NONLINEAR STABILITY OF SULICIU MODEL FOR DYNAMIC PHASE TRANSITIONS*

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Abstract. We study the nonlinear stability of subsonic phase boundary solutions in the Suliciu model for dynamic phase transitions under tri-linear structural relation. This is established through energy estimates with the help of Laplace transform to resolve the boundary terms. Furthermore, nonlinear stability is obtained for the relaxed system.

Key words. dynamic phase transition, nonlinear stability, Laplace transform

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1. Introduction. Phase transition is a challenging topic of great importance, arising widely in sciences and engineering. For instance, in a viscoelasticity system

$$\begin{aligned} u_t + v_x &= 0, \\ v_t + \sigma(u)_x &= 0, \end{aligned} \tag{1.1}$$

with u, v the strain and velocity, instability occurs when the stress tensor $\sigma(u)$ becomes non-monotone. Similarly, the dynamics of a van der Waals fluid is governed by

$$\begin{aligned} v_t - u_x &= 0, \\ u_t + p(v)_x &= 0, \end{aligned} \tag{1.2}$$

in Lagrangian coordinates, with v and u the specific volume and velocity, respectively. At a subcritical temperature, the pressure $p(v)$ is non-monotone, leading to instability of ellipticity [8]. A closely related difficulty is the non-uniqueness. Given a specific structural relation or pressure law, multiple solutions may appear in general. It was therefore proposed that the material should further supply a kinetic relation for subsonic phase boundaries, and a nucleation criterion [1]. While materials behave differently due to different micro-structures, a big variety of kinetic relations and nucleation criteria are expected. A fundamental concern then is to clarify which kind of kinetic relations and nucleation criteria may yield stable solutions, in particular, stable subsonic phase boundaries.

During the past two decades, quite some efforts have been put on resolving the instability and nonuniqueness. There are mainly two type of approaches. One is to add dissipations, high order or low order [13, 12]. The other is to specify a Riemann solver with a generalized entropy condition [7, 5]. For the latter approach, there are some interesting results on stability, typically applied to phase transitions near the Maxwellian states of equal area law [3, 6]. Yet not many generalized entropy conditions have been constructed. On the other hand, the dissipation approach, particularly the low order discrete BGK approximations, suits better the physical intuitions, and has a great potential in providing kinetic relations and nucleation criteria that may be physically relevant [12]. However, the nonlinear stability is even more involved. Most energy estimate techniques fail, due to the 'wrong' sign in the stress gradient $\sigma'(u)$ [11].

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While linear stability has been investigated in the literature, the nonlinear stability results are quite limited [2][4].

In this paper, we shall discuss the Suliciu model [14], which is a special case of discrete BGK approximations. More precisely, we study a system that approximates (1.1)

$$
\begin{aligned}
u_t + v_x &= 0,\\
v_t + w_x &= 0,\\
w_t + \lambda^2 v_x &= \frac{\sigma(u) - w}{\varepsilon}.
\end{aligned}
\tag{1.3}
$$

Under a rescaling

$$(t, x; u, v, w) \Longrightarrow (\varepsilon t, \varepsilon \lambda x; u, \lambda v, \lambda^2 w),$$

it becomes

$$
\begin{aligned}
u_t + v_x &= 0,\\
v_t + w_x &= 0,\\
w_t + v_x &= \frac{\sigma(u)}{\lambda^2} - w.
\end{aligned}
\tag{1.4}
$$

We assume a standard tri-linear structural relation

$$
\frac{\sigma(u)}{\lambda^2} = \begin{cases} \alpha_1 u, & \text{for} \quad u < u_1,\\ \dfrac{\alpha_2 u_2 - \alpha_1 u_1}{u_2 - u_1} u + \dfrac{\alpha_2 + \alpha_1}{u_2 - u_1} u_1 u_2, & \text{for} \quad u_1 < u < u_2,\\ \alpha_2 u, & \text{for} \quad u > u_2, \end{cases}
\tag{1.5}
$$

with $\alpha_1 > \alpha_2 > 0$. Moreover, λ is selected big enough to keep linear stability in the stable phases, i.e. a *subcharacteristic condition* holds [10]

$$\alpha_1 < 1, \quad \alpha_2 < 1. \tag{1.6}$$

It is known that a subsonic phase boundary must be stationary [12], namely, up to a translation,

$$
(u_s, v_s, w_s)(x,t) = \begin{cases} (u_+, \bar{v}, \bar{w}), & \text{for} \quad x < 0,\\ (u_-, \bar{v}, \bar{w}), & \text{for} \quad x > 0. \end{cases}
\tag{1.7}
$$

Here $u_+, u_-, \bar{v}$ and $\bar{w}$ are constant, with $\sigma(u_+) = \sigma(u_-) = \bar{w}$ for u_+ and u_- in different stable phases, i.e. either $u_+ < u_1 < u_2 < u_-$, or $u_- < u_1 < u_2 < u_+$.

We perturb the phase boundary by a continuous function $(u_0(x), v_0(x), w_0(x))$. The resulting deviation $(\widetilde{u}, \widetilde{v}, \widetilde{w})$ evolves according to

$$
\begin{aligned}
\widetilde{u}_t + \widetilde{v}_x &= 0,\\
\widetilde{v}_t + \widetilde{w}_x &= 0,\\
\widetilde{w}_t + \widetilde{v}_x &= \alpha \widetilde{u} - \widetilde{w},
\end{aligned}
\tag{1.8}
$$

provided that $\widetilde{u}(x,t)$ keeps of moderate amplitude to maintain the system in the stable phases. Here $\alpha = \alpha_\pm$ takes the value of α_1 at one side of $x = 0$, and α_2 at the other. We shall prove that the deviation keeps to be small under certain restrictions on initial perturbation.

In the following sections, we shall establish nonlinear stability through energy estimates. For resolving the boundary terms, we apply Laplace transform to the deviation equations (1.8) and make necessary estimates. Finally we describe the nonlinear stability of the relaxed scheme, i.e. $\varepsilon = 0$.

