma < \rho_{1,0} \leq 1$ then $\rho_1 \neq \sigma$, $\hat{\rho}_1 \neq \sigma$, and we have

$$1 \notin \Phi_J\big(\rho(\bar{t}-, \cdot)\big), \qquad (\rho_1, \rho_{1,0}) \in G^1\big(\rho(\bar{t}-, \cdot)\big),$$

hence

$$\hat{\rho}_1 = \rho_1 \Longrightarrow 1 \in \Phi_J\big(\rho(\bar{t}+, \cdot)\big),\ (\rho_1, \hat{\rho}_1) \notin G^1\big(\rho(\bar{t}+, \cdot)\big),$$

$$\sigma < \hat{\rho}_1 \leq 1 \Longrightarrow 1 \notin \Phi_J\big(\rho(\bar{t}+, \cdot)\big),\ (\rho_1, \hat{\rho}_1) \in G^1\big(\rho(\bar{t}+, \cdot)\big).$$

We conclude

$$N\big(\rho(\bar{t}+, \cdot)\big) = N\big(\rho(\bar{t}-, \cdot)\big).$$

The conclusion can be obtained in the same way if the wave is arriving to the junction from another road. Moreover, when two waves interact on the same road then there is a cancellation or gluing of waves and it is easy to check that N is constant or decreases. The proof is concluded. $\square$

6 Estimates on Flux Variation

This Section is dedicated to the estimation of the total variation of the flux along a solution. We assume that every junction has at most two incoming roads and two outgoing ones. This hypothesis is crucial, because, as shown in the Appendix A, the presence of more complicate junctions provokes increase of the total variation of the flux.

Lemma 6.1 *Let $f : [0,1] \to I\!R$ satisfy $(\mathcal{F}1)$. Consider a network $(\mathcal{I},\mathcal{J})$ in which each junction has at most two incoming road and two outgoing ones. Let ρ be a piecewise constant wave front tracking approximate solution. Then the map*

$$t > 0 \mapsto \mathit{Tot.Var.}\big(f(\rho(t,\cdot))\big),$$

is not increasing.

PROOF. First of all we consider a single junction J with $n \le 2$ roads with incoming traffic and $m \le 2$ roads with outgoing traffic as in Section 3. It suffices to study the case $n = m = 2$, the other ones are simpler. Let $(\rho_{1,0},...,\rho_{4,0})$ be an equilibrium configuration in the junction J. Assume that a wave comes to the junction at the time $\bar{t}$, we claim that

$$\text{Tot.Var.}\big(f(\rho(\bar{t}+,\cdot))\big) = \text{Tot.Var.}\big(f(\rho(\bar{t}-,\cdot))\big). \tag{6.17}$$

We begin assuming that the wave is on an incoming road, for example the first one, and that it is given by the values $(\rho_1, \rho_{1,0})$. Let us define the incoming flux

$$f^{in}(y) \doteq \begin{cases} f(y) & \text{if } 0 \le y \le \sigma, \\ f(\sigma) & \text{if } \sigma \le y \le 1, \end{cases} \tag{6.18}$$

and the outgoing flux

$$f^{out}(y) \doteq \begin{cases} f(\sigma) & \text{if } 0 \le y \le \sigma, \\ f(y) & \text{if } \sigma \le y \le 1. \end{cases} \tag{6.19}$$

Clearly, since the wave on the first road has positive velocity, we have

$$0 \le \rho_1 \le \sigma, \qquad f(\rho_1) < f^{out}(\rho_{1,0}). \tag{6.20}$$

Let $(\hat{\rho}_1,...,\hat{\rho}_4)$ the solution of the Riemann Problem in the junction J with initial data $(\rho_1,...,\rho_{4,0})$ (see Theorem 3.1). By definition $\big(f(\rho_{1,0}), f(\rho_{2,0})\big)$ is the maximum of the map E on the domain

$$\Omega_0 \doteq \Big\{(\gamma_1,\gamma_2) \in \Omega_{1,0} \times \Omega_{2,0} \big| A \cdot (\gamma_1,\gamma_2)^T \in \Omega_{3,0} \times \Omega_{4,0}\Big\},$$

and $\left(f(\hat{\rho}_1), f(\hat{\rho}_2)\right)$ is the maximum of the map E on the domain

$$\hat{\Omega} \doteq \left\{(\gamma_1, \gamma_2) \in \Omega_1 \times \Omega_{2,0} \middle| A \cdot (\gamma_1, \gamma_2)^T \in \Omega_{3,0} \times \Omega_{4,0}\right\},$$

where

$$\Omega_{j,0} \doteq \begin{cases} [0, f^{in}(\rho_{j,0})] & \text{if } j = 1, 2, \\ [0, f^{out}(\rho_{j,0})] & \text{if } j = 3, 4, \end{cases}$$

and ,by (6.20),

$$\Omega_1 \doteq [0, f^{in}(\rho_1)] = [0, f(\rho_1)].$$

It is also clear that

$$\left(f(\rho_{1,0}), f(\rho_{2,0})\right) \in \partial\Omega_0, \qquad \left(f(\hat{\rho}_{1,0}), f(\hat{\rho}_{2,0})\right) \in \partial\hat{\Omega}.$$

To simplify the notations, define

$$\alpha_1 \doteq \alpha_{31}, \qquad \alpha_2 \doteq \alpha_{32}$$

then, by (1.3),

$$1 - \alpha_1 = \alpha_{41}, \qquad 1 - \alpha_2 = \alpha_{42}$$

We distinguish two cases. First we suppose that

$$f(\rho_1) < f(\rho_{1,0}), \tag{6.21}$$

(equality can not happen in the previous equation because the wave would have velocity zero). Then there results $\hat{\Omega} \subset \Omega_0$, hence

$$f(\hat{\rho}_1) \leq f(\rho_1) < f(\rho_{1,0}), \qquad f(\hat{\rho}_1) + f(\hat{\rho}_2) \leq f(\rho_{1,0}) + f(\rho_{2,0}), \tag{6.22}$$

where the first inequality is due to the fact that the wave $(\rho_1, \hat{\rho}_1)$ has negative velocity. We claim that

$$f(\rho_{2,0}) \leq f(\hat{\rho}_2), \tag{6.23}$$

and

$$f(\hat{\rho}_3) \leq f(\rho_{3,0}), \qquad f(\hat{\rho}_4) \leq f(\rho_{4,0}). \tag{6.24}$$

