Evaluation of the Lopatinski Determinant for Multi-dimensional Euler Equations

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1 The Lopatinski condition for general systems

There is a large body of literature on the problem of stability for inviscid shock waves and in particular for gas dynamics, see [2, 16, 3, 5, 6, 8, 9, 15, 4, 7, 20, 11, 12, 1, 13, 14, 23].

The purpose of this appendix is to calculate the Lopatinski determinant, or "stability function," for the Euler equations of compressible gas dynamics. We offer two approaches to this problem. In the first we use a change of variables to simplify the computation. In the second method we take advantage of the Galilean invariance of the Euler equations and exploit a relationship between the eigenvectors of a particular pair of matrices. As a preliminary step for this first technique, we discuss how the Lopatinski determinant behaves under a change of coordinates.

Consider an inviscid system of n hyperbolic conservation laws in d space dimensions,

$$w_t + \sum_{j=1}^{d} f^j(w)_{x_j} = 0, \tag{1}$$

where $x = (x_1, \dots, x_d) \in \mathbf{R}^d$, $w \in \mathbf{R}^n$. A planar shock wave solution moving along the x_1-axis is a solution of the form $w(x,t) = W(x_1 - st)$ where W is piecewise constant with a single jump discontinuity at the origin:

$$W(y) = \begin{cases} W^- & \text{for } y < 0 \\ W^+ & \text{for } y > 0. \end{cases} \tag{2}$$

The Rankine-Hugoniot condition relating the right and left states is,

$$s[w] = [f^1(w)], \tag{3}$$

where $[q] := q^+ - q^-$ for any quantity q. By a linear change of independent variables we can assume that the shock is stationary, *i.e.* $s = 0$.

The linearized stability of planar shock waves for (1) can be studied through the Lopatinski determinant which we now define. We use the notation

$$f^\xi := \sum_{j=1}^d \xi_j f^j, \quad \xi = (\xi_1, \ldots, \xi_d)^{tr}, \quad \tilde{\xi} := (\xi_2, \ldots, \xi_d)^{tr},$$

$$A^\xi := \sum_{j=1}^d \xi_j A^j, \qquad A^{\tilde{\xi}} := A^{(0,\tilde{\xi})},$$

and define

$$\mathcal{A}(\tilde{\xi}, \lambda) := (\lambda I + iA^{\tilde{\xi}})(A^1)^{-1}. \tag{4}$$

The Lopatinski determinant (stability function) for a Lax p-shock is then defined to be

$$\Delta(\tilde{\xi}, \lambda) := \det(r_1^-, \ldots, r_{p-1}^-, \lambda[w] + i[f^{\tilde{\xi}}(w)], r_{p+1}^+, \ldots, r_n^+), \tag{5}$$

where $\{r_1^-, \ldots, r_{p-1}^-\}, \{r_{p+1}^+, \ldots, r_n^+\}$ are bases for stable and unstable subspaces of $\mathcal{A}(\tilde{\xi}, \lambda)$, respectively, and the jumps are given by the Rankine-Hugoniot relation for the unperturbed shock. The expression in (5) is obtained through a Fourier-Laplace analysis and a zero $(\tilde{\xi}, \lambda)$ of Δ corresponds to a mode of the form $\sim e^{\lambda t + i\tilde{\xi}\cdot\tilde{x}}$ for the equations linearized about $W(x)$. The function Δ is jointly analytic in $(\tilde{\xi}, \lambda)$ for $\tilde{\xi} \in \mathbf{R}^{d-1}$ and $\mathrm{Re}\lambda > 0$ and Δ is homogeneous of degree 1. These features will prove useful below. See [23, 22, 17, 12] for details.

Definition 1 *Given a planar shock wave W of the form (2), and the corresponding Lopatinski determinant $\Delta(\tilde{\xi}, \lambda)$ defined by (5). If Δ has a zero $(\tilde{\xi}, \lambda)$ with $Re\lambda > 0$ the shock wave is referred to as* strongly unstable. *If all zeros of Δ have $Re\lambda < 0$, the shock is called* uniformly stable. *Finally, the intermediate case where Δ has some root with $Re\lambda = 0$ but no root with $Re\lambda > 0$ is referred to as a* weakly stable *shock.*

We observe that in the case of an extreme shock, i.e. $p = 1$ or $p = n$, the expression for Δ simplifies to

$$\Delta = l \cdot (\lambda[w] + i[f^{\tilde{\xi}}(w)]), \tag{6}$$

where l is a left eigenvector of $\mathcal{A}$ evaluated at the right or left state corresponding to whether $p = 1$ or $p = n$. However, the calculation of these left eigenvectors for specific systems such as gas dynamics may be rather involved, the main problem being to invert the matrix A^1.

We next present two alternatives for calculating Δ for extreme shocks, one by a change of variables and the other by utilizing the relationship between left eigenvectors and eigenvalues of $\mathcal{A}$ and A^{ξ}.

Method 1: An appropriate change of coordinates may yield a straightforward and explicit calculation of Δ. Indeed Erpenbeck used this approach in [4] for gas dynamics, the first work in which this calculation was completely carried out. We begin by briefly considering how Δ in general behaves under a change of coordinates. We then apply the analysis to the Euler equations written in terms of specific volume, velocity, and specific entropy in the next section.

The system (1) can be written in quasilinear form

$$w_t + \sum_{j=1}^{d} A^j w_{x_j} = 0, \tag{7}$$

where $A^j(w) := Df^j(w)$. Consider a change of coordinates in (1) given by $w = \varphi(\bar{w})$. The corresponding quasilinear system is thus

$$\bar{w}_t + \sum_{j=1}^{d} M^j \bar{w}_{x_j} = 0, \tag{8}$$

where $M^j(\bar{w}) = C^{-1}A^j(\varphi(\bar{w}))C$ and $C = D\varphi$. Using notation as above,

$$\mathcal{A}(\tilde{\xi}, \lambda) = C(\lambda I + iM^{\tilde{\xi}})(M^1)^{-1}C^{-1}.$$

A proper choice of coordinates simplifies the calculation of the desired eigenvectors. We write $\mathcal{B} = (\lambda I + iM^{\tilde{\xi}})(M^1)^{-1}$. Since $\bar{l}$ is a left eigenvector of $\mathcal{B}$ with corresponding eigenvalue β if and only if $\bar{l}C^{-1}$ is a left eigenvector of $\mathcal{A}$ with eigenvalue β, we have

$$\Delta = \bar{l}^{tr}C^{-1}(\lambda[w] + i[f^{\tilde{\xi}}(w)]). \tag{9}$$

The point is that by properly choosing coordinates, the matrix M^1 may be easy to invert. In the case of gas dynamics, this gives an explicit formulation of Δ in terms of λ and $\tilde{\xi}$.

