On the Attainable set for Temple Class Systems with Boundary Controls

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Abstract

Consider the initial-boundary value problem for a strictly hyperbolic, genuinely nonlinear, Temple class system of conservation laws

$$u_t + f(u)_x = 0, \qquad u(0,x) = \overline{u}(x), \qquad \begin{cases} u(t,a) = \widetilde{u}_a(t), \\ u(t,b) = \widetilde{u}_b(t), \end{cases} \tag{1}$$

on the domain $\Omega = \{(t,x) \in \mathbb{R}^2 : t \geq 0,\ a \leq x \leq b\}$. We study the mixed problem (1) from the point of view of control theory, taking the initial data $\overline{u}$ fixed, and regarding the boundary data $\widetilde{u}_a$, $\widetilde{u}_b$ as control functions that vary in prescribed sets $\mathcal{U}_a$, $\mathcal{U}_b$, of $\mathbf{L}^\infty$ boundary controls. In particular, we consider the family of configurations

$$\mathcal{A}(T) \doteq \big\{u(T,\cdot)\ ;\quad u \text{ is a sol. to } (1), \quad \widetilde{u}_a \in \mathcal{U}_a,\ \ \widetilde{u}_b \in \mathcal{U}_b\big\}$$

that can be attained by the system at a given time $T > 0$, and we give a description of the attainable set $\mathcal{A}(T)$ in terms of suitable Oleinik-type conditions. We also establish closure and compactness of the set $\mathcal{A}(T)$ in the $\mathbf{L}^1$ topology.

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1 Introduction

Consider the initial-boundary value problem for a strictly hyperbolic, genuinely nonlinear, system of conservation laws in one space dimension

$$u_t + f(u)_x = 0, \tag{1.1}$$

$$u(0,x) = \overline{u}(x), \tag{1.2}$$

$$u(t,a) = \widetilde{u}_a(t), \tag{1.3}$$

$$u(t,b) = \widetilde{u}_b(t), \tag{1.4}$$

on the strip $\Omega = \{(t,x) \in \mathbb{R}^2 \ ; \ t \geq 0, x \in [a,b]\}$. Here, $u = u(t,x) \in \mathbb{R}^n$ is the vector of the conserved quantities, $\widetilde{u}_a$, $\widetilde{u}_b$ are measurable, bounded boundary data, and the flux function $f : U \mapsto \mathbb{R}^n$ is a smooth vector field defined on some open set $U \subseteq \mathbb{R}^n$, that belongs to a class of fields introduced by Temple [26, 25] for which rarefaction and Hugoniot curves coincide. We recall that, for problems of this type, classical solutions may develop discontinuities in finite time, no matter of the regularity of the initial and boundary data. Hence, it is natural to consider weak solutions in the sense of distributions. Moreover, since, in general, the Dirichlet conditions (1.3)-(1.4) cannot be fulfilled pointwise a.e. (see [6, 18]), different weaker formulations of the boundary condition have been considered in the literature (see [1, 20, 24] and references therein). Here, following F. Dubois, P.G. LeFloch [18], we will adopt a formulation of (1.3)-(1.4) based on the definition of a time-dependent set of *admissible boundary data*, that is related to the notion of Riemann problem.

In the present paper, having in mind applications of Temple systems to problems of oil reservoir simulation, multicomponent chromatography, as well as in traffic flow models, we study the effect of the boundary conditions (1.3)-(1.4) on the solution of (1.1)-(1.2) from the point of view of control theory. Namely, following the same approach adopted in [3, 4] for scalar conservation laws, we fix an initial data $\overline{u} \in \mathbf{L}^\infty([a,b])$ and we consider the family of configurations

$$\mathcal{A}(T; \mathcal{U}_a, \mathcal{U}_b) \doteq \big\{ u(T,\cdot) \ ; \ \ u \ \text{ is a sol. to } \ (1.1)-(1.4), \quad \widetilde{u}_a \in \mathcal{U}_a, \ \ \widetilde{u}_b \in \mathcal{U}_b \big\} \tag{1.5}$$

that can be attained at a given time $T > 0$ by solutions to (1.1)-(1.4), with boundary data $\widetilde{u}_a, \widetilde{u}_b$ that vary in prescribed sets $\mathcal{U}_a, \mathcal{U}_b \subset \mathbf{L}^\infty(\mathbb{R}^+)$ of *admissible boundary controls.* In the case of scalar, convex conservation laws, it was proved in [3], by using the theory of generalized characteristics [16], that the profiles $w(x)$ which can be attained at a fixed time $T > 0$ are only those for which the map $x \mapsto \frac{f'(w(x)}{x}$ is non increasing. Under the assumption that $f'(u) \geq 0$ for all u, and for solutions of the mixed problem (1.1)-(1.4) on the region Ω, this condition is equivalent to the Oleinik-type inequalities

$$D^+ w(x) \leq \frac{f'(w(x))}{(x-a)\, f''(w(x))} \qquad \text{for a.e. } \ x \in [a,b]\,, \tag{1.6}$$

($D^+ w$ denoting the upper Dini derivative of w). For general $n \times n$ systems, a complete characterization of the attainable set does not seem possible, due to the complexity of repeated wave-front interactions. However, in the particular case of Temple systems, wave interactions can only change the speed of wave-fronts, without modifying their amplitudes, due to the special geometric features of such systems. Therefore, the only restriction to boundary controllability is the decay due to genuine nonlinearity. We then consider here a convex, compact set $\Gamma \subset U$, and provide a description of the attainable set

$$\mathcal{A}(T) \doteq \mathcal{A}(T; \mathcal{U}^\infty, \mathcal{U}^\infty), \qquad \mathcal{U}^\infty \doteq \mathbf{L}^\infty([0,T],\Gamma)\,,$$

in terms of certain Oleinik-type conditions. We also establish the compactness of $\mathcal{A}(T)$ in the $\mathbf{L}^1$ topology. Applications to calculus of variations and problems of optimization (where the cost functional depends on the profile of the solution at a fixed time T) motivate the study of topological properties of $\mathcal{A}(T)$.

The paper is organized as follows. Section 2 contains the basic definitions and the statement of the main results. We also provide in this section a review of the existence and well-posedness theory for the mixed problem (1.1)-(1.4), and a description of a front tracking algorithm that will be used throughout the paper. In Section 3 we establish some preliminary estimates, and a regularity result concerning the global structure of solutions to the mixed problem (1.1)-(1.4) generated by a front tracking algorithm. The proof of the main results is contained in Section 4.

2 Preliminaries and statement of the main results

2.1 Formulation of the problem

Let $f : U \mapsto \mathbb{R}^n$ be the flux function of the strictly hyperbolic system (1.1) defined on a neighborhood of the origin $U \subseteq \mathbb{R}^n$. Denote by $\lambda_1(u) < \cdots < \lambda_n(u)$ the eigenvalues of the Jacobian matrix $Df(u)$, and let $\{r_1(u), \dots, r_n(u)\}$ be a basis of right eigenvectors of $Df(u)$. By possibly considering a sufficiently small restriction of the domain U, we may assume that the following *uniform* strict hyperbolicity condition holds.

(SH1) For every $u, v \in U$, the characteristic speeds at these points satisfy

$$\lambda_i(u) < \lambda_j(v), \qquad \forall\ 1 \le i < j \le n\,. \tag{2.1}$$

We also assume that there is a fixed set of characteristic lines entering the interior of the strip $[a,b] \times \mathbb{R}^+$ at the boundaries $x = a$, $x = b$, i.e. that, for some index $p \in \{1, \dots, n\}$, there holds

$$\lambda_p(u) < 0 < \lambda_{p+1}(u), \qquad \forall\ u \in U, \tag{2.2}$$

and we let $\lambda^{\min}$, $\lambda^{\max}$ denote the minimum and maximum characteristic speed so that there holds

$$0 < \lambda^{\min} \le |\lambda_i(u)| \le \lambda^{\max}, \qquad \forall\ u \in U. \tag{2.3}$$

Moreover, we assume that each i-th characteristic field r_i is *genuinely nonlinear* in the sense of Lax [21], and that system (1.1) is of Temple class according with the following.

Definition 2.1 *A system of conservation laws is of Temple class if there exists a system of coordinates $w = (w_1, \dots, w_n)$ consisting of Riemann invariants, and such that the level sets $\{u \in U;\ w_i(u) = constant\}$ are hyperplanes (see [25]).*

By possibly performing a translation of coordinates, it is not restrictive to assume that the Riemann invariants are chosen so that $\partial_i \lambda_i(w) > 0$, $i = 1, \dots n$, for all $w = w(u)$, $u \in U$. Throughout the paper, we will often write $w_i(t,x) \doteq w_i\big(u(t,x)\big)$ to denote the i-th Riemann coordinate of a solution $u = u(t,x)$ to (1.1). We recall that, for a Temple class system, Hugoniot curves and rarefaction curves coincide [26]. Moreover, as observed in [2], thanks to the existence of Riemann coordinates one can show that the assumption **SH1** implies the invertibility of the map $f : U \mapsto f(U)$.

We next introduce a definition of weak solution to (1.1)-(1.4) which includes an entropy admissibility condition of Oleinik type on the decay of positive waves, so to achieve uniqueness. The boundary conditions (1.3)-(1.4) are formulated in terms of the weak trace of the flux $f(u)$ at the the boundaries $x = a$, $x = b$, and are related to the notion of Riemann problem in the same spirit of [18]. To this purpose, letting $u(t,x) = W(\xi = x/t;\, u_L, u_R)$, $u_L, u_R \in U$, denote the self-similar solution of the Riemann problem for (1.1) with initial data

$$u(0,x) = \begin{cases} u_L & \text{if } x < 0, \\ u_R & \text{if } x > 0, \end{cases}$$

for any given boundary state $\widetilde{u} \in U$, we define the set of *admissible states at the boundaries*

$$\begin{aligned} \mathcal{V}_a(\widetilde{u}) &\doteq \big\{W(0+;\, \widetilde{u}, u_R)\ ;\ u_R \in U\big\}, \\ \mathcal{V}_b(\widetilde{u}) &\doteq \big\{W(0-;\, u_L, \widetilde{u})\ ;\ u_L \in U\big\}. \end{aligned} \tag{2.4}$$

Definition 2.2 *A function $u : [0,T] \times [a,b] \mapsto U$ is an entropy weak solution of the initial-boundary value problem (1.1)-(1.4) on $\Omega_T \doteq [0,T] \times [a,b]$, if it is continuous as a function from $]0,T]$ into $\mathbf{L}^1$, and the following properties hold:*

(i) u is a distributional solution to the Cauchy problem (1.1)-(1.2) on Ω_T in the sense that, for every test function $\phi \in \mathcal{C}^1_c$ with compact support contained in the set $\{(t,x) \in \mathbb{R}^2;\ a < x < b,\, t < T\}$, there holds

$$\int_0^T \int_a^b \big(u(t,x) \cdot \phi_t(t,x) + f(u(t,x)) \cdot \phi_x(t,x)\big) dx\ dt + \int_a^b \overline{u}(x) \cdot \phi(0,x) dx = 0\,;$$

(ii) the flux $f(u)$ admits weak traces at the boundaries $x = a$, $x = b$, i.e. there exist two measurable functions $\Psi_a, \Psi_b : [0,T] \mapsto \mathbb{R}^n$ such that*

$$f(u(\cdot,x)) \underset{x \to a^+}{\overset{*}{\rightharpoonup}} \Psi_a, \qquad f(u(\cdot,x)) \underset{x \to b^-}{\overset{*}{\rightharpoonup}} \Psi_b \qquad in \quad \mathbf{L}^\infty([0,T]), \tag{2.5}$$

and the boundary conditions (1.3)-(1.4) are satisfied in the following sense

$$\Psi_a(t) \in f\big(\mathcal{V}_a(\widetilde{u}_a(t))\big), \qquad \Psi_b(t) \in f\big(\mathcal{V}_b(\widetilde{u}_b(t))\big) \qquad \textit{for a.e.} \quad 0 \le t \le T; \tag{2.6}$$

(iii) u satisfies the following entropy conditions on the decay of positive waves in time and in space. There exists some constant $C > 0$, depending only on the system (1.1), so that