2. Two functionals. The Riemann variables

$$\begin{aligned} p &= \widetilde{w}+\widetilde{v}, \\ q &= \widetilde{w}-\widetilde{v}, \\ r &= \widetilde{w}-\widetilde{u}, \end{aligned} \tag{2.1}$$

solve a diagonal system

$$\begin{aligned} p_t+p_x &= -\frac{1-\alpha}{2}(p+q)-\alpha r, \\ q_t-q_x &= -\frac{1-\alpha}{2}(p+q)-\alpha r, \\ r_t &= -\frac{1-\alpha}{2}(p+q)-\alpha r. \end{aligned} \tag{2.2}$$

There exists a Lyapunov functional

$$\mathcal{E}(t)=\int_{I\!R} p^2+q^2+\frac{2\alpha}{1-\alpha}r^2\ \mathrm{d}x. \tag{2.3}$$

Across the stationary phase boundary $x=0$, the Rankine-Hugoniot condition for (1.4) yields

$$[\![v]\!]=0, \quad [\![w]\!]=0. \tag{2.4}$$

Here $[\![\eta]\!]=\eta(0+)-\eta(0-)$ denotes the jump across $x=0$. From (1.7), this means that $\widetilde{w}$ and $\widetilde{v}$, therefore p and q, are continuous across the phase boundary. Noticing that the discontinuous term α appears only in the coefficient of r, whose evolution is independent of its x-derivative, we may integrate by parts to get

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}=-\int_{I\!R}(1-\alpha)\left(p+q+\frac{2\alpha r}{1-\alpha}\right)^2\ \mathrm{d}x\le 0. \tag{2.5}$$

Due to the linearity, the derivatives $(P,Q,R)=(p_x,q_x,q_x)$ solve, in $I\!R^-\cup I\!R^+$,

$$\begin{aligned} P_t+P_x &= -\frac{1-\alpha}{2}(P+Q)-\alpha R, \\ Q_t-Q_x &= -\frac{1-\alpha}{2}(P+Q)-\alpha R, \\ R_t &= -\frac{1-\alpha}{2}(P+Q)-\alpha R. \end{aligned} \tag{2.6}$$

We may define a similar functional

$$\mathcal{E}_1(t)=\int_{I\!R} P^2+Q^2+\frac{2\alpha}{1-\alpha}R^2\ \mathrm{d}x. \tag{2.7}$$

In general, P and Q are discontinuous across the phase boundary $x=0$. As $\widetilde{w}$ and $\widetilde{v}$ are continuous, the last two equations in (1.8) imply

$$[\![\widetilde{w}_x]\!]=0, \quad [\![\widetilde{v}_x]\!]=[\![\alpha\widetilde{u}]\!]. \tag{2.8}$$

A direct computation shows

$$\frac{\mathrm{d}\mathcal{E}_1}{\mathrm{d}t}=-\int_{I\!R}(1-\alpha)\left(P+Q+\frac{2\alpha R}{1-\alpha}\right)^2\ \mathrm{d}x+\left[\!\!\left[P^2-Q^2\right]\!\!\right]\le 4\widetilde{w}_x(0,t)[\![\alpha\widetilde{u}]\!]. \tag{2.9}$$

In the next two sections, we shall find the expressions of the Laplace transforms for $\widetilde{w}_x(0,t)$ and $[\![\alpha\widetilde{u}]\!]$, and make estimates.

3. Exact solution via Laplace transform. The Laplace transform of $\widetilde{u}(x,t)$ is defined by

$$U(x,s) = \int_0^{+\infty} \widetilde{u}(x,t) e^{-st} \, \mathrm{d}t,$$

and similar for other variables. The time derivative u_t is transformed to $-u_0(x) + sU$. Denoting $' = \dfrac{\partial}{\partial x}$, we have ordinary differential equations on (U, V, W) for $x < 0$ and $x > 0$ respectively,

$$\begin{aligned} -u_0 + sU + V' &= 0, \\ -v_0 + sV + W' &= 0, \\ -w_0 + sW + V' &= \alpha_\pm U - W. \end{aligned} \tag{3.1}$$

The Rankine-Hugoniot relation at the phase boundary requires V and W to be continuous across $x = 0$. Moreover, we shall consider initial perturbations with $\mathcal{E}(0)$ bounded, therefore $\mathcal{E}(t)$ bounded. The conditions stated later will keep $\mathcal{E}_1(t)$ bounded as well. These together ensure

$$\lim_{x\to\pm\infty} p = \lim_{x\to\pm\infty} q = \lim_{x\to\pm\infty} r = 0.$$

Accordingly, at given $s > 0$,

$$\lim_{x\to\pm\infty} U = \lim_{x\to\pm\infty} V = \lim_{x\to\pm\infty} W = 0.$$

Reformulation of (3.1) gives

$$\begin{aligned} U &= \frac{s+1}{s+\alpha_\pm} W + \frac{u_0 - w_0}{s+\alpha_\pm}, \\ W'' - \frac{s^2(s+1)}{s+\alpha_\pm} W &= v_0' - \frac{s^2 w_0 + s\alpha_\pm u_0}{s+\alpha_\pm}. \end{aligned} \tag{3.2}$$

We deduce from the first equation

$$[\![\alpha U]\!] = \frac{s[\![\alpha]\!]}{(s+\alpha_+)(s+\alpha_-)} \left[(s+1)W(0,s) + u_0(0) - w_0(0)\right]. \tag{3.3}$$

Now let

$$a_\pm = \sqrt{\frac{s+1}{s+\alpha_\pm}}, \qquad f(x,s) = \frac{v_0'}{s} - \frac{sw_0 + \alpha_\pm u_0}{s+\alpha_\pm} = \frac{v_0'}{s} - \frac{a_\pm^2 (sw_0 + \alpha_\pm u_0)}{s+1}.$$

First, for $x \geq 0$, the general solution of the homogeneous equation takes the form of $C_1 e^{-sa_+x} + \widetilde{C_1} e^{sa_+x}$. Together with a specific solution, we know

$$W = C_1 e^{-sa_+x} + \widetilde{C_1} e^{sa_+x} - \frac{1}{2a_+} \int_0^{+\infty} e^{-sa_+|x-y|} f(y,s) \, \mathrm{d}y.$$

To make W bounded at $x \to +\infty$, $\widetilde{C_1}$ has to be zero. Then

$$W' = -sa_+ C_1 e^{-sa_+x} + \frac{s}{2} \left[\int_0^x e^{-sa_+(x-y)} f(y,s) \, \mathrm{d}y - \int_x^{+\infty} e^{-sa_+(y-x)} f(y,s) \, \mathrm{d}y \right].$$

