A Posteriori Error Estimate for Front-Tracking for Nonlinear Systems of Conservation Laws

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Abstract

We demonstrate an a posteriori error estimate in the L^1 norm for front-tracking approximate solutions to hyperbolic systems of nonlinear conservation laws. Extending the L^1-stability result of Bressan, Liu, and Yang we use their L^1-equivalent functional for pairs of front-tracking approximations and identify the leading order contribution to the numerical error. This leading term is closely related to the residual of the approximation and determines an a posteriori bound of the error. We demonstrate the estimate for the front-tracking approximations of Risebro, which are extensions to systems of Dafermos' polygonal approximations.

1 Introduction

We present an a posteriori error estimate for front-tracking approximations for nonlinear conservation laws,

$$u_t + f(u)_x = 0 \ , \tag{1.1}$$

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which improves on the L^1-stability estimate of Bressan, Liu, and Yang [4]. A posteriori error estimates are bounds on the error of approximate solutions which can be computed entirely from the approximation, the initial data, and the equation. They are therefore potentially useful in practical applications as rigorous upper bounds, as criteria for local mesh refinement strategies, and generally as guides for the construction of more efficient numerical approximations. Although such estimates are well-established for elliptic and parabolic equations [1], for hyperbolic equations and in particular for conservation laws, the theory is still in development.

For nonlinear scalar conservation laws, a number of results have already been obtained. Among these results, in one space dimension we mention the estimates in a dual Lipschitz norm of Nessyahu and Tadmor [20] and the L^1 norm estimates for E-schemes of Gosse and Makridakis [9]. In several space dimensions, Kröner and Ohlberger [14] have demonstrated estimates in the L^1 norm for upwind finite-volume schemes while Cockburn and Gau [6] have presented a general a posteriori error estimate for piecewise smooth approximations. Many of the known a posteriori error estimates [6, 9, 14] are based on the original estimate of Kuznetsov [15], which as Cockburn has remarked, is already a posteriori. Most of these estimates were developed for entropy solutions obtained in the limit of vanishing viscosity. As we showed in [16], our estimates restricted to scalar conservation laws coincide with those of Cockburn and Gau but with a simple proof closely related to Keyfitz's L^1-stability result [13].

Johnson and Szepessy [12] initiated the study of a posteriori error estimates for nonlinear systems of conservation laws, when they demonstrated an estimate in the L^2 norm for conservation laws with viscosity for approximations generated by the streamline diffusion finite element method. Despite their efforts to make their residual-based error estimate computable, it involved a constant which depended on the exact solution and therefore could not formally be computed a posteriori. Moreover, the size of their constant was only exhibited for weak non-interacting shock and rarefaction waves. For parabolic perturbations of linear systems of hyperbolic equations, many results have already been obtained for the finite element method [11, 24]. A more complete history of this problem can be found in references [5, 12, 14].

Although Bressan, Liu, and Yang do not appear to have interpreted their result as such, their L^1-stability estimate is in fact a posteriori. Our goal is therefore to relate the approximation error explicitly to their stability bound and to extend their estimate to a wider class of approximations. We demonstrate our estimate for a wide class of approximations derived from Risebro's front-tracking method [22]. These are extensions to systems of Dafermos' original polygonal approximation for scalar conservation laws [7]. Di Perna [21] first proposed an extension of Dafermos' method to 2×2 systems and Bressan [3] and Risebro [22] independently constructed algorithms for general $n \times n$ systems. The front-tracking method was developed into a numerical method for hyperbolic conservation laws by LeVeque [18], Hoegh-Krohn and Risebro [10], among others. Results on the uniqueness, stability, and long-term asymptotics of the limits of these approximations already exist [3, 22].

The main tool in our proof is the L^1-equivalent functional $\Phi(\cdot,\cdot)$ of Liu and Yang [19], defined for pairs of piecewise constant approximations of small total variation. They showed in [4] that Φ is decreasing in time modulo an error of order ϵt, while here, we identify the leading order contribution as the integral of some local measure of the error. Following Bressan, Liu and Yang [4] we use a simplification of the original functional of Liu and Yang.

The functional can be written as

$$\Phi\big(u(t), v(t)\big) = \sum_{i=1}^{n} \int_{-\infty}^{\infty} \big|q_i(x)\big| W_i(x)\ dx$$

where the parameters q_i measure the distance between the two approximate solutions in shock strength coordinates and

$$W_i(x) = \Big[1 + \kappa_1\Big(\text{waves potentially interacting with } q_i\Big) \\ + \kappa_2\Big(\text{interaction potential of } u \text{ and } v\Big)\Big]$$

with $\kappa_1, \kappa_2 \in \mathbb{R}^+$. Let $S_k(\cdot)(u^-)$ parameterize the k-th family of shock states u^+, starting at u^-. Thus u^+ and u^- satisfy the Rankine-Hugoniot condition for the k-th family of Df [23]. The $q_i(x)$ are the shock wave coordinates connecting u to v. They are defined implicitly by the relation

$$v(t,x) = S_n\big(q_n(x)\big) \circ \cdots \circ S_1\big(q_1(x)\big)\big(u(t,x)\big), \tag{1.2}$$

where the time dependence of the q_i has been suppressed. The parameter values $q_k < 0$ correspond to entropy (physical) shocks, while the values $q_k > 0$ correspond to rarefaction (non-physical) shocks. For u, v of small total variation and κ_1, κ_2 appropriately chosen, W_i is uniformly bounded in time and Φ is finite. Under these conditions, Φ is equivalent to the L^1 norm $\int ||u(t,x) - v(t,x)||\ dx$.

The residual $R(v) = v_t + f(v)_x$ of a piecewise constant approximation v is not a convenient measure of the error for hyperbolic conservation laws. In particular, the residual cannot distinguish between rarefaction shocks and continuous rarefaction fans. Moreover, it measures error in phase-space while the functional Φ measures distance along shock curves. Given a discontinuity at $x = z$, let V_r be the exact self-similar Riemann solution with this discontinuity as Cauchy data at time $t = s$. Let

$$V(\xi) = V\Big(\frac{x-z}{t-s}\Big)$$

be the self-similar solution, around (s, z), for the discontinuity propagating with its' time s velocity $\dot{z}$. The residual of the approximation v is given by

$$R(v) = \int_{-\infty}^{\infty} V(\xi) - V_r(\xi)\ d\xi.$$

We improve on this quantity by measuring the distance between V and V_r in the wave coordinates $\widetilde{p}_1, \ldots, \widetilde{p}_n$, see section 2.3, and integrating. We define the discrepancy to be

$$D_z(s) = \sum_k \int_{-\infty}^{\infty} |\widetilde{p}_k(\xi)|\ d\xi.$$

The discrepancy can be computed a posteriori since it depends only on the neighboring states of the discontinuity at (s, z) and the solution of the Riemann problem.

As shown in [4], Φ decreases at the discrete set of times when interactions occur. At the remaining times Φ is smooth and the derivative is a sum of separate contributions from each discontinuity. We show that each of these local contributions to the derivative of Φ is the sum of negative interaction terms and of discrepancies. The interaction terms are shown to be negative when the constant κ_1 is taken larger than some lower bound computable entirely from the system (1.1). If $\mathcal{D}\big(u(t)\big)$ denotes the set of the positions of discontinuities in $u(t, \cdot)$, then integrating the local estimates gives the desired global bound

$$\Phi\big(u(t), v(t)\big) \leq \Phi\big(u(0), v(0)\big) + C \int_0^t \Big(\sum_{\substack{z \in \mathcal{D}(u(s)) \bigcup \\ \mathcal{D}(v(s))}} D_z(s) \Big)\ ds. \tag{1.3}$$

In the process of demonstrating our a posteriori estimate, we extend and unify the techniques of Bressan, Liu, and Yang. Our proof distinguishes between the entropy production of the exact solution, controlled by the unique choice of the functional Φ, and the error between the approximation and the exact solution. We draw upon the notation of [4]. In Section 2, we describe our class of front-tracking approximations, construct the functional Φ, and provide a description of discrepancies. In Section 3, we state our results and explain the reduction of the global problem to a local one, namely through local bounds on the derivative of Φ. in (1.3). In the last section, we derive the local bounds in D_z for waves from genuinely nonlinear and linearly degenerate families.

2 Preliminaries

2.1 Front-Tracking

Consider an $n \times n$ system of conservation laws

$$u_t + f(u)_x = 0 \tag{2.1.1}$$

where the Jacobian of the smooth function f has n distinct real eigenvalues $\lambda_1(u) < \ldots < \lambda_n(u)$ for u inside some neighborhood of the origin $\Omega \subset \mathbb{R}^n$. Given initial data $\bar{u} : \mathbb{R} \to \Omega$ we shall say that the bounded measurable function $u : \mathbb{R}^+ \times \mathbb{R} \to \Omega$ is a weak solution of the system (2.1.1) if $u(0,x) = \bar{u}(x)$ and if

$$\int_0^\infty \int_{-\infty}^\infty \big(\phi_t u + \phi_x f(u)\big)\ dxdt + \int_{-\infty}^\infty \phi(x,0)u(x,0)\ dx = 0 \tag{2.1.2}$$

for every smooth ϕ with compact support in $t \geq 0$.

Let $r_1, \ldots, r_n$ be the eigenvectors of $Df(u)$, given as smooth functions of $u \in \Omega$. Define the k-th rarefaction curve through the point u^- to be the unique solution $R_i(\cdot)(u^-)$ of

$$\begin{aligned} \frac{d}{d\sigma} R_i(\sigma)(u^-) &= r_i\big(R_i(\sigma)(u^-)\big), \\ R_i(0)(u^-) &= u^-. \end{aligned}$$

Given two states u^+ and u^-, the matrix

$$A(u^+, u^-) = \int_0^1 Df((1-\theta)u^- + \theta u^+)\ d\theta \tag{2.1.3}$$

satisfies the relation

$$A(u^+,u^-)(u^+-u^-)=f(u^+)-f(u^-). \tag{2.1.4}$$

Define the k-th shock curve through u^- to be the set of states $u^+=S_k(\sigma)(u^-)$ satisfying the Rankine-Hugoniot condition

$$f(u^+)-f(u^-)=s(u^+-u^-),$$

and such that $s\in\mathbb{R}$ is the k-th eigenvalue of $A(u^+,u^-)$. Let $s_k(u^+,u^-)$ denote the k-th eigenvalue of $A(u^+,u^-)$. An explicit parameterization of the smooth shock curve $S_k(\cdot)$ is chosen below.

We assume that each family $k\in\{1,\dots,n\}$ is either *genuinely nonlinear*

$$r_k\cdot\nabla\lambda_k\neq 0, \tag{2.1.5}$$

or *linearly degenerate*

$$r_k\cdot\nabla\lambda_k=0. \tag{2.1.6}$$

If the k-th family is genuinely nonlinear then the shock and rarefaction curves can be parameterized so that

$$\begin{array}{rclcrcl} \frac{d}{d\sigma}\lambda_k\big(S_k(\sigma)(u)\big) &=& 1, & \quad & \frac{d}{d\sigma}\lambda_k\big(R_k(\sigma)(u)\big) &=& 1,\\ \lambda_k\big(S_k(\sigma)(u)\big)-\lambda_k(u) &=& \sigma, & & \lambda_k\big(R_k(\sigma)(u)\big)-\lambda_k(u) &=& \sigma. \end{array} \tag{2.1.7}$$

We parameterize a linearly degenerate family by arc-length. For a genuinely nonlinear family k define

$$T_k(\sigma)(u^-)=\begin{cases} R_k(\sigma)(u^-), \sigma\geq 0\\ S_k(\sigma)(u^-), \sigma<0\end{cases} \tag{2.1.8}$$

and for a linearly degenerate family let $T_k(\sigma)(u^-)=S_k(\sigma)(u^-)$. The curve T_k is smooth for $\sigma\neq 0$ with two continuous derivatives at $\sigma=0$.

A Riemann problem is an initial value problem for (2.1.1) consisting of piecewise constant initial data along the $t=0$ axis formed of only two constant states u^- and u^+ separated at the origin.

Theorem 2.1 *[17] For a given value of $u^-\in\Omega$, there exists a neighborhood $\Omega'\subset\Omega$ such that for any $u^+\in\Omega'$, the Riemann problem has a unique self-similar solution composed of $n+1$ constant states $u_0=u^-,u_1,\dots,u_n=u^+$ satisfying $u_k=T_k(p_k)(u_{k-1})$ for real numbers $p_k(u^-,u^+)$. The solution is piecewise constant between:*

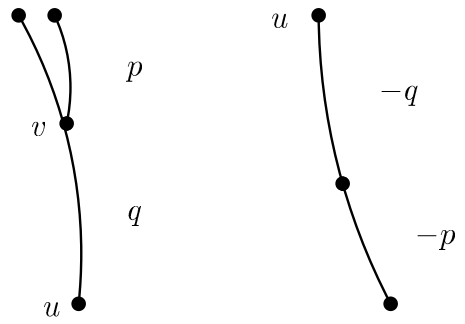

Figure 1: Figure illustrating the states in inequalities (2.1.13) and (2.1.14).

i) discontinuities traveling with the Rankine-Hugoniot speed and connecting

$$u_k = S_k\big(p_k(u^-, u^+)\big)(u_{k-1});$$

ii) continuous regions where the solution $u\big(x/t\big)$ satisfies $\lambda_k\big(u(x/t)\big) = x/t$ and

$$u_k = R_k\big(p_k(u^-, u^+)\big)(u_{k-1}).$$

A similar result also holds with the wave curves T_k replaced by the shock curves S_k.

Theorem 2.2 *Given a state $u \in \Omega$, there exists a neighborhood Ω' of u such that for any $v \in \Omega'$ there exists a unique sequence of shock wave coordinates $q_1(u,v), \dots, q_n(u,v)$ satisfying*

$$v = S_n\big(q_n(u,v)\big) \circ \cdots \circ S_1\big(q_1(u,v)\big)(u). \tag{2.1.9}$$

Henceforth, we assume that Ω has been replaced by the smaller set Ω' so that Riemann problems can be solved.

Let Λ be the positive constant satisfying

$$\sup_{\substack{u\in\Omega \\ i=1,\dots,n}} \big|\lambda_i(u)\big| = \Lambda. \tag{2.1.10}$$

We introduce the constant

$$d = \min\{d_1, d_2, d_3\} \tag{2.1.11}$$

depending on the three distinct constants described below. At the cost of possibly reducing the size of Ω, we assume that there exists a positive constant d_1 such that

$$d_1 < \inf \Big\{ |l_i - l_j| \,\Big|\, i \neq j \text{ and for } k = i, j,\ l_k \in \lambda_k(\Omega),\ \text{or} \quad l_k = s_k(u,v)\ \forall u, v \in \Omega \Big\}. \tag{2.1.12}$$

The genuine nonlinearity of f implies that for any $u \in \Omega$, any k, and any positive quantities p, q such that both $v \equiv S_k(q)(u)$ and $S_k(p+q)(u)$ belong to Ω, then there exists a positive constant d_2 such that

$$d_2|q| < s_k\big(v, S_k(p)(v)\big) - s_k\big(u, S_k(p+q)(u)\big). \tag{2.1.13}$$

For any $u \in \Omega$, any k, and any positive quantities p, q such that both $S_k(-q)(u)$ and $S_k(-p-q)(u)$ belong to Ω, then there exists a positive constant d_3 such that

$$d_3|p| < s_k\big(u, S_k(-q)(u)\big) - s_k\big(u, S_k(-p-q)(u)\big). \tag{2.1.14}$$

Figure 1 illustrates the states in (2.1.13) and (2.1.14). These constants have been grouped together under d since they will serve the same purpose later in the proof.

Front-tracking approximations are piecewise constant approximations containing a finite number of discontinuities. This method is a modification of Glimm's scheme [8] and convergence results are demonstrated in essentially the same manner as for the latter. The key point in the construction is to control the total number of discontinuities so that the approximation may be extended to all times. We present Risebro's version of the front-tracking algorithm, omitting Bressan's [3] and Biati and Jenssen [2]. We begin with a few preliminary definitions which will help to clarify the construction. For convenience, given a function u of time t and space x let $u(t)$ denote the function of space $u(t)(x) = u(t,x)$.

Definition 2.1 *A wave in a piecewise constant function $u(t) : \mathbb{R} \to \Omega$ is a pair $(z,k) \in \mathbb{R} \times \{1, \ldots, n\}$ such that the states u^- and u^+ separating the discontinuity at $x = z$ satisfy $p_k(u^-, u^+) \neq 0$. If $\alpha = (z,k)$ is a wave then we say that α belongs to the discontinuity at z and let*

i) $k_\alpha \equiv k$ denote its characteristic type, *ii)* $\sigma_\alpha \equiv p_k(u^-, u^+)$ *denote its strength, and*

iii) $x_\alpha \equiv z$ *denote its position.*

The set of all waves in $u(t)$ *will be denoted by* $\mathcal{W}\big(u(t)\big)$.

Definition 2.2 *A wave* α *is called isolated if it is the only wave located at* $x_\alpha \in \mathbb{R}$. *If a wave belonging to a genuinely nonlinear family satisfies* $\sigma_\alpha < 0$, *it will be called a shock wave; otherwise it will be called a rarefaction wave. A wave will be called linearly degenerate if the family* k_α *is linearly degenerate.*

Let $\mathcal{D}\big(u(t)\big)$ *denote the set of points in space where the approximation* $u(t)$ *has discontinuities.*

We note that for an isolated wave α, the left and right sided limits of u at x_α, denoted u^- and u^+, are related by

$$u^+ = T_{k_\alpha}(\sigma_\alpha)(u^-).$$

Definition 2.3 *Let* $\alpha = (z, k) \in \mathcal{W}(u(t))$ *be a wave, belonging to a discontinuity at* $x = z$, *with neighboring states* u^- *and* u^+. *We define the left and right hand states of* α *to be*

$$u_\alpha^- = T_{k-1}\big(p_{k-1}(u^-, u^+)\big) \circ \cdots \circ T_1\big(p_1(u^-, u^+)\big)(u^-), \qquad (2.1.15)$$
$$u_\alpha^+ = T_k(\sigma_\alpha)(u_\alpha^-).$$

Definition 2.4 *We say that a wave* α *is dominant if it travels at a speed*

$$\dot{x}_\alpha \in \lambda_{k_\alpha}(\Omega). \qquad (2.1.16)$$

We now construct a front-tracking approximation for (2.1.1) with initial data $\bar{u}$ of bounded variation. Choose $u(0, \cdot)$ a piecewise constant function with a finite number of discontinuities and such that $||\bar{u} - u(0, \cdot)||_{L^1}$ is finite. Suppose that each discontinuity in the approximation $u(0, \cdot)$ separated by states u^- and u^+, is of the form

$$u^+ = T_k\big(p_k(u^-, u^+)\big)(u^-). \qquad (2.1.17)$$

If we propagate one of these initial discontinuities forward in time along a smooth trajectory $z(t)$ then we will assume that

$$\dot{z} \in \lambda_k(\Omega) \qquad (2.1.18)$$

for the integer k in (2.1.17). Therefore initially, each discontinuity corresponds to exactly one isolated wave.

When two discontinuities meet, we use an approximate Riemann solver in the class of piecewise constant functions. Suppose that two discontinuities, separated by three consecutive states u_l, u_m, and u_r meet at time t_1. Let $u_0 = u_l, u_1, \ldots, u_n = u_r$ be the $n+1$ states of Theorem 2.1 for the Riemann problem at time t_1. In a neighborhood of the collision, construct a piecewise constant function composed of n regions separated by the states $u_0, \ldots, u_n$ and such that

i) if $u_k = S_k\big(p_k(u_l, u_r)\big)(u_{k-1})$ then these states are separated by an isolated wave α traveling at a speed

$$\dot{x}_\alpha \in \lambda_k(\Omega); \tag{2.1.19}$$

ii) if $u_k = R_k\big(p_k(u_l, u_r)\big)(u_{k-1})$ then there exists a finite number of isolated rarefaction waves $\alpha(1), \ldots, \alpha(j)$ satisfying

$$p_k(u_l, u_r) = \sum_{i=1}^{j} \sigma_{\alpha(i)}, \tag{2.1.20}$$

$$\dot{x}_{\alpha(i-1)} < \dot{x}_{\alpha(i)}, \tag{2.1.21}$$

and

$$\dot{x}_{\alpha(i)} \in \lambda_k(\Omega), \forall i. \tag{2.1.22}$$

For consistency, we also require that neighboring isolated shock waves not collide. There is clearly more than one possible solution which satisfies these conditions. In particular, there is freedom in the choice of the speeds $\dot{x}_\alpha$, the number and the strength of the rarefaction waves $\alpha(1), \ldots, \alpha(j)$.

Using the approximate Riemann solver described above two different situations may occur. Either the approximation will blow-up in finite time, in the L^1, L^∞, or total variation norm, or the approximation will develop an infinite number of discontinuities in finite time. Both of these difficulties require us to examine the global interaction patterns within the approximation. We say that two waves $\alpha, \beta \in \mathcal{W}\big(u(t)\big)$ are approaching if $x_\alpha < x_\beta$ and $k_\alpha > k_\beta$, or if $k_\alpha = k_\beta$ and at least one of the waves is a shock. Define the interaction potential to be

$$Q\big(u(t)\big) = \sum_{\alpha \text{ approaches } \beta} |\sigma_\alpha \cdot \sigma_\beta|, \tag{2.1.23}$$

and consider the following functional

$$V(u(t)) = \sum_{\alpha \in \mathcal{W}\big(u(t)\big)} |\sigma_\alpha|, \tag{2.1.24}$$

equivalent to the total variation norm $||u(t)||_{\mathrm{TV}}$. Let t_k denote the time at which the k-th interaction occurred and define

$$P_k = \sum_{\alpha,\beta} |\sigma_\alpha \cdot \sigma_\beta| \tag{2.1.25}$$

where the sum occurs over pairs of waves colliding at time t_k.

Following the argument of Glimm [8], Risebro showed that if the total variation of the initial data is smaller than some a priori bound then there exists a constant K for which the functional

$$V\big(u(t)\big) + KQ\big(u(t)\big) \tag{2.1.26}$$

is decreasing in time. This implies that the approximations remain total variation bounded uniformly in time. This argument also shows that for fixed positive Γ_u, there exists an integer N_1 such that

$$P_k < \Gamma_u, \, \forall k \geq N_1. \tag{2.1.27}$$

Fix the constant Γ_u and let t_{N_1} be the time satisfying (2.1.27) at which two isolated waves collide, say α and β with $k_\alpha < k_\beta$. Let $u_0, u_1, \ldots, u_n$ be the $n+1$ states of Theorem 2.1 for the Riemann problem at the collision of α and β. Instead of several outgoing isolated waves, we regroup these waves into two discontinuities traveling at the speeds of the k_α and k_β families. Choose the state to the right of the outgoing α-discontinuity to be u_n and choose the state to the left of the outgoing β-discontinuity to be u_0. Pick any integer i, satisfying $k_\alpha < i \leq k_\beta$, and let the state between both discontinuities be u_i. If there were several colliding pairs of discontinuities at time t_{N_1}, the same operation would apply to each pair. More generally, we now require that neighboring dominant shock waves not collide. This construction is illustrated in Figure 2

Having constructed a new approximation u for the times immediately following t_{N_1}, we return to our earlier approximate Riemann solver to resolve all new interactions at times after t_{N_1}. When a time t_{N_2} is reached such that

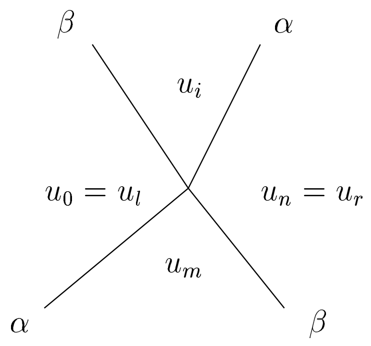

Figure 2: The interaction of the discontinuities α and β when condition (2.1.27) is satisfied.

$P_k < \Gamma_u$, for all $k \geq N_2$, then construct the approximation using non-isolated waves.

A time will be reached when (2.1.27) is always satisfied and no new discontinuities are created. Since the system is strictly hyperbolic, eventually all discontinuities will cross paths and the approximation will consist of a finite number of discontinuities spreading apart. The approximation will therefore be defined for all times. This method satisfies the CFL condition

$$|\dot{x}_\alpha| \leq \Lambda, \, \forall \alpha \in \mathcal{W}\big(u(t)\big), \tag{2.1.28}$$

with Λ defined in (2.1.10) We summarize the result of Risebro.

Lemma 2.1 *For any Γ_u, there exists positive constants δ and K, depending on the system* (2.1.1) *but independent of Γ_u, such that any front-tracking approximation u satisfying*

$$||u(0)||_{L^\infty} + ||u(0)||_{\mathrm{TV}} < \delta, \tag{2.1.29}$$

is defined for all t and has values in Ω. Moreover, for this approximation the functional

$$V\big(u(t)\big) + KQ\big(u(t)\big), \tag{2.1.30}$$

is decreasing in time.

The algorithm outlined above does not generate a unique front-tracking approximation and in fact when Γ_u is large the approximation may be very different from the exact solution. In the original construction of Risebro, a narrower class of approximations was considered so that existence results for the initial value problem (2.1.1) could be achieved. Each of the approximations of Risebro depended on a small parameter ϵ and therefore will be called an ϵ-approximation.

In the construction of Risebro, each ϵ-approximation u for equation (2.1.1) with initial data $\bar{u} \in L^1(\mathbb{R}, \mathbb{R}^n)$ satisfied the following conditions:

i) for any dominant wave $\alpha = (z, k) \in \mathcal{W}\big(u(t)\big)$,

$$\left|\dot{x}_\alpha - s_k(u_\alpha^-, u_\alpha^+)\right| < \epsilon,$$

ii) for any isolated rarefaction wave $\alpha \in \mathcal{W}\big(u(t)\big)$, $\sigma_\alpha < \epsilon$,

iii) $\Gamma_u < \epsilon$, and

iv) $||u(0) - \bar{u}||_{L^1} < \epsilon$.

Under these conditions, he showed the following.

Theorem 2.3 *[22] Given the system* (2.1.1) *there exists a constant* $\delta_0 > 0$ *such that for any initial data* $\bar{u}$ *satisfying* $||\bar{u}||_{L^\infty} + ||\bar{u}||_{\mathrm{TV}} < \delta_0$*, there exists a sequence* $\{u_\epsilon\}$ *of* ϵ*-approximations which converge in* $L^1_{loc}([0,\infty) \times \mathbb{R})$ *as* $\epsilon \to 0$ *to a weak solution of the system with initial data* $\bar{u}$*.*

2.2 The Liu-Yang Functional

Following [4], we construct the functional Φ of Liu and Yang. Let u and v be two front-tracking approximations and assume that $u(0) - v(0) \in L^1$. Since the total variation of both solutions remains bounded, $u(t) - v(t) \in L^1$ for all times t. The distance between both approximate solutions $u(t)$ and $v(t)$ will be measured in the shock coordinates of Theorem 2.2. Since the shock coordinates will only be used for this purpose and the time t will be fixed throughout, we will abbreviate

$$q_k\big(u(t,x), v(t,x)\big) = q_k(x).$$

Thus $q_i(x) < 0$ indicates that to reach v from u, a shock wave of the i-th family is used while for $q_i(x) > 0$, a rarefaction shock is used.

