Relaxation to isentropic gas dynamics for a BGK system with single kinetic entropy

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Abstract

We introduce a family of vector kinetic BGK equations leading to isentropic gas dynamics in the relaxation limit, that have only one entropy at the kinetic level. These models possess the generic structure of kinetic relaxation models. By a sharp adaptation of averaging lemmas to BGK models that have a dissipative entropy, we establish an estimate in the inverse of the square root of the relaxation parameter on the L^2 norm of the gradient of the approximations. This estimation is new in the context of kinetic equations, and it allows, by the method of DiPerna, to establish the convergence towards weak solutions of isentropic gas dynamics that satisfy a single entropy inequality.

Key-words: Relaxation limit – kinetic BGK model – kinetic entropy – isentropic gas dynamics – averaging – compensated compactness

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Contents

1 Introduction

The method developed by DiPerna [12] using Tartar's compensated compactness [25] for establishing the convergence of approximate solutions to one-dimensional nonlinear hyperbolic systems of conservation laws is well-known. It has been used mainly when dealing with viscosity or numerical approximations. However, the problem of establishing similar results to relaxation models such as those introduced in [11] is not solved in general. Apart from the scalar case, for which we refer to the review [27], until now, only particular relaxation structures have been treated in [26], [24], [3]. In [24] and [3], the main ingredient that is used is that many entropies have a kinetic extension, in the sense defined in [11]. A general method for finding such extensions in kinetic BGK models can be found in [6]. In physically relevant models however, only one entropy has an extension, and therefore one has to introduce other tools. This was done in [26], where from particular energy inequalities, the author establishes an estimate on the gradient of the approximate solutions

$$\|\partial_x U_\varepsilon\|_{L^2_{t,x}} \leq \frac{C}{\sqrt{\varepsilon}}, \tag{1.1}$$

where ε is the relaxation parameter. This is indeed exactly the estimate used by DiPerna in the viscosity approximation.

The purpose of this paper is to explain how this approach can indeed be used generically when dealing with kinetic BGK models, even if we have a single entropy extension. The new ingredient for establishing (1.1) is an averaging lemma which is particularly adapted to the BGK relaxation term. With different scalings, related appropriate sharp estimates for relaxation limits leading to incompressible models of gas dynamics in multidimension have recently been the object of great progress, see [10], [1], [19], [23], [15].

In order to be more specific, we shall deal with the one-dimensional system of isentropic gas dynamics

$$\begin{cases} \partial_t\rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + \kappa\rho^\gamma) = 0, \end{cases} \qquad \text{in }]0,\infty[_t \times \mathbb{R}_x, \tag{1.2}$$

with $\rho(t,x) \geq 0$, $u(t,x) \in \mathbb{R}$ and $\kappa > 0$, $1 < \gamma < 3$. Approximate solutions are obtained via a kinetic BGK model

$$\partial_t f_\varepsilon + \xi \partial_x f_\varepsilon = \frac{M(\rho_\varepsilon, u_\varepsilon, \xi) - f_\varepsilon}{\varepsilon} \qquad \text{in }]0,\infty[_t \times \mathbb{R}_x \times \mathbb{R}_\xi, \tag{1.3}$$

with

$$(\rho_\varepsilon, \rho_\varepsilon u_\varepsilon) = U_\varepsilon = \int f_\varepsilon \, d\xi. \tag{1.4}$$

Here $f_\varepsilon(t,x,\xi) \in \mathbb{R}^2$ as in the general framework of [6], and the maxwellian equilibrium M has to satisfy the consistency relations

$$\int M(\rho,u,\xi)\, d\xi = (\rho, \rho u), \qquad \int \xi M(\rho,u,\xi)\, d\xi = F \equiv (\rho u, \rho u^2 + \kappa \rho^\gamma). \tag{1.5}$$

The stability theory for (1.2), that has been completed in [21], [20], uses the entropy inequalities

$$\partial_t \, \eta(U) + \partial_x \, G(U) \leq 0, \tag{1.6}$$

for a whole family of couples entropy-entropy fluxes (η, G), with η convex, the so called weak entropies. According to [6], a kinetic entropy associated to η is a function $H_\eta(f,\xi)$ which is convex in f, verifies the consistency relation

$$\int H_\eta(M(\rho,u,\xi),\xi)\, d\xi = \eta(\rho,u), \tag{1.7}$$

and satisfies a minimization principle

$$\int H_\eta(M(\rho,u,\xi),\xi)\, d\xi \leq \int H_\eta(f(\xi),\xi)\, d\xi, \qquad (\rho,\rho u) = \int f(\xi) d\xi, \tag{1.8}$$

for any function $f(\xi) \in L^1_\xi$ taking values in suitable convex sets D_ξ, in the sense that $f(\xi) \in D_\xi$. The model used in [2], [3] (see also [4] and [5] for related problems) possesses kinetic entropies H_η for all entropies η. Here we shall consider more general models that have a single kinetic entropy H corresponding to the physical energy,

$$\eta^e(\rho,u) = \rho u^2/2 + \frac{\kappa}{\gamma - 1}\rho^\gamma, \qquad G^e(\rho,u) = (\eta^e(\rho,u) + \kappa\rho^\gamma)u. \tag{1.9}$$

Therefore, we shall obtain in the relaxation limit weak solutions $U = (\rho, \rho u)$ satisfying only

$$\partial_t \, \eta(U) + \partial_x \, G(U) = m_\eta \in \mathcal{M}_{loc} \tag{1.10}$$

for any (weak) entropy η, and the measure m_η is nonpositive only for $\eta = \eta^e$. This is indeed the approach of [26], [17], [16]. Thus we look for approximate entropy equations

$$\partial_t \, \eta(U_\varepsilon) + \partial_x \, G(U_\varepsilon) = m_{\eta,\varepsilon} + R_{\eta,\varepsilon}, \tag{1.11}$$

where $R_{\eta,\varepsilon} \to 0$ as $\varepsilon \to 0$ in $W^{-1,p}_{loc}$ for any $1 < p < \infty$, $m_{\eta,\varepsilon}$ are measures locally bounded uniformly in ε, and $m_{\eta^e,\varepsilon} \leq 0$. This is sufficient to apply the compensated compactness method. The measures $m_{\eta,\varepsilon}$ are indeed bounded via the estimate (1.1), as in the viscosity approximation. The fact that the energy has a kinetic extension comes into play only via an a priori estimate coming from the entropy dissipation,

$$\|M_\varepsilon - f_\varepsilon\|_{L^2_{tx\xi}} \leq C\sqrt{\varepsilon}, \tag{1.12}$$

which enables to establish (1.1). However, to get this we need coercivity of the kinetic entropy, and this holds in our model only for $1 < \gamma \leq 5/3$. We shall assume also some a priori L^∞ bound on ρ_ε, u_ε and that the vacuum does not appear. These are the main restrictions of our work. The convergence result can be stated as follows.

