ONE SIDED ESTIMATES AND UNIQUENESS FOR HYPERBOLIC SYSTEMS OF BALANCE LAWS

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ABSTRACT. Uniqueness of solutions of genuinely nonlinear $n \times n$ strictly hyperbolic systems of balance laws is established moving from a refined version of the Oleĭnik-type decay estimates found in [9]. As middle step, the result relies on the fulfillment of a condition which controls the local oscillation of the solution in a forward neighborhood of each point in the t-x plane.

1. INTRODUCTION

The *semigroup approach* introduced by Bressan in [2] and then developed in [3] provides a general method to obtain a well-posedness theory for hyperbolic systems of conservation laws of the form

$$u_t + f(u)_x = 0, \qquad x \in \mathbb{R}, t > 0.$$

Namely, in order to prove the uniqueness of the solutions to a given Cauchy problem with initial data u_o suitably small, it suffices to show that every entropy weak solution $u = u(t,x)$, that satisfies some mild assumption either on the local boundedness of the total variation or on the oscillation, actually coincides with the corresponding (unique) trajectory of the semigroup S:

$$u(t,\cdot) = S_t u_o \qquad \forall\, t \geq 0\,. \tag{1.1}$$

The first regularity condition which implies the identity (1.1) was introduced by Bressan and LeFloch in [6]. This condition, called *Tame Variation*, controls the total variation of an entropy weak solution u along space-like segments in the t-x plane, and is clearly satisfied by solutions constructed by the Glimm scheme or by a front tracking algorithm.

In [4] a weaker regularity assumption is introduced, the so-called *Tame Oscillation*, which controls the local oscillation of u in a forward neighborhood of each point in the t-x plane, and is sufficient to guarantee uniqueness of entropy weak solutions. In turn, this result yields the uniqueness of weak solutions which satisfy a decay estimate on positive waves for genuinely nonlinear systems, thus extending a classical result proved by Oleĭnik [14] in the scalar case. In fact, this decay property implies both the entropy admissibility condition and the Tame Oscillation

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condition. From the point of view of practical applications, such uniqueness result via one-sided estimates is very useful for instance in the context of multiphase Geometric Optics computations [10].

An alternative regularity assumption sufficient to guarantee uniqueness has been given by Bressan and Lewicka in [7]. This condition requires that the solution $u = u(t, x)$ has bounded variation along a suitable family of space-like curves. In all cases, the uniqueness is established within the same class of functions where an existence theorem can be proved.

Aim of the present paper is to establish a uniqueness theorem valid for **BV** solutions of the following inhomogeneous $n \times n$ system of equations

$$u_t + f(u)_x = g(x, u), \qquad x \in I\!R,\ t > 0, \tag{1.2}$$

endowed with a (suitably small) initial data $u_o \in \mathbf{L}^1 \cap \mathbf{BV}(I\!R; I\!R^n)$. Here $u(t, x) \in I\!R^n$ is the unknown function and $f : \Omega \to I\!R^n$ is a smooth $\mathcal{C}^2$ vector field defined on an open neighborhood Ω of the origin in $I\!R^n$. We will assume that the system (1.2) is strictly hyperbolic, with each characteristic field either genuinely nonlinear or linearly degenerate in the sense of Lax [12]. Moreover we assume the following Caratheodory-type conditions for the source term g:

(P1) $g : I\!R \times \Omega \to I\!R^n$ is measurable w.r.t. x, for any $u \in \Omega$, and is $\mathcal{C}^2$ w.r.t. u, for any $x \in I\!R$;
(P2) $\|g(x, \cdot)\|_{\mathcal{C}^2}$ is bounded over Ω, uniformly in $x \in I\!R$;
(P3) there exists a function $\omega \in \mathbf{L}^1(I\!R) \cap \mathbf{L}^\infty(I\!R)$ such that $|g(x, u)| \leq \omega(x)$, $\|\nabla_u g(x, u)\| \leq \omega(x)$ for all $(x, u) \in I\!R \times \Omega$.

In addition, we require that a *non-resonance* condition holds, that is, the characteristic speeds are bounded away from zero: for some $p \in \{1, \ldots, n\}$ and some $c > 0$ one has

$$\begin{cases} \lambda_i(u) \leq -c & \text{if } i \leq p, \\ \lambda_i(u) \geq c & \text{if } i > p, \end{cases} \tag{1.3}$$

for all $u \in \Omega$, where $\lambda_i(u)$ denote the eigenvalues of the Jacobian matrix $Df(u)$.

Under these assumptions, it was proved in [1] that there exists a family of entropy weak solutions to (1.2) continuously depending on the initial data. More precisely, if the $\mathbf{L}^1$-norm of ω is sufficiently small, there exist a closed domain $\mathcal{D} \subset \mathbf{L}^1(I\!R; I\!R^n)$ of functions with sufficiently small total variation, a constant L and a unique semigroup $P : [0, +\infty) \times \mathcal{D} \to \mathcal{D}$ with the properties:

(i) For all $u, v \in \mathcal{D}$ and $t, s \geq 0$ one has $\|P_s u - P_t v\|_{\mathbf{L}^1} \leq L\big(|t-s| + \|u-v\|_{\mathbf{L}^1}\big)$.
(ii) For all $u_o \in \mathcal{D}$ the function $u(t, \cdot) = P_t u_o$ is a weak entropy solution of the Cauchy problem (1.2), $u(t = 0, .) = u_o$.