Method 2: An alternative way to deal with the problem of inverting A^1 in (4) is to observe that l is a left eigenvector of $\mathcal{A}(\tilde{\xi}, \lambda)$ with eigenvalue β if and only if l is a left eigenvector of $A^{(i\beta, \tilde{\xi})}$ with eigenvalue $i\lambda$. Thus, in this approach it is not necessary to invert A^1, provided one can compute the left eigenvectors and eigenvalues of A^{ξ}. For gas dynamics these are straightforward (but tedious) to compute explicitly for any ξ. Furthermore, the relevant left eigenvector l for a 1-shock, say, is the unique left eigenvector of $\mathcal{A}^{+}$ with stable eigenvalue β, $\mathrm{Re}\beta < 0$. Now, a calculation shows that

$$i\lambda = i\beta u_1 \pm c\sqrt{|\tilde{\xi}|^2 - \beta^2}, \tag{10}$$

(c denoting the sound speed), so that the eigenvector l is expressed in terms of β. Thus, to obtain Δ explicitly in terms of λ and $\tilde{\xi}$, we solve (10) for β, substitute into the expression for l and then calculate Δ. This approach works particularly well for isentropic gas dynamics, see [17] and [21]. The calculations for the full Euler equations are given below.

2 Calculation for full Euler equations

The Euler equations in d space dimensions takes the form (1) with $w = (\rho, m_1, \ldots, m_d, \mathcal{E})^{tr}$ as follows

$$\begin{aligned} \rho_t + \sum_{j=1}^{d} (m_j)_{x_j} &= 0 && && (11)\\ (m_i)_t + \sum_{j=1}^{d} \Big(\frac{m_i m_j}{\rho} + \delta_{ij} p\Big)_{x_j} &= 0 && i = 1, 2, \ldots, d && (12)\\ \mathcal{E}_t + \sum_{j=1}^{d} \Big(\frac{m_j \mathcal{E}}{\rho} + \frac{m_j p}{\rho}\Big)_{x_j} &= 0. && && (13) \end{aligned}$$

Here ρ, m_j, p, and $\mathcal{E}$ denote density, momentum in the jth coordinate direction, pressure, and total energy, respectively. We have $m_j = \rho u_j$, where u_j is velocity in the jth coordinate direction, p is a given function of the thermodynamic variables, and the total energy is given by $\mathcal{E} = \rho(e + (u_1^2 + \cdots + u_d^2)/2)$ where e denotes the internal energy, $de = TdS - pdv$, $v = 1/\rho$ is specific volume, S is specific entropy, and T is temperature.

2.1 Method 1

In our first calculation we use the relation (9) to calculate Δ. It is computationally more convenient to write the Euler equations in terms of the (nonconservative) variables $\bar{w} = (v, u_1, \ldots, u_d, S)^{tr}$:

$$v_t + \sum_{j=1}^{d} (v_{x_j} u_j - v u_{j,x_j}) = 0 \tag{14}$$

$$(u_i)_t + \sum_{j=1}^{d} u_j u_{i,x_j} + \frac{p_{x_i}}{\rho} = 0 \qquad i = 1, 2, \ldots, d \tag{15}$$

$$S_t + \sum_{j=1}^{d} u_j S_{x_j} = 0 \tag{16}$$

(For a calculation in the one dimensional case using (ρ, u, e) as dependent variables, see [18].) Letting $u^{tr} = (u_1, \cdots, u_d)$, the change of coordinates $\phi : \mathbf{R}^{1+d+1} \to \mathbf{R}^{1+d+1}$ is given by

$$\phi \begin{pmatrix} v \\ u \\ S \end{pmatrix} = \begin{pmatrix} v^{-1} \\ v^{-1} u \\ \mathcal{E} \end{pmatrix},$$

where

$$\mathcal{E} = v^{-1}(e(v, S) + |u|^2/2).$$

The matrix $C = D\phi$ is thus given as,

$$C = \begin{pmatrix} -v^{-2} & 0 & 0 \\ -v^{-2} u & v^{-1} I_d & 0 \\ -v^{-1}(\mathcal{E} + p) & v^{-1} u^{tr} & v^{-1} T \end{pmatrix},$$

with inverse

$$C^{-1} = \begin{pmatrix} -v^2 & 0 & 0 \\ -vu & vI_d & 0 \\ vT^{-1}(|u|^2 - v(\mathcal{E} + p)) & -vu^{tr} T^{-1} & vT^{-1} \end{pmatrix},$$

where I_d denotes the $d \times d$ identity matrix. Note that both C and C^{-1} above as well as the matrices M^j below have the block structure

$$\left(\begin{array}{c|c|c} 1 \times 1 & 1 \times d & 1 \times 1 \\ \hline d \times 1 & d \times d & d \times 1 \\ \hline 1 \times 1 & 1 \times d & 1 \times 1 \end{array} \right).$$

The matrices M^j are given by

$$M^j = \begin{pmatrix} u_j & -ve_j^{tr} & 0 \\ vp_v e_j & u_j I_d & vp_S e_j \\ 0 & 0 & u_j \end{pmatrix},$$

where e_j denotes the jth standard unit vector. The sound speed c is defined as

$$c := \sqrt{-v^2 p_v},$$

and we let

$$\mu := u_1^2 - c^2.$$

Taking advantage the simple structure of M^1 we can easily find its inverse,

$$(M^1)^{-1} = \begin{pmatrix} \frac{u_1}{\mu} & \frac{v}{\mu} e_1^{tr} & \frac{-v^2 p_S}{u_1 \mu} \\ \frac{c^2 v^{-1}}{\mu} e_1 & \frac{1}{u_1} I_d + \frac{c^2}{u_1 \mu} e_1 e_1^{tr} & \frac{-v p_S}{\mu} e_1 \\ 0 & 0 & \frac{1}{u_1} \end{pmatrix}.$$