(a) For any $0 < t \le T$, and for a.e. $a < x < y < b$, there holds

$$w_i(t,y) - w_i(t,x) \le C \cdot \left\{ \frac{y-x}{t} + \log\left(\frac{y-b}{x-b}\right) \right\} \qquad \textit{if} \qquad i \in \{1, \dots, p\}, \tag{2.7}$$

$$w_i(t,y) - w_i(t,x) \le C \cdot \left\{ \frac{y-x}{t} + \log\left(\frac{y-a}{x-a}\right) \right\} \qquad \textit{if} \qquad i \in \{p+1, \dots, n\}; \tag{2.8}$$

(b) For a.e. $a < x < b$, and for a.e. $0 < \tau_1 < \tau_2 \le T$, there holds

$$w_i(\tau_2,x) - w_i(\tau_1,x) \le C \cdot \left\{ \frac{\tau_2-\tau_1}{x-b} + \log\left(\frac{\tau_2}{\tau_1}\right) \right\} \qquad \textit{if} \qquad i \in \{1, \dots, p\}, \tag{2.9}$$

$$w_i(\tau_2,x) - w_i(\tau_1,x) \le C \cdot \left\{ \frac{\tau_2-\tau_1}{x-a} + \log\left(\frac{\tau_2}{\tau_1}\right) \right\} \qquad \textit{if} \qquad i \in \{p+1, \dots, n\}. \tag{2.10}$$

Remark 2.1 The set of admissible flux values at the boundaries $x = a$, $x = b$, can be expressed in Riemann coordinates as

$$\begin{aligned} f\big(\mathcal{V}_a(\widetilde{u})\big) &= \Big\{ f(u) \;;\; w_i(u) = w_i(\widetilde{u}) \qquad \forall\, i = p+1, \dots, n \Big\}, \\ f\big(\mathcal{V}_b(\widetilde{u})\big) &= \Big\{ f(u) \;;\; w_i(u) = w_i(\widetilde{u}) \qquad \forall\, i = 1, \dots, p \Big\}. \end{aligned} \tag{2.11}$$

Hence, by the invertibility of the map $f : U \mapsto f(U)$, the above boundary conditions (2.6) are equivalent to the set of equalities

$$\begin{aligned} w_i\big(f^{-1}(\Psi_a(t))\big) &= w_i\big(\widetilde{u}_a(t)\big) \qquad \text{for a.e.} \quad 0 \le t \le T, \qquad i = p+1, \dots, n, \\ w_i\big(f^{-1}(\Psi_b(t))\big) &= w_i\big(\widetilde{u}_b(t)\big) \qquad \text{for a.e.} \quad 0 \le t \le T, \qquad i = 1, \dots, p. \end{aligned} \tag{2.12}$$

This means that the boundary conditions (2.6) guarantee that, at almost every time $t \in [0,T]$, the solution to the Riemann problem for (1.1), having left and right initial states $u^L = \widetilde{u}_a(t)$, $u^R = f^{-1}(\Psi_a(t))$, contains only waves with negative speeds, while the solution to the Riemann problem with initial states $u^L = f^{-1}(\Psi_b(t))$, $u^R = \widetilde{u}_b(t)$, contains only waves with positive speeds. Thus, in particular, such solutions do not contain any front entering the domain $[t, +\infty[\, \times]a, b[$.

In the present paper we regard the boundary data as admissible controls and, in connection with a fixed convex, compact set $\Gamma \subset U$ having the form

$$\Gamma = \Big\{ u \in U; \quad w_i(u) \in [\alpha_i,\ \beta_i], \qquad i = 1, \dots, n \Big\}, \tag{2.13}$$

we study the basic properties of the *attainable set* for (1.1)-(1.2), i.e. of the set

$$\mathcal{A}(T) \doteq \{ u(T, \cdot)\ ;\ \ u \ \text{ is a sol. to } \ (1.1) - (1.4), \quad \widetilde{u}_a,\, \widetilde{u}_b \in \mathbf{L}^\infty([0,T],\Gamma) \} \tag{2.14}$$

which consists of all profiles that can be attained at a fixed time $T > 0$, by entropy weak solutions of (1.1)-(1.4) (according with Definition 2.2) with a fixed initial data $\overline{u} \in \mathbf{L}^\infty([a,b],\Gamma)$, and boundary data $\widetilde{u}_a$, $\widetilde{u}_b$ that vary in

$$\mathcal{U}_T^\infty \doteq \mathbf{L}^\infty([0,T],\Gamma). \tag{2.15}$$

We will establish a characterization of (2.14) in terms of certain Oleinik type estimates on the decay of positive waves, and we will prove the compactness of (2.14) in the $\mathbf{L}^1$ topology.

2.2 Statements of the main results

For any $\rho > 0$, consider the set of maps

$$K^\rho \doteq \left\{ \varphi \in \mathbf{L}^\infty([a,b],\Gamma)\ ; \quad \begin{array}{ll} \dfrac{w_i(\varphi(y)) - w_i(\varphi(x))}{y - x} \le \dfrac{\rho}{x - a} & \begin{cases} \text{for a.e.} \quad a < x < y < b, \\ \text{if} \quad i \in \{p+1, ..., n\} \end{cases} \\ \dfrac{w_i(\varphi(y)) - w_i(\varphi(x))}{y - x} \le \dfrac{\rho}{b - y} & \begin{cases} \text{for a.e.} \quad a < x < y < b, \\ \text{if} \quad i \in \{1, ..., p\} \end{cases} \end{array} \right\}. \tag{2.16}$$

The inequalities in (2.16) reflect the fact that positive waves entering through the boundaries $x = a$, $x = b$ decay in time. Therefore, their density (expressed in terms of Riemann coordinates) is inversely proportional to their distance from their entrance point on the boundary.

Theorem 2.1 *Let (1.1) be a system of Temple class with all characteristic fields genuinely nonlinear, and assume that the strict hyperbolicity condition* (**SH1**) *is verified. Then, for every fixed* $\overline{\tau} > 0$, *there exists* $\rho = \rho(\overline{\tau}) > 0$ *such that*

$$\mathcal{A}(\tau) \subseteq K^\rho \qquad \forall\, \tau \ge \overline{\tau}. \tag{2.17}$$

Moreover, there exist $T > 0$ *and* $\rho' < \rho(T)$, *such that*

$$K^{\rho'} \subseteq \mathcal{A}(\tau) \qquad \forall\, \tau > T. \tag{2.18}$$

Remark 2.2 Observe that, given $\varphi \in K^\rho$, any map $x \mapsto w_i(\varphi(x))$, $i \in \{1, \dots, n\}$, is essentially bounded and has finite total increasing variation on subsets of $[a,\, b]$ bounded away from the end points a, b. Hence, any map $x \mapsto w_i(\varphi(x))$, $i \in \{1, \dots, n\}$, has also finite total variation on such sets and, in particular, it admits left and right limits in any point $x \in]a, b[$. Moreover, since an element φ of K^ρ is defined up to $\mathbf{L}^1$ equivalence, we may always assume that there is a right continuous representative of $w_i(\varphi)$, $i \in \{1, \dots, n\}$, that satisfies the inequalities appearing in the definition of K^ρ.

Theorem 2.2 *Under the same assumptions of Theorem 1, the set $\mathcal{A}(T)$ is a compact subset of $\mathbf{L}^1([a, b], \Gamma)$ for each $T > 0$.*

Indeed, we will prove in Section 4 that the compactness of the attainable set $\mathcal{A}(T)$ holds even in the case where $\mathcal{A}(T)$ is defined as the set of all configurations that can be reached at time T only by solutions of the mixed problem for (1.1) that admit a strong $\mathbf{L}^1$ trace at the boundaries $x = a$, $x = b$ (as the ones generated by a front tracking algorithm).

2.3 Existence and uniqueness of solutions

We describe here a front tracking algorithm that generates approximate solutions to (1.1) on the strip $[a, b] \times \mathbb{R}^+$ continuously depending on the initial and boundary data, which represents a natural extension of [2, 12]. Fix an integer $\nu \geq 1$ and consider the discrete set of points in Γ whose coordinates are integer multiples of $2^{-\nu}$:

$$\Gamma^\nu \doteq \Big\{ u \in \Gamma\,;\ w_i(u) \in 2^{-\nu}\mathbb{Z}, \quad i = 1, \dots, n \Big\}. \tag{2.19}$$

Moreover, consider the domain

$$\mathcal{D}^\nu \doteq \Big\{ (u, u', u'');\quad u \in \mathbf{L}^\infty([a, b], \Gamma^\nu),\ u', u'' \in \mathbf{L}^\infty(\mathbb{R}^+, \Gamma^\nu),\quad u,\, u',\, u'' \ \text{ are piecewise constant} \Big\}. \tag{2.20}$$

On $\mathcal{D}^\nu$ we now construct a flow map E^ν whose trajectories are front tracking approximate solutions of (1.1). To this end, we first describe how to solve a Riemann problem with left and right initial states $u^L, u^R \in \Gamma^\nu$. In Riemann coordinates, assume that

$$w(u^L) \doteq w^L = (w_1^L, \dots, w_n^L), \qquad w(u^R) \doteq w^R = (w_1^R, \dots, w_n^R).$$

Consider the intermediate states

$$z^0 \doteq u^L, \qquad \dots, \qquad z^i \doteq u(w_1^R, \dots, w_i^R, w_{i+1}^L, \dots, w_n^L), \qquad \dots, \qquad z^n \doteq u^R. \tag{2.21}$$

The solution to the Riemann problem (u^L, u^R) is constructed by piecing together the solutions to the simple Riemann problems (z^{i-1}, z^i), $i = 1, \dots, n$. If $w_i^R < w_i^L$, the solution of the Riemann problems (z^{i-1}, z^i) will contain a single i-shock, connecting the states z^{i-1}, z^i, and traveling with the Rankine-Hugoniot speed $\lambda_i(z^{i-1}, z^i)$. Here and in the sequel, by $\lambda_i(u, u')$ we denote the i-th eigenvalue of the averaged matrix

$$A(u, u') \doteq \int_0^1 Df\big(\theta u + (1 - \theta)u'\big)\, d\theta. \tag{2.22}$$

If $w_i^R > w_i^L$, the exact solution of the Riemann problem (z^{i-1}, z^i) would contain a centered rarefaction wave. This is approximated by a rarefaction fan as follows. If $w_i^R = w_i^L + p_i\, 2^{-\nu}$ we insert the states

$$z^{i,\ell} \doteq (w_1^R, \ldots, w_i^L + \ell\, 2^{-\nu}, w_{i+1}^L, \ldots, w_n^L), \qquad \ell = 0, \ldots, p_i, \tag{2.23}$$

so that $z^{i,0} = z^{i-1}$, $z^{i,p_i} = z^i$. Our front tracking solution will then contain p_i fronts of the i-th family, each connecting a couple of states $z^{i,\ell-1}$, $z^{i,\ell}$ and traveling with speed $\lambda_i\big(z^{i,\ell-1},\ z^{i,\ell}\big)$.

For any given triple of (piecewise constant) initial and boundary data $(\,\overline{u},\, \widetilde{u}_a\,,\, \widetilde{u}_b\,) \in \mathcal{D}^\nu$, the approximate solution $u(t,\cdot) \doteq E_t^\nu(\,\overline{u},\, \widetilde{u}_a\,,\, \widetilde{u}_b\,)$ is now constructed as follows. At time $t = 0$, for $a < x < b$ we solve the initial Riemann problems determined by the jumps in $\overline{u}$ according to the above procedure, while at $x = a$ we construct the solution to the Riemann problem with left and right initial states $u^L = \widetilde{u}_a(0+)$, $u^R = \overline{u}(a+)$ and take its restriction to the interior of the domain Ω. In the same way, at $x = b$, we take the restriction to the interior of Ω of the solution to the Riemann problem with initial states $u^L = \overline{u}(b-)$, $u^R = \widetilde{u}_b(0+)$. This yields a piecewise constant function with finitely many fronts, traveling with constant speeds. The solution is then prolonged up to the first time t_1 at which one of the following events takes place:

a) two or more discontinuities interact in the interior of Ω;
b) one or more discontinuities hit the boundary of Ω;
c) the boundary data $\widetilde{u}_a$ has a jump;
d) the boundary data $\widetilde{u}_b$ has a jump.