The characterization of the trajectories of the (unique) semigroup P given in [1, Theorem 8] provides a tool for proving that a weak entropy solution u to (1.2) is unique, that is, it suffices to show that it coincides with the corresponding semigroup trajectory:

$$u(t, \cdot) = P_t u_o \qquad \text{for all} \quad t \geq 0. \tag{1.4}$$

As in the homogeneous case [6, 4], the identity (1.4) will be established by means of the error estimate [1, Lemma 11 and Remark 5]

$$\|u(t) - P_t u(0)\|_{\mathbf{L}^1} \leq L \cdot \int_0^t \left\{ \liminf_{\eta \to 0} \frac{\|u(s+\eta) - P_\eta u(s)\|_{\mathbf{L}^1}}{\eta} \right\} d\eta \,, \tag{1.5}$$

valid for every Lipschitz continuous map $u : [0,T] \mapsto \mathcal{D}$. By showing that the integrand on the right hand side of (1.5) vanishes for almost every time t, one can thus establish (1.4), and hence uniqueness of the entropy weak solution. In order to control the right hand side of (1.4), as in [4], we require a control on the oscillation of u on a forward neighborhood of each given point in the t-x plane.

For clarity, a complete set of assumptions is listed below.

(A1) **(Balance Equations)** The function $u = u(t,x)$ is a weak solution of the Cauchy problem for (1.2), taking values within the domain $\mathcal{D}$ of a semigroup P. More precisely, $u : [0,T] \mapsto \mathcal{D}$ is continuous w.r.t. the $\mathbf{L}^1$ distance and satisfies (1.2) in the sense of distributions, i.e.

$$\int\!\!\int \{u\phi_t + f(u)\phi_x + g(x,u)\phi\}\, dx\, dt = 0 \tag{1.6}$$

for every function $\phi \in \mathcal{C}^1$ with compact support contained inside the open strip $]0,T[\times I\!R$.

(A2) **(Lax Entropy Condition)** Let u have an approximate jump discontinuity at some point $(\tau,\xi) \in]0,T[\times I\!R$. More precisely, let there exist states $u^-, u^+ \in \Omega$ and a speed $\lambda \in I\!R$ such that, calling

$$U(t,x) \doteq \begin{cases} u^- & \text{if } x < \xi + \lambda(t-\tau), \\ u^+ & \text{if } x > \xi + \lambda(t-\tau), \end{cases} \tag{1.7}$$

there holds

$$\lim_{\rho \to 0+} \frac{1}{\rho^2} \int_{\tau-\rho}^{\tau+\rho} \int_{\xi-\rho}^{\xi+\rho} |u(t,x) - U(t,x)|\, dx\, dt = 0\,. \tag{1.8}$$

Then, for some $i \in \{1,\dots,n\}$, one has the entropy inequality:

$$\lambda_i(u^-) \geq \lambda \geq \lambda_i(u^+).$$

(A3) **(Tame Oscillation Condition)** There exist constants $C, \delta > 0$ such that the following holds. For every $t > 0$ and every $a < b$, the oscillation of u over the triangle

$$\Delta \doteq \{(s,y) : s \geq t,\ a + (s-t)/\delta < y < b - (s-t)/\delta\}$$

satisfies

$$\text{Osc.}\{u;\Delta\} \leq C \cdot \left(\text{Tot.Var.}\{u(t,\cdot);\ ]a,b[\,\} + \|\omega\|_{\mathbf{L}^1([a,b])}\right). \tag{1.9}$$

Above, the oscillation of a function u on a set Δ is defined as

$$\text{Osc.}\{u;\Delta\} \doteq \sup_{(s,y),(s',y')\in\Delta} |u(s,y) - u(s',y')|.$$

Observe that in (1.9) we are taking into account the possible contribution of g. In fact, in contrast with the homogeneous case, oscillations may appear even if $u(t,\cdot)$ is constant on $]a,b[$, due to the presence of the source term (consider the h-Riemann solver [1] with initial states $u_L = u_R$).

Now we are ready to state the uniqueness theorem.

Theorem 1.1. (Uniqueness) *Let $P : [0,+\infty)\times\mathcal{D} \to \mathcal{D}$ be the semigroup generated by* (1.2). *Let $u : [0,T] \mapsto \mathcal{D}$ be continuous as a map with values into $\mathbf{L}^1$. Then, for every $u_o \in \mathcal{D}$, $T > 0$, the Cauchy problem for* (1.2) *has a unique weak solution $u : [0,T] \mapsto \mathcal{D}$ satisfying assumptions* $(A1)$, $(A2)$ *and* $(A3)$. *Indeed, these conditions imply the identity* (1.4).

In the second part of the paper we show that conditions $(A2)$ and $(A3)$ can be recovered, in the case of genuinely nonlinear systems, from an Oleĭnik-type estimate on the decay of positive waves, providing an additional uniqueness theorem and thus generalizing to inhomogeneous systems of the form (1.2) the results available for homogeneous conservation laws [4].