The functional of Liu and Yang

$$\Phi\big(u(t), v(t)\big) = \sum_{i=1}^{n} \int_{-\infty}^{\infty} |q_i(x)| W_i(x)\ dx \tag{2.2.1}$$

is finite if the W_i are uniformly bounded. It is convenient to require

$$1 \leq W_i(x) \leq 2,\ \forall x \in \mathbb{R}, \tag{2.2.2}$$

uniformly in time. Let

$$W_i(x) = 1 + \kappa_1 \Big(A_i(x) + B_i(x)\Big) + \kappa_2 \Big(Q\big(u(t)\big) + Q\big(v(t)\big)\Big) \tag{2.2.3}$$

where the terms A_i and B_i are given below and the constants κ_1, κ_2 will be chosen to satisfy (2.2.2). Let

$$A_i(x) = \Big(\sum_{\substack{\alpha \in \mathcal{W}(u) \bigcup \mathcal{W}(v) \\ x_\alpha < x,\, k_\alpha > i}} + \sum_{\substack{\alpha \in \mathcal{W}(u) \bigcup \mathcal{W}(v) \\ x_\alpha > x,\, k_\alpha < i}} \Big) |\sigma_\alpha| \tag{2.2.4}$$

measure the strength of all waves α, $k_\alpha \neq i$, with the potential to interact with the virtual wave q_i. When the i-th family is linearly degenerate, $B_i \equiv 0$. If the i-th family is genuinely nonlinear then let

$$B_i(x) = \begin{cases} \Big(\sum_{\substack{\alpha \in \mathcal{W}(u) \\ x_\alpha < x,\, k_\alpha = i}} + \sum_{\substack{\alpha \in \mathcal{W}(v) \\ x_\alpha > x,\, k_\alpha = i}} \Big) |\sigma_\alpha| & \text{if } q_i(x) < 0, \\ \Big(\sum_{\substack{\alpha \in \mathcal{W}(v) \\ x_\alpha < x,\, k_\alpha = i}} + \sum_{\substack{\alpha \in \mathcal{W}(u) \\ x_\alpha > x,\, k_\alpha = i}} \Big) |\sigma_\alpha| & \text{if } q_i(x) > 0, \end{cases} \tag{2.2.5}$$

measure the strength of the waves of the family $k_\alpha = i$ with the potential to interact with q_i. We will return to study B_i after the next lemma. Many of our later arguments will be localized around a specific wave α and it will therefore be natural, when it is clear from the context, to suppress the symbol α. For a quantity G, we will then write

$$G^{\pm} = \lim_{x \to x_\alpha \pm} G(x).$$

We also introduce the notation $[G]_\alpha = G^+ - G^-$ to denote the jump in a quantity G across a wave located at x_α.

Lemma 2.2 *If $\alpha \in \mathcal{W}\big(v(t)\big)$ is an isolated wave then for $i \neq k_\alpha$,*

$$[W_i]_\alpha = \begin{cases} \kappa_1|\sigma_\alpha| & \text{if } i < k_\alpha, \\ -\kappa_1|\sigma_\alpha| & \text{if } i > k_\alpha. \end{cases} \tag{2.2.6}$$

If α is linearly degenerate then $[W_{k_\alpha}]_\alpha = 0$ and otherwise

$$[W_{k_\alpha}]_\alpha = \begin{cases} \kappa_1|\sigma_\alpha| & \text{if } q^-_{k_\alpha} \text{ and } q^+_{k_\alpha} \text{ are both positive,} \\ -\kappa_1|\sigma_\alpha| & \text{if } q^-_{k_\alpha} \text{ and } q^+_{k_\alpha} \text{ are both negative,} \\ \kappa_1[B_{k_\alpha}]_\alpha & \text{otherwise.} \end{cases} \tag{2.2.7}$$

Suppose that α is a genuinely nonlinear wave. Then if $q^+_{k_\alpha} < 0 < q^-_{k_\alpha}$,

$$B^-_{k_\alpha} + B^+_{k_\alpha} + |\sigma_\alpha| = \Big(\sum_{\substack{\beta \in \mathcal{W}(u) \\ k_\beta = k_\alpha}} + \sum_{\substack{\beta \in \mathcal{W}(v) \\ k_\beta = k_\alpha}} \Big) |\sigma_\beta|, \tag{2.2.8}$$

and if $q^-_{k_\alpha} < 0 < q^+_{k_\alpha}$,

$$B^-_{k_\alpha} + B^+_{k_\alpha} = |\sigma_\alpha| + \Big(\sum_{\substack{\beta \in \mathcal{W}(u) \\ k_\beta = k_\alpha}} + \sum_{\substack{\beta \in \mathcal{W}(v) \\ k_\beta = k_\alpha}} \Big| \sigma_\beta|. \tag{2.2.9}$$

Proof We first consider identity (2.2.6). It is easy to verify that

$$[W_i]_\alpha = \kappa_1\Big([A_i]_\alpha + [B_i]_\alpha \Big). \tag{2.2.10}$$

For the first term $[A_i]_\alpha$, it suffices to consider in equation (2.2.4) only waves belonging to $\mathcal{W}\big(u(t)\big)$. If $i < k_\alpha$, the wave α appears in neither sum of A^-_i but does appear in the first sum of A^+_i. We therefore have $\kappa_1[A_i]_\alpha = \kappa_1|\sigma_\alpha|$. Similarly, when $i > k_\alpha$ the wave α belongs to the second term in A^-_i but in neither of the sums in A^+_i. In this case $\kappa_1[A_i]_\alpha = -\kappa_1|\sigma_\alpha|$. To verify that $[B_i]_\alpha = 0$, we notice that across a discontinuity of the k_α-th family the quantity q_i can only change by a second-order quantity. Since q_i is first order, it is greater than this change. This implies that the same sums appear in both B^-_i and B^+_i.

We now turn to (2.2.7). The jump $[A_{k_\alpha}]_\alpha = 0$ since it only involves waves of families other than k_α. If α is linearly degenerate then $B_{k_\alpha} = 0$ and (2.2.10) shows that $[W_{k_\alpha}]_\alpha = 0$. We now assume that α is genuinely nonlinear and that both $q^-_{k_\alpha}$ and $q^+_{k_\alpha}$ are positive. In this case, the term

$B^+_{k_\alpha}$ contains a sum over all k_α-waves in v located to left of x_α, including α. $B^-_{k_\alpha}$ has the same sum but without the wave α. The jump across α is therefore $[W_{k_\alpha}]_\alpha = \kappa_1[B_{k_\alpha}]_\alpha = \kappa_1|\sigma_\alpha|$. The case when both $q^\pm_{k_\alpha}$ are negative is similar. When the sign of q_{k_α} changes across the wave α, the waves in the sums $B^\pm_{k_\alpha}$ are distinct, except possibly for the occurrence of α. Therefore, there is no cancellation between their difference and we measure this jump to be $\kappa_1[B_{k_\alpha}]_\alpha$.

For the demonstration of (2.2.8), assume that α is a genuinely nonlinear wave and that $q^+_{k_\alpha} < 0 < q^-_{k_\alpha}$. From the definition (2.2.5) of $B_{k_\alpha}(x)$, in a neighborhood of $\dot{x}_\alpha$ we have

$$B^-_{k_\alpha} = \Bigg(\sum_{\substack{\beta\in\mathcal{W}(v)\\ x_\beta<x_\alpha, k_\beta=k_\alpha}} + \sum_{\substack{\beta\in\mathcal{W}(u)\\ x_\beta\geq x_\alpha, k_\beta=k_\alpha}} \Bigg)|\sigma_\beta|, \tag{2.2.11}$$

$$B^+_{k_\alpha} = \Bigg(\sum_{\substack{\beta\in\mathcal{W}(u)\\ x_\beta\leq x_\alpha, k_\beta=k_\alpha}} + \sum_{\substack{\beta\in\mathcal{W}(v)\\ x_\beta> x_\alpha, k_\beta=k_\alpha}} \Bigg)|\sigma_\beta|, \tag{2.2.12}$$

where the sums occur over disjoint sets of waves, none of which include the shock wave α. To prove (2.2.8), it suffices to sum (2.2.11) and (2.2.12).

Definition 2.2.5 states that for $q^-_{k_\alpha} < 0 < q^+_{k_\alpha}$,

$$B^-_{k_\alpha} = \sum_{\substack{\beta\in\mathcal{W}(u)\\ x_\beta<x_\alpha,\, k_\beta=k_\alpha}} |\sigma_\beta| + \sum_{\substack{\beta\in\mathcal{W}(v)\\ x_\beta\geq x_\alpha,\, k_\beta=k_\alpha}} |\sigma_\beta|, \tag{2.2.13}$$

$$B^+_{k_\alpha} = \sum_{\substack{\beta\in\mathcal{W}(v)\\ x_\beta\leq x_\alpha,\, k_\beta=k_\alpha}} |\sigma_\beta| + \sum_{\substack{\beta\in\mathcal{W}(u)\\ x_\beta> x_\alpha,\, k_\beta=k_\alpha}} |\sigma_\beta|. \tag{2.2.14}$$

Notice that the wave α occurs in both the second sum of (2.2.13) and the first sum of (2.2.14). Besides the wave α appearing twice, all other waves of the k_α family in u and v appear exactly once in either $B^-_{k_\alpha}$ or $B^+_{k_\alpha}$. This demonstrate (2.2.9). ∎

We now support the claim that $B_i(x)$ represents the sum of the strength of the waves α of the i-th family, $i \neq k_\alpha$, which have the potential to interact with q_i. For definiteness, choose a rarefaction wave $\alpha \in \mathcal{W}\big(v(t)\big)$ for which $0 \leq q^-_{k_\alpha} \leq q^+_{k_\alpha}$. This situation is described in Figure 3. For each integer k, let $w_k(x)$ be the intermediate states defined by

$$w_k(x) = S_k\big(q_k(x)\big) \circ \cdots \circ S_1\big(q_1(x)\big)\big(u(t,x)\big). \tag{2.2.15}$$

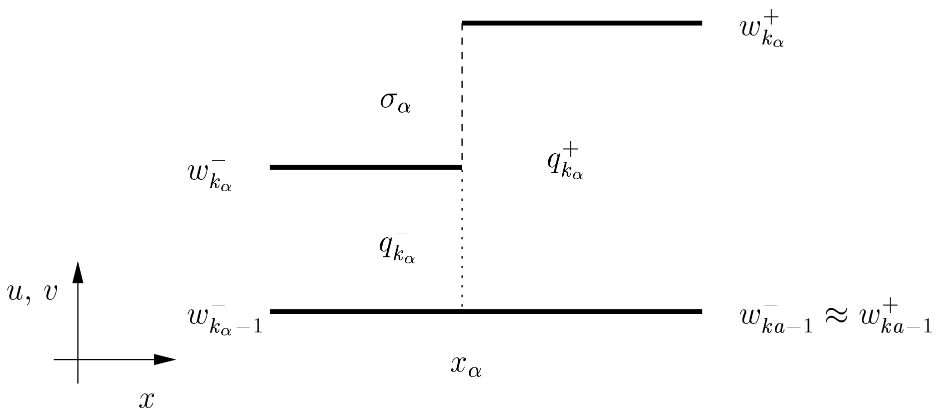

Figure 3: Figure illustrating the potential interaction between α and the virtual waves $q^-_{k_\alpha}$ and $q^+_{k_\alpha}$. The bold curve represents the graph of u and v in the k_α-th shock coordinate.

If the k_α-th shock of strength q_{k_α} joining u and v is interpreted as a virtual wave, then its virtual speed can be defined as $s_{k_\alpha}\big(w_{k_\alpha}(x), w_{k_\alpha - 1}(x)\big)$. Following this analogy, along the k_α-th shock coordinate the wave α has a virtual speed $s_{k_\alpha}(w^-_{k_\alpha}, w^+_{k_\alpha})$, faster than the speeds of the virtual waves $q^-_{k_\alpha}$ and $q^+_{k_\alpha}$. More specifically, the virtual speeds of $q^-_{k_\alpha}$, $q^+_{k_\alpha}$, and α are respectively

$$s_{k_\alpha}(w^-_{k_\alpha - 1}, w^-_{k_\alpha}) < s_{k_\alpha}(w^+_{k_\alpha - 1}, w^+_{k_\alpha}) < s_{k_\alpha}(w^-_{k_\alpha}, w^+_{k_\alpha}). \tag{2.2.16}$$

Therefore, as long as $q_{k_\alpha}(x)$ remains positive the waves to the left of x will approach it. The positivity of $[W_{k_\alpha}]_\alpha$ is essentially a consequence of the inequalities (2.2.16). Similarly, when $q^-_{k_\alpha} \leq q^+_{k_\alpha} \leq 0$ the virtual speed of α is slower than those of either $q^-_{k_\alpha}$ and $q^+_{k_\alpha}$,

$$s_{k_\alpha}(w^-_{k_\alpha}, w^+_{k_\alpha}) < s_{k_\alpha}(w^-_{k_\alpha - 1}, w^-_{k_\alpha}) < s_{k_\alpha}(w^+_{k_\alpha - 1}, w^+_{k_\alpha}). \tag{2.2.17}$$

In other words, only the waves to the right of x approach q_{k_α}. In this case, (2.2.7) predicts a negative contribution to $[W_{k_\alpha}]_\alpha$.

Assuming that κ_1 is some known but large constant, the next lemma states that condition (2.2.2) can be satisfied. The value of κ_1 will be determined later in the proof.

Lemma 2.3 *For any fixed value of κ_1, there exists δ_0 and κ_2, depending only on κ_1 and the system* (2.1.1)*, such that any front-tracking approximations u*

and v satisfying

$$||u(0)||_{L^\infty} + ||u(0)||_{\mathrm{TV}},\ ||v(0)||_{L^\infty} + ||v(0)||_{\mathrm{TV}} < \delta_0, \qquad (2.2.18)$$
$$u(0) - v(0) \in L^1(\mathbb{R}, \Omega),$$

are defined for all t and have values in Ω. For these approximations, the functional Φ exists, condition (2.2.2) holds, and the weights W_i decrease in time.

Proof With the constants δ and K of Lemma 2.1, take any front-tracking approximation u satisfying

$$||u(0)||_{L^\infty} + ||u(0)||_{\mathrm{TV}} < \delta. \qquad (2.2.19)$$

Lemma 2.1 assures us that u is defined for all t, takes values in Ω, and that the functional

$$V\big(u(t)\big) + KQ\big(u(t)\big) \qquad (2.2.20)$$

decreases in time. Remark that $Q(\cdot) \leq V(\cdot)^2$ and that there exists a constant C such that

$$V(\cdot) \leq C||\cdot||_{\mathrm{TV}}. \qquad (2.2.21)$$

We assume that

$$\delta_0 \leq \min\left\{\delta, \frac{1}{4C\kappa_1}, \frac{1}{CK}\right\}. \qquad (2.2.22)$$

and that the approximations u and v satisfy (2.2.18). Under these conditions, we have

$$\begin{aligned} V\big(u(t)\big) + KQ\big(u(t)\big) &\leq V\big(u(0)\big) + KQ\big(u(0)\big) \\ &\leq C||u(0)||_{\mathrm{TV}} + KC^2||u(0)||^2_{\mathrm{TV}} \\ &\leq 2C||u(0)||_{\mathrm{TV}} \leq \frac{1}{2\kappa_1}. \end{aligned} \qquad (2.2.23)$$

An identical estimate also holds for the approximation v. In conclusion, if u and v both satisfy conditions (2.2.19) and (2.2.22), then summing inequalities (2.2.23) for u and v we find

$$\Big(V\big(u(t)\big) + V\big(v(t)\big)\Big) + K\Big(Q\big(u(t)\big) + Q\big(v(t)\big)\Big) \leq \frac{1}{\kappa_1}, \forall t. \qquad (2.2.24)$$

Since the sums in A_i and B_i occur over subsets of those in $V\big(u(t)\big)+V\big(v(t)\big)$, the functional

$$\Big(A_i(x)+B_i(x)\Big)+K\Big(Q\big(u(t)\big)+Q\big(v(t)\big)\Big) \tag{2.2.25}$$

is also uniformly bounded in time by $1/\kappa_1$. Moreover, since the decay property of $V+KQ$ is in fact a consequence of the monotonicity of the interaction potential Q and of the smallness of V, the functional (2.2.25) also inherits this property.

Multiplying (2.2.24) by κ_1, defining $\kappa_2=\kappa_1 K$, and adding one, we notice that condition (2.2.2) is satisfied. Since the approximations are total variation bounded for all time, the difference $u(t)-v(t)$ remains in L^1 for all time and the functional Φ is finite. ■

Definition 2.5 *Any front-tracking approximation which satisfies conditions* (2.2.18) *of Lemma 2.3 will be called admissible.*

At times when no interactions occur, the functional Φ is smooth in time and the functions $W_i(x)$ are constant in the regions which do not intersect the trajectories of waves. On the other hand, at a discrete time t_0 when an interaction does occur at $x_0\in\mathbb{R}$, Lemma 2.3 implies that

$$\lim_{t\to t_0+} W_i(x)\le \lim_{t\to t_0-} W_i(x), \qquad x\ne x_0. \tag{2.2.26}$$

Since the terms W_i decrease during interactions, so does the functional Φ. The main step will therefore be an analysis of Φ during those times when it is smooth.

2.3 Residuals and Discrepancies

The residual $R(v)=v_t+f(v)_x$ of an approximation v is not a sufficient measure of the error for weak solutions to hyperbolic conservation laws. We therefore propose a closely related measure which can also be computed from the approximate solution, the initial data, and the flux.

The primary difficulty with the residual $R(v)$ is that it cannot distinguish between an unphysical rarefaction shock solution and a physical rarefaction fan. Moreover, it is technically inconvenient as it measures the error in phase-space coordinates while the functional Φ measures distance along shock coordinates. To resolve these difficulties, we introduce the notion of discrepancies for waves.

We begin by observing that for a smooth function V of x and t the residual coincides with the integral form of the conservation law

$$\begin{aligned}\int_{-l}^{l} R(V)\ dx &= \int_{-l}^{l} V_t + f(V)_x\ dx \\ &= \frac{d}{dt}\int_{-l}^{l} V\ dx + f(V)\Big|_{-l}^{l}.\end{aligned} \tag{2.3.1}$$

Since the differential expression $V_t + f(V)_x$ for conservation laws is derived from the integral form of the conservation law, we will assume that the residual for non-smooth functions is also given by identity (2.3.1). Consider now an isolated discontinuity located at $z(t)$ in the approximation v, with neighboring states v^- and v^+, and which for convenience we assume is initially located at the origin $z(0) = 0$. We now construct a discontinuity traveling with the $t = 0$ velocity of the discontinuity, namely

$$V(t,x) = \begin{cases} v^- & \text{for } x < \dot{z}(0)t, \\ v^+ & \text{for } x \geq \dot{z}(0)t. \end{cases}$$

Let V_r be the Riemann solution of (2.1.1) with initial data $V(0,\cdot)$. For any $l > 0$, the weak solution V_r satisfies the integral form of the conservation law

$$\int_{-l}^{l} R(V_r)\ dx = \frac{d}{dt}\int_{-l}^{l} V_r(t,x)\ dx - f\big(V_r(t,\cdot)\big)\Big|_{-l}^{l} = 0, \tag{2.3.2}$$

while in general

$$\int_{-l}^{l} R(V)\ dx = \frac{d}{dt}\int_{-l}^{l} V(t,x)\ dx - f\big(V(t,\cdot)\big)\Big|_{-l}^{l} \neq 0. \tag{2.3.3}$$

If l is sufficiently large then for $t > 0$ the fluxes $f(V)$ and $f(V_r)$ agree at the boundaries of $[-l, l]$. Subtracting (2.3.2) from (2.3.3), extending the domain of integration to the real line, and then performing a change of variable into self-similar coordinates $\xi = x/t$, we obtain

$$
\begin{aligned}
\int_{-l}^{l} R(V)\, dx &= \frac{d}{dt}\int_{-l}^{l} V(t,x)\, dx - f\big(V(t,\cdot)\big)\Big|_{-l}^{l} \\
&= \frac{d}{dt}\int_{-l}^{l} \Big(V(t,x) - V_r(t,x)\Big)\, dx \\
&= \frac{d}{dt}\int_{-\infty}^{\infty} \Big(V(t,x) - V_r(t,x)\Big)\, dx \qquad (2.3.4) \\
&= \frac{d}{dt}\int_{-\infty}^{\infty} \Big(V(\xi) - V_r(\xi)\Big)\, t\, d\xi \\
&= \int_{-\infty}^{\infty} \Big(V(\xi) - V_r(\xi)\Big)\, d\xi.
\end{aligned}
$$

The absolute value of the residual $|\int R\; dx|$ is clearly zero for isolated rarefaction waves in front-tracking approximations. For a discontinuity at an arbitrary position $z \in \mathbb{R}$, by a change of variable, the discontinuity can be centered at the origin and the self-similar functions V and V_r can be constructed from the knowledge of the states of v neighboring the origin. The following can therefore be computed entirely from the approximation v, i.e. the quantity is computable a posteriori.

Definition 2.6 *Define the phase-space discrepancy of the discontinuity located at z at time t, to be*

$$
D_z^*(t) = \int_{-\infty}^{\infty} ||V(\xi) - V_r(\xi)||\; d\xi, \qquad (2.3.5)
$$

where $||\cdot||$ is the euclidean norm in phase-space.

To measure the discrepancy in wave coordinates, introduce the coordinates $\widetilde{p}_k(\xi)$ satisfying

$$
\begin{aligned}
V_r(\xi) &= T_n\big(\widetilde{p}_n(\xi)\big) \circ \cdots \circ T_1\big(\widetilde{p}_1(\xi)\big)\big(V(\xi)\big), \text{ for } \xi < \dot{z}(0), \qquad (2.3.6) \\
V(\xi) &= T_n\big(\widetilde{p}_n(\xi)\big) \circ \cdots \circ T_1\big(\widetilde{p}_1(\xi)\big)\big(V_r(\xi)\big), \text{ for } \xi \geq \dot{z}(0).
\end{aligned}
$$

To treat shock waves consistently, their strengths must be measured along the shock curves S_k both before and after the point $\xi = \cot z$. This explains why the order in which we measure the distance between V_r and V must be reversed as we cross the line $\xi = \dot{z}$. This procedure also identifies the components of the discontinuity belonging to each underlying wave.

Definition 2.7 *Define the discrepancy of the wave* $\alpha = (z, k)$ *to be*

$$D_\alpha(t) = \int_{-\infty}^{\infty} |\widetilde{p}_k(\xi)| \, d\xi. \tag{2.3.7}$$

The topology defined by the phase-space and wave coordinates are equivalent. More precisely, there exists constants c_1 and c_2 such that for any two states, u and v, we have

$$\begin{aligned} c_1||u-v|| &\leq \sum_{k=1}^{n} |p_k(u,v)| \leq c_2||u-v||, \\ c_1||u-v|| &\leq \sum_{k=1}^{n} |q_k(u,v)| \leq c_2||u-v||. \end{aligned} \tag{2.3.8}$$

These bounds imply that for any discontinuity at $z \in \mathbb{R}$,

$$c_1 D_z^*(t) \leq \sum_{k=1}^{n} D_{(z,k)}(t) \leq c_2 D_z^*(t). \tag{2.3.9}$$

In fact, the integrand $\widetilde{p}_k$ in (2.3.7) can be computed as if α was an isolated wave. If v_α^- and v_α^+ are respectively the left and right hand states of the wave $\alpha = (z, k)$, as given in Definition 2.3, then define

$$\widetilde{V}(t,x) = \begin{cases} v_\alpha^- & \text{for } x < \dot{z}(0)t, \\ v_\alpha^+ & \text{for } x \geq \dot{z}(0)t, \end{cases}$$

and let $\widetilde{V}_r$ be the solution to the Riemann problem with initial data $\widetilde{V}(0,\cdot)$. It is now easy to verify that the integrand $\widetilde{p}_k$, originally introduced in 2.3.6, also satisfies

$$\begin{aligned} \widetilde{V}_r(\xi) &= T_k\big(\widetilde{p}_k(\xi)\big)\big(\widetilde{V}(\xi)\big), \text{ for } \xi < \dot{z}(0), \\ \widetilde{V}(\xi) &= T_k\big(\widetilde{p}_k(\xi)\big)\big(\widetilde{V}(\xi)\big), \text{ for } \xi \geq \dot{z}(0). \end{aligned} \tag{2.3.10}$$

Lemma 2.4 *If* $\alpha \in \mathcal{W}(v(t))$ *is a shock wave or a linearly degenerate wave then*

$$D_\alpha(t) = |\sigma_\alpha|\big|\dot{x}_\alpha - s_{k_\alpha}(v_\alpha^-, v_\alpha^+)\big|. \tag{2.3.11}$$

If α *is a rarefaction wave then*

$$D_\alpha(t) = \begin{cases} \frac{1}{2}\big(\lambda_{k_\alpha}(v_\alpha^-) - \dot{x}_\alpha\big)^2 + \frac{1}{2}\big(\dot{x}_\alpha - \lambda_{k_\alpha}(v_\alpha^+)\big)^2, & \textit{if } \dot{x}_\alpha \in [\lambda_{k_\alpha}(v_\alpha^-), \lambda_{k_\alpha}(v_\alpha^+)] \\ |\sigma_\alpha|\big|\dot{x}_\alpha - \frac{1}{2}\big(\lambda_{k_\alpha}(v_\alpha^-) + \lambda_{k_\alpha}(v_\alpha^+)\big)\big|, & \textit{otherwise.} \end{cases} \tag{2.3.12}$$

Proof Without loss of generality, we may assume that $\big(x_\alpha(t), t\big) = (0,0)$ and that α is isolated, i.e. $v^+ = v^+_\alpha$ and $v^- = v^-_\alpha$. Let α be a shock wave. The parameterization (2.1.7) implies that for $s_{k_\alpha}(v^-, v^+) < \dot{x}_\alpha$,

$$\widetilde{p}_{k_\alpha}(\xi) = \begin{cases} 0, & \text{if } \xi < s_{k_\alpha}(v^-, v^+) \\ \sigma_\alpha, & \text{if } s_{k_\alpha}(v^-, v^+) < \xi < \dot{x}_\alpha \\ 0, & \text{if } \dot{x}_\alpha < \xi. \end{cases}$$

From this, we verify that

$$D_\alpha(t) = \int_{-\infty}^{\infty} |\widetilde{p}_{k_\alpha}(\xi)| \, d\xi = |\sigma_\alpha| \big| \dot{x}_\alpha - s_{k_\alpha}(v^-, v^+) \big|.$$

A similar calculation demonstrates the result when $\dot{x}_\alpha \leq s_{k_\alpha}(v^-, v^+)$.