If the case a) occurs, we then solve the resulting Riemann problems applying again the above procedure, while in the other three cases b)-c)-d) we construct the solution to the Riemann problem with left and right initial states $u^L = \widetilde{u}_a(t_1+)$, $u^R = u(t_1, a+)$, or $u^L = u(t_1, b-)$, $u^R = \widetilde{u}_b(t_1+)$, and take its restriction to the interior of the domain Ω. This determines the solution $u(t,\cdot)$ until the time $t_2 > t_1$ where one of the events a),b),c) again takes place, etc... Notice that at any time where case b) occurs but c) or d) do not take place, no new wave is generated. Therefore, waves entering the domain Ω at the boundaries $x = a$, $x = b$ are produced only by the jumps of the boundary data $\widetilde{u}_a$, $\widetilde{u}_b$.

As in [2, 12], one checks that the approximate solution u constructed with this algorithm is well defined for all times $t \geq 0$. Indeed, the following properties hold.

- The total variation of $u(t,\cdot)$, measured w.r.t. the Riemann coordinates $w_1(t,\cdot), \ldots, w_n(t,\cdot)$, is non-increasing in time.

- The number of wave-fronts in $u(t,\cdot)$ is non-increasing at each interaction. Hence, the total number of wave-fronts in $u(t,\cdot)$ remains finite.

It is then possible to define a flow map

$$\mathbf{p} \mapsto E_t^\nu \mathbf{p}, \qquad \mathbf{p} \doteq (\,\overline{u},\, \widetilde{u}_a\,,\, \widetilde{u}_b\,) \in \mathcal{D}^\nu, \quad t \geq 0 \tag{2.24}$$

of approximate solutions of (1.1). By construction, each trajectory $t \mapsto E_t^\nu \mathbf{p}$ is a weak solution of (1.1) (because all fronts of $u(t,\cdot) \doteq E_t^\nu \mathbf{p}$ satisfy the Rankine-Hugoniot conditions), but may contain discontinuities that do not satisfy the usual Lax stability conditions (due to the presence of rarefaction fronts). On the other hand, one can verify as in [2, Lemma 4.4] that, due to genuine nonlinearity, the amount of positive waves in $u(t,\cdot)$, measured w.r.t. the Riemann coordinates $w_1(t,\cdot),\dots,w_n(t,\cdot)$, decays in time and in space. Hence, for a.e. $a<x<y<b$, one obtains the Oleinik type estimates

$$\begin{aligned} w_i(t,y)-w_i(t,x) &\leq C\cdot\left\{\frac{y-x}{t}+\log\left(\frac{y-b}{x-b}\right)\right\}+N_\nu\,2^{-\nu} \quad \text{if} \quad i\in\{1,\dots,p\},\\ w_i(t,y)-w_i(t,x) &\leq C\cdot\left\{\frac{y-x}{t}+\log\left(\frac{y-a}{x-a}\right)\right\}+N_\nu\,2^{-\nu} \quad \text{if} \quad i\in\{p+1,\dots,n\}, \end{aligned} \tag{2.25}$$

where N_ν denotes the maximum number of shocks of each family present in the initial data $\overline{u}$, and in the boundary data $\widetilde{u}_a$, $\widetilde{u}_b$. Similarly, one can check that along the x-sections, for a.e. $0<\tau_1<\tau_2$, there holds

$$\begin{aligned} w_i(\tau_2,x)-w_i(\tau_1,x) &\leq C\cdot\left\{\frac{\tau_2-\tau_1}{x-b}+\log\left(\frac{\tau_2}{\tau_1}\right)\right\}+N_\nu\,2^{-\nu} \quad \text{if} \quad i\in\{1,\dots,p\},\\ w_i(\tau_2,x)-w_i(\tau_1,x) &\leq C\cdot\left\{\frac{\tau_2-\tau_1}{x-a}+\log\left(\frac{\tau_2}{\tau_1}\right)\right\}+N_\nu\,2^{-\nu} \quad \text{if} \quad i\in\{p+1,\dots,n\}. \end{aligned} \tag{2.26}$$

Remark 2.3 Observe that, if $u(t,x)$ is a front tracking solution of the Cauchy problem for (1.1) (with initial data $\overline{u}(x) \doteq u(0,x)$) constructed by the algorithm in [12] on the upper half plane $\mathbb{R}^+\times\mathbb{R}$, then the restriction of $u(t,\cdot)$ to the interval $[a,b]$ coincides with the front tracking solution $E_t^\nu(\overline{u},\widetilde{u}_a,\widetilde{u}_b)$ of the mixed problem for (1.1), with boundary data $\widetilde{u}_a(t) \doteq u(t,a)$, $\widetilde{u}_b(t) \doteq u(t,b)$.

As $\nu\to\infty$, the domains $\mathcal{D}^\nu$ become dense in

$$\mathcal{D} \doteq \Big\{(\overline{u},\,\widetilde{u}_a\;\widetilde{u}_b)\;;\;\overline{u}\in\mathbf{L}^\infty([a,b],\Gamma),\;\;\widetilde{u}_a,\,\widetilde{u}_b\in\mathbf{L}^\infty(\mathbb{R}^+,\Gamma)\Big\}. \tag{2.27}$$

Thus, following the same technique adopted in [2], one can define a flow map E_t on $\mathcal{D}$ as a suitable limit of the flows E_t^ν in (2.24), that depends Lispschitz continuously on the initial and boundary data. Namely, the following holds.

Theorem 2.3 *Let (1.1) be a system of Temple class with all characteristic fields genuinely nonlinear, and assume that the strict hyperbolicity condition* (**SH1**) *holds. Then, there exists a continuous map*

$$(t,\overline{u},\widetilde{u}_a,\widetilde{u}_b)\mapsto E_t(\overline{u},\widetilde{u}_a,\widetilde{u}_b) \qquad t\geq 0,\quad (\overline{u},\widetilde{u}_a,\widetilde{u}_b)\in\mathcal{D}, \tag{2.28}$$

and some constant $C>0$ depending only on the system (1.1) and on the domain Γ, so that, *for every fixed* $0 < \delta < (b-a)/2$, *and for all* $\mathbf{p}_1 \doteq (\overline{u}, \widetilde{u}_a, \widetilde{u}_b)$, $\mathbf{p}_2 \doteq (\overline{v}, \widetilde{v}_a, \widetilde{v}_b) \in \mathcal{D}$, *letting* $L_t \doteq L_t(\delta) = C(1+\log(t/\delta))$, *there holds*

$$\left\| E_t \mathbf{p}_1 - E_t \mathbf{p}_2 \right\|_{\mathbf{L}^1([a+\delta,\, b-\delta])} \leq L_t \cdot \left\{ \left\| \overline{u} - \overline{v} \right\|_{\mathbf{L}^1([a,b])} + \left\| \widetilde{u}_a - \widetilde{v}_a \right\|_{\mathbf{L}^1([0,\,t])} + \left\| \widetilde{u}_b - \widetilde{v}_b \right\|_{\mathbf{L}^1([0,\,t])} \right\} \tag{2.29}$$

for all $t \geq \delta$. *Moreover, the map* $(t,x) \mapsto E_t(\overline{u}, \widetilde{u}_a, \widetilde{u}_b)(x)$ *yields an entropy weak solution (in the sense of Definition 2.2) to the initial-boundary value problem (1.1)-(1.4) on* Ω, *that admits strong* $\mathbf{L}^1$ *traces at the boundaries* $x = a$ *and* $x = b$, *i.e. there exist two measurable maps* $\psi_a, \psi_b : \mathbb{R}^+ \mapsto U$ *such that*

$$\begin{aligned} &\lim_{x \to a+} \int_0^\tau \left| E_t(\overline{u}, \widetilde{u}_a, \widetilde{u}_b)(x) - \psi_a(t) \right| \, dt = 0, \\ &\lim_{x \to b-} \int_0^\tau \left| E_t(\overline{u}, \widetilde{u}_a, \widetilde{u}_b)(x) - \psi_b(t) \right| \, dt = 0, \end{aligned} \qquad \forall\, \tau \geq 0. \tag{2.30}$$

The proof of Theorem 2.3 can be obtained with entirely similar arguments to those used to establish [2, Theorem 2.1], where a continuous flow of solutions to (1.1) is constructed in the case of a mixed problem on the quarter of plane $\{(t,x) \in \mathbb{R}^2 \ ; \ t \geq 0,\ x \geq 0\}$, with a single boundary at $x = 0$.

Concerning uniqueness, with the same arguments in [2] one obtains the following result which is the extension of [2, Theorem 2.2] to the present case of a domain Ω with two boundaries at $x = a$ and at $x = b$.

Theorem 2.4 *Let (1.1) be a system of Temple class satisfying the same assumptions as in Theorem 2.3. Let* $u = u(t,x)$ *be an entropy weak solution to the mixed problem (1.1)-(1.4) on the region* $\Omega_T \doteq [0,\, T] \times [a,b]$ *(in the sense of Definition 2.2). Assume that the following conditions hold.*

(i) The map $(t,x) \to \big(u(t,\cdot),\, u(\cdot,x)\big)$ *takes values within the domain*

$$\mathcal{D}_T \doteq \Big\{ (\overline{u}, \widetilde{u}_a, \widetilde{u}_b) \ ; \ \overline{u} \in \mathbf{L}^\infty([a,b],\Gamma),\ \widetilde{u}_a,\, \widetilde{u}_b \in \mathbf{L}^\infty([0,T],\Gamma) \Big\}. \tag{2.31}$$

(ii) There holds

$$ess \sup_{t \to 0+} \int_a^b \left| u(t,x) - \overline{u}(x) \right| dx = 0\,. \tag{2.32}$$

(iii) There holds

$$ess \sup_{x \to a+} \int_0^T \left| w_i\big(u(t,x)\big) - w_i(\widetilde{u}_a(t)) \right| dt = 0 \qquad \forall\ i = p+1, \ldots, n, \tag{2.33}$$

$$ess \sup_{x \to b-} \int_0^T \left| w_i\big(u(t,x)\big) - w_i(\widetilde{u}_b(t)) \right| dt = 0 \qquad \forall\ i = 1, \ldots, p. \tag{2.34}$$

Then, u coincides with the corresponding trajectory of the flow map E_t provided by Theorem 2.3, namely one has

$$u(t,\cdot) = E_t(\overline{u},\, \widetilde{u}_a\,,\, \widetilde{u}_b)(\cdot), \qquad \forall\ 0 \leq t \leq T. \tag{2.35}$$

The next result shows that the conditions (2.32)-(2.34) are certainly satisfied by entropy weak solutions to the mixed problem (1.1)-(1.4) obtained as limit of front tracking approximations.

Theorem 2.5 *Let (1.1) be a system of Temple class satisfying the same assumptions as in Theorem 2.3. Consider a sequence $u^\nu(t,\cdot) : [a,b] \mapsto \Gamma^\nu$ of wave-front tracking approximate solutions of the mixed problem for (1.1) (constructed with the above algorithm) that converges in $\mathbf{L}^1$, as $\nu \to \infty$, to some function $u(t,\cdot) : [a,b] \mapsto \Gamma$, for every $t \in [0,T]$. Then, there exist the right limit at $x = a$, and the left limit at $x = b$, of the map $x \to u(t,x)$ for every $t \in [0,T]$, and the right limit at $t = 0$ of the map $t \to u(t,x)$ for every $x \in [a,b]$. Moreover, there is a countable set $\mathcal{N} \subset \mathbb{R}$ such that $u(t,a) = u(t,a+)$, $u(t,b) = u(t,b-)$ for all $t \in [0,T] \setminus \mathcal{N}$, and $u(0,x) = u(0+,x)$ for all $x \in [a,b] \setminus \mathcal{N}$, and, setting $\overline{u} \doteq u(0,\cdot)$, $\widetilde{u}_a \doteq u(\cdot,a)$, $\widetilde{u}_b \doteq u(\cdot,b)$, there holds (2.35).*

Remark 2.4 It was shown in [2, Lemma 2.1] that an alternative way to prove the essential limits (2.33)-(2.34), is to employ the distributional entropy inequalities associated to the "boundary entropy pairs" for (1.1), introduced by G.-Q. Chen and H. Frid in [14, 15]. However, in order to apply [2, Lemma 2.1] to a function u obtained as a limit of approximate solutions u^ν, it is necessary to know the $\mathbf{L}^1$ convergence of the sequence of the corresponding boundary data $\widetilde{u}_a^\nu$, $\widetilde{u}_b^\nu$. Instead, the result provided here by Theorem 2.5 allows to derive the limits (2.33)-(2.34) requiring only the $\mathbf{L}^1$ convergence of the sequence of the approximate solutions $u^\nu(t,\cdot)$, for all t. This property will be crucial to establish the main result of the paper stated in Theorems 2.1-2.2.