The points $\left(f(\rho_{1,0}), f(\rho_{2,0})\right)$, $\left(f(\hat{\rho}_{1,0}), f(\hat{\rho}_{2,0})\right)$ are on the boundaries of Ω_0, $\hat{\Omega}$ respectively, where E is maximum, hence they are on one of the curves

$$\alpha_1\gamma_1 + \alpha_2\gamma_2 = f^{out}(\rho_{3,0}), \qquad (1 - \alpha_1)\gamma_1 + (1 - \alpha_2)\gamma_2 = f^{out}(\rho_{4,0}), \qquad \gamma_2 = f^{in}(\rho_{2,0}).$$

Using (6.21), we immediately get (6.23). Let us assume that the two points are on the same curve, the general case being similar, for example on

$$\alpha_1\gamma_1 + \alpha_2\gamma_2 = f^{out}(\rho_{3,0}).$$

From (6.21) it follows that the map E is increasing on the curve

$$\gamma_1 \mapsto \Big(\gamma_1,\, f^{out}(\rho_{3,0}) - \frac{\alpha_1}{\alpha_2}\gamma_1\Big),$$

otherwise we contradict the maximality of E at $\big(f(\rho_{1,0}), f(\rho_{2,0})\big)$. Thus $\alpha_1 < \alpha_2$, $\hat\rho_1 = \rho_1$, and

$$f(\hat\rho_1) = f(\rho_1), \qquad f(\hat\rho_3) = f(\rho_{3,0}) = f^{out}(\rho_{3,0}).$$

On the other hand, by (6.22) and (6.23), we have

$$f(\hat\rho_4) = (1-\alpha_1)f(\hat\rho_1) + (1-\alpha_2)f(\hat\rho_2) \leq$$

$$\leq (1-\alpha_1)\big(f(\rho_{1,0}) + f(\rho_{2,0}) - f(\hat\rho_2)\big) + (1-\alpha_2)f(\hat\rho_2) =$$

$$= (1-\alpha_1)\big(f(\rho_{1,0}) + f(\rho_{2,0})\big) + \big(\alpha_1 - \alpha_2\big)f(\hat\rho_2) \leq$$

$$\leq (1-\alpha_1)\big(f(\rho_{1,0}) + f(\rho_{2,0})\big) + \big(\alpha_1 - \alpha_2\big)f(\rho_{2,0}) = f(\rho_{4,0}).$$

Using the Rankine Hogoniot Condition (2.7) in the junction (6.22), (6.23) and (6.24), we get

$$\mathrm{Tot.Var.}\big(f(\rho(\bar t+,\cdot))\big) = |f(\hat\rho_1)-f(\rho_1)|+|f(\hat\rho_2)-f(\rho_{2,0})|+|f(\hat\rho_3)-f(\rho_{3,0})|+|f(\hat\rho_4)-f(\rho_{4,0})| =$$

$$= \big(f(\hat\rho_2) - f(\rho_{2,0})\big) + \big(f(\rho_{3,0}) - f(\hat\rho_3)\big) + \big(f(\rho_{4,0}) - f(\hat\rho_4)\big) =$$

$$= f(\rho_{1,0}) - f(\hat\rho_1) = f(\rho_{1,0}) - f(\rho_1) = \mathrm{Tot.Var.}\big(f(\rho(\bar t-,\cdot))\big).$$

Suppose now that

$$f(\rho_{1,0}) < f(\rho_1),$$

then $\Omega_0 \subset \hat\Omega$ and using the previous arguments

$$f(\hat\rho_1) = f(\rho_1), \quad f(\hat\rho_2) \leq f(\rho_{2,0}), \quad f(\rho_{3,0}) \leq f(\hat\rho_3), \quad f(\rho_{4,0}) \leq f(\hat\rho_4).$$

By the Rankine Hogonot Condition in the junction (see (2.7)), we have

$$\mathrm{Tot.Var.}\big(f(\rho(\bar t+,\cdot))\big) = |f(\hat\rho_1)-f(\rho_1)|+|f(\hat\rho_2)-f(\rho_{2,0})|+|f(\hat\rho_3)-f(\rho_{3,0})|+|f(\hat\rho_4)-f(\rho_{4,0})| =$$

$$= \big(f(\rho_{2,0}) - f(\hat\rho_2)\big) + \big(f(\hat\rho_3) - f(\rho_{3,0})\big) + \big(f(\hat\rho_4) - f(\rho_{4,0}))\big) =$$

$$= \big(f(\hat\rho_1) - f(\rho_{1,0}))\big) + = f(\rho_1) - f(\rho_{1,0}) = \mathrm{Tot.Var.}\big(f(\rho(\bar t-,\cdot))\big).$$

In the general case we have only to observe that the total variation of ρ does not increase on the roads (see [6, Chapter 6]) and when a wave approaches a junction we can use the previous argument, so the proof is concluded. □

7 Solutions with a finite number of Big Waves

In this section we prove existence and stability of solutions with a finite number of big waves.

Definition 7.1 *We call $\mathcal{D}_n$ the set of all maps $\bar{\rho} : \cup_i I_i \mapsto I\!R$ defined on the network with bounded variation such that there exists a sequence $\{\bar{\rho}_\nu\}$ of piecewise constant maps satisfying*

$$Tot.Var.(\bar{\rho}_\nu) \leq Tot.Var.(\bar{\rho}), \qquad N(\bar{\rho}_\nu) \leq n, \tag{7.25}$$

for each $\nu \in I\!N$ and

$$\bar{\rho}_\nu \longrightarrow \bar{\rho} \quad \text{in } L^1. \tag{7.26}$$

Notice that if $\bar{\rho} \in \mathcal{D}_n$ then

$$\text{Tot.Var.}(\bar{\rho}) \leq \left\| \frac{1}{f'} \right\|_{L^\infty} \text{Tot.Var.}\big(f(\bar{\rho})\big) + n. \tag{7.27}$$

The existence of solutions with values in the domain $\mathcal{D}_n$ is ensured by next Theorem.