Suppose now that α is a rarefaction wave. We begin by assuming that $\dot{x}_\alpha \in \big[\lambda_{k_\alpha}(v^-), \lambda_{k_\alpha}(v^+)\big]$. The parameterization of rarefaction curves (2.1.7) implies that when $\lambda_{k_\alpha}(v^-) \leq \xi \leq \dot{x}_\alpha$

$$\widetilde{p}_{k_\alpha}(\xi) = \lambda_{k_\alpha}\big(V_r(\xi)\big) - \lambda_{k_\alpha}(v^-) = \xi - \lambda_{k_\alpha}(v^-),$$

and when $\dot{x}_\alpha \leq \xi \leq \lambda_{k_\alpha}(v^+)$

$$\widetilde{p}_{k_\alpha}(\xi) = \lambda_{k_\alpha}(v^+) - \lambda_{k_\alpha}\big(V_r(\xi)\big) = \lambda_{k_\alpha}(v^+) - \xi.$$

With these identities, we compute

$$\begin{aligned} D_{k_\alpha}(t) &= \int_{-\infty}^{\infty} |\widetilde{p}_{k_\alpha}(\xi)| \, d\xi \\ &= \int_{\lambda_{k_\alpha}(v^-)}^{\dot{x}_\alpha} |\xi - \lambda_{k_\alpha}(v^-)| \, d\xi + \int_{\dot{x}_\alpha}^{\lambda_{k_\alpha}(v^+)} |\lambda_{k_\alpha}(v^+) - \xi| \, d\xi \\ &= \frac{1}{2}(\dot{x}_\alpha - \lambda_{k_\alpha}(v^-))^2 + \frac{1}{2}(\lambda_{k_\alpha}(v^+) - \dot{x}_\alpha)^2. \end{aligned}$$

Suppose now that $\dot{x}_\alpha < \lambda_{k_\alpha}(v^-)$.

$$\begin{aligned} D_{k_\alpha}(t) &= \int_{-\infty}^{\infty} |\widetilde{p}_{k_\alpha}(\xi)| \, d\xi \\ &= \int_{\dot{x}_\alpha}^{\lambda_{k_\alpha}(v^-)} |\sigma_\alpha| \, d\xi + \int_{\lambda_{k_\alpha}(v^-)}^{\lambda_{k_\alpha}(v^+)} \big|\lambda_{k_\alpha}(v^+) - \xi\big| \, d\xi \end{aligned}$$

$$
\begin{aligned}
&= |\sigma_\alpha|\big|\dot{x}_\alpha - \lambda_{k_\alpha}(v^-)\big| + \frac{1}{2}\big(\lambda_{k_\alpha}(v^-) - \lambda_{k_\alpha}(v^+)\big)^2 \\
&= |\sigma_\alpha|\big|\dot{x}_\alpha - \lambda_{k_\alpha}(v^-)\big| + |\sigma_\alpha|\Big(\frac{1}{2}\big(\lambda_{k_\alpha}(v^-) + \lambda_{k_\alpha}(v^+)\big) - \lambda_{k_\alpha}(v^-)\Big) \\
&= |\sigma_\alpha|\Big|\dot{x}_\alpha - \frac{1}{2}\big(\lambda_{k_\alpha}(v^-) + \lambda_{k_\alpha}(v^+)\big)\Big| \qquad (2.3.13)
\end{aligned}
$$

A similar proof holds when $\lambda_{k_\alpha}(v^+) \leq \dot{x}_\alpha$. ■

As we remarked earlier in (2.1.28), the wave speeds satisfy $|\dot{x}_\alpha| \leq \Lambda$, $\forall\alpha$. Using our explicit forms of the discrepancy, we can verify that

$$D_\alpha(t) = \mathcal{O}(1)|\sigma_\alpha|, \forall\alpha \in \mathcal{W}, \qquad (2.3.14)$$

where $\mathcal{O}(1)$ is a function with an upper bound in Ω computable entirely from the system (2.1.1). For rarefaction waves, we will need the more precise estimates of the next lemma.

Lemma 2.5 *Assume that α is a rarefaction wave. For any speed $\dot{x}_\alpha$ we have*

$$|\sigma_\alpha|\Big|\dot{x}_\alpha - \frac{1}{2}\big(\lambda_{k_\alpha}(v_\alpha^-) + \lambda_{k_\alpha}(v_\alpha^+)\big)\Big| \leq D_\alpha(t) \qquad (2.3.15)$$

with equality precisely when $\dot{x}_\alpha \notin \big(\lambda_{k_\alpha}(v_\alpha^-), \lambda_{k_\alpha}(v_\alpha^+)\big)$. Moreover,

$$\inf_{\dot{x}_\alpha} D_\alpha(t) = \frac{|\sigma_\alpha|^2}{4}. \qquad (2.3.16)$$

Proof

We will assume that the wave is isolated and hence, that $v^+ = v_\alpha^+$ and $v^- = v_\alpha^-$. Refer to Figure 4 for an illustration of the discrepancy $D_\alpha(t)$. From Lemma 2.4, we verify that the left and right hand sides of (2.3.15) coincide when $\dot{x}_\alpha \notin \big(\lambda_{k_\alpha}(v^-), \lambda_{k_\alpha}(v^+)\big)$. For $\dot{x}_\alpha \in \big(\lambda_{k_\alpha}(v^-), \lambda_{k_\alpha}(v^+)\big)$ we evaluate the derivative of both sides of expression (2.3.15)

$$
\begin{aligned}
\frac{d}{d\dot{x}_\alpha}\Big(|\sigma_\alpha|\Big|\dot{x}_\alpha - \frac{1}{2}\big(\lambda_{k_\alpha}(v^-) + \lambda_{k_\alpha}(v^+)\big)\Big|\Big) &= \pm|\sigma_\alpha|, \qquad (2.3.17) \\
\frac{d}{d\dot{x}_\alpha}D_\alpha(t) &= 2\Big(\dot{x}_\alpha - \frac{1}{2}\big(\lambda_{k_\alpha}(v^-) + \lambda_{k_\alpha}(v^+)\big)\Big). \qquad (2.3.18)
\end{aligned}
$$

For $\lambda_{k_\alpha}(v^-) < \dot{x}_\alpha < \big(\lambda_{k_\alpha}(v^-) + \lambda_{k_\alpha}(v^+)\big)/2$, these derivatives evaluate to

$$-|\sigma_\alpha| < \frac{d}{d\dot{x}_\alpha}D_\alpha(t)$$

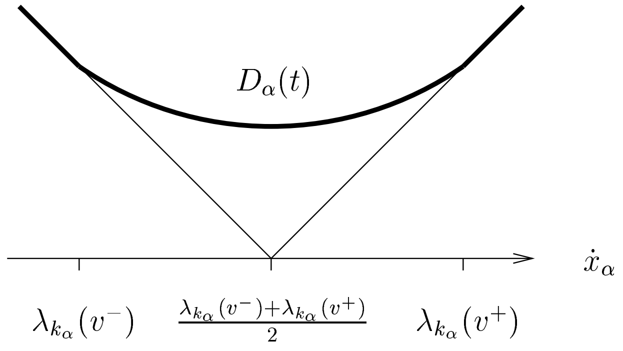

Figure 4: The bold line is the graph of $D_\alpha(t)$ as a function of $\dot{x}_\alpha$.

and the desired inequality (2.3.15) is clearly strict. For $\big(\lambda_{k_\alpha}(v^-)+\lambda_{k_\alpha}(v^+)\big)/2 < \dot{x}_\alpha < \lambda_{k_\alpha}(v^+)$, the derivative of the discrepancy is strictly smaller than the derivative of the left hand side of (2.3.15),

$$\frac{d}{d\dot{x}_\alpha} D_\alpha(t) < |\sigma_\alpha|. \tag{2.3.19}$$

Since both terms agree at $\dot{x}_\alpha = \lambda_{k_\alpha}(v^+)$, inequality (2.3.15) must be strict or else, by the mean value theorem, inequality (2.3.19) would be violated for at least one value of $\dot{x}_\alpha$.

According to formula (2.3.18) the minimum of $D_\alpha(t)$ occurs at

$$\dot{x}_\alpha = \big(\lambda_{k_\alpha}(v^-) + \lambda_{k_\alpha}(v^+)\big)/2.$$

Explicitly evaluating equation (2.3.12) at this minimum demonstrates (2.3.16). ■

3 Main Result

In this section we provide a proof the global a posteriori error estimate while assuming certain local estimates. Section 4 is dedicated to the proof of those local estimates. Since the global to local reduction is straightforward, it is convenient to introduce it before the technical proof of the local estimates.

Let u and v be two admissible front-tracking approximations of Lemma 2.3 and fix a time $t > 0$. Recall from Definition 2.2 that $\mathcal{D}\big(u(t)\big)$ is the set of all points of discontinuity in the approximation u at time t. The functional Φ is discontinuous but decreasing when the discontinuities interact. As mentioned in section 2.2, at times when discontinuities meet the weights W_i decrease and therefore so does Φ. When no interactions occur, Φ is a differentiable function of time and we compute

$$\begin{aligned} &\frac{d}{dt}\Big(\Phi\big(u(t),v(t)\big)\Big) \\ &\qquad = \sum_{\substack{z\in\mathcal{D}(u(t))\bigcup \\ \mathcal{D}(v(t))}} \sum_{k=1}^{n} \Big\{ \big|q_k(z-)\big| W_k(z-) - \big|q_k(z+)\big| W_k(z+) \Big\} \cdot \dot{z} \\ &\qquad = - \sum_{\substack{\delta\in\mathcal{D}(u(t))\bigcup \\ \mathcal{D}(v(t))}} \sum_{k=1}^{n} \big[\ |q_k| W_k\, \dot{z}\ \big]_z . \end{aligned} \tag{3.1}$$

Recall the definition of the intermediate states $w_0(x) = u(t,x)$,

$$w_k(x) = S_k\big(q_k(x)\big) \circ \cdots \circ S_1\big(q_1(x)\big)\big(u(t,x)\big),$$

and write

$$s_k(x) = s_k\big(w_k(x), w_{k-1}(x)\big) \tag{3.2}$$

where $s_k(\cdot,\cdot)$ is the k-th eigenvalue of the matrix $A(\cdot,\cdot)$ defined in (2.1.3). For simplicity, we have suppressed the time dependence. Consider two neighboring discontinuities located at $z < y$. The approximations are constant over the interval (z, y) and for every x in that interval we have

$$\big|q_k(z+)\big| s_k(z+) W_k(z+) = \big|q_k(x)\big| s_k(x) W_k(x) = \big|q_k(y-)\big| s_k(y-) W_k(y-). \tag{3.3}$$

For each interval between two neighboring discontinuities and for each k, add and subtract the terms in (3.3) to the sum (3.1). Then for each discontinuity, collect one term from each neighboring interval to obtain a contribution $\big[|q_k| s_k W_k\big]_z$. Since there exists a finite number of discontinuities, this leaves two terms unaccounted for over the intervals extending respectively to positive and negative infinity. Over these two intervals, $q_k(x)$ must be identically zero since the difference of the approximations $u(t) - v(t)$ belong to $L^1(\mathbb{R}, \Omega)$. The two remaining terms are therefore zero. Following this procedure, (3.1) can be rewritten as

$$\frac{d}{dt}\Big(\Phi\big(u(t),v(t)\big)\Big) = \sum_{z\in\mathcal{D}(u(t))\bigcup\mathcal{D}(v(t))}\sum_{k=1}^{n}\big[\ |q_k|W_k(s_k-\dot{z})\ \big]_z$$
$$= \sum_{z\in\mathcal{D}(u(t))\bigcup\mathcal{D}(v(t))}\sum_{k=1}^{n}E_k(z) \tag{3.4}$$

where we have introduced the terms

$$E_k(z) = \big|q_k(z+)\big|W_k(z+)\big(s_k(z+)-\dot{z}\big)-\big|q_k(z-)\big|W_k(z-)\big(s_k(z-)-\dot{z}\big). \tag{3.5}$$

We now present upper bounds for the local quantities $E_k(z)$ which can be computed a posteriori. In the following, $\mathcal{O}(1)$ will denote a quantity whose absolute value satisfies a uniform bound depending only on the system (2.1.1) and Ω.

Lemma 3.1 *Consider the strictly hyperbolic system* (2.1.1) *with genuinely nonlinear or linearly degenerate families in a neighborhood Ω of the origin. There exist positive constants $\delta_0, \kappa_1, \kappa_2$, depending only on the system* (2.1.1) *and Ω, with the following property. For any pair of front-tracking approximations u and v satisfying*

$$||u(0)||_{\mathrm{TV}},\ ||v(0)||_{\mathrm{TV}} < \delta_0,\ \textit{and}$$
$$u(0),\ v(0)\in L^1(\mathbb{R},\Omega), \tag{3.6}$$

and any position z and time t, we have

$$\sum_{k=1}^{n}E_k(z)\leq\mathcal{O}(1)\sum_{\{\alpha|x_\alpha=z\}}D_\alpha(t). \tag{3.7}$$

In contract to [4], where the local estimate for ϵ-approximations is of order $\mathcal{O}(1)\epsilon|\sigma_\alpha|$, we explicitly identify the approximation error and relate it to the residuals of u and v. Our global a posteriori error estimate is an immediate consequence of this local estimate.

Theorem 3.1 *Under the hypothesis of Lemma 3.1, there exists a constant C, depending only on the system* (2.1.1) *and on Ω, such that at any time t*

$$\Phi\big(u(t),v(t)\big)\leq\Phi\big(u(0),v(0)\big)+C\int_0^t\Big[\sum_{\substack{\alpha\in\mathcal{W}(u(s))\bigcup\\ \mathcal{W}(v(s))}}D_\alpha(s)\Big]\,ds. \tag{3.8}$$

Proof The coefficient $\mathcal{O}(1)$ in (3.7) is bounded over all waves by some constant C depending only on (2.1.1). Replace $\mathcal{O}(1)$ by C in inequality (3.7). Use (3.7) in equation (3.4) and integrate over time to obtain the final result. ■

Theorem 3.1 can be translated into a bound in the L^1 norm for the difference between an approximate solution and a weak solution.

Theorem 3.2 *Consider the strictly hyperbolic system* (2.1.1) *with genuinely nonlinear or linearly degenerate families in a neighborhood* Ω *of the origin. There exists positive constants* δ_0 *and* C*, depending only on* (2.1.1) *and* Ω*, with the following property. If* u *is any front-tracking approximation with* $u(0) \in L^1(\mathbb{R}, \Omega)$ *and* v *is any weak solution in* $L^\infty\big([0,t], L^1(\mathbb{R}, \Omega)\big)$ *which is the limit in* L^1 *of a sequence of* ϵ*-approximations and moreover, if both* u *and* v *satisfy*

$$||u(0)||_{\mathrm{TV}},\ ||v(0)||_{\mathrm{TV}} < \delta_0, \tag{3.9}$$

then we have the estimate

$$||u(t) - v(t)||_{L^1} \le C||u(0) - v(0)||_{L^1} + C\int_0^t \Big[\sum_{z \in \mathcal{D}(u(s))} D_z^*(s)\Big]\, ds. \tag{3.10}$$

Proof Let δ_0 be the constant given in Lemma 3.1. Let $\{v_\epsilon\}$ be the sequence of ϵ-approximations which converge in L^1 to the weak solution v. Since $||v(0)||_{\mathrm{TV}} < \delta_0$, we can assume that the approximations also satisfy this total variation bound. According to Theorem 3.1, for all ϵ we have

$$\Phi\big(u(t), v_\epsilon(t)\big) \le \Phi\big(u(0), v_\epsilon(0)\big) + C\int_0^t \Big[\sum_{\substack{\alpha \in \mathcal{W}(u(s)) \bigcup \\ \mathcal{W}(v_\epsilon(s))}} D_\alpha(s)\Big]\, ds. \tag{3.11}$$

Recall the bound (2.2.2) and the definition of the positive constants c_i in (2.3.8). Using this in (3.11), we find

$$c_1||u(t) - v_\epsilon(t)||_{L^1} \le 2c_2||u(0) - v_\epsilon(0)||_{L^1} + c_2 C\int_0^t \Big[\sum_{\substack{z \in \mathcal{D}(u(s)) \bigcup \\ \mathcal{D}(v_\epsilon(s))}} D_z^*(s)\Big]\, ds. \tag{3.12}$$

To demonstrate that the sum of the discrepancies of v_ϵ vanish as $\epsilon \to 0$, recall the form of the upper estimate in (3.7) originally found in [4],

$$\mathcal{O}(1) \sum_{\alpha \in \mathcal{W}(v_\epsilon(s))} D_\alpha(s) = \mathcal{O}(1) \sum_{\alpha \in \mathcal{W}(v_\epsilon(s))} \epsilon|\sigma_\alpha| \le \mathcal{O}(1)\epsilon||v_\epsilon(s)||_{\mathrm{TV}}.$$

Taking the limit of (3.12) as $\epsilon \to 0$, the ϵ-approximations v_ϵ converge to v and the discrepancies of v_ϵ vanish. Dividing by c_1 completes the result. ■

4 Local Estimates

The goal of this section is to demonstrate Lemma 3.1. In Section 4.1, we present estimates for wave interactions in shock coordinates and some lemmas characterizing quadratic quantities appearing during the proof. The proof of the lemma for isolated waves is split into estimates for $E_k(\alpha)$ when $k \neq k_\alpha$ and $k = k_\alpha$. These estimates are treated in sections 4.2 and 4.3. In the last section, general discontinuities with non-isolated waves are discussed.

Throughout section 4, we will assume that the discontinuities under discussion belong to the approximate solution v, omitting the similar arguments for those in u. Since the time t and the wave α will be fixed for most of these computations, whenever convenient, we will remove the subscripts and superscripts involving the wave α. In particular, our notation $q_i\big(x_\alpha(t)\pm, t\big) = q_i^{\alpha\pm}$ will be simplified to $q_i^\pm$.

4.1 Interaction Estimates

We begin with a lemma from Bressan, Liu, and Yang [4] concerning wave interactions.

Lemma 4.1 *If α is an isolated shock wave then*

$$\begin{aligned} |q_{k_\alpha}^+ - q_{k_\alpha}^- - \sigma_\alpha| &+ \sum_{i \neq k_\alpha} |q_i^+ - q_i^-| \\ &= \mathcal{O}(1)\Big(|\sigma_\alpha||q_{k_\alpha}^-|\big(|\sigma_\alpha| + |q_{k_\alpha}^-|\big) + |\sigma_\alpha| \sum_{i \neq k_\alpha} |q_i^-|\Big) \qquad (4.1.1) \\ &= \mathcal{O}(1)\Big(|\sigma_\alpha||q_{k_\alpha}^+|\big(|\sigma_\alpha| + |q_{k_\alpha}^+|\big) + |\sigma_\alpha| \sum_{i \neq k_\alpha} |q_i^+|\Big). \qquad (4.1.2) \end{aligned}$$

If α is an isolated rarefaction wave then

$$\begin{aligned} |q_{k_\alpha}^+ - q_{k_\alpha}^- - \sigma_\alpha| &+ \sum_{i \neq k_\alpha} |q_i^+ - q_i^-| \\ &= \mathcal{O}(1)\Big(|\sigma_\alpha|^3 + |\sigma_\alpha||q_{k_\alpha}^-|\big(|\sigma_\alpha| + |q_{k_\alpha}^-|\big) + |\sigma_\alpha| \sum_{i \neq k_\alpha} |q_i^-|\Big) \qquad (4.1.3) \\ &= \mathcal{O}(1)\Big(|\sigma_\alpha|^3 + |\sigma_\alpha||q_{k_\alpha}^+|\big(|\sigma_\alpha| + |q_{k_\alpha}^+|\big) + |\sigma_\alpha| \sum_{i \neq k_\alpha} |q_i^+|\Big). \qquad (4.1.4) \end{aligned}$$

If α is an isolated linearly degenerate wave then

$$\begin{aligned} |q_{k_\alpha}^+ - q_{k_\alpha}^- - \sigma_\alpha| + \sum_{i \neq k_\alpha} |q_i^+ - q_i^-| &= \mathcal{O}(1)|\sigma_\alpha| \sum_{i \neq k_\alpha} |q_i^-| \qquad (4.1.5) \\ &= \mathcal{O}(1)|\sigma_\alpha| \sum_{i \neq k_\alpha} |q_i^+|. \qquad (4.1.6) \end{aligned}$$

The following is a slight improvement to the previous estimates and can be derived in the same manner.

Lemma 4.2 *If α is an isolated shock wave and $-\sigma_\alpha \geq q_{k_\alpha}^- \geq 0$ then*

$$\begin{aligned} |q_{k_\alpha}^+ - q_{k_\alpha}^- - \sigma_\alpha| &+ \sum_{i \neq k_\alpha} |q_i^+ - q_i^-| \\ &= \mathcal{O}(1)\Big(|\sigma_\alpha||q_{k_\alpha}^-|\big(|\sigma_\alpha| - |q_{k_\alpha}^-|\big) + |\sigma_\alpha| \sum_{i \neq k_\alpha} |q_i^-|\Big). \end{aligned} \qquad (4.1.7)$$

If α is an isolated rarefaction wave and $\sigma_\alpha \geq -q_{k_\alpha}^- \geq 0$ then

$$\begin{aligned} |q_{k_\alpha}^+ - q_{k_\alpha}^- - \sigma_\alpha| &+ \sum_{i \neq k_\alpha} |q_i^+ - q_i^-| \\ &= \mathcal{O}(1)\Big(|\sigma_\alpha|^3 + |\sigma_\alpha||q_{k_\alpha}^+|\big(|\sigma_\alpha| - |q_{k_\alpha}^+|\big) + |\sigma_\alpha| \sum_{i \neq k_\alpha} |q_i^+|\Big). \end{aligned} \qquad (4.1.8)$$

The next estimate follows directly from (4.1.1) of Lemma 4.1. This weak form of the interaction estimates will be used in Lemma 4.4.

Lemma 4.3 *For any states $u, v, w \in \Omega$, we have*

$$\sum_{i=1}^n \big|q_i(u,v)\big| \leq \mathcal{O}(1)\Big(\sum_{i=1}^n \big|q_i(u,w)\big| + \big|q_i(u,w)\big|\Big). \qquad (4.1.9)$$

During the proof of Lemma 3.1 the possibly large value of κ_1 will be mitigated by other factors appearing in the expansions.

Lemma 4.4 *For any positive value of κ_1, there exists a positive constant δ_1, depending only on κ_1 and the system* (2.1.1)*, such that if u and v are admissible front-tracking approximations satisfying*

$$||u(0)||_{L^\infty} + ||v(0)||_{\mathrm{TV}},\ ||v(0)||_{L^\infty} + ||v(0)||_{\mathrm{TV}} < \delta_1, \tag{4.1.10}$$

then

$$\begin{aligned}
\kappa_1 \sum_{i=1}^{n} |q_i(x)| &\leq 1, \qquad \forall x \in \mathbb{R}, && (4.1.11)\\
\kappa_1 |\sigma_\alpha| &\leq 1, \qquad \forall \alpha \in \mathcal{W}\big(v(t)\big). && (4.1.12)
\end{aligned}$$

Proof Let δ and K be the positive constants of Lemma 2.1. Let δ_0 be the constant of Lemma 2.3, which depends only on the given κ_1 and the system (2.1.1). We begin by assuming that condition (4.1.10) holds for

$$\delta_1 \leq \min\{\delta, \delta_0\}. \tag{4.1.13}$$

Fix the time t, the position x, and abbreviate $u(t,x) = u$, $v(t,x) = v$. From (4.1.9) and the equivalence of the p_i and q_i coordinates, we have

$$\begin{aligned}
\kappa_1 \sum_{i=1}^{n} \big|q_i(x)\big| &= \kappa_1 \sum_{i=1}^{n} \big|q_i(u,v)\big| \\
&\leq \kappa_1 \mathcal{O}(1) \left(\sum_{i=1}^{n} \big|q_i(u,0)\big| + \big|q_i(0,v)\big| \right) \\
&= \kappa_1 \mathcal{O}(1) \left(\sum_{i=1}^{n} \big|p_i(u,0)\big| + \big|p_i(0,v)\big| \right). \qquad (4.1.14)
\end{aligned}$$

Notice that since both approximations belong to L^1, they must vanish at infinity. The distance in wave coordinates between u and 0 is therefore bounded by the total variation of the approximation between $-\infty$ and x. Introduce a constant C_1 bounding the $\mathcal{O}(1)$ coefficient in (4.1.14) and depending only on the system (2.1.1). Applying our first observation and using C_1, we obtain

$$
\begin{aligned}
&\kappa_1 \mathcal{O}(1)\Big(\sum_{i=1}^{n}\big|p_i(u,0)\big| + \big|p_i(0,v)\big|\Big) \\
&\qquad\qquad \le \kappa_1 C_1\Big(\sum_{i=1}^{n}\big|p_i(u,0)\big| + \big|p_i(0,v)\big|\Big) \\
&\qquad\qquad \le \kappa_1 C_1\Big(V\big(u(t)\big) + V\big(v(t)\big)\Big) \\
&\qquad\qquad \le \kappa_1 C_1\Big[\Big(V\big(u(t)\big) + V\big(v(t)\big)\Big) + K\Big(Q\big(u(t)\big) + Q\big(v(t)\big)\Big)\Big]
\end{aligned}
\tag{4.1.15}
$$

Note that there exists a constant C_2 such that $V(\cdot) \le C_2||\cdot||_{\mathrm{TV}}$ and that $V(\cdot) \le Q(\cdot)^2$. We now further restrict the value of δ_1 by setting

$$
\delta_1 = \min\Big\{\delta, \delta_0, \frac{1}{KC_2}, \frac{1}{2\kappa_1 C_2}, \frac{1}{4\kappa_1 C_1 C_2}\Big\}. \tag{4.1.16}
$$

According to Lemma 2.1, $V + KQ$ is a decreasing functional. Continuing from (4.1.15) and using the bounds (4.1.10) and (4.1.16), we conclude that

$$
\begin{aligned}
\kappa_1 \sum_{i=1}^{n}\big|q_i(x)\big| &\le \kappa_1 C_1\Big[\Big(V\big(u(t)\big) + V\big(v(t)\big)\Big) + K\Big(Q\big(u(t)\big) + Q\big(v(t)\big)\Big)\Big] \\
&\le \kappa_1 C_1\Big[\Big(V\big(u(0)\big) + V\big(v(0)\big)\Big) + K\Big(Q\big(u(0)\big) + Q\big(v(0)\big)\Big)\Big] \\
&\le \kappa_1 C_1\Big[\Big(V\big(u(0)\big) + V\big(v(0)\big)\Big) + K\Big(V\big(u(0)\big)^2 + V\big(v(0)\big)\Big)^2\Big] \\
&\le \kappa_1 C_1\Big[C_2\Big(\big|\big|u(0)\big|\big|_{\mathrm{TV}} + KC_2\big|\big|u(0)\big|\big|_{\mathrm{TV}}^2\Big) \\
&\qquad\qquad + C_2\Big(\big|\big|v(0)\big|\big|_{\mathrm{TV}} + KC_2\big|\big|v(0)\big|\big|_{\mathrm{TV}}^2\Big)\Big] \\
&\le 2\kappa_1 C_1 C_2\Big[\big|\big|u(0)\big|\big|_{\mathrm{TV}} + \big|\big|v(0)\big|\big|_{\mathrm{TV}}\Big] \le \frac{1}{2} + \frac{1}{2} = 1.
\end{aligned}
\tag{4.1.17}
$$

To demonstrate (4.1.12), it suffices to notice that

$$\begin{aligned}\kappa_1|\sigma_\alpha| &\le \kappa_1 V\big(v(t)\big)\\ &\le \kappa_1\Big(V\big(v(0)\big)+KQ\big(v(0)\big)\Big)\\ &\le \kappa_1\Big(\big|\big|v(0)\big|\big|_{\mathrm{TV}}+KC_2^2\big|\big|v(0)\big|\big|_{\mathrm{TV}}^2\Big)\\ &\le 2\kappa_1 C_2\big|\big|v(0)\big|\big|_{\mathrm{TV}}\le 1. \end{aligned}\tag{4.1.18}$$

■

The next three lemmas characterize similar expressions appearing in the proof of Lemma 3.1. The first estimate in Lemma 4.5 is taken from [4], with a slight improvement when the factors $\sigma+\sigma'=0$. Lemmas 4.6 and 4.7 explicitly relate quadratic expressions in the proof of Lemma 3.1 to the discrepancy of shock and rarefaction waves.

