STRONGLY DEGENERATE PARABOLIC-HYPERBOLIC SYSTEMS MODELING POLYDISPERSE SEDIMENTATION WITH COMPRESSION

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Abstract. We show how existing models for the sedimentation of monodisperse flocculated suspensions and of polydisperse suspensions of rigid spheres differing in size can be combined to yield a new theory of sedimentation processes of polydisperse suspensions forming compressible sediments ('sedimentation with compression' or 'sedimentation-consolidation process'). For N solid particle species, this theory reduces in one space dimension to an $N \times N$ coupled system of quasilinear degenerate convection-diffusion equations. Analyses of the characteristic polynomials of the Jacobian of the convective flux vector and of the diffusion matrix show that this system is of strongly degenerate parabolic-hyperbolic type for arbitrary N and particle size distributions. Bounds for the eigenvalues of both matrices are derived. The mathematical model for $N = 3$ is illustrated by a numerical simulation obtained by the Kurganov-Tadmor central difference scheme for convection-diffusion problems. The numerical scheme exploits the derived bounds on the eigenvalues to keep the numerical diffusion at a minimum.

1. Introduction

In recent years we have seen an increased interest in quasilinear systems of partial differential equations that in one space dimension can be written as

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \boldsymbol{\varphi}(\mathbf{u})}{\partial x} = \frac{\partial}{\partial x}\left(\mathbf{D}(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x}\right), \quad x \in \mathbb{R},\ t > 0, \tag{1.1}$$

where $\mathbf{u} : \mathbb{R} \times \mathbb{R}^+ \to \mathcal{D} \subset \mathbb{R}^N$ is the sought solution vector, $\boldsymbol{\varphi} : \mathcal{D} \to \mathbb{R}^N$ is a flux vector and $\mathbf{D} : \mathcal{D} \to \mathbb{R}^{N\times N}$ is a diffusion matrix. We allow the system to be degenerate in the sense that $\mathbf{D}(\mathbf{u}) = 0$ for $\mathbf{u} \in \mathcal{D}' \subset \mathcal{D}$, i.e. the system reduces to first order on $\mathcal{D}'$. The system is called *strongly degenerate* if $\mathcal{D}'$ is of nonzero N-dimensional measure. Moreover, we recall that the system (1.1) is *strictly parabolic* at a point $\mathbf{u}_0 \in \mathcal{D}$ if $\mathbf{D}(\mathbf{u}_0) > 0$, i.e. the matrix $\mathbf{D}(\mathbf{u}_0)$ has only positive eigenvalues. On the other hand, if $\mathbf{u}_0$ is chosen such that $\mathbf{D}(\mathbf{u}_0) = 0$ then, according to the usual terminology for conservation laws, the system (1.1) is called *hyperbolic* if the Jacobian $\mathcal{J}\boldsymbol{\varphi}(\mathbf{u}_0)$ has N real eigenvalues, and *strictly hyperbolic* if these eigenvalues are moreover pairwise distinct. Finally, we shall call (1.1) a *strongly degenerate parabolic-hyperbolic system* if at any point $\mathbf{u}_0$ belonging to the interior $\mathcal{D}^0$ of $\mathcal{D}$, the system (1.1) is either strictly parabolic or strictly hyperbolic in the sense given above, and the set $\mathcal{D}^0 \cap \mathcal{D}'$ on which the system is strictly hyperbolic is of nonzero N-dimensional measure. We emphasize here that points $\mathbf{u} \in \mathcal{D}\backslash\mathcal{D}^0$, which are on the boundary of the physically relevant region $\mathcal{D}$, do not enter the type characterization [41]. Note that a strictly hyperbolic first-order system of conservation laws, for which the right-hand side of (1.1) vanishes identically, is included as a special case. Of course, solutions of (1.1) are in general

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discontinuous, even for smooth initial data. We refer to Section 5 for a further discussion of the properties of strongly degenerate parabolic problems.

The main objectives of this paper are firstly to show how two existing models, one for sedimentation-consolidation processes of monodisperse flocculated suspensions, which are described by *scalar* strongly degenerate parabolic-hyperbolic equations, and one for the sedimentation of polydisperse suspensions of rigid spheres differing in size, which lead to *first-order systems* of conservation laws, can be combined into a model of sedimentation of polydisperse suspensions of particles (or flocs) forming compressible sediments. Secondly, we explicitly prove that this model gives rise to strongly degenerate parabolic-hyperbolic systems of partial differential equations in the sense given above. This type characterization is valid for arbitrary numbers N of particle sizes. The application considered thus provides provably strongly degenerate parabolic-hyperbolic systems of arbitrary size. Finally, for $N = 3$ we illustrate the model by a numerical example using a high-resolution finite difference scheme [46].

In the sequel, we give an outline of the paper and at the same time put it in the proper perspective of the existing literature.

In Section 2 we derive a set of spatially multidimensional model equations describing the sedimentation of polydisperse suspensions forming compressible sediments (also called a sedimentation-consolidation process). The modeling starts from the usual mass and linear momentum balance equations for the N solids species (each regarded as one phase) and the fluid. The generic material properties of the suspension are introduced by constitutive assumptions concerning the solid and fluid stress tensors and the solid-fluid interaction forces. In particular the solid phase pressures and the fluid pressure are replaced by the effective solid stress σ_e and the pore pressure. Here we assume that σ_e is a function of the total solids concentration $\phi := \phi_1 + \dots + \phi_N$ only, where ϕ_i is the concentration of species i having diameter d_i and density ϱ_i. The particular dependence of σ_e on the local concentrations of the solids species determines the resulting diffusion matrix of the degenerate system discussed above. Specifying the solid-fluid interaction force for each species and finally performing a dimensional analysis, which permits neglecting several terms from the linear momentum balance equations, we obtain explicit expressions for the solid-fluid relative velocity (or slip velocity) of each species as a function of $\Phi := (\phi_1, \dots, \phi_N)^\mathrm{T}$ and $\nabla\Phi$, which in turn yield the fluxes of the continuity equations. The final (spatially multidimensional) model equations form a strongly degenerate system of N convection-diffusion equations for $\phi_1, \dots, \phi_N$ coupled to the divergence-free condition of the volume-average mixture velocity and a three-component equation for the motion of the mixture. These last two equations account for viscous effects and reduce for $\Phi \equiv 0$ to the Stokes system for an incompressible fluid. Finally, we check that setting $N = 1$ reduces the strongly degenerate parabolic-hyperbolic system to the scalar equation governing the settling of a monodisperse suspension [22]. For an overview of the analysis, numerics and applications of scalar strongly degenerate parabolic equations we refer to Section 5.

The (controlled) sedimentation of polydisperse suspensions forms an important process in many industrial applications [2, 36, 58, 59, 63, 73]. However, the effect of compressible sediment has been studied only infrequently [61, 64]. Unfortunately, these treatments are incomplete in that they are not embedded in the appropriate mathematical PDE framework or are limited to $N = 2$. Moreover, the transition between the hindered settling zone (where $\phi \leq \phi_\mathrm{c}$) and the compression region (where $\phi > \phi_\mathrm{c}$) is introduced in [64] by an artificial moving boundary condition, which is avoided by the concept of a degenerate diffusion equation (see also the discussion in [18]). The model derived in Section 2 also describes the relative movement of the solids species against each other within the sediment under the influence of the effective solid stress. This is unlike any previous treatment. These few remarks are stated to emphasize that our model is new and therefore derived completely in Section 2. On the other hand, we essentially combine arguments that have been discussed extensively in previous works that focus on modeling either flocculated monodisperse [22, 23, 29] or non-flocculated polydisperse suspensions [12, 17, 20]. Thus the presentation in Section 2 is fairly concise and we refer to the cited papers for additional details and justification.

If the multidimensional model equations are restricted to one space dimension, the motion of the mixture is determined by the velocity at one end of the computational domain, and is zero for sedimentation in a closed vessel. In this case, only the degenerate system for $\phi_1, \dots, \phi_N$ needs to be solved. These equations can then be written as a system of second-order PDEs in the form

$$\frac{\partial \Phi}{\partial t}+\frac{\partial \mathbf{f}(\Phi)}{\partial z}=\frac{\partial}{\partial z}\left(\mathbf{A}(\Phi)\frac{\partial \Phi}{\partial z}\right), \tag{1.2}$$

where t is time and z is height. The mixture forms a compressible sediment layer whenever the cumulative solids concentration ϕ exceeds a critical value (or 'gel point') ϕ_{c}. In our application, we have $\mathbf{A}(\Phi)=0$ for $\phi\leq\phi_{\mathrm{c}}$. In this case, the system (1.2) is reduced to the first-order system

$$\frac{\partial \Phi}{\partial t}+\frac{\partial \mathbf{f}(\Phi)}{\partial z}=0. \tag{1.3}$$

We consider the system (1.2) for vectors $\Phi\in\mathcal{D}_{\phi_{\max}}$, where $0<\phi_{\max}\leq 1$ denotes the maximum admissible cumulative solids concentration and we define for $0<\phi_{\mathrm{M}}\leq 1$

$$\mathcal{D}_{\phi_{\mathrm{M}}}:=\bigl\{\Phi=(\phi_1,\dots,\phi_N)\in\mathbb{R}^N:\phi_1\geq 0,\dots,\phi_N\geq 0;\ \phi_1+\dots+\phi_N\leq\phi_{\mathrm{M}}\bigr\}.$$

Moreover, we denote by $\mathcal{D}^0_{\phi_{\mathrm{M}}}$ the interior of $\mathcal{D}_{\phi_{\mathrm{M}}}$, that is

$$\mathcal{D}^0_{\phi_{\mathrm{M}}}:=\bigl\{\Phi=(\phi_1,\dots,\phi_N)\in\mathbb{R}^N:\phi_1> 0,\dots,\phi_N> 0;\ \phi_1+\dots+\phi_N<\phi_{\mathrm{M}}\bigr\}.$$

Obviously, the type of (1.2) is determined by the properties of $\mathbf{A}(\Phi)$ for $\phi_{\mathrm{c}}<\phi<\phi_{\max}$ and by those of the vector $\mathbf{f}(\Phi)$ (more precisely, of its Jacobian $\mathcal{J}_{\mathbf{f}}(\Phi)$) for $0<\phi\leq\phi_{\mathrm{c}}$ and $\phi=\phi_{\max}$. Since every component of $\mathbf{f}(\Phi)=(f_1(\Phi),\dots,f_N(\Phi))^{\mathrm{T}}$ depends nonlinearly on every component of Φ, and $\mathcal{J}_{\mathbf{f}}(\Phi)$ is unsymmetric, it is by no means obvious that the system (1.3) is strictly hyperbolic. Since all entries of $\mathbf{A}(\Phi)$ are nonzero on $\mathcal{D}^0_{\phi_{\max}}\backslash\mathcal{D}_{\phi_{\mathrm{c}}}$, it is not apparent either that the system (1.2) is strictly parabolic for $\phi\in\mathcal{D}^0_{\phi_{\max}}\backslash\mathcal{D}_{\phi_{\mathrm{c}}}$. The core of this paper is formed by Sections 3 and 4, where these properties are established by analyzing the characteristic polynomials of $\mathcal{J}_{\mathbf{f}}(\Phi)$ and $\mathbf{A}(\Phi)$, respectively. This treatment has in part been inspired by Rosso and Sona's recent analysis of equations modeling the separation of oil-water dispersions [57]. In Section 3 we discuss the properties of the system (1.3) with $\mathbf{f}(\Phi)=\mathbf{f}^{\mathrm{M}}(\Phi)$, i.e. the system

$$\frac{\partial \phi_i}{\partial t}+\frac{\partial f_i^{\mathrm{M}}(\Phi)}{\partial z}=0,\quad i=1,\dots,N, \tag{1.4}$$

which arises from the model derived in Section 2 by considering one space dimension and a closed settling vessel and assuming that the effective solid stress vanishes ($\sigma_{\mathrm{e}}\equiv 0$). The 'M' indicates that the constitutive assumptions in Section 2 have been chosen according to the Masliyah-Lockett-Bassoon (MLB) approach [20, 48, 50]. Alternate equations for $\mathbf{f}(\Phi)$ are discussed in [12, 20].

The analysis of Section 3 leads to the type of the system (1.2) for $\phi\leq\phi_{\mathrm{c}}$, and fully determines its type if we choose $\sigma_{\mathrm{e}}\equiv 0$, that is if we consider a suspension of rigid particles as in [12, 17, 20]. To put Section 3 into the proper context, we mention that the MLB model equations (1.4) are in general not hyperbolic for suspensions in which two or more species have different densities, as was shown in [20]. In particular, for $N=2$ the equations are of mixed hyperbolic-elliptic type. The degeneracy into non-hyperbolic type is a criterion for the possible ocurrence of horizontal structures like fingers, columns or blobs during sedimentation. These instability phenomena, which have been observed in experiments (see, e.g., [69]), are particularly likely to occur in polydisperse suspensions including one species that is heavier and one that is lighter than the fluid. The precise shape of the instability region usually has to be determined numerically.

On the other hand, in [20] it was proved that the system (1.4), with $N=2$, the hindered settling function $V(\phi)=(1-\phi)^{n-2}$ with $n>2$, and spheres of the same density but different size, is always strictly hyperbolic on $\mathcal{D}^0_1$. For equal-density spheres, we indicated in [20] that numerical tests with $N=3$ never produced an instability region, and conjectured that the MLB equations were hyperbolic for arbitrary N. The main result proved in Section 3 is that in this equal-density case, the system (1.4) is indeed strictly hyperbolic for all $\Phi\in\mathcal{D}^0_{\phi_{\max}}$ for $0<\phi_{\max}\leq 1$. Strict hyperbolicity not only holds for all N, but also for arbitrary ratios $d_1/d_N=\sqrt{\delta_1/\delta_N}$ of the maximum to the minimum particles sizes. This contrasts, for example, with the properties of model equations proposed by Davis and Gecol [31], which for $N=2$ turn out to be hyperbolic only for small values of d_1/d_2 (typically, $d_1/d_2\leq 5$), see [20].

In Section 4 we turn attention to the right-hand side of (1.2) using the diffusion matrix $\mathbf{A}(\Phi)$ derived in Section 2. While it is obvious that $\mathbf{A}(\Phi) = 0$ on $\mathcal{D}_{\phi_c}$, it is by no means apparent that $\mathbf{A}(\Phi)$ is positive definite on $\mathcal{D}_1^0 \backslash \mathcal{D}_{\phi_c}$. The hyperbolicity and parabolicity properties of (1.2) associated with $\mathcal{J}_{\mathbf{f}^{\mathrm{M}}}(\Phi)$ and $\mathbf{A}(\Phi)$ are controlled by the independent model functions $V(\phi)$ and $\sigma_{\mathrm{e}}(\phi)$, but their entries are analogous. Thus, the formula for the characteristic polynomial of $\mathcal{J}_{\mathbf{f}^{\mathrm{M}}}(\Phi)$ derived (with some effort) in Section 3 also provides (after substitutions) a formula for the characteristic polynomial of $\mathbf{A}(\Phi)$. It is then straightforward to prove that $\mathbf{A}(\Phi)$ has N distinct nonnegative eigenvalues, which are positive if $\phi_c < \phi < 1$.

Section 5 is devoted to a short overview of the existing literature on mathematical and numerical theory for strongly degenerate parabolic problems.

In Section 6 we illustrate the (new) model of sedimentation of polydisperse suspensions forming compressible sediments by a numerical example with $N = 3$ and compare the results with simulations of the two (conventional) models of settling of monodisperse flocculated and polydisperse rigid-sphere suspensions (where the effective solid stress σ_{e} is identically zero), choosing equivalent parameters wherever possible. To this end, Equation (1.2) is discretized according to a finite difference scheme described in [46], which is a combination of a high-resolution central difference (Riemann solver free) scheme for the convection part (corresponding to the first-order equation (1.4)) with a central difference discretization of the parabolic parts (the right-hand side of (1.2)).

A brief closing discussion (Section 7) includes the possible extension of the model to polydisperse suspensions with particles differing in density, a physical interpretation of one of the eigenvalue bounds derived in Section 3, and applications in which the sediment compressibility is important.

2. Derivation of the model of polydisperse sedimentation with compression

2.1. Mass and linear momentum balance equations. The local mass balance equations of the solid species and of the fluid can be written as

$$\frac{\partial \phi_i}{\partial t} + \nabla \cdot \big(\phi_i \mathbf{v}_i\big) = 0, \quad i = 1, \dots, N, \quad -\frac{\partial \phi}{\partial t} + \nabla \cdot \big((1-\phi)\mathbf{v}_{\mathrm{f}}\big) = 0, \tag{2.1}$$

where we recall that $\phi = \phi_1 + \dots + \phi_N$. Defining the volume-average velocity of the mixture $\mathbf{q} := (1-\phi)\mathbf{v}_{\mathrm{f}} + \phi_1 \mathbf{v}_1 + \dots + \phi_N \mathbf{v}_N$ and the relative velocities or slip velocities $\mathbf{u}_i := \mathbf{v}_i - \mathbf{v}_{\mathrm{f}}$ for $i = 1, \dots, N$, we derive easily that

$$\phi_i \mathbf{v}_i = \phi_i \big(\mathbf{u}_i + \mathbf{q} - (\phi_1 \mathbf{u}_1 + \dots + \phi_N \mathbf{u}_N)\big), \quad i = 1, \dots, N; \tag{2.2}$$

hence the mass balance equations (2.1) can be rewritten in terms of $\mathbf{q}$ and $\mathbf{u}_1, \dots, \mathbf{u}_N$ as

$$\frac{\partial \phi_i}{\partial t} + \nabla \cdot \big(\phi_i \mathbf{u}_i + \phi_i \mathbf{q} - \phi_i(\phi_1 \mathbf{u}_1 + \dots + \phi_N \mathbf{u}_N)\big) = 0, \quad i = 1, \dots, N. \tag{2.3}$$

The sum of all equations in (2.1) produces the simple mass balance of the mixture, $\nabla \cdot \mathbf{q} = 0$. The momentum balance equations for the N solid species and the fluid are

$$\varrho_i \phi_i \frac{\mathrm{D}\mathbf{v}_i}{\mathrm{D}t} = \nabla \cdot \mathbf{T}_i + \varrho_i \phi_i \mathbf{b} + \mathbf{m}_i^{\mathrm{f}} + \mathbf{m}_i^{\mathrm{s}}, \quad i = 1, \dots, N, \tag{2.4}$$

$$\varrho_{\mathrm{f}}(1-\phi) \frac{\mathrm{D}\mathbf{v}_{\mathrm{f}}}{\mathrm{D}t} = \nabla \cdot \mathbf{T}_{\mathrm{f}} + \varrho_{\mathrm{f}}(1-\phi)\mathbf{b} - (\mathbf{m}_1^{\mathrm{f}} + \dots + \mathbf{m}_N^{\mathrm{f}}). \tag{2.5}$$

Here ϱ_{f} is the mass density of the fluid, $\mathbf{T}_i$ denotes the stress tensor of particle species i, $i = 1, \dots, N$, $\mathbf{T}_{\mathrm{f}}$ that of the fluid, $\mathbf{b}$ is the body force, $\mathbf{m}_i^{\mathrm{f}}$ and $\mathbf{m}_{ij}^{\mathrm{s}}$ are the interaction forces per unit volume between solid species i and the fluid and between the solid species i and j, respectively, $\mathbf{m}_i^{\mathrm{s}} := \mathbf{m}_{i1}^{\mathrm{s}} + \dots + \mathbf{m}_{iN}^{\mathrm{s}}$ is the particle-particle interaction term of species i, and we use the standard notation $\mathrm{D}\mathbf{v}/\mathrm{D}t := \partial \mathbf{v}/\partial t + (\mathbf{v} \cdot \nabla)\mathbf{v}$.