Theorem 1.1 *We assume that $1 < \gamma \leq 5/3$, and that the solution f_ε to (2.1)-(2.10) with initial data f^0_ε such that (2.22)-(2.24) and (4.2) hold, satisfies (2.35)-(2.36) and (4.1). We set $U_\varepsilon = (\rho_\varepsilon, \rho_\varepsilon u_\varepsilon) = \int_{\mathbb{R}} f_\varepsilon \, d\xi$. Then there exists a subsequence such that $\rho_\varepsilon \to \rho$ and $\rho_\varepsilon u_\varepsilon \to \rho u$ a.e., where $(\rho, \rho u)$ is a weak solution to (1.2) with initial data $(\rho^0, \rho^0 u^0) = w\text{-}\lim \int f^0_\varepsilon \, d\xi$, and (1.10) holds.*

The outline of the paper is as follows. In Section 2, we introduce our BGK model and establish the kinetic entropy dissipation estimate that leads to (1.12). In Section 3, we prove the key averaging lemma. Finally, in Section 4 we establish the relaxation limit by combining the averaging lemma and the dissipation result, in order to obtain (1.1), (1.11), and Theorem 1.1.

2 Kinetic equation

In this section, we introduce a family of vector kinetic BGK models for isentropic gas dynamics, parametrized by a scalar $0 < \theta \leq 1$. The value $\theta = (\gamma - 1)/2$ gives indeed the model of [2], [3], but for other values of θ, only the energy of the system has a kinetic extension. This is the case in particular for the most physical model, obtained for $\theta = 1$. We establish the estimates supplied by the corresponding dissipation in Theorem 2.5.

2.1 BGK model

The model we consider is written as

$$\partial_t f + \xi \partial_x f = \frac{M[f] - f}{\varepsilon} \quad \text{in }]0, \infty[\times \mathbb{R} \times \mathbb{R}, \tag{2.1}$$

where $f = f(t, x, \xi) = (f_0(t, x, \xi), f_1(t, x, \xi)) \in \mathbb{R}^2$, $t \geq 0$, $x \in \mathbb{R}$, $\xi \in \mathbb{R}$,

$$f(t, x, \xi) \in D_\xi, \tag{2.2}$$

$$M[f](t, x, \xi) = M\left(\rho(t, x), u(t, x), \xi\right), \tag{2.3}$$

$$\rho(t, x) = \int_{\mathbb{R}} f_0(t, x, \xi) \, d\xi, \;\; \rho(t, x) u(t, x) = \int_{\mathbb{R}} f_1(t, x, \xi) \, d\xi, \tag{2.4}$$

and

$$M(\rho, u, \xi) = \Big(M_0(\rho, u, \xi), M_1(\rho, u, \xi) = ((1 - \theta)u + \theta\xi) M_0(\rho, u, \xi)\Big), \tag{2.5}$$

$$M_0(\rho,u,\xi)=c_\theta\left(\frac{2\gamma\kappa}{(\gamma-1)\theta}\rho^{\gamma-1}-(\xi-u)^2\right)_+^\lambda, \tag{2.6}$$

$$\lambda=\frac{1}{\gamma-1}-\frac{1}{2},\ c_\theta=\frac{1}{J_\lambda}\left(\frac{2\gamma\kappa}{(\gamma-1)\theta}\right)^{-1/(\gamma-1)}, \tag{2.7}$$

$$J_\lambda=\int_{-1}^{1}(1-z^2)^\lambda\,dz=\sqrt{\pi}\Gamma(\lambda+1)/\Gamma(\lambda+3/2). \tag{2.8}$$

The parameter θ must satisfy

$$0<\theta\leq 1, \tag{2.9}$$

and the model of [2], [3] corresponds to the choice $\theta=(\gamma-1)/2$. The sets D_ξ of (2.2) are defined as

$$\begin{aligned} D_\xi=D=\{(f_0,f_1)\in\mathbb{R}^2,\ f_0>0 \text{ or } f_1=f_0=0\}\quad \text{if } \theta<1,\\ D_\xi=\{(f_0,f_1)\in\mathbb{R}^2,\ f_0\geq 0,\ f_1=\xi f_0\}\quad \text{if } \theta=1. \end{aligned} \tag{2.10}$$

The kinetic entropy for (2.1) is defined as follows. In the case $\theta<1$, for $f=(f_0,f_1)\in D\backslash\{(0,0)\}$,

$$\begin{aligned} H(f,\xi)&=\frac{\theta}{1-\theta}\frac{\xi^2}{2}f_0+\frac{\theta}{2c_\theta^{1/\lambda}}\frac{f_0^{1+1/\lambda}}{1+1/\lambda}+\frac{1}{1-\theta}\frac{1}{2}\frac{f_1^2}{f_0}-\frac{\theta}{1-\theta}\xi f_1\\ &=\frac{1}{1-\theta}\frac{1}{2}\left(\frac{f_1}{\sqrt{f_0}}-\theta\xi\sqrt{f_0}\right)^2+\frac{\theta}{2c_\theta^{1/\lambda}}\frac{f_0^{1+1/\lambda}}{1+1/\lambda}+\theta\frac{\xi^2}{2}f_0\geq 0, \end{aligned} \tag{2.11}$$

and $H(0,\xi)=0$, while in the case $\theta=1$, for $f\in D_\xi$,

$$H(f,\xi)=\frac{\xi^2}{2}f_0+\frac{1}{2c_1^{1/\lambda}}\frac{f_0^{1+1/\lambda}}{1+1/\lambda}. \tag{2.12}$$

Indeed, making $f_1=\xi f_0$ in (2.11) and letting $\theta\to 1$ gives (2.12).
The value of the moments of M are

$$\int_{\mathbb{R}}M\,d\xi=(\rho,\rho u),\qquad \int_{\mathbb{R}}\xi M\,d\xi=(\rho u,\rho u^2+\kappa\rho^\gamma), \tag{2.13}$$

$$\int_{\mathbb{R}}\xi^2M\,d\xi=\left(\rho u^2+\frac{\kappa\rho^\gamma}{\theta},\left(\rho u^2+(2+\frac{1}{\theta})\kappa\rho^\gamma\right)u\right), \tag{2.14}$$

for every $\rho\geq 0$ and $u\in\mathbb{R}$. These relations enable to compute the formal Chapman-Enskog expansion for this model, which is, up to terms in ε^2,

$$\partial_t U+\partial_x F(U)=\varepsilon\partial_x(D(U)\partial_x U), \tag{2.15}$$

with

$$D(U)=\begin{pmatrix} (1/\theta-1)\gamma\kappa\rho^{\gamma-1} & 0\\ ((\gamma-1)/\theta-2)\kappa\rho^{\gamma-1}u & (2-\gamma+1/\theta)\kappa\rho^{\gamma-1}\end{pmatrix}, \tag{2.16}$$

and $U = (\rho, \rho u)$. We observe that the least dissipative model is obtained for $\theta = 1$, where $D(U)$ becomes singular. This model is degenerate also because $M(\rho, u, \xi)$ is then colinear to $(1, \xi)$, thus we have to look for solutions $f = (f_0, \xi f_0)$. This is the most physically relevant model, that is already mentioned in [6]. Our analysis also works in this case.