As usual, let $A(u) = Df(u)$ be the Jacobian matrix of f, and call $\lambda_i(u)$, $l_i(u)$, $r_i(u)$ respectively the eigenvalues and the left and right eigenvectors of $A(u)$. Let $u : I\!\!R \to \Omega$ have bounded variation in x and satisfy (1.2). The distributional derivative $\mu \doteq D_x u$ is a vector measure. For $i = 1, \ldots, n$ we can now define μ^i as

$$\int \phi \, d\mu^i = \int \phi \, \tilde{l}_i \cdot D_x u, \qquad \phi \in \mathcal{C}^0_c,$$

where $\tilde{l}_i(x) = l_i(u(x))$ at points where u is continuous, while $\tilde{l}_i(x_\alpha)$ is some vector which satisfies

$$|\tilde{l}_i(x_\alpha) - l_i(u(x_\alpha))| = \mathcal{O}(1) \cdot |u(x_\alpha+) - u(x_\alpha-)|$$

$$\tilde{l}_i(x_\alpha) \cdot \big(u(x_\alpha+) - u(x_\alpha-)\big) = \sigma^i_\alpha,$$

where with σ^i_α we denote the strength of the i-th wave generated by the resolution of the corresponding discontinuity in x_α.

We denote by μ^{i+}, μ^{i-} the positive and negative parts of μ^i, then we have

$$\mu^i = \mu^{i+} - \mu^{i-}, \qquad |\mu^i| = \mu^{i+} + \mu^{i-}.$$

Decay estimates for (1.2) were proved in [9]. In this paper (see Section 4) we present a slightly modified estimate, in order to fit the following

(A4) (Decay Assumption) There exist a constant $K > 0$ and a non-increasing function Z such that

$$\mu^{i+}_t([a,b]) \leq K \cdot \frac{b-a}{t-s} + [Z(s) - Z(t)] + K\,(b-a)\,. \tag{1.10}$$

for every interval $[a,b]$ and all $i = 1, \ldots, n$, $t > s \geq 0$.

Theorem 1.2. *Assume that each characteristic field of system* (1.2) *is genuinely nonlinear and let $P : [0,+\infty) \times \mathcal{D} \to \mathcal{D}$ be the associated semigroup. Then, for every $u_o \in \mathcal{D}$, $T > 0$, the Cauchy problem for* (1.2) *has a unique weak solution $u : [0,T] \mapsto \mathcal{D}$ satisfying assumptions* $(A1)$ *and* $(A4)$ *for some constant K and some non-increasing function Z. Indeed, these conditions imply the identity* (1.4).

The paper is organized as follows. In Section 2 we recall some preliminary results that will be used in the subsequent proof of Theorem 1.1, which is presented in Section 3. Since the Oleĭnik type estimate proved in [9] does not fit the Decay Assumption (A4), in Section 4 we present a refined version which has the same behavior of the OSLC in (1.10). The proof of Theorem 1.2 is then sketched at the end of the section.

2. Some preliminary results

In this section we collect some technical lemmas, which will be used to establish the main result.

Lemma 2.1. *Let $u : [0,T] \times \mathbb{R} \to \mathbb{R}^n$ be a function which satisfies the Balance Equations $(A1)$. In particular Tot.Var.$u(t,\cdot) \le M$ for all $t \in [0,T]$.*

Then the map $t \mapsto u(t,\cdot)$ is Lipschitz continuous, i.e.

$$\|u(t,\cdot) - u(s,\cdot)\|_{\mathbf{L}^1} \le L' \, |t-s| \tag{2.1}$$

for some constant L' and all $t, s \in [0,T]$.

Proof. Fix any $t > s > 0$ and consider the smooth approximation of the characteristic function of the interval $[s,t]$ defined as follows. Given any smooth nondecreasing function $\alpha : \mathbb{R} \to [0,1]$ such that

$$\alpha(x) = \begin{cases} 0 & \text{if } x \le -1, \\ 1 & \text{if } x \ge 1, \end{cases}$$

define $\alpha_h(x) \doteq \alpha(x/h)$, so that α_h converges (point-wise a.e.) to the Heaviside function as $h \to 0+$. Consider any smooth function $\psi = \psi(x)$ with compact support, and define

$$\phi_h(\tau, x) = [\alpha_h(\tau - s) - \alpha_h(\tau - t)]\psi(x).$$

Using ψ_h in (1.6) and letting $h \to 0+$, thanks to the $\mathbf{L}^1$-continuity of the function $t \mapsto u(t,\cdot)$ we obtain

$$\int \psi(x)[u(t,x) - u(s,x)]\, dx + \int_s^t \int \psi_x(x) f(u)\, dx\, d\tau + \int_s^t \int \psi(x) g(x,u)\, dx\, d\tau = 0\,.$$

By assumption, $u(t,\cdot) \in \mathcal{D}$ for all t. As a consequence, one has the uniform bounds

$$\text{Tot.Var.}\{u(t,\cdot)\} \le M, \qquad |u(t,\cdot)| \le M, \tag{2.2}$$

for some constant M. It thus follow

$$\begin{aligned} \|u(t,\cdot) - u(s,\cdot)\|_{\mathbf{L}^1} &= \sup_{\psi \in \mathcal{C}^1_c,\, |\psi| \le 1} \int \psi(x)[u(t,x) - u(s,x)]\, dx \\ &\le \int_s^t \text{Tot.Var.}\{f(u(\tau,\cdot))\}\, d\tau + \int_s^t \int |g(x,u)|\, dx\, d\tau \\ &\le \big(M \cdot Lip(f) + \|\omega\|_{\mathbf{L}^1}\big) \cdot |t-s|\,, \end{aligned}$$

where by $Lip(f)$ we denote the Lipschitz constant of the function f on the domain $|u| \le M$. □