2.2. Solid and fluid stress tensors. We assume that the stress tensors of the solid and fluid phases can be written as $\mathbf{T}_i = -p_i \mathbf{I} + \mathbf{T}_i^{\mathrm{E}}$ for $i = 1, \dots, N$ and $\mathbf{T}_{\mathrm{f}} = -p_{\mathrm{f}} \mathbf{I} + \mathbf{T}_{\mathrm{f}}^{\mathrm{E}}$, respectively, where p_i denotes the phase pressure of particle species i, p_{f} that of the fluid, $\mathbf{I}$ denotes the identity tensor, and $\mathbf{T}_i^{\mathrm{E}}$ and $\mathbf{T}_{\mathrm{f}}^{\mathrm{E}}$ are the extra (or viscous) stress tensors of particle species i and the fluid, respectively, all of which could be given by expressions that correspond, for example, to a viscous-linear fluid. Since the focus here is on the continuity equations for the solid species, and we assume that viscous effects due to the motion of the mixture are not dominant, all viscous effects are assigned to the fluid extra-stress tensor. To make this simplification visible in the dimensional analysis, we assume that ν_0^{f} and $\nu_0^{\mathrm{s}} < \nu_0^{\mathrm{f}}$ are characteristic viscosities associated with the fluid and the solid species, respectively.

2.3. **Partial pressures, pore pressure, and effective solid stress.** The phase pressures $p_1, \dots, p_N$ and p_{f} are theoretical variables, which cannot be measured experimentally. As in [22], they are replaced by the pore pressure p and the effective solid stress σ_{e}, which are measurable. We assume here that σ_{e} is given by a constitutive equation $\sigma_{\mathrm{e}} = \sigma_{\mathrm{e}}(\Phi)$, that is, as a function of the local composition of the sediment only. To our knowledge, and as has been noted in [64], no suitable relationship valid in the polydisperse case has been derived either theoretically or empirically so far. However, most researchers utilize formulas that relate the effective stress σ_{e} to the porosity of the sediment or, equivalently, to the total volumetric solids concentration ϕ [51, 61]. The generic assumptions on σ_{e} are the following (a specific example is given in Section 6):

$$\sigma_{\mathrm{e}}(\phi) \begin{cases} = 0 & \text{for } \phi \le \phi_{\mathrm{c}}, \\ > 0 & \text{for } \phi > \phi_{\mathrm{c}}, \end{cases} \qquad \sigma_{\mathrm{e}}'(\phi) := \frac{d\sigma_{\mathrm{e}}(\phi)}{d\phi} \begin{cases} = 0 & \text{for } \phi \le \phi_{\mathrm{c}}, \\ > 0 & \text{for } \phi > \phi_{\mathrm{c}}. \end{cases} \tag{2.6}$$

We now relate the fluid phase pressure p_{f} and the solid species phase pressures to the effective solid stress σ_{e} and to the pore pressure p. While the pore pressure p is defined within the fluid filling the interstices of the solids, the partial fluid pressure p_{f} is defined in the fluid component occupying the whole volume of the mixture. Let S be the cross-section of a settling column and $S_{\mathrm{f}} \subset S$ be its part that is filled out by the fluid in the porous medium, and let ϵ denote the surface porosity $\epsilon := |S_{\mathrm{f}}|/|S|$, i.e. $dS_{\mathrm{f}} = \epsilon dS$. Then the surface forces exerted on the fluid in a cross section of the sediment are

$$\int_S p_{\mathrm{f}}\, dS = \int_{S_{\mathrm{f}}} p\, dS_{\mathrm{f}} = \int_S p(\epsilon dS). \tag{2.7}$$

Since we may assume that the surface porosity equals the volume porosity [20], we may replace ϵ by $1-\phi$, and as a consequence of the localization theorem [38], we obtain from (2.7) $p_{\mathrm{f}} = (1-\phi)p$.

The total stress of the mixture, p_{t}, can be decomposed in two different ways:

$$p_{\mathrm{t}} = p_{\mathrm{f}} + p_1 + \dots + p_N = p + \sigma_{\mathrm{e}}(\phi). \tag{2.8}$$

The effective solid stress σ_{e} is that part of the total stress which acts on the porous network formed by the solid particles. Assuming that the cross-sectional surface area fraction of each solids species equals its volume fraction [20], we may conclude that $(\phi_i/\phi)\sigma_{\mathrm{e}}(\phi)$ is that part of the effective solid stress which acts on species i. Since in view of $p_{\mathrm{f}} = (1-\phi)p$, equation (2.8) may be rewritten as

$$p_1 + \dots + p_N = \phi p + \frac{\phi_1 + \dots + \phi_N}{\phi}\sigma_{\mathrm{e}}(\phi),$$

the phase pressure p_i is related to p and σ_{e} by $p_i = (\phi_i/\phi)(\phi p + \sigma_{\mathrm{e}}(\phi))$ for $i = 1, \dots, N$.

2.4. **Body force, solid-fluid and particle-particle interaction forces.** We assume that the only body force is gravity, $\mathbf{b} = -g\mathbf{k}$, where g is the acceleration of gravity and $\mathbf{k}$ is the upwards-pointing unit vector. Furthermore, for a monodisperse suspension, the interaction force $\mathbf{m}$ between the fluid and the unique solid phase can be modeled by $\mathbf{m} = \alpha(\phi)\mathbf{u} + \beta\nabla\phi$, where α is the resistance coefficient, $\mathbf{u} := \mathbf{v}_{\mathrm{s}} - \mathbf{v}_{\mathrm{f}}$ is the solid-fluid relative velocity (or slip velocity), and the coefficient β can be shown to coincide with the pore pressure p [22]. In the present case, we analogously assume that the solid-fluid interaction term $\mathbf{m}_i^{\mathrm{f}}$ corresponding to species i is given by $\mathbf{m}_i^{\mathrm{f}} = \alpha_i(\Phi)\mathbf{u}_i + \beta\nabla\phi_i$, where α_i denotes the resistance coefficient related to the transfer of momentum between the fluid and solid phase species i, $i = 1, \dots, N$.

The interaction force between the different solid particle species could be specified by a formula like that of Nakamura and Capes [1, 52, 61]:

$$\mathbf{m}_{ij}^{\mathrm{s}} = \frac{3}{2}\varphi_{\mathrm{e}} \frac{\varrho_i \varrho_j \phi_i \phi_j (d_i + d_j)^2}{\varrho_i d_i^3 + \varrho_j d_j^3} \|\mathbf{v}_i - \mathbf{v}_j\| (\mathbf{v}_i - \mathbf{v}_j), \quad i, j = 1, \dots, N, \quad i \ne j,$$

where the parameter φ_{e} accounts for non-head-on-collisions [61] and its value depends on whether these are plastic or elastic. Typical values of φ_{e} vary between zero and 5 [1, 52], and numerical simulations have not turned out to be sensitive to φ_{e} [1]. Nevertheless, the elimination of the term $\mathbf{m}_i^{\mathrm{s}} = \mathbf{m}_{i1}^{\mathrm{s}} + \dots + \mathbf{m}_{iN}^{\mathrm{s}}$ due to the dimensional analysis (see Sect. 2.5) is not dependent on any particular formula, since there is considerable experimental and theoretical evidence (summarized in [20]) that $\mathbf{m}_{ij}^{\mathrm{s}}$ can be neglected at the very low Reynolds numbers considered here.

Inserting the present constitutive assumptions into equations (2.4) and (2.5) and considering the mixture at equilibrium ($t \to \infty$) in a settling column, i.e. when $\mathbf{v}_{\mathrm{f}} = 0$, $\mathbf{u}_1 = \dots = \mathbf{u}_N = 0$ and $\nabla p = -\varrho_{\mathrm{f}} g \mathbf{k}$, we obtain $\beta = p$ [20, 22]. The linear momentum balances now read

$$\varrho_i \phi_i \frac{\mathrm{D}\mathbf{v}_i}{\mathrm{D}t} = -\varrho_i \phi_i g \mathbf{k} + \nabla \cdot \mathbf{T}_i^{\mathrm{E}} - \phi_i \nabla p + \alpha_i(\Phi)\mathbf{u}_i + \mathbf{m}_i^{\mathrm{s}} - \nabla\left(\frac{\phi_i}{\phi}\sigma_{\mathrm{e}}(\phi)\right), \quad i = 1,\dots,N, \tag{2.9}$$

$$\nabla p = -\varrho_{\mathrm{f}} g \mathbf{k} - \frac{1}{1-\phi}\big(\alpha_1(\Phi)\mathbf{u}_1 + \dots + \alpha_N(\Phi)\mathbf{u}_N\big) - \varrho_{\mathrm{f}}\frac{\mathrm{D}\mathbf{v}_{\mathrm{f}}}{\mathrm{D}t} + \frac{1}{1-\phi}\nabla \cdot \mathbf{T}_{\mathrm{f}}^{\mathrm{E}}. \tag{2.10}$$

2.5. **Dimensional analysis.** We introduce dimensionless variables by referring all densities to ϱ_{f}, all velocities to the velocity U, all lengths to a typical length L, all solid and fluid viscosities to ν_0^{s} and ν_0^{f}, respectively, and all pressures to the hydrostatic pressure $\varrho_{\mathrm{f}} g L$. Here, we assume that U is the settling velocity of a single particle of the fastest settling species in an unbounded medium, and L is the depth of the settling vessel. A characteristic time is then given by $T = L/U$. All dimensionless variables are marked by a star. A dimensionless gradient of a variable u is defined by $\nabla^* u = L\nabla u$ and a dimensionless time derivative by $\partial u/\partial t^* = T\partial u/\partial t = (L/U)\partial u/\partial t$.

Using the Froude number of the flow $\mathrm{Fr} := U^2/(gL)$ and the sedimentation Reynolds number $\mathrm{Re} := dU/\nu_0^{\mathrm{f}}$, where d is the size of the largest particles, we obtain from (2.9) and (2.10) the dimensionless equations

$$\begin{aligned} \varrho_i^* \phi_i \mathrm{Fr}\frac{\mathrm{D}\mathbf{v}_i^*}{\mathrm{D}t^*} ={}& -\varrho_i^* \phi_i \mathbf{k} + \frac{d}{L}\frac{\nu_0^{\mathrm{s}}}{\nu_0^{\mathrm{f}}}\frac{\mathrm{Fr}}{\mathrm{Re}}\nabla^* \cdot \big(\mathbf{T}_i^{\mathrm{E}}\big)^* - \phi_i \nabla^* p^* + \alpha_i^*(\Phi)\mathbf{u}_i^* \\ &+ \frac{L}{d}\mathrm{Fr}(\mathbf{m}_i^{\mathrm{s}})^* - \nabla^*\left(\frac{\phi_i}{\phi}\sigma_{\mathrm{e}}^*(\phi)\right), \quad i = 1,\dots,N, \end{aligned} \tag{2.11}$$

$$\nabla^* p^* = -\mathbf{k} - \frac{1}{1-\phi}\big(\alpha_1^*(\Phi)\mathbf{u}_1^* + \dots + \alpha_N^*(\Phi)\mathbf{u}_N^*\big) - \mathrm{Fr}\frac{\mathrm{D}\mathbf{v}_{\mathrm{f}}^*}{\mathrm{D}t^*} + \frac{1}{1-\phi}\frac{d}{L}\frac{\mathrm{Fr}}{\mathrm{Re}}\nabla^* \cdot \big(\mathbf{T}_{\mathrm{f}}^{\mathrm{E}}\big)^*. \tag{2.12}$$

The values $d = 10^{-4}\,\mathrm{m}$, $g = 10\,\mathrm{m/s^2}$, $L = 1\,\mathrm{m}$ (height of a settling vessel), $U = 10^{-4}\,\mathrm{m/s}$ (settling velocity of a particle of the fastest species in an unbounded fluid), and $\nu_0^{\mathrm{f}} = 10^{-6}\,\mathrm{m^2/s}$ (kinematic viscosity of water) are typical for the particulate systems considered here, and imply $\mathrm{Fr} = 10^{-9}$, $\mathrm{Re} = 10^{-2}$ and $d/L = 10^{-4}$. Since all viscous effects have been moved onto the fluid extra-stress tensor, we can assume $\nu_0^{\mathrm{s}}/\nu_0^{\mathrm{f}} \ll 1$. We assume that all dimensionless variables are of the order of magnitude $\mathcal{O}(1)$. Then we obtain, by discarding from Eqns. (2.11) all terms that have a coefficient which is 10^{-5} or smaller, and discarding the advective acceleration term from (2.12) but retaining the viscous term, the following simplified linear momentum balances:

$$\alpha_i(\Phi)\mathbf{u}_i = \varrho_i \phi_i g \mathbf{k} + \phi_i \nabla p + \nabla\left(\frac{\phi_i}{\phi}\sigma_{\mathrm{e}}(\phi)\right), \quad i = 1,\dots,N, \tag{2.13}$$

$$\nabla p = -\varrho_{\mathrm{f}} g \mathbf{k} - \frac{1}{1-\phi}\big(\alpha_1(\Phi)\mathbf{u}_1 + \dots + \alpha_N(\Phi)\mathbf{u}_N\big) + \frac{1}{1-\phi}\nabla \cdot \mathbf{T}_{\mathrm{f}}^{\mathrm{E}}, \tag{2.14}$$

which are written again in their dimensional forms. Inserting (2.14) into (2.13), we obtain

$$\begin{aligned} &\frac{\alpha_i(\Phi)(1-\phi)}{\phi_i}\mathbf{u}_i + \alpha_1(\Phi)\mathbf{u}_1 + \dots + \alpha_N(\Phi)\mathbf{u}_N = \mathbf{r}_i, \\ &\mathbf{r}_i := (1-\phi)\left[(\varrho_i - \varrho_{\mathrm{f}})\, g\mathbf{k} + \frac{1}{\phi_i}\nabla\left(\frac{\phi_i}{\phi}\sigma_{\mathrm{e}}(\phi)\right)\right] + \nabla \cdot \mathbf{T}_{\mathrm{f}}^{\mathrm{E}}, \quad i = 1,\dots,N. \end{aligned} \tag{2.15}$$

2.6. **Explicit formula for the slip velocities $\mathbf{u}_i$.** The linear system (2.15) for the unknowns $\mathbf{u}_1, \dots, \mathbf{u}_N$ can be solved explicitly by the Sherman-Morrison formula [20]. The solution is

$$\mathbf{u}_i = \frac{\phi_i}{\alpha_i(\Phi)(1-\phi)}\bigl(\mathbf{r}_i - (\phi_1\mathbf{r}_1 + \dots + \phi_N\mathbf{r}_N)\bigr), \quad i = 1, \dots, N. \tag{2.16}$$

Let $\varrho(\Phi) := (1-\phi)\varrho_{\mathrm{f}} + \phi_1\varrho_1 + \dots + \phi_N\varrho_N$ denote the local density of the mixture and note that $\phi_1(\varrho_1 - \varrho_{\mathrm{f}}) + \dots + \phi_N(\varrho_N - \varrho_{\mathrm{f}}) = \varrho(\Phi) - \varrho_{\mathrm{f}}$. Inserting $\mathbf{r}_1, \dots, \mathbf{r}_N$ into (2.16) and neglecting the viscous term $\nabla \cdot \mathbf{T}_{\mathrm{f}}^{\mathrm{E}}$ leads to an explicit equation for the slip velocities $\mathbf{u}_i$ as functions of Φ, which after rearranging reads

$$\mathbf{u}_i = \frac{\phi_i}{\alpha_i(\Phi)}\biggl[\bigl(\varrho_i - \varrho(\Phi)\bigr)g\mathbf{k} + \frac{\sigma_{\mathrm{e}}(\phi)}{\phi_i}\nabla\Bigl(\frac{\phi_i}{\phi}\Bigr) + \frac{1-\phi}{\phi}\nabla\sigma_{\mathrm{e}}(\phi)\biggr], \quad i = 1, \dots, N. \tag{2.17}$$

