A discrete BGK approximation for strongly degenerate parabolic problems with boundary conditions

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Abstract

We consider a class of BGK systems with a finite number of velocities, depending on a positive relaxation parameter, that approximate strongly degenerate hyperbolic–parabolic equations with initial boundary condition. We prove a priori estimates for the solutions of the systems, showing that these functions converge towards the entropy solutions of strongly degenerate problems when the parameter let to zero.

Key-words: entropy solutions – strongly degenerate parabolic equations – singular perturbation problems – BGK models

1 Introduction

In this paper we are interested in the approximation of the following parabolic equation

$$\partial_t u + \partial_x A(u) = \partial_{xx}[B(u)], \qquad (x,t) \in \mathbb{R}^+ \times (0,T), \tag{1.1}$$

with initial data

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^+ \ , \tag{1.2}$$

and boundary condition

$$u(0,t) = a_0(t) \qquad t \in (0,T) \ ; \tag{1.3}$$

here $u = u(x,t) \in \mathbb{R}$ with $(x,t) \in \mathbb{R}^+ \times (0,T)$.

We assume that A, B are locally lipschitz functions, u_0 and a_0 are BV functions and the function $B(u)$ is not decreasing.

This assumption allows that the diffusion function B can be constant for some intervals of the state function u; for these values the problem is completely hyperbolic and it is necessary to give an entropy formulation for it (see [21]). Obviously the situation is different for the solutions that take values in the intervals in which B' is strictly separate from zero; in such case the problem is purely parabolic and the solution is classic. In general the equation is of parabolic–hyperbolic type; such phenomena appears in many application models, (see [5], [6], [7]).

It needs a formulation that considers both the behaviors of the problem. In our case there is also the difficulty related to the boundary conditions, in fact it is well known that for first order hyperbolic problems this condition is not achieved in a classical sense and should be interpreted as a consistent condition (see [12], [16]).

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An entropy formulation for the problem (1.1)–(1.3) was given in [13] in the multidimensional case for homogeneous boundary data, proving that this is well-posed.

More recently it was given an entropy formulation for general boundary data (see Definition 1.1) and existence and uniqueness results were proved (see [26], [28]).

In particular in [26] was proved that the entropy solution of (1.1)–(1.3) can be obtained as limit of solutions of regularized equations of non degenerate parabolic type (the diffusion function B is approximated by functions B^ϵ strictly increasing). Other results are contained in [7], for 1-dimensional case with homogeneous boundary data and in [5] and [6] for some application model with different boundary value problems.

In this paper we are interested in the approximation of the above problem by means of a sequence of semilinear systems of conservation laws with source and initial-boundary conditions.

We introduce the following system

$$\partial_t f_k^\epsilon + \gamma_k^\epsilon \partial_x f_k^\epsilon = \frac{1}{\epsilon}(M_k(u^\epsilon,\epsilon)) - f_k^\epsilon) \qquad \text{in } \mathbb{R}^+ \times (0,T), \qquad k = 1 \cdots N \tag{1.4}$$

with $\gamma_k^\epsilon = \lambda_k + \frac{\theta_k}{\epsilon}$, $\epsilon > 0$ and

$$u^\epsilon(x,t) = \sum_{i=1}^{N} f_i^\epsilon(x,t) \ , \tag{1.5}$$

where the functions M_i (called maxwellian functions) satisfy the following properties

(M_1) $\sum_{i=1}^N M_i(w,\epsilon) = w$ for all $\epsilon \in]0,1]$ and for all $w \in I$;

(M_2) $\sum_{i=1}^N \gamma_i^\epsilon M_i(w,\epsilon) = A(w)$ for all $\epsilon \in]0,1]$ and for all $w \in I$;

(M_3) $\sum_{i=1}^N \theta_i^2 M_i(w,0) = B(w)$ for all $w \in I$;

(M_4) $M_i(w,\epsilon) \to M_i(w,0)$, when $\epsilon \to 0$, uniformly for w in I ;

(M_5) $M_i'(\cdot,\epsilon) \geq 0$ in I for all $\epsilon \in]0,1]$;

here I is an interval of $\mathbb{R}$.

These properties assure that systems (1.4) approximate the problem (1.1). In fact it is easy to see, formally, that if the sequence $\{u^\epsilon\}$ converges to some limit function u strongly in $C([0,T]; L^1_{loc}(\mathbb{R}^+))$, the function u is a weak solution of equation (1.1) [11].

In particular the condition (M_5) is a stability condition and is crucial in proving the comparison result for the solutions of the system (1.4) and compactness properties for the sequence $\{f_i^\epsilon\}$. In fact we will show that, under the stability condition (M_5), our approximation (1.4)–(1.7) keeps the monotonicity properties which are typical of equation (1.1).

We complement the system by the initial conditions

$$f_i^\epsilon(x,0) = M_i(u_0(x),0) \ , \qquad i = 1 \cdots N \tag{1.6}$$

and the boundary condition for the functions satisfying the equation in (1.4) with positive transport velocity,

$$f_i^\epsilon(0,t) = M_i(a_0(t),0) \qquad \text{for } i \in \{1 \cdots N\} \text{ such that } \gamma_i^\epsilon > 0. \tag{1.7}$$

System (1.4), (1.6), (1.7) can be considered as a discrete BGK model [3].