We have the following relations, that can be obtained with straightforward computations.

Proposition 2.1 *The energy η^e and the kinetic entropy H are related through*

$$\int_{\mathbb{R}} H(M(\rho, u, \xi), \xi)\, d\xi = \eta^e(\rho, u) \quad \text{for every } \rho \geq 0,\ u \in \mathbb{R}, \tag{2.17}$$

$$\int_{\mathbb{R}} \xi H(M(\rho, u, \xi), \xi)\, d\xi = G^e(\rho, u) \quad \text{for every } \rho \geq 0,\ u \in \mathbb{R}, \tag{2.18}$$

and if $\theta < 1$,

$$\begin{cases} \dfrac{\partial H}{\partial f_0}(M(\rho, u, \xi), \xi) = \dfrac{\partial \eta^e}{\partial \rho}(\rho, u) = \dfrac{\gamma}{\gamma - 1}\kappa \rho^{\gamma-1} - u^2/2, \\ \dfrac{\partial H}{\partial f_1}(M(\rho, u, \xi), \xi) = \dfrac{\partial \eta^e}{\partial(\rho u)}(\rho, u) = u, \end{cases} \tag{2.19}$$

for every $\rho > 0$, $u \in \mathbb{R}$ and $\xi \in \mathbb{R}$ such that $M_0(\rho, u, \xi) > 0$.

With the same proofs as in [2], one can deduce the following results.

Proposition 2.2 (Subdifferential inequality) *For any $\rho \geq 0$, $u, \xi \in \mathbb{R}$ and $f \in D_\xi$, we have*

$$H(f, \xi) \geq H(M(\rho, u, \xi), \xi) + \left(\frac{\gamma\kappa}{\gamma - 1}\rho^{\gamma-1} - \frac{u^2}{2}, u \right) \cdot (f - M(\rho, u, \xi)). \tag{2.20}$$

Proposition 2.3 (Minimization principle) *Consider $f \in L^1(\mathbb{R}_\xi)$ such that $f(\xi) \in D_\xi$ a.e. and $\int_{\mathbb{R}} H(f(\xi), \xi)\, d\xi < \infty$. Then*

$$\int_{\mathbb{R}} H(M[f](\xi), \xi)\, d\xi \leq \int_{\mathbb{R}} H(f(\xi), \xi)\, d\xi. \tag{2.21}$$

2.2 Dissipation estimates

We take an initial data $f_\varepsilon^0 \in L^1(\mathbb{R}_x \times \mathbb{R}_\xi)$ which satisfies

$$f_\varepsilon^0(x, \xi) \in D_\xi \quad \text{a.e. in } \mathbb{R} \times \mathbb{R}, \tag{2.22}$$

$$\iint_{\mathbb{R}\times\mathbb{R}} |f_\varepsilon^0(x, \xi)|\, dx d\xi \leq C_I^0 < \infty, \tag{2.23}$$

$$\iint_{\mathbb{R}\times\mathbb{R}} H(f_\varepsilon^0(x, \xi), \xi)\, dx d\xi \leq C_H^0 < \infty. \tag{2.24}$$

Then with the same proof as in [2], there exists a solution f_ε to the BGK model (2.1)-(2.10), with initial data f_ε^0, satisfying

$$f_\varepsilon \in C_t \cap L_t^\infty([0,\infty[, L^1(\mathbb{R}_x \times \mathbb{R}_\xi)), \tag{2.25}$$

$$\forall t \geq 0, \quad f_\varepsilon(t,x,\xi) \in D_\xi \text{ a.e. in } \mathbb{R}\times\mathbb{R}, \tag{2.26}$$

$$\forall t \geq 0, \quad \iint\limits_{\mathbb{R}\times\mathbb{R}} f_\varepsilon(t,x,\xi)\,dxd\xi = \iint\limits_{\mathbb{R}\times\mathbb{R}} f_\varepsilon^0(x,\xi)\,dxd\xi, \tag{2.27}$$

$$\forall t \geq 0, \iint\limits_{\mathbb{R}\times\mathbb{R}} H(f_\varepsilon(t,x,\xi),\xi)\,dxd\xi \leq C_H^0, \tag{2.28}$$

$$\partial_t \int f_\varepsilon\,d\xi + \partial_x \int \xi f_\varepsilon\,d\xi = 0. \tag{2.29}$$

As in [2], we have the following identity.

Proposition 2.4 (Entropy variation) *The solution f_ε to (2.1)-(2.10), with initial data f_ε^0 satisfying (2.22)-(2.24), verifies*

$$\begin{aligned}&\left[\iint\limits_{\mathbb{R}\times\mathbb{R}} H(f_\varepsilon(t,x,\xi),\xi)\,dxd\xi\right]_0^T \\ &= \frac{1}{\varepsilon}\iiint\limits_{]0,T[\times\mathbb{R}\times\mathbb{R}} \left(H'(f_\varepsilon,\xi) - (\frac{\gamma\kappa}{\gamma-1}\rho_\varepsilon^{\gamma-1} - \frac{u_\varepsilon^2}{2}, u_\varepsilon)\right)\cdot(M[f_\varepsilon]-f_\varepsilon)\,dtdxd\xi.\end{aligned} \tag{2.30}$$

Moreover,

$$\left(H'(f_\varepsilon,\xi) - (\frac{\gamma\kappa}{\gamma-1}\rho_\varepsilon^{\gamma-1} - \frac{u_\varepsilon^2}{2}, u_\varepsilon)\right)\cdot(M[f_\varepsilon]-f_\varepsilon) \leq 0 \quad a.e. \tag{2.31}$$

In the case $\theta < 1$, the derivative $H'(f,\xi)$ of $H(f,\xi)$ with respect to f is given for $f \neq 0$ by

$$H'(f,\xi) = \left(\frac{\theta}{1-\theta}\frac{\xi^2}{2} + \frac{\theta}{2c_\theta^{1/\lambda}}f_0^{1/\lambda} - \frac{1}{1-\theta}\frac{1}{2}\frac{f_1^2}{f_0^2}, -\frac{\theta}{1-\theta}\xi + \frac{1}{1-\theta}\frac{f_1}{f_0}\right), \tag{2.32}$$

while if $\theta = 1$, we set by convention

$$H'(f,\xi) = \left(\frac{\xi^2}{2} + \frac{1}{2c_1^{1/\lambda}}f_0^{1/\lambda}, 0\right). \tag{2.33}$$