Lemma 2.2. *Let $w :]a,b[\mapsto \mathbb{R}^n$ be an integrable function such that, for some measure μ, one has*

$$\left| \int_{\xi_1}^{\xi_2} w(x)\, dx \right| \le \mu([\xi_1, \xi_2])\,, \qquad \textit{for } a < \xi_1 < \xi_2 < b.$$

Then

$$\int_a^b |w(x)|\ dx \le \mu(]a,b[).$$

Lemma 2.3. *Let $u : [0,T] \mapsto \mathcal{D}$ be Lipschitz continuous. At a given point (τ, ξ) let the conditions* (1.7) *and* (1.8) *hold, for some u^-, $u^+ \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$. Then for each $\hat{\lambda} > 0$ one has*

$$\lim_{\rho \to 0+} \sup_{|h| \leq \rho} \frac{1}{\rho} \int_{\xi - \hat{\lambda}\rho}^{\xi + \hat{\lambda}\rho} |u(\tau + h, x) - U(\tau + h, x)| \, dx = 0\,. \tag{2.3}$$

For a proof of the above statements see [4, 3]. We remark here that the states u^-, u^+ in (1.7) – (1.8) satisfy

$$u^- = \lim_{x \to \xi-} u(\tau, x), \qquad u^+ = \lim_{x \to \xi+} u(\tau, x)$$

Lemma 2.4. *Let u be a weak solution of* (1.2). *Let the conditions* (1.7) *and* (1.8) *hold at some point (τ, ξ). Then the states u^-, u^+ and the speed λ satisfy the Rankine-Hugoniot equations*

$$\lambda(u^+ - u^-) = f(u^+) - f(u^-).$$

We conclude this section by recalling two local integral estimates that characterize the trajectories of the semigroup P. We begin by introducing some notations: given a function $u = u(t, x)$ and a point (τ, ξ), we denote by $U^\sharp_{(u;\tau,\xi)}$ the solution of the homogeneous Riemann problem centered in ξ with data

$$u_L = \lim_{x \to \xi-} u(\tau, \xi)\,, \qquad u_R = \lim_{x \to \xi+} u(\tau, \xi)\,.$$

Moreover we define $U^\flat_{(u;\tau,\xi)}$ as the solution of the linear hyperbolic Cauchy problem with constant coefficients

$$w_t + \tilde{A} w_x = \tilde{g}(x)\,, \qquad w(\tau, x) = u(\tau, x)\,, \tag{2.4}$$

with $\tilde{A} = Df(u(\tau, \xi))$, $\tilde{g}(x) = g(x, u(\tau, \xi))$.

We will rely on the following fundamental result [1, Theorem 8].

Lemma 2.5. *Let $P : [0, +\infty) \times \mathcal{D} \to \mathcal{D}$ be the semigroup generated by* (1.2) *and let $\hat{\lambda}$ be an upper bound for all wave speeds. Then every trajectory $u(t, \cdot) = P_t u_o$, $u_o \in \mathcal{D}$, satisfies the following conditions at every $\tau \geq 0$.*

(i) *For every ξ one has*

$$\lim_{\theta \to 0} \frac{1}{\theta} \int_{\xi - \theta\hat{\lambda}}^{\xi + \theta\hat{\lambda}} \left| u(\tau + \theta, x) - U^\sharp_{(u;\tau,\xi)}(\theta, x) \right| dx = 0. \tag{2.5}$$

(ii) *There exists a constant C such that, for every $a < \xi < b$ and $0 < \theta < \frac{b-a}{2\hat{\lambda}}$ one has*

$$\frac{1}{\theta} \int_{a + \theta\hat{\lambda}}^{b - \theta\hat{\lambda}} \left| u(\tau + \theta, x) - U^\flat_{(u;\tau,\xi)}(\tau + \theta, x) \right| dx \leq C \big[\textit{Tot.Var.}\{u(\tau);]a, b[\,\} + (b - a) \big]^2. \tag{2.6}$$

Vice-versa let $u : [0, T] \to \mathcal{D}$ be Lipschitz continuous as a map with values in $\mathbf{L}^1(\mathbb{R}, \mathbb{R}^n)$ and assume that the conditions (i), (ii) hold at almost every time τ. Then $u(t, \cdot)$ coincides with the corresponding trajectory of the semigroup P.

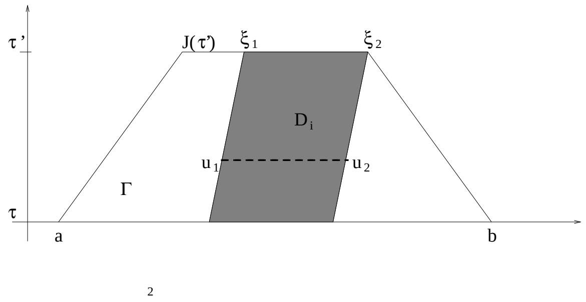

FIGURE 1

3. Proof of Theorem 1.1

Let $\hat{\lambda} > 0$ be an upper bound for all wave speeds. Given any function u satisfying the hypotheses of Theorem 1.1, in order to prove (1.4) it is sufficient to show that u satisfies conditions (i) and (ii) in Lemma 2.5.