These kind of models were introduced in the kinetic theory of the gases in order to simplify the Bolzmann equation [8], [14], [15]. In the BGK model the collision term is replaced by the quantity $Q(f) = \frac{1}{\epsilon}(M(f) - f)$, where M is a Maxwellian distribution and ϵ is the mean free path of the molecules.

In particular Euler and incompressible Nevier–Stokes equations can be obtained by the BGK equation introducing suitable scaling for the x, t variables and letting the parameter ϵ go to zero [15], [14], [36].

This paper was inspired by the results in [11], [32] and [19]. The first paper considers the approximation of the Cauchy problem for the equation (1.1) in the multidimensional case, with a BGK model similar to that in (1.4); in the second one the authors prove the convergence of a similar relaxation system, with two transport coefficients independent from ϵ, towards an initial-boundary value problem for a conservation law; in the last paper the authors consider a particular system (1.4) with three velocities that converges to a weakly parabolic problem with initial and boundary condition (diffusion B is not allowed to be constant in a interval).

BGK approximations to initial-boundary value problems for conservation laws is also studied in [29]; here the author introduces a new technique to estimate the boundary terms which allows to consider a general BGK model, without constrains on the number of velocities. This technique will be crucial in the present paper to treat strongly degenerate parabolic problems.

We recall that this kind of discrete velocities approximations was introduced in [31] for conservation laws in $\mathbb{R}^d \times [0,T]$. The study of diffusive limit of Cauchy problems for discrete velocities hyperbolic systems can be found in [22, 27, 25, 24, 36] and in [20, 30, 18, 4] for numerical results. Finally, relaxation approximations for boundary value problems are studied in [34, 33, 38, 37] (besides [32]) in the setting of hyperbolic equations.

Let us state more precisely the convergence result proved in this paper.

We consider the following definition for entropy solution of strongly degenerate problem (1.1)–(1.3) (see [26], [28])

Definition 1.1 *Let $T > 0$. Let $u_0 \in BV(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ $a_0 \in BV((0,T)) \cap C((0,T))$ a function $u \in L^\infty(\Omega \times (0,T))$ is said entropy solution of problem (1.1)–(1.3) if and only if*

(i) (regularity) *:*

$$B(u) \in L^2(0,T; H^1(\mathbb{R}^+)),$$

and $B(u)_{|\{0\}\times(0,T)} = B(a_0)$;

(ii) (entropy condition) *Let $K_x^\pm(u,c) = H^\pm(u-c)(A(u) - A(c) - \partial_x B(u))$ then*

$$\int_Q \left\{ [u-c]_\pm \phi_t + K_x^\pm(u,c)\phi_x \right\} dxdt + \int_0^\infty [u_0 - c]_\pm \, dx \geq 0, \tag{1.8}$$

for every $\phi \in H^1(Q)$, $\phi \geq 0$, such that $\phi H^\pm(a_0 - c)_{|\{0\}\times(0,T)} = 0$.

Where $H^\pm(s)$ are the Heaviside functions $\frac{sgn(s)\pm 1}{2}$ and $[s]_\pm$ denote respectively positive and negative part of s.

We have the following definition for the weak solution to the relaxation system (1.4).

Definition 1.2 *Let $u_0 \in BV(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ $a_0 \in BV((0,T)) \cap C((0,T))$. The N-ple $(f_1^\epsilon \cdots, f_N^\epsilon) \in (BV(\mathbb{R}^+ \times (0,T)) \cap L^\infty(\mathbb{R}^+ \times (0,T)))^N$ is a weak solution to problem (1.4), (1.6), (1.7) if*

i) for every $\phi \in C_0^1(\mathbb{R}^+ \times (0,T))$

$$\int_0^T \int_{\mathbb{R}^+} f_i^\epsilon (\phi_t + \gamma_i^\epsilon \phi_x) + \frac{1}{\epsilon}(M_i(u^\epsilon, \epsilon) - f_i^\epsilon) \, dxdt = 0, \qquad i = 1 \cdots N;$$

ii) $f_i^\epsilon(x,0) = M_i(u_0(x),0)$ for almost every $x \in \mathbb{R}^+, i = 1 \cdots N$;

iii) $f_i^\epsilon(0,t) = M_i(a_0(t),0)$ for almost every $t \in (0,T)$, for $i \in \{1 \cdots N\}$ such that $\gamma_i^\epsilon > 0$.

In this paper we prove comparison and stability results with respect to the data for the weak solutions of problem (1.4)-(1.7), moreover we prove a priori estimates for it.

Then we prove the main result.

Theorem 1.1 *Let $u_0 \in BV(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ and $a_0 \in BV((0,T)) \cap C((0,T))$, such that $u_0(\cdot) \in I$ and $a_0(\cdot) \in I$. Let $(f_1^\epsilon \cdots, f_N^\epsilon)$ be the solution of problem (1.4), (1.6), (1.7) in $(\mathbb{R}^+ \times (0,T])$, $u^\epsilon = \sum_{i=1}^N f_i^\epsilon$ and let u be the solution of the problem (1.1)–(1.3) in $\mathbb{R}^+ \times (0,T))$. Then*

$$\lim_{\epsilon \to 0} u^\epsilon = u \qquad \text{in } C([0,T]; L^1_{loc}(\mathbb{R}^+)) \ .$$

In this case the main difficulty is to treat the boundary terms that appear in this problem, in particular we have to verified that the weak boundary condition proposed in Definition (1.1) is achieved for the limit of solutions of the relaxation approximation (1.4), (1.6), (1.7).