Similarly, we have for $x < 0$,

$$\begin{aligned} W &= C_2 e^{sa_- x} - \frac{1}{2a_-} \int_{-\infty}^{0} e^{-sa_-|x-y|} f(y,s)\, \mathrm{d}y, \\ W' &= sa_- C_2 e^{sa_- x} + \frac{s}{2} \left[\int_{-\infty}^{x} e^{-sa_-(x-y)} f(y,s)\, \mathrm{d}y - \int_{x}^{0} e^{-sa_-(y-x)} f(y,s)\, \mathrm{d}y \right]. \end{aligned}$$

The continuity of V across $x = 0$ is equivalent to the continuity in W'. The matching conditions require

$$\begin{aligned} C_1 - \frac{1}{2a_+} \int_{0}^{+\infty} e^{-sa_+ y} f(y,s)\, \mathrm{d}y &= C_2 - \frac{1}{2a_-} \int_{-\infty}^{0} e^{sa_- y} f(y,s)\, \mathrm{d}y, \\ -sa_+ C_1 - \frac{s}{2} \int_{0}^{+\infty} e^{-sa_+ y} f(y,s)\, \mathrm{d}y &= sa_- C_2 + \frac{s}{2} \int_{-\infty}^{0} e^{sa_- y} f(y,s)\, \mathrm{d}y. \end{aligned}$$

They are solved by

$$\begin{aligned} C_1 &= \frac{-1}{a_+ + a_-} \int_{-\infty}^{0} e^{sa_- y} f(y,s)\, \mathrm{d}y + \frac{a_- - a_+}{2a_+(a_+ + a_-)} \int_{0}^{+\infty} e^{-sa_+ y} f(y,s)\, \mathrm{d}y, \\ C_2 &= \frac{a_+ - a_-}{2a_-(a_+ + a_-)} \int_{-\infty}^{0} e^{sa_- y} f(y,s)\, \mathrm{d}y - \frac{1}{a_+ + a_-} \int_{0}^{+\infty} e^{-sa_+ y} f(y,s)\, \mathrm{d}y. \end{aligned}$$

Therefore, we obtain an explicit expression of W. Other variables can be derived accordingly.

$$\begin{aligned} W(x,s) = \; & -\frac{1}{2a_+} \int_{0}^{+\infty} e^{-sa_+|x-y|} f(y,s)\, \mathrm{d}y \\ & -\frac{1}{a_+ + a_-} \int_{-\infty}^{0} e^{s(a_- y - a_+ x)} f(y,s)\, \mathrm{d}y \\ & +\frac{a_- - a_+}{2a_+(a_+ + a_-)} \int_{0}^{+\infty} e^{-sa_+(x+y)} f(y,s)\, \mathrm{d}y, \qquad \text{for} \quad x > 0, \end{aligned} \tag{3.4}$$

$$\begin{aligned} W(x,s) = \; & -\frac{1}{2a_-} \int_{-\infty}^{0} e^{-sa_-|x-y|} f(y,s)\, \mathrm{d}y \\ & -\frac{1}{a_+ + a_-} \int_{0}^{+\infty} e^{s(a_- x - a_+ y)} f(y,s)\, \mathrm{d}y, \\ & +\frac{a_+ - a_-}{2a_-(a_+ + a_-)} \int_{-\infty}^{0} e^{sa_-(x+y)} f(y,s)\, \mathrm{d}y, \qquad \text{for} \quad x < 0. \end{aligned} \tag{3.5}$$

These expressions have clear physical meanings. For instance, in (3.4), the first term $-\frac{1}{2a_+} \int_0^{+\infty} e^{-sa_+|x-y|} f(y,s)\, \mathrm{d}y$ describes the wave propagating in the domain $x > 0$, with both diffusive and dispersive effects. The second term $-\frac{1}{a_+ + a_-} \int_{-\infty}^{0} e^{s(a_- y - a_+ x)} f(y,s)\, \mathrm{d}y$ stands for the wave transmitted from $x < 0$ across the phase boundary. The last term $\frac{a_- - a_+}{2a_+(a_+ + a_-)} \int_0^{+\infty} e^{-sa_+(x+y)} f(y,s)\, \mathrm{d}y$ is the wave reflected from the phase boundary.

Therefore, we have

$$W(0,s) = -\frac{1}{a_+ + a_-} \left[\int_{-\infty}^{0} e^{sa_- y} f(y,s)\, \mathrm{d}y + \int_{0}^{+\infty} e^{-sa_+ y} f(y,s)\, \mathrm{d}y \right], \tag{3.6}$$

$$W'(0,s) = \frac{sa_+}{a_+ + a_-}\int_{-\infty}^{0} e^{sa_- y} f(y,s)\,\mathrm{d}y - \frac{sa_-}{a_+ + a_-}\int_0^{+\infty} e^{-sa_+ y} f(y,s)\,\mathrm{d}y. \tag{3.7}$$

Substitute (3.6) into (3.3), we have

$$\begin{aligned}
\frac{[\![\alpha U]\!]}{[\![\alpha]\!]} = &\ \frac{s}{(s+\alpha_+)(s+\alpha_-)}(u_0(0) - w_0(0)) \\
&- \frac{s+1}{(s+\alpha_+)(s+\alpha_-)(a_+ + a_-)}\int_{-\infty}^{0} e^{sa_- y} v_0'(y)\mathrm{d}y \\
&- \frac{s+1}{(s+\alpha_+)(s+\alpha_-)(a_+ + a_-)}\int_0^{+\infty} e^{-sa_+ y} v_0'(y)\mathrm{d}y \\
&+ \frac{s^2(s+1)}{(s+\alpha_+)(s+\alpha_-)^2(a_+ + a_-)}\int_{-\infty}^{0} e^{sa_- y} w_0(y)\,\mathrm{d}y \\
&+ \frac{s^2(s+1)}{(s+\alpha_+)^2(s+\alpha_-)(a_+ + a_-)}\int_0^{\infty} e^{-sa_+ y} w_0(y)\,\mathrm{d}y \\
&+ \frac{\alpha_- s(s+1)}{(s+\alpha_+)(s+\alpha_-)^2(a_+ + a_-)}\int_{-\infty}^{0} e^{sa_- y} u_0(y)\,\mathrm{d}y \\
&+ \frac{\alpha_+ s(s+1)}{(s+\alpha_+)^2(s+\alpha_-)(a_+ + a_-)}\int_0^{\infty} e^{-sa_+ y} u_0(y)\,\mathrm{d}y.
\end{aligned} \tag{3.8}$$