Lemma 4.5 *Choose $\overline{w}\in\Omega,\sigma,\sigma'\in\mathbb{R}$, $\quad k\in\{1,\ldots,n\}$. Define the states and the wave speeds*

$$\begin{array}{rclrcl} w &=& S_k(\sigma)(\overline{w}), & s &=& s_k(\overline{w},w),\\ w' &=& S_k(\sigma')(w), & s' &=& s_k(w,w'),\\ w'' &=& S_k(\sigma+\sigma')(\overline{w}), & s'' &=& s_k(\overline{w},w''). \end{array}$$

If the family k is genuinely nonlinear then

$$\big|(\sigma+\sigma')(s''-s')-\sigma(s-s')\big|=\mathcal{O}(1)|\sigma\sigma'|\cdot|\sigma+\sigma'|.\tag{4.1.19}$$

Proof Define the C^2 function

$$Q(\sigma,\sigma')\equiv(\sigma+\sigma')s''-\sigma s-\sigma's'=(\sigma+\sigma')(s''-s')-\sigma(s-s').$$

Since

$$Q(\sigma,0)=0\qquad\text{and}\qquad Q(0,\sigma')=0,\qquad\forall\sigma,\sigma',$$

only mixed derivatives occur in Taylor's expansion of Q. We compute the first mixed derivative

$$\frac{\partial Q}{\partial\sigma'}=s''+(\sigma+\sigma')\frac{\partial s''}{\partial\sigma'}-s'-\sigma'\frac{\partial s'}{\partial\sigma'},\tag{4.1.20}$$

$$\frac{\partial^2 Q}{\partial\sigma\partial\sigma'}=\frac{\partial s''}{\partial\sigma}+\frac{\partial s''}{\partial\sigma'}+(\sigma+\sigma')\frac{\partial^2 s''}{\partial\sigma\partial\sigma'}-\frac{\partial s'}{\partial\sigma}-\sigma'\frac{\partial^2 s'}{\partial\sigma\partial\sigma'}.\tag{4.1.21}$$

Using the identity (2.1.7) and the fact that the shock speed is to second order the average of the characteristic speeds of its neighboring states, we find

$$\begin{aligned}\frac{\partial s''}{\partial\sigma} &= \frac{\partial}{\partial\sigma}\Big[s_{k_\alpha}\big(\overline{w}, S_{k_\alpha}(\sigma+\sigma')(\overline{w})\big)\Big]\\ &= \frac{\partial}{\partial\sigma}\Big[\frac{1}{2}\Big(\lambda_{k_\alpha}(\overline{w}) + \lambda_{k_\alpha}\big(S_{k_\alpha}(\sigma+\sigma')(\overline{w})\big)\Big) + \mathcal{O}(1)|\sigma+\sigma'|\Big]\\ &= \frac{1}{2} + \frac{\partial}{\partial\sigma}\big(\mathcal{O}(1)|\sigma+\sigma'|^2\big). \end{aligned} \tag{4.1.22}$$

Evaluating at the origin,

$$\frac{\partial s''}{\partial\sigma}(0,0) = \frac{1}{2}. \tag{4.1.23}$$

Similarly,

$$\frac{\partial s''}{\partial\sigma'}(0,0) = \frac{1}{2}. \tag{4.1.24}$$

When $\sigma' = 0$, then $w' = S_{k_\alpha}(\sigma')(w) = w$. Combining this fact with the parameterization (2.1.7), we verify that

$$\begin{aligned}\frac{\partial s'}{\partial\sigma}(0,0) &= \frac{d}{d\sigma}\Big[s_{k_\alpha}(w,w')\big|_{\sigma'=0}\Big]\\ &= \frac{d}{d\sigma}\Big[s_{k_\alpha}(w,w)\Big]\\ &= \frac{d}{d\sigma}\Big[\lambda_{k_\alpha}\big(S_{k_\alpha}(\sigma)(\overline{w})\big)\Big] = 1. \end{aligned} \tag{4.1.25}$$

Evaluating the mixed derivative (4.1.21) at the origin, identities (4.1.23), (4.1.24), and (4.1.25) imply

$$\frac{\partial^2 Q}{\partial\sigma\partial\sigma'} = 0.$$

Integrate this derivative to compute

$$\begin{aligned}Q(\sigma,\sigma') = \int_0^{\sigma}\int_0^{\sigma'} \frac{\partial^2 Q}{\partial\sigma\partial\sigma'}(r,s)\ drds &= \mathcal{O}(1)\int_0^{|\sigma|}\int_0^{|\sigma'|} |rs|\ drds\\ &= \mathcal{O}(1)|\sigma\sigma'|\big(|\sigma|+|\sigma'|\big).\end{aligned}$$

In fact, when $\sigma+\sigma' = 0$ then $s - s' = 0$ and Q vanishes. Q is therefore

$$Q(\sigma,\sigma') = \mathcal{O}(1)|\sigma\sigma'|\cdot|\sigma+\sigma'|.$$

■

We now introduce two quadratic quantities, similar to (4.1.19), associated to shock and rarefaction waves. First, choose a shock wave α, a state $v^- \in \Omega$, and define

$$\widehat{v} = S_{k_\alpha}(-q)(v^-), \text{ and } v^* = S_{k_\alpha}(q+\sigma_\alpha)(\widehat{v}).$$

The quadratic shock remainder is the function

$$Q_{\rm shk}(q,s) \equiv -(q+\sigma_\alpha)\big(s_{k_\alpha}(\widehat{v},v^*) - s\big) - q\big(s_{k_\alpha}(v^-,\widehat{v}) - s\big), \tag{4.1.26}$$

for $(q,s) \in [0,|\sigma_\alpha|] \times \mathbb{R}$. If α is a rarefaction wave, v^+ is some fixed state of Ω and

$$\overline{v} = R_{k_\alpha}(-q)(v^+), \qquad v^- = R_{k_\alpha}(-\sigma_\alpha)(v^+),$$

then we define the quadratic rarefaction remainder

$$\begin{aligned} Q_{\rm rar}(q,s) \equiv & q\Big(\frac{1}{2}\big(\lambda_{k_\alpha}(v) + \lambda_{k_\alpha}(\overline{v})\big) - s\Big) \\ & - (\sigma_\alpha - q)\Big(\frac{1}{2}\big(\lambda_{k_\alpha}(\overline{v}) + \lambda_{k_\alpha}(v^-)\big) - s\Big), \end{aligned} \tag{4.1.27}$$

for $(q,s) \in [0,\sigma_\alpha] \times \mathbb{R}$. Notwithstanding the weight W_{k_α}, these remainders and (4.1.19) are the contribution from the k_α-th coordinate to the term $E_{k_\alpha}(\alpha)$.

Lemma 4.6 *There exists a positive constant δ_2, depending only on the system* (2.1.1)*, such that if $|\sigma_\alpha| < \delta_2$ then*

$$\sup_{q\in[0,|\sigma_\alpha|]} Q_{shk}(q,\dot{x}_\alpha) = D_\alpha(t). \tag{4.1.28}$$

Proof Figure 5 illustrates the states in the construction of $Q_{\rm shk}$. Introduce the function

$$M(q) = \frac{1}{2}(q+\sigma_\alpha)s_{k_\alpha}(\widehat{v},v^*) + \frac{q}{2}s_{k_\alpha}(v^-,\widehat{v}) \tag{4.1.29}$$

for which

$$Q_{\rm shk}(q,s) = 2\Big[(q+\frac{1}{2}\sigma_\alpha)s - M(q)\Big]. \tag{4.1.30}$$

We begin by demonstrating that M is a concave function of q over the interval $\big[0,|\sigma_\alpha|\big]$. The parameterization (2.1.7) along shock curves provides two facts. First

$$\frac{d}{dq}\Big(\lambda_{k_\alpha}\big(\widehat{v}(q)\big)\Big) = \frac{d}{dq}\Big(\lambda_{k_\alpha}\big(S_{k_\alpha}(-q)(v^-)\big)\Big) = -1. \tag{4.1.31}$$

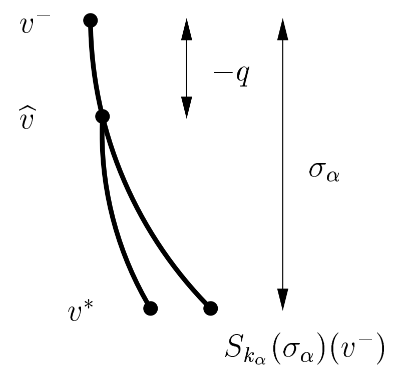

Figure 5: Illustration of the states defining Q_{shk}.

Secondly,

$$\begin{aligned}
\lambda_{k_\alpha}(\widehat{v}) - \lambda_{k_\alpha}(v^-) &= -q, & (4.1.32)\\
\lambda_{k_\alpha}(v^*) - \lambda_{k_\alpha}(\widehat{v}) &= q + \sigma_\alpha, & (4.1.33)\\
\lambda_{k_\alpha}(v^+) - \lambda_{k_\alpha}(v^-) &= \sigma_\alpha. & (4.1.34)
\end{aligned}$$

Adding equation (4.1.32) to (4.1.33) and then comparing the sum to (4.1.34), we conclude that

$$\lambda_{k_\alpha}(v^+) = \lambda_{k_\alpha}(v^*). \tag{4.1.35}$$

Using (4.1.35) and the fact that the Rankine-Hugoniot speeds are approximately the average of characteristic speeds of the end states, we obtain

$$\begin{aligned}
M(q) = \frac{1}{4}(q+\sigma_\alpha)\Big(\lambda_{k_\alpha}(\widehat{v}) + \lambda_{k_\alpha}(v^+)\Big) + \mathcal{O}(1)(q+\sigma_\alpha)^3\\
+ \frac{q}{4}\Big(\lambda_{k_\alpha}(v^-) + \lambda_{k_\alpha}(\widehat{v})\Big) + \mathcal{O}(1)q^3.
\end{aligned} \tag{4.1.36}$$

Concentrating on the lower order terms and using (4.1.31), (4.1.32), and (4.1.33) we have

$$\begin{aligned}
&\frac{d^2}{dq^2}\left[\frac{1}{4}(q+\sigma_\alpha)\Big(\lambda_{k_\alpha}(\widehat{v}) + \lambda_{k_\alpha}(v^+)\Big) + \frac{q}{4}\Big(\lambda_{k_\alpha}(v^-) + \lambda_{k_\alpha}(\widehat{v})\Big)\right]\\
&\quad = \frac{d}{dq}\left[\frac{1}{4}\big(\lambda_{k_\alpha}(v^-) + 2\lambda_{k_\alpha}(\widehat{v}) + \lambda_{k_\alpha}(v^+)\big) - \frac{1}{4}(q+\sigma_\alpha) - \frac{q}{4}\right]\\
&\quad = -1.
\end{aligned}$$

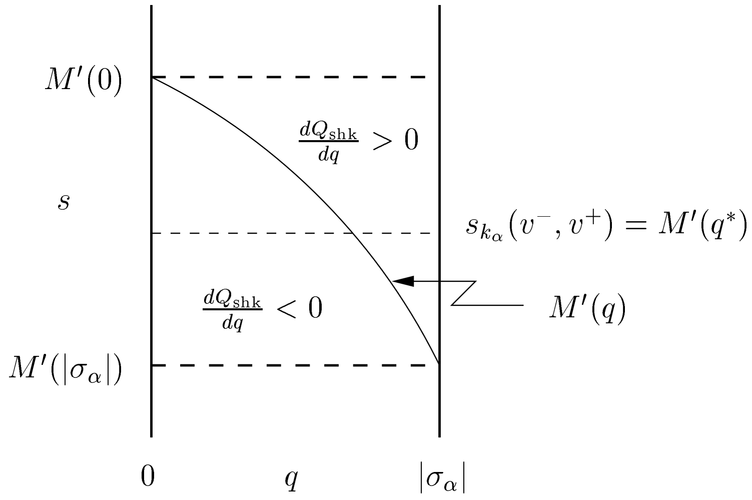

Figure 6: Behavior of Q_{shk} over the domain $\big[0, |\sigma_\alpha|\big] \times \mathbb{R}$.

Since the higher order terms are cubic in q and $q + \sigma_\alpha$, both of which are bounded by $|\sigma_\alpha|$, for $|\sigma_\alpha|$ sufficiently small, the concavity of M results from the concavity of the lower order terms. Notice that the bound on $|\sigma_\alpha|$ depends entirely on the $\mathcal{O}(1)$ coefficient which has upper bounds computable from the system (2.1.1). Thus M is concave when $|\sigma_\alpha| \leq \mathcal{O}(1)$, as asserted.

The derivative of Q_{shk} computed from identity (4.1.30) is

$$\frac{dQ_{\mathrm{shk}}}{dq} = 2\Big[s - \frac{dM}{dq}\Big]. \tag{4.1.37}$$

If M is concave then M' is monotonic decreasing over $\big[0, |\sigma_\alpha|\big]$. We complete the proof by considering three cases. First suppose $s \in \big[M'\big(|\sigma_\alpha|\big), M'(0)\big]$. Then by (4.1.37) the function $Q_{\mathrm{shk}}(\cdot, s)$ has a critical point. This critical point must be a global minimum in $q \in \big[0, |\sigma_\alpha|\big]$ and the maximum of the function $Q_{\mathrm{shk}}(\cdot, s)$ must occur at one of the boundaries. This situation is illustrated in Figure 6. Note that $\widehat{v}(0) = v^-$, $v^*(0) = v^+$ while $\widehat{v}\big(|\sigma_\alpha|\big) = v^+$. Using these identities to evaluate $Q_{\mathrm{shk}}(\cdot, s)$ at the boundary, we find that

$$\begin{aligned}\sup_{q \in [0, |\sigma_\alpha|]} Q_{\mathrm{shk}}(q, s) &= \max\Big\{Q_s(0, s), Q_{\mathrm{shk}}(|\sigma_\alpha|, s)\Big\} \\ &= \max\Big\{\sigma_\alpha\big(s - s_{k_\alpha}(v^-, v^+)\big), |\sigma_\alpha|\big(s - s_{k_\alpha}(v^-, v^+)\big)\Big\} \\ &= |\sigma_\alpha|\big|s - s_{k_\alpha}(v^-, v^+)\big|.\end{aligned}$$

This quantity is equal to $D_\alpha(t)$ when $s = \dot{x}_\alpha$.

When $s \in \big(M'(0), +\infty\big)$, equation (4.1.37) shows that the supremum of $Q_{\text{shk}}(\cdot, s)$ occurs at the boundary $q = |\sigma_\alpha|$. Computing this, we obtain

$$\sup_{q\in[0,|\sigma_\alpha|]} Q_{\text{shk}}(q,s) = Q_{\text{shk}}(|\sigma_\alpha|, s) = |\sigma_\alpha|\big(s - s_{k_\alpha}(v^-, v^+)\big). \tag{4.1.38}$$

We need to show that this quantity is in fact positive and therefore equal to $D_\alpha(t)$ when $s = \dot{x}_\alpha$. According to the mean value theorem, there exists $q^* \in \big(0, |\sigma_\alpha|\big)$ such that

$$\begin{aligned} M'(q^*) &= \frac{1}{|\sigma_\alpha|}\Big(M\big(|\sigma_\alpha|\big) - M(0)\Big) \\ &= \frac{1}{|\sigma_\alpha|}\Big(\frac{|\sigma_\alpha|}{2} - \frac{\sigma_\alpha}{2}\Big) s_{k_\alpha}(v^-, v^+) \\ &= s_{k_\alpha}(v^-, v^+). \end{aligned}$$

Since M' is monotonic decreasing, $s_{k_\alpha}(v^-, v^+) = M'(q^*) < M'(0) \leq s$ and therefore the supremum in (4.1.38) is positive and equal to the discrepancy. With a similar argument, we verify the lemma when $s \in \big(-\infty, M'(|\sigma_\alpha|)\big)$. This completes the proof. ■

Lemma 4.7 *With the definition* (4.1.27)*, we have*

$$\sup_{q\in[0,\sigma_\alpha]} Q_{rar}(q, \dot{x}_\alpha) = D_\alpha(t). \tag{4.1.39}$$

Proof To compute the maximum of Q_{rar} we begin by examining its derivative. The parameterization (2.1.7) along rarefaction curves and the definition of $\overline{v}$ imply that

$$\frac{d}{dq}\Big(\lambda_{k_\alpha}(\overline{v})\Big) = \frac{d}{dq}\Big(\lambda_{k_\alpha}\big(R_{k_\alpha}(-q)(v^+)\big)\Big) = -1.$$

Using this fact, we proceed to compute

$$\begin{aligned} \frac{\partial Q_{\text{rar}}}{\partial q} =& \Big[\frac{1}{2}\big(\lambda_{k_\alpha}(v^+) + \lambda_{k_\alpha}(\overline{v})\big) - s\Big] + q\Big(-\frac{1}{2}\Big) \\ &+ \Big[\frac{1}{2}\big(\lambda_{k_\alpha}(\overline{v}) + \lambda_{k_\alpha}(v^-)\big) - s\Big] - (\sigma_\alpha - q)\Big(-\frac{1}{2}\Big) \\ =& 2\Big[\frac{1}{4}\big(\lambda_{k_\alpha}(v^+) + 2\lambda_{k_\alpha}(\overline{v}) + \lambda_{k_\alpha}(v^-)\big) + \frac{1}{2}\big(\frac{1}{2}\sigma_\alpha - q\big) - s\Big]. \end{aligned} \tag{4.1.40}$$

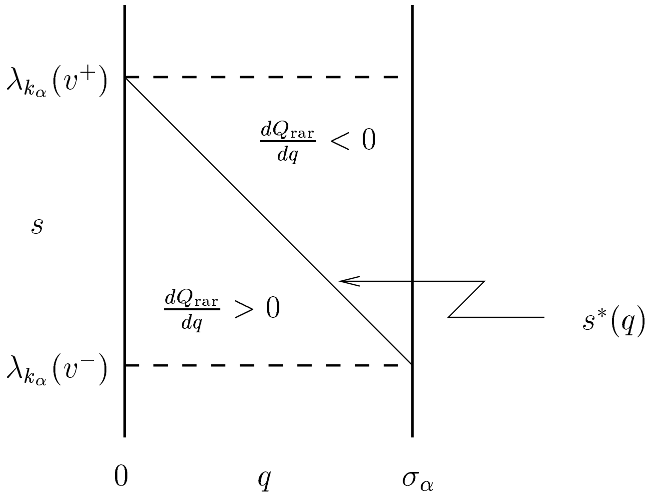

Figure 7: Behavior of $Q_{\rm rar}$ over the domain $\big[0, \sigma_\alpha\big] \times \mathbb{R}$.

Taking another derivative gives

$$\frac{\partial^2 Q_{\rm rar}}{\partial q^2} = -2. \tag{4.1.41}$$

With the help of the identity $\lambda_{k_\alpha}(v^+) - \lambda_{k_\alpha}(v^-) = \sigma_\alpha$, we compute

$$\begin{aligned}\frac{\partial Q_{\rm rar}}{\partial q}(0) &= \frac{1}{2}\Big(\lambda_{k_\alpha}(v^+) + 2\lambda_{k_\alpha}(v^+) + \lambda_{k_\alpha}(v^-)\Big) + \frac{1}{2}\sigma_\alpha - 2s \\ &= \frac{1}{2}\Big(\lambda_{k_\alpha}(v^+) + 2\lambda_{k_\alpha}(v^+) + \lambda_{k_\alpha}(v^+) - \sigma_\alpha\Big) + \frac{1}{2}\sigma_\alpha - 2s \\ &= 2\lambda_{k_\alpha}(v^+) - 2s. \end{aligned} \tag{4.1.42}$$

From (4.1.41) and (4.1.42), we conclude

$$\frac{\partial Q_{\rm rar}}{\partial q} = 2\lambda_{k_\alpha}(v^+) - 2s - 2q. \tag{4.1.43}$$

We need to find the supremum of $Q_{\rm rar}$ for all $q \in [0, \sigma_\alpha]$ and all $s \in \mathbb{R}$. Formula (4.1.43) shows that

$$\frac{\partial Q_{\rm rar}}{\partial q} > 0 \text{ when } s < \lambda_{k_\alpha}(v^+) - q, \tag{4.1.44}$$

and

$$\frac{\partial Q_{\text{rar}}}{\partial q} < 0 \text{ when } \lambda_{k_\alpha}(v^+) - q < s. \tag{4.1.45}$$

Since q is contained in $[0, \sigma_\alpha]$, we have

$$\lambda_{k_\alpha}(v^-) \leq \lambda_{k_\alpha}(v^+) - q \leq \lambda_{k_\alpha}(v^+). \tag{4.1.46}$$

We divide the remainder of the proof into three cases. Refer to Figure 7 for a representation of these regions in the (q, s) plane. If $\lambda_{k_\alpha}(v^+) < s$ then (4.1.45) and (4.1.46) show that the maximum of $Q_{\text{rar}}(\cdot, s)$ occurs when $q = 0$. For $s = \dot{x}_\alpha$, this is maximum is

$$\begin{aligned}\sup_{q\in[0,\sigma_\alpha]} Q_{\text{rar}}(q, \dot{x}_\alpha) = Q_{\text{rar}}(0, \dot{x}_\alpha) &= -\sigma_\alpha\Big(\frac{1}{2}\big(\lambda_{k_\alpha}(v^+) + \lambda_{k_\alpha}(v^-)\big) - \dot{x}_\alpha\Big)\\ &= D_\alpha(t).\end{aligned}$$

If $s = \dot{x}_\alpha < \lambda_{k_\alpha}(v^-)$, then (4.1.44) and (4.1.46) imply that

$$\begin{aligned}\sup_{q\in[0,\sigma_\alpha]} Q_{\text{rar}}(q, \dot{x}_\alpha) = Q_{\text{rar}}(\sigma_\alpha, \dot{x}_\alpha) &= \sigma_\alpha\Big(\frac{1}{2}\big(\lambda_{k_\alpha}(v^+) + \lambda_{k_\alpha}(v^-)\big) - \dot{x}_\alpha\Big)\\ &= D_\alpha(t).\end{aligned}$$

If $s \in I \equiv \big[\lambda_{k_\alpha}(v^-), \lambda_{k_\alpha}(v^+)\big]$, then the supremum of $Q_{\text{rar}}(\cdot, s)$ occurs when

$$\lambda_{k_\alpha}(v^+) - q = s.$$

Let $s^*(q) \equiv \lambda_{k_\alpha}(v^+) - q$. Notice that s^* parameterizes the interval I as q traverses $[0, \sigma_\alpha]$. We can therefore say that for $(q, s) \in [0, \sigma_\alpha] \times I$ the supremum of Q_{rar} occurs along the diagonal $\big(q, s^*(q)\big)$. Using identity (4.1.43), the derivative along the diagonal is

$$\frac{d}{dq}\Big(Q_{\text{rar}}\big(q, s^*(q)\big)\Big) = 2(q - \frac{1}{2}\sigma_\alpha). \tag{4.1.47}$$

We also verify that

$$\begin{aligned}Q_{\text{rar}}\big(0, s^*(0)\big) &= -\sigma_\alpha\Big(\frac{1}{2}\big(\lambda_{k_\alpha}(v^+) + \lambda_{k_\alpha}(v^-)\big) - \lambda_{k_\alpha}(v^+)\Big)\\ &= \frac{\sigma_\alpha^2}{2}.\end{aligned} \tag{4.1.48}$$

Consider now the discrepancy of the rarefaction wave α for $\dot{x}_\alpha \in I$, which in Lemma 2.4 was found to be

$$D_\alpha(t) = \frac{1}{2}\big(\lambda_{k_\alpha}(v_\alpha^-) - \dot{x}_\alpha\big)^2 + \frac{1}{2}\big(\dot{x}_\alpha - \lambda_{k_\alpha}(v_\alpha^+)\big)^2. \tag{4.1.49}$$

Parameterizing the $\dot{x}_\alpha \in I$ by $s^*(q)$ and using (4.1.49), we have

$$\begin{aligned}
\frac{d}{dq}D_\alpha(t) &= \frac{d}{ds^*}\Big(D_\alpha(t)\Big)\frac{ds^*}{dq} \\
&= \frac{d}{ds^*}\Big[\frac{1}{2}\big(\lambda_{k_\alpha}(v_\alpha^-) - s^*\big)^2 + \frac{1}{2}\big(s^* - \lambda_{k_\alpha}(v_\alpha^+)\big)^2\Big](-1) \\
&= \Big[(\lambda_{k_\alpha}(v^-) - \lambda_{k_\alpha}(v^+) + q) + (\lambda_{k_\alpha}(v^+) - \lambda_{k_\alpha}(v^+) + q)\Big] \\
&= 2(q - \frac{1}{2}\sigma_\alpha).
\end{aligned} \tag{4.1.50}$$

Finally, when $\dot{x}_\alpha = s^*(0)$

$$D_\alpha(t) = \frac{\sigma_\alpha^2}{2}. \tag{4.1.51}$$

As a function of q, $Q_{\text{rar}}\big(q, s^*(q)\big)$ and $D_\alpha(t)$ agree when $q = 0$, their derivatives agree throughout $[0, \sigma_\alpha]$, and therefore they must be equal. ■

4.2 Isolated Waves: $E_k(\alpha)$ when $k \neq k_\alpha$

In this section we demonstrate a preliminary estimate for the terms $E_k(\alpha)$ defined in (3.5). We assume that the waves are isolated and belong to the approximation v. In this case, their left and right hand states, as defined in 2.3, coincide with the left and right sided limits of v at $\dot{x}_\alpha$, denoted v^- and v^+. Lemmas 4.4 and 4.6 introduced two upper bounds on the total variation, δ_1 and δ_2. We therefore combine these bounds with the admissibility requirements of Lemma 2.3 and assume throughout this section that our approximations satisfy

$$||u(0)||_{L^\infty} + ||u(0)||_{\text{TV}},\ ||v(0)||_{L^\infty} + ||v(0)||_{\text{TV}} < \min\big\{\delta_0, \delta_1, \delta_2\big\}. \tag{4.2.1}$$

In Section 4.5, where non-isolated waves will be examined, traveling discontinuities contain n waves only one of which travels at a speed satisfying condition (2.1.19), $\dot{x}_\alpha \in \lambda_{k_\alpha}(\Omega)$. In anticipation of this fact, we present the results of this section for isolated waves α whose wave speeds are only required to satisfy

$$|\dot{x}_\alpha| \le \Lambda, \tag{4.2.2}$$

for the constant Λ of (2.1.28). Recall the definition (2.1.11) of the lower bound d on wave speed separation and on genuine nonlinearity.