Following [20] and being consistent with the Masliyah [50] and Lockett and Bassoon [48] models, we choose $\phi_i/\alpha_i(\Phi) = -d_i^2 V(\Phi)/(18\mu_{\mathrm{f}})$, where μ_{f} is the viscosity of the pure fluid, and the hindered settling factor $V(\Phi)$ can, for example, be chosen as $V(\Phi) = (1-\phi)^{n(\Phi)-2}$ [56]. Since the dependence of n on Φ is through wall effects, which are small when d is very small compared to the diameter of the settling vessel, we may limit the analysis to formulas of the type $V(\Phi) = V(\phi)$, and obtain

$$\mathbf{u}_i = -\frac{d_i^2}{18\mu_{\mathrm{f}}}V(\phi)\biggl[\bigl(\varrho_i - \varrho(\Phi)\bigr)g\mathbf{k} + \frac{\sigma_{\mathrm{e}}(\phi)}{\phi_i}\nabla\Bigl(\frac{\phi_i}{\phi}\Bigr) + \frac{1-\phi}{\phi}\nabla\sigma_{\mathrm{e}}(\phi)\biggr], \quad i = 1, \dots, N. \tag{2.18}$$

The generic assumption to ensure hyperbolicity, which is satisfied by $V(\phi) = (1-\phi)^{n-2}$, $n > 2$, is

$$V(\phi) > 0, \quad V'(\phi) < 0 \quad \text{for } 0 < \phi < \phi_{\max}. \tag{2.19}$$

2.7. **Final form of the model equations.** The final model equations are the continuity equations of the solids species and of the mixture ($\nabla \cdot \mathbf{q} = 0$), the linear momentum balance of the fluid (2.14), and the equations (2.18) for the slip velocities $\mathbf{u}_i$, which have been obtained from the linear momentum balances of the solid species. To state these equations, we need explicit expressions for the fluxes $\phi_1\mathbf{v}_1, \dots, \phi_N\mathbf{v}_N$. To this end we introduce the reduced densities $\bar{\varrho}_{\mathrm{s}} := \varrho_{\mathrm{s}} - \varrho_{\mathrm{f}}$, $\bar{\varrho}_i := \varrho_i - \varrho_{\mathrm{f}}$, $i = 1, \dots, N$, the vector $\bar{\boldsymbol{\varrho}} := (\bar{\varrho}_1, \dots, \bar{\varrho}_N)^{\mathrm{T}}$, and the parameters $\mu := -gd_1^2/(18\mu_{\mathrm{f}})$ and $\delta_i := d_i^2/d_1^2$, $i = 1, \dots, N$, such that (2.18) reads

$$\mathbf{u}_i = \mu\delta_i V(\phi)\biggl[(\bar{\varrho}_i - \bar{\boldsymbol{\varrho}}^{\mathrm{T}}\Phi)\mathbf{k} + \frac{\sigma_{\mathrm{e}}(\phi)}{g\phi_i}\nabla\Bigl(\frac{\phi_i}{\phi}\Bigr) + \frac{1-\phi}{g\phi}\nabla\sigma_{\mathrm{e}}(\phi)\biggr], \quad i = 1, \dots, N. \tag{2.20}$$

In view of (2.2), we get $\phi_i\mathbf{v}_i = f_i^{\mathrm{M}}(\Phi)\mathbf{k} + \phi_i\mathbf{q} - \mathbf{a}_i(\Phi, \nabla\Phi)$ for $i = 1, \dots, N$, where the components of the flux vector $\mathbf{f}^{\mathrm{M}}(\Phi)$ (corresponding to the Masliyah-Lockett-Bassoon (MLB) model for polydisperse suspensions of rigid spheres) are given by

$$f_i(\Phi) = f_i^{\mathrm{M}}(\Phi) = \mu V(\phi)\phi_i\left[\delta_i(\bar{\varrho}_i - \bar{\boldsymbol{\varrho}}^{\mathrm{T}}\Phi) - \sum_{k=1}^{N}\delta_k\phi_k(\bar{\varrho}_k - \bar{\boldsymbol{\varrho}}^{\mathrm{T}}\Phi)\right], \quad i = 1, \dots, N. \tag{2.21}$$

If we let $\boldsymbol{\delta} := (\delta_1, \dots, \delta_N)^{\mathrm{T}}$, then the vectors $\mathbf{a}_i(\Phi, \nabla\Phi)$ are given by

$$\begin{aligned}\mathbf{a}_i(\Phi, \nabla\Phi) = -\frac{\mu V(\phi)}{g}\biggl[&\frac{(1-\phi)\phi_i}{\phi}(\delta_i - \boldsymbol{\delta}^{\mathrm{T}}\Phi)\nabla\sigma_{\mathrm{e}}(\phi)\\ &+ \sigma_{\mathrm{e}}(\phi)\biggl(\delta_i\nabla\Bigl(\frac{\phi_i}{\phi}\Bigr) - \phi_i\sum_{k=1}^{N}\delta_k\nabla\Bigl(\frac{\phi_k}{\phi}\Bigr)\biggr)\biggr], \quad i = 1, \dots, N.\end{aligned} \tag{2.22}$$

The continuity equations for the solids, i.e. for the N unknowns ϕ_1 to ϕ_N, can then be written as

$$\frac{\partial\phi_i}{\partial t} + \nabla \cdot \bigl(\phi_i\mathbf{q} + f_i^{\mathrm{M}}(\Phi)\mathbf{k}\bigr) = \nabla \cdot \mathbf{a}_i(\Phi, \nabla\Phi), \quad i = 1, \dots, N. \tag{2.23}$$

Due to the property (2.6), $a(\Phi, \nabla\Phi) = 0$ wherever $\phi \le \phi_{\mathrm{c}}$. At these concentrations, the system (2.23) turns into one of the first-order systems of N scalar equations analyzed in [20].

The final system of model equations, valid in several space dimensions, can be stated as

$$\frac{\partial \phi_i}{\partial t} + \nabla \cdot \big(\phi_i \mathbf{q} + f_i(\Phi)\mathbf{k}\big) = \nabla \cdot \mathbf{a}_i(\Phi, \nabla\Phi), \quad i = 1, \dots, N, \tag{2.24}$$

$$\nabla \cdot \mathbf{q} = 0, \tag{2.25}$$

$$\nabla p = -(\varrho_{\mathrm{f}} + \bar{\boldsymbol{\varrho}} \cdot \Phi) g \mathbf{k} + \frac{1}{1-\phi} \nabla \cdot \mathbf{T}_{\mathrm{f}}^{\mathrm{E}} \equiv -\varrho(\Phi) g \mathbf{k} + \frac{1}{1-\phi} \nabla \cdot \mathbf{T}_{\mathrm{f}}^{\mathrm{E}}. \tag{2.26}$$

Noting that $\mathbf{v}_{\mathrm{f}} = \mathbf{q} - (\phi_1 \mathbf{u}_1 + \dots + \phi_N \mathbf{u}_N)$, we can rewrite $\mathbf{T}_{\mathrm{f}}^{\mathrm{E}}$ in terms of the mixture velocity $\mathbf{q}$ and the slip velocities $\mathbf{u}_i$, which are now given functions of Φ. For example, if we use the expression $\mathbf{T}_{\mathrm{f}}^{\mathrm{E}} = \mu(\phi)[\nabla \mathbf{v}_{\mathrm{f}} + (\nabla \mathbf{v}_{\mathrm{f}})^{\mathrm{T}} - (2/3)(\nabla \cdot \mathbf{v}_{\mathrm{f}})\mathbf{I}]$ corresponding to a standard viscous-linear fluid with a concentration-dependent viscosity function, then Eq. (2.26) can be rewritten in the form

$$\nabla p = -\varrho(\Phi) g \mathbf{k} + \frac{1}{1-\phi} \Big[\big(\nabla \mu(\phi)\big)^{\mathrm{T}} \big(\nabla \mathbf{q} + (\nabla \mathbf{q})^{\mathrm{T}}\big) + \mu(\phi) \Delta \mathbf{q} \Big] + \mathbf{g}(\Phi, \nabla\Phi, \nabla^2 \Phi),$$

where $\mathbf{g}$ is a function depending on Φ and the derivatives of its components of up to second order. For pure fluid, i.e. when $\Phi \equiv 0$ (and thus $\mathbf{q} \equiv \mathbf{v}_{\mathrm{f}}$), Eqns. (2.25) and (2.26) form the Stokes system for an incompressible fluid for the velocity $\mathbf{q}$ and the pressure p.

In the rest of this paper we limit the discussion to one space dimension. In this case, we get $\partial q/\partial z = 0$, and only Eq. (2.24) needs to be solved, since the quantity q is given by boundary conditions and Eq. (2.26) turns into an equation for the pore pressure p, which permits us to calculate this quantity a posteriori from the concentrations $\phi_1, \dots, \phi_N$.

Finally, we set $N = 1$ to check consistency with the model of sedimentation of monodisperse flocculated suspensions [22]. With

$$f^{\mathrm{M}}(\phi) = -\frac{g d^2 \bar{\varrho}_{\mathrm{s}}}{18 \mu_{\mathrm{f}}} V(\phi) \phi (1-\phi)^2, \quad \mathbf{a}(\phi, \nabla\phi) = a(\phi) \nabla \phi = -\frac{f^{\mathrm{M}}(\phi)}{\bar{\varrho}_{\mathrm{s}} g \phi} \nabla \sigma_{\mathrm{e}}(\phi),$$

Eq. (2.23) indeed reduces to the scalar equation

$$\frac{\partial \phi}{\partial t} + \nabla \cdot \big(\phi \mathbf{q} + f^{\mathrm{M}}(\phi) \mathbf{k}\big) = \nabla \cdot \big(a(\phi) \nabla \phi\big) \tag{2.27}$$

for monodisperse flocculated suspensions derived in [22], where $a(\phi) = -f^{\mathrm{M}}(\phi) \sigma_{\mathrm{e}}'(\phi)/(\bar{\varrho}_{\mathrm{s}} g \phi)$. It is easy to see that Eq. (2.27) is first-order hyperbolic for $\phi \leq \phi_{\mathrm{c}}$ and $\phi = 1$, and second-order parabolic for $\phi_{\mathrm{c}} < \phi < 1$, and therefore a strongly degenerate parabolic equation.

2.8. **Initial and boundary conditions in one space dimension.** In a closed one-dimensional vessel, the mixture velocity at the bottom vanishes, hence $q \equiv 0$ and the remaining equations that actually have to be solved are the system of convection-diffusion equations

$$\frac{\partial \phi_i}{\partial t} + \frac{\partial f_i^{\mathrm{M}}(\Phi)}{\partial z} = \frac{\partial}{\partial z} \left[a_i \left(\Phi, \frac{\partial \Phi}{\partial z} \right) \right], \quad i = 1, \dots, N, \tag{2.28}$$

together with an initial concentration distribution

$$\Phi(z, 0) = \Phi^0(z) \in \mathcal{D}_{\phi_{\max}}, \quad 0 \leq z \leq L, \tag{2.29}$$

and the zero flux boundary conditions $\phi_1 v_1 = \dots = \phi_N v_N = 0$ at $z = 0$ and $z = L$, i.e.

$$f_i^{\mathrm{M}}(\Phi) - a_i \left(\Phi, \frac{\partial \Phi}{\partial z} \right) = 0 \quad \text{for } z = 0 \text{ and } z = L,\ i = 1, \dots, N. \tag{2.30}$$

3. Hyperbolicity of the first-order system

We now assume $\varrho_1 = \dots = \varrho_N = \varrho_{\mathrm{s}}$, so that the components of $\mathbf{f}^{\mathrm{M}}(\Phi)$ are

$$f_i^{\mathrm{M}}(\Phi) = \mu V(\phi)(1-\phi) \bar{\varrho}_{\mathrm{s}} (\delta_i - \boldsymbol{\delta}^{\mathrm{T}} \Phi) \phi_i, \quad i = 1, \dots, N, \tag{3.1}$$

and denote by $P(\lambda)$ the characteristic polynomial of $\mathbf{J} := (\mu \bar{\varrho}_{\mathrm{s}})^{-1} \mathcal{J}_{\mathbf{f}^{\mathrm{M}}}(\Phi)$, where $\mathcal{J}_{\mathbf{f}^{\mathrm{M}}}(\Phi) = (\partial f_i^{\mathrm{M}}(\Phi)/\partial \phi_j)_{i,j=1,\dots,N}$ is the Jacobian of $\mathbf{f}^{\mathrm{M}}(\Phi)$. We now derive a closed algebraic expression for $P(\lambda)$. We can write $(\mu \bar{\varrho}_{\mathrm{s}})^{-1} \partial f_i^{\mathrm{M}}(\Phi)/\partial \phi_j = \gamma_j^i(\Phi) \phi_i + \gamma^i(\Phi) \delta_{ij}$ for $i, j = 1, \dots, N$, where

$$\gamma^i(\Phi) := V(\phi)(1-\phi)(\delta_i - \boldsymbol{\delta}^{\mathrm{T}} \Phi), \quad i = 1, \dots, N, \tag{3.2}$$

$$\gamma_j^i(\Phi) := \big(V(\phi)(1-\phi)\big)'(\delta_i - \boldsymbol{\delta}^{\mathrm{T}}\Phi) - V(\phi)(1-\phi)\delta_j,\ i,j=1,\dots,N. \tag{3.3}$$

The characteristic polynomial can be written as

$$P(\lambda) := \det(\mathbf{J} - \lambda\mathbf{I}) = \begin{vmatrix} \gamma_1^1\phi_1 + \gamma^1 - \lambda & \gamma_2^1\phi_1 & \cdots & \gamma_N^1\phi_1 \\ \gamma_1^2\phi_2 & \gamma_2^2\phi_2 + \gamma^2 - \lambda & \cdots & \gamma_N^2\phi_2 \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_1^N\phi_N & \gamma_2^N\phi_N & \cdots & \gamma_N^N\phi_N + \gamma^N - \lambda \end{vmatrix}. \tag{3.4}$$

In the sequel we omit the argument Φ and, for later use, note that $\gamma^i - \gamma^l = V(\phi)(1-\phi)(\delta_i - \delta_l)$, which due to $\delta_1 > \delta_2 > \cdots > \delta_N$ implies $\gamma^N < \gamma^{N-1} < \cdots < \gamma^1$. Moreover, we observe that

$$\begin{aligned} \gamma_j^1 - \gamma_k^1 &= \cdots = \gamma_j^N - \gamma_k^N = -V(\phi)(1-\phi)(\delta_j - \delta_k), \quad j,k = 1,\dots,N, \\ \gamma_1^j - \gamma_1^k &= \cdots = \gamma_N^j - \gamma_N^k = (V(\phi)(1-\phi))'(\delta_j - \delta_k), \quad j,k = 1,\dots,N. \end{aligned}$$

The common values of $\gamma_j^i - \gamma_k^i$ and $\gamma_i^j - \gamma_i^k$ for all $i = 1,\dots,N$, $j,k = 1,\dots,N$, will be denoted by $\gamma_{j,k}$ and $\gamma^{j,k}$, respectively. Since the matrices $\mathcal{J}_{\mathbf{f}^{\mathrm{M}}}(\Phi)$ and $\mathbf{A}(\Phi)$ have similar structure and therefore similar characteristic polynomials, it is convenient for later use to prove the following lemma separately.

Lemma 1. *The polynomial $P(\lambda)$ defined in* (3.4) *satisfies*

$$P(\lambda) = \left\{ 1 + \sum_{m=1}^N \frac{\phi_m \gamma_m^m}{\gamma^m - \lambda} - \sum_{m=1}^N \frac{\phi_m}{\gamma^m - \lambda} \sum_{l=1}^N \frac{\phi_l \gamma_{l,N} \gamma^{l,m}}{\gamma^l - \lambda} \right\} \prod_{k=1}^N (\gamma^k - \lambda). \tag{3.5}$$

Proof. In this proof we merely use the definitions of $\gamma_{j,k}$ and $\gamma^{j,k}$ in terms of the γ_j^i's and γ_j's. Subtracting column N from columns 1 to $N-1$ in (3.4) yields

$$P(\lambda) = \begin{vmatrix} \gamma_{1,N}\phi_1 + \gamma^1 - \lambda & \cdots & \gamma_{N-1,N}\phi_1 & \gamma_N^1\phi_1 \\ \vdots & \ddots & \vdots & \vdots \\ \gamma_{1,N}\phi_{N-1} & \cdots & \gamma_{N-1,N}\phi_{N-1} + \gamma^{N-1} - \lambda & \gamma_N^{N-1}\phi_{N-1} \\ \gamma_{1,N}\phi_N - \gamma^N + \lambda & \cdots & \gamma_{N-1,N}\phi_N - \gamma^N + \lambda & \gamma_N^N\phi_N + \gamma^N - \lambda \end{vmatrix}. \tag{3.6}$$

Expanding this determinant on the last row, we get

$$P(\lambda) = X + \sum_{m=1}^{N-1} (-1)^{N+1-m} (\gamma^N - \lambda) Y_m, \tag{3.7}$$

where X and Y_m are the determinants obtained from (3.6) by replacing the last row by

$$(\gamma_{1,N}\phi_N, \dots, \gamma_{N-1,N}\phi_N, \gamma_N^N\phi_N + \gamma^N - \lambda)$$

and by deleting the last row and the mth column, $m = 1,\dots,N-1$, respectively. Multiplying the last row in X by $(-\phi_i/\phi_N)$ and adding the result to the ith row, $i = 1,\dots,N-1$ leads to

$$X = \begin{vmatrix} \gamma^1 - \lambda & \cdots & 0 & \phi_1\big(\gamma^{1,N} - (\gamma^N - \lambda)/\phi_N\big) \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \gamma^{N-1} - \lambda & \phi_{N-1}\big(\gamma^{N-1,N} - (\gamma^N - \lambda)/\phi_N\big) \\ \gamma_{1,N}\phi_N & \cdots & \gamma_{N-1,N}\phi_N & \gamma_N^N\phi_N + \gamma^N - \lambda \end{vmatrix}.$$