To compute Δ as given by (9) we need the eigenvectors and eigenvalues of the matrix $\mathcal{B}$. To find these it is convenient to write M^j in the form

$$M^j = u_j I_{d+2} + \bar{M}^j,$$

where

$$\bar{M}^j = \begin{pmatrix} 0 & -v e_j^{tr} & 0 \\ v p_v e_j & 0 & v p_S e_j \\ 0 & 0 & 0 \end{pmatrix},$$

Then

$$M^{\tilde{\xi}} := \sum_{j=2}^{d} \xi_j M^j = (\tilde{u} \cdot \tilde{\xi}) I_{d+2} + \bar{M}^{\tilde{\xi}},$$

where the matrix $\bar{M}^{\tilde{\xi}}$ is given as

$$\bar{M}^{\tilde{\xi}} = \begin{pmatrix} 0 & 0 & -v \tilde{\xi}^{tr} & 0 \\ 0 & 0 & 0 & 0 \\ v p_v \tilde{\xi} & 0 & 0 & v p_S \tilde{\xi} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It follows that the matrix $\mathcal{B}$ is given as

$$\mathcal{B} = (\alpha I_{d+2} + i \bar{M}^{\tilde{\xi}})(M^1)^{-1},$$

where $\alpha = \lambda + i(\tilde{u} \cdot \tilde{\xi})$. We thus have

$$\mathcal{B} \;=\; \begin{pmatrix} \frac{\alpha u_1}{\mu} & \frac{\alpha v}{\mu} & -i \frac{v}{u_1} \tilde{\xi}^{tr} & \frac{-\alpha v^2 p_S}{u_1 \mu} \\ \frac{\alpha c^2 v^{-1}}{\mu} & \frac{\alpha u_1}{\mu} & 0 & \frac{-\alpha v p_S}{\mu} \\ i p_v \frac{v u_1}{\mu} \tilde{\xi} & i p_v \frac{v^2}{\mu} \tilde{\xi} & \frac{\alpha}{u_1} I_{d-1} & i p_S \frac{v u_1}{\mu} \tilde{\xi} \\ 0 & 0 & 0 & \frac{\alpha}{u_1} \end{pmatrix}$$

which has the block structure

$$\left(\begin{array}{c|c|c|c} 1\times 1 & 1\times 1 & 1\times(d-1) & 1\times 1 \\ \hline 1\times 1 & 1\times 1 & 1\times(d-1) & 1\times 1 \\ \hline (d-1)\times 1 & (d-1)\times 1 & (d-1)\times(d-1) & (d-1)\times 1 \\ \hline 1\times 1 & 1\times 1 & 1\times(d-1) & 1\times 1 \end{array}\right).$$

We proceed to find the eigenvalues of $\mathcal{B}$. The determinant $\det(\mathcal{B}-\beta I_{d+2})$ has the form

$$\det(\mathcal{B}-\beta I_{d+2}) = \begin{vmatrix} a_1 & a_2 & \delta\tilde{\xi}^{tr} & * \\ a_3 & a_1 & 0 & * \\ a_4\tilde{\xi} & a_5\tilde{\xi} & a_6 I_{d-1} & * \\ 0 & 0 & 0 & a_6 \end{vmatrix},$$

which is easily reduced to

$$\det(\mathcal{B}-\beta I_{d+2}) = a_6^{d-1}(a_6(a_1^2 - a_2 a_3) + \delta|\tilde{\xi}|^2(a_3 a_5 - a_1 a_4)).$$

Substituting the appropriate values for $a_1, \ldots, a_6$ and simplifying, one obtains

$$\det(\mathcal{B}-\beta I_{d+2}) = (\frac{\alpha}{u_1} - \beta)^d \left[\left(\frac{\alpha u_1}{\mu} - \beta\right)^2 + \frac{c^2}{\mu^2}(\mu|\tilde{\xi}|^2 - \alpha^2)\right],$$

which yields three different eigenvalues $\beta = \frac{\alpha}{u_1}$ (a $d-$fold eigenvalue), and

$$\beta_\pm = \frac{\alpha u_1}{\mu} \pm \frac{c}{\mu}\sqrt{\alpha^2 - \mu|\tilde{\xi}|^2}. \tag{17}$$

The complex square root in (17) is chosen to have its branch cut along the negative real axis. Recalling that $\alpha = \lambda + i(\tilde{u}\cdot\tilde{\xi})$ this ensures that β varies analytically as λ varies over the right complex half plane. The eigenvalues of interest for the stability analysis of (extreme) gas dynamical shocks are given by (17). We next find the eigenvectors $\bar{l}_\pm$ of $\mathcal{B}$ corresponding to $\beta\pm$. Letting $\bar{l}_\pm^{tr} = (l_{1\,\pm}, l_{2\,\pm}, \tilde{l}_\pm, l_{*\,\pm})$ with $l_{1\,\pm} = \alpha - u_1\beta_\pm$ and then multiplying on the right by C^{-1}, we obtain the desired left eigenvector $l_\pm$ of $\mathcal{A}_\pm$,

$$l_\pm = \begin{pmatrix} \alpha - u_1\beta_\pm \\ -v\beta_\pm \\ iv\tilde{\xi} \\ \frac{v^2 p_S}{\mu}\left(2\alpha + \frac{u_1^2|\tilde{\xi}|^2 - \alpha^2}{\alpha - u_1\beta_\pm}\right) \end{pmatrix}.$$

To compute (9), we now need only the jumps which are easily calculated from the Rankine-Hugoniot conditions.