Theorem 7.1 *Let $f : [0,1] \to I\!R$ satisfying ($\mathcal{F}1$). Consider a road network $(\mathcal{I}, \mathcal{J})$ in which all junctions have at most two incoming roads and two outgoing ones. Given $n \in I\!N$ and $\bar{\rho} \in \mathcal{D}_n$, there exists an admissible solution ρ in the sense of Definition 2.2 such that $\rho(0,\cdot) = \bar{\rho}$, and $\rho(t,\cdot) \in \mathcal{D}_n$, for each $t \geq 0$.*

Proof. Let $\{\bar{\rho}_\nu\} \subset \mathcal{D}_n$ be a sequence of piecewise constant maps such that

$$\text{Tot.Var.}(\bar{\rho}_\nu) \leq \text{Tot.Var.}(\bar{\rho}), \qquad \bar{\rho}_\nu \longrightarrow \bar{\rho} \text{ in } L^1,$$

and ρ_ν the wave front tracking approximate solutions such that $\rho_\nu(0,\cdot) = \bar{\rho}_\nu$. By Lemma 5.1 we have

$$\rho_\nu(t,\cdot) \in \mathcal{D}_n, \qquad t \geq 0, \ \nu \in I\!N$$

and, by (7.27) and Lemma 6.1,

$$\text{Tot.Var.}(\rho_\nu(t,\cdot)) \leq \left\| \frac{1}{f'} \right\|_{L^\infty} \text{Tot.Var.}\big(f(\rho_\nu(t,\cdot))\big) + n \leq$$

$$\leq \left\| \frac{1}{f'} \right\|_{L^\infty} \text{Tot.Var.}\big(f(\bar{\rho}_\nu)\big) + n \leq \left\| \frac{1}{f'} \right\|_{L^\infty} \cdot \|f'\|_{L^\infty} \text{Tot.Var.}(\bar{\rho}) + n.$$

Since the wave speeds are bounded, the maps ρ_ν are uniformly Lipschitz continuous from $[0, +\infty[$ into L^1_{loc} for the L^1 norm on every compact set, and are obviously uniformly bounded. Then by Helly Theorem (see [6, Theorem 2.4]), ρ_ν converge to some continuous map $\rho \in L^1_{loc}([0, +\infty[\times \cup_i I_i, I\!R)$ such that, up to redefining ρ on a set of zero measure, $\rho(t,\cdot)$ has bounded variation. Since for every t we can obtain $\rho(t,\cdot)$ as limits of $\rho_\nu(t_n,\cdot)$ for some $t_n \to t$, we get $\rho(t,\cdot) \in \mathcal{D}_n$ for every $t \geq 0$.

It is a standard argument, see [6], to prove that ρ solves the conservation law on the road network. Moreover, $\rho(t,\cdot) \in BV$ and one can easily check the other properties to guarantee that ρ is an admissible solution. The proof is concluded. $\square$

Regarding stability of solutions we have:

Theorem 7.2 *Let $f : [0,1] \to I\!R$ satisfy $(\mathcal{F}1)$. Consider a network in which all junctions have at most two incoming roads and two outgoing ones. Let $n \in I\!N$, ρ and $\tilde\rho$ be admissible solutions such that*

$$\rho(t,\cdot), \tilde\rho(t,\cdot) \in \mathcal{D}_n \cap L^1, \qquad t \geq 0.$$

There results

$$\|\rho(t,\cdot) - \tilde\rho(t,\cdot)\|_{L^1} \leq \|\rho(0,\cdot) - \tilde\rho(0,\cdot)\|_{L^1},$$

for each $t \geq 0$.

We begin proving a lemma.

Lemma 7.1 *Let $f : [0,1] \to I\!R$ satisfy $(\mathcal{F}1)$ and J be a junction with two incoming roads and two outgoing ones as in Section 3. Let $\rho_1, \rho_{1,0}, ..., \rho_{4,0} \in [0,1]$, $\xi \in I\!R$ and $\bar x < b_1$ be such that*

$$\bar x + \xi < b_1, \qquad \rho_1 \in \begin{cases} [0,\sigma] \setminus \{\rho_{1,0}\} & \text{if } 0 \leq \rho_{1,0} \leq \sigma, \\ [0, \tau(\rho_{1,0})] & \text{if } \sigma \leq \rho_{1,0} \leq 1. \end{cases}$$

Let ρ and ρ^ξ be the wave front tracking approximate solutions of (1.1) on J such that

$$\rho_1(0,\cdot) = \rho_1 \cdot \chi_{]-\infty,\bar x]} + \rho_{1,0} \cdot \chi_{]\bar x, b_1]}, \quad \rho_1^\xi(0,\cdot) = \rho_1 \cdot \chi_{]-\infty,\bar x+\xi]} + \rho_{1,0} \cdot \chi_{]\bar x+\xi, b_1]},$$

$$\rho_i(0,\cdot) \equiv \rho_i^\xi(0,\cdot) \equiv \rho_{i,0}, \qquad i = 2,3,4.$$

There results

$$\|\rho_1^\xi(t,\cdot) - \rho_1(t,\cdot)\|_{L^1} \leq |\xi| \cdot |\rho_{1,0} - \rho_1| = \|\rho_1^\xi(0,\cdot) - \rho_1(0,\cdot)\|_{L^1},$$

for each $t \geq 0$.

Proof. We begin assuming that $(\rho_{1,0}, ..., \rho_{4,0})$ is an equilibrium configuration. By possibly changing the notations, we can assume that $\xi > 0$. Since we approximate the rarefaction fronts with many small shocks we have only to study the case in which the Riemann Problem $(\rho_1, \rho_{1,0})$ on the first road and the one $(\rho_{1,0}, ..., \rho_{4,0})$ in the junction generate only shocks. Let $(\hat\rho_1, ..., \hat\rho_4)$ be the solution of the Riemann Problem with initial data $(\rho_1, \rho_{2,0} ..., \rho_{4,0})$ (see

Theorem 3.1) and $\lambda_{1,0}, \hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\lambda}_4$ the velocities of the shocks generated by the Riemann Problems $(\rho_1, \rho_{1,0}), (\rho_1, \hat{\rho}_1), (\rho_2, \hat{\rho}_2), (\hat{\rho}_3, \rho_{3,0}), (\hat{\rho}_4, \rho_{4,0})$ respectively.