In order to prove Theorem 2.5, we will show in the next section that, for Temple systems, solutions of the mixed problem (1.1)-(1.4) with possibly unbounded variation enjoy the same regularity property (of being continuous outside a countable number of Lipschitz curves) possessed by solutions with small total variation of a general system, thus extending the regularity results obtained under the smallness assumption of the total variation by DiPerna [17] and Liu [22] (for solutions constructed by the Glimm scheme) and by Bressan and LeFloch [13] (for solutions generated by a front tracking algorithm).

Proposition 2.1 *In the same setting as Theorem 2.5, consider a sequence $u^\nu(t,\cdot) : [a,b] \mapsto \Gamma^\nu$ of wave-front tracking approximate solutions of the mixed problem for (1.1) (constructed with the above algorithm) that converges in $\mathbf{L}^1$, as $\nu \to \infty$, to some function $u(t,\cdot) : [a,b] \mapsto \Gamma$, for every $t \in [0,T]$. Then, there exist a countable set of interaction points $\Theta \doteq \big\{(\tau_l,\, x_l);\ l \in \mathbb{N}\big\} \subset \Omega_T \doteq [0,\, T] \times [a,\, b]$, and a countable family of Lipschitz continuous shock curves $\Upsilon \doteq \big\{x = y_m(t);\ t \in\,]r_m,\, s_m[,\ m \in \mathbb{N}\big\}$, such that the following hold.*

(i) *For each $m \in \mathbb{N}$, and for any $\tau \in]r_m, s_m[$ with $(\tau, y_m(\tau)) \notin \Theta$, there exist the derivative $\dot{y}_m(\tau)$ and the left and right limits*

$$\lim_{(s,y)\to(\tau,y_m(\tau)),\, y<y_m(\tau)} u(s,y) \doteq u^-, \qquad \lim_{(s,y)\to(\tau,y_m(\tau)),\, y>y_m(\tau)} u(s,y) \doteq u^+ . \tag{2.36}$$

Moreover, these limits satisfy the Rankine Hugoniot relations

$$\dot{y}_m(\tau)\cdot(u^+ - u^-) = f(u^+) - f(u^-) \tag{2.37}$$

and, for some $i \in \{1, ..., n\}$, there hold the Lax entropy inequalities

$$\lambda_i(u^+) < \dot{y}_m(t) < \lambda_i(u^-) . \tag{2.38}$$

(ii) *The map u is continuous outside the set $\Theta \cup \Upsilon$.*

3 Preliminary results

In this section we first provide some estimates on the distance between two rarefaction fronts of a front tracking solution (constructed by the algorithm described in Section 2.3) similar to [12, Lemma 4], [7, Prop. 4.5]. We next show how to approximate the profile $u(t,\cdot)$ of a solution of the mixed problem (1.1)-(1.4), with a function taking values in the discrete set Γ^ν defined at (2.19), which enjoys the same type of estimates on the positive waves as $u(t,\cdot)$. We conclude the section establishing the regularity result stated in Proposition 2.1 on the global structure of solutions to the mixed problem for (1.1), which in turn yields Theorem 2.5.

Lemma 3.1 *There exists some constant $C_1 > 0$ depending only on the system (1.1) such that the following holds. Consider a front tracking solution $u(t,x)$ with values in Γ^ν, constructed by the algorithm of Section 2.3 on the region $[\tau,\tau']\times[a,b]$. Then, given any two adjacent rarefaction fronts of u located at $x(t) \le y(t)$, $t \in [\tau, \tau']$, and belonging to the same family, there holds*

$$\big|y(\tau') - x(\tau')\big| \le \big|y(\tau) - x(\tau)\big| + C_1(\tau' - \tau)\,2^{-\nu} . \tag{3.1}$$

Proof. Consider two adjacent rarefaction fronts of the k-th family $x(t) \le y(t)$, $t \in [\tau, \tau']$, and let $\tau_1 < ... < \tau_N$ be the interaction times of $x(t)$ in the interval $[\tau, \tau']$. Set $\tau_0 \doteq \tau$, $\tau_{N+1} \doteq \tau'$, and fix $\alpha \in \{0, ..., N\}$. Let $t \to z(t; s, x)$ be the characteristic curve of the k-th family starting at (s, x), i.e. the solution to the ODE

$$\dot{z} = \lambda_k(u(t,z)), \qquad z(s; s, x) = x.$$

Notice that, although the above ODE has discontinuous right hand-side (because of the discontinuities in the front tacking solution u), its solution $z(\cdot; s, x)$ is unique and depends Lipschitz continuously on the initial data x since it crosses only a finite number of jumps (see [9]). Choose

$t_0 < t_1 < \tau_{\alpha+1}$ so that the characteristic curve $z(\cdot; t_0, x(t_0))$ does not cross any wave-front of the other families in the interval $[t_0, t_1]$, and then, by induction, define a sequence of times $\{t_i\}_{i\in\mathbb{Z}} \subset \,]\tau_\alpha, \tau_{\alpha+1}[$ so that

$$\begin{gathered}\tau_\alpha < t_{-i-1} < t_{-i} \le t_0 \le t_i < t_{i+1} < \tau_{\alpha+1}, \quad i \in \mathbb{N},\\ \lim_{i\to-\infty} t_i = \tau_\alpha, \qquad \lim_{i\to+\infty} t_i = \tau_{\alpha+1},\end{gathered} \tag{3.2}$$

with the properties that the characteristic curve of the k-th family starting at $(t_i, x(t_i))$, does not cross any wave-front of the other families in the interval $[t_i, t_{i+1}]$, for each $i \in \mathbb{Z}$. Thus, setting

$$u_i^+ \doteq u(t_i, x(t_i)+), \qquad u_i^- \doteq u(t_i, x(t_i)-),$$

and observing that, by construction, one has $|w(u_i^+) - w(u_i^-)| < 2^{-\nu}$, we derive

$$\begin{aligned}\big|z(t_{i+1}; t_i, x(t_i)) - x(t_{i+1})\big| &\le (t_{i+1} - t_i) \cdot \big|\lambda_k(u_i^+) - \lambda_k(u_i^+, u_i^-)\big|\\ &\le c \cdot (t_{i+1} - t_i) \cdot \big|w(u_i^+) - w(u_i^-)\big|\\ &\le c \cdot (t_{i+1} - t_i) \cdot 2^{-\nu}\end{aligned} \tag{3.3}$$

for some constant $c > 0$ depending only on the system. Relying on (3.3), and since $z(\tau'; t_{i+1}, x)$ depends Lipschits continuously on the initial data x, we deduce that there exists some other constant $c' > 0$, depending only on the system and on the set Γ, so that there holds

$$\begin{aligned}\big|z(\tau'; t_i, x(t_i)) - z(\tau'; t_{i+1}, x(t_{i+1}))\big| &\le c' \cdot \big|z(t_{i+1}; t_i, x(t_i)) - x(t_{i+1})\big|\\ &\le c' \cdot c \cdot (t_{i+1} - t_i) \cdot 2^{-\nu}\end{aligned} \tag{3.4}$$

for any $i \in \mathbb{Z}$. Thus, by (3.2), and thanks to (3.4), we obtain

$$\begin{aligned}\big|z(\tau'; \tau_\alpha, x(\tau_\alpha)) - z(\tau'; \tau_{\alpha+1}, x(\tau_\alpha))\big| &\le \sum_{i\in\mathbb{Z}} \big|z(\tau'; t_i, x(t_i)) - z(\tau'; t_{i+1}, , x(t_{i+1}))\big|\\ &\le c' \cdot c \cdot (\tau_{\alpha+1} - \tau_\alpha) \cdot 2^{-\nu}.\end{aligned} \tag{3.5}$$

Repeating this computation for every interval $]\tau_\alpha, \tau_{\alpha+1}[$, $\alpha \in \{0, ..., N\}$, we get

$$\begin{aligned}\big|z(\tau'; \tau, x(\tau)) - x(\tau')\big| &\le \sum_{\alpha=0}^{N} \big|z(\tau'; \tau_\alpha, x(\tau_\alpha)) - z(\tau'; \tau_{\alpha+1}, x(\tau_\alpha))\big|\\ &\le c' \cdot c \cdot (\tau' - \tau) \cdot 2^{-\nu}.\end{aligned} \tag{3.6}$$

Clearly, one obtains the same type of estimate as (3.6) for the other rarefaction front $y(t)$, i.e. there holds

$$\big|z(\tau'; \tau, y(\tau)) - y(\tau')\big| \le c' \cdot c \cdot (\tau' - \tau) \cdot 2^{-\nu}. \tag{3.7}$$

On the other hand, by (2.3), we have

$$\big|z(\tau'; \tau, x(\tau)) - z(\tau'; \tau, y(\tau))\big| \le \big|x(\tau) - y(\tau)\big| + 2\,\lambda^{\max} \cdot (\tau' - \tau). \tag{3.8}$$

Thus, (3.6)-(3.8) together yield (3.1), concluding the proof. □

In the following, in connection with any (right continuous) piecewise constant map $\psi : [a,b] \mapsto 2^{-\nu}\mathbb{Z}$, we will let $\pi(\psi) = \{x_0 = a < x_1 < \cdots < x_{\overline{\ell}} = b\}$ denote the partition of $[a,b]$ induced by ψ, in the sense that $\psi(x)$ is constant on every interval $[x_\ell, x_{\ell+1}[$, $0 \le \ell < \overline{\ell}$. Then, given $\rho > 0$, for any $\nu \ge 1$, consider the set of piecewise constant maps

$$K_\nu^\rho \doteq \left\{ \varphi : [a,b] \mapsto \Gamma^\nu ; \quad \begin{array}{ll} \dfrac{w_i(\varphi(x_k)) - w_i(\varphi(x_h))}{x_k - x_h} \le \dfrac{5\rho}{x_h - a} & \left\{ \begin{array}{ll} \text{for} & \begin{array}{l} a < x_h < x_k < b, \\ x_h,\, x_k \in \pi(w_i \circ \varphi), \end{array} \\ \text{if} & i \in \{p+1, ..., n\} \end{array} \right. \\ & \\ \dfrac{w_i(\varphi(x_k)) - w_i(\varphi(x_h))}{x_k - x_h} \le \dfrac{5\rho}{b - x_k} & \left\{ \begin{array}{ll} \text{for} & \begin{array}{l} a < x_h < x_k < b, \\ x_h,\, x_k \in \pi(w_i \circ \varphi), \end{array} \\ \text{if} & i \in \{1, ..., p\} \end{array} \right. \end{array} \right\}. \tag{3.9}$$

The next lemma shows that we can approximate in $\mathbf{L}^1$ any map $\varphi \in K^\rho$ with a piecewise constant function $\varphi_\nu \in K_\nu^\rho$.