Meanwhile, we rewrite $W'(0,s)$ in terms of u_0, v_0, w_0 as

$$\begin{aligned}
W'(0,s) = &\ \frac{a_+}{a_+ + a_-}\int_{-\infty}^{0} e^{sa_- y} v_0'(y)\,\mathrm{d}y - \frac{a_-}{a_+ + a_-}\int_0^{+\infty} e^{-sa_+ y} v_0'(y)\,\mathrm{d}y \\
&+ \frac{sa_+a_-}{(s+1)(a_+ + a_-)}\left[\int_{-\infty}^{0} e^{sa_- y} w_0'(y)\,\mathrm{d}y + \int_0^{+\infty} e^{-sa_+ y} w_0'(y)\,\mathrm{d}y\right] \\
&- \frac{\alpha_- sa_+}{(s+\alpha_-)(a_+ + a_-)}\int_{-\infty}^{0} e^{sa_- y} u_0(y)\,\mathrm{d}y \\
&+ \frac{\alpha_+ sa_-}{(s+\alpha_+)(a_+ + a_-)}\int_0^{+\infty} e^{-sa_+ y} u_0(y)\,\mathrm{d}y.
\end{aligned} \tag{3.9}$$

4. Main estimates. Most following estimates hold for both a_+, α_+ and a_-, α_-, so we shall not distinguish them unless necessary.

We recall some rules which will be used extensively in the estimates.

$$\mathcal{L}^{-1}(F(s+s_0)) = e^{-s_0 t}\mathcal{L}^{-1}(F(s)), \tag{4.1}$$

$$\mathcal{L}^{-1}(sF(s)) = \lim_{t\to 0}\mathcal{L}^{-1}(F(s))\delta(t) + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}^{-1}(F(s)), \tag{4.2}$$

$$\mathcal{L}^{-1}(F(ps)) = p^{-1}\mathcal{L}^{-1}_{t/p}(F(s)), \tag{4.3}$$

$$\mathcal{L}^{-1}(e^{-ps}F(ps)) = H(t-p)\mathcal{L}^{-1}_{t-p}(F(s)), \quad \text{for } p > 0, \tag{4.4}$$

$$|f(t) * g(t)| \le F(t) * G(t), \quad \text{if } |f(t)| \le F(t),\ |g(t)| \le G(t), \tag{4.5}$$

where $H(t)$ is the Heaviside function, the time variable corresponding to s is t/p for $\mathcal{L}^{-1}_{t/p}$, and that to s is $t-p$ for $\mathcal{L}^{-1}_{t-p}$.

First, for two non-negative constants s_1 and s_2, it holds

$$\mathcal{L}^{-1}\left(\sqrt{\frac{s+s_1}{s+s_2}}\right) = \frac{e^{-s_1 t}}{\pi}\mathcal{L}^{-1}\left(s\sqrt{\frac{\pi}{s}}\sqrt{\frac{\pi}{s+s_2-s_1}}\right).$$

Using the fact $\mathcal{L}(t^{-1/2}) = \pi/\sqrt{s}$, we compute convolution as

$$\begin{aligned}\mathcal{L}^{-1}\left(\sqrt{\frac{\pi}{s}}\sqrt{\frac{\pi}{s+s_2-s_1}}\right) &= \int_0^t \frac{1}{\sqrt{t-\tau}}\left(e^{-(s_2-s_1)\tau}\frac{1}{\sqrt{\tau}}\right)\mathrm{d}\tau \\ &= 2\int_0^{\pi/2} e^{-(s_2-s_1)t\sin^2\theta}\,\mathrm{d}\theta.\end{aligned}$$

After some calculations, we reach at

$$\begin{aligned}&\mathcal{L}^{-1}\left(\sqrt{\frac{s+s_1}{s+s_2}}\right)\\ =\;& \frac{e^{-s_1 t}}{\pi}\left(2\int_0^{\pi/2} 1\,\mathrm{d}\theta\delta(t) + 2\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{\pi/2} e^{-(s_2-s_1)t\sin^2\theta}\,\mathrm{d}\theta\right)\\ =\;& \delta(t) + \frac{2(s_1-s_2)}{\pi}\int_0^{\pi/2} e^{-(s_1\cos^2\theta+s_2\sin^2\theta)t}\sin^2\theta\,\mathrm{d}\theta.\end{aligned} \tag{4.6}$$

We denote $K(\theta;s_1,s_2) = s_1\cos^2\theta + s_2\sin^2\theta$. A rough estimate holds

$$\frac{\pi}{4}e^{-\max(s_1,s_2)t} \le \int_0^{\pi/2} e^{-K(\theta;s_1,s_2)t}\sin^2\theta\,\mathrm{d}\theta \le \frac{\pi}{4}e^{-\min(s_1,s_2)t}. \tag{4.7}$$

There are two special cases

$$\begin{aligned}\mathcal{L}^{-1}(a) - \delta(t) &= \frac{2(1-\alpha)}{\pi}\int_0^{\pi/2} e^{-K(\theta;1,\alpha)t}\sin^2\theta\,\mathrm{d}\theta \\ &\in \left[\frac{1-\alpha}{2}e^{-t}, \frac{1-\alpha}{2}e^{-\alpha t}\right],\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{1}{a}\right) - \delta(t) &= \frac{2(\alpha-1)}{\pi}\int_0^{\pi/2} e^{-K(\theta;\alpha,1)t}\sin^2\theta\,\mathrm{d}\theta \\ &\in \left[\frac{\alpha-1}{2}e^{-\alpha t}, \frac{\alpha-1}{2}e^{-t}\right].\end{aligned} \tag{4.9}$$

Secondly, it is observed that

$$\frac{1}{a_+ + a_-} = \frac{1}{\alpha_+ - \alpha_-}\left[s\left(\frac{1}{a_-} - \frac{1}{a_+}\right) + \frac{\alpha_+}{a_-} - \frac{\alpha_-}{a_+}\right].$$