For simplicity and concreteness we carry out the remaining calculations for a Lax 1-shock in dimension $d = 2$. Due to Galilean invariance we may without loss of generality assume that the transversal velocity is zero, i.e. $\tilde{u} = 0$. As explained in the preceding introductory discussion, β, the eigenvalue relevant for a 1-shock, is a stable eigenvalue of $\mathcal{A}_+$, i.e. β should have *negative* real part. By considering the one-dimensional case where $\xi_2 = 0$ and letting λ be a positive real number, it follows that we must choose the $+$ sign in (17).

In the following calculation of Δ all quantities without a subscript are evaluated on the right of (i.e. behind) the shock ($v = v_+$, $p = p_+$ etc.), and u will denote u_1^+. A $-$ subscript denotes evaluation on the left of (i.e. in front of) the shock. We note at this point that, as a consequence of the entropy inequalities for Lax shocks, the flow is supersonic in front of the shock and subsonic behind the shock.

In this setting the Rankine-Hugoniot conditions for a stationary gas dynamical shock take the form

$$[m_1] = 0 \tag{18}$$

$$\left[\frac{m_1^2}{\rho} + p\right] = 0 \tag{19}$$

$$\left[\frac{m_1 m_2}{\rho}\right] = 0 \tag{20}$$

$$\left[\frac{m_1(\mathcal{E} + p)}{\rho}\right] = 0, \tag{21}$$

so that

$$\lambda[w] + i[f^{\tilde{\xi}}(w)] = \begin{pmatrix} \lambda[\rho] \\ 0 \\ i\xi_2[p] \\ \lambda[\mathcal{E}] \end{pmatrix}. \tag{22}$$

Substituting into (9) we finally get the desired expression for the stability function Δ,

$$\Delta = v[\rho]\left\{\lambda v(\beta u - l_1) - v\xi_2^2\frac{[p]}{[\rho]} + \frac{\lambda}{T}l_*\left(\frac{[\mathcal{E}]}{[\rho]} + u^2 - v(\mathcal{E} + p)\right)\right\} \tag{23}$$

To simplify this expression we first observe that (21) implies

$$\left[\frac{\mathcal{E}}{\rho}\right] = -\left[\frac{p}{\rho}\right],$$

so that

$$[e] + \left[\frac{p}{\rho}\right] + \frac{m^2}{2}\left[\frac{1}{\rho^2}\right] = 0, \tag{24}$$

where $m = m_1$. From (19) it follows that

$$m^2 = -\frac{[p]}{[1/\rho]},$$

whence

$$\left[\frac{p}{\rho}\right] + \frac{m^2}{2}\left[\frac{1}{\rho^2}\right] = \frac{p_+ + p_-}{2}\left[\frac{1}{\rho}\right] = < p > \left[\frac{1}{\rho}\right].$$

Thus (24) takes the form

$$[e] + < p > \left[\frac{1}{\rho}\right] = 0,$$

and we obtain

$$\frac{[\mathcal{E}]}{[\rho]} + u^2 - v(\mathcal{E} + p) = \frac{m^2}{\rho}\left[\frac{1}{\rho}\right].$$

Before using this relation to simplify (23), we slightly change notation to emphasize the important physical quantities

$$M = \frac{|u|}{c} \text{ the Mach number}, \quad r = \frac{u}{u_-} = \frac{v}{v_-} = \frac{\rho_-}{\rho} \text{ the compression ratio.}$$

We let

$$\omega = \frac{\lambda}{u}, \quad \text{and introduce} \quad s(\omega) = \omega + \sqrt{\xi_2^2(1 - M^2) + M^2\omega^2},$$

where the square root has positive real part. With this notation we have

$$l_1 = \frac{u}{1 - M^2} s(\omega),$$

and

$$l_* = \frac{uv^2 p_S}{c^2(1 - M^2)}\left(2\frac{\lambda}{u} + \frac{(1 - M^2)(\xi_2^2 - \omega^2)}{s(\omega)}\right) = \frac{uv^2 p_S}{c^2(1 - M^2)} s(\omega),$$

where we have used that $(1 - M^2)\xi_2^2 = (s(\omega) - \omega)^2 - \omega^2 M^2$. Substituting into the expression for Δ, including the compression ratio, and using Rankine-Hugoniot conditions to simplify the resulting expression we find

$$\Delta(\xi_2, \omega) = -\frac{\lambda v^2 u[\rho]}{1 - M^2}\left\{rk\omega s(\omega) + (1 - M^2)(\xi_2^2 - r\omega^2)\right\}, \tag{25}$$

where

$$k := 2 - \frac{p_S}{T}(v_- - v)M^2. \tag{26}$$

We summarize our findings:

Theorem 2.1 *Consider the Euler equations (11)-(13) in two space dimensions. The Lopatinski determinant (5) for a 1-shock is given by (25)-(26), where all unmarked quantities are evaluated to the right of the shock.*

2.2 Method 2

In this section we give an alternative computation of Δ in (6) for a gas dynamical shock of the first family. As explained above the key observation is that l is a left eigenvector of $\mathcal{A}(\tilde{\xi}, \lambda)$ with eigenvalue β if and only if l is a left eigenvector of $A^{(i\beta,\tilde{\xi})}$ with eigenvalue $i\lambda$. We begin by outlining how to effectively compute the eigenvalues and left eigenvectors of A^{ξ}, for any ξ. We then use the result to express Δ as given by (6).

Writing the Euler equations (11)-(13) in quasilinear form (7) we obtain that

$$A^{\xi} = \begin{pmatrix} 0 & \xi^{tr} & 0 \\ \begin{array}{c} -(u\cdot\xi)u \\ +\xi(P_\rho + P_e e_\rho) \end{array} & \begin{array}{c} (u\cdot\xi)I_d \\ +u\xi^{tr} - \frac{P_e}{\rho}\xi u^{tr} \end{array} & \frac{P_e}{\rho}\xi \\ (u\cdot\xi)(-E - \frac{P}{\rho} + P_\rho + P_e e_\rho) & \begin{array}{c} -(u\cdot\xi)\frac{P_e}{\rho}u^{tr} \\ +\left(E + \frac{P}{\rho}\right)\xi^{tr} \end{array} & u\cdot\xi\left(1 + \frac{P_e}{\rho}\right) \end{pmatrix}.$$