Lemma 3.2 *For any given $\varphi \in K^\rho$, there exists a sequence of right continuous maps $\varphi_\nu \in K_\nu^\rho$, $\nu \ge 1$, such that:*

a) for every $i \in \{1, \dots, n\}$, and for any $x_h \in \pi(w_i \circ \varphi_\nu)$, there holds

$$w_i(\varphi_\nu(x_{h+1})) > w_i(\varphi_\nu(x_h)) \qquad \Longrightarrow \qquad w_i(\varphi(x_{h+1})) = w_i(\varphi(x_h)) + 2^{-\nu}; \tag{3.10}$$

b) there holds

$$\varphi_\nu \to \varphi \qquad \text{in} \qquad \mathbf{L}^1([a,b]). \tag{3.11}$$

1. First observe that, by Remark 2.2, any map $x \mapsto w_i(\varphi(x))$, $i \in \{1, \dots, n\}$ has finite total variation on $[a+\varepsilon,\, b-\varepsilon]$, $\varepsilon > 0$. Hence, we may assume that $w_i(\varphi(\cdot))$ admits left and right limits in any point $x \in]a, b[$, and that $w_i(\varphi(x)) = w_i(\varphi(x^+)) \doteq \lim_{\xi \to x^+} w_i(\varphi(\xi))$, for all $i \in \{1, \dots, n\}$. Let $\{y_{i,m}\ ;\ m \in \mathbb{N}\}$ be the countable set of discontinuities of $w_i(\varphi(\cdot))$, $i \in \{1, \dots, n\}$. Then, we can find a partition $\xi_{i,m}^1 = y_{i,m} < \xi_{i,m}^2 < \cdots < \xi_{i,m}^{\ell_{i,m}} = y_{i,m'}$ of each interval $[y_{i,m},\, y_{i,m'}[$ where $x \mapsto w_i(\varphi(x))$ is continuous, so that:

i) for every $1 < \ell < \ell_{i,m}$ there holds

$$w_i(\varphi(\xi_{i,m}^\ell)) \in 2^{-\nu}\,\mathbb{Z}\,; \tag{3.12}$$

ii) for every $1 \le \ell < \ell_{i,m}$ one has

$$\left| w_i(\varphi(x)) - w_i(\varphi(\xi_{i,m}^\ell)) \right| \le 2^{-\nu} \qquad \forall\, x \in [\xi_{i,m}^\ell,\, \xi_{i,m}^{\ell+1}[\,. \tag{3.13}$$

Notice that the Oleinik type conditions stated in the definition of K^ρ imply that, at any discontinuity point $y_{i,m}$ of $w_i(\varphi(\cdot))$, one has

$$\lim_{\xi\to y_{i,m}^-} w_i(\varphi(\xi)) > w_i(\varphi(y_{i,m}))\,. \tag{3.14}$$

2. Let $\varphi_\nu : [a,b] \mapsto \Gamma^\nu$ be the piecewise constant, right continuous map defined by setting, for every $i \in \{1,\dots,n\}$, and for any interval $[y_{i,m},\, y_{i,m'}[$ where $w_i(\varphi(\cdot))$ is continuous,

$$w_i(\varphi_\nu(x)) \doteq \begin{cases} 2^{-\nu}\lfloor 2^\nu\, w_i(\varphi(\xi^1_{i,m}))\rfloor & \text{if} \quad \begin{cases} x\in[\xi^1_{i,m},\,\xi^2_{i,m}[\,, \quad \text{and} \\ w_i(\varphi_\nu(\xi^1_{i,m})) \le 2^{-\nu}\big(\lfloor 2^\nu\, w_i(\varphi(\xi^1_{i,m}))\rfloor + 2^{-1}\big), \end{cases} \\[2ex] 2^{-\nu}\big(\lfloor 2^\nu\, w_i(\varphi(\xi^1_{i,m}))\rfloor + 1\big) & \text{if} \quad \begin{cases} x\in[\xi^1_{i,m},\,\xi^2_{i,m}[\,, \quad \text{and} \\ w_i(\varphi_\nu(\xi^1_{i,m})) > 2^{-\nu}\big(\lfloor 2^\nu\, w_i(\varphi(\xi^1_{i,m}))\rfloor + 2^{-1}\big), \end{cases} \\[2ex] w_i(\varphi(\xi^\ell_{i,m})) & \text{if} \quad x\in[\xi^\ell_{i,m},\,\xi^{\ell+1}_{i,m}[\,, \qquad 1<\ell<\ell_{i,m}\,, \end{cases} \tag{3.15}$$

where $\lfloor\cdot\rfloor$ denotes the integer part. Notice that, by construction, and because of (3.12)-(3.13), (3.14), the map $\varphi_\nu : [a,b]\mapsto\Gamma^\nu$ enjoys the following property

$$\left.\begin{aligned} & w_i(\varphi_\nu(x_k)) > w_i(\varphi_\nu(x_h)) \\ & x_h < x_k \in \pi(w_i\circ\varphi_\nu) \end{aligned}\right\} \quad\Longrightarrow\quad w_i(\varphi(x_k)) > w_i(\varphi(x_h)) + 2^{-(\nu+1)}\,. \tag{3.16}$$

Therefore, since $\varphi \in K^\rho$, relying on (3.13), (3.16), we deduce that, for every $w_i(\varphi_\nu(\cdot))$, $i \in \{p+1,\dots,n\}$, and for any $x_h < x_k \in \pi(w_i\circ\varphi_\nu)$ such that $w_i(\varphi_\nu(x_k)) > w_i(\varphi_\nu(x_h))$, there holds

$$\begin{aligned} \frac{w_i(\varphi_\nu(x_k)) - w_i(\varphi_\nu(x_h))}{x_k - x_h} &\le \frac{w_i(\varphi(x_k)) - w_i(\varphi(x_h)) + 2^{-(\nu-1)}}{x_k - x_h} \\ &\le \frac{5\,\big(w_i(\varphi(x_k)) - w_i(\varphi(x_h))\big)}{x_k - x_h} \\ &\le \frac{5\,\rho}{x_h - a}\,. \end{aligned} \tag{3.17}$$

Clearly, with the same computations, we can show that, for every $w_i(\varphi_\nu(\cdot))$, $i\in\{1,\dots,p\}$, and for any $x_h < x_k \in \pi(w_i\circ\varphi_\nu)$, there holds

$$\frac{w_i(\varphi_\nu(x_k)) - w_i(\varphi_\nu(x_h))}{x_k - x_h} \le \frac{5\,\rho}{b - x_k}\,. \tag{3.18}$$

The estimates (3.17)-(3.18), together, imply that $\varphi_\nu \in K^\rho_\nu$, while (3.13) yields (3.11). On the other hand observe that, by construction, and because of (3.14), the map φ_ν satisfies condition (3.10), which completes the proof of the lemma. □

We now provide a further estimate on the distance between two rarefaction fronts of a front tracking solution that, at a fixed time τ, attains a profile belonging to the set (3.9).

Lemma 3.3 *Consider a front tracking solution $u(t,x)$ with values in Γ^ν, $\nu \geq 1$,constructed by the algorithm of Section 2.3 on the region $[\tau,\tau'] \times [a,b]$. Assume that $u(\tau',\cdot)$ is right-continuous, verifies condition a) of Lemma 3.2, and satisfies*

$$u(\tau',\cdot) \in K_\nu^{\rho'}, \qquad \rho' \doteq \frac{\lambda^{\min}}{6\,C_1}, \tag{3.19}$$

where $\lambda^{\min}$, C_1, are the minimum speed in (2.3), and the constant of Lemma 3.1. Then, given any two adjacent rarefaction fronts of u located at $x(t) \leq y(t)$, $t \in [\tau,\tau']$, and belonging to the same family, there holds

$$x(\tau) < y(\tau). \tag{3.20}$$

Proof. To fix the ideas, assume that $x(t) \leq y(t)$ are the locations of two adjacent rarefaction fronts of the $k \in \{p+1,\dots,n\}$ - th family, and hence, by (2.2), have positive speeds. Observe that, by condition a) of Lemma 3.2, one has

$$w_k(u(\tau',y(\tau'))) - w_k(u(\tau',x(\tau'))) = 2^{-\nu}. \tag{3.21}$$

Moreover, since u is a front tracking solution constructed by the algorithm of Section 2.3 on the region $[\tau,\tau'] \times [a,b]$, we can apply Lemma 3.1. Thus, using (2.3), (3.1), (3.21), and recalling the definition (3.9) of $K_\nu^{\rho'}$, we deduce

$$\begin{aligned}
y(\tau') - x(\tau') &\leq y(\tau) - x(\tau) + C_1(\tau'-\tau)\,2^{-\nu}\\
&\leq y(\tau) - x(\tau) + C_1\frac{x(\tau')-x(\tau)}{\lambda^{\min}}\cdot\big(w_k(\varphi_\nu(y(\tau'))) - w_k(\varphi_\nu(x(\tau')))\big)\\
&\leq y(\tau) - x(\tau) + C_1\frac{5\rho'}{\lambda^{\min}}\cdot\big(y(\tau') - x(\tau')\big)
\end{aligned}$$

which, because of (3.19), implies

$$y(\tau) - x(\tau) \geq \left(1 - C_1\frac{5\rho'}{\lambda^{\min}}\right)\cdot\big(y(\tau') - x(\tau')\big) > 0\,,$$

proving (3.20). □

We next derive a regularity property enjoyed by general BV solutions of Temple systems defined as limit of front tracking approximations, which allows us to establish Proposition 2.1. This is an extension of the regularity results obtained in [17, 22, 13] for solution with small total variation of general systems. The arguments of the proof are quite similar as for the corresponding result in [13], but we will repeat some of them for completeness, referring to [13] (see also [8, Theorem 10.4]) for further details.

Lemma 3.4 *Let (1.1) be a system of Temple class satisfying the same assumptions as in Theorem 2.3. Consider a sequence $u^\nu(t,\cdot) : [c,d] \mapsto \Gamma^\nu$, $t \in [r,s]$, of front tracking approximate solutions of the mixed problem for (1.1) (constructed by the algorithm of Section 2.3),* *that converges in* $\mathbf{L}^1$, *as* $\nu \to \infty$, *to some function* $u(t,\cdot) : [c,\, d] \mapsto \Gamma$, *for every* $t \in [r,\, s] \subset \mathbb{R}^+$. *Assume that*

$$Tot.Var.(u^\nu(t,\cdot)) \le M, \qquad Tot.Var.(u^\nu(\cdot,x)) \le M \qquad \forall\; t, x, \nu\,, \tag{3.22}$$

for some constant $M > 0$. *Then, there exist a countable set of interaction points* $\Theta \doteq \big\{(\tau_l,\, x_l);\ l \in \mathbb{N}\big\} \subset D \doteq [r,\, s] \times [c,\, d]$, *and a countable family of Lipschitz continuous shock curves* $\Upsilon \doteq \big\{x = y_m(t);\ t \in]r_m,\, s_m[,\ m \in \mathbb{N}\big\}$, *such that the following hold.*

- *(i) For each* $m \in \mathbb{N}$, *and for any* $\tau \in]r_m,\, s_m[$ *with* $(\tau, y_m(\tau)) \notin \Theta$, *there exist the left and right limits (2.36) of* u *at* $(\tau, y_m(\tau))$ *and the shock speed* $\dot y_m(\tau)$. *Moreover, these limits satisfy the Rankine Hugoniot relations (2.37) and the Lax entropy inequality (2.38), for some* $i \in \{1, ..., n\}$.
- *(ii) The map* u *is continuous outside the set* $\Theta \cup \Upsilon$.

PROOF.

1. To establish (i) we need to recall some technical tools introduced in [13] (see also [8, Theorem 10.4]). For every front tracking solution u^ν, we define the *interaction and cancellation measure* μ_ν^{IC} that is a positive, purely atomic measure on D, concentrated on the set of points P where two or more wave-fronts of u^ν interact. Namely, if the incoming fronts at P have size $\sigma_1, \dots, \sigma_\ell$ (w.r.t. the Riemann coordinates), and belong to the families $i_1, \dots, i_\ell$ respectively, we set

$$\mu_\nu^{IC}(P) \doteq \sum_{\alpha,\beta} \big|\sigma_\alpha\, \sigma_\beta\big| + \sum_i \Bigg(\sum_{\{i_\alpha\,;\, i_\alpha = i\}} |\sigma_\alpha| - \Bigg| \sum_{\{i_\alpha\,;\, i_\alpha = i\}} \sigma_\alpha \Bigg| \Bigg). \tag{3.23}$$

Since μ_ν^{IC} have a uniformly bounded total mass, by possibly taking a subsequence we can assume the weak convergence

$$\mu_\nu^{IC} \rightharpoonup \mu^{IC} \tag{3.24}$$

for some positive, purely atomic measure μ^{IC} on D. Call Θ the countable set of atoms of μ^{IC}, i.e. set

$$\Theta \doteq \{P \in D;\ \mu^{IC}(P) > 0\}.$$

For every approximate solution u^ν taking values in Γ^ν, $\nu \ge 1$, and for any fixed $\varepsilon \ge 2^{-\nu}$, by an $\varepsilon-$*shock front of the* $i-th$ *family* in u^ν we mean a polygonal line in D, with nodes $(\tau_0, x_0), ..., (\tau_N, x_N)$, having the following properties.

- (I) The nodes (τ_h, x_h) are interaction points or lie on the boundary of D, and the sequence of times is increasing $\tau_0 < \tau_1 < \dots < \tau_N$.
- (II) Along each segment joining (τ_{h-1}, x_{h-1}) with (τ_h, x_h), the function u^ν has an $i-$shock with strength $|\sigma_h| \ge \varepsilon$.
- (III) For $h < N$, if two (or more) incoming $i-$shocks of strength $\ge \varepsilon$ interact at the node (τ_h, x_h), then the shock coming from (τ_{h-1}, x_{h-1}) has the larger speed, i.e. is the one coming from the left.