By Lemma 2.1 and the assumption $u(t,\cdot) \in \mathcal{D}$, we know that $u = u(t,x)$ is a **BV** function on the strip $[0,T]\times I\!R$ (for a detailed proof, see [3, Theorem 2.6]). Moreover, there exists a set $\mathcal{N} \subset]0,T[$ of measure zero such that, for every $(\tau,\xi) \in]0,T[\times I\!R$ with $\tau \notin \mathcal{N}$, either u is approximately continuous, or it has an approximate jump discontinuity, in the sense that (1.7)-(1.8) hold for the states $u^- \doteq \lim_{x\to\xi-} u(\tau,x)$, $u^+ \doteq \lim_{x\to\xi+} u(\tau,x)$ and some $\lambda \in I\!R$.

At this stage, (2.5) follows exactly as in [4, 3] and we will not repeat the proof here.

In order to prove (2.6), for any given point (τ,ξ), call $U^\flat = U^\flat_{(u;\tau,\xi)}$ the solution of the linear Cauchy problem (2.4). Consider any open interval $]a,b[$ containing the point ξ, define the open intervals

$$J(t) \doteq \,]a + \hat{\lambda}(t-\tau), b - \hat{\lambda}(t-\tau)[\,,$$

for $t \geq \tau$, and consider the region (Fig. 1)

$$\Gamma \doteq \{(t,x);\ 0 \leq t \in [\tau,\tau'],\ x \in J(t)\}\,.$$

With the above notation we claim that, for every $\tau' \geq \tau$ with $\tau' - \tau < (b-a)/2\hat{\lambda}$, one has

$$\begin{aligned}\int_{J(\tau')} |u(\tau',x) - U^\flat(\tau',x)|\,dx \;&=\; \mathcal{O}(1)\cdot\left(\sup_{(t,x)\in\Gamma} |u(t,x) - u(\tau,\xi)|\right)\\ &\quad\cdot\int_\tau^{\tau'} \left[\text{Tot.Var.}\{u(t,\cdot);\ J(t)\} + \|\omega\|_{\mathbf{L}^1(J(t))}\right]\,dt.\end{aligned}\tag{3.1}$$

To prove (3.1), call $\tilde{\lambda}_i$, $\tilde{l}_i$, $\tilde{r}_i$ respectively the i-th eigenvalues and left and right eigenvectors of the matrix $\tilde{A} \doteq Df(u(\tau,\xi))$. By (2.4) one has

$$\begin{aligned}\tilde{l}_i\cdot U^\flat(\tau',x) \;&=\; \tilde{l}_i\cdot U^\flat(\tau, x - \tilde{\lambda}_i(\tau'-\tau)) + \int_\tau^{\tau'} \tilde{l}_i\cdot\tilde{g}(x - \tilde{\lambda}_i(s-\tau))\,ds\\ &= \tilde{l}_i\cdot u(\tau, x - \tilde{\lambda}_i(\tau'-\tau)) + \int_\tau^{\tau'} \tilde{l}_i\cdot\tilde{g}(x - \tilde{\lambda}_i(s-\tau))\,ds\end{aligned}$$

Now fix any $\xi_1, \xi_2 \in J(\tau')$ and consider the quantity

$$
\begin{aligned}
E_i(\xi_1,\xi_2) &\doteq \tilde{l}_i \cdot \int_{\xi_1}^{\xi_2} \big(u(\tau',x) - U^\flat(\tau',x)\big)\,dx \\
&= \tilde{l}_i \cdot \int_{\xi_1}^{\xi_2} \big(u(\tau',x) - u(\tau, x - \tilde{\lambda}_i(\tau'-\tau))\big)\,dx \\
&\quad -\tilde{l}_i \cdot \int_{\tau}^{\tau'}\int_{\xi_1}^{\xi_2} \tilde{g}(x - \tilde{\lambda}_i(\tau'-t))\,dx\,dt
\end{aligned}
\tag{3.2}
$$

We can now apply the divergence theorem to the vector $(u, f(u))$ on the domain (Fig. 1)

$$
D_i \doteq \{(t,x);\ t\in[\tau,\tau'],\ \xi_1 + \tilde{\lambda}_i(t-\tau') \le x \le \xi_2 + \tilde{\lambda}_i(t-\tau')\}.
$$

Since u satisfies the balance equation (1.2), the difference between the integral of u at the top and at the bottom of the domain D_i is measured by the inflow from the left side minus the outflow from the right side of D_i, plus the integral of the source term on the domain. From (3.2) it thus follows that

$$
\begin{aligned}
E_i(\xi_1,\xi_2) &= \int_{\tau}^{\tau'} \tilde{l}_i \cdot \big((f(u) - \tilde{\lambda}_i u)(t, \xi_1 + \tilde{\lambda}_i(t-\tau'))\big)\,dt \\
&\quad - \int_{\tau}^{\tau'} \tilde{l}_i \cdot \big((f(u) - \tilde{\lambda}_i u)(t, \xi_2 + \tilde{\lambda}_i(t-\tau'))\big)\,dt \\
&\quad +\tilde{l}_i \cdot \int_{\tau}^{\tau'}\int_{\xi_1}^{\xi_2} g(x - \tilde{\lambda}_i(\tau'-t), u(t, x - \tilde{\lambda}_i(\tau'-t)))\,dx\,dt \\
&\quad -\tilde{l}_i \cdot \int_{\tau}^{\tau'}\int_{\xi_1}^{\xi_2} \tilde{g}(x - \tilde{\lambda}_i(\tau'-t))\,dx\,dt
\end{aligned}
\tag{3.3}
$$