In (2.30), $H'(0,\xi)$ needs not be defined since by applying the result of [7] to each component of f_ε, we have

$$M[f_\varepsilon] - f_\varepsilon = 0 \quad \text{a.e. } t,x,\xi \text{ where } f_\varepsilon = 0. \tag{2.34}$$

We shall now assume L^∞ bounds, because we are not able to prove them. It was possible in [3] because of the compatibility with all entropies, but here this property does not hold. Thus we assume that for some F, $A>0$ independent of ε,

$$|f_\varepsilon|,\ |M[f_\varepsilon]|\le F,\qquad |(f_\varepsilon)_1|\le A(f_\varepsilon)_0,\ |(M[f_\varepsilon])_1|\le A(M[f_\varepsilon])_0. \tag{2.35}$$

We assume also compact support in velocity, that is to say there exists a closed ball $\overline{B}(0,Q)$ such that

$$\mathrm{supp}_\xi f_\varepsilon,\ \mathrm{supp}_\xi M[f_\varepsilon]\subset \overline{B}(0,Q). \tag{2.36}$$

These assumptions imply obviously that ρ_ε, u_ε are bounded independently of ε. We are now able to establish the optimal rate of convergence for $M[f_\varepsilon]-f_\varepsilon$.

Theorem 2.5 *We assume that $\lambda\ge 1$, that is to say $1<\gamma\le 5/3$, and that the solution f_ε to (2.1)-(2.10) with initial data f_ε^0 such that (2.22)-(2.24) hold, satisfies (2.35). Then there exists $C>0$ independent of ε, such that*

$$\iiint_{]0,\infty[\times\mathbb{R}\times\mathbb{R}} |f_\varepsilon - M[f_\varepsilon]|^2\,dtdxd\xi\le C\varepsilon. \tag{2.37}$$

Proof. Since the case $\theta=1$ is the easiest and is performed with the same type of estimates, we let it to the reader, and consider only the case $\theta<1$. We start from the dissipation identity of Proposition 2.4,

$$\iiint_{]0,\infty[\times\mathbb{R}\times\mathbb{R}}\left(H'(f_\varepsilon,\xi)-(\frac{\gamma\kappa}{\gamma-1}\rho_\varepsilon^{\gamma-1}-\frac{u_\varepsilon^2}{2},u_\varepsilon)\right)\cdot(f_\varepsilon-M[f_\varepsilon])\,dtdxd\xi\le C_H^0\varepsilon. \tag{2.38}$$

We note that whenever $M[f_\varepsilon]\neq 0$,

$$(\frac{\gamma\kappa}{\gamma-1}\rho_\varepsilon^{\gamma-1}-\frac{u_\varepsilon^2}{2},u_\varepsilon)=H'(M[f_\varepsilon],\xi). \tag{2.39}$$

We also notice that for $X=(X_0,X_1)\in D\setminus\{0\}$, $Y=(Y_0,Y_1)\in\mathbb{R}^2$,

$$H''(X,\xi)\cdot Y\cdot Y=K_1X_0^{1/\lambda-1}Y_0^2+K_2(X_1Y_0-X_0Y_1)^2/X_0^3, \tag{2.40}$$

with $K_1=\theta/2\lambda c_\theta^{1/\lambda}$ and $K_2=1/(1-\theta)$. We deduce as in [3] that

$$\begin{aligned}&\mathbb{1}_{M[f_\varepsilon]\neq 0}(H'(M[f_\varepsilon],\xi)-H'(f_\varepsilon,\xi))\cdot(M[f_\varepsilon]-f_\varepsilon)\\ \ge\ & K_1F^{1/\lambda-1}((M[f_\varepsilon])_0-(f_\varepsilon)_0)^2\mathbb{1}_{M[f_\varepsilon]\neq 0},\end{aligned} \tag{2.41}$$

thus

$$\iiint_{M[f_\varepsilon]\neq 0}((f_\varepsilon)_0-(M[f_\varepsilon])_0)^2\,dtdxd\xi\le\frac{F^{1-1/\lambda}C_H^0}{K_1}\varepsilon. \tag{2.42}$$

We also have, recalling that by (2.34), $M[f_\varepsilon]=0$ a.e. where $f_\varepsilon=0$,

$$\begin{aligned}&\mathbb{1}_{M[f_\varepsilon]\neq 0}(H'(M[f_\varepsilon],\xi)-H'(f_\varepsilon,\xi))\cdot(M[f_\varepsilon]-f_\varepsilon)\\ \ge\ & \frac{K_2}{F}\left(\frac{(f_\varepsilon)_1(M[f_\varepsilon])_0-(f_\varepsilon)_0(M[f_\varepsilon])_1}{(f_\varepsilon)_0+(M[f_\varepsilon])_0}\right)^2\mathbb{1}_{M[f_\varepsilon]\neq 0}.\end{aligned} \tag{2.43}$$

In the case $(f_\varepsilon)_0 \le (M[f_\varepsilon])_0$, we write

$$\begin{aligned} &(f_\varepsilon)_1 - (M[f_\varepsilon])_1 \\ &= \frac{(f_\varepsilon)_1 (M[f_\varepsilon])_0 - (f_\varepsilon)_0 (M[f_\varepsilon])_1}{(M[f_\varepsilon])_0} - ((M[f_\varepsilon])_0 - (f_\varepsilon)_0) \frac{(M[f_\varepsilon])_1}{(M[f_\varepsilon])_0}, \end{aligned} \tag{2.44}$$

and in the case $(f_\varepsilon)_0 > (M[f_\varepsilon])_0$, we write

$$\begin{aligned} &(f_\varepsilon)_1 - (M[f_\varepsilon])_1 \\ &= \frac{(f_\varepsilon)_1 (M[f_\varepsilon])_0 - (f_\varepsilon)_0 (M[f_\varepsilon])_1}{(f_\varepsilon)_0} - ((M[f_\varepsilon])_0 - (f_\varepsilon)_0) \frac{(f_\varepsilon)_1}{(f_\varepsilon)_0}. \end{aligned} \tag{2.45}$$

In any case, we obtain

$$\begin{aligned} &\mathbb{1}_{M[f_\varepsilon]\neq 0} ((f_\varepsilon)_1 - (M[f_\varepsilon])_1)^2 \\ &\le 8 \mathbb{1}_{M[f_\varepsilon]\neq 0} \left(\frac{(f_\varepsilon)_1 (M[f_\varepsilon])_0 - (f_\varepsilon)_0 (M[f_\varepsilon])_1}{(f_\varepsilon)_0 + (M[f_\varepsilon])_0} \right)^2 \\ &\quad + 2A^2 \mathbb{1}_{M[f_\varepsilon]\neq 0} ((M[f_\varepsilon])_0 - (f_\varepsilon)_0)^2, \end{aligned} \tag{2.46}$$

thus we get

$$\iiint_{M[f_\varepsilon]\neq 0} ((f_\varepsilon)_1 - (M[f_\varepsilon])_1)^2 \, dt dx d\xi \le \left(\frac{8FC_H^0}{K_2} + \frac{2A^2 F^{1-1/\lambda} C_H^0}{K_1} \right) \varepsilon. \tag{2.47}$$