An $\varepsilon-$shock front which is maximal with respect to the set theoretical inclusion will be called a *maximal $\varepsilon-$shock front*. Observe that, because of (III), two maximal $\varepsilon-$shock fronts of the same family either are disjoint or coincide. Moreover, by (3.22), the number of maximal $\varepsilon-$shock front that starts at the boundary of D is uniformly bounded by $3M/\varepsilon$. On the other hand, the special geometric features of Temple class systems guarantee that no new shock front can arise in the interior of D. Indeed, the coinciding shock and rarefaction assumption together with the existence of Riemann invariants prevents the creation of shocks of other families than the ones of the incoming fronts at any interaction point. Therefore, for fixed $\varepsilon > 0$, and $i \in \{1, \ldots, n\}$, the number of maximal $\varepsilon-$shock front of the i-th family remains uniformly bounded by $M_\varepsilon \doteq 3M/\varepsilon$ in all u^ν, $\nu \geq 1$. Denote such curves by

$$y^\varepsilon_{\nu,m} : [t^{\varepsilon,-}_{\nu,m},\, t^{\varepsilon,+}_{\nu,m}] \mapsto \mathbb{R}, \qquad m = 1, ..., M_\varepsilon\,.$$

By possibly extracting a further subsequence, we can assume the convergence

$$y^\varepsilon_{\nu,m}(\cdot) \longrightarrow y^\varepsilon_m(\cdot), \qquad t^{\varepsilon,\pm}_{\nu,m} \longrightarrow t^{\varepsilon,\pm}_m, \qquad m = 1, ..., M_\varepsilon\,,$$

for some Lipschitz continuous paths $y^\varepsilon_m : [t^{\varepsilon,-}_m,\, t^{\varepsilon,+}_m] \mapsto \mathbb{R}$, $m = 1, ..., M_\varepsilon$. Repeating this construction in connection with a sequence $\varepsilon_k \to 0$, and taking the union of all the paths thus obtained, we find, for each characteristic family $i \in \{1, \ldots, n\}$, a countable family of Lipschitz continuous curves $y_m : [t^-_m, t^+_m] \mapsto \mathbb{R}$, $m \in \mathbb{N}$. Call Υ the union of all such curves.

2. Consider now a point $P = (\tau, y_m(\tau)) \notin \Theta$ along a curve $y_m \in \Upsilon$ of a family $i \in \{1, \ldots, n\}$. Notice that, by construction, and because of (3.24), no curve in Υ can cross y_m at P. Moreover, by (3.22), the function $u(\tau, \cdot)$ has bounded variation, and hence there exist the limits

$$\lim_{x \to y_m(\tau)-} u(\tau, x) \doteq u^-, \qquad \lim_{x \to y_m(\tau)+} u(\tau, x) \doteq u^+. \tag{3.25}$$

We claim that also the limits (2.36) exist, and thus coincide with those in (3.25). To this end observe that, by construction, there exist a sequence of shocks curves $y_{\nu,m}$ of the i-th family converging to y_m, along which each approximate solution u^ν has a jump of strength $\geq \varepsilon^*$, for some $\varepsilon^* > 0$. Then, relying on the assumption

$$\mu^{IC}(\{P\}) = 0\,, \tag{3.26}$$

and letting $B(P, r)$ denote the ball centerd at P with radius r, one can establish the limits

$$\lim_{r \to 0+} \limsup_{\nu \to +\infty} \left(\sup_{\substack{(t,x) \in B(P,r) \\ x < y_{\nu,m}(t)}} \left| u^\nu(t,x) - u^- \right| \right) = 0, \tag{3.27}$$

$$\lim_{r \to 0+} \limsup_{\nu \to +\infty} \left(\sup_{\substack{(t,x) \in B(P,r) \\ x > y_{\nu,m}(t)}} \left| u^\nu(t,x) - u^- \right| \right) = 0, \tag{3.28}$$

which clearly yield (2.36). Indeed, if for example (3.27) do not hold, by possibly taking a subsequence we would find $\varepsilon > 0$ and points $P_\nu \doteq (t_\nu,\, \xi_\nu) \to P$ on the left of $y_{\nu,m}$ such that

$$\big|u^\nu(t_\nu,\, \xi_\nu) - u^-\big| \geq \varepsilon \qquad \forall\ \nu.$$

On the other hand, by the first limit in (3.25), and since $u^\nu(\tau, x) \to u(\tau, x)$ for a.e. $x \in [\alpha, \beta]$, we could also find points $Q_\nu \doteq (\tau,\, \xi'_\nu) \to P$ on the left of $y_{\nu,m}$ such that

$$u^\nu(\tau,\, \xi'_\nu) \to u^-, \qquad \frac{|\xi_\nu - \xi'_\nu|}{|t_\nu - \tau|} > \lambda^{\max} \qquad \forall\ \nu\,,$$

where $\lambda^{\max}$ denotes the maximum speed at (2.3). But then, for each solution u^ν, the segment $\overrightarrow{P_\nu\, Q_\nu}$ would be crossed by an amount of waves of strength $\geq \varepsilon$. Hence, by strict hyperbolicity and genuine nonlinearity, this would generate a uniformly positive amount of interaction and cancellation within an arbitrary small neighborhood of P (see. [8, Theorem 10.4-Step 5]) which, by the definition (3.23), and because of (3.24), contradicts the assumption (3.26).

To complete the proof of (i) observe that, by construction, the states $u^-_{\nu,m}(\tau)$, $u^+_{\nu,m}(\tau)$ to the left and to the right of the jump in u^ν at $y_{\nu,m}(\tau)$ satisfy the Rankine Hugoniot conditions. Thus, relying on (3.27)-(3.28), and on the convergence $y_{\nu,m} \to y_\nu$, one deduces (2.37). The proof of (ii) can be established with the same type of arguments (cfr. [8, Theorem 10.4-Step 8]). □

As an immediate consequence of Lemma 3.4, we derive Proposition 2.1 stated in Section 2.3.

Proof of Proposition 2.1. Consider a sequence $u^\nu(t,\cdot) : [a,b] \mapsto \Gamma^\nu$ of front tracking approximate solutions of the mixed problem for (1.1) on the region $\Omega_T \doteq [0,T] \times [a,b]$, that converges in $\mathbf{L}^1$, as $\nu \to \infty$, to some function $u(t,\cdot) : [a,b] \mapsto \Gamma$, for every $t \in [0,T]$. Observe that, by Theorem 2.3 one can find another sequence $\{v^\nu\}_{\nu\geq 1}$ of approximate solutions of (1.1) on the region Ω_T, whose initial and boundary data have a number of shocks $N_\nu \leq \nu$ for each characteristic family, and such that

$$\big\|u^\nu(t,\cdot) - v^\nu(t,\cdot)\big\|_{\mathbf{L}^1([a,b])} \leq 1/\nu \qquad \forall\ t \in [1/\nu,\, T]\,.$$

Then, thanks to the Oleinik estimates (2.25)-(2.26), and because all v^ν take values in the compact set (2.13), there will be, for every fixed $\varepsilon > 0$, some constant $M_\varepsilon > 0$ such that

$$\begin{aligned} \text{Tot.Var.}\big\{v^\nu(t,\,\cdot)\ ;\ [a+\varepsilon,\, b-\varepsilon]\big\} &\leq\ M_\varepsilon \qquad \forall\ t \in [\varepsilon,\, T]\,, \\ \text{Tot.Var.}\big\{v^\nu(\cdot,\, x)\ ;\ [\varepsilon,\, T]\big\} &\leq M_\varepsilon \qquad \forall\ x \in [a+\varepsilon,\, b-\varepsilon]\,, \end{aligned} \qquad \forall\ \nu \in \mathbb{N}. \tag{3.29}$$

Thus, writing Ω_T as the countable union

$$\Omega_T = \cup_k D_k, \qquad D_k \doteq [1/k,\, T] \times [a + (1/k),\, b - (1/k)],$$

and applying Lemma 3.4 to each sequence of maps $v^\nu_k \doteq v^\nu \restriction_{D_k}$, $\nu \geq 1$, defined as the restriction of v^ν to the domain D_k, we clearly reach the conclusion of Proposition 2.1. □

We are now in the position to establish Theorem 2.5, relying on Proposition 2.1 and on Theorem 2.4.

PROOF OF THEOREM 2.5. Let $u^\nu(t,\cdot) : [a,b] \mapsto \Gamma^\nu$ be a sequence of front tracking approximate solutions of the mixed problem for (1.1) on the region $\Omega_T \doteq [0,T] \times [a,b]$, that converges in $\mathbf{L}^1$, as $\nu \to \infty$, to some function $u(t,\cdot) : [a,b] \mapsto \Gamma$, for every $t \in [0,T]$. Since, by construction, each u^ν is a weak solution of (1.1), and because $u^\nu(0,\cdot) \to u(0,\cdot) = \overline{u}$, also the limit function u is a weak solution of the Cauchy problem (1.1)-(1.2) on the region Ω_T. Moreover, applying Proposition 2.1, we deduce that u admits at $t = 0$ and at $x = a$, $x = b$ the left and right limits stated in Theorem 2.5. On the other hand, by the same arguments used in the proof of Proposition 2.1, we may assume that the initial and boundary data of each approximate solution u^ν have at most $N_\nu \leq \nu$ shocks for every characteristic family. Then, letting $\nu \to \infty$ in (2.25)-(2.26), by the lower semicontinuity of the total variation we find that u satisfies the entropy conditions (2.7)-(2.10) on the decay of positive waves. It follows that u is an entropy weak solution of the mixed problem (1.1)-(1.4) according with Definition 2.2. Hence, observing that by constructin the map $(t,x) \to \big(u(t,\cdot),\, u(\cdot,x)\big)$ takes values within the domain $\mathcal{D}_T$ defined in (2.31), and applying Theorem 2.4, we deduce that (2.35) is verified. □

4 Proof of Theorems 2.1-2.2

PROOF OF THEOREM 2.1. We shall first prove that, for every fixed $\overline{\tau} > 0$, there exists some constant $\rho = \rho(\overline{\tau}) > 0$ so that (2.17) holds. Given $\widetilde{u}_a \in \mathcal{U}_\tau^\infty$, $\widetilde{u}_b \in \mathcal{U}_\tau^\infty$, $\tau \geq \overline{\tau}$, let $u = u(t,x)$ be an entropy weak solution of (1.1)-(1.4) on the region $[0,\tau]\times[a,b]$ according with Definition 2.2. Then, the Oleinik-type estimates (2.8) on the decay of positive waves imply that, for $i \in \{p+1,...,n\}$, $\tau \geq \overline{\tau}$, and for a.e. $a < x < y < b$, there holds

$$\begin{aligned}
\frac{w_i(\tau,y) - w_i(\tau,x)}{y-x} &\leq C \cdot \left\{ \frac{y-x}{\tau} + \log\left(\frac{y-a}{x-a}\right) \right\} \\
&\leq (b-a)\, C \cdot \left\{ \frac{1}{\overline{\tau}} + \frac{1}{x-a} \right\} \\
&\leq \frac{C\,(b-a)\big((b-a)+\overline{\tau}\big)}{\overline{\tau}} \cdot \frac{1}{x-a}.
\end{aligned} \tag{4.1}$$

Clearly, with the same computations, relying on the Oleinik-type estimates (2.7), we deduce that, for $i \in \{1,...,p\}$, $\tau \geq \overline{\tau}$, and for a.e. $a < x < y < b$, there holds

$$\frac{w_i(\tau,y) - w_i(\tau,x)}{y-x} \leq \frac{C\,(b-a)\big((b-a)+\overline{\tau}\big)}{\overline{\tau}} \cdot \frac{1}{b-y}. \tag{4.2}$$

Hence, taking

$$\rho \geq \frac{C\,(b-a)\big((b-a)+\overline{\tau}\big)}{\overline{\tau}} \tag{4.3}$$

from (4.1)-(4.2) we derive $u(\tau,\cdot) \in K^\rho$, which proves (2.17).