If $t \geq \dfrac{b_1 - \bar{x} - \xi}{\lambda_{1,0}}$ by the first part of the proof of Lemma 6.1 and the Rankine Hugoniot Conditions on the roads (see [6, Lemma 4.2]) we have

$$\|\rho_1^\xi(t,\cdot) - \rho_1(t,\cdot)\|_{L^1} = \frac{|\xi|}{|\lambda_{1,0}|}\Big(|\rho_1 - \hat{\rho}_1| \cdot |\hat{\lambda}_1| + \sum_{i=2}^{4} |\rho_{i,0} - \hat{\rho}_i| \cdot |\hat{\lambda}_i|\Big) =$$

$$= \frac{|\xi|}{|\lambda_{1,0}|}\Big(|f(\rho_1) - f(\hat{\rho}_1)| + \sum_{i=2}^{4} |f(\rho_{i,0}) - f(\hat{\rho}_i)|\Big) = \frac{|\xi|}{|\lambda_{1,0}|}\text{Tot.Var.}\big(f(\rho(t,\cdot))\big) =$$

$$= \frac{|\xi|}{|\lambda_{1,0}|}\text{Tot.Var.}\big(f(\rho(0,\cdot))\big) = \frac{|\xi|}{|\lambda_{1,0}|}\big|f(\rho_1) - f(\rho_{1,0})\big| = |\xi| \cdot |\rho_1 - \rho_{1,0}| = \|\rho_1^\xi(0,\cdot) - \rho_1(0,\cdot)\|_{L^1}.$$

In the case in which $(\rho_{1,0}, ..., \rho_{4,0})$ is not an equilibrium configuration we have only to recall that the L^1-distance between the solutions decreases on each roads (see of [6, Corollary 6.1]) and use the same arguments.

This concludes the proof. □

Proof of Theorem 7.2. Let ρ^k and $\tilde{\rho}^k$ be front tracking approximate solutions such that

$$\|\rho^k(0,\cdot) - \rho(0,\cdot)\|_{L^1} \leq \frac{1}{k}, \qquad \|\tilde{\rho}^k(0,\cdot) - \tilde{\rho}(0,\cdot)\|_{L^1} \leq \frac{1}{k}, \qquad k \in \mathbb{N},$$

$$\text{Tot.Var.}(\rho^k(0,\cdot)) \leq \text{Tot.Var.}(\rho(0,\cdot)), \quad \text{Tot.Var.}(\tilde{\rho}^k(0,\cdot)) \leq \text{Tot.Var.}(\tilde{\rho}(0,\cdot)), \quad k \in \mathbb{N}.$$

Now consider finitely many wave front tracking approximate solutions $\rho^{k,0}, ..., \rho^{k,N}$

$$\rho^{k,0} \equiv \rho^k, \qquad \rho^{k,N} \equiv \tilde{\rho}^k$$

where $\rho^{k,h}$ is obtained by $\rho^{k,h-1}$ shifting and rescaling only one jump as in [7] and [8]. Precisely denoting

$$\rho_i^{k,h-1}(0,\cdot) = \sum_{l=0}^{N-1} \alpha_l \cdot \chi_{[\beta_l,\beta_{l+1}[},$$

there exist $\lambda, \xi \in \mathbb{R}$ and $\bar{l} \in \{1, ..., N-1\}$ such that

$$\rho_i^{k,h}(0,\cdot) = \sum_{l=0}^{\bar{l}-2} \alpha_l \cdot \chi_{[\beta_l,\beta_{l+1}[} + \alpha_{\bar{l}-1} \cdot \chi_{[\beta_{\bar{l}-1},\beta_{\bar{l}}+\xi[} + \lambda \cdot \chi_{[\beta_{\bar{l}}+\xi,\beta_{\bar{l}+1}[} + \sum_{l=\bar{l}+1}^{N+1} \alpha_l \cdot \chi_{[\beta_l,\beta_{l+1}[},$$

with

$$\beta_{\bar{l}} - \beta_{\bar{l}-1} \leq \xi \leq \beta_{\bar{l}+1} - \beta_{\bar{l}},$$

for each $i = 1, ..., N$. In this way we have

$$\sum_{h=1}^{N} \|\rho^{k,h}(0,\cdot) - \rho^{k,h+1}(0,\cdot)\|_{L^1} = \|\rho^k(0,\cdot) - \tilde{\rho}^k(0,\cdot)\|_{L^1}. \tag{7.28}$$

Since the distance between solutions decreases on each road (see [6, Corollary 6.1]) and by the previous lemma, we have

$$\|\rho^{k,h}(t,\cdot) - \rho^{k,h+1}(t,\cdot)\|_{L^1} \le \|\rho^{k,h}(0,\cdot) - \rho^{k,h+1}(0,\cdot)\|_{L^1}.$$

So, by (7.28),

$$\|\rho^k(t,\cdot) - \tilde{\rho}^k(t,\cdot)\|_{L^1} \le \sum_{h=1}^{N} \|\rho^{k,h}(t,\cdot) - \rho^{k,h+1}(t,\cdot)\|_{L^1} \le$$

$$\le \sum_{h=1}^{N} \|\rho^{k,h}(0,\cdot) - \rho^{k,h+1}(0,\cdot)\|_{L^1} = \|\rho^k(0,\cdot) - \tilde{\rho}^k(0,\cdot)\|_{L^1}.$$

Moreover there exists a decreasing sequence $\{k_n\} \subset I\!N$ such that $\rho^{k_n} \to \rho$ and $\tilde{\rho}^{k_n} \to \tilde{\rho}$ in L^1 as $k_n \to +\infty$. Hence

$$\|\rho(t,\cdot) - \tilde{\rho}(t,\cdot)\|_{L^1} \le \|\rho(0,\cdot) - \tilde{\rho}(0,\cdot)\|_{L^1},$$

as to be proved. □

8 Existence and Stability of Solutions in L^1

Let us first consider the case in which $(\mathcal{F}1)$ holds true.