Lemma 4.8 *For any isolated wave $\alpha \in \mathcal{W}\big(v(t)\big)$, any $k \ne k_\alpha$, and either sign $s \in \{+,-\}$, we have*

$$\begin{aligned} E_k(\alpha) \le& -\kappa_1 d|\sigma_\alpha||q_k^s| + \mathcal{O}(1)D_\alpha(t) \\ &+ \mathcal{O}(1)\Big(|\sigma_\alpha||q_{k_\alpha}^s|\big(|\sigma_\alpha| + |q_{k_\alpha}^s|\big) + |\sigma_\alpha|\sum_{i\ne k_\alpha}|q_i^s|\Big). \end{aligned} \tag{4.2.3}$$

If the isolated wave satisfies $q_{k_\alpha}^- q_{k_\alpha}^+ < 0$ then there exists a sign s for which

$$\begin{aligned} E_k(\alpha) \le& -\kappa_1 d|\sigma_\alpha||q_k^s| + \mathcal{O}(1)D_\alpha(t) \\ &+ \mathcal{O}(1)\Big(|\sigma_\alpha||q_{k_\alpha}^s|\big(|\sigma_\alpha| - |q_{k_\alpha}^s|\big) + |\sigma_\alpha|\sum_{i\ne k_\alpha}|q_i^s|\Big). \end{aligned} \tag{4.2.4}$$

If the wave belongs to a linearly degenerate family then,

$$E_k(\alpha) \le -\kappa_1 d|\sigma_\alpha||q_k^-| + D_\alpha(t) + \mathcal{O}(1)|\sigma_\alpha|\sum_{i\ne k_\alpha}|q_i^-|. \tag{4.2.5}$$

For these estimates, the wave speed $\dot{x}_\alpha$ is only required to satisfy (4.2.2).

Proof We begin with the proof of equation (4.2.3) for a shock wave α and s positive. The other cases are similar and will be discussed later. Since the wave is isolated, its left and right hand states are the neighboring states of the discontinuity v^- and v^+. Rewrite the contribution of the wave α to the derivative of Φ as

$$\begin{aligned} E_k(\alpha) =& |q_k^+|W_k^+(s_k^+ - \dot{x}_\alpha) - |q_k^-|W_k^-(s_k^- - \dot{x}_\alpha) \\ =& |q_k^+|W_k^+(s_k^+ - s_k^-) + \big(|q_k^+| - |q_k^-|\big)W_k^-(s_k^- - \dot{x}_\alpha) \\ &+ |q_k^+|(W_k^+ - W_k^-)(s_k^- - \dot{x}_\alpha). \end{aligned} \tag{4.2.6}$$

When $k \ne k_\alpha$, the first term is handled by combining the bound $W_k^+ \le 2$ with the simple bound $|s_k^+ - s_k^-| = \mathcal{O}(1)|\sigma_\alpha|$. This gives

$$|q_k^+|W_k^+(s_k^+ - s_k^-) = \mathcal{O}(1)|q_k^+||\sigma_\alpha|. \tag{4.2.7}$$

Since the absolute value of the speed $\dot{x}_\alpha$ is bounded by Λ which can be determined entirely from the system, the quantity $(s^-_{k_\alpha} - \dot{x}_\alpha)$ is $\mathcal{O}(1)$. Combining this with the bound (4.1.2) of Lemma 4.1, we obtain an estimate for the second term in (4.2.6).

$$\begin{aligned}\big(|q_k^+| - |q_k^-|\big) W_k^-(s_k^- - \dot{x}_\alpha) \leq & \mathcal{O}(1)|q_k^+ - q_k^-| \\ \leq & \mathcal{O}(1)\Big(|\sigma_\alpha||q^+_{k_\alpha}|\big(|\sigma_\alpha| + |q^+_{k_\alpha}|\big) + |\sigma_\alpha| \sum_{i\neq k_\alpha} |q_i^+|\Big)\end{aligned} \tag{4.2.8}$$

The third term of (4.2.6) remains to be bounded. We rewrite it as

$$\begin{aligned}|q_k^+|(W_k^+ - W_k^-)(s_k^- - \dot{x}_\alpha) = & |q_k^+|(W_k^+ - W_k^-)\big(s_k^- - s_{k_\alpha}(v^-, v^+)\big) \\ & + |q_k^+|(W_k^+ - W_k^-)\big(s_{k_\alpha}(v^-, v^+) - \dot{x}_\alpha\big)\end{aligned} \tag{4.2.9}$$

When $k_\alpha > k$, equations (2.1.12) and (2.2.6) imply that

$$s_k^- + d < s_{k_\alpha}(v^-, v^+) \qquad \text{and} \qquad W_k^+ - W_k^- = \kappa_1|\sigma_\alpha| \ ,$$

while for $k_\alpha < k$, they give

$$s_{k_\alpha}(v^-, v^+) + d < s_k^- \qquad \text{and} \qquad W_k^+ - W_k^- = -\kappa_1|\sigma_\alpha| \ .$$

For both choices of k, the first term in (4.2.9) is negative. Using the estimate of Lemma 4.4, $\kappa_1|q_k^+| \leq 1$, (2.2.6), and (2.3.11) shows that the second term is bounded by the discrepancy. The last term in (4.2.6) therefore satisfies

$$|q_k^+|(W_k^+ - W_k^-)(s_k^- - \dot{x}_\alpha) \leq -\kappa_1 d|\sigma_\alpha||q_k^+| + D_\alpha(t). \tag{4.2.10}$$

To obtain the final estimate for $k \neq k_\alpha$, sum (4.2.7), (4.2.8), and (4.2.10). We have shown that

$$\begin{aligned}E_k(\alpha) \leq & -\kappa_1 d|\sigma_\alpha||q_k^+| + D_\alpha(t) \\ & + \mathcal{O}(1)\Big(|\sigma_\alpha||q^+_{k_\alpha}|\big(|\sigma_\alpha| + |q^+_{k_\alpha}|\big) + |\sigma_\alpha| \sum_{i\neq k_\alpha} |q_i^+|\Big).\end{aligned} \tag{4.2.11}$$

To demonstrate the result for a rarefaction wave, still with s positive, follow the same procedure as for shock waves. Estimates (4.2.7) and (4.2.10) remain the same. When estimating the second term in (4.2.6), use equation (4.1.4) of Lemma 4.1 to obtain the following replacement to (4.2.8)

$$\begin{aligned}(|q_k^+| - |q_k^-|)W_k^-(s_k^- - \dot{x}_\alpha) \leq& \mathcal{O}(1)|q_k^+ - q_k^-| \\ \leq& \mathcal{O}(1)\Big(|\sigma_\alpha|^3 + |\sigma_\alpha||q_{k_\alpha}^+|(|\sigma_\alpha| + |q_{k_\alpha}^+|) \\ &+ |\sigma_\alpha| \sum_{i \neq k_\alpha} |q_i^+|\Big). \end{aligned} \tag{4.2.12}$$

Summing (4.2.7), (4.2.9), and (4.2.12) leads to the introduction of a cubic term $|\sigma_\alpha|^3$ in the estimate

$$\begin{aligned} E_k(\alpha) \leq& -\kappa_1 d|\sigma_\alpha||q_k^+| + D_\alpha(t) \\ &+ \mathcal{O}(1)\Big(|\sigma_\alpha|^3 + |\sigma_\alpha||q_{k_\alpha}^+|\big(|\sigma_\alpha| + |q_{k_\alpha}^+|\big) + |\sigma_\alpha| \sum_{i \neq k_\alpha} |q_i^+|\Big). \end{aligned} \tag{4.2.13}$$

To obtain the form (4.2.3) of the result, we show how to control the cubic term with the discrepancy. In Lemma 2.5 we showed that the infimum of $D_\alpha(t)$ was $\sigma_\alpha^2/2$. Using this we have

$$\begin{aligned} \mathcal{O}(1)|\sigma_\alpha|^3 =& \left[\frac{\mathcal{O}(1)|\sigma_\alpha|^3}{D_\alpha(t)}\right] D_\alpha(t) \\ \leq& \mathcal{O}(1)|\sigma_\alpha| D_\alpha(t) = \mathcal{O}(1) D_\alpha(t). \end{aligned} \tag{4.2.14}$$

Using this in (4.2.13) proves (4.2.3).

We now outline the argument for a linearly degenerate wave α when s is positive. Starting from the expansion (4.2.6), we can obtain for the first term an identical estimate (4.2.7). With the help of equation (4.1.6), estimate (4.2.8) is now replaced by

$$\big(|q_k^+| - |q_k^-|\big)W_k^-(s_k^- - \dot{x}_\alpha) \leq \mathcal{O}(1)|\sigma_\alpha| \sum_{i \neq k_\alpha} |q_i^+|.$$

For linearly degenerate waves, (4.2.10) still holds. Together, these estimates demonstrate (4.2.5).

For a minus sign s, replace identity (4.2.6) by

$$\begin{aligned} E_k(\alpha) =& |q_k^-|W_k^-(s_k^+ - s_k^-) + \big(|q_k^+| - |q_k^-|\big)W_k^+(s_k^+ - \dot{x}_\alpha) \\ &+ |q_k^-|(W_k^+ - W_k^-)(s_k^+ - \dot{x}_\alpha). \end{aligned} \tag{4.2.15}$$

Follow the procedure outlined above for either shock, rarefaction, or linearly degenerate waves and use the interaction estimates (4.1.1), (4.1.3), and (4.1.5) of Lemma 4.1. The rest of the proof is identical.

The estimate of this lemma when $q^-_{k_\alpha} q^+_{k_\alpha} < 0$ also follows the same procedure but with the estimates of Lemma 4.2 now replacing those of Lemma 4.1. ■

4.3 Isolated Waves: $E_{k_\alpha}(\alpha)$ Bounds

In this section we demonstrate on estimate for $E_{k_\alpha}(\alpha)$ when α is an isolated wave. We continue to require the total variation bounds (4.2.1) and the condition (4.2.2) on the wave speeds.

Lemma 4.9 *For an isolated genuinely nonlinear wave $\alpha \in \mathcal{W}\big(v(t)\big)$ satisfying $q^-_{k_\alpha} q^+_{k_\alpha} \geq 0$, and for some sign s,*

$$\begin{aligned} E_{k_\alpha}(\alpha) \leq & -\kappa_1 d|\sigma_\alpha||q^s_{k_\alpha}|\big(|\sigma_\alpha| + |q^s_{k_\alpha}|\big) + \mathcal{O}(1)D_\alpha(t) \\ & + \mathcal{O}(1)\Big(|q^s_{k_\alpha}|\big(|\sigma_\alpha| + |q^s_{k_\alpha}|\big) + |\sigma_\alpha| \sum_{i \neq k_\alpha} |q^s_i|\Big). \end{aligned} \tag{4.3.1}$$

For an isolated genuinely nonlinear wave satisfying $q^-_{k_\alpha} q^+_{k_\alpha} < 0$, and some sign s,

$$\begin{aligned} E_{k_\alpha}(\alpha) \leq & -\kappa_1 d|\sigma_\alpha||q^s_{k_\alpha}|\big(|\sigma_\alpha| - |q^s_{k_\alpha}|\big) + \mathcal{O}(1)D_\alpha(t) \\ & + \mathcal{O}(1)\Big(|q^s_{k_\alpha}|\big(|\sigma_\alpha| - |q^s_{k_\alpha}|\big) + |\sigma_\alpha| \sum_{i \neq k_\alpha} |q^s_i|\Big). \end{aligned} \tag{4.3.2}$$

For an isolated linearly degenerate wave,

$$E_{k_\alpha}(\alpha) \leq 2D_\alpha(t) + \mathcal{O}(1)|\sigma_\alpha| \sum_{i \neq k_\alpha} |q^-_i|. \tag{4.3.3}$$

For these estimates, the wave speed $\dot{x}_\alpha$ is only required to satisfy (4.2.2).

We divide the proof between linearly degenerate and genuinely nonlinear waves while the latter are further subdivided into cases depending on the type of the wave and the relative ordering of the virtual waves $q^-_{k_\alpha}$, $q^+_{k_\alpha}$. This division is summarized in Table 4.3. We demonstrate the lemma for five of these cases, omitting cases 3 and 6 whose proofs are similar to cases 2 and 5.

Case	wave type	relative ordering	s	σ_α	Section
1	degenerate	NA	$-$	NA	4.3.1
2	shock	$0 < q^+_{k_\alpha} < q^-_{k_\alpha}$	$+$	$-$	4.3.2
3	shock	$q^+_{k_\alpha} < q^-_{k_\alpha} < 0$	$-$	$-$	NA
4	shock	$q^+_{k_\alpha} < 0 < q^-_{k_\alpha}$	$-$	$-$	4.3.3
5	rarefaction	$0 < q^-_{k_\alpha} < q^+_{k_\alpha}$	$-$	$+$	4.3.4
6	rarefaction	$q^-_{k_\alpha} < q^+_{k_\alpha} < 0$	$+$	$+$	NA
7	rarefaction	$q^-_{k_\alpha} < 0 < q^+_{k_\alpha}$	$+$	$+$	4.3.5

Table 1: Case structure for Lemma 4.9

4.3.1 Case 1: linearly degenerate

We assume that α is a wave belonging to a linearly degenerate family k_α. Refer to Figure 8 for a representation in phase-space of the important states. In this case, the characteristic speeds λ_{k_α} are constant along shock curves and therefore so are the Rankine-Hugoniot shock speeds. We use this observation to demonstrate the following two estimates. First,

$$\begin{aligned}\left|s^-_{k_\alpha} - s_{k_\alpha}(v^-, v^+)\right| =& \left|s_{k_\alpha}(w^-_{k_\alpha-1}, w^-_{k_\alpha}) - s_{k_\alpha}(v^-, v^+)\right| \\ =& \left|\lambda_{k_\alpha}(w^-_{k_\alpha}) - \lambda_{k_\alpha}(v^-)\right| \\ =& \mathcal{O}(1)|w^-_{k_\alpha} - v^-| \\ =& \mathcal{O}(1)\sum_{i\neq k_\alpha}|q^-_i|. \end{aligned} \tag{4.3.1.1}$$

Secondly, with the help of Lemma 4.1, we have

$$\begin{aligned}|s^+_{k_\alpha} - s^-_{k_\alpha}| =& \left|s_{k_\alpha}(w^+_{k_\alpha-1}, w^+_{k_\alpha}) - s_{k_\alpha}(w^-_{k_\alpha-1}, w^-_{k_\alpha})\right| \\ =& \left|\lambda_{k_\alpha}(w^+_{k_\alpha-1}) - \lambda_{k_\alpha}(w^-_{k_\alpha-1})\right| \\ =& \mathcal{O}(1)|w^+_{k_\alpha-1} - w^-_{k_\alpha-1}| \\ =& \mathcal{O}(1)|\sigma_\alpha|\sum_{i\neq k_\alpha}|q^-_i|. \end{aligned} \tag{4.3.1.2}$$

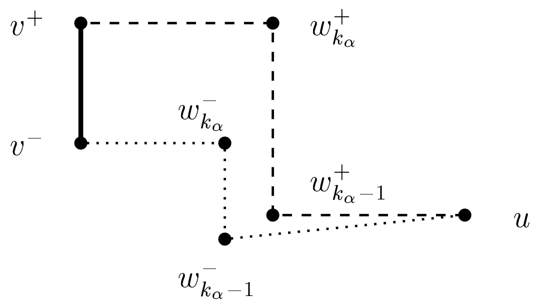

Figure 8: Illustration of the definitions for Case 1. The bold line represents the wave α. The dotted line represents the virtual waves q_i^- while the dashed lines represent the virtual waves q_i^+.

Using the identity $W^+_{k_\alpha} = W^-_{k_\alpha}$, we write

$$\begin{aligned}
E_{k_\alpha}(\alpha) =& W^+_{k_\alpha}|q^+_{k_\alpha}|(s^+_{k_\alpha} - \dot{x}_\alpha) - W^-_{k_\alpha}|q^-_{k_\alpha}|(s^-_{k_\alpha} - \dot{x}_\alpha) \\
=& W^-_{k_\alpha}\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha} - \dot{x}_\alpha) - |q^-_{k_\alpha}|(s^-_{k_\alpha} - \dot{x}_\alpha)\Big\} \\
=& W^-_{k_\alpha}\Big\{|\sigma_\alpha + q^-_{k_\alpha}|\Big[\big(s^+_{k_\alpha} - s^-_{k_\alpha}\big) + \big(s^-_{k_\alpha} - s_{k_\alpha}(v^-, v^+)\big) \\
& \qquad + \big(s_{k_\alpha}(v^-, v^+) - \dot{x}_\alpha\big)\Big] \\
& - |q^-_{k_\alpha}|\Big[\big(s^-_{k_\alpha} - s_{k_\alpha}(v^-, v^+)\big) + \big(s_{k_\alpha}(v^-, v^+) - \dot{x}_\alpha\big)\Big] \\
& + \big(|q^+_{k_\alpha}| - |\sigma_\alpha + q^-_{k_\alpha}|\big)(s^+_{k_\alpha} - \dot{x}_\alpha)\Big\} \\
=& W^-_{k_\alpha}\Big\{|\sigma_\alpha + q^-_{k_\alpha}|(s^+_{k_\alpha} - s^-_{k_\alpha}) + \big(|q^+_{k_\alpha}| - |\sigma_\alpha + q^-_{k_\alpha}|\big)(s^+_{k_\alpha} - \dot{x}_\alpha) \\
& + \big(|\sigma_\alpha + q^-_{k_\alpha}| - |q^-_{k_\alpha}|\big)(s^-_{k_\alpha} - s_{k_\alpha}(v^-, v^+)) \\
& + \big(|\sigma_\alpha + q^-_{k_\alpha}| - |q^-_{k_\alpha}|\big)(s_{k_\alpha}(v^-, v^+) - \dot{x}_\alpha)\Big\}.
\end{aligned} \tag{4.3.1.3}$$

To estimate these terms, observe $W^-_{k_\alpha} \leq 2$ by (2.2.2). According to Lemma 2.3, the approximations u and v remain in Ω for all t and therefore both $|\sigma_\alpha|$ and $|q^-_{k_\alpha}|$ are $\mathcal{O}(1)$ quantities. Combining these bounds with (4.3.1.2), we can estimate the first term in the braces as

$$|\sigma_\alpha + q^-_{k_\alpha}|(s^+_{k_\alpha} - s^-_{k_\alpha}) \leq \mathcal{O}(1)|\sigma_\alpha| \sum_{i \neq k_\alpha} |q_i^-|. \tag{4.3.1.4}$$

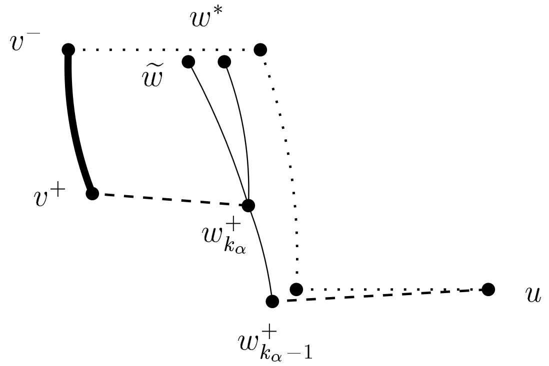

Figure 9: Illustration of the definitions (4.3.2.1) for Case 2. The solid line represents the wave curves joining the states in definitions (4.3.2.1).

In the second term, the quantity $(s^+_{k_\alpha} - \dot{x}_\alpha)$ is bounded since $|\dot{x}_\alpha| \leq \Lambda$. Estimate (4.1.5) of Lemma 4.1 then suffices to bound the second term.

$$\begin{aligned}\big(|q^+_{k_\alpha}| - |\sigma_\alpha + q^-_{k_\alpha}|\big)(s^+_{k_\alpha} - \dot{x}_\alpha) \leq & \mathcal{O}(1)|q^+_{k_\alpha} - q^-_{k_\alpha} - \sigma_\alpha| \\ = & \mathcal{O}(1)|\sigma_\alpha| \sum_{i \neq k_\alpha} |q^-_i|. \end{aligned} \tag{4.3.1.5}$$

The third term can be estimated with a combination of (4.3.1.1) and the triangle inequality

$$\big||\sigma_\alpha + q^-_{k_\alpha}| - |q^-_{k_\alpha}|\big| \leq |\sigma_\alpha|. \tag{4.3.1.6}$$

Using the same triangle inequality, and the explicit form of the residuals in Lemma 2.4, then after multiplying by W^-_k we can bound the last term by $2D_\alpha(t)$. Summarizing, we have shown that

$$E_{k_\alpha}(\alpha) \leq 2D_\alpha(t) + \mathcal{O}(1)|\sigma_\alpha| \sum_{i \neq k_\alpha} |q^-_i|. \tag{4.3.1.7}$$

This completes the proof of the first case.

4.3.2 Case 2: shock with $0 < q^+_{k_\alpha} < q^-_{k_\alpha}$

Assume that α is a shock wave with $0 < q^+_{k_\alpha} < q^-_{k_\alpha}$. We note that since the wave is isolated, v^- and v^+ equal the states to the left and right of the wave

α. As a preliminary for the first estimate we introduce the states and shock speeds

$$\begin{aligned} \widetilde{w} &= S_{k_\alpha}(q^+_{k_\alpha} - \sigma_\alpha)(w^+_{k_\alpha - 1}), & \widetilde{s} &= s_{k_\alpha}(w^+_{k_\alpha-1}, \widetilde{w}), \\ w^* &= S_{k_\alpha}(-\sigma_\alpha)(w^+_{k_\alpha}), & s^* &= s_{k_\alpha}(w^+_{k_\alpha}, w^*). \end{aligned} \tag{4.3.2.1}$$

Figure 9 illustrates these definitions in phase-space. The estimates we obtain involve q^+_i, i.e. the sign s is positive. The wave speeds $s^-_{k_\alpha} = s_{k_\alpha}(w^-_{k_\alpha-1}, w^-_{k_\alpha})$ and $\widetilde{s} = s_{k_\alpha}(w^+_{k_\alpha-1}, \widetilde{w})$ differ by the distance between $w^-_{k_\alpha-1}$ and $w^+_{k_\alpha-1}$ and by the length of the shock curves connecting these states to $w^-_{k_\alpha}$ and $\widetilde{w}$ respectively. This observation and the estimates of Lemma 4.1 demonstrate

$$\begin{aligned} |s^-_{k_\alpha} - \widetilde{s}| &= \mathcal{O}(1)\Big(|w^+_{k_\alpha-1} - w^-_{k_\alpha-1}| + |q^+_{k_\alpha} - q^-_{k_\alpha} - \sigma_\alpha|\Big) \\ &\le \mathcal{O}(1)\Big(|\sigma_\alpha||q^+_{k_\alpha}|\big(|\sigma_\alpha| + |q^+_{k_\alpha}|\big) + |\sigma_\alpha| \sum_{i \ne k_\alpha} |q^+_i|\Big). \end{aligned} \tag{4.3.2.2}$$

The shock speeds $s^* = s_{k_\alpha}(w^+_{k_\alpha}, w^*)$ and $s_{k_\alpha}(v^-, v^+)$ are both associated to states separated by shock curves of length $|\sigma_\alpha|$. We can therefore use the distance between the states $w^+_{k_\alpha}$ and v^+ to bound the difference

$$\big|s^* - s_{k_\alpha}(v^+, v^-)\big| = \mathcal{O}(1)|w^+_{k_\alpha} - v^+| = \mathcal{O}(1) \sum_{i \ne k_\alpha} |q^+_i|. \tag{4.3.2.3}$$

Recall property (2.1.13) of the positive constant d. If $q = q^+_{k_\alpha}$, $p = \sigma_\alpha$, $u = w^+_{k_\alpha-1}$, then that inequality becomes

$$d|q^+_{k_\alpha}| < s^* - \widetilde{s}. \tag{4.3.2.4}$$

Making use of the identity $W^+_{k_\alpha} - W^-_{k_\alpha} = \kappa_1 |\sigma_\alpha|$ from Lemma 2.2 and the fact that $q^-_{k_\alpha}$, $q^+_{k_\alpha}$ are positive and σ_α is negative, we expand

$$\begin{aligned} E_{k_\alpha}(\alpha) &= W^+_{k_\alpha}|q^+_{k_\alpha}|(s^+_{k_\alpha} - \dot{x}_\alpha) - W^-_{k_\alpha}|q^-_{k_\alpha}|(s^-_{k_\alpha} - \dot{x}_\alpha) \\ &= W^+_{k_\alpha}\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha} - \dot{x}_\alpha) - |q^-_{k_\alpha}|(s^-_{k_\alpha} - \dot{x}_\alpha)\Big\} \\ &\quad + \kappa_1|\sigma_\alpha||q^-_{k_\alpha}|(s^-_{k_\alpha} - \dot{x}_\alpha) \end{aligned}$$

$$
\begin{aligned}
&= W^+_{k_\alpha}\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha}-\dot{x}_\alpha)-|q^-_{k_\alpha}|(s^-_{k_\alpha}-\dot{x}_\alpha)\Big\}\\
&\qquad+\kappa_1|\sigma_\alpha|\big(|q^-_{k_\alpha}|-|\sigma_\alpha|-|q^+_{k_\alpha}|\big)(s^-_{k_\alpha}-\dot{x}_\alpha)\\
&\qquad+\kappa_1|\sigma_\alpha|\big(|\sigma_\alpha|+|q^+_{k_\alpha}|\big)(s^-_{k_\alpha}-\dot{x}_\alpha)\\
&\le W^+_{k_\alpha}\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha}-\dot{x}_\alpha)-|q^-_{k_\alpha}|(s^-_{k_\alpha}-\dot{x}_\alpha)\Big\}\\
&\qquad+\kappa_1|\sigma_\alpha|\big|q^+_{k_\alpha}-q^-_{k_\alpha}-\sigma_\alpha\big||s^-_{k_\alpha}-\dot{x}_\alpha|\\
&\qquad+\kappa_1|\sigma_\alpha|\big(|\sigma_\alpha|+|q^+_{k_\alpha}|\big)\Big(\big(s^-_{k_\alpha}-\widetilde{s}\big)+\big(\widetilde{s}-s^*\big)\\
&\qquad+\big(s^*-s_{k_\alpha}(v^-,v^+)\big)+\big(s_{k_\alpha}(v^-,v^+)-\dot{x}_\alpha\big)\Big).
\end{aligned}
\tag{4.3.2.5}
$$

There are three terms in this upper bound on $E_{k_\alpha}(\alpha)$. On the second term, we use the bound $\kappa_1|\sigma_\alpha|\le 1$ from Lemma 4.4 and equation (4.1.2) of Lemma 4.1. Together these show that

$$
\begin{aligned}
&\kappa_1|\sigma_\alpha|\big|q^+_{k_\alpha}-q^-_{k_\alpha}-\sigma_\alpha\big||s^-_{k_\alpha}-\dot{x}_\alpha|\\
&\qquad\qquad\le\mathcal{O}(1)\Big(|\sigma_\alpha||q^+_{k_\alpha}|\big(|\sigma_\alpha|+|q^+_{k_\alpha}\big)+|\sigma_\alpha|\sum_{i\ne k_\alpha}|q^+_i|\Big).
\end{aligned}
\tag{4.3.2.6}
$$

Notice that Lemma 4.4 provides us with the upper bound $\kappa_1\big(|\sigma_\alpha|+|q^+_{k_\alpha}|\big)\le 2$. Use identities (4.3.2.2), (4.3.2.3), and (4.3.2.4) on the third term of (4.3.2.5) to yield

$$
\begin{aligned}
&\kappa_1|\sigma_\alpha|\big(|\sigma_\alpha|+|q^+_{k_\alpha}|\big)\times\\
&\qquad\Big(\big(s^-_{k_\alpha}-\widetilde{s}\big)+\big(\widetilde{s}-s^*\big)+\big(s^*-s_{k_\alpha}(v^-,v^+)\big)+\big(s_{k_\alpha}(v^-,v^+)-\dot{x}_\alpha\big)\Big)\\
&\qquad\qquad\le\mathcal{O}(1)\Big(|\sigma_\alpha||q^+_{k_\alpha}|\big(|\sigma_\alpha|+|q^+_{k_\alpha}\big)+|\sigma_\alpha|\sum_{i\ne k_\alpha}|q^+_i|\Big)
\end{aligned}
\tag{4.3.2.7}
$$

$$
-\kappa_1 d|\sigma_\alpha||q^+_{k_\alpha}|\big(|\sigma_\alpha|+|q^+_{k_\alpha}|\big)+2|\sigma_\alpha|\big|s_{k_\alpha}(v^-,v^+)-\dot{x}_\alpha\big|.
$$