Concerning the second statement of the theorem, we will show that, letting $\lambda^{\min}$, ρ', be the minimum speed in (2.3), and the constant (3.19) of Lemma 3.1, and taking

$$T \doteq \frac{4\,(b-a)}{\lambda^{\min}} \tag{4.4}$$

the relation (2.18) is verified, i.e. that, given $\varphi \in K^{\rho'}$, and $\tau > T$, there exist $\widetilde{u}_a \in \mathcal{U}_\tau^\infty$, $\widetilde{u}_b \in \mathcal{U}_\tau^\infty$, and a solution $u(t,x)$ of (1.1)-(1.4) on $[0,\tau] \times [a,b]$ (according with Definition 2.2), such that $u(\tau,\cdot) \equiv \varphi$. Notice that, by Remark 2.2, we may assume that $w_i(\varphi(x))$ admits left and right limits in any point $x \in]a,b[$, and that $w_i(\varphi(x)) = w_i(\varphi(x^+)) \doteq \lim_{\xi \to x^+} w_i(\varphi(\xi))$, for all $i \in \{1,\dots,n\}$. The proof is devided in two steps.

Step 1. Backward construction of front tracking approximations. Letting $\rho' > 0$ be the constant in (3.19), consider a sequence $\{\varphi_\nu\}_{\nu \geq 1}$ of (right continuous) piecewise constant maps in $K_\nu^{\rho'}$, satisfying the conditions a)-b) of Lemma 3.2, and take a piecewise constant approximation $\overline{u}^\nu : [a,b] \mapsto \Gamma^\nu$ of the initial data $\overline{u}$, so that $\overline{u}^\nu \to \overline{u}$ in $\mathbf{L}^1$. Given $\tau > T$ (T being the time defined in (4.4)), for each $\nu \geq 1$, we will construct here a front tracking solution $u^\nu(t,x)$ of (1.1) on the region $[0,\tau] \times [a,b]$, with initial data $u^\nu(0,\cdot) = \overline{u}^\nu$, so that

$$u^\nu(\tau,\cdot) = \varphi_\nu\,. \tag{4.5}$$

This goal is accomplished by proving the following two lemmas.

Lemma 4.1 *Let T, $\rho' > 0$ be the constants in (4.4) and (3.19). Then, for every (right continuous) $\varphi_\nu \in K_\nu^{\rho'}$, $\nu \geq 1$, satisfying the condition a) of Lemma 3.2, and for any $\tau > T$, there exists a front tracking solution $u^\nu(t,x)$ of (1.1) on the region $[(3/4)T,\, \tau] \times [a,b]$, with boundary data $\widetilde{u}_a^\nu \doteq u^\nu(\cdot,\, a)$, $\widetilde{u}_b^\nu \doteq u^\nu(\cdot,\, b) \in \mathbf{L}^\infty([(3/4)T,\, \tau],\, \Gamma^\nu)$, so that*

$$u^\nu\big((3/4)T,\, x\big) \equiv \omega, \qquad u^\nu\big(\tau,\, x\big) = \varphi_\nu(x)\,, \qquad \forall\, x \in [a,b]\,, \tag{4.6}$$

for some constant state $\omega \in \Gamma^\nu$.

Proof. Given $\tau > T$, and $\varphi_\nu \in K_\nu^{\rho'}$, $\nu \geq 1$, satisfying the condition a) of Lemma 3.2, we will use the algorithm described in Section 2.3 to construct backward in time a front tracking solution that takes value φ_ν at time τ. To this end, we first observe that according with the algorithm of Section 2.3 we can always construct the backward solution of a Riemann problem with terminal data

$$u(t,x) = \begin{cases} u^L & \text{if} \quad x < \xi\,, \\ u^R & \text{if} \quad x > \xi\,, \end{cases} \tag{4.7}$$

if the the terminal states u^L, $u^R \in \Gamma^\nu$ have Riemann coordinates

$$w(u^L) \doteq w^L = (w_1^L,\dots,w_n^L), \qquad w(u^R) \doteq w^R = (w_1^R,\dots,w_n^R)$$

that satisfy

$$w_i^L < w_i^R \qquad \Longrightarrow \qquad w_i^R = w_i^L + 2^{-\nu} \qquad \forall\, i\,. \tag{4.8}$$

Indeed, if we consider the intermediate states

$$z^i = \begin{cases} u^L & \text{if} \quad i = 0\,, \\ u(w_1^L, \dots, w_{n-i}^L\,,\, w_{n-i+1}^R, \dots, w_n^R) & \text{if} \quad 0 < i < n\,, \\ u^R & \text{if} \quad i = n\,, \end{cases} \tag{4.9}$$

we realize that, because of (4.8), the solution of every Riemann problem with initial states (z^{i-1}, z^i) (defined as in Section 2.3) contains only a single front. Thus, we can construct the solution to the Riemann problem with terminal data (4.7) in a backward neighborhood of (t, ξ) by piecing together the solutions to the simple Riemann problems (z^{i-1}, z^i), $i = 1, \dots, n$.

A front tracking solution u^ν can now be constructed backward in time starting at $t = \tau$, and piecing together the backward solutions of the Riemann problems determined by the jumps in φ_ν. The resulting piecewise constant function $u^\nu(\tau-, \cdot)$ is then prolonged for $t < \tau$ tracing backward the incoming fronts at $t = \tau$, up to the first time $\tau_1 < \tau$ at which two or more discontinuities cross in the interior of Ω. Observe that, since u^ν is a front tracking solution constructed by the algorithm of Section 2.3 on the region $[\tau_1, \tau] \times [a, b]$, we can apply Lemma 3.3. Hence, it follows that the left and right states of the jumps occuring in $u^\nu(\tau_1, \cdot)$ satisfy condition (4.8), because (3.20) guarantees that two (or more) adjacent rarefaction fronts of the same family cannot cross at time τ_1. We then solve backward the resulting Riemann problems applying again the above procedure. This determines the solution $u^\nu(t, \cdot)$ untill the time $\tau_2 < \tau_1$ at which another intersection between its fronts takes place in the interior of Ω, and so on (see figure 1a).

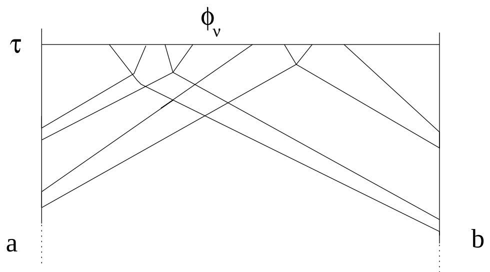

figure 1a

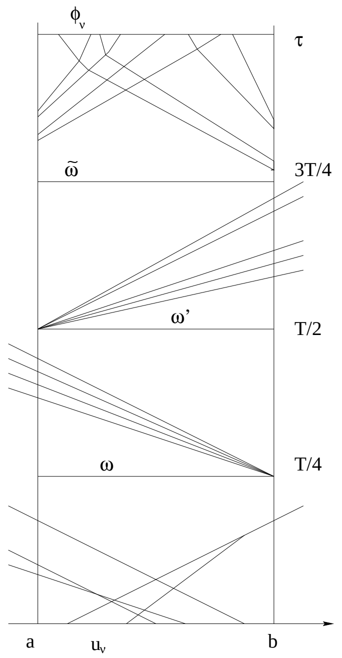

figure 1b

With this construction we define a front tracking solution $u^\nu(t, x)$ on the whole region $[(3/4)T,\,\tau]\times[a,b]$, that verifies the first equality in (4.6), and corresponds to the boundary data $\widetilde{u}^\nu_a \doteq u^\nu(\cdot,a)$, $\widetilde{u}^\nu_b \doteq u^\nu(\cdot,b) \in \mathbf{L}^\infty([(3/4)T,\,\tau],\,\Gamma^\nu)$. Clearly, the total number of wave-fronts in $u^\nu(t,\,\cdot)$ decreases, as $t \downarrow (3/4)T$, whenever a (backward) front crosses the boundary points $x = a$, $x = b$. Since (2.3) implies that the maximum time taken by fronts of u^ν to cross the interval $[a,b]$ is $(b-a)/\lambda^{\min}$, the definition (4.4) of T guarantees that all the (backward) fronts of u^ν will hit the boundaries $x = a$, $x = b$ within some time $\tau' \in](3/4)T,\,\tau[$, which shows that also the second equality in (4.6) is verified, thus completing the proof. □

Lemma 4.2 *Let $T > 0$ be the constant in (4.4). Then, for any piecewise constant function $\overline{u}^\nu \in \mathbf{L}^\infty([a,b],\Gamma^\nu)$, and for every state $\omega \in \Gamma^\nu$, there exists a front tracking solution $u^\nu(t,x)$ of (1.1) on the region $[0,\,(3/4)\,T] \times [a,b]$, corresponding to some boundary data $\widetilde{u}^\nu_a,\,\widetilde{u}^\nu_b \in \mathbf{L}^\infty([0,\,(3/4)\,T],\Gamma^\nu)$, so that*

$$u^\nu(0,\,x) = \overline{u}^\nu(x), \qquad u^\nu\big((3/4)\,T,\,x\big) \equiv \omega\,, \qquad \forall\, x \in [a,b]\,. \tag{4.10}$$

Proof. The approximate solution u^ν is constructed as follows. By Remark 2.3, for $t \in [0,T/4]$, we can define $u^\nu(t,x)$ as the restriction to the region $[0,T/4] \times [a,b]$ of the front tracking solution to the Cauchy problem for (1.1), with initial data

$$\overline{u}(x) = \begin{cases} \overline{u}^\nu(a+) & \text{if} \quad x < a, \\ \overline{u}^\nu(x) & \text{if} \quad a \le x \le b, \\ \overline{u}^\nu(b-) & \text{if} \quad x > b, \end{cases}$$

(constructed as in [12] with the same type of algorithm described in Section 2.3). Observe that, since u^ν contains only fronts originated at the points of the segment $\{(0,x)\,;\, x \in [a,b]\}$, because of (2.3), (4.4) these wave-fronts cross the whole interval $[a,b]$ and exit from the boundaries $x = a$, $x = b$ before time $T/4$ (see figure 1b). Hence, there will be some state $\omega' \in \Gamma^\nu$ such that

$$u^\nu\big(T/4,\,x\big) \equiv \omega' \qquad \forall\, x \in [a,b]\,. \tag{4.11}$$

Thus, introducing the intermediate state

$$\widetilde{\omega} \doteq \big(\omega_1,\ldots,\omega_p,\,\omega'_{p+1},\ldots,\omega'_n\big)$$

between ω' and ω, we will define $u^\nu(t,x)$, for $t \in [T/4,\,T/2]$, as the restriction to the region $[T/4,T/2] \times [a,b]$ of the approximate solution to the Riemann problem for (1.1), with initial data

$$u^\nu(T/4,x) = \begin{cases} u(\omega') & \text{if} \quad x < b\,, \\ u(\widetilde{\omega}) & \text{if} \quad x > b\,, \end{cases} \tag{4.12}$$

while, for $t \in [T/2,\,(3/4)T]$, we will let $u^\nu(t,x)$ be the restriction to the region $[T/2,\,(3/4)T]\times[a,b]$ of the approximate solution to the Riemann problem for (1.1), with initial data

$$u^\nu(T/2,x) = \begin{cases} u(\omega) & \text{if} \quad x < a\,, \\ u(\widetilde{\omega}) & \text{if} \quad x > a\,. \end{cases} \tag{4.13}$$

By the definition of $\widetilde{\omega}$, and because of (2.3), (4.4), on $[T/4,\, T/2]$ the solution of the Riemann problems with initial data (4.12) contains only wave-fronts originated at the point $(T/4,\, b)$, that cross the whole interval $[a,b]$ and exit from the boundary $x = a$ before time $T/2$. Similarly, still by (2.3), (4.4), for $t \in [T/2,\, (3/4)T]$ the solution of the Riemann problem with initial data (4.13), contains only wave-fronts originated at $(T/2, a)$, that cross the whole interval $[a,b]$, and exit from the boundary $x = b$ before time $(3/4)T$ (see figure 1b). Hence, $u^\nu(t,x)$ is a front-tracking solution defined on the whole region $[0,\, (3/4)\,T] \times [a,b]$, that corresponds to the boundary data $\widetilde{u}_a^\nu \doteq u^\nu(\cdot, a)$, $\widetilde{u}_b^\nu \doteq u^\nu(\cdot, b) \in \mathbf{L}^\infty([0,\, (3/4)\,T], \Gamma^\nu)$, and verifies the conditions (4.10). □