Expanding X on the last row yields

$$X = \left(\gamma_N^N\phi_N + \gamma^N - \lambda - \phi_N \sum_{m=1}^{N-1} \frac{\phi_m \gamma_{m,N}}{\gamma^m - \lambda} \left(\gamma^{m,N} - \frac{\gamma^N - \lambda}{\phi_N} \right) \right) \prod_{k=1}^{N-1} (\gamma^k - \lambda). \tag{3.8}$$

Furthermore, we have $Y_m = (-1)^{N-1-m}\tilde{Y}_m$, where

$$\tilde{Y}_m = \begin{vmatrix} \begin{matrix}\gamma_{1,N}\phi_1 \\ +\gamma^1-\lambda\end{matrix} & \cdots & \gamma_{m-1,N}\phi_1 & \gamma_{m+1,N}\phi_1 & \cdots & \gamma_{N-1,N}\phi_1 & \gamma_N^1\phi_1 \\ \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots \\ \gamma_{1,N}\phi_{m-1} & \cdots & \begin{matrix}\gamma_{m-1,N}\phi_{m-1} \\ +\gamma^{m-1}-\lambda\end{matrix} & \gamma_{m+1,N}\phi_{m-1} & \cdots & \gamma_{N-1,N}\phi_{m-1} & \gamma_N^{m-1}\phi_{m-1} \\ \gamma_{1,N}\phi_{m+1} & \cdots & \gamma_{m-1,N}\phi_{m+1} & \begin{matrix}\gamma_{m+1,N}\phi_{m+1} \\ +\gamma^{m+1}-\lambda\end{matrix} & \cdots & \gamma_{N-1,N}\phi_{m+1} & \gamma_N^{m+1}\phi_{m+1} \\ \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots \\ \gamma_{1,N}\phi_{N-1} & \cdots & \gamma_{m-1,N}\phi_{N-1} & \gamma_{m+1,N}\phi_{N-1} & \cdots & \begin{matrix}\gamma_{N-1,N}\phi_{N-1} \\ +\gamma^{N-1}-\lambda\end{matrix} & \gamma_N^{N-1}\phi_{N-1} \\ \gamma_{1,N}\phi_m & \cdots & \gamma_{m-1,N}\phi_m & \gamma_{m+1,N}\phi_m & \cdots & \gamma_{N-1,N}\phi_m & \gamma_N^m\phi_m \end{vmatrix},$$

which implies

$$Y_m = (-1)^{N-1-m}\frac{\phi_m}{\gamma^m-\lambda}\left(\gamma_N^m - \sum_{l=1}^{N-1}\frac{\gamma_{l,N}\gamma^{l,m}\phi_l}{\gamma^l-\lambda}\right)\prod_{k=1}^{N-1}(\gamma^k-\lambda), \quad m=1,\dots,N-1. \tag{3.9}$$

Inserting (3.8) and (3.9) into (3.7), we get

$$\begin{aligned} P(\lambda) = & \Biggl\{\gamma_N^N\phi_N + \gamma^N - \lambda - \phi_N\sum_{m=1}^{N-1}\frac{\phi_m}{\gamma^m-\lambda}\gamma_{m,N}\left(\gamma^{m,N} - \frac{\gamma^N-\lambda}{\phi_N}\right) \\ & + (\gamma^N-\lambda)\sum_{m=1}^{N-1}\frac{\phi_m}{\gamma^m-\lambda}\left(\gamma_N^m - \sum_{l=1}^{N-1}\phi_l\frac{\gamma_{l,N}\gamma^{l,m}}{\gamma^l-\lambda}\right)\Biggr\}\prod_{k=1}^{N-1}(\gamma^k-\lambda) \\ = & \Biggl\{1 - \frac{\phi_N}{\gamma^N-\lambda}\sum_{m=1}^{N-1}\frac{\phi_m\gamma_{m,N}\gamma^{m,N}}{\gamma^m-\lambda} + \sum_{m=1}^{N-1}\frac{\phi_m\gamma_{m,N}}{\gamma^m-\lambda} \\ & + \sum_{m=1}^{N}\frac{\phi_m\gamma_N^m}{\gamma^m-\lambda} - \sum_{m=1}^{N-1}\frac{\phi_m}{\gamma^m-\lambda}\sum_{l=1}^{N-1}\frac{\phi_l\gamma_{l,N}\gamma^{l,m}}{\gamma^l-\lambda}\Biggr\}\prod_{l=1}^{N}(\gamma^l-\lambda). \end{aligned} \tag{3.10}$$

The upper index of summation in the second sum in the second equation of (3.10) can be changed to N since $\gamma_{N,N}=0$ and the second and third sum can be combined into one using $\gamma_{m,N}+\gamma_N^m = \gamma_m^m$. Furthermore, the first and the fourth sum can be combined into one by changing the upper index of summation for m in the fourth sum from $N-1$ to N, from which we obtain (3.5). □

We can now prove the following lemma.

Lemma 2. *Let $\lambda\in\mathbb{R}$ and $\delta(\lambda) := (V(\phi)(1-\phi))^{-1}\lambda + \boldsymbol{\delta}^{\mathrm{T}}\Phi$. Then $P(\lambda)$ is given by*

$$\begin{aligned} P(\lambda) = & \Biggl\{V(\phi)(1-\phi) + \sum_{m=1}^{N}\frac{\phi_m}{\delta_m-\delta(\lambda)}\biggl[-\delta_m V(\phi)(1-\phi) + \bigl(V(\phi)(1-\phi)\bigr)'\times \\ & \times\left(\delta_m - \boldsymbol{\delta}^{\mathrm{T}}\Phi + \sum_{l=1}^{N}\frac{\delta_l\phi_l(\delta_l-\delta_m)}{\delta_l-\delta(\lambda)}\right)\biggr]\Biggr\}\bigl(V(\phi)(1-\phi)\bigr)^{N-1}\prod_{k=1}^{N}\bigl(\delta_k-\delta(\lambda)\bigr). \end{aligned} \tag{3.11}$$

This expression is also well-defined for $\lambda\in\{\gamma^1,\dots,\gamma^N\}$ and reads for $k=1,\dots,N$

$$P(\gamma^k) = \phi_k\delta_k\Bigl\{\bigl(V(\phi)(1-\phi)\bigr)' - V(\phi)\Bigr\}\bigl(V(\phi)\bigr)^{N-1}(1-\phi)^N\prod_{\substack{l=1\\ l\neq k}}^{N}(\delta_l-\delta_k). \tag{3.12}$$

Proof. Using that $\gamma^m - \lambda = (\delta_m - \delta(\lambda))V(\phi)(1-\phi)$ and inserting the definitions of $\gamma_{l,N}$ and $\gamma^{l,m}$, we get

$$
\begin{aligned}
P(\lambda) =& \Biggl\{ 1 + \sum_{m=1}^{N} \frac{\phi_m[(V(\phi)(1-\phi))'(\delta_m - \boldsymbol{\delta}^{\mathrm{T}}\Phi) - V(\phi)(1-\phi)\delta_m]}{V(\phi)(1-\phi)(\delta_m - \delta(\lambda))} + \frac{(V(\phi)(1-\phi))'}{V(\phi)(1-\phi)} \times \\
&\times \sum_{m=1}^{N}\sum_{l=1}^{N} \frac{\phi_m\phi_l(\delta_l^2 - \delta_l\delta_m - \delta_l\delta_N + \delta_m\delta_N)}{(\delta_m - \delta(\lambda))(\delta_l - \delta(\lambda))} \Biggr\} \bigl(V(\phi)(1-\phi)\bigr)^N \prod_{k=1}^{N}(\delta_k - \delta(\lambda)) \\
=& \Biggl\{ V(\phi)(1-\phi) + \sum_{m=1}^{N} \frac{\phi_m}{\delta_m - \delta(\lambda)} \Biggl[-V(\phi)(1-\phi)\delta_m + \bigl(V(\phi)(1-\phi)\bigr)' \times \\
&\times \Biggl(\delta_m - \boldsymbol{\delta}^{\mathrm{T}}\Phi + \sum_{l=1}^{N} \frac{\delta_l\phi_l(\delta_l - \delta_m)}{\delta_l - \delta(\lambda)} \Biggr) \Biggr] \Biggr\} \bigl(V(\phi)(1-\phi)\bigr)^{N-1} \prod_{k=1}^{N}(\delta_k - \delta(\lambda)),
\end{aligned} \tag{3.13}
$$

which is Eq. (3.11). This expression can be rewritten as

$$
\begin{aligned}
P(\lambda) =& \prod_{l=1}^{N}(\gamma^l - \lambda) + \Biggl\{ \sum_{m=1}^{N} \phi_m \Bigl[-\delta_m V(\phi)(1-\phi) + \bigl(V(\phi)(1-\phi)\bigr)'(\delta_m - \boldsymbol{\delta}^{\mathrm{T}}\Phi) \Bigr] \prod_{\substack{l=1\\ l\neq m}}^{N} \bigl(\delta_l - \delta(\lambda)\bigr) \\
&+ \bigl(V(\phi)(1-\phi)\bigr)' \sum_{m=1}^{N}\sum_{l=1}^{N} \phi_m\phi_l\delta_l(\delta_l - \delta_m) \prod_{\substack{n=1\\ n\neq m,l}}^{N} \bigl(\delta_n - \delta(\lambda)\bigr) \Biggr\} V(\phi)(1-\phi)^{N-1}.
\end{aligned}
$$

For $\lambda = \gamma^k$ the first product vanishes and in the first sum only the summand with $m = k$ and in the second sum only the summands with $m = k$ or $l = k$ do not vanish. This implies

$$
\begin{aligned}
P(\gamma^k) =& \Biggl\{ \phi_k \Bigl[-\delta_k V(\phi)(1-\phi) + \bigl(V(\phi)(1-\phi)\bigr)'(\delta_k - \boldsymbol{\delta}^{\mathrm{T}}\Phi) \Bigr] \prod_{\substack{l=1\\ l\neq k}}^{N}(\delta_l - \delta_k) \\
&+ \bigl(V(\phi)(1-\phi)\bigr)' \Biggl[\phi_k \sum_{l=1}^{N} \Biggl(\delta_l\phi_l(\delta_l - \delta_k) \prod_{\substack{n=1\\ n\neq k,l}}^{N} (\delta_n - \delta_k) \Biggr) \\
&+ \sum_{m=1}^{N} \Biggl(\phi_m\delta_k\phi_k(\delta_k - \delta_m) \prod_{\substack{n=1\\ n\neq k,m}}^{N} (\delta_n - \delta_k) \Biggr) \Biggr] \Biggr\} \bigl(V(\phi)(1-\phi)\bigr)^{N-1} \\
=& \Bigl\{ -\delta_k\phi_k V(\phi)(1-\phi) + \delta_k\phi_k\bigl(V(\phi)(1-\phi)\bigr)' - \phi_k\boldsymbol{\delta}^{\mathrm{T}}\Phi\bigl(V(\phi)(1-\phi)\bigr)' \\
&+ \phi_k\boldsymbol{\delta}^{\mathrm{T}}\Phi\bigl(V(\phi)(1-\phi)\bigr)' - \delta_k\phi_k\phi\bigl(V(\phi)(1-\phi)\bigr)' \Bigr\} \bigl(V(\phi)(1-\phi)\bigr)^{N-1} \prod_{\substack{l=1\\ l\neq k}}^{N}(\delta_l - \delta_k),
\end{aligned}
$$

from which (3.12) can be read off immediately. □

Theorem 1. *If $\varrho_1 = \cdots = \varrho_N = \varrho_{\mathrm{s}}$, $\delta_1 > \delta_2 > \cdots > \delta_N$, and $\Phi \in \mathcal{D}^0_{\phi_{\max}}$, then the system (1.4) is strictly hyperbolic, i.e. the Jacobian $\mathcal{J}_{\mathbf{f}^{\mathrm{M}}}(\Phi)$ has N distinct real eigenvalues.*

Proof. From (3.12) we see that

$$
P(\gamma^k) = C_k\bigl(V(\phi)\bigr)^{N-1}(1-\phi)^N \prod_{\substack{m=1\\ m\neq k}}^{N} (\delta_m - \delta_k), \quad k = 1, \dots, N
$$

with $C_k := \delta_k\phi_k(V'(\phi)(1-\phi) - 2V(\phi))$. Since $C_k < 0$ on $\mathcal{D}^0_{\phi_{\max}}$ for all $k = 1, \dots, N$ due to (2.19), we have

$$
\operatorname{sgn}\bigl(P(\gamma^k)\bigr) = -\operatorname{sgn}\Biggl(\prod_{\substack{m=1\\ m\neq k}}^{N} (\delta_m - \delta_k) \Biggr) = -\operatorname{sgn}\Biggl(\prod_{m=k+1}^{N} (\delta_m - \delta_k) \Biggr) = (-1)^{N-k+1}.
$$

Consequently, we have shown that $\operatorname{sgn}(P(\gamma^i)) = (-1)^{N+1-i}$ for $i = 1, \dots, N$. Whether N is even or odd, we have $P(\lambda) \to \infty$ as $\lambda \to -\infty$ and $P(\gamma^N) < 0$. In view of $\gamma^N < \gamma^{N-1} < \dots < \gamma^1$ and since $P(\gamma^N) < 0$, there exists a number $\lambda_N < \gamma^N$ with $P(\lambda_N) = 0$. Furthermore, $\operatorname{sgn}(P(\gamma^i)) = (-1)^{N+1-i}$ implies that there exist $N-1$ numbers $\lambda_i \in (\gamma^{i+1}, \gamma^i)$, $i = 1, \dots N-1$, with $P(\lambda_i) = 0$. This shows that $P(\lambda) = \det(\mathbf{J} - \lambda \mathbf{I})$ has N roots $\lambda_1, \dots, \lambda_N$ satisfying

$$\lambda_N < \gamma^N < \lambda_{N-1} < \gamma^{N-1} < \dots < \lambda_2 < \gamma^2 < \lambda_1 < \gamma^1. \tag{3.14}$$

Thus the system (1.4) is strictly hyperbolic for all $\Phi \in \mathcal{D}^0_{\phi_{\max}}$. Theorem 1 is proved. □

The statement of Theorem 1 can still be improved. In fact, it is desirable to have lower and upper bounds for all eigenvalues of $\mathcal{J}_{\mathbf{f}^{\mathrm{M}}}(\Phi)$. However, in (3.14) a lower bound for the eigenvalue λ_N of $\mathbf{J}$ is still lacking. The following theorem shows that by evaluating $P(\lambda)$ at a suitable number $\gamma^\infty < \gamma^1$ it is indeed possible to provide that bound.

Theorem 2. *Under the conditions of Theorem 1, the eigenvalues $\nu_1(\Phi), \dots, \nu_N(\Phi)$ of $\mathcal{J}_{\mathbf{f}^{\mathrm{M}}}(\Phi)$ satisfy*

$$\nu_i(\Phi) \in \bigl(\mu\bar{\varrho}_{\mathrm{s}} V(\phi)(1-\phi)(\delta_i - \boldsymbol{\delta}^{\mathrm{T}}\Phi), \mu\bar{\varrho}_{\mathrm{s}} V(\phi)(1-\phi)(\delta_{i+1} - \boldsymbol{\delta}^{\mathrm{T}}\Phi)\bigr), \quad i = 1, \dots, N-1, \tag{3.15}$$

$$\nu_N(\Phi) \in \bigl(\mu\bar{\varrho}_{\mathrm{s}} V(\phi)(1-\phi)(\delta_N - \boldsymbol{\delta}^{\mathrm{T}}\Phi), \mu\bar{\varrho}_{\mathrm{s}}\gamma^\infty\bigr), \tag{3.16}$$

where $\gamma^\infty := -2\boldsymbol{\delta}^{\mathrm{T}}\Phi V(\phi)(1-\phi) + (V(\phi)(1-\phi))'(\boldsymbol{\delta}^{\mathrm{T}}\Phi + \phi)$.