Here the pressure P is given as a function of ρ and e, i.e. P is related to p above by

$$p(v, S) = P(v^{-1}, e(v, S)).$$

In computing the eigenvalues eigenvectors of A^{ξ} it is convenient to conjugate with the "shift" matrix S given by

$$S = \begin{pmatrix} 1 & 0 & 0 \\ u & I_d & 0 \\ \frac{|u|^2}{2} & u^{tr} & 0 \end{pmatrix},$$

whose inverse is

$$S^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -u & I_d & 0 \\ \frac{|u|^2}{2} & -u^{tr} & 0 \end{pmatrix}.$$

This yields

$$S^{-1}A^\xi S = (u \cdot \xi) I_{d+2} + \bar{A}^\xi,$$

where

$$\bar{A}^\xi = \begin{pmatrix} 0 & \xi^{tr} & 0 \\ \left(P_\rho - \frac{P_e}{\rho} e\right) & 0 & \frac{P_e}{\rho}\xi \\ 0 & \left(e + \frac{P}{\rho}\right) & 0 \end{pmatrix}.$$

We denote the eigenvalues of $\bar{A}^\xi$ by $\bar{a}_j$, the right eigenvectors by $\bar{r}_j$ and the left eigenvectors by $\bar{l}_j$. If a_j, r_j, and l_j denote the corresponding quantities for A^ξ, then

$$a_j = u \cdot \xi + \bar{a}_j, \qquad r_j = S\bar{r}_j, \qquad l_j = \bar{l}_j S^{-1}.$$

Now the eigenvalues and eigenvectors of $\bar{A}^\xi$ are straightforward to compute and we get that the eigenvalues of A^ξ are,

$$a_1 = u \cdot \xi - c|\xi|, \tag{27}$$

$$a_j = u \cdot \xi, \qquad j = 1, \ldots, d+1, \tag{28}$$

$$a_{d+2} = u \cdot \xi + c|\xi|, \tag{29}$$

where $c = P_\rho + \frac{PP_e}{\rho^2}$ is the sound speed. To write down the eigenvectors we introduce the coefficients

$$\theta = P_\rho - \frac{P_e}{\rho} e, \qquad \eta = \frac{P_e}{\rho}, \qquad \zeta = e + \frac{P}{\rho}.$$

The left eigenvectors are found to be

$$l_1 = \left(\theta + \frac{c(u \cdot \xi)}{|\xi|} + \frac{\eta |u|^2}{2}, -\frac{c\xi}{|\xi|} - \eta u, \eta\right)^{tr} \tag{30}$$

$$l_2 = \left(-\zeta + \frac{|u|^2}{2}, -u, 1\right)^{tr} \tag{31}$$

$$l_j = \left(-u \cdot \xi^\perp, \xi^\perp, 0\right)^{tr}, \qquad j = 3, \ldots, d+1 \tag{32}$$

$$l_{d+2} = \left(\theta - \frac{c(u \cdot \xi)}{|\xi|} + \frac{\eta |u|^2}{2}, \frac{c\xi}{|\xi|} - \eta u, \eta\right)^{tr}, \tag{33}$$

where $\xi^\perp$ is orthogonal to ξ. We note here that the right eigenvectors r_j are also easily calculated from $\bar{r}_j$, the corresponding right eigenvectors of $\bar{A}^\xi$. As we have no use for these eigenvectors in what follows, we omit them.

We next have to decide which eigenvector to use in the computation of Δ. From the discussion above we know that the left eigenvector of $\mathcal{A}(\tilde{\xi}, \lambda)$ to be used in (6) is the same as the left eigenvector of $A^{(i\beta,\tilde{\xi})}$ with eigenvalue $i\lambda$, with $\mathrm{Re}\beta < 0$ and $\mathrm{Re}\lambda > 0$. Thus $i\lambda$ must be equal to either $u \cdot \xi \pm c|\xi|$ or $u \cdot \xi$, where $\xi = (i\beta, \tilde{\xi})$. In the special case $\tilde{u} = 0$ the last choice yields $\lambda = \beta u_1$. Since $u_1 > 0$ for a 1-shock while $\mathrm{Re}\beta < 0$ and $\mathrm{Re}\lambda > 0$, this is not the right choice. It remains to decide which sign and which branch of the square root is to be used in the relation

$$i\lambda = i\beta u_1 \pm c\sqrt{|\tilde{\xi}|^2 - \beta^2}. \tag{34}$$

To do so we use the fact that (34) must give λ as a continuous function of β as β varies over the left complex half plane and $\tilde{\xi}$ varies over $\mathbf{R}^{d-1}$. In particular, for $\tilde{\xi} = 0$, $\sqrt{-\beta^2}$ must be continuous in $\{\mathrm{Re}\beta < 0\}$. It follows that the square root in (34) is the branch with positive imaginary part. For $\tilde{\xi} = 0$ we thus have $\lambda = \beta u_1 \pm c|\beta|$. Taking real parts and using the fact that $c > u$, $\mathrm{Re}\lambda > 0$ and $\mathrm{Re}\beta < 0$, we conclude that the correct sign in (34) is the $+$, i.e. we must use l_{d+2} (and not l_1) in the computation of Δ for shock of the first family.

For the actual computation of Δ we restrict ourselves as before to the two-dimensional case and we assume without loss of generality that $\tilde{u} = 0$. From the preceding discussion and the expression (33) thus have that the left eigenvector in (6) is given as

$$l = \left(\theta - \frac{ic\beta u}{\sqrt{\xi_2^2 - \beta^2}} + \frac{\eta u^2}{2}, \frac{ic\beta}{\sqrt{\xi_2^2 - \beta^2}} - \eta u, \frac{c\xi_2}{\sqrt{\xi_2^2 - \beta^2}}, \eta \right)^{tr},$$

where $u = u_1$, and β with $\mathrm{Re}\beta < 0$ is related to λ through

$$c\sqrt{\xi_2^2 - \beta^2} = i(\lambda - \beta u). \tag{35}$$

Taking the inner product with $\lambda[w] + i[f^{\tilde{\xi}}(w)]$ as given in (22), and using (35), we see that Δ is given by