To avoid lengthy expressions, we let

$$K_+ = K(\theta;\alpha_+,1), \quad K_- = K(\theta;\alpha_-,1).$$

From (4.9), we have

$$\mathcal{L}^{-1}\left(\frac{1}{a_-} - \frac{1}{a_+}\right) = \frac{2}{\pi}\int_0^{\pi/2}\left[(\alpha_- - 1)e^{-K_- t} - (\alpha_+ - 1)e^{-K_+ t}\right]\sin^2\theta\,\mathrm{d}\theta,$$

and then

$$\begin{aligned}\mathcal{L}^{-1}\left[s\left(\frac{1}{a_-} - \frac{1}{a_+}\right)\right] =\;& \frac{\alpha_- - \alpha_+}{2}\delta(t)\\ &+\frac{2}{\pi}\left[(\alpha_+ - 1)\int_0^{\pi/2} e^{-K_+ t}\sin^2\theta(\alpha_+\cos^2\theta + \sin^2\theta)\,\mathrm{d}\theta\right.\\ &\left.-(\alpha_- - 1)\int_0^{\pi/2} e^{-K_- t}\sin^2\theta(\alpha_-\cos^2\theta + \sin^2\theta)\,\mathrm{d}\theta\right].\end{aligned}$$

This leads to

$$
\begin{aligned}
&\mathcal{L}^{-1}\left(\frac{1}{a_+ + a_-}\right) - \frac{1}{2}\delta(t) \\
(4.10) = \quad &\frac{2}{\pi}\left\{\int_0^{\pi/2} \left[(\alpha_+ - 1)e^{-K_+ t} + (\alpha_- - 1)e^{-K_- t}\right]\sin^2\theta \,\mathrm{d}\theta\right. \\
&\left. + \frac{1}{\alpha_+ - \alpha_-}\int_0^{\pi/2}\left[-(1-\alpha_+)^2 e^{-K_+ t} + (1-\alpha_-)^2 e^{-K_- t}\right]\sin^4\theta \,\mathrm{d}\theta\right\}.
\end{aligned}
$$

To get an estimate, we let

$$\alpha_0 = \frac{\alpha_+ + \alpha_-}{2}, \quad k = \frac{\alpha_+ - \alpha_-}{\alpha_+ + \alpha_-}.$$

Or,

$$\alpha_+ = (1+k)\alpha_0, \quad \alpha_- = (1-k)\alpha_0.$$

Then we rewrite

$$
\begin{aligned}
&\frac{2}{(\alpha_+ - \alpha_-)\pi}\int_0^{\pi/2}\left[-(1-\alpha_+)^2 e^{-K_+ t} + (1-\alpha_-)^2 e^{-K_- t}\right]\sin^4\theta \,\mathrm{d}\theta \\
= \quad &\frac{1}{\pi\alpha_0 k}\int_0^{\pi/2}\Big[-(1-(1+k)\alpha_0)^2 e^{-K(\theta;(1+k)\alpha_0,1)t} \\
&\qquad + (1-(1-k)\alpha_0)^2 e^{-K(\theta;(1-k)\alpha_0,1)t}\Big]\sin^4\theta \,\mathrm{d}\theta.
\end{aligned}
$$

Because the integral is zero at $k=0$, and $K_\pm = (1\pm k)\alpha_0\cos^2\theta + \sin^2\theta$, there exists $\tilde{k} \in [k,0]\cup[0,k]$, such that

$$
\begin{aligned}
&\frac{2}{(\alpha_+ - \alpha_-)\pi}\int_0^{\pi/2}\left[-(1-\alpha_+)^2 e^{-K_+ t} + (1-\alpha_-)^2 e^{-K_- t}\right]\sin^4\theta \,\mathrm{d}\theta \\
= \quad &\frac{1}{\pi}\int_0^{\pi/2}\Big\{[2(1-(1+\tilde{k})\alpha_0) + (1-(1+\tilde{k})\alpha_0)^2 t\cos^2\theta]e^{-K(\theta;(1+\tilde{k})\alpha_0,1)t} \\
&\qquad + [2(1-(1-\tilde{k})\alpha_0) + (1-(1-\tilde{k})\alpha_0)^2 t\cos^2\theta]e^{-K(\theta;(1-\tilde{k})\alpha_0,1)t}\Big\}\sin^4\theta \,\mathrm{d}\theta \\
\in \quad &\left[e^{-t}\left(\frac{3(1-\alpha_1)}{4} + \frac{(1-\alpha_1)^2 t}{16}\right), e^{-\alpha_2 t}\left(\frac{3(1-\alpha_2)}{4} + \frac{(1-\alpha_2)^2 t}{16}\right)\right].
\end{aligned}
$$

Here we have used the facts that $(1\pm\tilde{k})\alpha_0 \in [\alpha_2, \alpha_1]$, and $\int_0^{\pi/2}\sin^4\theta \,\mathrm{d}\theta = 3\pi/16$, $\int_0^{\pi/2}\sin^4\theta\cos^2\theta \,\mathrm{d}\theta = \pi/32$. Be reminded that α_1, α_2 come from the structural relation (1.5).

We end up with the following estimate

$$
\begin{aligned}
&\mathcal{L}^{-1}\left(\frac{1}{a_+ + a_-}\right) - \frac{1}{2}\delta(t) \in \left[e^{-t}\left(\frac{3(1-\alpha_1)}{4} + \frac{(1-\alpha_1)^2 t}{16}\right) - e^{-\alpha_2 t}(1-\alpha_2),\right. \\
(4.11) \quad &\qquad\qquad \left. e^{-\alpha_2 t}\left(\frac{3(1-\alpha_2)}{4} + \frac{(1-\alpha_2)^2 t}{16}\right) - e^{-t}(1-\alpha_1)\right].
\end{aligned}
$$

Next, we consider e^{-sap} for $p \geq 0$ independent of s. For this purpose, (4.8) is recast into

$$\mathcal{L}^{-1}(1-a) = -\frac{2(1-\alpha)}{\pi}\int_0^{\pi/2} e^{-K(\theta;1,\alpha)t}\sin^2\theta \,\mathrm{d}\theta.$$