Proof. We first evaluate $P(\lambda)$ assuming that $\delta(\lambda) < 0$. Moreover, to estimate the factor in curled brackets in the second equation of (3.13), we use that $(V(\phi)(1-\phi))'/V(\phi)(1-\phi) < 0$ justifies deleting $-\delta_m$ and the replacement of δ_l^2 by δ_l in the last sum. Furthermore, we use that $1/(\delta_m - \delta(\gamma^\infty)) < -1/\delta(\gamma^\infty)$ in several instances, which leads to

$$\begin{aligned} &1 + \sum_{m=1}^N \frac{\phi_m}{\delta_m - \delta(\lambda)} \left[-\delta_m + \frac{(V(\phi)(1-\phi))'}{V(\phi)(1-\phi)} \left(\delta_m - \boldsymbol{\delta}^{\mathrm{T}}\Phi + \sum_{l=1}^N \frac{\delta_l \phi_l (\delta_l - \delta_m)}{\delta_l - \delta(\lambda)} \right) \right] \\ &\geq 1 - \frac{\boldsymbol{\delta}^{\mathrm{T}}\Phi}{\delta(\lambda)} \left(\frac{(V(\phi)(1-\phi))'}{V(\phi)(1-\phi)} - 1 \right) + \frac{(V(\phi)(1-\phi))'}{V(\phi)(1-\phi)} \left(\frac{\phi}{\delta(\lambda)^2} \sum_{l=1}^N \delta_l \phi_l - \boldsymbol{\delta}^{\mathrm{T}}\Phi \sum_{m=1}^N \frac{\phi_m}{\delta_m - \delta(\lambda)} \right). \end{aligned}$$

If we delete the last sum, the left-hand part of this inequality will remain positive whenever

$$\delta^2(\lambda) + \delta(\lambda)\boldsymbol{\delta}^{\mathrm{T}}\Phi \left[1 - \frac{(V(\phi)(1-\phi))'}{V(\phi)(1-\phi)} \right] + \boldsymbol{\delta}^{\mathrm{T}}\Phi\phi \frac{(V(\phi)(1-\phi))'}{V(\phi)(1-\phi)} > 0.$$

This can be achieved by letting $\lambda = \gamma^\infty$, such that

$$\delta(\gamma^\infty) = \boldsymbol{\delta}^{\mathrm{T}}\Phi \left[\frac{(V(\phi)(1-\phi))'}{V(\phi)(1-\phi)} - 1 \right] + \frac{(V(\phi)(1-\phi))'}{V(\phi)(1-\phi)}\phi.$$

In fact, making obvious simplifications, we then obtain $\phi(\phi + \boldsymbol{\delta}^{\mathrm{T}}\Phi) > 0$, and the inequality is proved. Since $\gamma^\infty < \gamma^N$, $P(\gamma^N) < 0$ and $P(\gamma^\infty) > 0$, the smallest eigenvalue λ_N of $\mathbf{J}$ satisfies $\gamma^\infty < \lambda_N < \gamma^N$. Combining this with (3.14) and recalling that the eigenvalues ν_i of $\mathcal{J}_{\mathbf{f}^{\mathrm{M}}}(\Phi)$ are given by $\nu_i = \mu\bar{\varrho}_{\mathrm{s}}\lambda_i$, $i = 1, \dots, N$, we obtain the statement of Theorem 2. □

4. Properties of the diffusion matrix

Using

$$\frac{\partial}{\partial z}\left(\frac{\phi_i}{\phi}\right) = \frac{1}{\phi}\frac{\partial \phi_i}{\partial z} - \frac{\phi_i}{\phi^2}\sum_{j=1}^N \frac{\partial \phi_j}{\partial z} = \frac{1}{\phi}\left(\frac{\partial \phi_i}{\partial z} - \frac{\phi_i}{\phi}\sum_{j=1}^N \frac{\partial \phi_j}{\partial z}\right), \quad i = 1, \dots, N$$

and defining $W(\phi) := -\mu V(\phi)/(g\phi)$ and

$$\eta_{ij}(\Phi) := W(\phi)\left\{ (1-\phi)\phi_i(\delta_i - \boldsymbol{\delta}^{\mathrm{T}}\Phi)\sigma_{\mathrm{e}}'(\phi) + \left[\delta_i\delta_{ij} - \delta_j\phi_i - \frac{\phi_i}{\phi}(\delta_i - \boldsymbol{\delta}^{\mathrm{T}}\Phi) \right]\sigma_{\mathrm{e}}(\phi) \right\}$$

for $i,j=1,\dots,N$, where $\delta_{ij}=1$ if $i=j$ and $\delta_{ij}=0$ otherwise, we get from Eq. (2.22)

$$a_i\left(\Phi,\frac{\partial\Phi}{\partial z}\right)=\sum_{j=1}^{N}\eta_{ij}(\Phi)\frac{\partial\phi_j}{\partial z},\quad i=1,\dots,N. \tag{4.1}$$

Defining the matrix $\mathbf{A}(\Phi):=(\eta_{ij}(\Phi))_{1\leq i,j\leq N}$, and taking $\mathbf{f}=\mathbf{f}^{\mathrm{M}}$ we can rewrite the equations (2.28) in the form (1.2). We show that the eigenvalues of $\mathbf{A}(\boldsymbol{\Phi})$ are positive and pairwise distinct on $\mathcal{D}^0_{\phi_{\max}}$ by evaluating the characteristic polynomial in a fashion similar to Section 3. To this end, we first provide an explicit expression for $S(\lambda):=\det(W(\phi)^{-1}\mathbf{A}(\Phi)-\lambda\mathbf{I})$.

Lemma 3. *Let $\delta^*(\lambda):=\lambda/\sigma_{\mathrm{e}}(\phi)$. Then the polynomial $S(\lambda)$ is given by*

$$\begin{aligned}S(\lambda)=&\Biggl\{\sigma_{\mathrm{e}}(\phi)+\sum_{m=1}^{N}\frac{\phi_m}{\delta_m-\delta^*(\lambda)}\Biggl[-\delta_m\sigma_{\mathrm{e}}(\phi)+\left((1-\phi)\sigma_{\mathrm{e}}'(\phi)-\frac{\sigma_{\mathrm{e}}(\phi)}{\phi}\right)\times\\ &\times\left(\delta_m-\boldsymbol{\delta}^{\mathrm{T}}\Phi+\sum_{l=1}^{N}\frac{\phi_l\delta_l(\delta_l-\delta_m)}{\delta_l-\delta^*(\lambda)}\right)\Biggr]\Biggr\}\bigl(\sigma_{\mathrm{e}}(\phi)\bigr)^{N-1}\prod_{k=1}^{N}\bigl(\delta_k-\delta^*(\lambda)\bigr).\end{aligned} \tag{4.2}$$

Proof. We write $\eta_{ij}(\Phi)/W(\phi)=s^i_j\phi_i+s^i\delta_{ij}$ for $1\leq i,j\leq N$, where we define

$$\begin{aligned}s^i_j&:=(1-\phi)(\delta_i-\boldsymbol{\delta}^{\mathrm{T}}\Phi)\sigma_{\mathrm{e}}'(\phi)-\left(\delta_j+\frac{1}{\phi}(\delta_i-\boldsymbol{\delta}^{\mathrm{T}}\Phi)\right)\sigma_{\mathrm{e}}(\phi),\quad i,j=1,\dots,N,\\ s^i&:=\sigma_{\mathrm{e}}(\phi)\delta_i,\ i=1,\dots,N.\end{aligned}$$

Consequently, $S(\lambda)$ can be written as

$$S(\lambda)=\begin{vmatrix}s^1_1\phi_1+s^1-\lambda & s^1_2\phi_1 & \dots & s^1_N\phi_1\\ s^2_1\phi_2 & s^2_2\phi_2+s^2-\lambda & \dots & s^2_N\phi_2\\ \vdots & \vdots & \ddots & \vdots\\ s^N_1\phi_N & s^N_2\phi_N & \dots & s^N_N\phi_N+s^N-\lambda\end{vmatrix}.$$

Observe that the numbers s^i_j satisfy

$$\begin{aligned}s^i_j-s^i_k&=-\sigma_{\mathrm{e}}(\phi)(\delta_j-\delta_k),\quad i=1,\dots,N,\\ s^j_i-s^k_i&=(\delta_j-\delta_k)\left[(1-\phi)\sigma_{\mathrm{e}}'(\phi)-\frac{\sigma_{\mathrm{e}}(\phi)}{\phi}\right],\quad i=1,\dots,N,\end{aligned} \tag{4.3}$$

i.e., the right-hand parts of (4.3) do not depend on i. Therefore, we may introduce

$$s_{j,k}:=s^1_j-s^1_k=\dots=s^N_j-s^N_k,\quad s^{j,k}:=s^j_1-s^k_1=\dots=s^j_N-s^k_N,\qquad j,k=1,\dots,N.$$

We can now easily provide an explicit expression for $S(\lambda)$ in terms of the s's, since the rules for the s's correspond to those for the γ's. Thus, replacing $V(\phi)(1-\phi)$ by $\sigma_{\mathrm{e}}(\phi)$, $(V(\phi)(1-\phi))'$ by $(1-\phi)\sigma_{\mathrm{e}}'(\phi)-\sigma_{\mathrm{e}}(\phi)/\phi$, $\delta(\lambda)$ by $\delta^*(\lambda)=\lambda/\sigma_{\mathrm{e}}(\phi)$, we obtain (4.2) by closely following the proofs of Lemma 1 and of Eq. (3.11) in Lemma 2. □

To localize the eigenvalues of $\mathbf{A}(\Phi)$ we need to evaluate $S(\lambda)$ at $\lambda=0$ and $\lambda=s^1,\dots,s^N$. Using the analogy between $P(\lambda)$ and $S(\lambda)$, we can easily prove the following lemma.

Lemma 4. *The determinant of $\mathbf{A}(\Phi)$ is given by $\det(\mathbf{A}(\Phi))=(W(\phi))^N S(0)$, where*

$$S(0)=\delta_1\cdots\delta_N\bigl(\sigma_{\mathrm{e}}(\phi)\bigr)^{N-1}\sigma_{\mathrm{e}}'(\phi)\phi(1-\phi)^2\quad\text{for } 0\leq\phi\leq\phi_{\max}\text{ and } N\geq 2. \tag{4.4}$$

Moreover, for $k=1,\dots,N$ we have

$$S(s^k)=\phi_k\delta_k\left\{(1-\phi)\left[(1-\phi)\sigma_{\mathrm{e}}'(\phi)-\frac{\sigma_{\mathrm{e}}(\phi)}{\phi}\right]-\sigma_{\mathrm{e}}(\phi)\right\}\bigl(\sigma_{\mathrm{e}}(\phi)\bigr)^{N-1}\prod_{\substack{m=1\\ m\neq k}}^{N}(\delta_m-\delta_k). \tag{4.5}$$

Proof. Setting $\lambda = \delta^*(\lambda) = 0$ in (4.2), we obtain

$$
\begin{aligned}
S(0) =& \Biggl\{ \sigma_{\mathrm{e}}(\phi) + \sum_{m=1}^{N} \frac{\phi_m}{\delta_m} \biggl[-\delta_m \sigma_{\mathrm{e}}(\phi) + \biggl((1-\phi)\sigma_{\mathrm{e}}'(\phi) - \frac{\sigma_{\mathrm{e}}(\phi)}{\phi} \biggr) \times \\
& \times \bigl(\delta_m - \boldsymbol{\delta}^{\mathrm{T}}\Phi + \boldsymbol{\delta}^{\mathrm{T}}\Phi - \delta_m \phi\bigr) \Biggr\} \bigl(\sigma_{\mathrm{e}}(\phi)\bigr)^{N-1} \delta_1 \cdots \delta_N \\
=& \Biggl\{ \sigma_{\mathrm{e}}(\phi) - \phi\sigma_{\mathrm{e}}(\phi) + (1-\phi)\phi \biggl((1-\phi)\sigma_{\mathrm{e}}'(\phi) - \frac{\sigma_{\mathrm{e}}(\phi)}{\phi} \biggr) \Biggr\} \bigl(\sigma_{\mathrm{e}}(\phi)\bigr)^{N-1} \delta_1 \cdots \delta_N,
\end{aligned}
\tag{4.6}
$$

which immediately leads to (4.4). Equation (4.5) can be derived by closely following the proof of Eq. (3.12) in Lemma 2. □

Theorem 3. *Let $G(\phi) := \phi(1-\phi)^2\sigma_{\mathrm{e}}'(\phi) - \sigma_{\mathrm{e}}(\phi)$ and assume that $V(\phi) \neq 0$ for $\phi < \phi_{\max}$ and $V(\phi) = 0$ otherwise. Then, for all $\Phi \in \mathcal{D}^0_{\phi_{\max}} \backslash \mathcal{D}_{\phi_{\mathrm{c}}}$, the matrix $\mathbf{A}(\Phi)$ has N distinct positive eigenvalues $\Lambda_1, \dots, \Lambda_N$, i.e., the system (1.2) is strictly parabolic on $\mathcal{D}^0_{\phi_{\max}} \backslash \mathcal{D}_{\phi_{\mathrm{c}}}$. Moreover, we have that:*

a) *If Φ is chosen such that $G(\phi) > 0$, then these eigenvalues satisfy*

$$
\begin{aligned}
&0 < W(\phi)\sigma_{\mathrm{e}}(\phi)\delta_N < \Lambda_N < W(\phi)\sigma_{\mathrm{e}}(\phi)\delta_{N-1} < \Lambda_{N-1} < \dots \\
&\cdots < W(\phi)\sigma_{\mathrm{e}}(\phi)\delta_1 < \Lambda_1 < W(\phi)\delta_1\phi(1-\phi)^2\sigma_{\mathrm{e}}'(\phi).
\end{aligned}
\tag{4.7}
$$

b) *At those points Φ where $G(\phi) < 0$, we have*

$$
\begin{aligned}
&0 < W(\phi)\delta_N\phi(1-\phi)^2\sigma_{\mathrm{e}}'(\phi) < \Lambda_N < W(\phi)\sigma_{\mathrm{e}}(\phi)\delta_N < \Lambda_{N-1} \\
&< W(\phi)\sigma_{\mathrm{e}}(\phi)\delta_{N-1} < \cdots < \Lambda_1 < W(\phi)\sigma_{\mathrm{e}}(\phi)\delta_1.
\end{aligned}
\tag{4.8}
$$

c) *If $G(\phi) = 0$, then the eigenvalues are given by $\Lambda_i = W(\phi)\sigma_{\mathrm{e}}(\phi)\delta_i$ for $i = 1, \dots, N$.*

Proof. Using the function $G(\phi)$, we can rewrite (4.5) as

$$
S(s^k) = \frac{\phi_k}{\phi}\delta_k G(\phi)\bigl(\sigma_{\mathrm{e}}(\phi)\bigr)^{N-1} \prod_{\substack{m=1\\ m\neq k}}^{N} (\delta_m - \delta_k), \quad k = 1, \dots, N.
\tag{4.9}
$$

This implies

$$
\operatorname{sgn}\bigl(S(s^k)\bigr) = \operatorname{sgn}\bigl(G(\phi)\bigr) \cdot \operatorname{sgn}\Biggl(\prod_{\substack{m=1\\ m\neq k}}^{N} (\delta_m - \delta_k) \Biggr) = (-1)^{N-k} \operatorname{sgn}\bigl(G(\phi)\bigr), \quad k = 1, \dots, N.
\tag{4.10}
$$

Recall first that for $\phi > \phi_{\mathrm{c}}$ and due to $\delta_1 > \delta_2 > \cdots > \delta_N$, we have $0 < s^N < s^{N-1} < \cdots < s^1$.

If $\operatorname{sgn}(G(\phi)) = 1$, then $S(s^N) > 0$, $S(s^{N-1}) < 0$, and so on, until we obtain $S(s^2) > 0$ and $S(s^1) < 0$ if N is even and $S(s^2) < 0$ and $S(s^1) > 0$ if N is odd. Thus there exist $N-1$ values

$$
0 < s^N < \lambda_N < s^{N-1} < \lambda_{N-1} < s^{N-2} < \cdots < \lambda_3 < s^2 < \lambda_2 < s^1 < \lambda_1
\tag{4.11}
$$

with $S(\lambda_2) = \cdots = S(\lambda_N) = 0$. Moreover, $S(\lambda) \to \infty$ for $\lambda \to \infty$ if N is even and $S(\lambda) \to -\infty$ if N is odd. Thus there exists an Nth number $\lambda_1 > s^1$ with $S(\lambda_1) = 0$. Since the determinant of a matrix is the product of its eigenvalues, which are all positive here, equation (4.6) implies

$$
\lambda_1 = \frac{S(0)}{\lambda_2 \cdots \lambda_N} < \frac{S(0)}{s^2 \cdots s^N} = \frac{\phi(1-\phi)^2\sigma_{\mathrm{e}}'(\phi)(\sigma_{\mathrm{e}}(\phi))^{N-1}\delta_1 \cdots \delta_N}{\delta_2 \cdots \delta_N(\sigma_{\mathrm{e}}(\phi))^{N-1}} = \delta_1\phi(1-\phi)^2\sigma_{\mathrm{e}}'(\phi).
\tag{4.12}
$$

If $\operatorname{sgn}(G(\phi)) = -1$, then $S(s^N) < 0$, $S(s^{N-1}) > 0$, and so on, and $S(s^2) < 0$, $S(s^1) > 0$ if N is even and $S(s^2) > 0$, $S(s^1) < 0$ if N is odd. This means we have $N-1$ values

$$
s^N < \lambda_{N-1} < s^{N-1} < \lambda_{N-2} < s^{N-2} < \cdots < \lambda_1 < s^1
\tag{4.13}
$$

with $S(\lambda_1) = \cdots = S(\lambda_{N-1}) = 0$. Since $S(s^N) < 0$ but $S(0) > 0$ due to Lemma 4, there exists an Nth value $\lambda_N \in (0, s_N)$ satisfying $S(\lambda_N) = 0$, and we have

(4.14)
$$\lambda_N = \frac{S(0)}{\lambda_1 \cdots \lambda_{N-1}} > \frac{S(0)}{s^1 \cdots s^{N-1}} = \frac{\phi(1-\phi)^2 \sigma_{\mathrm{e}}'(\phi)(\sigma_{\mathrm{e}}(\phi))^{N-1} \delta_1 \cdots \delta_N}{\delta_1 \cdots \delta_{N-1} (\sigma_{\mathrm{e}}(\phi))^{N-1}} = \delta_N \phi (1-\phi)^2 \sigma_{\mathrm{e}}'(\phi).$$

The eigenvalues of $\mathbf{A}(\Phi)$ are given by $\Lambda_i = W(\phi)\lambda_i$, $i = 1, \ldots, N$. Thus parts (a) and (b) of Theorem 3 follow from (4.11) to (4.14). Part (c) is the common limit for $G(\phi) \uparrow 0$ and $G(\phi) \downarrow 0$. $\square$

Since the eigenvalues $\Lambda_1, \ldots, \Lambda_N$ are positive independent of the sign of $G(\phi)$, we see that the system (1.2) is strictly parabolic for all Φ satisfying $\phi_{\mathrm{c}} < \phi < 1$, although due to the properties of σ_{e} and W, at least $N-1$ of these eigenvalues approach zero as $\phi \downarrow \phi_{\mathrm{c}}$ or $\phi \uparrow \phi_{\max}$.

5. Strongly degenerate parabolic problems

We have demonstrated that polydisperse sedimentation models taking into account compression effects give rise to strongly degenerate parabolic (also known as mixed hyperbolic-parabolic) systems of partial differential equations. The general theory of *uniformly parabolic* systems is an old subject and is by now well developed, see [32, 45, 66]. One can consult [62] for some special uniformly parabolic systems as well as [30, 40, 55] for some results on parabolic systems with weaker parabolicity conditions. The general mathematical theory of *hyperbolic* systems is also fairly well developed (at least in one spatial dimension), see for example [27] and the references therein. On the other hand, to date there exists no general theory for *strongly degenerate parabolic* systems. However, the mathematical theory for *scalar* strongly degenerate parabolic equations has advanced significantly in the recent few years. It is well known that nonlinear degenerate parabolic equations exhibit "hyperbolic phenomena" like finite speed of propagation or appearance of interfaces. These effects are consequences of the partial loss of parabolicity. *Strongly* degenerate parabolic equations (e.g., those arising in the theory of sedimentation-consolidation processes) exhibit even more novel hyperbolic features such as the appearance of shock waves, loss of uniqueness, and the need of entropy conditions. Recall that a simple example of a strongly degenerate equation is a hyperbolic equation. Hence, strongly degenerate parabolic equations will in general possess discontinuous (weak) solutions. Moreover, discontinuous solutions are not uniquely determined by their initial (and boundary) data. In fact, an additional condition — the entropy condition — is needed to single out the physically relevant weak solution of the problem.