$$\Delta(\xi_2, \lambda) = \lambda[\rho] \left(\theta - \frac{ic\beta u}{\sqrt{\xi_2^2 - \beta^2}} + \frac{\eta u^2}{2} \right) + \frac{c\xi_2^2}{\sqrt{\xi_2^2 - \beta^2}}[P] + \eta\lambda[\mathcal{E}]$$

$$= \frac{i}{c\sqrt{\xi_2^2 - \beta^2}} \left\{ \lambda(\lambda - \beta u) \left([\rho](\theta + \frac{\eta u^2}{2}) + \eta[\mathcal{E}] \right) + c^2\xi_2^2[P] - \lambda[\rho]c^2\beta u \right\},$$

where all unmarked quantities are evaluated on the right of the shock. Making use of the identities that follow from the Rankine-Hugoniot relation in the same way as above, we see that

$$[\rho](\theta + \frac{\eta u^2}{2}) + \eta[\mathcal{E}] = [\rho]\left(c^2 + \eta u^2(1 - \frac{1}{r})\right),$$

r being the compression ratio as defined above. Also, $\eta = P_e/\rho = vp_S/T$, and $[P]/[\rho] = uu_-$ so that the expression for Δ reduces to

$$\begin{aligned}\Delta(\xi_2, \lambda) &= \frac{ic[\rho]}{\sqrt{\xi_2^2 - \beta^2}}\left\{(\lambda^2 - \beta u\lambda)(k-1) - \beta\lambda u + \xi_2^2\frac{u^2}{r}\right\} \\ &= \frac{icu^2[\rho]}{\sqrt{\xi_2^2 - \beta^2}}\left\{(k-1)\omega^2 - k\beta\omega + \xi_2^2\frac{1}{r}\right\},\end{aligned}$$

where $\omega = \lambda/u$ and k is given by (26) above. Now, solving (35) for β shows that

$$\beta = \frac{-u - \sqrt{\lambda^2 + \xi_2^2(c^2 - u^2)}}{c^2 - u^2} = -\frac{M^2\omega + \sqrt{\xi_2^2(1 - M^2) + M^2\omega^2}}{1 - M^2}. \quad (36)$$

The branch and sign of the square root are determined as in (17), i.e. the square root in (36) is the standard one with positive real part. Using the notation from above we have that

$$\beta = \omega - \frac{s(\omega)}{1 - M^2}.$$

Substituting back into the expression for Δ and simplifying finally gives

$$\Delta(\xi_2, \lambda) = \frac{icu^2[\rho]}{r(1 - M^2)\sqrt{\xi_2^2 - \beta^2}}\left\{rk\omega s(\omega) + (1 - M^2)(\xi_2^2 - r\omega^2)\right\},$$

which agrees with the expression in (25) up to a multiplicative factor, which is due to a different normalization of the left eigenvector l in (6).

3 The Stability Conditions

We now derive conditions under which a planar 1-shock is uniformly stable, weakly stable, or strongly unstable. According to Definition 1 this amounts to determining the sign of $\mathrm{Re}\lambda$ whenever $\Delta(\xi_2, \lambda) = 0$.

Here we exploit the analyticity of Δ with respect to λ, and as in [4], we apply the argument principle to count the number of roots of Δ in the right complex half-plane, see [10]. Without loss of generality we may set $\xi_2 = 1$ by the homogeneity of Δ. Also we introduce polar coordinates (R, θ) and define the function

$$G(R,\theta) = G(\omega) := -\frac{1-M^2}{\lambda v^2 u[\rho]}\Delta(1,\omega), \qquad \omega = Re^{i\theta}.$$

Thus the zeros of Δ and G are the same. We consider $G(\omega)$ as ω varies counterclockwise along the closed contour C consisting of a semicircle together with a vertical segment joining the ends, see Figure 1. Special care must be taken during the mapping of the imaginary axis to avoid the branch cuts arising from the square root in $s(\omega)$. To consider the change in argument

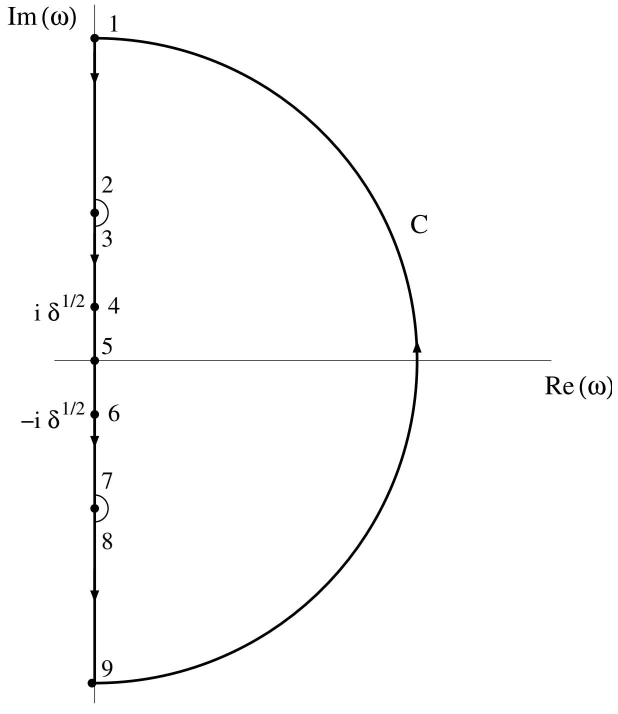

Figure 1: The path C in the ω plane

of G along the circular portion of the contour C we consider the limiting behavior as $R \to +\infty$. For large R, we find

$$G \approx (1+M)r(k-1+M)R^2 e^{i2\theta}. \tag{37}$$

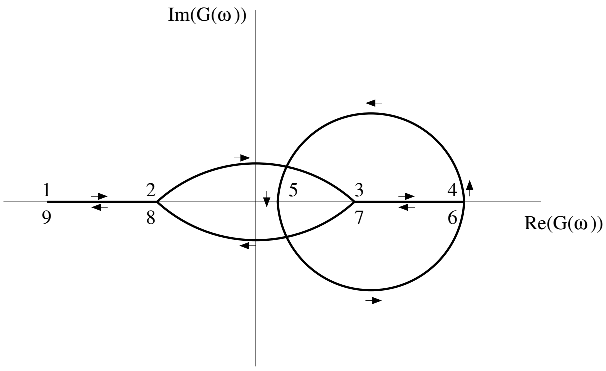

Figure 2: Case (Ia), the change in $Arg(G(\omega))$ is -2π.