This implies

$$\begin{aligned}
&\mathcal{L}^{-1}(ps(1-a))\\
=\;& -\frac{(1-\alpha)p}{2}\delta(t) - p\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{2(1-\alpha)}{\pi}\int_0^{\pi/2} e^{-K(\theta;1,\alpha)t}\sin^2\theta\,\mathrm{d}\theta\right]\\
=\;& -\frac{(1-\alpha)p}{2}\left\{\delta(t) - \frac{1}{\pi}\int_0^{\pi/2} e^{-K(\theta;1,\alpha)t}[\sin^2(2\theta)+\alpha(1-\cos(2\theta))^2]\,\mathrm{d}\theta\right\}.
\end{aligned}$$

Under the subcharacteristic condition $\alpha \le 1$, it is readily shown that

$$\mathcal{L}^{-1}\left(ps(1-a) + \frac{(1-\alpha)p}{2}\right) \in \frac{(1-\alpha)(1+3\alpha)p}{8}\left[e^{-t}, e^{-\alpha t}\right].$$

Here we use the notation $A[l, L] = [Al, AL]$ for $A \ge 0$. As all terms involved are non-negative, it may be proven through convolutions that

$$\mathcal{L}^{-1}\left[\left(ps(1-a) + \frac{(1-\alpha)p}{2}\right)^n\right] \in \left(\frac{(1-\alpha)(1+3\alpha)p}{8}\right)^n \frac{t^{n-1}}{(n-1)!}\left[e^{-t}, e^{-\alpha t}\right].$$

This results in

$$\begin{aligned}
&\mathcal{L}^{-1}\left(e^{ps(1-a)+\frac{(1-\alpha)p}{2}}\right) - \delta(t)\\
=\;& \sum_{n=1}^{\infty}\frac{\mathcal{L}^{-1}\left[\left(ps(1-a)+\frac{(1-\alpha)p}{2}\right)^n\right]}{n!}\\
\in\;& \frac{(1-\alpha)(1+3\alpha)p}{8}\sum_{n=0}^{\infty}\frac{\left[\frac{(1-\alpha)(1+3\alpha)pt}{8}\right]^n}{(n+1)!n!}\left[e^{-t}, e^{-\alpha t}\right].
\end{aligned}$$

For $k_1, k_2 > 0$ satisfying $k_1k_2 = (1-\alpha)(1+3\alpha)/8$, multiplication of Taylor series shows

$$1 \le \sum_{n=0}^{\infty}\frac{\left[\frac{(1-\alpha)(1+3\alpha)pt}{8}\right]^n}{(n+1)!n!} \le \frac{1}{k_2p}(e^{k_2p}-1)e^{k_1t}.$$

It then follows that

$$\mathcal{L}^{-1}\left(e^{ps(1-a)+\frac{(1-\alpha)p}{2}}\right) - \delta(t) \in \left[\frac{(1-\alpha)(1+3\alpha)p}{8}e^{-t}, k_1e^{(k_1-\alpha)t+k_2p}\right].$$

Noticing

$$\mathcal{L}^{-1}\left(e^{-psa}\right) = e^{-\frac{(1-\alpha)p}{2}}H(t-p)\mathcal{L}^{-1}_{t-p}\left(e^{ps(1-a)+\frac{(1-\alpha)p}{2}}\right),$$

we obtain an estimate

$$\begin{aligned}
&\mathcal{L}^{-1}\left(e^{-psa}\right) - e^{-\frac{(1-\alpha)p}{2}}H(t-p)\delta(t-p)\\
\in\;& H(t-p)\left[\frac{(1-\alpha)(1+3\alpha)p}{8}e^{-t+\frac{(1+\alpha)}{2}p}, k_1e^{(k_1-\alpha)t+(k_2-k_1-\frac{1+\alpha}{2})p}\right].
\end{aligned} \tag{4.12}$$

For a function $\eta(p)$ decaying fast enough at infinity, e.g.

$$|\eta(p)| \le Me^{-k_0 p}, \qquad \text{for} \quad p \in I\!R,$$

with $k_0 > k_2 - k_1 - (1+\alpha)/2$, we may easily derive

$$\begin{aligned}
&\left|\mathcal{L}^{-1}\left(\int_0^{+\infty} e^{-psa}\eta(p)\,\mathrm{d}p\right) - e^{-\frac{1-\alpha}{2}t}\eta(t)\right| \\
\le\ & k_1 \int_0^t e^{(k_1-\alpha)t + (k_2 - k_1 - \frac{1+\alpha}{2})p}|\eta(p)|\,\mathrm{d}p \\
\le\ & CMe^{(k_1-\alpha)t},
\end{aligned}$$

with C independent of $\eta(p)$.

So, if we specify $k_1 \in (0, \alpha_2)$, and take correspondingly

$$k_2 = \frac{(1-\alpha)(1+3\alpha)}{8k_1},$$

then there exists a constant

$$k^* = k_2 - k_1 - (1+\alpha_2)/2,$$

such that if the initial perturbation satisfies the following decay condition

$$|u_0(x)|, |u_0'(x)|, |v_0'(x)|, |w_0(x)|, |w_0'(x)| \le Me^{-k_0|x|}, \qquad \text{for} \quad x \in I\!R, \tag{4.13}$$

with $k_0 > k^*$, then the absolute value for Laplace inverse transform of each integral in (3.8) and (3.9) is bounded by $CMe^{(k_1-\alpha)t}$ for certain C independent of initial data.

Inverse transform for other terms in (3.8) and (3.9) can be put as following, with coefficients C_i determined purely by $\alpha_\pm$.