According to Lemma 2.4

$$
|\sigma_\alpha|\big|s_{k_\alpha}(v^-,v^+)-\dot{x}_\alpha\big|=D_\alpha(t). \tag{4.3.2.8}
$$

Using this fact in (4.3.2.7) and then combining the result with (4.3.2.6), we obtain

$$\begin{aligned}E_{k_\alpha}(\alpha) \leq & W^+_{k_\alpha}\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha} - \dot{x}_\alpha) - |q^-_{k_\alpha}|(s^-_{k_\alpha} - \dot{x}_\alpha)\Big\} \\ & + \mathcal{O}(1)\Big(|\sigma_\alpha||q^+_{k_\alpha}|\big(|\sigma_\alpha| + |q^+_{k_\alpha}|\big) + |\sigma_\alpha|\sum_{i\neq k_\alpha}|q^+_i|\Big) \\ & - \kappa_1 d|\sigma_\alpha||q^+_{k_\alpha}|\big(|\sigma_\alpha| + |q^+_{k_\alpha}|\big) + 2D_\alpha(t) \end{aligned} \tag{4.3.2.9}$$

From our knowledge of the signs of $q^+_{k_\alpha}, q^-_{k_\alpha}$, and σ_α, we expand the first term in expression (4.3.2.9).

$$\begin{aligned} & W^+_{k_\alpha}\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha} - \dot{x}_\alpha) - |q^-_{k_\alpha}|(s^-_{k_\alpha} - \dot{x}_\alpha)\Big\} \\ & \quad = W^+_{k_\alpha}\Big\{q^+_{k_\alpha}\Big[\big(s^+_{k_\alpha} - s^*\big) + \big(s^* - s_{k_\alpha}(v^-, v^+)\big) + \big(s_{k_\alpha}(v^-,v^+) - \dot{x}_\alpha\big)\Big] \\ & \qquad - (q^+_{k_\alpha} - \sigma_\alpha)\Big[\big(s^-_{k_\alpha} - \widetilde{s}\big) + \big(\widetilde{s} - s^*\big) \\ & \qquad + \big(s^* - s_{k_\alpha}(v^-, v^+)\big) + \big(s_{k_\alpha}(v^-,v^+) - \dot{x}_\alpha\big)\Big] \\ & \qquad - (q^-_{k_\alpha} - q^+_{k_\alpha} + \sigma_\alpha)(s^-_{k_\alpha} - \dot{x}_\alpha)\Big\} \\ & \quad = W^+_{k_\alpha}\Big\{q^+_{k_\alpha}(s^+_{k_\alpha} - s^*) - (q^+_{k_\alpha} - \sigma_\alpha)(\widetilde{s} - s^*)\Big\} \\ & \qquad + W^+_{k_\alpha}\Big\{ - (q^+_{k_\alpha} - \sigma_\alpha)(s^-_{k_\alpha} - \widetilde{s}) \\ & \qquad\quad + \sigma_\alpha\Big[\big(s^* - s_{k_\alpha}(v^-, v^+)\big) + \big(s_{k_\alpha}(v^-,v^+) - \dot{x}_\alpha\big)\Big] \\ & \qquad\quad - (q^-_{k_\alpha} - q^+_{k_\alpha} + \sigma_\alpha)(s^-_{k_\alpha} - \dot{x}_\alpha)\Big\} \end{aligned} \tag{4.3.2.10}$$

Lemma 4.5, with $\overline{w} = w^+_{k_\alpha - 1}$, $\sigma = q^+_{k_\alpha}$, and $\sigma' = -\sigma_\alpha$, followed by the bound (2.2.2), $W^+_{k_\alpha} \leq 2$, imply a bound on the first term in (4.3.2.10):

$$|W^+_{k_\alpha}|\Big|q^+_{k_\alpha}(s^+_{k_\alpha} - s^*) - (q^+_{k_\alpha} - \sigma_\alpha)(\widetilde{s} - s^*)\Big| \leq \mathcal{O}(1)|\sigma_\alpha||q^+_{k_\alpha}|\big(|\sigma_\alpha| + |q^+_{k_\alpha}|\big). \tag{4.3.2.11}$$

Applying Lemma 4.1, the bounds (4.3.2.2), (4.3.2.3), (4.3.2.11), (4.1.2), and our characterization (4.3.2.8) of the discrepancy, we can construct an upper

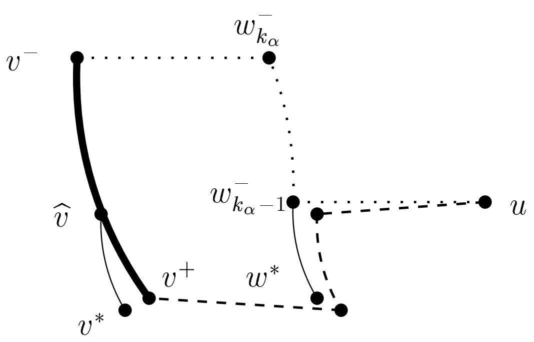

Figure 10: Illustration of the definitions (4.3.3.1) for case 4.

bound for (4.3.2.10),

$$\begin{aligned} W^+_{k_\alpha}\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha} - \dot{x}_\alpha) - |q^-_{k_\alpha}|(s^-_{k_\alpha} - \dot{x}_\alpha)\Big\} \\ \le \mathcal{O}(1)\Big(|\sigma_\alpha||q^+_{k_\alpha}|\big(|\sigma_\alpha| + |q^+_{k_\alpha}|\big) + |\sigma_\alpha|\sum_{i\neq k_\alpha}|q^+_i|\Big) + 2D_\alpha(t). \quad (4.3.2.12) \end{aligned}$$

Estimates (4.3.2.9) and (4.3.2.12) give

$$\begin{aligned} E_{k_\alpha}(\alpha) \le &- \kappa_1 d|\sigma_\alpha||q^+_{k_\alpha}|\big(|\sigma_\alpha| + |q^+_{k_\alpha}|\big) + 4D_\alpha(t) \\ &+ \mathcal{O}(1)\Big(|\sigma_\alpha||q^+_{k_\alpha}|\big(|\sigma_\alpha| + |q^+_{k_\alpha}|\big) + |\sigma_\alpha|\sum_{i\neq k_\alpha}|q^+_i|\Big). \quad (4.3.2.13) \end{aligned}$$

4.3.3 Case 4: shock with $q^+_{k_\alpha} < 0 < q^-_{k_\alpha}$

Assume that α is a shock wave for which $q^+_{k_\alpha} < 0 < q^-_{k_\alpha}$. The desired estimate will be obtained with a negative sign s.

We introduce the following states obtained from shock curves of the k_α-th family

$$\begin{aligned} w^* &= S_{k_\alpha}(q^-_{k_\alpha} + \sigma_\alpha)(w^-_{k_\alpha-1}), & s^* &= s_{k_\alpha}(w^-_{k_\alpha-1}, w^*), \\ \widetilde{w} &= S_{k_\alpha}(\sigma_\alpha)(w^-_{k_\alpha}), & \widetilde{s} &= s_{k_\alpha}(w^-_{k_\alpha}, \widetilde{w}), \\ \widehat{v} &= S_{k_\alpha}(-q^-_{k_\alpha})(v^-), & \widehat{\eta} &= s_{k_\alpha}(v^-, \widehat{v}), \\ v^* &= S_{k_\alpha}(q^-_{k_\alpha} + \sigma_\alpha)(\widehat{v}), & \eta^* &= s_{k_\alpha}(\widehat{v}, v^*). \end{aligned} \quad (4.3.3.1)$$

For the pairs of shock speeds used below, the states defining them are separated by the same distance along shock curves. Using this observation, we have

$$\begin{aligned}\left|\widetilde{s}-s_{k_\alpha}(v^-,v^+)\right| &= \left|s_{k_\alpha}(w^-_{k_\alpha},\widetilde{w})-s_{k_\alpha}(v^-,v^+)\right| \\ &= \mathcal{O}(1)|w^-_{k_\alpha}-v^-| \\ &= \mathcal{O}(1)\sum_{i\neq k_\alpha}|q_i^-|, \end{aligned} \tag{4.3.3.2}$$

$$\begin{aligned}|s^-_{k_\alpha}-\widehat{\eta}| &= \left|s_{k_\alpha}(w^-_{k_\alpha-1},w^-_{k_\alpha})-s_{k_\alpha}(v^-,\widehat{v})\right| \\ &= \mathcal{O}(1)|w^-_{k_\alpha}-v^-| \\ &= \mathcal{O}(1)\sum_{i\neq k_\alpha}|q_i^-|, \end{aligned} \tag{4.3.3.3}$$

$$\begin{aligned}|s^*-\eta^*| &= \left|s_{k_\alpha}(w^-_{k_\alpha-1},w^*)-s_{k_\alpha}(\widehat{v},v^*)\right| \\ &= \mathcal{O}(1)|w^-_{k_\alpha-1}-\widehat{v}| \\ &= \mathcal{O}(1)|w^-_{k_\alpha}-v^-| \\ &= \mathcal{O}(1)\sum_{i\neq k_\alpha}|q_i^-|. \end{aligned} \tag{4.3.3.4}$$

The same observation combined with the bounds of Lemma 4.2 leads to

$$\begin{aligned}|s^+_{k_\alpha}-s^*| &= \mathcal{O}(1)\Big(|w^+_{k_\alpha-1}-w^-_{k_\alpha-1}|+|q^+_{k_\alpha}-q^-_{k_\alpha}-\sigma_\alpha|\Big) \\ &= \mathcal{O}(1)\Big(|\sigma_\alpha||q^-_{k_\alpha}|\big(|\sigma_\alpha|-|q^-_{k_\alpha}|\big)+|\sigma_\alpha|\sum_{i\neq k_\alpha}|q_i^-|\Big). \end{aligned} \tag{4.3.3.5}$$

Consider property (2.1.14) of the constant d when $q=q^-_{k_\alpha}$, $p=-\sigma_\alpha-q^-_{k_\alpha}$, and $u=w^-_{k_\alpha}$. We find

$$d|\sigma_\alpha+q^-_{k_\alpha}|=d\big(|\sigma_\alpha|-|q^-_{k_\alpha}|\big)\leq s^-_{k_\alpha}-\widetilde{s}. \tag{4.3.3.6}$$

The approximations $u(t)$ are in L^1 for all t and therefore, being piecewise constant, the approximations must vanish outside some compact set. In the k_α-th coordinate, the sum of the increasing variation is then equal to the sum of the decreasing variation. In particular, for the wave α, we have

$$2|\sigma_\alpha|\leq \sum_{\substack{\beta\in\mathcal{W}(v)\\ k_\beta=k_\alpha}}|\sigma_\beta|. \tag{4.3.3.7}$$

Using this bound with (2.2.8), we find that

$$|\sigma_\alpha| \leq B^-_{k_\alpha} + B^+_{k_\alpha}. \tag{4.3.3.8}$$

We present a preliminary decomposition for $E_{k_\alpha}(\alpha)$ into three terms. Note that $A^+_{k_\alpha} = A^-_{k_\alpha}$ since the terms in (2.2.4) involve only waves from families other than k_α. Suppressing the time dependence, we have

$$\begin{aligned}
E_{k_\alpha}(\alpha) =& W^+_{k_\alpha}|q^+_{k_\alpha}|(s^+_{k_\alpha} - \dot{x}_\alpha) - W^-_{k_\alpha}|q^-_{k_\alpha}|(s^-_{k_\alpha} - \dot{x}_\alpha) \\
=& \Big(1 + \kappa_1\big(A^+_{k_\alpha} + B^+_{k_\alpha}\big) + \kappa_2\big(Q(u) + Q(v)\big)\Big)|q^+_{k_\alpha}|(s^+_{k_\alpha} - \dot{x}_\alpha) \\
&- \Big(1 + \kappa_1\big(A^-_{k_\alpha} + B^-_{k_\alpha}\big) + \kappa_2\big(Q(u) + Q(v)\big)\Big)|q^-_{k_\alpha}|(s^-_{k_\alpha} - \dot{x}_\alpha) \\
=& \Big(1 + \kappa_1 A^-_{k_\alpha} + \kappa_2\big(Q(u) + Q(v)\big)\Big)\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha} - \dot{x}_\alpha) - |q^-_{k_\alpha}|(s^-_{k_\alpha} - \dot{x}_\alpha)\Big\} \\
&- \kappa_1|q^-_{k_\alpha}|(s^-_{k_\alpha} - \dot{x}_\alpha)\Big(B^-_{k_\alpha} + B^+_{k_\alpha} - B^+_{k_\alpha}\Big) \\
&+ \kappa_1|q^+_{k_\alpha}|(s^+_{k_\alpha} - \dot{x}_\alpha)B^+_{k_\alpha} \\
=& \Big(1 + \kappa_1 A^-_{k_\alpha} + \kappa_2\big(Q(u) + Q(v)\big)\Big)\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha} - \dot{x}_\alpha) - |q^-_{k_\alpha}|(s^-_{k_\alpha} - \dot{x}_\alpha)\Big\} \\
&- \kappa_1|q^-_{k_\alpha}|(s^-_{k_\alpha} - \dot{x}_\alpha)\Big(B^-_{k_\alpha} + B^+_{k_\alpha}\Big) \\
&+ \kappa_1 B^+_{k_\alpha}\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha} - \dot{x}_\alpha) + |q^-_{k_\alpha}|(s^-_{k_\alpha} - \dot{x}_\alpha)\Big\}.
\end{aligned} \tag{4.3.3.9}$$

We begin by remarking that the bound $W^+_{k_\alpha} \leq 2$, from equation (2.2.2), implies that

$$\Big(1 + \kappa_1 A^-_{k_\alpha} + \kappa_2\big(Q(u) + Q(v)\big)\Big) \leq 2. \tag{4.3.3.10}$$

For the other half of the first term, using $q^+_{k_\alpha} \leq 0 \leq q^-_{k_\alpha}$, and $\sigma_\alpha < 0$, we have

$$\begin{aligned}
&|q^+_{k_\alpha}|(s^+_{k_\alpha} - \dot{x}_\alpha) - |q^-_{k_\alpha}|(s^-_{k_\alpha} - \dot{x}_\alpha) \\
&\quad = -(q^-_{k_\alpha} + \sigma_\alpha)\big((s^+_{k_\alpha} - s^*) + (s^* - \eta^*) + (\eta^* - \dot{x}_\alpha)\big) \\
&\qquad - q^-_{k_\alpha}\big((s^-_{k_\alpha} - \widehat{\eta}) + (\widehat{\eta} - \dot{x}_\alpha)\big) + (-q^+_{k_\alpha} + q^-_{k_\alpha} + \sigma_\alpha)(s^+_{k_\alpha} - \dot{x}_\alpha) \\
&\quad = -(q^-_{k_\alpha} + \sigma_\alpha)(\eta^* - \dot{x}_\alpha) - q^-_{k_\alpha}(\widehat{\eta} - \dot{x}_\alpha) \\
&\qquad - (q^-_{k_\alpha} + \sigma_\alpha)\big((s^+_{k_\alpha} - s^*) + (s^* - \eta^*)\big) \\
&\qquad - q^-_{k_\alpha}(s^-_{k_\alpha} - \widehat{\eta}) + (-q^+_{k_\alpha} + q^-_{k_\alpha} + \sigma_\alpha)(s^+_{k_\alpha} - \dot{x}_\alpha).
\end{aligned} \tag{4.3.3.11}$$

We recognize that the first two terms form the quadratic shock remainder $Q_{\mathrm{shk}}(q^-_{k_\alpha}, \dot{x}_\alpha)$ whose supremum over $q^-_{k_\alpha} \in [0, |\sigma_\alpha|]$ was shown in Lemma 4.6

to be precisely $D_\alpha(t)$. Notice that $|q^-_{k_\alpha}|, |q^-_{k_\alpha}+\sigma_\alpha|$ are both bounded above by $|\sigma_\alpha|$. Using this fact and identities (4.3.3.3), (4.3.3.4), and (4.3.3.5) we can produce upper bounds on the third and fourth terms. The last term in (4.3.3.11) can be bounded by the estimate (4.1.1) of Lemma 4.2. In this manner, we have demonstrated that the first term in (4.3.3.9) is bounded by

$$\Big(1+\kappa_1 A^-_{k_\alpha}+\kappa_2\big(Q(u)+Q(v)\big)\Big)\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha}-\dot{x}_\alpha)-|q^-_{k_\alpha}|(s^-_{k_\alpha}-\dot{x}_\alpha)\Big\}$$
$$\le 2D_\alpha(t)+\mathcal{O}(1)\Big(|\sigma_\alpha||q^-_{k_\alpha}|\big(|\sigma_\alpha|-|q^-_{k_\alpha}|\big)+|\sigma_\alpha|\sum_{i\neq k_\alpha}|q^-_i|\Big). \quad (4.3.3.12)$$

We now turn to the second term in (4.3.3.9) and rewrite it as

$$\begin{aligned}
&-\kappa_1|q^-_{k_\alpha}|(s^-_{k_\alpha}-\dot{x}_\alpha)\big(B^-_{k_\alpha}+B^+_{k_\alpha}\big)\\
&\quad=-\kappa_1|q^-_{k_\alpha}|(s^-_{k_\alpha}-\widetilde{s})\big(B^-_{k_\alpha}+B^+_{k_\alpha}\big)\\
&\quad-\kappa_1|q^-_{k_\alpha}|\Big(\big(\widetilde{s}-s_{k_\alpha}(v^+,v^-)\big)+\big(s_{k_\alpha}(v^+,v^-)-\dot{x}_\alpha\big)\Big)\big(B^-_{k_\alpha}+B^+_{k_\alpha}\big).
\end{aligned} \quad (4.3.3.13)$$

We note the inequalities $\kappa_1(B^-_{k_\alpha}+B^+_{k_\alpha})\le 2$ and $|q^-_{k_\alpha}|\le|\sigma_\alpha|$. Using these in conjunction with estimates (2.3.11), (4.3.3.2), (4.3.3.6), and (4.3.3.8), (4.3.3.13) can be estimated as

$$-\kappa_1|q^-_{k_\alpha}|(s^-_{k_\alpha}-\dot{x}_\alpha)\Big(B^-_{k_\alpha}+B^+_{k_\alpha}\Big)$$
$$\le -\kappa_1 d|\sigma_\alpha||q^-_{k_\alpha}|\big(|\sigma_\alpha|-|q^-_{k_\alpha}|\big)+2D_\alpha(t)+\mathcal{O}(1)|\sigma_\alpha|\sum_{i\neq k_\alpha}|q^-_i|. \quad (4.3.3.14)$$

We expand the remaining term in (4.3.3.9) using our assumption that $q^-_{k_\alpha}, -q^+_{k_\alpha}$, and $-\sigma_\alpha$ are positive.

$$\begin{aligned}
&\kappa_1 B^+_{k_\alpha}\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha}-\dot{x}_\alpha)+|q^-_{k_\alpha}|(s^-_{k_\alpha}-\dot{x}_\alpha)\Big\}\\
&\quad=\kappa_1 B^+_{k_\alpha}\Big\{-(q^-_{k_\alpha}+\sigma_\alpha)\Big[\big(s^+_{k_\alpha}-s^*\big)+\big(s^*-\widetilde{s}\big)\\
&\qquad\qquad+\big(\widetilde{s}-s_{k_\alpha}(v^+,v^-)\big)+\big(s_{k_\alpha}(v^+,v^-)-\dot{x}_\alpha\big)\Big]\\
&\qquad+q^-_{k_\alpha}\Big[\big(s^-_{k_\alpha}-\widetilde{s}\big)+\big(\widetilde{s}-s_{k_\alpha}(v^+,v^-)\big)+\big(s_{k_\alpha}(v^+,v^-)-\dot{x}_\alpha\big)\Big]\\
&\qquad-(q^+_{k_\alpha}-q^-_{k_\alpha}-\sigma_\alpha)(s^+_{k_\alpha}-\dot{x}_\alpha)\Big\}
\end{aligned}$$

$$
\begin{aligned}
=&\kappa_1 B^+_{k_\alpha}\Big\{ -(q^-_{k_\alpha}+\sigma_\alpha)(s^*-\widetilde{s}) + q^-_{k_\alpha}(s^-_{k_\alpha}-\widetilde{s}) \\
&\quad -(q^-_{k_\alpha}+\sigma_\alpha)(s^+_{k_\alpha}-s^*) - (q^+_{k_\alpha}-q^-_{k_\alpha}-\sigma_\alpha)(s^+_{k_\alpha}-\dot{x}_\alpha) \\
&\quad -\sigma_\alpha\Big[\big(\widetilde{s}-s_{k_\alpha}(v^+,v^-)\big)+\big(s_{k_\alpha}(v^+,v^-)-\dot{x}_\alpha\big)\Big]\Big\} \qquad (4.3.3.15)
\end{aligned}
$$

According to Lemma 2.3, the coefficient $\kappa_1 B^+_{k_\alpha}$ multiplying this expression is bounded by 2. Ignoring this coefficient, the first two terms are precisely expression (4.1.19) from Lemma 4.5 with $\overline{w}=w^-_{k_\alpha-1}$, $\sigma=q^-_{k_\alpha}$, and $\sigma'=\sigma_\alpha$. This shows that

$$
\big|-(q^-_{k_\alpha}+\sigma_\alpha)(s^*-\widetilde{s})+q^-_{k_\alpha}(s^-_{k_\alpha}-\widetilde{s})\big| = \mathcal{O}(1)|\sigma_\alpha||q^-_{k_\alpha}|\big(|\sigma_\alpha|-|q^-_{k_\alpha}|\big). \quad (4.3.3.16)
$$

The next three terms in (4.3.3.15) are controlled by (4.3.3.2), (4.3.3.5), and Lemma 4.2. The remaining term is the discrepancy of the wave α. Together, these demonstrate that

$$
\begin{aligned}
&\kappa_1 B^+_{k_\alpha}\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha}-\dot{x}_\alpha)+|q^-_{k_\alpha}|(s^-_{k_\alpha}-\dot{x}_\alpha)\Big\} \\
&\qquad \le 2D_\alpha(t)+\mathcal{O}(1)\Big(|\sigma_\alpha||q^-_{k_\alpha}|\big(|\sigma_\alpha|-|q^-_{k_\alpha}|\big)+|\sigma_\alpha|\sum_{i\neq k_\alpha}|q^-_i|\Big). \quad (4.3.3.17)
\end{aligned}
$$

The final estimate for $E_{k_\alpha}(\alpha)$ is the sum of the estimates (4.3.3.12), (4.3.3.14), and (4.3.3.17) for the three terms in (4.3.3.9). We have shown that in case 4, the estimate is

$$
\begin{aligned}
E_{k_\alpha}(\alpha) \le &-\kappa_1 d|\sigma_\alpha||q^-_{k_\alpha}|\big(|\sigma_\alpha|-|q^-_{k_\alpha}|\big)+6D_\alpha(t) \\
&+\mathcal{O}(1)\Big(|\sigma_\alpha||q^-_{k_\alpha}|\big(|\sigma_\alpha|-|q^-_{k_\alpha}|\big)+|\sigma_\alpha|\sum_{i\neq k_\alpha}|q^-_i|\Big). \qquad (4.3.3.18)
\end{aligned}
$$

4.3.4 Case 5: rarefaction with $0<q^-_{k_\alpha}<q^+_{k_\alpha}$

Suppose that α is a pure rarefaction wave with $0<q^-_{k_\alpha}<q^+_{k_\alpha}$. We define the intermediate states and speeds

$$
\begin{aligned}
\widetilde{w} &= S_{k_\alpha}(q^-_{k_\alpha}+\sigma_\alpha)(w^-_{k_\alpha-1}), & \widetilde{s} &= s_{k_\alpha}(w^-_{k_\alpha-1},\widetilde{w}), \\
w^* &= S_{k_\alpha}(\sigma_\alpha)(w^-_{k_\alpha}), & s^* &= s_{k_\alpha}(w^-_{k_\alpha},w^*),
\end{aligned} \qquad (4.3.4.1)
$$

and

$$
v^* = S_{k_\alpha}(\sigma_\alpha)(v^-). \qquad (4.3.4.2)
$$

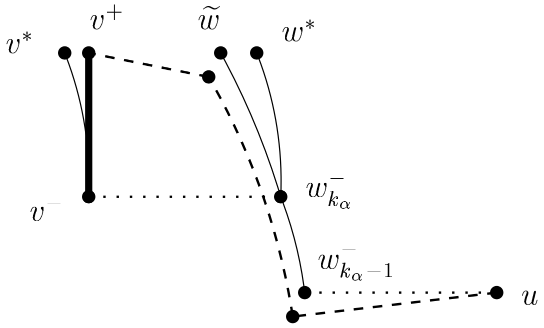

Figure 11: Illustration of definitions (4.3.4.1) for the fifth case.