An entropy condition for strongly degenerate parabolic equations was first proposed in the work [70], which also established existence of an entropy solution by passing to the limit in a parabolic regularization. In the one-dimensional case, uniqueness of the entropy solution was proved in [71, 72], see also [5, 6, 7]. Uniqueness of entropy solutions for multi-dimensional equations is obtained in the recent work [24] for a particular homogeneous boundary value problem. Extensions of this uniqueness result to the initial value problem can be found in [43, 44] for bounded entropy solutions (of more general equations). Uniqueness for unbounded entropy solutions and kinetic solutions are studied in [25, 26] respectively. The inhomogeneous Dirichlet boundary value problem is treated in [49]. Some other boundary value problems arising in the theory of sedimentation-consolidation processes are studied in [15, 19, 21]. Weakly coupled systems of (strongly) degenerate parabolic equations are treated in [39].

Following up the recent development of a well-posedness theory for scalar strongly degenerate parabolic equations, there has also been a lot of activity on the design and analysis of numerical methods for such equations. Most of this activity can be seen as natural extentions of ideas and techniques from the hyperbolic numerical literature. Let us here mention the studies on monotone finite difference schemes [34], operator splitting methods (see [33] for an overview), finite volume schemes [35, 54], central finite difference schemes [46], the local discontinuous Galerkin method [28], and BGK schemes [3, 11]. Numerical methods for parabolic systems (with weak degeneracy) are studied and analyzed in [42, 47]. Applications of operator splitting methods and finite difference schemes to scalar sedimentation-consolidation models can be found in [16, 18] respectively.

In the next section, we will present and apply certain numerical schemes for systems of strongly degenerate parabolic equations. Except for [3], the available numerical literature has so far dealt with *scalar* strongly degenerate parabolic equations. Let us add that the generality in [3] is such that it does not include systems of the form considered in the present paper.

6. Numerical results

In this section we give an exposition of the numerical scheme used in this paper, namely the high-resolution difference scheme due to Kurganov and Tadmor [46]. We refer to [12, 17] for the application of similar numerical schemes to first-order systems of conservation laws modeling sedimentation of polydisperse suspensions without compression effects. The basic new element introduced in [12, 17] is the use of boundary conditions.

The Kurganov-Tadmor (KT) scheme [46] can be regarded as a refinement of the essentially non-oscillatory Nessyahu-Tadmor scheme [53], where the improvement is based on local estimates of the propagation velocities of the Riemann fan emerging from the cell boundaries during each time step. Thus, the accuracy of the resulting scheme depends on how accurately the eigenvalues of the Jacobian of the flux vector are determined. Since it is possible only for small systems to determine these eigenvalues exactly, it is important for large N that sharp estimates can be obtained with low computational effort. The analysis of Section 3 implies that sharp estimates are indeed available for the first-order system of equations. Given the importance of these analytical results in the light of the KT scheme, we give in what follows a rather compressed but complete description of this scheme. A general introduction to central schemes for systems of conservation laws is given in the lecture notes by Tadmor [65].

6.1. **General difference scheme.** We consider the computational domain $Q_T := [0,1] \times [0,T]$ and introduce a rectangular grid by setting $z_j := j\Delta z$, $j = 0,\dots,J$, where J is an even integer and $\Delta z := 1/J$ denotes the width of a half-cell. (Thus the scheme is introduced in such a way that all grid-point indices are integers.) Moreover, let $t_n := n\Delta t$, $n = 0,\dots,\mathcal{N}$, where $\Delta t := T/\mathcal{N}$, $\mathcal{N} \in \mathbb{N}$. In the remainder of this paper we denote by $\lambda := \Delta t/(2\Delta z)$ the fixed mesh-size ratio. The (approximate) cell average of ϕ_i, $i = 1,\dots,N$, with respect to the cell $[z_j, z_{j+2}]$, $j = 0,\dots,J-1$, at time t_n is denoted by $\bar{\phi}_{i,j}^n$, $j = 1,3,\dots,J-1$, and we define the approximate cell-average vector $\bar{\Phi}_j^n := (\bar{\phi}_{1,j}^n,\dots,\bar{\phi}_{N,j}^n)^{\mathrm{T}}$, $j = 1,3,\dots,J-1$, $n = 0,1,\dots,\mathcal{N}$.

To outline the scheme, assume that at time $t = t_n$, $n = 0,1,\dots,\mathcal{N}-1$, the vector $\bar{\Phi}_j^n$ has either been calculated from the previous time-step (for $n \geq 1$) or is given by the discretization of the initial condition,

$$\bar{\phi}_{i,j}^0 = \frac{1}{2\Delta z}\int_{z_{j-1}}^{z_{j+1}} \phi_i^0(\zeta)\,d\zeta, \quad j = 1,3,\dots,J-1, \quad i = 1,\dots,N.$$

For the interior cells, the general scheme is of the type

$$\bar{\Phi}_j^{n+1} = \bar{\Phi}_j^n - \lambda(\mathbf{h}_{j+1}^n - \mathbf{h}_{j-1}^n) + \lambda(\mathbf{p}_{j+1}^n - \mathbf{p}_{j-1}^n), \quad j = 3,5,\dots,J-3, \quad n = 0,\dots,\mathcal{N}-1, \tag{6.1}$$

where $\mathbf{h}_{j\pm1}^n$ and $\mathbf{p}_{j\pm1}^n$ are approximations of the 'hyperbolic' and 'parabolic' fluxes $\mathbf{f}^{\mathrm{M}}$ and $\mathbf{a}$, respectively, through the boundaries of cell $I_j := [z_{j-1}, z_{j+1}]$ at time t_n. The detailed computation of these fluxes from the solution values at time t_n is described in Section 6.2.

While the interior scheme (6.1) approximates the field equation (1.2), the corresponding boundary conditions (2.30) prescribed at $z = 0$ and $z = L$ are discretized by setting $\mathbf{h}_0^n - \mathbf{p}_0^n = 0$ and $\mathbf{h}_J^n - \mathbf{p}_J^n = 0$ for $n = 0,\dots,\mathcal{N}-1$. Inserting this into (6.1), where we set $j = 1$ and $j = J-1$, we obtain the following update formulas for the boundary cells (boundary scheme):

$$\bar{\Phi}_1^{n+1} = \bar{\Phi}_1^n - \lambda\mathbf{h}_1^n + \lambda\mathbf{p}_1^n, \quad \bar{\Phi}_{J-1}^{n+1} = \bar{\Phi}_{J-2}^n + \lambda\mathbf{h}_{J-2}^n - \lambda\mathbf{p}_J^n, \quad n = 0,\dots,\mathcal{N}-1. \tag{6.2}$$

The extension of the CFL stability condition for the explicit KT scheme stated in [46] for scalar equations to the present case of a strongly degenerate parabolic-hyperbolic problem reads

$$\frac{\Delta t}{\Delta z}\max_{\mathcal{D}_{\phi_{\max}}} \rho\big(\mathcal{J}_{\mathbf{f}}(\Phi)\big) + \frac{\Delta t}{2\Delta z^2}\max_{\mathcal{D}_{\phi_{\max}}} \rho\big(\mathbf{A}(\Phi)\big) \leq \frac{1}{4}. \tag{6.3}$$

It is emphasized that we view (6.3) as a *necessary* condition for the present explicit KT scheme to produce a physically relevant numerical result, and that no rigorous convergence result is associated with (6.3). For that matter, an existence and uniqueness theory for the system (1.1) is still lacking.

6.2. **Computation of the numerical fluxes.** Given the vectors $\bar{\Phi}_j^n$, $j = 1, 3, \dots, J-1$, we calculate a piecewise linear reconstruction of the solution values at time t_n by determining the slope vector $\Phi'_j = (\phi'_{1,j}, \dots, \phi'_{N,j})^{\mathrm{T}}$, $j = 1, 3, \dots, J-1$, whose components are defined by

$$\phi'_{i,j} := \begin{cases} 0 & \text{for } j=1 \text{ and } j = J-1, \\ \mathrm{MM}\bigl(\theta(\bar{\phi}_{i,j}^n - \bar{\phi}_{i,j-2}^n), (\bar{\phi}_{i,j+2}^n - \bar{\phi}_{i,j-2}^n)/2, \theta(\bar{\phi}_{i,j+2}^n - \bar{\phi}_{i,j}^n)\bigr) & \text{for } j = 3,5,\dots,J-3 \end{cases}$$

for $i = 1, \dots, N$. Here $\mathrm{MM}(\cdot,\cdot,\cdot)$ is the minmod function given by $\mathrm{MM}(a,b,c) = \min(a,b,c)$ if $a,b,c > 0$, $\mathrm{MM}(a,b,c) = \max(a,b,c)$ if $a,b,c < 0$ and $\mathrm{MM}(a,b,c) = 0$ otherwise. The extrapolated values of Φ at the cell boundaries z_j, $j = 2, 4, \dots, J-2$, are then given by

$$\Phi_j^- := \bar{\Phi}_{j-1}^n + \frac{1}{2}\Phi'_{j-1}, \quad \Phi_j^+ := \bar{\Phi}_{j+1}^n - \frac{1}{2}\Phi'_{j+1}, \quad j = 2,4,\dots,J-2,$$

and are used to calculate the local speeds of propagation

$$a_j^n := \max\left\{\rho\left(\mathcal{J}_{\mathbf{f}}(\Phi_j^-)\right), \rho\left(\mathcal{J}_{\mathbf{f}}(\Phi_j^+)\right)\right\}, \quad j = 2,4,\dots,J-2, \tag{6.4}$$

where $\rho(\cdot)$ denotes the spectral radius. Of course, it is feasible only for small N to use exact eigenvalues here. However, the analysis of Section 3 provides estimates of the eigenvalues that can be used here (see Section 6.3). Observe that for each cell I_j the solution of (1.4) with the piecewise linear initial data defined by $\bar{\Phi}_j^n$ and the slope vectors Φ'_j remains smooth for $t_n \leq t \leq t_{n+1}$ in the subinterval $[z_{j-1} + a_{j-1}^n \Delta t, z_{j+1} - a_{j+1}^n \Delta t]$ for $j = 1, \dots, J$. Equipped with the numbers a_j^n and the vectors Φ'_j we next calculate the vectors

$$\Phi_{j,\mathrm{L}}^n := \bar{\Phi}_{j-1}^n + \left(\frac{1}{2} - \lambda a_j^n\right)\Phi'_{j-1}, \quad \Phi_{j,\mathrm{R}}^n := \bar{\Phi}_{j+1}^n - \left(\frac{1}{2} - \lambda a_j^n\right)\Phi'_{j+1}, \quad j = 2,4,\dots,J-2,$$

which represent the parts of the cell averages pertaining to the left and right half cell adjacent to $z = z_j$ that are mapped onto a smooth solution. The vectors $\Phi_{j,\mathrm{L}}^n$ and $\Phi_{j,\mathrm{R}}^n$ are used to calculate the flux slope vectors

$$\mathbf{f}'(\Phi_{j,\mathrm{L}}^n) = \bigl(f'_1(\Phi_{j,\mathrm{L}}^n), \dots, f'_N(\Phi_{j,\mathrm{L}}^n)\bigr)^{\mathrm{T}}, \ \mathbf{f}'(\Phi_{j,\mathrm{R}}^n) = \bigl(f'_1(\Phi_{j,\mathrm{R}}^n), \dots, f'_N(\Phi_{j,\mathrm{R}}^n)\bigr)^{\mathrm{T}}, \ j = 2,4,\dots,J-2,$$

whose components are defined by $f'_i(\Phi_{2,\mathrm{L}}^n) = f'_i(\Phi_{J-2,\mathrm{L}}^n) = 0$, $f'_i(\Phi_{2,\mathrm{R}}^n) = f'_i(\Phi_{J-2,\mathrm{R}}^n) = 0$ and

$$f'_i\bigl(\Phi_{j,c}^n\bigr) := \mathrm{MM}\Bigl(\theta\bigl(f_i(\Phi_{j,c}^n) - f_i(\Phi_{j-2,c}^n)\bigr), \bigl(f_i(\Phi_{j+2,c}^n) - f_i(\Phi_{j-2,c}^n)\bigr)/2, \theta\bigl(f_i(\Phi_{j+2,c}^n) - f_i(\Phi_{j,c}^n)\bigr)\Bigr)$$

for $c = \mathrm{L}, \mathrm{R}$, $i = 1, \dots, N$ and $j = 2, 4, \dots, J-4$. We then calculate the predictor solution values

$$\Phi_{j,\mathrm{L}}^{n+1/2} := \Phi_{j,\mathrm{L}}^n - \frac{\lambda}{2}\mathbf{f}'\bigl(\Phi_{j,\mathrm{L}}^n\bigr), \quad \Phi_{j,\mathrm{R}}^{n+1/2} := \Phi_{j,\mathrm{R}}^n - \frac{\lambda}{2}\mathbf{f}'\bigl(\Phi_{j,\mathrm{R}}^n\bigr), \quad j = 2,4,\dots,J-2,$$

at which the flux vector $\mathbf{f}$ is evaluated in order to calculate the new approximate values $\bar{\Psi}_j^{n+1}$, $j = 2, 3, \dots, J-1, J$ of the solution at time t_{n+1}, which are referred to a non-uniform grid as follows. For $j = 2, 4, \dots J-2$, approximate cell averages $\bar{\Psi}_j^{n+1}$ referring to the intervals $[z_j - a_j^n \Delta t, z_j + a_j^n \Delta t]$, $j = 2, 4, J-2$ are calculated by

$$\bar{\Psi}_j^{n+1} = \frac{1}{2}(\bar{\Phi}_{j-1}^n + \bar{\Phi}_{j+1}^n) + \frac{1 - \lambda a_j^n}{4}(\Phi'_{j-1} - \Phi'_{j+1}) - \frac{1}{2a_j^n}\Bigl[\mathbf{f}\bigl(\Phi_{j,\mathrm{R}}^{n+1/2}\bigr) - \mathbf{f}\bigl(\Phi_{j,\mathrm{L}}^{n+1/2}\bigr)\Bigr],$$

while the second family of approximate cell averages $\bar{\Psi}_j^{n+1}$ refers to the non-uniform cells $[z_{j-1} + a_{j-1}^n \Delta, z_{j+1} - a_{j+1}^n \Delta t] \subset I_j$, $j = 3, 5, \dots, J-3$, and is calculated by

$$\bar{\Psi}_j^{n+1} = \bar{\Phi}_j^n - \frac{\lambda}{2}(a_{j+1}^n - a_{j-1}^n)\Phi'_j - \frac{\lambda}{1 - \lambda\bigl(a_{j-1}^n - a_{j+1}^n\bigr)}\Bigl[\mathbf{f}(\Phi_{j+1,\mathrm{L}}^{n+1/2}) - \mathbf{f}(\Phi_{j-1,\mathrm{R}}^{n+1/2})\Bigr].$$

Using both families of non-uniform approximate cell averages, we determine the vector of discrete derivatives $\Psi'_j = (\Psi'_{1,j}, \dots, \Psi'_{N,j})^{\mathrm{T}}$ for $j = 2, 4, \dots, J-2$ setting $\Psi'_2 = \Psi'_{J-2} = 0$ and

$$\Psi'_{i,j} = \frac{2}{\Delta z} \mathrm{MM}\left(\theta \frac{\bar{\Psi}^{n+1}_{i,j} - \bar{\Psi}^{n+1}_{i,j-1}}{1 + \lambda\big(a^n_j - a^n_{j-2}\big)}, \frac{\bar{\Psi}^{n+1}_{i,j+1} - \bar{\Psi}^{n+1}_{i,j-1}}{2 + \lambda\big(2a^n_j - a^n_{j-2} - a^n_{j+2}\big)}, \theta \frac{\bar{\Psi}^{n+1}_{i,j+1} - \bar{\Psi}^{n+1}_{i,j}}{1 + \lambda\big(a^n_j - a^n_{j+2}\big)} \right)$$

for $i = 1, \dots, N$ and $j = 4, 6, \dots, J-4$, where $\theta \in [0, 2]$ is a parameter. Finally, we can calculate the desired numerical flux vectors

$$\begin{aligned} \mathbf{h}^n_j = & \frac{1}{2}\Big[\mathbf{f}\big(\Phi^{n+1/2}_{j,\mathrm{R}}\big) + \mathbf{f}\big(\Phi^{n+1/2}_{j,\mathrm{L}}\big)\Big] - \frac{a^n_j}{2}\big(\bar{\Phi}^n_{j+1} - \bar{\Phi}^n_{j-1}\big) \\ & + \frac{a^n_j}{4}\big(1 - \lambda a^n_j\big)\big(\Phi'_{j-1} + \Phi'_{j+1}\big) + \frac{\lambda \Delta z}{2}\big(a^n_j\big)^2 \Psi'_j, \quad j = 2, 4, \dots, J-2. \end{aligned}$$

For the diffusion part, we approximate $\partial\Phi/\partial z(z_j, t_n)$ by the slope vector $\widetilde{\Phi}'_j = (\tilde{\phi}'_{1,j}, \dots, \tilde{\phi}'_{N,j})^{\mathrm{T}}$, $j = 2, 4, \dots, J-2$, whose components are defined by $(\bar{\phi}^n_{i,j+1} - \bar{\phi}^n_{i,j-1})/(2\Delta z)$ for $j = 2, 4, \dots, J-2$ and $i = 1, \dots, N$. The numerical diffusion vectors are then calculated by

$$\mathbf{p}^n_j = \frac{1}{2}\big(\mathbf{a}(\bar{\Phi}_{j-1}, \widetilde{\Phi}'_j) + \mathbf{a}(\bar{\Phi}_{j+1}, \widetilde{\Phi}'_j)\big), \quad j = 2, 4, \dots, J-2, \tag{6.5}$$

where the components $a_1, \dots, a_N$ of the diffusion vector $\mathbf{a} = \mathbf{a}(\Phi, \partial\Phi/\partial z)$ are given by (4.1).