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{s}{(s+\alpha_+)(s+\alpha_-)}\right) &= \frac{\alpha_- e^{-\alpha_+ t} - \alpha_+ e^{-\alpha_- t}}{\alpha_- - \alpha_+}, \\
\mathcal{L}^{-1}\left(\frac{s+1}{(s+\alpha_+)(s+\alpha_-)}\right) &= \frac{(1-\alpha_+)e^{-\alpha_+ t} - (1-\alpha_-)e^{-\alpha_- t}}{\alpha_- - \alpha_+}, \\
\mathcal{L}^{-1}\left(\frac{s^2(s+1)}{(s+\alpha_+)(s+\alpha_-)^2}\right) &= \delta(t) + (C_1 + C_2 t)e^{-\alpha_- t} + C_3 e^{-\alpha_+ t}, \\
\mathcal{L}^{-1}\left(\frac{s^2(s+1)}{(s+\alpha_+)^2(s+\alpha_-)}\right) &= \delta(t) + C_4 e^{-\alpha_- t} + (C_5 + C_6 t)e^{-\alpha_+ t}, \\
\mathcal{L}^{-1}\left(\frac{s(s+1)}{(s+\alpha_+)(s+\alpha_-)^2}\right) &= (C_7 + C_8 t)e^{-\alpha_- t} + C_9 e^{-\alpha_+ t}, \\
\mathcal{L}^{-1}\left(\frac{s(s+1)}{(s+\alpha_+)^2(s+\alpha_-)}\right) &= C_{10} e^{-\alpha_- t} + (C_{11} + C_{12} t)e^{-\alpha_+ t}, \\
\mathcal{L}^{-1}\left(\frac{s}{s+1}\right) &= \delta(t) - e^{-t}, \\
\mathcal{L}^{-1}\left(\frac{s}{s+\alpha_-}\right) &= \delta(t) - \alpha_- e^{-\alpha_- t}, \\
\mathcal{L}^{-1}\left(\frac{s}{s+\alpha_+}\right) &= \delta(t) - \alpha_+ e^{-\alpha_+ t}.
\end{aligned}$$

5. Nonlinear stability. Using the estimates in the previous section, we may estimate each term in (3.8) and (3.9) with the following results on convolution.

$$\delta(t) * \theta(t) = \theta(t),$$
$$[P_1(t)e^{-m_1 t}] * [P_2(t)e^{-m_2 t}] = P_3(t)e^{-m_1 t} + P_4(t)e^{-m_2 t},$$
$$[P_5(t)e^{-mt}] * [P_6(t)e^{-mt}] = P_7(t)e^{-mt}.$$

Here $m, m_1 \neq m_2$ are constants, $\theta(t)$ is any function that has a Laplace transform, and each $P_i(t)$ is a polynomial of degree d_i. These are proved readily by taking Laplace transform. Moreover, we have

$$d_3 = d_1, \quad d_4 = d_2, \quad d_7 = d_5 + d_6 + 1.$$

Now with $k_1 < \alpha_2$, and under the decay condition (4.13), by convolutions we find that

$$|[\![\alpha u]\!]|, |\widetilde{w}(0,t)| \leq C(t) M e^{(k_1 - \alpha)t},$$

with $C(t)$ a polynomial independent of initial data. Integrate (2.9) on t, we have a bound on $\mathcal{E}_1(t)$

$$\mathcal{E}_1(t) \leq \mathcal{E}_1(0) + 4\int_0^t |\widetilde{w}_x(0,\tau)|\, |[\![\alpha\widetilde{u}]\!]|\, \mathrm{d}\tau \leq \mathcal{E}_1(0) + CM^2. \tag{5.1}$$

Up to this point, we have furnished the assumption that $\mathcal{E}_1$ is bounded, and therefore the Laplace transforms make sense. We should furnish another important assumption made before, i.e. the perturbation maintains at a moderate amplitude to keep the system in the stable phases. In another word, the perturbation equations (1.8) hold. In fact, the boundedness of $\mathcal{E}(t)$ and $\mathcal{E}_1(t)$ ensures that the L^∞ norm of $(\widetilde{u}(x,t), \widetilde{v}(x,t), \widetilde{w}(x,t))$ are bounded. Due to the linearity, if the initial data $(u_0(x), v_0(x), w_0(x))$ decreases by a factor, so does the deviation at any time. Therefore, we conclude in the theorem as follows.

THEOREM 5.1. *A subsonic phase boundary solution (1.7) of the Suliciu model (1.4) with tri-linear structure relation (1.5) is stable, in the sense that* $\forall\Delta > 0$ *which may not necessarily be small, and select* $k_1 \in (0, \alpha_2)$*, there exist* $\delta, M > 0$*, if*

1. initial data is of moderate amplitude, i.e.

$$\|(u_0(x), v_0(x), w_0(x))\|_{H^1} \leq \delta; \tag{5.2}$$

2. initial data decay exponentially at infinity

$$|u_0(x)|, |u_0'(x)|, |v_0'(x)|, |w_0(x)|, |w_0'(x)| \leq M e^{-k_0|x|}, \tag{5.3}$$

with $k_0 < k^* = \dfrac{(1-\alpha)(1+3\alpha)}{8k_1} - k_1 - (1+\alpha_2)/2,$
then we have

$$\|(\widetilde{u}(x,t), \widetilde{v}(x,t), \widetilde{w}(x,t))\|_{H^1} \leq \Delta; \tag{5.4}$$

and in particular $u(x,t)$ *keeps in the same stable phases as it is initially.*

REMARK 5.1. *The restriction on* $v_0(x)$ *may be relaxed. Because only the derivatives of* v *appear in the Suliciu model, a translation of* v *or* $\widetilde{v}$ *does not affect the* *evolution of u and w. This corresponds to the change of the inertial reference coordinate.*

REMARK 5.2. *In the special case $\alpha_+ = \alpha_-$, the boundary terms in (2.9) disappear. The decay of initial perturbation at infinity is then not necessary for the nonlinear stability.*

REMARK 5.3. *If we go back to the original Suliciu model (1.3) with relaxation parameter ϵ, the restriction on H^1-norm and decay can be relaxed. The smallness of $\|(u_0(x), v_0(x), w_0(x))\|_{L^2} + \epsilon\|(u_0'(x), v_0'(x), w_0'(x))\|_{L^2}$, and decay of $|u_0(x)|, \epsilon|u_0'(x)|, \epsilon|v_0'(x)|, |w_0(x)|, \epsilon|w_0'(x)| \le Me^{-k_0|x|}$ is enough.*

REMARK 5.4. *If either of u_+, u_- is closer to the stable-unstable transition points u_1, u_2, then the deviation $\widetilde{u}(x,t)$ should be smaller to maintain the system in the same stable phases, therefore the stability is weaker in the sense that the amplitude of perturbation must be smaller.*

REMARK 5.5. *A special case that ensures exponential decay at infinity is that the initial perturbation has compact support.*

6. Nonlinear stability of the relaxed Suliciu model. By relaxed Suliciu model, we mean to take $\epsilon \to 0$ in (1.3). In another word, one solves (1.1) by elementary waves that have profiles in Suliciu model. It has been proven that under the tri-linear structural relation, any Riemann problem is uniquely soluble [12].