We remark that the results proved in this paper can be useful to implement numerical schemes for the approximation of the solution of the strongly degenerate parabolic boundary value problem.

The paper is organized in two further sections. In the next section we state comparison results for problems (1.4), (1.6), (1.7) and existence and uniqueness theorem. Moreover we prove a set of a priori estimates which ensure relatively compactness in $C([0,T]; L^1_{loc}(\mathbb{R}^+))$ for the sequences $\{f_i^\epsilon\}$ and $\{u^\epsilon\}$. The last section is devoted to prove that the limit of the sequence $\{u^\epsilon\}$ is in fact the entropy solution of problem (1.1)-(1.3).

2 A priori estimates

The goal of this section is to establish stability, comparison, existence and uniqueness results and some a priori estimates for the solutions of the system (1.4). Most of the proofs are the extension of those in [19] for a particular choise of the Maxwellian functions, to a general BGK system of type (1.4). On the contrary the result in Proposition 2.4 is proved by means a different technique which does not involve BV estimates for the traces $f_i^\epsilon(0,t)$; in [19] the proofs of such boundary estimates for i such that $\theta_i = 0$ are based on the assumption of parabolicity for the equation (1.1).

We will make use of the following lemma [21].

Lemma 2.1 *Let u be a weak solution to the linear Cauchy problem*

$$\begin{cases} \partial_t u + \theta \partial_x u = g(x,t) \ , \\ u(x,0) = u_0(x) \ , \end{cases}$$

for $x \in \mathbb{R}^+$ and $t \in (0,T)$ $(T > 0)$, and v the solution of the above problem with a function h in place of g and the initial data v_0 in place of u_0. Then for every $\phi \in C_0^1(\mathbb{R}^+ \times (0,T))$, with $\phi \geq 0$, there holds

$$\int_0^T \int_{\mathbb{R}^+} [u-v]_+ \partial_t \phi + \theta [u-v]_+ \partial_x \phi \ dx \ dt \geq - \int_0^T \int_{\mathbb{R}^+} H(u-v)(g-h)\phi \ dx \ dt \ , \tag{2.1}$$

where H is the Heaviside function.

In the following $f_i^\epsilon(0,t), u^\epsilon(0,t)$ denote the traces of the functions $f_i^\epsilon(x,t), u^\epsilon(x,t)$ on the boundary $x = 0$.

We assume ϵ varying in a suitable small interval $(0,\overline{\epsilon})$ where the maxwellian functions are increasing in u and $\operatorname{sgn} \gamma_i^\epsilon = \operatorname{sgn} \theta_i$ for all i such that $\theta_i \neq 0$. Moreover we set $Z^+ = \{i = 1, \ldots, N : \theta_i > 0\}$, $Z^- = \{i = 1, \ldots, N : \theta_i < 0\}$, $Z_0^+ = \{i = 1, \ldots, N : \gamma_i^\epsilon \geq 0, \theta_i = 0\}$, $Z_0^- = \{i = 1, \ldots, N : \gamma_i^\epsilon < 0, \theta_i = 0\}$.

Proposition 2.1 *Assume that $u_0, \overline{u_0} \in BV(\mathbb{R}^+)$, $a_0, \overline{a_0} \in BV((0,T))$. Let $(f_i^\epsilon)_{i=1,\ldots,N}$ be a solution of problem (1.4)-(1.7) and $u^\epsilon \in I$ for almost every $(x,t) \in \mathbb{R}^+ \times (0,T)$. Let $(\overline{f}_i)_{i=1,\ldots,N}$ be another* *solution corresponding to the initial-boundary conditions* $(\overline{u_0}, \overline{a_0})$ *and* $\overline{u^\epsilon} = \sum_{i=1}^N \overline{f_i^\epsilon} \in I$ *for almost every* $(x,t) \in \mathbb{R}^+ \times (0,T)$ *. Let* $\hat\gamma = \max\{|\gamma_i^\epsilon| : \gamma_i^\epsilon < 0, i = 1,\ldots,N\}$*. Then for every* $K \in \mathbb{R}^+$ *and for almost every* $t \in (0,T)$ *the following inequalities hold*

$$\begin{aligned}&\textstyle\int_0^K \sum_{i=1}^N [f_i^\epsilon(x,t) - \overline{f_i^\epsilon}(x,t)]_+ dx \le \int_0^{K+\hat\gamma t}[u_0(x) - \overline{u_0}(x)]_+ dx \\ &+ \max\{\gamma_i^\epsilon : i = 1,\ldots,N\} \textstyle\int_0^t [a_0(s) - \overline{a_0}(s)]_+ ds \ ,\end{aligned} \tag{2.2}$$

$$\begin{aligned}&\textstyle\sum_{Z^- \cup Z_0^-} |\gamma_i^\epsilon| \int_0^t [f_i^\epsilon(0,s) - \overline{f_i^\epsilon}(0,s)]_+ ds \\ &\le \textstyle\int_0^{\hat\gamma t}[u_0(x) - \overline{u_0}(x)]_+ dx + \max\{\gamma_i^\epsilon : i = 1,\ldots,N\} \int_0^t [a_0(s) - \overline{a_0}(s)]_+ ds \ .\end{aligned} \tag{2.3}$$