To estimate the quantity in (3.3), consider the states

$$
u_1(t) \doteq u(t, \xi_1 + \tilde{\lambda}_i(t-\tau')), \qquad u_2(t) \doteq u(t, \xi_2 + \tilde{\lambda}_i(t-\tau')), \qquad \tilde{u} \doteq u(\tau,\xi).
$$

We then have

$$
\begin{aligned}
\tilde{l}_i \cdot [f(u_2) - f(u_1) - \tilde{\lambda}_i(u_2-u_1)] &= \tilde{l}_i \cdot [Df(\tilde{u})\cdot(u_2-u_1) - \tilde{\lambda}_i(u_2-u_1)] \\
&\quad +\tilde{l}_i \cdot A^* \cdot (u_2-u_1),
\end{aligned}
\tag{3.4}
$$

where A^* is defined by

$$
A^* \doteq \int_0^1 [Df(su_2 + (1-s)u_1) - Df(\tilde{u})]\,ds.
$$

Since the first term on the right hand side of (3.4) vanishes, we thus obtain

$$
\begin{aligned}
&\big|\tilde{l}_i \cdot [f(u_2) - f(u_1) - \tilde{\lambda}_i(u_2-u_1)]\big| \\
&\quad = \mathcal{O}(1)\cdot|u_2-u_1|\cdot\big(|u_2-\tilde{u}| + |u_1-\tilde{u}|\big) \\
&\quad = \mathcal{O}(1)\cdot \text{Tot.Var.}\{u(t,\cdot);\ [\xi_1 + \tilde{\lambda}_i(t-\tau'),\ \xi_2 + \tilde{\lambda}_i(t-\tau')]\} \\
&\qquad \cdot \sup_{(t,x)\in\Gamma} |u(t,x) - u(\tau,\xi)|.
\end{aligned}
\tag{3.5}
$$

Consider next the last two terms in the right hand side of (3.3). Call $x(t) \doteq x - \tilde{\lambda}_i(\tau' - t)$ and compute

$$
\begin{aligned}
\left|\tilde{l}_i \cdot [g(x(t), u(t, x(t))) - g(x(t), \tilde{u})]\right| &= \mathcal{O}(1) \cdot \sup_u \left|\nabla_u g(x(t), u)\right| \cdot |u(t, x(t)) - \tilde{u}| \\
&= \mathcal{O}(1) \cdot \omega(x(t)) \cdot \sup_{(t,x)\in\Gamma} |u(t, x) - u(\tau, \xi)|.
\end{aligned} \tag{3.6}
$$

Using (3.5) and (3.6) in (3.3) we obtain the estimate

$$
\begin{aligned}
|E_i(\xi_1, \xi_2)| \leq C_i \cdot &\sup_{(t,x)\in\Gamma} |u(t, x) - u(\tau, \xi)| \\
&\cdot \int_\tau^{\tau'} \Big(\text{Tot.Var.}\{u(t, \cdot);\ [\xi_1 + \tilde{\lambda}_i(t - \tau'),\ \xi_2 + \tilde{\lambda}_i(t - \tau')]\} \\
&\qquad + \|\omega\|_{\mathbf{L}^1([\xi_1+\tilde{\lambda}_i(t-\tau'),\ \xi_2+\tilde{\lambda}_i(t-\tau')])}\Big)\, dt
\end{aligned}
$$

for some constant C_i. Define the measures μ_i on the interval $J(\tau')$ by setting

$$
\begin{aligned}
\mu_i([a, b]) \doteq C_i' \cdot \int_\tau^{\tau'} &\Big(\text{Tot.Var.}\{u(t, \cdot);\ [a + \tilde{\lambda}_i(t - \tau'),\ b + \tilde{\lambda}_i(t - \tau')]\} \\
&+ \|\omega\|_{\mathbf{L}^1([a+\tilde{\lambda}_i(t-\tau'),\ b+\tilde{\lambda}_i(t-\tau')])}\Big)\, dt,
\end{aligned}
$$

where we have set

$$
C_i' \doteq C_i \cdot \sup_{(t,x)\in\Gamma} |u(t, x) - u(\tau, \xi)|.
$$

By Lemma 2.2 we thus have

$$
\begin{aligned}
\int_{J(\tau')} &\left|\tilde{l}_i \cdot \big(u(\tau', x) - U^\flat(\tau', x)\big)\right| dx \\
&\leq \mu_i(J(\tau')) \\
&= C_i \cdot \sup_{(t,x)\in\Gamma} |u(t, x) - u(\tau, \xi)| \cdot \int_\tau^{\tau'} \Big(\text{Tot.Var.}\{u(t, \cdot);\ J(t)\} + \|\omega\|_{\mathbf{L}^1(J(t))}\Big)\, dt
\end{aligned}
$$

for all $i = 1, \ldots, n$ and thus (3.1) is proved.

Finally, by the Tame Oscillation Condition (1.9) there holds

$$
\sup_{(t,x)\in\Gamma} |u(t, x) - u(\tau, \xi)| \leq C \cdot \big(\text{Tot.Var.}\{u(t, \cdot);\ ]a, b[\,\} + \|\omega\|_{\mathbf{L}^1([a,b])}\big)
$$

and (2.6) is thus verified.

4. A localized version of OSLC estimates

In this section we briefly recall and modify the procedure applied in [9] in order to obtain one-sided Lipschitz estimates that fit the Decay Assumption (A4). For the definition of the h-Riemann solver and the construction of wave-front tracking approximations we refer to [1].