6.3. **Application to the model of polydisperse sedimentation with compression.** In the sequel we illustrate the polydisperse sedimentation model by a choice of numerical examples. We consider the hindered settling factor

$$V(\phi) = (1-\phi)^{n-2}, \quad n > 2, \tag{6.6}$$

due to Richardson and Zaki [56] and the widely used power-law effective solid-stress formula

$$\sigma_{\mathrm{e}}(\phi) = \begin{cases} 0 & \text{for } \phi \le \phi_{\mathrm{c}}, \\ \sigma_0\big((\phi/\phi_{\mathrm{c}})^k - 1\big) & \text{for } \phi > \phi_{\mathrm{c}}, \end{cases} \quad \text{i.e.} \quad \sigma'_{\mathrm{e}}(\phi) = \begin{cases} 0 & \text{for } \phi < \phi_{\mathrm{c}}, \\ (\sigma_0/\phi^k_{\mathrm{c}}) k \phi^{k-1} & \text{for } \phi > \phi_{\mathrm{c}}, \end{cases} \tag{6.7}$$

with parameters $\sigma_0 > 0$ and $k \ge 1$. For values of σ_0, ϕ_{c} and k for a variety of real materials we refer, for example, to [14, 67].

For our choice of $V(\phi)$, we may significantly sharpen the upper bound for the eigenvalues of $\mathcal{J}_{\mathbf{f}^{\mathrm{M}}(\phi)}(\Phi)$ compared with the bound given by Theorem 2.

Lemma 5. *For the hindered settling function* (6.6) *and* $\Phi \in \mathcal{D}^0_{1/n}$*, i.e.* $\phi < 1/n$*, the eigenvalues* $\nu_1(\Phi)$ *to* $\nu_N(\Phi)$ *of* $\mathcal{J}_{\mathbf{f}^{\mathrm{M}}}(\Phi)$ *satisfy* (3.15) *and*

$$\nu_N(\Phi) \in \big(\mu \bar{\varrho}_{\mathrm{s}} V(\phi)(1-\phi)(\delta_N - \boldsymbol{\delta}^{\mathrm{T}}\Phi), -\mu \bar{\varrho}_{\mathrm{s}} V(\phi)(1-\phi)\boldsymbol{\delta}^{\mathrm{T}}\Phi\big). \tag{6.8}$$

Proof. We set $\widetilde{\gamma}^\infty := -V(\phi)(1-\phi)\boldsymbol{\delta}^{\mathrm{T}}\Phi$, such that $\delta(\widetilde{\gamma}^\infty) = 0$. Using (3.11) we get

$$\begin{aligned} P(\widetilde{\gamma}^\infty) = & \left\{ V(\phi)(1-\phi) + \sum_{m=1}^N \frac{\phi_m}{\delta_m}\Big[-\delta_m V(\phi)(1-\phi) + \big(V(\phi)(1-\phi)\big)'(1-\phi)\delta_m\Big] \right\} \times \\ & \times \big(V(\phi)(1-\phi)\big)^{N-1} \delta_1 \cdots \delta_N \\ = & \Big\{ V(\phi)(1-\phi) - \phi V(\phi)(1-\phi) + \big(V(\phi)(1-\phi)\big)' \phi (1-\phi) \Big\} \big(V(\phi)(1-\phi)\big)^{N-1} \delta_1 \cdots \delta_N \\ = & \Big\{ V(\phi)(1-\phi) + \big(V(\phi)(1-\phi)\big)' \phi \Big\} (1-\phi)^N \big(V(\phi)\big)^{N-1} \delta_1 \cdots \delta_N. \end{aligned}$$

For $V(\phi) = (1-\phi)^{n-2}$ the expression in curled brackets is given by

$$(1-\phi)^{n-1} - (n-1)(1-\phi)^{n-2}\phi = (1-\phi)^{n-2}\big((1-\phi) - (n-1)\phi\big) = (1-\phi)^{n-2}(1-n\phi),$$

which is positive if and only if $\phi < 1/n$. In this case we thus have $P(\widetilde{\gamma}^\infty) > 0$. Since $\widetilde{\gamma}^\infty < \gamma^N$, $P(\gamma^N) < 0$ and we have now shown that $P(\widetilde{\gamma}^\infty) > 0$, the smallest eigenvalue λ_N of $\mathbf{J}$ satisfies $\widetilde{\gamma}^\infty < \lambda_N < \gamma^N$, which implies the statement of the lemma. $\square$

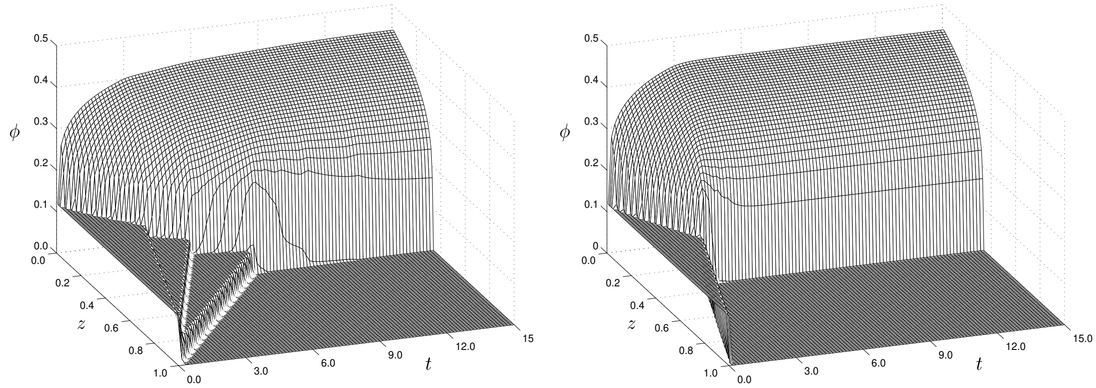

FIGURE 1. Simulations of the sedimentation-consolidation process of a tridisperse suspension (left) and a monodisperse suspension (right) using the same model functions $V(\phi)$ and $\sigma_e(\phi)$: plots of the (total) solids concentration ϕ.

Wherever the mixture is dilute enough, Lemma 5 is useful to estimate the local speeds of propagation (6.4) in the numerical method since, for our choice of $V(\phi)$, we can conclude that

$$\rho\left(\mathcal{J}_{\mathbf{f}}(\Phi)\right) \leq \mu \bar{\varrho}_s (1-\phi)^{n-1} \max\{\boldsymbol{\delta}^{\mathrm{T}}\Phi, 1-\boldsymbol{\delta}^{\mathrm{T}}\Phi\} \quad \text{for } \phi < 1/n.$$

Similarly, the eigenvalue bounds of Theorem 3 can be utilized to estimate the term $\max \rho(\mathcal{J}_{\mathbf{f}})$ required in the CFL condition (6.3), which limits the step size ratio.

6.4. **Numerical results.** The numerical scheme is now employed to simulate the sedimentation-consolidation process of a tridisperse ($N = 3$) suspension forming compressible sediment. We consider here a (hypothetical) mixture described by the model functions (6.6) with $n = 4.7$ [60] and (6.7) with the parameter values $\sigma_0 = 180\,\mathrm{Pa}$, $\phi_c = 0.2$ and $k = 6$. The remaining parameters are $\mu_f = 10^{-3}\mathrm{Pa}\cdot\mathrm{s}$ (the dynamic viscosity of water), $d_1 = 7.14\times 10^{-4}\,\mathrm{m}$, $\bar{\varrho}_s = 1800\,\mathrm{kg/m^3}$ and $g = 9.81\,\mathrm{m/s^2}$. Moreover, we let $d_2/d_1 = \sqrt{0.5}$ and $d_3/d_1 = 0.5$, such that $\boldsymbol{\delta} = (1, 0.5, 0.25)^{\mathrm{T}}$.

We consider an initially homogeneous suspension with $\Phi^0 = (0.04, 0.04, 0.04)$ in a vessel of height $L = 1\,\mathrm{m}$. For the numerical simulation of the sedimentation-consolidation process we chose $J = 600$ and $\lambda = 0.03$ (where z is measured in meters and t in hours). The left diagram of Figure 1 shows the total volumetric solids concentration $\phi = \phi_1 + \phi_2 + \phi_3$ as a function of z and t, while Figure 2 displays the corresponding concentrations of the individual species.

To make the numerical results comparable to those obtained from the two existing models for monodisperse flocculated suspensions and for polydisperse suspensions of rigid spheres, we show in the right diagram of Figure 1 a simulation of the settling of a monodisperse suspension with $\phi_0 = 0.12$ and in Figure 3 the simulation of a tridisperse suspension of rigid particles (forming a sediment without compressibility effects) having the same parameters as the previously discussed case but $\sigma_0 = 0$. The simulation shown in Figure 3 was made with $\lambda = 0.35$ and $J = 1200$. Note that the visual grid used in all diagrams is much coarser than the computational grid.

In the left plot of Figure 1, three distinct zones are formed by the downwards-propagating concentration discontinuities and, as expected, the concentration ϕ in the sediment bed increases more slowly than in the monodisperse case. Comparing ϕ in the two tridisperse cases, we see that the zones formed in the first stages of sedimentation are still visible in the upper left plot of Figure 3, but have been entirely smoothed out in the left plot of Figure 1. The values of ϕ and ϕ_1 are very high at the bottom in the top plots of Figure 3, which is essentially due to the fact that $V(\phi)$ has not been cut off at a maximum concentration smaller than one. The two bottom plots of Figure 3 show the expected layering caused by differential sedimentation and the consequent enhancement of ϕ_2 and ϕ_3 above the lowest zone. Figure 2 shows that the additional terms in the equation for suspensions forming compressible sediments result in the upward diffusion of the largest spheres and the downward diffusion of the smallest. Additional numerical examples

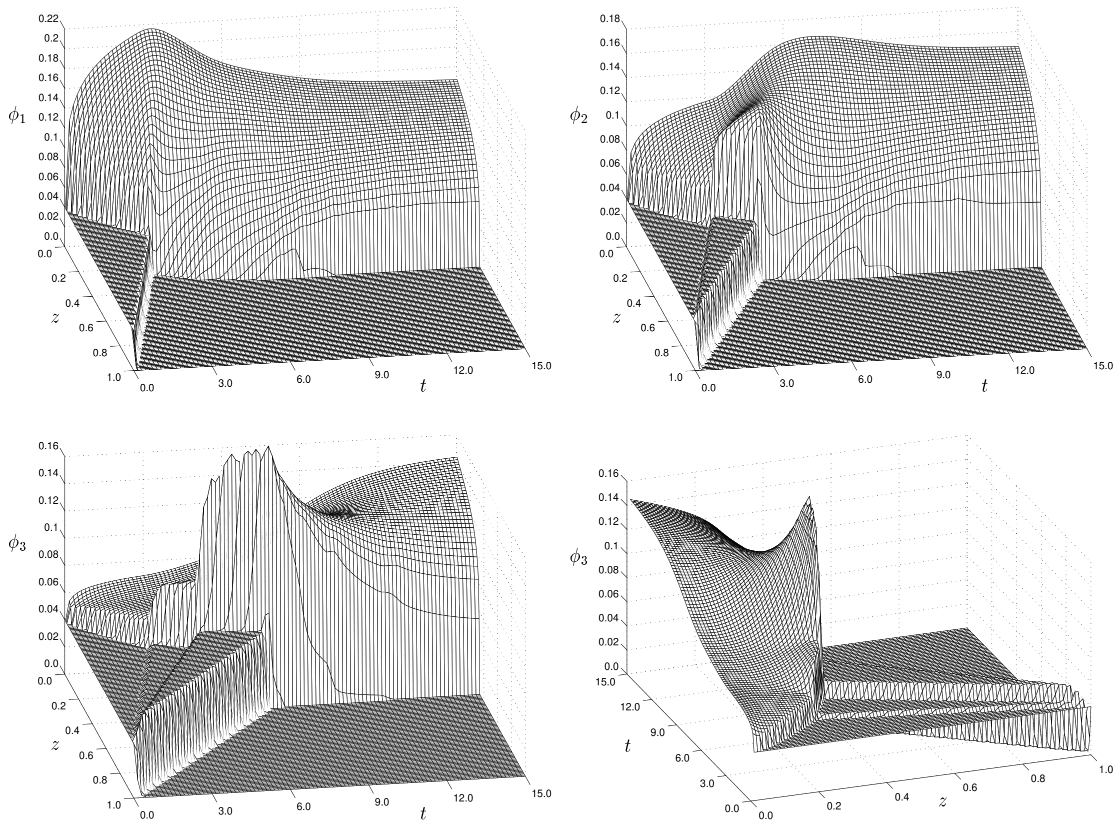

FIGURE 2. Simulation of the sedimentation-consolidation process of a tridisperse suspension with compression: plots of the concentrations ϕ_1 and ϕ_2 of the largest and second largest species (top left and top right) and two different views of the concentrations ϕ_3 of the smallest species.

illustrating the conventional model of sedimentation of suspensions of rigid spheres (when $\sigma_{\mathrm{e}} = 0$) are given in [12, 17] (see also [37]).

7. Discussion

The model equations established in Section 2 admit that the solids species differ both in size and density. The analysis of Section 3 is valid for the case of equal-density particles only, while the matrix $\mathbf{A}(\Phi)$ is independent of the particle densities. In view of the results of [20], where it was demonstrated that different densities lead to hyperbolic-elliptic or (for $N \geq 3$) non-hyperbolic systems, it is tempting to conclude that the model framework of Section 2 leads to systems having even more interesting properties (like a second-order parabolic system for $N = 2$ degenerating into a first-order hyperbolic-elliptic one). However, since particles of different densities consist of different materials, the assumption $\sigma_{\mathrm{e}}(\Phi) = \sigma_{\mathrm{e}}(\phi)$, stating that the effective stress depends on the porosity $1 - \phi$ of the sediment layer only, is very unlikely to remain valid.

Furthermore, we mention that Lemma 5 has an interesting physical explanation. To this end, we recall that $f_i^{\mathrm{M}}(\Phi) = \phi_i v_i$, where v_i is the phase velocity of species i, that is the settling velocity of a particle of species i. In view of (3.1), Theorem 2 states that the eigenvalues ν_1 to ν_{N-1} of $\mathcal{J}_{\mathbf{f}^{\mathrm{M}}}(\Phi)$ satisfy $v_i \leq \nu_i \leq v_{i+1}$. In other words, the propagation of the characteristic information associated with the eigenvalue ν_i is bounded by the physical velocities of particles of species i and $i+1$ for $i = 1, \dots, N-1$. The upper bound ν_N defined by the parameter γ^∞ of Theorem 2, which is valid for any admissible hindered settling function $V(\phi)$, has no obvious physical interpretation, but

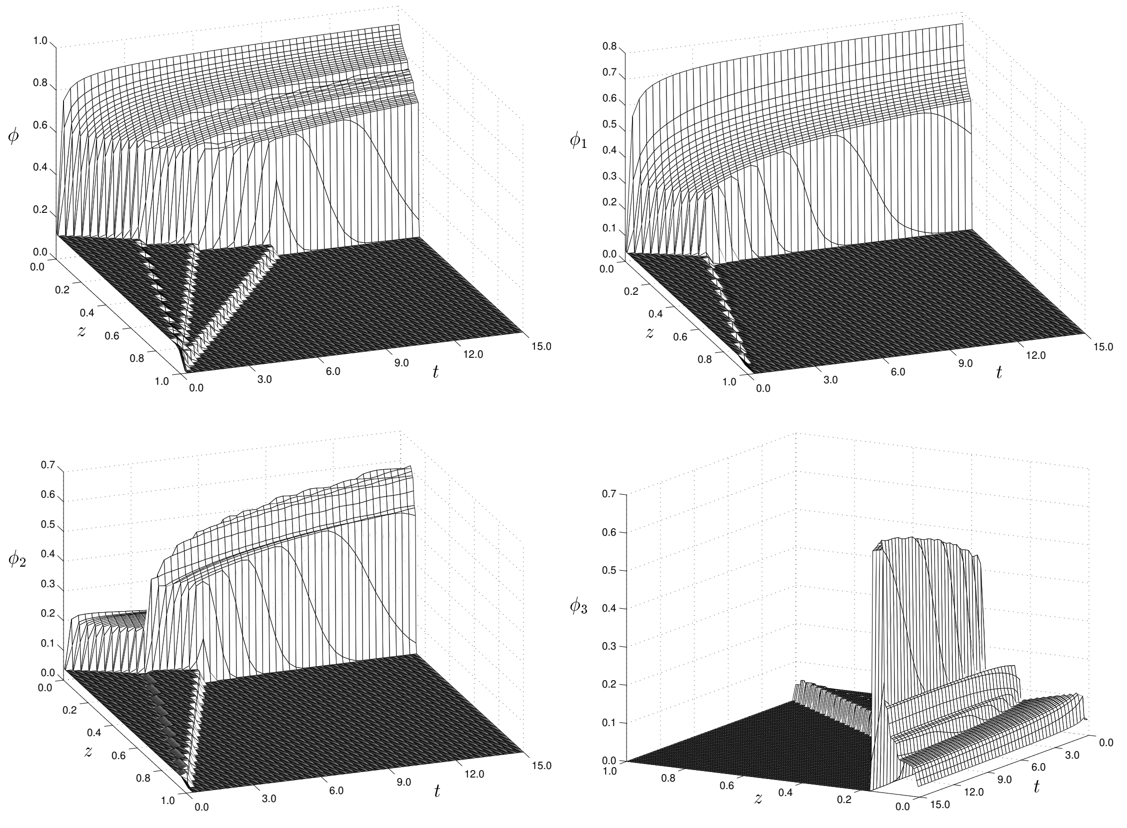

FIGURE 3. Simulation of the sedimentation of a tridisperse suspension of rigid particles (without compression, $\sigma_{\mathrm{e}} \equiv 0$): plots of the cumulative concentration ϕ and the concentration ϕ_1 of the largest species (top left and top right), and of the second largest and smallest species (bottom left and bottom right).