Theorem 8.1 *Let $f : [0,1] \to I\!R$ satisfy $(\mathcal{F}1)$. Consider a road network in which each junction has at most two incoming roads and two outgoing ones. Let $\bar{\rho}$ be an initial data in L^1_{loc} and fix $T > 0$. Then there exists a unique admissible solution ρ defined on $[0,T]$, obtained as limit of wave front tracking approximate solutions such that $\rho(0,\cdot) = \bar{\rho}$. Moreover if $\bar{\rho} \in L^1$ then $\rho(t,\cdot) \in L^1$.*

If ρ and $\tilde{\rho}$ are admissible solutions obtained as limit of wave front tracking approximate solutions such that $\rho(t,\cdot)$, $\tilde{\rho}(t,\cdot) \in L^1$, for every $t \ge 0$, then for each $t \ge 0$

$$\|\rho(t,\cdot) - \tilde{\rho}(t,\cdot)\|_{L^1} \le \|\rho(0,\cdot) - \tilde{\rho}(0,\cdot)\|_{L^1}. \tag{8.29}$$

PROOF. We begin proving the existence of a solution for $\bar{\rho} \in L^1$. There exists $\{\bar{\rho}_n\}$ sequence of piecewise constant maps defined on the network such that

$$\bar{\rho}_n \in \mathcal{D}_n \cap L^1, \qquad \bar{\rho}_n \longrightarrow \bar{\rho} \quad \text{in } L^1. \tag{8.30}$$

Let ρ_n be the wave front tracking approximate solutions with $\rho_n(0,\cdot) = \bar{\rho}_n$. Fix $n < m$, by Lemma 5.1, there results $\rho_n(t,\cdot) \in \mathcal{D}_n \subset \mathcal{D}_m, \rho_m(t,\cdot) \in \mathcal{D}_m$ and by Theorem 7.2,

$$\|\rho_n(t,\cdot) - \rho_m(t,\cdot)\|_{L^1} \leq \|\bar{\rho}_n - \bar{\rho}_m\|_{L^1}.$$

Hence $\{\rho_n(t,\cdot)\}$ is a Cauchy sequence in $L^1\big([0,T] \times I\!R, I\!R\big)$. Then there exists ρ such that $\rho_n(t,\cdot) \longrightarrow \rho(t,\cdot)$ in $L^1\big([0,T] \times I\!R, I\!R\big)$. It is easy to check that ρ is as admissible solution. The case of L^1_{loc} can be obtained by localization.

Now we prove (8.29). Let $\{\rho_n\}$, $\{\tilde{\rho}_n\}$ be sequences of wave front tracking approximate solutions such that $\rho_n(t,\cdot)$, $\tilde{\rho}_n(t,\cdot) \in \mathcal{D}_n \cap L^1$ and $\rho_n \longrightarrow \rho$, $\tilde{\rho}_n \longrightarrow \tilde{\rho}$ in $L^1\big([0,T] \times I\!R, I\!R\big)$. By Theorem 7.2, we have

$$\|\rho_n(t,\cdot) - \tilde{\rho}_n(t,\cdot)\|_{L^1} \leq \|\rho_n(0,\cdot) - \tilde{\rho}_n(0,\cdot)\|_{L^1}.$$

Therefore (8.29) is proved and uniqueness holds true. $\square$

We now relax the assumption $(\mathcal{F}1)$, namely we suppose that f satisfies $(\mathcal{F})$.

Let $\{f_\nu\}$ be a sequence of maps satisfying $(\mathcal{F}1)$ such that

$$f_\nu(\sigma) = \max_{\rho \in [0,1]} f_\nu(\rho), \qquad \nu \in I\!N \tag{8.31}$$

and

$$f_\nu \longrightarrow f \text{ in } L^\infty([0,1]) \quad \text{and} \quad {f_\nu}' \longrightarrow f' \text{ in } L^\infty([0,1]). \tag{8.32}$$

Moreover let $\bar{\rho}$ be an initial data in L^1_{loc}. We know that there exists a unique $\rho_\nu = \rho_\nu(t,x)$ admissible solution to the Cauchy Problem on the network (see Theorem 8.1) obtained as limit of front tracking approximate solutions for

$$\rho_t + f_\nu(\rho)_x = 0, \qquad \rho(0,\cdot) \equiv \bar{\rho}, \qquad \nu \in I\!N. \tag{8.33}$$

Theorem 8.2 *Let $f : [0,1] \to I\!R$ satisfy $(\mathcal{F})$. Consider a road network in which all the junction have at most two incoming roads and two outgoing ones. Let $\bar{\rho}$ be an initial data in L^1_{loc} and fix $T > 0$. Then there exists a unique admissible solution ρ defined on $[0,T]$, with $\rho(0,\cdot) = \bar{\rho}$, obtained as limit of solutions to (8.33). The limit does not depend on the choice of the functions f_ν and if $\bar{\rho} \in L^1$ then $\rho(t,\cdot) \in L^1$.*

If ρ and $\tilde{\rho}$ are such admissible solutions and satisfy $\rho(t,\cdot)$, $\tilde{\rho}(t,\cdot) \in L^1$, for every $t \geq 0$, then

$$\|\rho(t,\cdot) - \tilde{\rho}(t,\cdot)\|_{L^1} \leq \|\rho(0,\cdot) - \tilde{\rho}(0,\cdot)\|_{L^1}. \tag{8.34}$$

The Theorem can be proved exactly as Theorem 8.1 from next Lemmas.

Lemma 8.1 *Let $f : [0,1] \to I\!R$ satisfy $(\mathcal{F})$. Consider a road network in which all junctions have at most two incoming roads and two outgoing ones. Let $\bar{\rho}$ be an initial data in $\mathcal{D}_n \cap L^1$ and fix $T > 0$. Then there exists a unique admissible solution ρ defined on $[0,T]$, with $\rho(0,\cdot) = \bar{\rho}$, obtained as limit of solutions to (8.33). The limit does not depend on the choice of the functions f_ν and and $\rho(t,\cdot) \in L^1$.*

PROOF. Let $\{f_\nu\}$ be a sequence of maps satisfying $(\mathcal{F}1)$, (8.31) and (8.32) and ρ_ν be the admissible solutions for the Cauchy problems associated to f_ν. By Theorem 7.1 we have

$$\rho_\nu(t,\cdot) \in \mathcal{D}_n, \qquad \nu \in I\!N, \quad 0 \le t \le T. \tag{8.35}$$