Proof. Setting $w_i = f_i^\epsilon - \overline{f_i^\epsilon}$ and using inequality (2.1) we obtain

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}^+} \sum_{i=1}^N [w_i]_+ \left(\phi_t + \gamma_i^\epsilon \phi_x\right) dxdt \\ &\qquad\qquad \ge -\frac{1}{\epsilon}\int_0^T \int_{\mathbb{R}^+} \sum_{i=1}^N H(w_i)(M_i(u^\epsilon,\epsilon) - M_i(\overline{u^\epsilon},\epsilon) - w_i) \ \phi \ dxdt \ ,\end{aligned}$$

for every $\phi \in C_0^1(\mathbb{R}^+ \times (0,T))$, $\phi \ge 0$. If $H(w_i) = 1$ for $i = 1,\ldots,N$ or $H(w_i) = 0$ for $i = 1,\ldots,N$ then the right hand side of the above inequality is zero. In order to prove that it is always nonnegative set $J_+ = \{i : H(w_i) = 1\}$; then, thanks to (M_1) and (M_5)

$$\sum_{J_+} (M_i(u^\epsilon,\epsilon) - M_i(\overline{u^\epsilon},\epsilon) - w_i) \le \left(\sum_{J_+} M_j'(\eta,\epsilon) - 1\right) \sum_{J_+} w_j \le 0 \ ,$$

where η is a suitable intermediate value.

Hence

$$\int_0^T \int_{\mathbb{R}^+} \sum_{i=1}^N [f_i^\epsilon - \overline{f_i^\epsilon}]_+ \left(\phi_t + \gamma_i^\epsilon \phi_x\right) dxdt \ge 0 \ .$$

Now, choosing a sequence of test functions approximating the characteristic function of the set $\{(x,s) \in \mathbb{R}^+ \times (0,t) : 0 \le x \le K + \hat\gamma(t-s)\}$ we obtain

$$\begin{aligned}&\int_0^K \sum_{i=1}^N [f_i^\epsilon(x,t) - \overline{f_i^\epsilon}(x,t)]_+ dx \le \\ &\int_0^{K+\hat\gamma t} \sum_{i=1}^N [M_i(u_0,\epsilon) - M_i(\overline{u_0},\epsilon)]_+ dx + \int_0^t \sum_{i=1}^N \gamma_i^\epsilon [f_i^\epsilon(0,s) - \overline{f_i^\epsilon}(0,s)]_+ ds \\ &+ \int_0^t \sum_{i=1}^N (-\hat\gamma - \gamma_i^\epsilon) \left[f_i^\epsilon(K + \hat\gamma(t-s), s) - \overline{f_i^\epsilon}(K + \hat\gamma(t-s), s)\right]_+ ds\end{aligned}$$

and (2.2) follows since $-\hat\gamma - \gamma_i^\epsilon \le 0$ for $i = 1,\ldots,N$.

In order to prove inequality (2.3) we introduce a sequence of test functions approximating the characteristic function of the set $\{(x,s) \in \mathbb{R}^+ \times (0,t) : 0 \le x \le \hat\gamma(t-s)\}$ and, with the same technique used in the first part of the proof, we obtain the claim. □

As a consequence of the previous proposition we have the following two results.

Corollary 2.1 *Assume that* $u_0, \overline{u_0} \in BV(\mathbb{R}^+)$, $a_0, \overline{a_0} \in BV((0,T))$. *Let* $(f_i^\epsilon)_{i=1,\dots,N}$ *be a solution of problem (1.4)-(1.7) and* $u^\epsilon \in I$ *for almost every* $(x,t) \in \mathbb{R}^+ \times (0,T)$. *Let* $(\overline{f}_i)_{i=1,\dots,N}$ *be another solution corresponding to the initial-boundary conditions* $(\overline{u_0}, \overline{a_0})$ *and* $\overline{u^\epsilon} = \sum_{i=1}^N \overline{f_i^\epsilon} \in I$ *for almost every* $(x,t) \in \mathbb{R}^+ \times (0,T)$. *Assume that*

$$u_0 \leq \overline{u}_0 \qquad \text{almost everywhere in } \mathbb{R}^+ \;;$$

$$a_0 \leq \overline{a}_0 \qquad \text{almost everywhere in } (0,T) \;.$$

Then

$$f_i^\epsilon \leq \overline{f}_i^\epsilon \qquad \text{almost everywhere in } \mathbb{R}^+ \times (0,T) \;\; \text{for } i = 1, \dots, N.$$

Corollary 2.2 *Assume that* $u_0 \in BV(\mathbb{R}^+)$, $a_0 \in BV((0,T))$ *and* $u_0, a_0 \in I$. *Let* $(f_i^\epsilon)_{i=1,\dots,N}$ *be a solution of problem (1.4)-(1.7) and* $u^\epsilon \in I$ *for almost every* $(x,t) \in \mathbb{R}^+ \times (0,T)$. *Then for almost every* $(x,t) \in \mathbb{R}^+ \times (0,T)$

$$M_i(\min\{essinf(u_0), essinf(a_0)\}, \epsilon) \leq f_i^\epsilon \leq M_i(\max\{esssup(u_0), esssup(a_0)\}, \epsilon)$$

for $i = 1, \dots, N$.