Let us now treat the set $\{M[f_\varepsilon] = 0; f_\varepsilon \neq 0\}$. On this set, we have

$$\begin{aligned} &\left(H'(f_\varepsilon, \xi) - \left(\frac{\gamma\kappa}{\gamma-1} \rho_\varepsilon^{\gamma-1} - \frac{u_\varepsilon^2}{2}, u_\varepsilon\right) \right) \cdot (f_\varepsilon - M[f_\varepsilon]) \\ = \quad &\left[\frac{\theta}{1-\theta} \frac{\xi^2}{2} + \frac{u_\varepsilon^2}{2} - \frac{\gamma\kappa}{\gamma-1} \rho_\varepsilon^{\gamma-1} \right] (f_\varepsilon)_0 + \frac{\theta}{2c_\theta^{1/\lambda}} (f_\varepsilon)_0^{1+1/\lambda} \\ &+ \frac{1}{1-\theta} \frac{1}{2} \frac{(f_\varepsilon)_1^2}{(f_\varepsilon)_0} - \left(\frac{\theta}{1-\theta}\xi + u_\varepsilon\right) (f_\varepsilon)_1 \\ = \quad &\frac{1}{2(1-\theta)} \left[\frac{(f_\varepsilon)_1}{\sqrt{(f_\varepsilon)_0}} - (\theta\xi + (1-\theta)u_\varepsilon) \sqrt{(f_\varepsilon)_0} \right]^2 + \frac{\theta}{2c_\theta^{1/\lambda}} (f_\varepsilon)_0^{1+1/\lambda} \\ &+ \left[\frac{\theta}{2} (\xi - u_\varepsilon)^2 - \frac{\gamma\kappa}{\gamma-1} \rho_\varepsilon^{\gamma-1} \right] (f_\varepsilon)_0. \end{aligned} \tag{2.48}$$

On the set $\{M[f_\varepsilon] = 0; f_\varepsilon \neq 0\}$, we have $\rho_\varepsilon > 0$, and $\frac{\theta}{2}(\xi - u_\varepsilon)^2 - \frac{\gamma\kappa}{\gamma-1}\rho_\varepsilon^{\gamma-1}$ is nonnegative by definition of M, thus (2.48) and (2.38) give

$$\iiint_{M[f_\varepsilon]=0} (f_\varepsilon)_0^{1+1/\lambda} \, dt dx d\xi \le \frac{2c_\theta^{1/\lambda} C_H^0}{\theta} \varepsilon. \tag{2.49}$$

Now, since $\lambda \geq 1$, we have

$$\begin{aligned}
\iiint_{M[f_\varepsilon]=0} ((f_\varepsilon)_0 - (M[f_\varepsilon])_0)^2 \, dtdxd\xi = & \iiint_{M[f_\varepsilon]=0} (f_\varepsilon)_0^2 \, dtdxd\xi \\
\leq & \; F^{1-1/\lambda} \iiint_{M[f_\varepsilon]=0} (f_\varepsilon)_0^{1+1/\lambda} \, dtdxd\xi \\
\leq & \; \frac{2c_\theta^{1/\lambda} C_H^0 F^{1-1/\lambda}}{\theta} \varepsilon, \qquad (2.50)
\end{aligned}$$

which gives the bound on the first component. Furthermore,

$$\begin{aligned}
\mathbb{1}_{M[f_\varepsilon]=0}((f_\varepsilon)_1 - (M[f_\varepsilon])_1)^2 &= \mathbb{1}_{M[f_\varepsilon]=0}(f_\varepsilon)_1^2 \\
&\leq A^2 \mathbb{1}_{M[f_\varepsilon]=0}(f_\varepsilon)_0^2 \\
&= A^2 \mathbb{1}_{M[f_\varepsilon]=0}((f_\varepsilon)_0 - (M[f_\varepsilon])_0)^2, \quad (2.51)
\end{aligned}$$

which concludes the proof. □

3 Averaging

The regularity theory for averages of kinetic equations has been developed in [14], [13], [9], [22], [10], [18], [8]. We introduce here a result that is particularly adapted to BGK right-hand sides. In order to treat correctly the initial data, we use a Fourier transform in the variable x only, in the spirit of [9]. The dual variable is denoted by k.

Proposition 3.1 *Let $a \in L^\infty_{loc}(\mathbb{R}^M, \mathbb{R}^N)$ and $\psi \in L^2(\mathbb{R}^M)$, such that for some $K \geq 0$ and $0 < \alpha \leq 1$, one has*

$$\forall \sigma \in S^{N-1}, \forall z \in \mathbb{R}, \forall \eta > 0, \qquad \int_{z<a(\xi)\cdot\sigma<z+\eta} |\psi(\xi)|^2 \, d\xi \leq K\eta^\alpha. \tag{3.1}$$

Let $f \in C([0,T], L^2(\mathbb{R}^N_x \times \mathbb{R}^M_\xi))$ solve

$$\partial_t f + \mathrm{div}_x[a(\xi) f] = \frac{h-f}{\varepsilon} + g \qquad \text{in }]0,T[\times \mathbb{R}^N_x \times \mathbb{R}^M_\xi, \tag{3.2}$$

for some $g, h \in L^2(]0,T[\times \mathbb{R}^N_x \times \mathbb{R}^M_\xi)$ and initial data $f(0,.) = f^0$, and define

$$\rho_\psi(t,x) = \int_{\mathbb{R}^M} f(t,x,\xi)\psi(\xi) \, d\xi \in C([0,T], L^2(\mathbb{R}^N_x)). \tag{3.3}$$

Then, for a.e. k,

$$\left(\int_0^T |\widehat{\rho_\psi}(t,k)|^2 \, dt \right)^{1/2}$$

$$
\begin{aligned}
\leq\ & \frac{C\sqrt{K}}{|k|^{\alpha/2}}\left[\varepsilon^{\frac{1-\alpha}{2}}\left(\int|\widehat{f^0}|^2\,d\xi\right)^{1/2}+\left(\int|\widehat{f^0}|^2\,d\xi\right)^{\alpha/2}\left(\int_0^T\!\!\int|\widehat{h}|^2\,dtd\xi\right)^{\frac{1}{2}(1-\alpha)}\right.\\
&+\varepsilon^{1-\frac{\alpha}{2}}\left(\int_0^T\!\!\int|\widehat{g}|^2\,dtd\xi\right)^{1/2}+\left(\int_0^T\!\!\int|\widehat{h}|^2\,dtd\xi\right)^{\frac{1}{2}(1-\frac{\alpha}{2})}\left(\int_0^T\!\!\int|\widehat{g}|^2\,dtd\xi\right)^{\alpha/4}\\
&\left.+\frac{1}{\varepsilon^{\alpha/2}}\left(\int_0^T\!\!\int|\widehat{h}|^2\,dtd\xi\right)^{\frac{1}{2}(1-\frac{\alpha}{2})}\left(\int_0^T\!\!\int|\widehat{h}-\widehat{f}|^2\,dtd\xi\right)^{\alpha/4}\right].
\end{aligned}
\tag{3.4}
$$