Moreover there results

$$\|\rho_\nu(t,\cdot) - \rho_\mu(t,\cdot)\|_{L^1} \le C \cdot \|{f_\nu}' - {f_\mu}'\|_{L^\infty} \cdot \text{Tot.Var.}(\bar{\rho}), \tag{8.36}$$

where C depends only on f. By (8.32), $\{\rho_\nu\}$ is a Cauchy sequence in L^1, then there exists $\rho \in L^1$ such that $\rho_\nu \to \rho$ in L^1. Moreover, ρ is an admissible solution and satisfies $\rho(0,\cdot) \equiv \bar{\rho}$. From (8.36) we have that ρ does not depend on the choice of $\{f_\nu\}$, so we are done. □

Lemma 8.2 *Let $f : [0,1] \to I\!R$ satisfy $(\mathcal{F})$. Consider a road network in which all junctions have at most two incoming roads and two outgoing ones. Fix $T > 0$ and let $\tilde{\rho}$, ρ be admissible solutions in L^1, obtained as limit of solutions to (8.33), defined on $[0,T]$. If $\rho(0,\cdot)$, $\tilde{\rho}(0,\cdot) \in \mathcal{D}_n$, then*

$$\|\rho(t,\cdot) - \tilde{\rho}(t,\cdot)\|_{L^1} \le \|\rho(0,\cdot) - \tilde{\rho}(0,\cdot)\|_{L^1},$$

for each $0 \le t \le T$.

PROOF. Let $\{f_\nu\}$ be a sequence of maps satisfying $(\mathcal{F}1)$, (8.31) and (8.32) and ρ_ν be the admissible solutions associated to f_ν such that

$$\rho_\nu(0,\cdot) \equiv \rho(0,\cdot), \qquad \tilde{\rho}_\nu(0,\cdot) \equiv \tilde{\rho}(0,\cdot). \tag{8.37}$$

By Lemma 8.1, we have $\rho_\nu \to \rho$ and $\tilde{\rho}_\nu \to \tilde{\rho}$ in L^1. By Theorem 8.1 and (8.37), for each $0 \le t \le T$ and $\nu \in I\!N$, there results

$$\|\rho(t,\cdot) - \tilde{\rho}(t,\cdot)\|_{L^1} \le \|\rho(t,\cdot) - \rho_\nu(t,\cdot)\|_{L^1} + \|\rho_\nu(t,\cdot) - \tilde{\rho}_\nu(t,\cdot)\|_{L^1} + \|\tilde{\rho}(t,\cdot) - \tilde{\rho}_\nu(t,\cdot)\|_{L^1} \le$$

$$\le \|\rho(t,\cdot) - \rho_\nu(t,\cdot)\|_{L^1} + \|\rho(0,\cdot) - \tilde{\rho}(0,\cdot)\|_{L^1} + \|\tilde{\rho}(t,\cdot) - \tilde{\rho}_\nu(t,\cdot)\|_{L^1} \longrightarrow \|\rho(0,\cdot) - \tilde{\rho}(0,\cdot)\|_{L^1}.$$

So the proof is concluded. □

9 Time Dependent Traffic

In this section we consider a model of traffic including cross lights and time dependent traffic. The latter means that the choice of drivers at junctions depends on the period of the day, so during the morning the traffic flows towards some direction and during afternoon it may change towards another direction. This means that the matrix A depends on time t (see Section 3).

Consider a single junction as in Section 3 with two incoming roads and two outgoing ones. Let $\alpha_1 = \alpha_1(t)$, $\alpha_2 = \alpha_2(t)$ be two piecewise constant periodic functions such that

$$\alpha_1(t) \neq \alpha_2(t), \tag{9.38}$$

for each $t \geq 0$. Moreover let $\chi_1 = \chi_1(t), \chi_2 = \chi_2(t)$ be piecewise constant periodic maps such that

$$\chi_1(t) + \chi_2(t) = 1, \qquad \chi_i(t) \in \{0, 1\}, \qquad i = 1, 2$$

for each $t \geq 0$. The two maps represent traffic lights, the value 0 corresponding to red light and the value 1 to green light.

Definition 9.1 *Consider $\rho = \rho(t, x_1, ..., x_4) = \big(\rho_1(t, x_1), ..., \rho_4(t, x_4)\big)$ with bounded variation. We say that ρ is a solution at the junction J if it satisfies (i), (ii), (iv) of Definition 2.1 and the following property holds:*

(v) $f(\rho_3(t, a_3+)) = \alpha_1(t)\chi_1(t) f(\rho_1(t, b_1-)) + \alpha_2(t)\chi_2(t) f(\rho_2(t, b_2-))$ *and* $f(\rho_4(t, a_4+)) = \big(1 - \alpha_1(t)\big)\chi_1(t) f(\rho_1(t, b_1+)) + \big(1 - \alpha_2(t)\big)\chi_2(t) f(\rho_2(t, b_2+))$ *for each* $t > 0$.

Assume that at time $\bar{t}$ one of the maps $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, $\chi_1(\cdot)$, $\chi_2(\cdot)$ jumps, then we have to solve a new Riemann Problem in the junction hence four waves are generated and

$$N\big(\rho(\bar{t}+, \cdot)\big) \leq N\big(\rho(\bar{t}-, \cdot)\big), \tag{9.39}$$

$$\text{Tot.Var.}\big(f(\rho(\bar{t}+, \cdot))\big) \leq \text{Tot.Var.}\big(f(\rho(\bar{t}-, \cdot))\big) + 4f(\sigma). \tag{9.40}$$

Then the map $N\big(\rho(t, \cdot)\big)$ is still non increasing while

$$\text{Tot.Var.}\big(f(\rho(t_2, \cdot))\big) \leq \text{Tot.Var.}\big(f(\rho(t_1, \cdot))\big) + 4f(\sigma)\Phi(t_1, t_2),$$

for each $0 < t_1 \leq t_2$, where

$$\Phi(t_1, t_2) \doteq \sum_{i=1,2} \Big(\#\big\{t_l | t_1 < t_l \leq t_2,\ \chi_i \text{ jumps in } t_l\big\} + \#\big\{t_l | t_1 < t_l \leq t_2,\ \alpha_i \text{ jumps in } t_l\big\}\Big).$$