Proof. Observe that for every $p \in \mathbb{R}$ the vector $(M_1(p,\epsilon), \dots, M_N(p,\epsilon))$ is a weak solution to system (1.4)-(1.7) with initial and boundary data identically equal to p. Then the claim follows by Corollary 2.1.□

In the following we set

$$I := \{u \in \mathbb{R}; \min\{essinf(u_0), essinf(a_0)\} \leq u \leq \max\{esssup(u_0), esssup(a_0)\}\}.$$

By using standard regularity properties, Corollary 2.2 and proceeding as in [31] we obtain the following result for the solutions of problem (1.4)-(1.7) (see [29]).

Theorem 2.1 *Let* $u_0 \in BV(\mathbb{R}^+), a_0 \in BV((0,T))$. *Then there exists a unique global weak solution* $(f_i^\epsilon)_{i=1,\dots,N}$ *to problem (1.4)-(1.7) and* $u^\epsilon \in I$. *Moreover, when the data* u_0, a_0 *are in the class* C^k $(k \geq 1)$ *with* $u_0(0) = a_0(0)$, $u_0'(0) = a_0'(0)$ *then* $f_i^\epsilon \in C^1(\mathbb{R}^+ \times [0,T)) \cap C^k(\mathbb{R}^+ \times (0,T) \setminus \cup_{Z^+ \cup Z_0^+} \Gamma_j)$, $i = 1, \dots, N$, *where* $\Gamma_j := \{(x,t) \in \mathbb{R}^+ \times [0,T) : x = \gamma_j^\epsilon t\}$

Now we prove some estimates for the solutions of problem (1.4) - (1.7) in the case of smooth data, which can be extended to the general BV case thanks to Proposition 2.1.

Proposition 2.2 *Let* $u_0 \in C^2(\overline{\mathbb{R}^+}) \cap BV(\mathbb{R}^+)$ *and* $a_0 \in C^2([0,T)) \cap BV((0,T))$ *with* $u_0(0) = a_0(0)$, $u_0'(0) = a_0'(0)$. *Let* $(f_i^\epsilon)_{i=1,\dots,N}$ *be the solution of problem (1.4) - (1.7). Then for every* $t \in [0,T)$ *the following estimate holds for* $i = 1, \dots, N$

$$\int_{\mathbb{R}^+} |\partial_t f_i^\epsilon(x,t)| dx \leq \max\{|\gamma_i^\epsilon| : i = 1, \dots, N\} TV(u_0) + \max\{\gamma_i^\epsilon : i = 1, \dots, N\} TV(a_0) \;. \tag{2.4}$$

Proof. We fix $L, t \in \mathbb{R}^+$ and we set $p_i = \partial_t f_i^\epsilon$. Now we first differentiate the equations with respect to t and multiply each equation by the corresponding $sgn(p_i)$, then taking the sum for $i = 1, \dots, N$ and integrating on the domain $D = \{(x,s) \in \mathbb{R}^+ \times (0,t) : 0 \leq x \leq L + \hat{\gamma}(t-s)\}$, where $\hat{\gamma}$ is defined in Proposition 2.1, we have

$$\int\int_D \partial_t \left(\sum_{i=1}^N |p_i(x,t)| \right) + \partial_x \left(\sum_{i=1}^N \gamma_i^\epsilon |p_i(x,t)| \right) dxdt = \frac{1}{\epsilon} \int\int_D F(x,t) dxdt$$

where

$$F(x,t) := \sum_{i=1}^{N} \left(\sum_{j=1}^{N} M_j'(u^\epsilon(x,t)) sgn(p_j) - sgn(p_i) \right) p_i(x,t) \ .$$

Since $F(x,t) \le 0$ (thanks to condition (M_1) and (M_5)), using the divergence theorem on the domain D we obtain

$$\int_0^L \sum_{i=1}^{N} |p_i(x,s)| dx \le \int_0^{L+\hat{\gamma}t} \sum_{i=1}^{N} |p_i(x,0)| dx + \int_0^t \sum_{i=1}^{N} \gamma_i^\epsilon |p_i(0,s)| ds \ ;$$

then, using the initial and the boundary conditions, we have the claim.□

The estimate proved in the above proposition is not uniform in ϵ, due to the expressions of γ_i^ϵ, however it allows to obtain the following uniform estimate for the BV norm of the traces of f_i^ϵ on the boundary, for each index i such that $\theta_i < 0$.

Proposition 2.3 *Let $u_0 \in C^2(\overline{\mathbb{R}^+}) \cap BV(\mathbb{R}^+)$ and $a_0 \in C^2([0,T)) \cap BV((0,T))$ with $u_0(0) = a_0(0)$, $u_0'(0) = a_0'(0)$. Let $(f_i^\epsilon)_{i=1,\dots,N}$ be the solution of problem (1.4) - (1.7). Then there exists a constant $K_1 = K_1(\lambda_i, \theta_i, TV(a_0), TV(u_0))$, not depending on ϵ, such that for every $h > 0$ small enough*

$$\int_0^{T-h} \sum_{Z^-} |f_i^\epsilon(0,t+h) - f_i^\epsilon(0,t)| dt \le K_1 h \ , \tag{2.5}$$