We perturb a subsonic phase boundary solution

$$(u_s, v_s)(x,t) = \begin{cases} (u_+, \bar{v}), & \text{for} \quad x < 0, \\ (u_-, \bar{v}), & \text{for} \quad x > 0, \end{cases} \tag{6.1}$$

with $\sigma(u_+) = \sigma(u_-)$ by a bounded perturbation $(u_0(x), v_0(x))$. The perturbation evolves, for $x > 0$,

$$\begin{aligned} \widetilde{u}_t + \widetilde{v}_x &= 0, \\ \widetilde{v}_t + \lambda^2 \alpha_+ \widetilde{u}_x &= 0, \end{aligned} \tag{6.2}$$

and for $x < 0$,

$$\begin{aligned} \widetilde{u}_t + \widetilde{v}_x &= 0, \\ \widetilde{v}_t + \lambda^2 \alpha_- \widetilde{u}_x &= 0, \end{aligned} \tag{6.3}$$

provided that $\widetilde{u}(x,t)$ keeps of moderate amplitude to maintain the system in the same stable phases. The matching conditions across the boundary are

$$[\![\alpha\widetilde{u}]\!] = [\![\widetilde{v}]\!] = 0. \tag{6.4}$$

Denoting sound speeds $c_\pm = \lambda\sqrt{\alpha_\pm}$, the Riemann variables $(\phi^\pm, \psi^\pm) = (\widetilde{v} + c_\pm\widetilde{u}, \widetilde{v} + c_\pm\widetilde{u})$ solve

$$\begin{aligned} \phi_t^\pm + c_\pm \phi_x^\pm &= 0, \\ \psi_t^\pm - c_\pm \psi_x^\pm &= 0. \end{aligned} \tag{6.5}$$

The matching conditions now read

$$\begin{aligned} \phi^+(0,t) + \psi^+(0,t) &= \phi^-(0,t) + \psi^-(0,t), \\ c_+(\phi^+(0,t) - \psi^+(0,t)) &= c_-(\phi^-(0,t) - \psi^-(0,t)). \end{aligned}$$

We know that

$$\begin{aligned} \psi^+(0,t) &= v_0(c_+t) - c_+u_0(c_+t), \\ \phi^-(0,t) &= v_0(-c_-t) + c_-u_0(-c_-t). \end{aligned}$$

Therefore, we have

$$
\begin{aligned}
\phi^{+}(0,t) &= \frac{2c_-}{c_+ + c_-}v_0(-c_-t) - \frac{2c_-^2}{c_+ + c_-}u_0(-c_-t) \\
&\quad + \frac{c_+ - c_-}{c_+ + c_-}v_0(c_+t) - \frac{c_+(c_+ - c_-)}{c_+ + c_-}u_0(c_+t), \\
\psi^{-}(0,t) &= \frac{c_- - c_+}{c_+ + c_-}v_0(-c_-t) + \frac{c_-(c_- - c_+)}{c_+ + c_-}u_0(-c_-t) \\
&\quad + \frac{2c_+}{c_+ + c_-}v_0(c_+t) - \frac{2c_+^2}{c_+ + c_-}u_0(c_+t).
\end{aligned}
$$

The system then can be solved explicitly in terms of $c_\pm$, $u_0(x)$ and $v_0(x)$. A nonlinear stability then follows.

THEOREM 6.1. *The subsonic phase boundary solution (u_s, v_s) is stable for the relaxed Suliciu model. More precisely, let $D_\pm = \min\{|u_\pm - u_1|, |u_\pm - u_2|\}$, if initial perturbation $(u_0(x), v_0(x))$ satisfies the following condition*

$$
\begin{aligned}
&\max_{x,y\geq 0} \Bigg\{ \Big|v_0(x) + c_+u_0(x) - v_0(y) + c_+u_0(y)\Big|, \Big|\frac{2c_-}{c_+ + c_-}v_0\left(-\frac{c_-x}{c_+}\right) \\
&\qquad - \frac{2c_-^2}{c_+ + c_-}u_0\left(-\frac{c_-x}{c_+}\right) + \frac{c_+ - c_-}{c_+ + c_-}v_0(x) - \frac{c_+(c_+ - c_-)}{c_+ + c_-}u_0(x) \\
&\qquad - v_0(y) + c_+u_0(y)\Big| \Bigg\} < 2c_+D_+, \\
&\max_{x,y\leq 0} \Bigg\{ \Big|v_0(x) + c_+u_0(x) - v_0(y) + c_+u_0(y)\Big|, \Big|v_0(x) + c_+u_0(x) \\
&\qquad - \frac{c_- - c_+}{c_+ + c_-}v_0(y) - \frac{c_-(c_- - c_+)}{c_+ + c_-}u_0(y) - \frac{2c_+}{c_+ + c_-}v_0\left(-\frac{c_+y}{c_-}\right) \\
&\qquad + \frac{2c_+^2}{c_+ + c_-}u_0\left(-\frac{c_+y}{c_-}\right)\Big| \Bigg\} < 2c_-D_-,
\end{aligned}
\tag{6.6}
$$

then $\widetilde{v}(x,t)$ keeps bounded, and the system lies in the same stable phases as it is at initial time, with $|\widetilde{u}(x,t)| < D_+$ for $x > 0$, and $|\widetilde{u}(x,t)| < D_-$ for $x < 0$. Moreover, the bounds of $|\widetilde{u}(x,t)|, |\widetilde{v}(x,t)|$ is proportional to that of the initial perturbation.

7. Conclusion. In this paper, we have established one of the first nonlinear stability results on dynamic phase transitions. This may shed an insight into this tough and striking topic. The technical novelty lies mainly in the reduction of a nonlinear problem to linear problems, and the combination of energy estimate with Laplace transform. Unfortunately, one of the major advantageous property of the Suliciu model lies in the fact the system is away from the unstable phase except at the transition point. Moreover, the Laplace transform relies on the linearity of the problem, therefore unlikely applies to more general structural relation, or other discrete BGK models, such as the Jin-Xin's relaxation model [9]. Nevertheless, the success for the current model suggests that new techniques is essential to overcome difficulties of standard energy estimations.

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