Step 2. Convergence of the approximate solutions. By Step 1, for a given $\varphi \in K^{\rho'}$ (with ρ' as in (3.19)), we have found a sequence of initial data $\overline{u}^\nu$, and of boundary data $\widetilde{u}_a^\nu,\, \widetilde{u}_b^\nu \in \mathcal{U}_\tau^\infty$, so that, letting $u^\nu(\tau, \cdot) \doteq E_\tau^\nu(\overline{u}^\nu,\, \widetilde{u}_a^\nu,\, \widetilde{u}_b^\nu)$ be the corresponding front tracking solution, there holds

$$\overline{u}^\nu \to \overline{u}, \qquad u^\nu(\tau, \cdot) \to \varphi \qquad \text{in} \qquad \mathbf{L}^1([a,b]). \tag{4.14}$$

By the same arguments used in the proof of Proposition 2.1, we may assume that the initial and boundary data of each approximate solution u^ν have at most $N_\nu \leq \nu$ shocks for every characteristic family. Then, thanks to the Oleinik-type estimates (2.25), and because u^ν are uniformly bounded since they take values in the compact set (2.13), for every fixed $\varepsilon > 0$, there will be some constant $C_\varepsilon > 0$ such that

$$\left.\begin{aligned} &\text{Tot.Var.}\big\{u^\nu(t,\cdot)\ ;\ [a+\varepsilon,\, b-\varepsilon]\big\} \leq C_\varepsilon \qquad \forall\, t \in [\varepsilon, \tau]\,, \\ &\int_{a+\varepsilon}^{b-\varepsilon} |u^\nu(t,x) - u^\nu(s,x)|\; dx \leq C_\varepsilon |t-s| \qquad \forall\, t,s \in [\varepsilon, \tau]\,, \end{aligned}\right\} \quad \forall\, \nu \in \mathbb{N}. \tag{4.15}$$

Hence, applying Helly's Theorem, we deduce that there exists a subsequence $\{u^{\nu_j}\}_{j\geq 0}$ that converges in $\mathbf{L}^1([a,b],\, \Gamma)$ to some function $u_\varepsilon(t,\cdot)$, for any $t \in [\varepsilon,\, \tau]$. Therefore, repeating the same construction in connection with a sequence $\varepsilon_k \to 0+$, and using a diagonal procedure, we obtain a subsequence $\{u^{\nu'}(t,\cdot)\}_{\nu'\geq 0}$ that converges in $\mathbf{L}^1([a,b],\Gamma)$ to some function $u(t,\cdot)$, for any $t \in [0,\, \tau]$. Then, by Theorem 2.5, there holds (2.35), with $\widetilde{u}_a \doteq u(\cdot, a)$, $\widetilde{u}_b \doteq u(\cdot, b) \in \mathcal{U}_\tau^\infty$, while (4.14) implies $u(\tau,\cdot) = \varphi$, which shows $\varphi \in \mathcal{A}(\tau)$. This completes the proof of Theorem 2.1. □

We next establish the compactness of the attainable set (2.14) stated in Theorem 2.2. The proof is quite similar to that of [2, Theorem 2.3]. We repeat it for completeness.

Proof of Theorem 2.2. Fix $T > 0$, and consider a sequence $\{u^\nu\}_{\nu\geq 0}$ of entropy weak solutions to the mixed problem for (1.1) on $\Omega_T \doteq [0,T] \times [a,b]$ (according with Definition 2.2), with a fixed initial data $\overline{u} \in \mathbf{L}^\infty([a,b],\Gamma)$. Since all u^ν are uniformly bounded, and because of the Oleinik-type estimates (2.7)-(2.8), one can find, for every $\varepsilon > 0$, some constant $C_\varepsilon > 0$ so that (4.15) holds. Thus, with the same arguments used in **Step 2** of the previous proof, we can construct a subsequence $\{u^{\nu'}\}_{\nu'\geq 0}$ so that, for any $t \in [0,\, T]$, $u^{\nu'}(t,\cdot)$ converges in $\mathbf{L}^1$ to some function $u(t,\cdot)$, which is continuous as a map from $]0,T]$ into $\mathbf{L}^1([a,b],\Gamma)$, and satisfies the entropy conditions

(2.7)-(2.10) on the decay of positive waves. On the other hand, the weak traces $\Psi_a^{\nu'}$, $\Psi_b^{\nu'}$ of the fluxes $f(u^{\nu'})$ at the boundaries $x = a$, $x = b$ are uniformly bounded, and hence are weak* relatively compact in $\mathbf{L}^\infty([0,T])$. Thus, by possibly taking a further subsequence, we have

$$\Psi_a^{\nu'} \overset{*}{\rightharpoonup} \Psi_a\,, \qquad \Psi_b^{\nu'} \overset{*}{\rightharpoonup} \Psi_b \qquad \text{in} \quad \mathbf{L}^\infty([0,T]), \tag{4.16}$$

for some maps $\Psi_a,\ \Psi_b \in \mathbf{L}^\infty([0,T])$. Notice that, by the properties of the Riemann invariants, the set $f(\Gamma)$ is closed and convex, and hence also the weak limits $\Psi_a,\ \Psi_b$ take values in $f(\Gamma)$. Moreover, since each u^ν is a distributional solution of (1.1)-(1.2) on Ω_T, also the limit function u is a distributional solution of the Cauchy problem (1.1)-(1.2) on the region Ω_T. Then, setting $\widetilde{u}_a \doteq f^{-1} \circ \Psi_a$, $\widetilde{u}_b \doteq f^{-1} \circ \Psi_b$, it follows that u is an entropy weak solution of the mixed problem (1.1)-(1.4) (with boundary data in $\mathcal{U}_T^\infty$) according with Definition 2.2, which shows that $u(T,\cdot) \in \mathcal{A}(T)$. This completes the proof of Theorem 2.2. □

If we take in consideration only solutions to the mixed problem (1.1)-(1.4) that are trajectories of the flow map E obtained in Theorem 2.3 (which, in particular, admit a strong $\mathbf{L}^1$ trace at the boundaries $x = a, x = b$), we are lead to study the set of attainable profiles

$$\mathcal{A}_E(T) \doteq \left\{ E_T(\overline{u},\, \widetilde{u}_a\,,\, \widetilde{u}_b\,)\ ; \qquad \widetilde{u}_a,\, \widetilde{u}_b \in \mathbf{L}^\infty([0,T],\Gamma) \right\}. \tag{4.17}$$

Since $\mathcal{A}_E(T) \subset \mathcal{A}(T)$, and by the proof of Theorem 2.1, it clearly follows that the characterization of the set $\mathcal{A}(T)$ provided by the inclusions (2.17)-(2.18) of Theorem 2.1 holds also for $\mathcal{A}_E(T)$. Concerning the compactness of the set $\mathcal{A}_E(T)$, observe that, given any sequence of exact solutions $u^\nu(t,\cdot) \doteq E_t(\overline{u}^\nu,\, \widetilde{u}_a^\nu\,,\, \widetilde{u}_b^\nu\,)$, $\nu \geq 1$, by Theorem 2.3 one can find another sequence of approximate solutions $v^\nu(t,\cdot)$ constructed by the front tracking algorithm of Section 2.3, so that

$$\left\| u^\nu(t,\cdot) - v^\nu(t,\cdot) \right\|_{\mathbf{L}^1([a,b])} \leq 1/\nu \qquad \forall\ t \in [1/\nu,\, T]\,.$$

Therefore, relying on the regularity property of a solution obtained as limit of front tracking approximations provided by Theorem 2.5, with the same arguments used in the proof of Theorem 2.2 one can establish also the compactness of the set $\mathcal{A}_E(T)$.

Theorem 4.1 *Under the same assumptions of Theorem 1, the set $\mathcal{A}_E(T)$ is a compact subset of $\mathbf{L}^1([a,b],\Gamma)$ for each $T > 0$.*

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References

[1] D. Amadori, Initial-boundary value problems for nonlinear systems of conservation laws, *NoDEA* **4** (1997), pp. 1-42.

[2] F. Ancona and P. Goatin, Uniqueness and Stability of $\mathbf{L}^\infty$ Solutions for Temple Class Systems with Boundary and Properties of the Attainable Sets, *to appear on SIAM Journal on Mathematical Analysis.*

[3] F. Ancona and A. Marson, On the attainable set for scalar non-linear conservation laws with boundary control, *SIAM Journal on Control and Optimization* **36** (1998), no. 1, pp. 290-312.

[4] F. Ancona and A. Marson, Scalar non-linear conservation laws with integrable boundary data, *Nonlinear Anal.* **35** (1999), pp. 687-710.

[5] P. Baiti and A. Bressan, The semigroup generated by a Temple class system with large data, *Differ. Integ. Equat.* **10** (1997), pp. 401-418.

[6] C. Bardos, A.Y. Leroux and J.C. Nedelec, First Order Quasilinear Equations with Boundary Conditions, *Comm. in P.D.E.,* 4 **9** (1979), pp. 1017-1034.

[7] S. Bianchini, Stability of $\mathbf{L}^\infty$ solutions for hyperbolic systems with coinciding shocks and rarefactions, *SIAM Journal on Mathematical Analysis*, **33** (2001), no. 4, pp. 959-981.

[8] A. Bressan, Hyperbolic Systems of Conservation Laws - The one-dimensional Cauchy problem, Oxford Univ. Press, 2000.

[9] A. Bressan, Uniques solutions for a class of discontinuous diferential equations, *Proc. Amer. Math. Soc.* **104** (1988), pp. 772-778.

[10] A. Bressan and R. M. Colombo, The semigroup generated by 2×2 conservation laws, *Arch. Rational Mech. Anal.* **133** (1995), pp. 1-75.

[11] A. Bressan and P. Goatin, Oleinik type estimates and uniqueness for $n \times n$ conservation laws, *J. Differential Equations* **156** (1999), pp. 26-49.

[12] A. Bressan and P. Goatin, Stability of $\mathbf{L}^\infty$ solutions of Temple class systems, *Differ. Integ. Equat.*, **13** (10-12), (2000), pp. 1503-1528.

[13] A. Bressan and P.G. LeFloch, Structural stability and regularity of entropy solutions to hyperbolic systems of conservation laws, *Indiana Univ. Math. J.*, **48**, (1999), no. 1, pp. 43-84.

[14] G.-Q. Chen and H. Frid, Divergence-measure fields and hyperbolic conservation laws, *Arch. Rational Mech. Anal.* **147** (1999), pp. 89-118.

[15] G.-Q. Chen and H. Frid, Vanishing viscosity limit for initial-boundary value problems for conservation laws, *Contemp. Math.* **238** (1999), pp. 35-51.

[16] C.M. Dafermos, Generalized characteristic and teh structure of solutions of hyperbolic conservation laws, *Indiana Univ. Math. J.*, **26**, (1977), no. 1, pp. 1097-1119.

[17] R. J. DiPerna, Singularities of solutions of nonlinear hyperbolic systems of conservation laws, *Arch. Rational Mech. Anal.* **60** (1976), pp. 75-100.

[18] F. Dubois and P.G. LeFloch, Boundary conditions for non-linear hyperbolic systems of conservation laws, *J. Differential Equations* **71** (1988), pp. 93-122.

[19] G.B. Folland, Real Analysis. Modern techniques and their applications, Pure & Appl. Math., John Wiley & Sons, New York, 1999.

[20] K.T. Joseph and P.G. LeFloch, Boundary layers in weak solutions of hyperbolic conservation laws, *Arch. Rational Mech. Anal.* **147** (1999), pp. 47-88.

[21] P. Lax, Hyperbolic systems of conservation laws II, *Comm. Pure Appl. Math.* **10** (1957), pp. 537-566.

[22] T.-P. Liu, Admissible solutions of hyperbolic conservation laws, *Amer. Math. Soc. Memoir* **240** (1981), Providence.

[23] F. Otto, Initial-boundary value problem for a scalar conservation law, *C.R. Acad. Sci. Paris,* Série I, **322** (1996), pp. 729-734.

[24] M. Sablé-Tougeron, Méthode de Glimm et probléme mixte, *Ann. Inst. Henri Poincaré* **10**, no. 4, (1993), pp. 423-443.

[25] D. Serre, Systemes de Lois de Conservation, Diderot Editeur, 1996.

[26] B. Temple, Systems of conservation laws with invariant submanifolds, *Trans. Amer. Math. Soc.* **280** (1983), pp. 781-795.