Proof. For $i = 1, \dots, N$ we set $w_i(x,t) = f_i^\epsilon(x,t+h) - f_i^\epsilon(x,t)$ and

$$C_i(x,t) := \frac{M_i(u^\epsilon(x,t+h)) - M_i(u^\epsilon(x,t))}{u^\epsilon(x,t+h) - u^\epsilon(x,t)} \ .$$

We consider now the equations verified by the functions w_i for $i = 1, \dots, N$

$$\partial_t w_i(x,t) + \gamma_i^\epsilon \partial_x w_i(x,t) = \frac{1}{\epsilon} \left((C_i(x,t) - 1) w_i(x,t) + C_i(x,t) \sum_{j \ne i} w_j(x,t) \right) \ ,$$

and we multiply each of them by the corresponding $sgn(w_i)$; summing up the equations and integrating on the domain $D := \{(x,t) \in \mathbb{R}^+ \times (0,T) : 0 < x < \hat{\gamma}(T-t)\}$ we obtain the equality

$$\int\int_D \partial_t \left(\sum_{i=1}^{N} |w_i| \right) + \partial_x \left(\sum_{i=1}^{N} \gamma_i^\epsilon |w_i| \right) dxdt = \frac{1}{\epsilon} \int\int_D F dxdt$$

where

$$F(x,t) := \sum_{i=1}^{N} \left(\sum_{j=1}^{3} C_j(x,t) sgn(w_j) - sgn(w_i) \right) w_i(x,t) \ .$$

We observe that the conditions (M_1) and (M_5) imply that $0 \le C_i(x,t) \le 1$ for $i = 1, \dots, N$ and $\sum_{i=1}^N C_i(x,t) = 1$ and then that $F(x,t) \le 0$; therefore, using the divergence theorem on the domain D we obtain

$$\sum_{Z^- \cup Z_0^-} |\gamma_i^\epsilon| \int_0^t |f_i^\epsilon(0,t+h) - f_i^\epsilon(0,t)| dt \le$$

$$\int_0^{\hat{\gamma}t} \sum_{i=1}^{N} |f_i^\epsilon(x,h) - f_i^\epsilon(x,0)| dx + \sum_{Z^+ \cup Z_0^+} \gamma_i^\epsilon \int_0^t |M_i(a_0(t+h)) - M_i(a_0(t))| dt \ ,$$

and then

$$\sum_{Z^-}\int_0^t |f_i^\epsilon(0,t+h)-f_i^\epsilon(0,t)|dt \le \frac{\sqrt{\epsilon}}{\min\{\sqrt{\epsilon}\lambda_i+\theta_i : i\in Z^-\}}\int_0^{\hat{\gamma}t}\sum_{i=1}^N |f_i^\epsilon(x,h)-f_i^\epsilon(x,0)|dx$$

$$+\frac{\max\{\sqrt{\epsilon}\lambda_i+\theta_i : i\in Z^+\cup Z_0^+\}}{\min\{\sqrt{\epsilon}\lambda_i+\theta_i : i\in Z^-\}}TV(a_0)h\ .$$

The claim follows using (2.4). □

Next lemma gives an estimate for the spatial derivatives of the functions f_i^ϵ on the boundary, in terms of the BV-norm of $a_0(t)$. The result is crucial to prove a further uniform estimate holding for the sequence $\{f_i^\epsilon\}$, which will be useful to derive compactness properties .

Lemma 2.2 *[29] Let $u_0\in C^2(\overline{\mathbb{R}^+})\cap BV(\mathbb{R}^+)$ and $a_0\in C^2([0,T))\cap BV((0,T))$ with $u_0(0)=a_0(0)$, $u_0'(0)=a_0'(0)$. Let $(f_i^\epsilon)_{i=1,\dots,N}$ be the solution of problem (1.4) - (1.7).Then*

$$\sum_{i=1}^N\int_0^T \gamma_i^\epsilon|\partial_x f_i^\epsilon(t,0)|dt \le \int_0^T |a_0'(t)|dt \tag{2.6}$$

Proof. Using the equations for f_i^ϵ we obtain the following inequality

$$\sum_{i=1}^N \gamma_i^\epsilon|\partial_x f_i^\epsilon(0,t)| \le \sum_{i\in Z^+\cup Z_0^+}\left(\frac{|M_i(u^\epsilon(0,t))-M_i(a_0(t)|}{\epsilon}+|\partial_t M_i(a_0(t))|\right)$$

$$-\sum_{i\in Z^-\cup Z_0^-}\left|\frac{M_i(u^\epsilon(0,t))-f_i^\epsilon(0,t)}{\epsilon}-\partial_t f_i^\epsilon(0,t)\right|$$

which can be rewritten as

$$\sum_{i=1}^N \gamma_i^\epsilon|\partial_x f_i^\epsilon(0,t)| \le \frac{|u^\epsilon(0,t)-a_0(t)|}{\epsilon}+|a_0'(t)|$$

$$-\textstyle\sum_{i\in Z^-\cup Z_0^-}\left(\frac{|M_i(u^\epsilon(0,t)-M_i(a_0(t))|}{\epsilon}+|M_i'(a_0(t))|+\left|\frac{M_i(u^\epsilon(0,t)-f_i^\epsilon(0,t)}{\epsilon}-\partial_t f_i^\epsilon(0,t)\right|\right)\ .$$