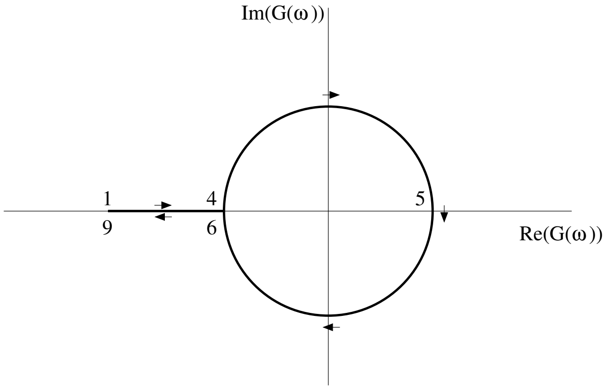

Figure 3: Case (Ib), the change in $Arg(G(\omega))$ is -2π.

It is then straightforward to see that the change in argument of G is 2π on this part of the contour.

Next consider the part of C on the imaginary axis where $\theta = \pm\pi/2$. Substitution in (25) yields

$$\begin{aligned} G(R, \pm\frac{\pi}{2}) &= (1 - M^2) + rR^2(1 - M^2 - k) \\ &\quad + rkMR \begin{cases} -\sqrt{R^2 - \delta} & \text{for } R^2 > \delta \\ \pm i\sqrt{\delta - R^2} & \text{for } R^2 < \delta, \end{cases} \end{aligned}$$

where we have put $\delta = (1 - M^2)/M^2 > 0$. We observe from this that G restricted to the imaginary axis has nonzero imaginary part only on that interval of the axis lying between $-\sqrt{\delta}$ and $+\sqrt{\delta}$. Thus any zero of G on the imaginary axis must be outside this interval. We also note that $G(0, 0) = 1 - M^2$ is real and positive, but for large R it's clear from (37)

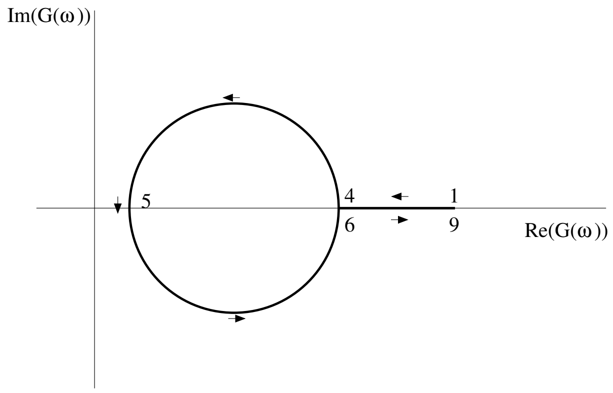

Figure 4: Case (II), the change in $Arg(G(\omega))$ is 0.

that the sign of the real part of $G(R, \pm\frac{\pi}{2})$ depends on the sign of the quantity $k - 1 + M$. There are thus two cases to consider:

(I) $1 - M - k < 0$ (in which case $\mathrm{Re}G(R, \pm\frac{\pi}{2}) < 0$ for large R)

(II) $1 - M - k > 0$ (in which case $\mathrm{Re}G(R, \pm\frac{\pi}{2}) > 0$ for large R).

Now, a straightforward computation shows that in case (I),

$$\frac{\partial G(R, \pm\frac{\pi}{2})}{\partial R} < 0 \qquad \text{for } R^2 > \delta.$$

Consequently G in case (I) has a single root on each half of the imaginary axis if and only if

$$G(\sqrt{\delta}, \pm\frac{\pi}{2}) = (1 - M^2) + (1 - M^2 - k)r\frac{1 - M^2}{M^2} > 0. \tag{38}$$

We thus further divide case (I) into two subcases depending on whether or not (38) holds. We label them as (Ia) if (38) is valid and (Ib) otherwise; Thus in subcase (Ib) G has no zeros on the imaginary axis. Also it is easily seen that

$$G(R, \pm\frac{\pi}{2}) > 0 \quad \text{for all } R > \sqrt{\delta}$$

in case (II), whence G has no root on the imaginary axis in that case either.

Finally, a careful analysis of G as ω moves down the vertical part of C shows that its argument changes by -2π in case (I), while it is unchanged in case (II). The mapping in case (Ia) is shown in Figure 2. We note that in this case the cut-outs between points 2 and 3 and points 7 and 8 in Figure 1 must be followed to avoid the roots of G on the imaginary axis. The change in the argument for the mapping of the imaginary axis is thus -2π and in combination with the 2π change in argument along the circular portion of the contour, we find that in case (Ia), there are no roots of G, and hence of Δ, for $\mathrm{Re}\lambda > 0$. By Definition 1 the pure imaginary roots indicate that the shock is *weakly stable* in this case. Combining the condition defining case (I) and (38), we find that a shock is weakly stable if and only if

$$1 - M < k < 1 - M^2 + M^2/r.$$

The mapping in case (Ib) is simpler and is shown in Figure 3. In this case there are no pure imaginary roots to contend with and the change in argument as the imaginary axis is mapped is again -2π. In combination with the 2π change in argument along the circular portion of the contour, it follows then that there are no roots of G, and again therefore of Δ, for $\mathrm{Re}\lambda > 0$. It follows then from Definition 1, that the shock is *uniformly stable*. In case (Ib) the inequality (38) is assumed not to hold in which case it follows that the shock under consideration is uniformly stable if and only if

$$k > 1 - M^2 + M^2/2.$$

Finally we examine the mapping of the imaginary axis in case (II). The mapping in this case is shown in Figure 4. There is no change in the argument from the mapping of the imaginary axis. It follows then from the 2π change in argument due to the mapping of the circular portion of the contour that there is a zero with positive real part and thus that the shock is *strongly unstable*. The condition for strong instability is thus the condition that characterizes case (II). A shock is strongly unstable if and only if

$$k < 1 - M.$$

Summing up the discussion above we have the following result.