The *total strength of waves* in u is defined as

$$
\mathbf{V}(u) \doteq \sum_{i=1}^n \mathbf{V}_i(u), \qquad \mathbf{V}_i(u) \doteq |\mu^i|(I\!R), \tag{4.1}
$$

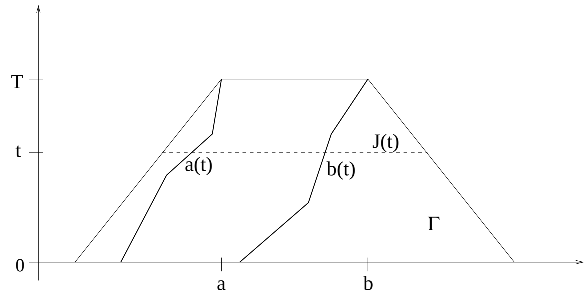

FIGURE 2

while the *interaction potential* is defined in terms of product measures on $I\!R^2$:

(4.2)
$$\mathbf{Q}(u) \doteq \sum_{i<j} \big(|\mu^j| \times |\mu^i|\big)\big(\{(x,y) : x < y\}\big) + \sum_{i\in\mathcal{GN}} \big(\mu^{i-} \times |\mu^i|\big)\big(\{(x,y) : x \neq y\}\big),$$

where $\mathcal{GN}$ denotes the set of genuinely nonlinear families.

Let $\hat\lambda > 0$ be an upper bound for all characteristic speeds. Given any open interval $]a,b[\, \in I\!R$ and a fixed time $t > 0$, for $s \leq t$ define the open intervals

$$J(s) \doteq \,]a - (t-s)\hat\lambda,\ b + (t-s)\hat\lambda[\,.$$

Following [3, Chapter 7.5], we define the Glimm functionals $\mathbf{V}(s) \doteq \mathbf{V}\big(u; J(s)\big)$ and $\mathbf{Q}(s) \doteq \mathbf{Q}\big(u; J(s)\big)$ respectively as the sum of strengths and the interaction potential of all fronts in u crossing the horizontal line $\{(s,x);\ x \in J(s)\}$.

We are now ready to state the following

Proposition 4.1. (One-Sided Lipschitz Condition) *Let the system* (1.2) *be strictly hyperbolic and let the i-th characteristic field be genuinely non-linear. Then there exists a constant C depending solely on f such that for every $0 \leq s < t$ and every solution u with small total variation obtained as limit of wave-front tracking approximations, the measure μ_t^{i+} of i-waves in $u(t,\cdot)$ satisfies*

$$\mu_t^{i+}(]a,b[) \leq C\cdot\left(\frac{b-a}{t-s} + \mathbf{Q}(s) - \mathbf{Q}(t) + \mathbf{V}(u_o)\cdot \|\omega\|_{\mathbf{L}^1([a-\hat\lambda(t-s),\ b+\hat\lambda(t-s)])}\right) \tag{4.3}$$

for every interval $]a,\ b[\subset I\!R$.

Sketch of the proof. Assume $s = 0$, $t = T$, and consider the domain (Fig. 2)

$$\Gamma \doteq \{(t,x);\ 0 \leq t \leq T,\ x \in J(t)\}\,.$$

Toward the proof of (4.3) one can proceed following [9], first proving the estimate for the front tracking ϵ, h-approximate solutions and then passing to the limit as $\epsilon \searrow 0$, by simply restricting the total variation and the interaction potential for the piecewise constant approximate solutions to the segment $J(t)$, i.e.

$$V_{\epsilon,h}(t) \doteq V_{\epsilon,h}\big(t; J(t)\big) = \sum_{\substack{\alpha\in\mathcal{J}\\ x_\alpha\in J(t)}} |\sigma_\alpha| = \sum_{\substack{\alpha\in\mathcal{S}\cup\mathcal{R}\cup\mathcal{NP}\cup\mathcal{Z}\\ x_\alpha\in J(t)}} |\sigma_\alpha|, \tag{4.4}$$

$$Q_{\epsilon,h}(t) \doteq Q_{\epsilon,h}\big(t; J(t)\big) = \sum_{\substack{\alpha,\beta\in\tilde{\mathcal{A}}\\ x_\alpha, x_\beta\in J(t)}} |\sigma_\alpha\sigma_\beta| \leq V_{\epsilon,h}(t)^2. \tag{4.5}$$

In fact, the role of these functionals consists essentially in estimating the quantity

$$M(t) \doteq \sum_{k_\alpha=i, x_\alpha\in I(t)} \sigma_\alpha = \mu_t^i\Big(I(t)\Big)\,,$$

that is, the total amount of (signed) i-waves in $u(t,\cdot)$ contained in $I(t) \doteq [a(t), b(t)[$. Here $a(t)$ (respectively $b(t)$) denotes the minimal i-characteristic passing through a (resp. b) at time T. Since $M(t)$ can change only when an interaction occurs within the interval $[a(t), b(t)]$, and no wave-front can enter the region Γ through the lateral boundaries, the estimate

$$M(\tau+) - M(\tau-) = \mathcal{O}(1)[Q(\tau-) - Q(\tau+)]\,, \tag{4.6}$$

still holds. Hence we obtain the usual estimate for the front tracking ϵ, h-approximate solution u

$$\mu_T^{i+}(]a,b[) \le C\left(\frac{b-a}{T} + (N+1)r(\epsilon) + [\mathbf{Q}_h(u(0)) - \mathbf{Q}_h(u(T))]\right) \tag{4.7}$$

for some constant C independent of ϵ, where the interaction potential $\mathbf{Q}_h(u(t))$ is defined as in [1], but restricted to the interval $J(t)$. In this case we have

$$\mathbf{Q}(u(t)) \le \mathbf{Q}_h(u(t)) \le \mathbf{Q}(u(t)) + \mathbf{V}(u(t)) \cdot \|\omega\|_{\mathbf{L}^1(J(t))}. \tag{4.8}$$