Now, it is readly seen that

$$\sum_{i=1}^N \gamma_i^\epsilon|\partial_x f_i^\epsilon(0,t)| \le \frac{|u^\epsilon(0,t)-a_0(t)|}{\epsilon}+|a_0'(t)|$$

$$-\left|\sum_{i\in Z^-\cup Z_0^-}\left(\frac{M_i(a_0(t))-f_i^\epsilon(0,t)}{\epsilon}+\partial_t(M_i(a_0(t))-f_i^\epsilon(0,t))\right)\right|$$

and using that $u^\epsilon(0,t)=\sum_{Z^+\cup Z_0^+}M_i(a_0(t))+\sum_{Z^-\cup Z_0^-}f_i^\epsilon(0,t)$ we derive the following inequality

$$\sum_{i=1}^N \gamma_i^\epsilon|\partial_x f_i^\epsilon(0,t)| \le \frac{|u^\epsilon(0,t)-a_0(t)|}{\epsilon}+|a_0'(t)|-\left|\frac{a_0(t)-u^\epsilon(0,t)}{\epsilon}+\partial_t(a_0(t)-u^\epsilon(0,t))\right|\ .$$

Thanks to convexity property of the absolute value function we obtain

$$\sum_{i=1}^N \gamma_i^\epsilon|\partial_x f_i^\epsilon(0,t)| \le -\operatorname{sgn}\left(\frac{a_0(t)-u^\epsilon(0,t)}{\epsilon}\right)\partial_t(a_0(t)-u^\epsilon(0,t))+|a_0'(t)|$$

$$=-\partial_t(|a_0(t)-u^\epsilon(0,t)|)+|a_0'(t)|\ .$$

Integrating on $(0,T)$ the above estimate we obtain

$$\sum \int_0^T \gamma_i^\epsilon |\partial_x f_i^\epsilon(0,t)| dt \leq - \int_0^T \left(\partial_t(|a_0(t) - u^\epsilon(0,t)|) + |a_0'(t)|\right) dt$$

which is the claim.

Now we use the previous lemma to estimate the spatial derivatives of f_i^ϵ in the interior domain.

Proposition 2.4 *Let $u_0 \in C^2(\overline{\mathbb{R}^+}) \cap BV(\mathbb{R}^+)$ and $a_0 \in C^2([0,T)) \cap BV((0,T))$ with $u_0(0) = a_0(0)$, $u_0'(0) = a_0'(0)$. Let $(f_i^\epsilon)_{i=1,\ldots,N}$ be the solution of problem (1.4) - (1.7). Then there exists a constant $K_2 = K_2(TV(a_0), TV(u_0))$, not depending on ϵ, such that for every $L > 0$ and for every $h > 0$ small enough*

$$\int_0^L |\partial_x f_i^\epsilon(x,t)| dx \leq K_2 \ , \qquad \textit{for } i = 1, \ldots, N \ . \tag{2.7}$$

Proof. We fix $L, t > 0$ and we set $m_i = \partial_x f_i^\epsilon$; applying the technique of the previous proofs we establish that

$$\int_0^L \sum_{i=1}^N |m_i(x,t)| dx \leq \int_0^{L+\hat{\gamma}t} \sum_{i=1}^N |m_i(x,0)| dx + \int_0^t \sum_{i=1}^N \gamma_i^\epsilon |m_i(0,s)| ds \ .$$

The last integral can be estimate using Lemma 2.2 and the proof is complete.□

Now we state a proposition which gives an estimate of the deviation from the equilibrium in the L^1 norm (see [11] for the proof). The result will be crucial in the proof of the last uniform estimate in this section and in the proof of consistency of our relaxation approximation in the next section.

Proposition 2.5 *Let $u_0 \in C^2(\overline{\mathbb{R}^+}) \cap BV(\mathbb{R}^+)$ and $a_0 \in C^2([0,T)) \cap BV((0,T))$ with $u_0(0) = a_0(0)$, $u_0'(0) = a_0'(0)$. Let $(f_i^\epsilon)_{i=1,\ldots,N}$ be the solution of problem (1.4) - (1.7). Then there exists a constant $K_3 = K_3(\lambda_i, \theta_i, TV(a_0), TV(u_0))$, not depending on ϵ, such that for every $L > 0$*

$$\sum_{i=1}^N \int_0^L |f_i^\epsilon(x,t) - M_i(u^\epsilon, \epsilon)| dx \leq \sqrt{\epsilon} K_3 \ .$$

The following proposition completes the set of estimates necessary to prove the convergence result in the next section.

Proposition 2.6 *Let $u_0 \in C^2(\overline{\mathbb{R}^+}) \cap BV(\mathbb{R}^+)$ and $a_0 \in C^2([0,T)) \cap BV((0,T))$ with $u_0(0) = a_0(0)$, $u_0'(0) = a_0'(0)$. Let $(f_i^\epsilon)_{i=1,\ldots,N}$ be the solution of problem (1.4) - (1.7). Then for every $L > 0$ there exist a positive constant h_0 and a continuous nondecreasing function $w : [0, h_0] \to \mathbb{R}$, not depending on ϵ, with $w(0) = 0$, such that, for $t \in (0, T-h)$ and $h \in (0, h_0)$*

$$\int_0^L |u^\epsilon(x,t+h) - u^\epsilon(x,t)| dx \leq w(h) \ .$$

Proof. We use Lemma 5 in [21]; in view of Proposition 2.4 we must only prove that there exists a constant C_L such that

$$\left| \int_0^L \partial_t u^\epsilon(x,t) \varphi(x) dx \right| \leq C_L \|\varphi\|_{C^1} \qquad \text{for every function } \varphi \in C_0^1((0,L)) \ .$$