Theorem 2 already provides further support for the MLB model wherever $\sigma_{\mathrm{e}} = 0$ since all waves should travel at finite speeds that are within a certain physical range and, for a given particle size distribution, γ^{∞} is uniformly bounded with respect to Φ. However, the upper bound of ν_N given in (6.8) also has a physical meaning. From (3.1), the total solids flux is

$$(7.1) \qquad f^{\mathrm{M}}(\Phi) := \phi_1 v_1 + \cdots + \phi_N v_N = f_1^{\mathrm{M}}(\Phi) + \cdots + f_N^{\mathrm{M}}(\Phi) = \mu V(\phi)(1-\phi)^2 \bar{\varrho}_{\mathrm{s}} \boldsymbol{\delta}^{\mathrm{T}} \Phi.$$

On the other hand, we recall from the definition of $\mathbf{q}$ that

$$v_{\mathrm{f}} = \frac{1}{1-\phi}\big(q - (\phi_1 v_1 + \cdots + \phi_N v_N)\big) = \frac{q - f^{\mathrm{M}}(\Phi)}{1-\phi},$$

where v_{f} is the fluid phase velocity. Since we consider the case $q = 0$, we obtain the relation $-\mu \bar{\varrho}_{\mathrm{s}} V(\phi)(1-\phi)\boldsymbol{\delta}^{\mathrm{T}}\Phi = v_{\mathrm{f}}$. Thus Lemma 5 states that, for relatively dilute suspensions (when $\phi < 1/n$), all eigenvalues (and therefore wave velocities) are bounded by the local velocities of the solid and fluid 'particles'. Note that in the numerical example in Section 6, we chose $1/n = 0.2128 > \phi_{\mathrm{c}} = 0.2$, such that the model equations are either hyperbolic with the sharp estimates of Lemma 5 holding or parabolic.

The focus of the present paper has been on the derivation of a mathematical model for the sedimentation of polydisperse suspensions forming compressible sediments and its type analysis, which confirms that the model is well-posed in that the one-dimensional system (1.2) is not of 'general' type but has desirable distinct algebraic properties. Published experimental information to which the numerical predictions could be compared is scarce [4, 8, 68], but includes the well-known paper by Been and Sills [4], who measured local changes in particle size distribution due to the relative movement of particles of different sizes under the influence of effective solid stress and at different initial concentrations. Been and Sills' experiments [4] were performed with estuarine mud, a natural mixture composed of many different materials for which the constitutive model equations are difficult to determine. More precise knowledge of these functions can be expected in chemical engineering applications, where the settling solids are usually formed by a product having more homogeneous material properties. The model outlined herein should thus be useful for simulations in any of the industrial applications cited in Section 1. In particular, the model can also be applied to centrifugal configurations and pressure filtration (see [13, 19] for the monodisperse cases), and thereby be employed to simulate the manufacturing and final composition of ceramic materials with functionally graded material properties, see [8, 9, 10]. Comparing our Figures 2 and 3 illustrates that the effective stress is a decisive factor when the variation of composition should be continuous.

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References

[1] H. Arastoopour, S.C. Lin and S.A. Weil, *Analysis of vertical pneumatic conveying of solids using multiphase flow models*, AIChE J., 28 (1982), pp. 467–473.
[2] L.G. Austin, C.H. Lee and F. Concha, *Hindered settling and classification partition curves*, Minerals & Metallurgical Process., 9 (1992), pp. 161–168.
[3] D. Aregba-Driollet, R. Natalini and S. Tang, *Diffusive kinetic explicit schemes for nonlinear degenerate parabolic systems.* Preprint, 2000.
[4] K. Been and G.C. Sills, *Self-weight consolidation of soft soils: An experimental and theoretical study*, Géotechnique, 31 (1981), pp. 519–535.
[5] P. Bénilan and H. Touré, *Sur l'équation générale* $u_t = \varphi(u)_{xx} - \psi(u)_x + v$, C. R. Acad. Sci. Paris Sér. I Math., 299 (1984), 919–922.
[6] P. Bénilan and H. Touré, *Sur l'équation générale* $u_t = a(\cdot, u, \phi(\cdot, u)_x)_x + v$ *dans* L^1*. I. Étude du problème stationnaire*, In: *Evolution equations (Baton Rouge, LA, 1992)*, Dekker, New York, 1995, pp. 35–62.
[7] P. Bénilan and H. Touré, *Sur l'équation générale* $u_t = a(\cdot, u, \phi(\cdot, u)_x)_x + v$ *dans* L^1*. II. Le problème d'évolution*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 12 (1995), pp. 727–761.
[8] P.M. Biesheuvel, *Particle segregation during pressure filtration for cast formation*, Chem. Eng. Sci., 55 (2000), pp. 2595–2606.
[9] P.M. Biesheuvel and H. Verweij, *Calculation of the composition profile of a functionally graded material by centrifugal casting*, J. Am. Ceram. Soc., 83 (2000), pp. 743–749.
[10] P.M. Biesheuvel, V. Breedveld, A.P. Higler and H. Verweij, *Graded membrane supports produced by centrifugal casting of a slightly polydisperse suspension*, Chem. Eng. Sci., 56 (2001), 3517–3525.
[11] F. Bouchut, F.R. Guarguaglini and R. Natalini, *Diffusive BGK approximations for nonlinear multidimensional parabolic equations*, Indiana Univ. Math. J., 49 (2000), pp. 723–749.
[12] R. Bürger, F. Concha, K.-K. Fjelde and K.H. Karlsen, *Numerical simulation of the settling of polydisperse suspensions of spheres*, Powder Technol., 113 (2000), pp. 30–54.
[13] R. Bürger, F. Concha and K.H. Karlsen, *Phenomenological model of filtration processes: 1. Cake formation and expression*, Chem. Eng. Sci., 56 (2001), pp. 4537–4553.
[14] R. Bürger, F. Concha and F.M. Tiller, *Applications of the phenomenological theory to several published experimental cases of sedimentation processes*, Chem. Eng. J., 80 (2000), pp. 105–117.
[15] R. Bürger, S. Evje and K.H. Karlsen, *On strongly degenerate convection-diffusion problems modeling sedimentation-consolidation processes*, J. Math. Anal. Appl., 247 (2000), pp. 517–556.
[16] R. Bürger, S. Evje, K. H. Karlsen and K.-A. Lie, *Numerical methods for the simulation of the settling of flocculated suspensions*, Chem. Eng. J., 80 (2000), 91–104.
[17] R. Bürger, K.-K. Fjelde, K. Höfler and K.H. Karlsen, *Central difference solutions of the kinematic model of settling of polydisperse suspensions and three-dimensional particle-scale simulations*, J. Eng. Math., 41 (2001), pp. 167–187.
[18] R. Bürger and K.H. Karlsen, *On some upwind difference schemes for the phenomenological sedimentation-consolidation model*, J. Eng. Math., 41 (2001), pp. 145–166.
[19] R. Bürger and K.H. Karlsen, *A strongly degenerate convection-diffusion problem modeling centrifugation of flocculated suspensions.* In: H. Freistühler and G. Warnecke (eds.), *Hyperbolic Problems: Theory,*

Numerics, Applications. Eighth International Conference in Magdeburg, February/March 2000, Volume I. Int. Ser. of Numer. Math. vol. 140, Birkhäuser Verlag, Basel 2001, pp. 207–216.

[20] R. Bürger, K.H. Karlsen, E.M. Tory and W.L. Wendland, *Model equations and instability regions for the sedimentation of polydisperse suspensions of spheres*, Z. Angew. Math. Mech., to appear.

[21] R. Bürger and W. L. Wendland, *Existence, uniqueness, and stability of generalized solutions of an initial-boundary value problem for a degenerating quasilinear parabolic equation*, J. Math. Anal. Appl., 218 (1998), pp. 207–239.

[22] R. Bürger, W.L. Wendland and F. Concha, *Model equations for gravitational sedimentation-consolidation processes*, Z. Angew. Math. Mech., 80 (2000), pp. 79–92.

[23] M.C. Bustos, F. Concha, R. Bürger and E.M. Tory, *Sedimentation and Thickening*, Kluwer Academic Publishers, Dordrecht 1999.

[24] J. Carrillo, *Entropy solutions for nonlinear degenerate problems*, Arch. Rat. Mech. Anal., 147 (1999), pp. 269–361.

[25] G.-Q. Chen and E. DiBenedetto, *Stability of entropy solutions to the Cauchy problem for a class of nonlinear hyperbolic-parabolic equations*, SIAM J. Math. Anal., 33 (2001), pp. 751–762.

[26] G.-Q. Chen and B. Perthame, *Well-posedness for non-isotropic degenerate parabolic-hyperbolic equations*, Preprint, 2001.

[27] G.-Q. Chen and D. Wang, *The Cauchy problem for the Euler equations for compressible fluids.* In: *Handbook of Mathematical Fluid Dynamics*, Volume 1. Elsevier Science, Amsterdam, to appear.

[28] B. Cockburn and C.-W. Shu, *The local discontinuous Galerkin method for time-dependent convection-diffusion systems*, SIAM J. Numer. Anal., 35 (1998), 2440–2463.

[29] F. Concha, M.C. Bustos and A. Barrientos, *Phenomenological theory of sedimentation.* In: E.M. Tory (ed.), *Sedimentation of Small Particles in a Viscous Fluid*, Computational Mechanics Publications, Southampton 1996, pp. 51–96.

[30] P. D'Ancona and S. Spagnolo, *The Cauchy problem for weakly parabolic systems*, Math. Ann., 309 (1997), 307–330.

[31] R.H. Davis and H. Gecol, *Hindered settling function with no empirical parameters for polydisperse suspensions*, AIChE J., 40 (1994), pp. 570–575.

[32] S.D. Eidel'man, *Parabolic Systems*, North-Holland, Amsterdam and London 1969.

[33] M.S. Espedal and K.H. Karlsen, *Numerical solution of reservoir flow models based on large time step operator splitting algorithms.* In: A. Fasano (ed.), *Filtration in Porous Media and Industrial Applications (Cetraro, Italy, 1998). Lecture Notes in Mathematics*, vol. 1734, Springer-Verlag, Berlin 2000, pp. 9–77.

[34] S. Evje and K. H. Karlsen, *Monotone difference approximations of BV solutions to degenerate convection-diffusion equations*, SIAM J. Numer. Anal., 37 (2000), 1838–1860.

[35] R. Eymard, T. Gallouët, R. Herbin and A. Michel, *Convergence of a finite volume scheme for nonlinear degenerate parabolic equations.* Preprint, 2000.

[36] D. Gidaspow, *Multiphase Flow and Fluidization*, Academic Press, San Diego 1994.

[37] X. Flotats, *Mathematical modeling of polydisperse suspensions sedimentation*, Hung. J. Ind. Chem., 23 (1995), pp. 215–221.

[38] M. Gurtin, *An Introduction to Continuum Mechanics*, Acadamic Press, San Diego 1981.

[39] H. Holden, K.H. Karlsen and N.H. Risebro, *On uniqueness and existence of entropy solutions of weakly coupled systems of nonlinear degenerate parabolic equations*, Preprint, Department of Mathematics, University of Bergen, 2002.

[40] F. Hubert, *Global existence for hyperbolic-parabolic systems with large periodic initial data*, Diff. Int. Eqns., 11 (1998), 69–83.

[41] B. L. Keyfitz, *Multiphase saturation equations, change of type and inaccessible regions.* In: J. Douglas and U. Hornung (eds.), *Flow in Porous Media*, Birkhäuser Verlag, Basel, Boston and Berlin 1993, pp. 103–116.

[42] K. H. Karlsen, K.-A. Lie, J. R. Natvig, H. F. Nordhaug, and H. K. Dahle, *Operator splitting methods for systems of convection-diffusion equations: nonlinear error mechanisms and correction strategies*, J. Comp. Phys., 173 (2001), 636–663.

[43] K. H. Karlsen and M. Ohlberger, *A note on the uniqueness of entropy solutions of nonlinear degenerate parabolic equations*, J. Math. Anal. Appl., to appear.

[44] K.H. Karlsen and N.H. Risebro, *On the uniqueness and stability of entropy solutions of nonlinear degenerate parabolic equations with rough coefficients*, Preprint, Department of Mathematics, University of Bergen, 2000.

[45] H.-O. Kreiss and J. Lorenz, *Initial-boundary value problems and the Navier-Stokes equations.* Academic Press Inc., Boston, MA, 1989.

[46] A. Kurganov and E. Tadmor, *New high-resolution central schemes for nonlinear conservation laws and convection-diffusion equations*, J. Comp. Phys., 160 (2000), pp. 241–282.

[47] C. Lattanzio and R. Natalini, *Convergence of diffusive BGK approximations for nonlinear strongly parabolic systems*, Proc. Royal Soc. of Edinburgh Sect. A: Math., to appear.

[48] M.J. Lockett and K.S. Bassoon, *Sedimentation of binary particle mixtures*, Powder Technol., 24 (1979), pp. 1–7.

[49] C. Mascia, A. Porretta and A. Terracina, *Non-homogeneous Dirichlet problems for degenerate parabolic-hyperbolic equations*, Arch. Rat. Mech. Anal., to appear.

[50] J.H. Masliyah, *Hindered settling in a multiple-species particle system*, Chem. Eng. Sci., 34 (1979), pp. 1166–1168.

[51] M. Massoudi, K.R. Rajagopal, J.M. Ekmann and M.P. Mathur, *Remarks on the modeling of fluidized systems*, AIChE J., 38 (1992), pp. 471–472.

[52] K. Nakamura and C.E. Capes, *Vertical pneumatic conveying of binary particle mixtures*. In: D.L. Keairns (ed.), *Fluidization Technology*, Vol. 2, Hemisphere Publ. Corp., Washington 1976, pp. 159–184.

[53] H. Nessyahu and E. Tadmor, *Non-oscillatory central differencing for hyperbolic conservation laws*, J. Comp. Phys., 87 (1990), pp. 408–463.

[54] M. Ohlberger, *A posteriori error estimates for vertex centered finite volume approximations of convection-diffusion-reaction equations*. Math. Model. Numer. Anal., 35 (2001), 355–387.

[55] M. Renardy, *A degenerate parabolic-hyperbolic system modeling the spreading of surfactants*, SIAM J. Math. Anal., 28 (1997), pp. 1048–1063.

[56] J.F. Richardson and W.N. Zaki, *Sedimentation and fluidization: Part I*, Trans. Instn. Chem. Engrs. (London), 32 (1954), pp. 35–53.

[57] F. Rosso and G. Sona, *Gravity-driven separation of oil-water dispersions*, Adv. Math. Sci. Appl., 11 (2001), pp. 127–151.

[58] A. Rushton, A.S. Ward and R.G. Holdich, *Solid-Liquid Filtration and Sedimentation Technology*, 2nd ed., Wiley-VCH, Weinheim 2000.

[59] W.K. Sartory, *Three-component analysis of blood sedimentation by the method of characteristics*, Math. Biosci., 33 (1977), pp. 145–165.

[60] W. Schneider, G. Anestis and U. Schaflinger, *Sediment composition due to settling of particles of different sizes*, Int. J. Multiphase Flow, 11 (1985), pp. 419–423.

[61] Y.T. Shih, D. Gidaspow and D.T. Wasan, *Hydrodynamics of sedimentation of multisized particles*, Powder Technol., 50 (1987), pp. 201–215.

[62] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, Second Ed., Springer-Verlag, New York 1994.

[63] A. Spannenberg and K.P. Galvin, *Continuous differential sedimentation of a binary suspension*, Chem. Eng. in Austral., 21 (1996), pp. 7–11.

[64] K. Stamatakis and C. Tien, *Batch sedimentation calculations—the effect of compressible sediment*, Powder Technol., 72 (1992), pp. 227–240.

[65] E. Tadmor, *Approximate solutions of nonlinear conservation laws*. In: B. Cockburn, C. Johnson, C.-W. Shu and E. Tadmor, *Advanced Numerical Approximation of Nonlinear Hyperbolic Equations*. Lecture Notes in Mathematics vol. 1697, Springer-Verlag, Berlin 1998, pp. 1–149.

[66] M. E. Taylor, *Partial Differential Equations III*, Springer-Verlag, New York 1996.

[67] F.M. Tiller, R. Lu, J.H. Kwon and D.J. Lee, *Variable flow rate in compactible filter cakes*, Wat. Res., 33 (1999), pp. 15–22.

[68] D.J. Wedlock, I.J. Fabris and J. Grimsey, *Sedimentation in polydisperse particulate suspensions*, Colloids Surf., 43 (1990), 67–81.

[69] R.H. Weiland, Y.P. Fessas and B.V. Ramarao, *On instabilities arising during sedimentation of two-component mixtures of solids*, J. Fluid Mech., 142 (1984), pp. 383–389.

[70] A. I. Vol'pert and S. I. Hudjaev, *Cauchy's problem for degenerate second order quasilinear parabolic equations*, Math. USSR Sb., 7 (1969), pp. 365–387.

[71] Z. Wu and J. Yin, *Some properties of functions in* BV_x *and their applications to the uniqueness of solutions for degenerate quasilinear parabolic equations*, Northeastern Math. J., 5 (1989), 395–422.

[72] Z. Wu, J. Zhao, J. Yin and H. Li, *Nonlinear Diffusion Equations*, World Scientific, Singapore 2001.

[73] Y. Zimmels, *Theory of density separation of particulate systems*, Powder Technol., 43 (1985), pp. 127–139.