The error term $r(\epsilon)$ tends to 0 as ϵ does, and comes from the difference between $Q_{\epsilon,h}(u)$ and $\mathbf{Q}_h(u)$, since in the latter there are no non-physical fronts. Passing to the limit in (4.7) as $\epsilon \searrow 0$ and using (4.8) we recover (4.3). □

We conclude the section by giving a sketch of the proof of Theorem 1.2. The procedure in the inhomogeneous case does not differ much from what is done for homogeneous conservation laws. Anyway, we will recall the main lines here, since the Time Oscillation condition was given in [4] in a slightly different form.

The Entropy Condition (A2) can be deduced from the Decay Assumption (A4) following exactly the proof presented in [4].

In order to prove the Tame Oscillation condition (A3), it is sufficient to prove that the following condition holds

(A3') There exist constants $C', \delta > 0$ such that, for every $t > 0$ and every $a < b$

$$|u(\hat{t},\xi) - u(t,\xi)| \le C' \cdot \left(\text{Tot.Var.}\{u(t,\cdot);\ [a,b]\} + \|\omega\|_{\mathbf{L}^1([a,b])}\right)\,, \tag{4.9}$$

where

$$\hat{t} \doteq t + \frac{\delta}{2}(b-a), \qquad \xi \doteq \frac{a+b}{2}.$$

Then it is easily seen that (1.9) holds with $C = 3\,C'$.

To prove (4.9) one can argue by contradiction as in [4, Section 4].

5. Concluding remark

The regularity assumptions (P1)–(P3) guarantee the global existence of a whole semigroup of solutions. On the contrary, uniqueness is established here by means of a local estimate (in the time as well as the space variables), and hence it is a local property. One can then expect the same result to hold for local solutions of (1.2), under weaker assumptions on g. In connection with this, we recall that the local existence of a unique Lipschitz semigroup of solutions was established by Crasta and Piccoli [5] for 2×2 systems of balance laws with Lipschitz source term g of the form

$$u_t + f(u)_x = g(t,x,u).$$

More precisely, $g : I\!R \times I\!R \times I\!R^n \to I\!R^n$ is required to be continuous w.r.t. to t, and to satisfy

$$|g(t,x,u) - g(t,y,v)| \le L \cdot \big(|x-y| + |u-v|\big), \qquad |g_x(t,x,u)| \le k(x),$$

for every t and every (x,u), (y,v), with $k \in \mathbf{L}^1(I\!R)$.

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References

1. D. Amadori, L. Gosse and G. Guerra, *Global* **BV** *entropy solutions and uniqueness for hyperbolic systems of balance laws*, to appear on Arch. Rational Mech. Anal. (2002).
2. A. Bressan, *The unique limit of the Glimm scheme*, Arch. Rational Mech. Anal. **130** (1995), 205–230.
3. A. Bressan, *Hyperbolic systems of conservation laws - The one dimensional Cauchy problem*, Oxford University Press, 2000.
4. A. Bressan and P. Goatin, *Oleĭnik type estimates and uniqueness for $n\times n$ conservation laws*, J. Differential Equations **156** (1999), 26–49.
5. G. Crasta and B. Piccoli, *Viscosity solutions and uniqueness for systems of inhomogeneous balance laws*, Discrete Contin. Dynam. Systems **3** (1997), 477-502.
6. A. Bressan and P. LeFloch, *Uniqueness of weak solutions to hyperbolic systems of conservation laws*, Arch. Rational Mech. Anal. **140** (1997), 301-317.
7. A. Bressan and M. Lewicka, *A uniqueness condition for hyperbolic systems of conservation laws*, Discrete Contin. Dyn. Syst. **6** (2000), 673-682.
8. J. Glimm, *Solutions in the large for nonlinear hyperbolic systems of equations*, Comm. Pure Appl. Math. **18** (1965), 697–715.
9. P. Goatin and L. Gosse, *Decay of Positive Waves for $n \times n$ Hyperbolic Systems of Balance Laws*, preprint (2001).
10. L. Gosse, *Using K-branch entropy solutions for multiphase geometric optics computations*, preprint (2001).
11. A.N. Kolmogorov and S.V. Fomin, *Introductory Real Analysis*, Dover, New York, 1975.
12. P.D. Lax, *Hyperbolic systems of conservation laws*, II, Comm. Pure Appl. Math. **10** (1957), 537–566.
13. O. Oleĭnik, *Discontinuous solutions of nonlinear differential equations*, Uspekhi Mat. Nauk **12** (1957), 3–73. [English transl. in Amer. Math. Soc. Transl. Ser. 2 **26** (1963), 95–172]
14. O. Oleĭnik, *On the uniqueness of the generalized solution of the Cauchy problem for a nonlinear system of equations occurring in mechanics*, Uspekhi Mat. Nauk **12** (1957), 169–176 (in Russian).

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