Therefore, for fixed $T > 0$, we have uniform bounds of the total variation on the interval $[0, T]$, and using arguments as in the previous sections we obtain existence and stability for the Cauchy Problem. However, the total variation of $f(\rho)$ does not depend continuously on the total variation of the maps $\alpha_1(\cdot)$, $\alpha_2(\cdot)$. Indeed consider a single junction with two incoming roads and two outgoing ones without traffic lights, i.e. $\chi_i \equiv 1$, and let

$$\alpha_1(t) = \begin{cases} \beta_1 & \text{if } 0 \le t \le \bar{t}, \\ \beta_2 & \text{if } \bar{t} \le t \le T, \end{cases} \quad , \quad \alpha_2(t) = \begin{cases} \beta_2 & \text{if } 0 \le t \le \bar{t}, \\ \beta_1 & \text{if } \bar{t} \le t \le T, \end{cases} \quad ,$$

where $0 < \beta_2 < \beta_1 < \frac{1}{2}$ and $0 < \bar{t} < T$. Consider the initial data $(\rho_{1,0}, \rho_{2,0}, \rho_{3,0}, \rho_{4,0})$, where

$$f(\rho_{1,0}) = f(\rho_{4,0}) = f(\sigma), \qquad f(\rho_{2,0}) = f(\rho_{3,0}) = \frac{\beta_1}{1-\beta_2} f(\sigma).$$

This is an equilibrium configuration for the choice $\alpha_i = \beta_i$, $i = 1, 2$, hence the solution of the Riemann Problem is identically equal to the initial data for $0 \le t \le \bar{t}$. At time $t = \bar{t}$ we have to solve a new Riemann Problem. Let $(\hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_3, \hat{\rho}_4)$ its solution, there results (see figure 2)

$$f(\hat{\rho}_2) = f(\hat{\rho}_4) = f(\sigma), \qquad f(\hat{\rho}_1) = f(\hat{\rho}_3) = \frac{\beta_2}{1-\beta_1} f(\sigma).$$

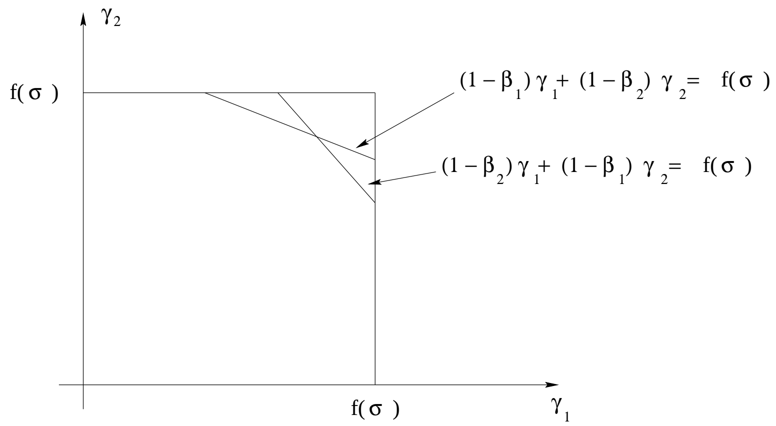

figure 2

Now, if $\beta_1 \to \beta_2$, then

$$\text{Tot.Var.}\big(\alpha_1; [0, T]\big) \longrightarrow 0, \qquad \text{Tot.Var.}\big(\alpha_2; [0, T]\big) \longrightarrow 0,$$

but

$$\big(f(\rho_{1,0}),\, f(\rho_{2,0})\big) \longrightarrow \left(f(\sigma),\, \frac{\beta_2}{1-\beta_2} f(\sigma)\right), \qquad \big(f(\hat{\rho}_1),\, f(\hat{\rho}_2)\big) \longrightarrow \left(\frac{\beta_2}{1-\beta_2} f(\sigma),\, f(\sigma)\right),$$

hence $\text{Tot.Var.}(f(\rho); [0, T])$ is bounded away from zero.

A Appendix: Total Variation of the Fluxes

In this section we show an example in which the total variation of the flux increases due to interactions of waves with junctions.

Consider a single junction with three incoming roads and three outgoing ones, the matrix

$$A \doteq \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{6} & 0 & \frac{1}{6} \end{pmatrix} \tag{A.41}$$

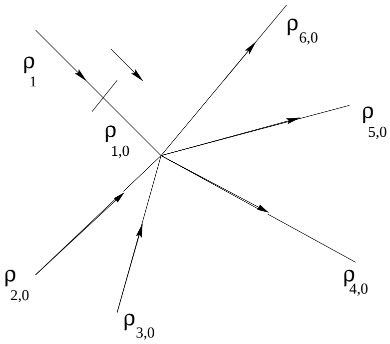

figure 3

and constants $\rho_1,\ \rho_{1,0}, ...,\ \rho_{6,0} \in [0,1]$ such that

$$\rho_{1,0} = \rho_{3,0} = \rho_{4,0} = \rho_{5,0} = \sigma,\ \ \sigma < \rho_{2,0} < 1,\ \ 0 < \rho_{6,0},\ \rho_1 < \sigma,\ \ f(\rho_{2,0}) = \frac{1}{3},\ \ f(\rho_{6,0}) = \frac{1}{3}.$$

Assume that $f(\sigma) = 1$, then $(\rho_{1,0}, ...,\ \rho_{6,0})$ is an equilibrium configuration and ρ given by

$$\rho_1(0,x) = \begin{cases} \rho_{1,0} & \text{if } x_1 \le x \le b_1, \\ \rho_1 & \text{if } x < x_1, \end{cases} \qquad \rho_i(0,\cdot) \equiv \rho_{i,0}, \quad i = 2, ..., 6,$$

is a solution. Moreover the plane

$$\frac{1}{6}\gamma_1 + \frac{1}{6}\gamma_3 = 1$$

does not intersect the cube $[0,1]^3$ and the point $\big(f(\rho_{1,0}), ...,\ f(\rho_{6,0})\big)$ is on the intersection of the planes

$$\frac{1}{2}\gamma_1 + \frac{1}{2}\gamma_2 + \frac{1}{3}\gamma_3 = 1, \qquad \frac{1}{3}\gamma_1 + \frac{1}{2}\gamma_2 + \frac{1}{2}\gamma_3 = 1,$$

that is the line described by the map

$$\gamma_1 \mapsto \Big(\gamma_1, 2 - \frac{5}{3}\gamma_1, \gamma_1\Big). \tag{A.42}$$