These states are illustrated in Figure 11. From the definition of w^* and v^*, we have

$$\begin{aligned}\big|s^* - s_{k_\alpha}(v^-, v^*)\big| &= \big|s_{k_\alpha}(w^-_{k_\alpha}, w^*) - s_{k_\alpha}(v^-, v^*)\big| \\ &= \mathcal{O}(1)|w^-_{k_\alpha} - v^-| \\ &= \mathcal{O}(1)\sum_{i\neq k_\alpha}|q^-_i|. \end{aligned} \tag{4.3.4.3}$$

The same method is now used in conjunction with the estimates (4.1.3) of Lemma 4.1 to yield

$$\begin{aligned}|s^+_{k_\alpha} - \widetilde{s}| &= \big|s_{k_\alpha}(w^+_{k_\alpha-1}, w^+_{k_\alpha}) - s_{k_\alpha}(w^-_{k_\alpha-1}, \widetilde{w})\big| \\ &= \mathcal{O}(1)\Big(|w^+_{k_\alpha-1} - w^-_{k_\alpha-1}| + |q^+_{k_\alpha} - q^-_{k_\alpha} - \sigma_\alpha|\Big) \\ &= \mathcal{O}(1)\Big(|\sigma_\alpha|^3 + |\sigma_\alpha||q^-_{k_\alpha}|\big(|\sigma_\alpha| + |q^-_{k_\alpha}|\big) + |\sigma_\alpha|\sum_{i\neq k_\alpha}|q^-_i|\Big). \end{aligned} \tag{4.3.4.4}$$

Property (2.1.13) of the constant d shows that for $q = q^-_{k_\alpha}$, $p = \sigma_\alpha$, and $u = w^-_{k_\alpha-1}$

$$d|q^-_{k_\alpha}| < s^* - \widetilde{s}. \tag{4.3.4.5}$$

Using the fact that, to second-order, the shock speed is the average of the characteristic speeds of the states on the left and right hand sides, we have

$$\left|s_{k_\alpha}(v^-, v^*) - \frac{1}{2}\big(\lambda_{k_\alpha}(v^-) + \lambda_{k_\alpha}(v^*)\big)\right| = \mathcal{O}(1)|\sigma_\alpha|^2. \tag{4.3.4.6}$$

For $v^+ = R_{k_\alpha}(\sigma_\alpha)(v^-)$ and $v^* = S_{k_\alpha}(\sigma_\alpha)(v^-)$, our choice of parameterization (2.1.7) implies that

$$\lambda_{k_\alpha}(v^+) - \lambda_{k_\alpha}(v^-) = \sigma_\alpha = \lambda_{k_\alpha}(v^*) - \lambda_{k_\alpha}(v^-). \tag{4.3.4.7}$$

In particular,

$$\lambda_{k_\alpha}(v^+) = \lambda_{k_\alpha}(v^*). \tag{4.3.4.8}$$

Lemma 2.2 states that under the assumptions of Case 5, $W^+_{k_\alpha} - W^-_{k_\alpha} = \kappa_1|\sigma_\alpha|$. Using this fact, identity (4.3.4.8), and the positivity of $q^+_{k_\alpha}$, $q^-_{k_\alpha}$, and σ_α, we expand the local contribution to the derivative of Φ

$$\begin{aligned}
E_{k_\alpha}(\alpha) &= W^+_{k_\alpha}|q^+_{k_\alpha}|(s^+_{k_\alpha} - \dot{x}_\alpha) - W^-_{k_\alpha}|q^-_{k_\alpha}|(s^-_{k_\alpha} - \dot{x}_\alpha) \\
&= \kappa_1|\sigma_\alpha||q^+_{k_\alpha}|(s^+_{k_\alpha} - \dot{x}_\alpha) + W^-_{k_\alpha}\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha} - \dot{x}_\alpha) - |q^-_{k_\alpha}|(s^-_{k_\alpha} - \dot{x}_\alpha)\Big\} \\
&\le \kappa_1|\sigma_\alpha|(q^-_{k_\alpha} + \sigma_\alpha)\Big[\big(s^+_{k_\alpha} - \widetilde{s}\big) + \big(\widetilde{s} - s^*\big) + \big(s^* - s_{k_\alpha}(v^-, v^*)\big) \\
&\qquad + \Big(s_{k_\alpha}(v^-, v^*) - \frac{1}{2}\big(\lambda_{k_\alpha}(v^-) + \lambda_{k_\alpha}(v^*)\big)\Big) \\
&\qquad + \Big(\frac{1}{2}\big(\lambda_{k_\alpha}(v^-) + \lambda_{k_\alpha}(v^+)\big) - \dot{x}_\alpha\Big)\Big] \\
&\quad + \kappa_1|\sigma_\alpha||q^+_{k_\alpha} - q^-_{k_\alpha} - \sigma_\alpha||s^+_{k_\alpha} - \dot{x}_\alpha| \\
&\quad + W^-_{k_\alpha}\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha} - \dot{x}_\alpha) - |q^-_{k_\alpha}|(s^-_{k_\alpha} - \dot{x}_\alpha)\Big\}.
\end{aligned} \tag{4.3.4.9}$$

In the first term, use identity (4.3.4.5) to obtain

$$\kappa_1|\sigma_\alpha|(q^-_{k_\alpha} + \sigma_\alpha)(\widetilde{s} - s^*) \le -\kappa_1 d|\sigma_\alpha||q^-_{k_\alpha}|\big(|\sigma_\alpha| + |q^-_{k_\alpha}|\big). \tag{4.3.4.10}$$

According to Lemma 4.4, $\kappa_1(q^-_{k_\alpha} + \sigma_\alpha) \le 2$. Using this bound and (4.3.4.3), (4.3.4.4), and (4.3.4.6) we can bound three other components of the first term. From Lemma 2.5, we recognize the lower bound on the discrepancy as the last component of the last term in equation (4.3.4.9). Finally, we can use the estimate $\kappa_1|\sigma_\alpha| \le 1$ from Lemma 4.4 and the bound on $|q^+_{k_\alpha} - q^-_{k_\alpha} - \sigma_\alpha|$ from Lemma 4.1 to control the second term in (4.3.4.9). Together, these remarks show that

$$\begin{aligned}
E_{k_\alpha}(\alpha) \le &- \kappa_1 d|\sigma_\alpha||q^-_{k_\alpha}|\big(|\sigma_\alpha| + |q^-_{k_\alpha}|\big) + 2D_\alpha(t) \\
&+ \mathcal{O}(1)\Big(|\sigma_\alpha|^3 + |\sigma_\alpha||q^-_{k_\alpha}|\big(|\sigma_\alpha| + |q^-_{k_\alpha}|\big) + |\sigma_\alpha|\sum_{i \ne k_\alpha}|q^-_i|\Big) \\
&+ W^-_{k_\alpha}\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha} - \dot{x}_\alpha) - |q^-_{k_\alpha}|(s^-_{k_\alpha} - \dot{x}_\alpha)\Big\}
\end{aligned} \tag{4.3.4.11}$$

To study the last term in (4.3.4.11), use the positivity of $q^-_{k_\alpha}$, $q^+_{k_\alpha}$, and σ_α to rewrite it as

$$
\begin{aligned}
&W^-_{k_\alpha}\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha}-\dot{x}_\alpha)-|q^-_{k_\alpha}|(s^-_{k_\alpha}-\dot{x}_\alpha)\Big\}\\
&\quad=W^-_{k_\alpha}\Bigg\{(q^-_{k_\alpha}+\sigma_\alpha)\Big[\big(s^+_{k_\alpha}-\widetilde{s}\big)+\big(\widetilde{s}-s^*\big)\\
&\qquad\qquad\qquad+\big(s^*-s_{k_\alpha}(v^-,v^*)\big)+\big(s_{k_\alpha}(v^-,v^*)-\dot{x}_\alpha\big)\Big]\\
&\qquad\quad-q^-_{k_\alpha}\Big[\big(s^-_{k_\alpha}-s^*\big)+\big(s^*-s_{k_\alpha}(v^-,v^*)\big)+\big(s_{k_\alpha}(v^-,v^*)-\dot{x}_\alpha\big)\Big]\\
&\qquad\quad+(q^+_{k_\alpha}-q^-_{k_\alpha}-\sigma_\alpha)(s^+_{k_\alpha}-\dot{x}_\alpha)\Bigg\}\\
&\quad=W^-_{k_\alpha}\Big\{(q^-_{k_\alpha}+\sigma_\alpha)(\widetilde{s}-s^*)-q^-_{k_\alpha}(s^-_{k_\alpha}-s^*)\Big\}\\
&\qquad+W^-_{k_\alpha}\Bigg\{(q^-_{k_\alpha}+\sigma_\alpha)(s^+_{k_\alpha}-\widetilde{s})+(q^+_{k_\alpha}-q^-_{k_\alpha}-\sigma_\alpha)(s^+_{k_\alpha}-\dot{x}_\alpha)\\
&\qquad\quad+\sigma_\alpha\big[\big(s^*-s_{k_\alpha}(v^-,v^*)\big)+\big(s_{k_\alpha}(v^-,v^*)-\dot{x}_\alpha\big)\big]\Bigg\} \qquad (4.3.4.12)
\end{aligned}
$$

Lemma 4.5, with $\overline{w}=w^-_{k_\alpha-1}$, $\sigma=q^-_{k_\alpha}$, and $\sigma'=\sigma_\alpha$ gives

$$
\big|(q^-_{k_\alpha}+\sigma_\alpha)(\widetilde{s}-s^*)-q^-_{k_\alpha}(s^-_{k_\alpha}-s^*)\big|=\mathcal{O}(1)|\sigma_\alpha||q^-_{k_\alpha}|\big(|\sigma_\alpha|+|q^-_{k_\alpha}|\big), \quad (4.3.4.13)
$$

which is precisely the first term of (4.3.4.12), up the constant $W^-_{k_\alpha}\leq 2$. Ignoring the multiple $W^-_{k_\alpha}$, we can control the third term with the help of identity (4.3.4.4). The fourth term, in combination with (4.3.4.3), is of the order of $|\sigma_\alpha|\sum|q^-_i|$. The last term in (4.3.4.12) can clearly be bounded by the interaction estimates of Lemma 4.1. Concerning the term involving the wave speed $\dot{x}_\alpha$, use equations (4.3.4.6), (4.3.4.8), and Lemma 2.5 to bound it as

$$
\begin{aligned}
\sigma_\alpha(s_{k_\alpha}(v^-,v^*)-\dot{x}_\alpha)&=\sigma_\alpha\Bigg[\Big(s_{k_\alpha}(v^-,v^*)-\frac{1}{2}\big(\lambda_{k_\alpha}(v^-)-\lambda_{k_\alpha}(v^*)\big)\Big)\\
&\qquad+\Big(\frac{1}{2}\big(\lambda_{k_\alpha}(v^-)-\lambda_{k_\alpha}(v^+)\big)-\dot{x}_\alpha\Big)\Bigg]\\
&\leq\mathcal{O}(1)|\sigma_\alpha|^3+D_\alpha(t). \qquad (4.3.4.14)
\end{aligned}
$$

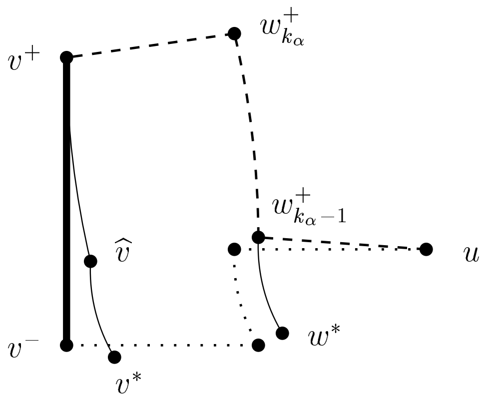

Figure 12: The virtual waves $q_i^\pm$, the rarefaction wave α, and the states in (4.3.5.1) for the seventh case.

These observations on the terms in (4.3.4.12) demonstrate that

$$
\begin{aligned}
W^-_{k_\alpha}&\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha}-\dot{x}_\alpha)-|q^-_{k_\alpha}|(s^-_{k_\alpha}-\dot{x}_\alpha)\Big\}\\
&\leq 2D_\alpha(t)+\mathcal{O}(1)\Big(|\sigma_\alpha|^3+|\sigma_\alpha||q^-_{k_\alpha}|\big(|\sigma_\alpha|+|q^-_{k_\alpha}|\big)+|\sigma_\alpha|\sum_{i\neq k_\alpha}|q_i^-|\Big).
\end{aligned}
\tag{4.3.4.15}
$$

Combining this with our earlier estimate (4.3.4.11) leads us to

$$
\begin{aligned}
E_{k_\alpha}(\alpha)\leq&-\kappa_1 d|\sigma_\alpha||q^-_{k_\alpha}|\big(|\sigma_\alpha|+|q^-_{k_\alpha}|\big)+4D_\alpha(t)\\
&+\mathcal{O}(1)\Big(|\sigma_\alpha|^3+|\sigma_\alpha||q^-_{k_\alpha}|\big(|\sigma_\alpha|+|q^-_{k_\alpha}|\big)+|\sigma_\alpha|\sum_{i\neq k_\alpha}|q_i^-|\Big).
\end{aligned}
\tag{4.3.4.16}
$$

In Lemma 2.5, we showed that for a fixed rarefaction wave, the infimum of the discrepancy $D_\alpha(t)$ is $\sigma_\alpha^2/4$. From this we deduce that $\mathcal{O}(1)|\sigma_\alpha|^3 \leq \mathcal{O}(1)D_\alpha(t)$. Using this in (4.3.4.16) we obtain the final form of the estimate.

4.3.5 Case 7: rarefaction with $q^-_{k_\alpha} < 0 < q^+_{k_\alpha}$

Assume that α is a pure rarefaction wave satisfying $q^-_{k_\alpha} < 0 < q^+_{k_\alpha}$. We demonstrate the estimate for terms in $|q_i^+|$.

Define the quantities

$$
\begin{aligned}
\widetilde{w} &= S_{k_\alpha}(-\sigma_\alpha)(w^+_{k_\alpha}), & \widetilde{s} &= s_{k_\alpha}(w^+_{k_\alpha}, \widetilde{w}),\\
w^* &= S_{k_\alpha}(q^+_{k_\alpha} - \sigma_\alpha)(w^+_{k_\alpha - 1}), & s^* &= s_{k_\alpha}(w^+_{k_\alpha - 1}, w^*),\\
\widetilde{v} &= S_{k_\alpha}(-\sigma_\alpha)(v^+), & \widetilde{\eta} &= s_{k_\alpha}(v^+, \widetilde{v}),\\
\widehat{v} &= S_{k_\alpha}(-q^+_{k_\alpha})(v^+), & \widehat{\eta} &= s_{k_\alpha}(v^+, \widehat{v}),\\
v^* &= S_{k_\alpha}(q^+_{k_\alpha} - \sigma_\alpha)(\widehat{v}), & \eta^* &= s_{k_\alpha}(\widehat{v}, v^*).
\end{aligned}
\tag{4.3.5.1}
$$

and

$$\overline{v} = R_{k_\alpha}(-q^+_{k_\alpha})(v^+).$$

The states $\widetilde{v}$ and v^- are both a distance $-\sigma_\alpha$ from v^+. From the parameterization (2.1.7) we conclude that

$$\lambda_{k_\alpha}(\widetilde{v}) = \lambda_{k_\alpha}(v^-). \tag{4.3.5.2}$$

We now compare pairs of shock speeds defined by states along the shock curve separated by equal distances. We verify that

$$
\begin{aligned}
|\widetilde{s} - \widetilde{\eta}| &= \left| s_{k_\alpha}(w^+_{k_\alpha}, \widetilde{w}) - s_{k_\alpha}(v^+, \widetilde{v}) \right|\\
&= \mathcal{O}(1)|w^+_{k_\alpha} - v^+|\\
&= \mathcal{O}(1) \sum_{i \neq k_\alpha} |q^+_i|,
\end{aligned}
\tag{4.3.5.3}
$$

$$
\begin{aligned}
|s^+_{k_\alpha} - \widehat{\eta}| &= \left| s_{k_\alpha}(w^+_{k_\alpha - 1}, w^+_{k_\alpha}) - s_{k_\alpha}(v^+, \widehat{v}) \right|\\
&= \mathcal{O}(1)|w^+_{k_\alpha} - v^+|\\
&= \mathcal{O}(1) \sum_{i \neq k_\alpha} |q^+_i|,
\end{aligned}
\tag{4.3.5.4}
$$

$$
\begin{aligned}
|s^* - \eta^*| &= |s_{k_\alpha}(w^+_{k_\alpha - 1}, w^*) - s_{k_\alpha}(\widehat{v}, v^*)|\\
&= \mathcal{O}(1)|w^+_{k_\alpha - 1} - \widehat{v}|\\
&= \mathcal{O}(1)|w^+_{k_\alpha} - v^+|\\
&= \mathcal{O}(1) \sum_{i \neq k_\alpha} |q^+_i|.
\end{aligned}
\tag{4.3.5.5}
$$

In the next estimate, the pairs of states defining the shock speeds $s^-_{k_\alpha}$ and s^* are separated along S_{k_α} by the distances $q^-_{k_\alpha}$ and $q^+_{k_\alpha} - \sigma_\alpha$. With estimate

(4.1.8) of Lemma 4.2, we find

$$\begin{aligned}|s_{k_\alpha}^- - s^*| =& \big|s_{k_\alpha}(w_{k_\alpha-1}^-, w_{k_\alpha}^-) - s_{k_\alpha}(w_{k_\alpha-1}^+, w^*)\big|\\ =&\mathcal{O}(1)\Big(|w_{k_\alpha-1}^+ - w_{k_\alpha-1}^-| + |q_{k_\alpha}^+ - q_{k_\alpha}^- - \sigma_\alpha|\Big)\\ =&\mathcal{O}(1)\Big(|\sigma_\alpha|^3 + |\sigma_\alpha||q_{k_\alpha}^+|\big(|\sigma_\alpha| - |q_{k_\alpha}^+|\big) + \sum_{i\neq k_\alpha}|\sigma_\alpha||q_i^+|\Big).\end{aligned} \tag{4.3.5.6}$$

Since the shock speeds are, to second-order, the average of the characteristic speeds of the waves at the end states, we have the following three estimates

$$\Big|\widetilde{\eta} - \frac{1}{2}\big(\lambda_{k_\alpha}(v^+) + \lambda_{k_\alpha}(\widetilde{v})\big)\Big| = \mathcal{O}(1)|\sigma_\alpha|^2, \tag{4.3.5.7}$$

$$\Big|\eta^* - \frac{1}{2}\big(\lambda_{k_\alpha}(v^*) + \lambda_{k_\alpha}(\widehat{v})\big)\Big| = \mathcal{O}(1)|q_{k_\alpha}^+ - \sigma_\alpha|^2, \tag{4.3.5.8}$$

$$\Big|\widehat{\eta} - \frac{1}{2}\big(\lambda_{k_\alpha}(v^+) + \lambda_{k_\alpha}(\widehat{v})\big)\Big| = \mathcal{O}(1)|q_{k_\alpha}^+|^2. \tag{4.3.5.9}$$

From the second-order tangency of shock and rarefaction curves, we deduce

$$\begin{aligned}|\widehat{v} - \overline{v}| =& \big|S_{k_\alpha}(-q_{k_\alpha}^+)(v^+) - R_{k_\alpha}(-q_{k_\alpha}^+)(v^+)\big|\\ =&\mathcal{O}(1)|q_{k_\alpha}^+|^3.\end{aligned} \tag{4.3.5.10}$$

Estimate (2.1.14) in the definition of the constant d, with $q = q_{k_\alpha}^+$, $p = \sigma_\alpha - q_{k_\alpha}^+$, and $u = w_{k_\alpha}^+$ states

$$d|\sigma_\alpha - q_{k_\alpha}^+| = d\big(|\sigma_\alpha| - |q_{k_\alpha}^+|\big) < s_{k_\alpha}^+ - \widetilde{s}. \tag{4.3.5.11}$$

From the definition of W_{k_α} and the estimate $W_{k_\alpha} \leq 2$, we have

$$\kappa_1\big(V(u) + V(v)\big) \leq W_{k_\alpha} - 1 \leq 1. \tag{4.3.5.12}$$

Using identity (2.2.9) and these inequalities, we find

$$\begin{aligned}\kappa_1\big(B_{k_\alpha}^- + B_{k_\alpha}^+\big) &\leq \kappa_1\big(|\sigma_\alpha| + V(u) + V(v)\big)\\ &\leq 2\kappa_1\big(V(u) + V(v)\big) \leq 2.\end{aligned} \tag{4.3.5.13}$$

We now consider the expression $E_{k_\alpha}(\alpha)$ which we expand using the identity $A^-_{k_\alpha} = A^+_{k_\alpha}$.

$$
\begin{aligned}
E_{k_\alpha}(\alpha) =& \Big(1 + \kappa_1\big(A^+_{k_\alpha} + B^+_{k_\alpha}\big) + \kappa_2\big(Q(u) + Q(v)\big)\Big)|q^+_{k_\alpha}|(s^+_{k_\alpha} - \dot{x}_\alpha) \\
& - \Big(1 + \kappa_1\big(A^-_{k_\alpha} + B^-_{k_\alpha}\big) + \kappa_2\big(Q(u) + Q(v)\big)\Big)|q^-_{k_\alpha}|(s^-_{k_\alpha} - \dot{x}_\alpha) \\
=& \Big(1 + \kappa_1 A^+_{k_\alpha} + \kappa_2\big(Q(u) + Q(v)\big)\Big)\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha} - \dot{x}_\alpha) - |q^-_{k_\alpha}|(s^-_{k_\alpha} - \dot{x}_\alpha)\Big\} \\
& + \kappa_1\big(B^-_{k_\alpha} + B^+_{k_\alpha}\big)|q^+_{k_\alpha}|(s^+_{k_\alpha} - \dot{x}_\alpha) \\
& - \kappa_1 B^-_{k_\alpha}\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha} - \dot{x}_\alpha) + |q^-_{k_\alpha}|(s^-_{k_\alpha} - \dot{x}_\alpha)\Big\}. \qquad (4.3.5.14)
\end{aligned}
$$

Beginning with the second term in (4.3.5.14), use identity (4.3.5.2) to write

$$
\begin{aligned}
&\kappa_1|q^+_{k_\alpha}|\big(B^-_{k_\alpha} + B^+_{k_\alpha}\big)(s^+_{k_\alpha} - \dot{x}_\alpha) \\
&\quad = \kappa_1|q^+_{k_\alpha}|\big(B^-_{k_\alpha} + B^+_{k_\alpha} + |\sigma_\alpha|\big)(s^+_{k_\alpha} - \widetilde{s}) - \kappa_1|q^+_{k_\alpha}||\sigma_\alpha|(s^+_{k_\alpha} - \widetilde{s}) \\
&\quad\quad + \kappa_1|q^+_{k_\alpha}|\big(B^-_{k_\alpha} + B^+_{k_\alpha}\big)\Big[(\widetilde{s} - \widetilde{\eta}) + \Big(\widetilde{\eta} - \frac{1}{2}\big(\lambda_{k_\alpha}(v^+) + \lambda_{k_\alpha}(v^-)\big)\Big) \\
&\quad\quad\quad + \Big(\frac{1}{2}\big(\lambda_{k_\alpha}(v^+) + \lambda_{k_\alpha}(v^-)\big) - \dot{x}_\alpha\Big)\Big]. \qquad (4.3.5.15)
\end{aligned}
$$

Our parameterization (2.1.7) along shock curves implies that

$$0 \le s^+_{k_\alpha} - \widetilde{s} \le \lambda_{k_\alpha}(w^+_{k_\alpha}) - \lambda_{k_\alpha}(\widetilde{w}) = |\sigma_\alpha|. \qquad (4.3.5.16)$$

In the following order, use the bounds (4.3.5.13), $|q^+_{k_\alpha}| \le |\sigma_\alpha|$, (4.3.5.16), and the bound on the discrepancy (2.3.16) to estimate the first term of (4.3.5.15)

$$
\begin{aligned}
\kappa_1|q^+_{k_\alpha}|\big(B^-_{k_\alpha} + B^+_{k_\alpha} + |\sigma_\alpha|\big)(s^+_{k_\alpha} - \widetilde{s}) \le& 3|q^+_{k_\alpha}||s^+_{k_\alpha} - \widetilde{s}| \\
\le& 3|\sigma_\alpha|^2 \\
\le& 6D_\alpha(t). \qquad (4.3.5.17)
\end{aligned}
$$

Apply estimate (4.3.5.11) to the second term in (4.3.5.15) to find

$$-\kappa_1|q^+_{k_\alpha}||\sigma_\alpha|(s^+_{k_\alpha} - \widetilde{s}) < -\kappa_1 d|\sigma_\alpha||q^+_{k_\alpha}|\big(|\sigma_\alpha| - |q^+_{k_\alpha}|\big). \qquad (4.3.5.18)$$

To estimate the third term, combine (4.3.5.3), (4.3.5.13), and $|q^+_{k_\alpha}| < |\sigma_\alpha|$. For the fourth term, use (4.3.5.7), (4.3.5.13), and $|q^+_{k_\alpha}| < |\sigma_\alpha|$. The sum of these two terms is therefore

$$\mathcal{O}(1)|\sigma_\alpha| \sum_{i \ne k_\alpha} |q^+_i| + \mathcal{O}(1)|\sigma_\alpha|^3. \qquad (4.3.5.19)$$

Referring to our explicit description of the discrepancy (2.3.12), we see that the last term in (4.3.5.15) is bounded above by $2D_\alpha(t)$. With estimates (4.3.5.17), (4.3.5.18), and (4.3.5.19) we have shown that

$$\begin{aligned}\kappa_1|q^+_{k_\alpha}|\big(B^-_{k_\alpha}+B^+_{k_\alpha}\big)(s^+_{k_\alpha}-\dot{x}_\alpha)\le &-\kappa_1 d|\sigma_\alpha||q^+_{k_\alpha}|\big(|\sigma_\alpha|-|q^+_{k_\alpha}|\big)+8D_\alpha(t)\\ &+\mathcal{O}(1)\Big(|\sigma_\alpha|^3+|\sigma_\alpha|\sum_{i\neq k_\alpha}|q^+_i|\Big).\end{aligned} \tag{4.3.5.20}$$

We expand the third term of (4.3.5.14)

$$\begin{aligned}-\kappa_1 B^-_{k_\alpha}&\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha}-\dot{x}_\alpha)+|q^-_{k_\alpha}|(s^-_{k_\alpha}-\dot{x}_\alpha)\Big\}\\
&=-\kappa_1 B^-_{k_\alpha}\Big\{q^+_{k_\alpha}(s^+_{k_\alpha}-\dot{x}_\alpha)+(\sigma_\alpha-q^+_{k_\alpha})(s^-_{k_\alpha}-\dot{x}_\alpha)\\
&\qquad+(q^+_{k_\alpha}-q^-_{k_\alpha}-\sigma_\alpha)(s^-_{k_\alpha}-\dot{x}_\alpha)\Big\}\\
&=-\kappa_1 B^-_{k_\alpha}\Big\{q^+_{k_\alpha}\big((s^+_{k_\alpha}-\widetilde{s})+(\widetilde{s}-\dot{x}_\alpha)\big)\\
&\qquad\qquad+(\sigma_\alpha-q^+_{k_\alpha})\big((s^-_{k_\alpha}-s^*)+(s^*-\widetilde{s})+(\widetilde{s}-\dot{x}_\alpha)\big)\\
&\qquad\qquad+(q^+_{k_\alpha}-q^-_{k_\alpha}-\sigma_\alpha)(s^-_{k_\alpha}-\dot{x}_\alpha)\Big\}\\
&=-\kappa_1 B^-_{k_\alpha}\Big\{(\sigma_\alpha-q^+_{k_\alpha})(s^*-\widetilde{s})+q^+_{k_\alpha}(s^+_{k_\alpha}-\widetilde{s})\Big\}\\
&\quad-\kappa_1 B^-_{k_\alpha}\Big\{(\sigma_\alpha-q^+_{k_\alpha})(s^-_{k_\alpha}-s^*)+\sigma_\alpha(\widetilde{s}-\dot{x}_\alpha)\\
&\qquad\qquad+(q^+_{k_\alpha}-q^-_{k_\alpha}-\sigma_\alpha)(s^-_{k_\alpha}-\dot{x}_\alpha)\Big\}.\end{aligned} \tag{4.3.5.21}$$

From (4.3.5.12), $\kappa_1 B^-_{k_\alpha}\le 1$, and we may ignore the coefficient in front of (4.3.5.21). The first and second terms in (4.3.5.21) give precisely the expression (4.1.19) of Lemma 4.5 with $\overline{w}=w^+_{k_\alpha-1}$, $\sigma=q^+_{k_\alpha}$, and $\sigma'=-\sigma_\alpha$

$$\big|(q^+_{k_\alpha}-\sigma_\alpha)(s^*-\widetilde{s})-q^+_{k_\alpha}(s^+_{k_\alpha}-\widetilde{s})\big|=\mathcal{O}(1)|\sigma_\alpha||q^+_{k_\alpha}|\big(|\sigma_\alpha|-|q^+_{k_\alpha}|\big). \tag{4.3.5.22}$$