Proof. Denote $\underline{g}=\frac{h-f}{\varepsilon}+g\in L^2(]0,T[\times\mathbb{R}^N_x\times\mathbb{R}^M_\xi)$. Then

$$
\partial_t f+\mathrm{div}_x[a(\xi)f]=\underline{g}\qquad\text{in }]0,T[\times\mathbb{R}^N_x\times\mathbb{R}^M_\xi,
\tag{3.5}
$$

and by estimate (3.23) in [9], the Fourier transform in x, $\widehat{\rho_\psi}(t,k)$ satisfies for a.e. k

$$
\begin{aligned}
\left(\int_0^T|\widehat{\rho_\psi}(t,k)|^2\,dt\right)^{1/2}\leq\frac{C\sqrt{K}}{|k|^{\alpha/2}}&\left[\frac{1}{\lambda^{\frac{1-\alpha}{2}}}\left(\int|\widehat{f^0}(k,\xi)|^2\,d\xi\right)^{1/2}\right.\\
&\left.+\frac{1}{\lambda^{1-\frac{\alpha}{2}}}\left(\int_0^T\!\!\int|\lambda\widehat{f}+\widehat{\underline{g}}|^2(t,k,\xi)\,dtd\xi\right)^{1/2}\right],
\end{aligned}
\tag{3.6}
$$

for any function $\lambda=\lambda(k)>0$. Since

$$
\lambda\widehat{f}+\widehat{\underline{g}}=\lambda\widehat{h}+(\frac{1}{\varepsilon}-\lambda)(\widehat{h}-\widehat{f})+\widehat{g},
\tag{3.7}
$$

we deduce that

$$
\begin{aligned}
&\left(\int_0^T|\widehat{\rho_\psi}(t,k)|^2\,dt\right)^{1/2}\\
\leq\ & \frac{C\sqrt{K}}{|k|^{\alpha/2}}\left[\frac{1}{\lambda^{\frac{1-\alpha}{2}}}\left(\int|\widehat{f^0}|^2\,d\xi\right)^{1/2}+\lambda^{\alpha/2}\left(\int_0^T\!\!\int|\widehat{h}|^2\,dtd\xi\right)^{1/2}\right.\\
&\left.+\frac{1}{\lambda^{1-\frac{\alpha}{2}}}\left(\int_0^T\!\!\int|\widehat{g}|^2\,dtd\xi\right)^{1/2}+\frac{\left|\frac{1}{\varepsilon}-\lambda\right|}{\lambda^{1-\frac{\alpha}{2}}}\left(\int_0^T\!\!\int|\widehat{h}-\widehat{f}|^2\,dtd\xi\right)^{1/2}\right].
\end{aligned}
\tag{3.8}
$$

By taking

$$
\lambda=\min\left(\frac{1}{\varepsilon},\frac{\frac{1}{\varepsilon}\left(\iint|\widehat{h}-\widehat{f}|^2\,dtd\xi\right)^{1/2}+\left(\iint|\widehat{g}|^2\,dtd\xi\right)^{1/2}}{\left(\iint|\widehat{h}|^2\,dtd\xi\right)^{1/2}}+\frac{\int|\widehat{f^0}|^2\,d\xi}{\iint|\widehat{h}|^2\,dtd\xi}\right),
\tag{3.9}
$$

we get the result. □

We can notice that in the important case $\alpha=1$, the last term of the second argument in the right-hand side of (3.9) can be omitted.

Corollary 3.2 *If in Proposition 3.1 we have* $h \in H^1_x(]0,T[\times\mathbb{R}^N_x \times \mathbb{R}^M_\xi)$ *and* $f^0 \in H^{1/2}_x \cap H^{1-\alpha/2}_x(\mathbb{R}^N_x \times \mathbb{R}^M_\xi)$, $g \in H^{1-\alpha/2}_x(]0,T[\times\mathbb{R}^N_x \times \mathbb{R}^M_\xi)$, *then* $\rho_\psi \in H^1_x(]0,T[\times\mathbb{R}^N_x)$, *with*

$$\|\rho_\psi\|_{\dot H^1_x} \leq C_N\sqrt{K}\Big(\varepsilon^{\frac{1-\alpha}{2}}\|f^0\|_{\dot H^{1-\alpha/2}_x} + \|f^0\|^{\alpha}_{\dot H^{1/2}_x}\|h\|^{1-\alpha}_{\dot H^1_x} + \varepsilon^{1-\alpha/2}\|g\|_{\dot H^{1-\alpha/2}_x} \\ + \|g\|^{\alpha/2}_{L^2}\|h\|^{1-\alpha/2}_{\dot H^1_x} + \|h\|^{1-\alpha/2}_{\dot H^1_x}\left\|\frac{h-f}{\varepsilon}\right\|^{\alpha/2}_{L^2}\Big). \tag{3.10}$$

4 Relaxation limit

This section is devoted to the proof of the a priori estimates that enable to apply compensated compactness to the approximate solutions ρ_ε, u_ε obtained in Section 2.

4.1 Estimate in H^1

We shall assume that vacuum does not appear. Since $\rho_\varepsilon \in L^1_x$, we cannot have a global lower bound, thus we assume a local lower bound,

$$\rho_\varepsilon(t,x) \geq \eta_{T,R} \qquad \text{for } 0 \leq t \leq T,\ -R < x < R, \tag{4.1}$$

for some $\eta_{T,R} > 0$ independent of ε.