At some time say $\bar{t}$ the wave $(\rho_1, \rho_{1,0})$ interacts with the junction. Let $(\hat{\rho}_1, ..., \hat{\rho}_6)$ be the solution of the Riemann Problem at the junction for the data $(\rho_1,\ \rho_{2,0}, ...,\ \rho_{6,0})$. Since the map E increases on the line described by (A.42), the point $\big(f(\hat\rho_1),....,f(\hat\rho_6)\big)$ is on the curve (A.42) and

$$f(\hat\rho_1)=f(\hat\rho_3)=f(\rho_1),\;\; f(\hat\rho_2)=2-\frac{5}{3}f(\rho_1),\;\; f(\hat\rho_4)=f(\hat\rho_5)=f(\sigma),\;\; f(\hat\rho_6)=\frac{1}{3}f(\rho_1).$$

We get

$$\mathrm{Tot.Var.}\big(f(\rho(\bar t-,\cdot))\big)=1-f(\rho_1),$$

while

$$\mathrm{Tot.Var.}\big(f(\rho(\bar t+,\cdot))\big)=4\big(1-f(\rho_1)\big)>\mathrm{Tot.Var.}\big(f(\rho(\bar t-,\cdot))\big).$$

B Appendix: Total Variation of the Densities

Consider a junction J with two incoming roads and two outgoing ones that we parameterize with the intervals $]-\infty,\, b_1]$, $]-\infty,\, b_2]$, $[a_3,\, +\infty[$, $[a_4,\, +\infty[$ respectively. Fix the constants α_1, α_2 such that $0<\alpha_1<\alpha_2<1/2$ and set

$$\alpha_{3,1}=\alpha_1,\quad \alpha_{3,2}=\alpha_2,\quad \alpha_{4,1}=1-\alpha_1,\quad \alpha_{3,2}=1-\alpha_2.$$

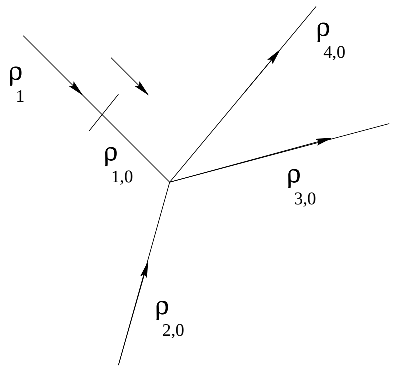

figure 4

Define a solution ρ by

$$\rho_1(0,x)=\begin{cases}\rho_{1,0} & \text{if } x_1\le x\le b_1,\\ \rho_1 & \text{if } x<x_1,\end{cases}\qquad \rho_2(0,x)=\rho_{2,0},\quad \rho_3(0,x)=\rho_{3,0},\quad \rho_4(0,x)=\rho_{4,0}, \tag{B.43}$$

where $\rho_1,\ \rho_{1,0},\ \rho_{2,0},\ \rho_{3,0},\ \rho_{4,0}$ are constants such that

$$\sigma\le\ \rho_{2,0},\,\rho_{3,0}\le 1,\quad 0\le\rho_1\le\sigma,\quad \rho_{1,0}=\rho_{4,0}=\sigma, \tag{B.44}$$

$$f(\rho_{1,0})=f(\rho_{4,0})=f(\sigma),\quad f(\rho_{2,0})=f(\rho_{3,0})=\frac{\alpha_1}{1-\alpha_2}f(\sigma),$$

so $(\rho_{1,0},\ \rho_{2,0},\ \rho_{3,0},\ \rho_{4,0})$ is an equilibrium configuration.

After some time the wave $(\rho_1, \rho_{1,0})$ interacts with the junction. Let $(\hat{\rho}_1,\ \hat{\rho}_2,\ \hat{\rho}_3,\ \hat{\rho}_4)$ be the solution of the Riemann Problem in the junction for the data $(\rho_1,\ \rho_{2,0},\ \rho_{3,0},\ \rho_{4,0})$. By (B.43) and (B.44) there results

$$f(\hat{\rho}_1) = f(\rho_1), \quad f(\hat{\rho}_2) = \frac{f(\sigma) - (1-\alpha_1) f(\rho_1)}{1-\alpha_2},$$

$$f(\hat{\rho}_3) = \frac{\alpha_1 - \alpha_2}{1-\alpha_2} f(\rho_1) + \frac{\alpha_2}{1-\alpha_2} f(\sigma), \quad f(\hat{\rho}_4) = f(\sigma)$$

and

$$0 \leq \hat{\rho}_3 \leq \sigma \leq \hat{\rho}_2 \leq 1. \tag{B.45}$$

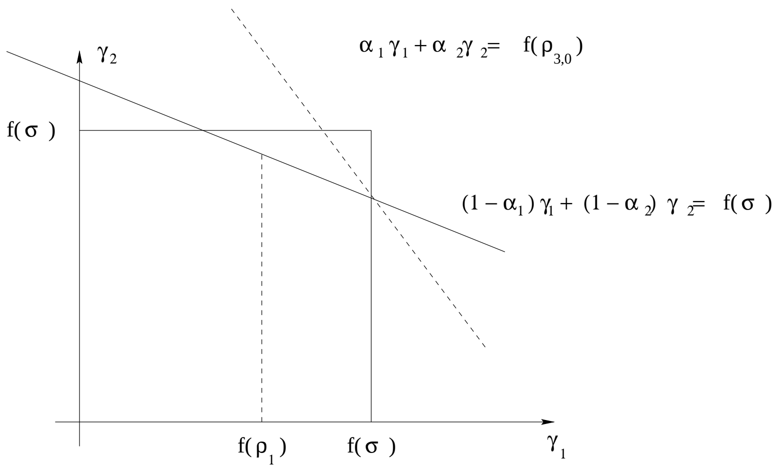

figure 5

Therefore if $\rho_1 \to \rho_{1,0} = \sigma$ then

$$f(\hat{\rho}_3) \longrightarrow \frac{\alpha_1}{1-\alpha_2} f(\sigma) = f(\rho_{3,0}),$$

and by (B.45), (B.44) we have $\hat{\rho}_3 \to \tau(\rho_{3,0})$. Therefore we are able to create on the third road a wave with strength bounded away from zero using an arbitrarily small wave on the first one.

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