The third term in (4.3.5.21) is handled by appealing to estimate (4.3.5.6). In the last term $(s^-_{k_\alpha}-\dot{x}_\alpha)$ is bounded since $|\dot{x}_\alpha|$ is bounded by Λ. Using this and equation (4.1.8), we bound the last term. Rewrite the fourth term as

$$\begin{aligned}\sigma_\alpha(\widetilde{s}-\dot{x}_\alpha)=\sigma_\alpha\Big[&(\widetilde{s}-\widetilde{\eta})+\Big(\widetilde{\eta}-\frac{1}{2}\big(\lambda_{k_\alpha}(v^+)+\lambda_{k_\alpha}(\widetilde{v})\big)\Big)\\
&+\Big(\frac{1}{2}\big(\lambda_{k_\alpha}(v^+)+\lambda_{k_\alpha}(\widetilde{v})\big)-\dot{x}_\alpha\Big)\Big].\end{aligned} \tag{4.3.5.23}$$

Use estimates (4.3.5.3), (4.3.5.7), and inequality (2.3.15) of Lemma 2.5 to demonstrate

$$\sigma_\alpha(\widetilde{s} - \dot{x}_\alpha) \le \mathcal{O}(1)|\sigma_\alpha| \sum_{i \ne k_\alpha} |q_i^+| + \mathcal{O}(1)|\sigma_\alpha|^3 + D_\alpha(t). \tag{4.3.5.24}$$

For the third term in (4.3.5.14), these observations show that

$$\begin{aligned} -\kappa_1 B_{k_\alpha}^- \Big\{ |q_{k_\alpha}^+|(s_{k_\alpha}^+ - \dot{x}_\alpha) + |q_{k_\alpha}^-|(s_{k_\alpha}^- - \dot{x}_\alpha) \Big\} \\ \le D_\alpha(t) + \mathcal{O}(1)\Big(|\sigma_\alpha|^3 + |\sigma_\alpha||q_{k_\alpha}^+|\big(|\sigma_\alpha| - |q_{k_\alpha}^+|\big) + \sum_{i \ne k_\alpha} |\sigma_\alpha||q_i^+| \Big). \end{aligned} \tag{4.3.5.25}$$

The coefficient of the first term in (4.3.5.14) satisfies

$$1 + \kappa_1 A_{k_\alpha}^+ + \kappa_2\big(Q(u) + Q(v)\big) \le 2. \tag{4.3.5.26}$$

For convenience, we expand the first term in (4.3.5.14) without this coefficient. Using the fact that $q_{k_\alpha}^-$, $q_{k_\alpha}^+$, and σ_α are positive, we write

$$\begin{aligned} |q_{k_\alpha}^+|(s_{k_\alpha}^+ - \dot{x}_\alpha) - |q_{k_\alpha}^-|(s_{k_\alpha}^- - \dot{x}_\alpha) \\ = q_{k_\alpha}^+ \Bigg[\big(s_{k_\alpha}^+ - \widehat{\eta}\big) + \Big(\widehat{\eta} - \frac{1}{2}\big(\lambda_{k_\alpha}(v^+) + \lambda_{k_\alpha}(\widehat{v})\big)\Big) \\ + \Big(\frac{1}{2}\big(\lambda_{k_\alpha}(v^+) + \lambda_{k_\alpha}(\widehat{v})\big) - \dot{x}_\alpha\Big)\Bigg] \\ - (\sigma_\alpha - q_{k_\alpha}^+)\Bigg[\big(s_{k_\alpha}^- - s^*\big) + \big(s^* - \eta^*\big) \\ + \Big(\eta^* - \frac{1}{2}\big(\lambda_{k_\alpha}(\widehat{v}) + \lambda_{k_\alpha}(v^*)\big)\Big) \\ + \Big(\frac{1}{2}\big(\lambda_{k_\alpha}(\widehat{v}) + \lambda_{k_\alpha}(v^*)\big) - \dot{x}_\alpha\Big)\Bigg] \\ - (q_{k_\alpha}^+ - q_{k_\alpha}^- - \sigma_\alpha)(s_{k_\alpha}^- - \dot{x}_\alpha). \end{aligned} \tag{4.3.5.27}$$

For the first term, combine $|q_{k_\alpha}^+| \le |\sigma_\alpha|$ with (4.3.5.4) to obtain the bound $\mathcal{O}(1)|\sigma_\alpha| \sum |q_i^+|$. Estimate (4.3.5.9) and $|q_{k_\alpha}^+| \le |\sigma_\alpha|$ imply that the second term is of order $\mathcal{O}(1)|\sigma_\alpha|^3$. Also note that $|\sigma_\alpha - q_{k_\alpha}^+| \le |\sigma_\alpha|$. Combining this bound with (4.3.5.5), (4.3.5.6), and (4.3.5.8) we find that the fourth, fifth and sixth terms are of order

$$\mathcal{O}(1)\Big(|\sigma_\alpha|^3 + |\sigma_\alpha||q_{k_\alpha}^+|\big(|\sigma_\alpha| - |q_{k_\alpha}^+|\big) + \sum_{i \ne k_\alpha} |\sigma_\alpha||q_i^+| \Big). \tag{4.3.5.28}$$

Equation (4.1.8) of Lemma 4.2 shows that the last term is also of this order. As for the terms involving $\dot{x}_\alpha$, notice that our parameterization along the shock curves provides us with the identities $\lambda_{k_\alpha}(\widehat{v}) = \lambda_{k_\alpha}(\overline{v})$ and $\lambda_{k_\alpha}(v^*) = \lambda_{k_\alpha}(v^-)$. Summarizing, we have shown that the first term in (4.3.5.14) satisfies

$$\begin{aligned}
\Big(1 + \kappa_1 A^+_{k_\alpha} + \kappa_2\big(Q(u) + Q(v)\big)\Big)&\Big\{|q^+_{k_\alpha}|(s^+_{k_\alpha} - \dot{x}_\alpha) - |q^-_{k_\alpha}|(s^-_{k_\alpha} - \dot{x}_\alpha)\Big\} \\
\leq 2\bigg| & q^+_{k_\alpha}\Big(\frac{1}{2}\big(\lambda_{k_\alpha}(v^+) + \lambda_{k_\alpha}(\overline{v})\big) - \dot{x}_\alpha\Big) \\
& - (\sigma_\alpha - q^+_{k_\alpha})\Big(\frac{1}{2}\big(\lambda_{k_\alpha}(\overline{v}) + \lambda_{k_\alpha}(v^-)\big) - \dot{x}_\alpha\Big)\bigg| \\
& + \mathcal{O}(1)\Big(|\sigma_\alpha|^3 + |\sigma_\alpha||q^+_{k_\alpha}|\big(|\sigma_\alpha| - |q^+_{k_\alpha}|\big) + \sum_{i \neq k_\alpha} |\sigma_\alpha||q^+_i|\Big).
\end{aligned} \tag{4.3.5.29}$$

The first term above is twice the quadratic rarefaction remainder $Q_{\mathrm{rar}}(q^+_{k_\alpha}, \dot{x}_\alpha)$ and by Lemma 4.7 is bounded by the discrepancy $2D_\alpha(t)$. The final estimate of $E_{k_\alpha}(\alpha)$ combines the estimates (4.3.5.20), (4.3.5.25), (4.3.5.29) for the three terms in (4.3.5.14) with $\mathcal{O}(1)|\sigma_\alpha|^3 = \mathcal{O}(1)D_\alpha(t)$. The conclusion is

$$\begin{aligned}
E_{k_\alpha}(\alpha) \leq & -\kappa_1 d|\sigma_\alpha||q^+_{k_\alpha}|\big(|\sigma_\alpha| - |q^+_{k_\alpha}|\big) + \mathcal{O}(1)D_\alpha(t) \\
& + \mathcal{O}(1)\Big(|\sigma_\alpha||q^+_{k_\alpha}|\big(|\sigma_\alpha| - |q^+_{k_\alpha}|\big) + |\sigma_\alpha|\sum_{i \neq k_\alpha}|q^+_i|\Big).
\end{aligned} \tag{4.3.5.30}$$

The proof of Lemma 4.9 is complete and contained in Subsections 4.3.1-4.3.5.

4.4 Isolated Waves: Proof of Lemma 3.1

We now present the proof of Lemma 3.1 under the assumption that $\alpha \in \mathcal{W}\big(v(t)\big)$ is an isolated wave.

Proof We must first demonstrate that Φ is well-defined under the conditions of this lemma.

Choose a value for κ_1, say $\kappa_1 = 1$. Let $\delta_0 = \delta_0(\kappa_1)$ be the positive constant of Lemma 2.3 and let δ_1, δ_2 be the constants of Lemmas 4.4 and 4.6. Define $\delta_0' = 1/2 \min\{\delta_0, \delta_1, \delta_2\}$ and suppose that the front-tracking approximations u and v satisfy

$$||u(0)||_{\mathrm{TV}}, ||v(0)||_{\mathrm{TV}} < \delta_0'. \tag{4.4.1}$$

Notice that because both $u(0)$ and $v(0)$ are assumed to be in $L^1(\mathbb{R}, \Omega)$,

$$u(0) - v(0) \in L^1(\mathbb{R}, \Omega). \tag{4.4.2}$$

Moreover, both $u(0)$ and $v(0)$ must vanish at infinity and therefore the total variation bound also implies an L^∞ bound

$$|| \cdot ||_{L^\infty} \leq || \cdot ||_{\text{TV}}. \tag{4.4.3}$$

Using (4.4.1) and (4.4.3), we have

$$||u(0)||_{L^\infty} + ||u(0)||_{\text{TV}} \leq 2||u(0)||_{\text{TV}} < 2\delta_0' \leq \delta_0. \tag{4.4.4}$$

Similarly, $||u(0)||_{L^\infty} + ||u(0)||_{\text{TV}} < \delta_0$.

Conditions (4.4.1), therefore imply the initial bounds (4.4.4) for both u and v. These are precisely the conditions of Lemma 2.3 ensuring that $\Phi\big(u(t), v(t)\big)$ is well-defined. In fact, the approximations u and v also satisfy the conditions on the total variation of Lemmas 4.4 and 4.6 and therefore satisfy the assumption (4.2.1) stated at the beginning of Section 4.2. The estimates of Lemmas 4.8 and 4.9 hold for u and v.

Suppose that α is shock wave satisfying $0 < q_{k_\alpha}^+ < q_{k_\alpha}^-$. Summing equation (4.2.3) over $k \neq k_\alpha$ with (4.3.1), all with the positive sign s, we obtain

$$\begin{aligned}\sum_{k=1}^{n} E_k(\alpha) \leq &- \kappa_1 d|\sigma_\alpha||q_{k_\alpha}^+|\big(|\sigma_\alpha| + |q_{k_\alpha}^+|\big) - \kappa_1 d|\sigma_\alpha| \sum_{i \neq k_\alpha} |q_i^+| + \mathcal{O}(1) D_\alpha(t) \\ &+ \mathcal{O}(1)\Big(|\sigma_\alpha||q_{k_\alpha}^+|\big(|\sigma_\alpha| + |q_{k_\alpha}^+|\big) + |\sigma_\alpha| \sum_{i \neq k_\alpha} |q_i^+|\Big).\end{aligned} \tag{4.4.5}$$

There exists κ_1, depending only on the system (2.1.1), such that

$$\mathcal{O}(1) \leq \kappa_1 d$$

where $\mathcal{O}(1)$ is the coefficient of the last term in (4.4.5). For this value of κ_1 recompute $\delta_0 = \delta_0(\kappa_1)$ of Lemma 2.3 and δ_0'. Assuming that both u and v satisfy the bounds (4.4.1) for this new value of δ_0', we have

$$\sum_{k=1}^{n} E_k(\alpha) \leq \mathcal{O}(1) D_\alpha(t). \tag{4.4.6}$$

If α is a shock wave satisfying either $q^+_{k_\alpha} < q^-_{k_\alpha} < 0$ or $q^+_{k_\alpha} < 0 < q^-_{k_\alpha}$, sum the estimates of Lemmas 4.8 and 4.9 to obtain another lower bound for κ_1 of the form $\mathcal{O}(1) \leq \kappa_1 d$. If α is a rarefaction or linearly degenerate wave, Lemmas 4.8 and 4.9 also imply a lower bound on the size of κ_1.

In all these cases, the lower bound on κ_1 is computable entirely from the system (2.1.1). In particular, the constant $\delta_0' = \delta_0'(\kappa_1)$ also depends only on (2.1.1). This demonstrates the lemma. ■

4.5 Non-Isolated Waves

In general, discontinuities in front-tracking approximations are composed of more than one wave. The construction of Section 2.1 gives rise to non-isolated waves when condition (2.1.27) forces the introduction of an approximation which limits the number of outgoing discontinuities. To demonstrate Lemma 3.1 for such discontinuities we construct an intermediate approximation where the single discontinuity is replaced locally by n isolated waves traveling within a distance ϵ of each other. We then show that our earlier estimates pass to the limit as $\epsilon \to 0$. This construction is formalized below.

Consider a front-tracking approximation u with a single discontinuity traveling along the trajectory $z(t)$. Since we only need to demonstrate Lemma 3.1 for times when discontinuities are not colliding, we can assume that there is only one discontinuity in a neighborhood of $\big(t_0, z(t_0)\big) = (t_0, z_0)$. More specifically, we assume that there exists ϵ such that the box

$$B_\epsilon(t_0, z_0) = \big[t_0 - \epsilon, t_0 + \epsilon\big] \times \big[z_0 - \Lambda\epsilon, z_0 + \Lambda\epsilon\big] \tag{4.5.1}$$

contains only one discontinuity. Since Λ is the CFL number of the front-tracking algorithm, defined in (2.1.28), the discontinuity enters and exits the box along the lower and upper edges.

We denote the n waves in $\mathcal{W}\big(v(t)\big)$ at located along $z(t)$ by $\alpha(1), \alpha(2), \ldots,$ $\alpha(n)$ and order them to satisfy $k_{\alpha(i)} < k_{\alpha(i+1)}$. Recall Definition 2.3 of the left and right hand states of a wave. It states that if v^- and v^+ are the left and right sided limits of v at $z(t)$, then

$$v^- = v^-_{\alpha(1)}, \qquad v^+ = v^+_{\alpha(n)}, \tag{4.5.2}$$

and

$$v^+_{\alpha(i)} = T_i(\sigma_{\alpha(i)})(v^-_{\alpha(i)}), \qquad v^+_{\alpha(i)} = v^-_{\alpha(i+1)}. \tag{4.5.3}$$

Note that the states $v^-_{\alpha(1)}, \ldots, v^-_{\alpha(n)}, v^+_{\alpha(n)}$ are the $n+1$ intermediate states in the solution of the Riemann problem between v^- and v^+.

We replace the discontinuity $z(t)$ in $B_\epsilon(t_0, z_0)$ by n isolated waves traveling along the trajectories $\tau_1(t), \ldots, \tau_n(t)$ given below. For $t \in \big[t_0-\epsilon, t_0+\epsilon\big]$, define

$$\tau_i(t) = z(t) + \epsilon\big(\epsilon - |t - t_0|\big) \cdot \left(\frac{2i - n - 1}{2n - 2}\right). \tag{4.5.4}$$

These curves are disjoint, $\tau_i(t) < \tau_{i+1}(t)$, except at the boundary of $B_\epsilon(t_0, z_0)$. In fact, they are separated by precisely

$$\big|\tau_i(t) - \tau_{i+1}(t)\big| = \frac{\epsilon}{n-1}\big(\epsilon - |t - t_0|\big).$$

In particular, the separation is of order ϵ^2 and for ϵ sufficiently small the trajectories τ_i are contained inside the box.

Definition 4.1 *The ϵ-splitting of v at (t_0, z_0) is the piecewise constant function*

$$V_\epsilon(t,x) = \begin{cases} v(t,x), & \text{if } (t,x) \notin B_\epsilon(t_0,z_0), \\ v^+_{\alpha(i)} = v^-_{\alpha(i+1)}, & \text{if } (t,x) \text{ is between the curves } \tau_i \text{ and } \tau_{i+1}, \\ v^-, & \text{if } (t,x) \text{ is to the left of } \tau_1, \\ v^+, & \text{if } (t,x) \text{ is to the right of } \tau_n. \end{cases} \tag{4.5.5}$$

The discontinuities in v and V_ϵ inside the domain $B_\epsilon(t_0, z_0)$ are illustrated in Figure 13.

The ϵ-splitting V_ϵ contains a single discontinuity at $z(t_0 - \epsilon)$ which separates inside $B_\epsilon(t_0, z_0)$ into n isolated waves. This process is consistent with the first approximate Riemann solver used in the front-tracking algorithm. When the waves merge back together at $z(t_0 + \epsilon)$, one is essentially appealing to the second approximate Riemann solver previously used under condition (2.1.27). The ϵ-splitting is therefore also a front-tracking approximation and the results of Lemmas 2.1 and 2.3 also hold. In particular $\Phi(u, V_\epsilon)$ is well-defined for the same total variation bounds as the admissible pair u and v.

Before proceeding, we introduce some notation to help distinguish between quantities appearing in the construction of the functionals $\Phi(u, v)$ and

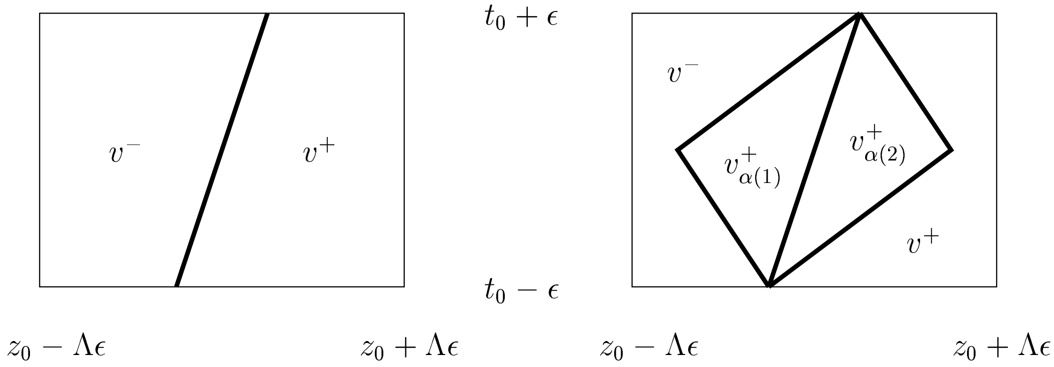

Figure 13: The bold line in the left box represents the trajectory z of the discontinuity in the approximation v. The right box represents the trajectories τ_i and the value of V_ϵ in each region bounded by these curves.

$\Phi(u, V_\epsilon)$. Slightly abusing our notation,let $\alpha(i)$ denote both the wave of the i-th family at $z(t)$ in $\mathcal{W}\big(v(t)\big)$ and the isolated wave in $\mathcal{W}\big(V_\epsilon(t)\big)$ traveling along the trajectory τ_i. Quantities used in the definition of the functional $\Phi(u, V_\epsilon)$ will be written with a further subscript ϵ. In particular, at any time t

$$\Phi\big(u(t), V_\epsilon(t)\big) = \sum_{k=1}^{n} \int_{-\infty}^{\infty} |q_{k,\epsilon}(x)| W_{k,\epsilon}(x)\ dx.$$

Recall the notation,

$$q_k^{\beta\pm} = \lim_{x \to x_\beta \pm} q_k(x),$$

which will also be used for any function of x.

Lemma 4.10

$$\lim_{\epsilon \to 0} \sum_{i=1}^{n} \sum_{k=1}^{n} E_{k,\epsilon}\big(\alpha(i)\big) = \sum_{k=1}^{n} E_k(z_0) \tag{4.5.6}$$

Proof For $\alpha(i)$ and $\alpha(i+1)$ compare the two contributions from the time derivative of $\Phi\big(u(t), V_\epsilon(t)\big)$

$$\begin{aligned} E_{k,\epsilon}\big(\alpha(i)\big) =& \big|q_{k,\epsilon}^{\alpha(i)+}\big| W_{k,\epsilon}^{\alpha(i)+}\big(s_{k,\epsilon}^{\alpha(i)+} - \dot{\tau}_i\big) \\ & - \big|q_{k,\epsilon}^{\alpha(i)-}\big| W_{k,\epsilon}^{\alpha(i)-}\big(s_{k,\epsilon}^{\alpha(i)-} - \dot{\tau}_i\big), \end{aligned} \tag{4.5.7}$$

$$\begin{aligned} E_{k,\epsilon}\big(\alpha(i+1)\big) =& \big|q_{k,\epsilon}^{\alpha(i+1)+}\big| W_{k,\epsilon}^{\alpha(i+1)+}\big(s_{k,\epsilon}^{\alpha(i+1)+} - \dot{\tau}_{i+1}\big) \\ & - \big|q_{k,\epsilon}^{\alpha(i+1)-}\big| W_{k,\epsilon}^{\alpha(i+1)-}\big(s_{k,\epsilon}^{\alpha(i+1)-} - \dot{\tau}_{i+1}\big). \end{aligned} \tag{4.5.8}$$

Using (4.5.3),

$$q_{k,\epsilon}^{\alpha(i)+} = q_k\big(u, v_{\alpha(i)}^+\big) = q_k\big(u, v_{\alpha(i+1)}^-\big) = q_{k,\epsilon}^{\alpha(i+1)-}. \tag{4.5.9}$$

The same procedure also shows that

$$s_{k,\epsilon}^{\alpha(i)+} = s_{k,\epsilon}^{\alpha(i+1)-}, \text{ and } W_{k,\epsilon}^{\alpha(i)+} = W_{k,\epsilon}^{\alpha(i+1)-}. \tag{4.5.10}$$

Moreover, notice that as $\epsilon \to 0$ both $\dot{\tau}_i$ and $\dot{\tau}_{i+1}$ approach $\dot{z}$. For fixed k, in the limit as $\epsilon \to 0$, the terms on the left hand side of (4.5.6) form a telescoping sum and only the first and last terms remain. The limit can therefore be reduced to

$$\begin{aligned} \lim_{\epsilon\to 0} \sum_{i=1}^{n}\sum_{k=1}^{n} E_{k,\epsilon}\big(\alpha(i)\big) =& \lim_{\epsilon\to 0} \sum_{k=1}^{n} \big|q_{k,\epsilon}^{\alpha(n)+}\big| W_{k,\epsilon}^{\alpha(n)+}\big(s_{k,\epsilon}^{\alpha(n)+} - \dot{z}\big) \\ & - \big|q_{k,\epsilon}^{\alpha(1)-}\big| W_{k,\epsilon}^{\alpha(1)-}\big(s_{k,\epsilon}^{\alpha(1)-} - \dot{z}\big). \end{aligned} \tag{4.5.11}$$

From the definition of the ϵ-splitting

$$\begin{aligned} q_{k,\epsilon}^{\alpha(1)-} &= q_k\big(u, v_{\alpha(1)}^-\big) = q_k\big(u, v^-\big) = q_k^{z-}, \\ q_{k,\epsilon}^{\alpha(n)+} &= q_k\big(u, v_{\alpha(n)}^+\big) = q_k\big(u, v^+\big) = q_k^{z+}. \end{aligned} \tag{4.5.12}$$

Similarly,

$$s_{k,\epsilon}^{\alpha(1)-} = s_k^{z-}, \text{ and } s_{k,\epsilon}^{\alpha(n)+} = s_k^{z+}. \tag{4.5.13}$$

We now examine the weight

$$W_{k,\epsilon}^{\alpha(1)-} = 1 + \kappa_1\big(A_{k,\epsilon}^{\alpha(1)-} + B_{k,\epsilon}^{\alpha(1)-}\big) + \kappa_2\big(Q(u) + Q(V_\epsilon)\big). \tag{4.5.14}$$

We know that $Q(V_\epsilon) = Q(v)$ since both approximations contain the same waves. From the definition of A_i, equation (2.2.4), we find

$$\begin{aligned}\lim_{\epsilon\to 0} A_{k,\epsilon}^{\alpha(1)-} =& \lim_{\epsilon\to 0}\Big(\sum_{\substack{\beta\in\mathcal{W}(u)\bigcup\mathcal{W}(V_\epsilon)\\ x_\beta<\tau_1,\, k_\beta>k}} + \sum_{\substack{\beta\in\mathcal{W}(u)\bigcup\mathcal{W}(V_\epsilon)\\ x_\beta\geq\tau_1,\, k_\beta<k}} \Big)|\sigma_\beta| \\ =& A_k^{z-}. \end{aligned} \tag{4.5.15}$$

One can also show that

$$\lim_{\epsilon\to 0} A_{k,\epsilon}^{\alpha(n)+} = A_k^{z+}. \tag{4.5.16}$$

If k is a genuinely nonlinear family, then

$$\begin{aligned}\lim_{\epsilon\to 0} B_{k,\epsilon}^{\alpha(1)-} =& \lim_{\epsilon\to 0}\begin{cases} \Big(\sum_{\substack{\beta\in\mathcal{W}(u)\\ x_\beta<\tau_1,\, k_\alpha=k}} + \sum_{\substack{\beta\in\mathcal{W}(V_\epsilon)\\ x_\beta\geq\tau_1,\, k_\alpha=k}}\Big)|\sigma_\beta| & \text{if } q_{k,\epsilon}^{\alpha(1)-} < 0, \\ \Big(\sum_{\substack{\beta\in\mathcal{W}(V_\epsilon)\\ x_\beta<\tau_1,\, k_\beta=k}} + \sum_{\substack{\beta\in\mathcal{W}(u)\\ x_\beta\geq\tau_1,\, k_\beta=k}}\Big)|\sigma_\beta| & \text{if } q_{k,\epsilon}^{\alpha(1)-} > 0, \end{cases} \\ =& B_k^{z-}. \end{aligned} \tag{4.5.17}$$

In the same manner, one verifies that

$$\lim_{\epsilon\to 0} B_{k,\epsilon}^{\alpha(n)+} = B_k^{z+}. \tag{4.5.18}$$

Applying identities (4.5.12)-(4.5.18) to equation (4.5.11) demonstrates the lemma. ■

Lemma 4.11

$$\lim_{\epsilon\to 0} D_{\alpha(i),\epsilon}(t_0) = D_{\alpha(i)}(t_0) \tag{4.5.19}$$

Proof If $\alpha(i)$ is shock wave or a linearly degenerate wave, then use formula (2.3.11) to compute

$$\begin{aligned}\lim_{\epsilon\to 0} D_{\alpha(i),\epsilon}(t_0) =& \lim_{\epsilon\to 0} |\sigma_{\alpha(i)}|\big|\dot{\tau}_i - s_i(v^-_{\alpha(i)}, v^+_{\alpha(i)})\big| \\ =& |\sigma_{\alpha(i)}|\big|\dot{z} - s_i(v^-_{\alpha(i)}, v^+_{\alpha(i)})\big| \\ =& D_{\alpha(i)}(t_0). \end{aligned} \tag{4.5.20}$$

If α is a rarefaction wave, repeat the calculation with formula (2.3.12). ■

Proof of Lemma 3.1 for non-isolated waves. It suffices to demonstrate this lemma at a point (t_0, z_0). Under the same conditions as those in the proof for isolated waves, we take two front-tracking approximations u and v and construct the ϵ-splitting V_ϵ for the discontinuity at (t_0, z_0). The lemma for isolated waves states that

$$\sum_{i=1}^{n} \sum_{k=1}^{n} E_{k,\epsilon}\big(\alpha(i)\big) \leq \mathcal{O}(1) \sum_{i=1}^{n} D_{\alpha(i),\epsilon}(t_0). \tag{4.5.21}$$

Lemmas 4.10 and 4.11 show that in the limit as ϵ goes to zero,

$$\sum_{k=1}^{n} E_k(z_0) \leq \mathcal{O}(1) \sum_{i=1}^{n} D_{\alpha(i)}(t_0). \tag{4.5.22}$$

■

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