Theorem 4.1 *We assume that* $\lambda \geq 1$, *that is to say* $1 < \gamma \leq 5/3$, *and that the solution* f_ε *to (2.1)-(2.10) with initial data* $f^0_\varepsilon \in H^{1/2}_x(\mathbb{R}_x \times \mathbb{R}_\xi)$ *such that (2.22)-(2.24) hold and*

$$\|f^0_\varepsilon\|_{H^{1/2}_x(\mathbb{R}_x\times\mathbb{R}_\xi)} \leq \frac{C}{\varepsilon^{1/2}}, \tag{4.2}$$

satisfies (2.35)-(2.36) and (4.1). We set $U_\varepsilon = (\rho_\varepsilon, \rho_\varepsilon u_\varepsilon) = \int f_\varepsilon\, d\xi$. *Then, there exists* $C_{T,R} > 0$, *independent of* ε, *such that for any* $0 < \varepsilon \leq \varepsilon_0$,

$$\iint\limits_{]0,T[\times]-R,R[} |\partial_x U_\varepsilon|^2\, dtdx \leq \frac{C_{T,R}}{\varepsilon}, \tag{4.3}$$

$$\iint\limits_{]0,T[\times]-R,R[} |\partial_t U_\varepsilon|^2\, dtdx \leq \frac{C_{T,R}}{\varepsilon}. \tag{4.4}$$

If moreover $f^0_\varepsilon \in H^1_x(\mathbb{R}_x \times \mathbb{R}_\xi)$ *with*

$$\|f^0_\varepsilon\|_{H^1_x(\mathbb{R}_x\times\mathbb{R}_\xi)} \leq \frac{C}{\varepsilon}, \tag{4.5}$$

then

$$\iiint\limits_{]0,T[\times]-R,R[\times\mathbb{R}} |\partial_x f_\varepsilon|^2\, dtdxd\xi \ + \iiint\limits_{]0,T[\times]-R,R[\times\mathbb{R}} |\partial_t f_\varepsilon|^2\, dtdxd\xi \leq \frac{C_{T,R}}{\varepsilon}. \tag{4.6}$$

Proof. Obviously, we can apply Proposition 3.1 to a vector function. We take $a(\xi) = \xi$ and $\psi \in C_c(\mathbb{R})$, $0 \leq \psi \leq 1$, such that $\psi = 1$ on the support in ξ of f_ε. Then (3.1) holds with $\alpha = 1$, $K = 1$, and we have $\rho_\psi = \int f_\varepsilon \psi(\xi)\, d\xi = U_\varepsilon$. Therefore we obtain with $h = M[f_\varepsilon]$ and $g = 0$

$$\|U_\varepsilon\|_{\dot{H}_x^{1/2}(]0,T[\times\mathbb{R})} \leq C\left[\|f_\varepsilon^0\|_{L^2_{x,\xi}} + \|M[f_\varepsilon]\|_{L^2_{t,x,\xi}}^{1/2}\left\|\frac{M[f_\varepsilon]-f_\varepsilon}{\varepsilon}\right\|_{L^2_{t,x,\xi}}^{1/2}\right]. \tag{4.7}$$

According to Theorem 2.5, we conclude that

$$\|U_\varepsilon\|_{\dot{H}_x^{1/2}(]0,T[\times\mathbb{R})} \leq C_T\, \varepsilon^{-1/4}. \tag{4.8}$$

Then, we estimate $M[f_\varepsilon] = M(U_\varepsilon, \xi)$. We localize in x by taking a function $\zeta_R(x) \in C_c^\infty$, $\zeta_R \geq 0$, such that $\zeta_R(x) = 1$ for $-R \leq x \leq R$. Since on the support of ζ_R, ρ_ε is lower bounded by (4.1), we have that there $M(.,\xi)$ is Lipschitz continuous, because $\lambda \geq 1$. Therefore, we deduce that

$$\|\zeta_R(x) M[f_\varepsilon]\|_{H_x^{1/2}(]0,T[\times\mathbb{R}_x\times\mathbb{R}_\xi)} \leq C_{T,R}\, \varepsilon^{-1/4}. \tag{4.9}$$

We then write the characteristics formula

$$f_\varepsilon(t,x,\xi) = f_\varepsilon^0(x-t\xi,\xi)e^{-t/\varepsilon} + \frac{1}{\varepsilon}\int_0^t e^{-s/\varepsilon} M[f_\varepsilon](t-s, x-s\xi, \xi)\, ds. \tag{4.10}$$

By multiplying by $\zeta_R(x)$ and by taking the $H_x^{1/2}$ norm, we obtain by using the translation invariance that for fixed t, ξ

$$\begin{aligned}
&\|\zeta_R(x) f_\varepsilon(t,x,\xi)\|_{H_x^{1/2}(\mathbb{R}_x)} \\
&\leq \|\zeta_R(x+t\xi) f_\varepsilon^0(x,\xi)\|_{H_x^{1/2}(\mathbb{R}_x)} e^{-t/\varepsilon} \\
&\quad + \frac{1}{\varepsilon}\int_0^t e^{-s/\varepsilon}\, \|\zeta_R(x+s\xi) M[f_\varepsilon](t-s,x,\xi)\|_{H_x^{1/2}(\mathbb{R}_x)}\, ds.
\end{aligned} \tag{4.11}$$

Since ξ remains in a bounded domain, we conclude with (4.9) that

$$\|\zeta_R(x) f_\varepsilon\|_{H_x^{1/2}(]0,T[\times\mathbb{R}_x\times\mathbb{R}_\xi)} \leq C_{T,R}\, \varepsilon^{-1/4}. \tag{4.12}$$

Next, we would like to use the estimate of Corollary 3.2, but we need to localize in x. We write

$$\partial_t(\zeta_R f_\varepsilon) + \xi \partial_x(\zeta_R f_\varepsilon) = \frac{\zeta_R M_\varepsilon - \zeta_R f_\varepsilon}{\varepsilon} + \xi \zeta_R' f_\varepsilon, \tag{4.13}$$

thus we can apply Corollary 3.2 to $f_\varepsilon^R = \zeta_R f_\varepsilon$ with $h = \zeta_R M_\varepsilon$ and $g = \xi \zeta_R' f_\varepsilon$. We obtain

$$\begin{aligned}
&\|\partial_x(\zeta_R U_\varepsilon)\|_{L^2_{t,x}} \\
&\leq C\Bigg[\|\zeta_R f_\varepsilon^0\|_{\dot{H}_x^{1/2}(\mathbb{R}_x\times\mathbb{R}_\xi)} + \varepsilon^{1/2}\|\xi\zeta_R' f_\varepsilon\|_{\dot{H}_x^{1/2}(]0,T[\times\mathbb{R}_x\times\mathbb{R}_\xi)} \\
&\qquad + \|\partial_x(\zeta_R M_\varepsilon)\|_{L^2_{tx\xi}}^{1/2}\left(\|\xi\zeta_R' f_\varepsilon\|_{L^2_{tx\xi}}^{1/2} + \left\|\zeta_R\frac{M_\varepsilon - f_\varepsilon}{\varepsilon}\right\|_{L^2_{tx\xi}}^{1/2}\right)\Bigg].
\end{aligned} \tag{4.14}$$