Theorem 3.1 *With the same assumptions as in Theorem 2.1 a shock of the first family is*

- *strongly unstable if and only if* $k < 1 - M$*,*
- *weakly (but not uniformly) stable if and only if* $1 - M < k < 1 - M^2 + M^2/r$*,*
- *uniformly stable if and only if* $k > 1 - M^2 + M^2/r$*,*

where $r = \frac{u}{u_-}$ is the compression ratio, k is given by (26), and $M = \frac{|u|}{c}$ is the Mach number, all of which are evaluated on the right of the 1-shock.

We point out that these results were (apart from the terminology) first proved by Erpenbeck [4], and later derived in the more comprehensive work of Majda [11, 12]. Serre [18] has shown (under certain conditions) that the transition from weak stability to strong instability in several space dimensions is signaled by instability with respect to one dimensional perturbations.

3.1 The Case of an Ideal Gas

We end by considering the special case of an ideal, polytropic gas; so the equation of state is given by

$$p = R\rho T,$$

and $e = c_v T$, and the pressure can also be written in terms of density and entropy as

$$p = K_1 \rho^\gamma \exp(S/c_v),$$

where c_v is the specific heat at constant volume, K_1 and R are positive constants, and $\gamma > 1$. The ratio $\frac{R}{c_v}$ is called the Gruneisen coefficient, and it will be denoted by Γ. Using the fact that $1 + \gamma = \Gamma$, the condition for uniform stability can be rewritten as

$$(\frac{1}{r} - 1)M^2 < \frac{1}{\Gamma + 1} = \frac{1}{\gamma}.$$

Then a straightforward manipulation of the Rankine-Hugoniot conditions yields the relationship

$$(\frac{1}{r} - 1)M^2 = (\frac{1}{\gamma})(1 - \frac{p}{p_-}).$$

Then combining the two expressions above, we see that for an ideal gas the uniform stability condition is

$$(\frac{1}{\gamma})(1 - \frac{p}{p_-}) < \frac{1}{\gamma}.$$

For a 1-shock $p > p_-$ by a standard calculation; see [19]. It follows then that the uniform stability condition is satisfied, and all ideal gas shocks are uniformly stable.

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References

[1] A. A. Barmin, S. A. Egorushkin, Stability of shock waves, *Adv. Mech.* **15** (1992), 3–37.

[2] H. A. Bethe, On the theory of shock waves for an arbitrary equation of state, Report no. 545 for the Office of Scientific Research and Development, Serial no. NDRC-B-237, (1942), in J. N. Johnson and R. Chéret (eds.), "Classic Papers in Shock Compression Science", Springer-Verlag, New York, 1998.

[3] S. P. D'yakov, On the stability of shock waves (Russian)*v Z. Eksper. Teoret. Fiz.* **27**, (1954) 288–295 (translation: Air Force Office of Scientific Research Report AFOSR-TN-56-406 (1956)).

[4] J. Erpenbeck, Stability of step shocks, *Physics of Fluids* **5** (1962), 1181–1187.

[5] N. C. Freeman, A theory of the stability of plane shock waves, *Proc. Roy. Soc. London. Ser. A.* **228** (1955), 341–362.

[6] N. C. Freeman, On the stability of plane shock waves, *J. Fluid Mech.* **2** (1957), 397–411.

[7] C. S. Gardner, Comment on "Stability of step shocks", *Physics of Fluids* **6** (1962), 1366–1367.

[8] V. M. Kontorovich, Stability of shock waves in relativistic hydrodynamics, *Soviet Physics JETP* **34** (1958) 127–132.

[9] V. M. Kontorovich, On the interaction between small disturbances and discontinuities in magnetohydrodynamics and on the stability of shock waves, *Soviet Physics. JETP* **35** (1959) 851–858 (1216–1225 Z. Eksper. Teoret. Fiz.).

[10] S. Lang Complex analysis, Springer-Verlag, New York, 1999.

[11] A. Majda,The stability of multidimensional shock fronts, *Memoirs of the AMS* **41** (1983) 1-95.

[12] A. Majda, Compressible Fluid Flows and Systems of Conservation Laws, Springer-Verlag, New York, 1983.

[13] G. Métivier, Stability of multi-dimensional weak shocks, *Comm. Partial Differential Equations* **15** (1990), 983–1028.

[14] G. Métivier, Stability of multidimensional shocks in "Advances in the theory of shock waves", 25–103, Progr. Nonlinear Differential Equations Appl., 47, Birkhäuser Boston, Boston, MA, 2001.

[15] R. Richtmyer, Taylor instability in shock acceleration of compressible fluids, *Comm. Pure Appl. Math.* **13** (1960) 297–319.

[16] A. E. Roberts, Stability of steady plane shock, Los Alamos Scientific Laboratory Report LA-299 (1945).

[17] D. Serre, Systems of conservation laws. 2. Geometric structures, oscillations, and initial-boundary value problems, Cambridge University Press, Cambridge, 2000.

[18] D. Serre, La transition vers l'instabilité pour les ondes de choc multi-dimensionnelles (French) [Transition to instability for multidimensional shock waves], *Trans. Amer. Math. Soc.* **353** (2001), 5071–5093.

[19] J. Smoller, Shock Waves and Reaction-Diffusion Equations, Springer-Verlag, New York, 1994.

[20] G. W. Swan, G. R. Fowles, Shock wave stability, *Physics of Fluids* **18** (1975), 28–35.

[21] K. Zumbrun, Multi-dimensional Stability for Isentropic Euler Equations, lecture notes, 1999.

[22] K. Zumbrun, D. Serre, Viscous and inviscid stability of multidimensional planar shock fronts, *Indiana Univ. Math. J.* **48** (1999), 937–992.

[23] K. Zumbrun, Multidimensional stability of planar shock waves, in "Advances in the theory of shock waves", 307–516, Progr. Nonlinear Differential Equations Appl., 47, Birkhäuser Boston, Boston, MA, 2001.