We notice that since $M[f_\varepsilon] = M(U_\varepsilon,\xi)$, we have $\partial_x M[f_\varepsilon] = M'(U_\varepsilon,\xi)\partial_x U_\varepsilon$. One can check that ρ_ε, u_ε being bounded, we have

$$\|M'(U_\varepsilon,\xi)\|_{L^2_\xi} \le C\rho_\varepsilon^{-\frac{3}{4}(\gamma-1)}, \tag{4.15}$$

this is true indeed for $1 < \gamma < 2$, and this is a bit more general than the Lipschitz argument used in (4.9). We deduce with (4.14), (4.12) and Theorem 2.5 that

$$\|\zeta_R \partial_x U_\varepsilon\|_{L^2_{t,x}} \le C_{T,R}\left[\varepsilon^{-1/2} + \varepsilon^{-1/4}\|\zeta_R\partial_x U_\varepsilon\|^{1/2}_{L^2_{t,x}}\right], \tag{4.16}$$

which gives (4.3). Then, we multiply (4.13) by ξ and apply again Corollary 3.2, which gives that $\|\zeta_R\partial_x \int \xi f_\varepsilon\, d\xi\|_{L^2_{tx}} \le C_{T,R}\varepsilon^{-1/2}$, and since $\partial_t U_\varepsilon + \partial_x \int \xi f_\varepsilon\, d\xi = 0$, we get (4.4). The last estimate (4.6) is finally obtained by replacing the $H^{1/2}$ norm by the H^1 norm in (4.11), and by expressing $\partial_t f_\varepsilon$ from the BGK equation with the estimate of Theorem 2.5. □

4.2 Entropy equations and compensated compactness

We conclude here the proof of Theorem 1.1, by establishing the strong convergence of U_ε. It is obtained by the method of DiPerna [12].

Proof of Theorem 1.1. We have from (2.29)

$$\partial_t U_\varepsilon + \partial_x \int_{\mathbb{R}} \xi f_\varepsilon\, d\xi = 0, \tag{4.17}$$

thus by (1.5)

$$\partial_t U_\varepsilon + \partial_x F(U_\varepsilon) = W_\varepsilon, \qquad \text{with } W_\varepsilon = \partial_x \int_{\mathbb{R}} \xi(M[f_\varepsilon] - f_\varepsilon)\, d\xi. \tag{4.18}$$

Take now a smooth entropy η, with entropy flux G. Recalling that by Theorem 4.1, $U_\varepsilon \in H^1_{tx}$ locally, we multiply (4.18) by $\eta'(U_\varepsilon)$, and get

$$\partial_t\, \eta(U_\varepsilon) + \partial_x\, G(U_\varepsilon) = \eta'(U_\varepsilon)\cdot W_\varepsilon. \tag{4.19}$$

Now

$$\begin{aligned}&\eta'(U_\varepsilon)\cdot W_\varepsilon\\ &= \partial_x\left[\eta'(U_\varepsilon)\cdot\int_{\mathbb{R}}\xi(M[f_\varepsilon]-f_\varepsilon)\, d\xi\right] - \partial_x\left[\eta'(U_\varepsilon)\right]\cdot\int_{\mathbb{R}}\xi(M[f_\varepsilon]-f_\varepsilon)\, d\xi,\end{aligned} \tag{4.20}$$

therefore we get

$$\partial_t\, \eta(\rho_\varepsilon,u_\varepsilon) + \partial_x\, G(\rho_\varepsilon,u_\varepsilon) = m_{\eta,\varepsilon} + R_{\eta,\varepsilon}, \tag{4.21}$$

with

$$m_{\eta,\varepsilon} = -\eta''(U_\varepsilon)\cdot\partial_x U_\varepsilon\cdot\int_{\mathbb{R}}\xi(M[f_\varepsilon]-f_\varepsilon)\, d\xi, \tag{4.22}$$

and

$$R_{\eta,\varepsilon} = \partial_x \left[\eta'(U_\varepsilon) \cdot \int_{\mathbb{R}} \xi(M[f_\varepsilon] - f_\varepsilon)\, d\xi \right]. \tag{4.23}$$

Using Theorem 2.5, we obtain

$$\eta'(U_\varepsilon) \cdot \int_{\mathbb{R}} \xi(M[f_\varepsilon] - f_\varepsilon)\, d\xi \to 0 \quad \text{in } L^2_{t,x}, \quad \text{as } \varepsilon \to 0, \tag{4.24}$$

and indeed also in L^p for $p < \infty$ since this is bounded in L^∞. Furthermore, by Theorems 4.1 and 2.5,

$$\begin{aligned} &\iint_{]0,T[\times]-R,R[} |\partial_x U_\varepsilon| \cdot \left| \int_{\mathbb{R}} \xi(M[f_\varepsilon] - f_\varepsilon)\, d\xi \right| dt dx \\ &\le C \|\partial_x U_\varepsilon\|_{L^2(]0,T[\times]-R,R[)} \|M[f_\varepsilon] - f_\varepsilon\|_{L^2_{t,x,\xi}} \le C_{T,R}. \end{aligned} \tag{4.25}$$

which yields that $m_{\eta,\varepsilon}$ is bounded in $\mathcal{M}_{loc}$ uniformly with respect to ε. We obtain therefore that $\partial_t\, \eta(\rho_\varepsilon, u_\varepsilon) + \partial_x\, G(\rho_\varepsilon, u_\varepsilon)$ is compact in H^{-1}_{loc}. We are now able to apply compensated compactness, and the result of [20] allows to conclude the compactness of the sequence $(\rho_\varepsilon, \rho_\varepsilon u_\varepsilon)$. By passing to the limit in (4.18), we obtain (1.2) together with the initial data. Finally, it remains to establish an inequality for the physical entropy. We use a differential form of (2.30), that is fully justified in [3],

$$\partial_t \int H(f_\varepsilon, \xi)\, d\xi + \partial_x \int \xi H(f_\varepsilon, \xi)\, d\xi \le 0, \tag{4.26}$$

which gives with (2.17), (2.18) that

$$\begin{aligned} &\partial_t\, \eta^e(\rho_\varepsilon, u_\varepsilon) + \partial_x\, G^e(\rho_\varepsilon, u_\varepsilon) \\ &\le \partial_t \int \left(H(M[f_\varepsilon], \xi) - H(f_\varepsilon, \xi)\right) d\xi + \partial_x \int \xi \left(H(M[f_\varepsilon], \xi) - H(f_\varepsilon, \xi)\right) d\xi \\ &\longrightarrow 0 \quad \text{in } W^{-1,p}_{loc}, \end{aligned} \tag{4.27}$$

and the desired result. $\square$

Remark. The above proof shows that the rate $\|M[f_\varepsilon] - f_\varepsilon\|_{L^2} \le C\varepsilon^{1/2}$ cannot be improved, otherwise we would get in the relaxation limit a solution that satisfies entropy equalities, which is not possible in general. The rates of Theorem 4.1 are therefore also sharp, since they say that all the terms of the BGK equation are of the same order in terms of ε. However, for other norms than L^2, the rate could be improved. For example, one can have $\|M[f_\varepsilon] - f_\varepsilon\|_{L^1} \le C\varepsilon$, this should be somehow related to a BV bound.

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