Using the equations for f_i^ϵ and the condition (M_2) we establish that

$$\left|\int_0^t \partial_t u^\epsilon(x,t)\varphi(x)dx\right| \leq \|\varphi\|_{C^1}\int_0^L \max\{|\gamma_i^\epsilon|\}\sum_{i=1}^N |f_i^\epsilon(x,t)\ - M_i(u^\epsilon,\epsilon)|dx$$

$$+\|\varphi\|_{C^1}\int_0^L |A(u^\epsilon)|dx$$

and using Proposition 2.5 we conclude the proof.□

Now the above propositions in combination with the comparison results (2.2)-(2.3) imply the following theorem.

Theorem 2.2 *Let $L>0$, $u_0 \in BV(\mathbb{R}^+)$ and $a_0 \in BV((0,T))$. Let $(f_i^\epsilon)_{i=1,\dots,N}$ be the solution of problem (1.4) - (1.7) in $(\mathbb{R}^+ \times (0,T))$. Then for every $L,h>0$ there exist two positive constants $h_0, C_1 = C_1(\lambda_i, \theta_i, TV(a_0), TV(u_0), L)$ and a continuous nondecreasing function $w:[0,h_0]\to\mathbb{R}$, not depending on ϵ, with $w(0)=0$, such that*

$$TV(f_i^\epsilon(\cdot,t),(0,L)) \leq C_1 \qquad \textit{for } t\in(0,T]\ , i=1,\dots,N;$$

$$\int_0^L |f_i^\epsilon(x,t) - M_i(u^\epsilon(x,t),\epsilon)|dx \leq C_1\sqrt{\epsilon} \qquad \textit{for } t\in(0,T]\ , i=1,\dots,N;$$

$$\sum_{Z^+}\int_0^{T-h} |f_i^\epsilon(0,t+h) - f_i^\epsilon(0,t)|dt \leq C_1 h\ ;$$

$$\int_0^L |u^\epsilon(x,t+h) - u^\epsilon(x,t)|dx \leq w(h) \qquad \textit{for } t\in(0,T-h)\ , h\in(0,h_0)\ .$$

Using the above results we can extract a subsequence from $\{u^\epsilon\}$ converging to some limit function u. In the next section we will show that such limit is the solution of problem (1.4)- (1.7), concluding the proof of Theorem 1.1 stated in the Introdution.

3 Proof of Theorem 1.1

The last step in the proof of Theorem 1.1 is to prove that the function u obtained as limit of a subsequence of $\{u^\epsilon\}$ verifies conditions $i)$ and $ii)$ of Definition 1.1.

As in [11] we are going to use the convex kinetic entropies $H_{\eta,k}^\epsilon$ associated to each convex entropy η. According to [9], they can be obtained as

$$H_{k,\eta}^\epsilon(f_k^\epsilon) = \int_{\mathbb{R}} \tfrac{1}{2}\left(|f_k^\epsilon - M_k^\epsilon(c)| - |M_k^\epsilon(c)|\right)\eta''(c)dc$$

$$+\tfrac{1}{2}f_k^\epsilon(\eta'(-\infty)+\eta'(\infty))$$

which is convex in f_k^ϵ and has a Lipschitz constant independant of ϵ.

As in [11] we make first a simplifying assumption: we assume that

$$M_k^\epsilon(\cdot) \text{ is strictly increasing in } I=\{u\in\mathbb{R} : |u|\leq\mu_\infty\} \tag{3.8}$$

where $\mu_\infty = max(\|u_0\|_{L^\infty(\mathbb{R}^+)}, \|a_0\|_{L^\infty(\mathbb{R}^+)})$ and let us consider only entropies $\eta \in C^2$. Now for such C^2 entropies, (3.8) ensures that $H^\epsilon_{k,\eta} \in C^1([M^\epsilon_k(-\mu_\infty), M^\epsilon_k(-\mu_\infty)])$. As in [11], using $(M_1) - (M_4)$, we have the following relationships

$$(H^\epsilon_{k,\eta}(M^\epsilon_k(w))' = \eta'(w), \ \forall w \in I \tag{3.9}$$

$$\sum_k H^\epsilon_{k,\eta}(M^\epsilon_k(w)) = \eta(w) - \eta(0)$$

$$\sum_k \gamma^\epsilon_k H^\epsilon_{k,\eta}(M^\epsilon_k(w)) = G_\eta(w) \qquad \text{for } G'_\eta = \eta' A', \ G_\eta(0) = 0 \tag{3.10}$$

$$\sum_k {\theta_k}^2 H^\epsilon_{k,\eta}(M^\epsilon_k(w)) = B_\eta(w) + o(1) \quad \text{for } B'_\eta = \eta' B', \ B_\eta(0) = 0.$$

In the following we set $Q = \mathbb{R}^+ \times (0,T)$.

The first result we need is an $L^2(Q)$-estimate of the quantities $(M_i(u^\epsilon) - f^\epsilon_i)$. In order to do this we will prove some preliminary results.

Let us multiply (1.4) by $H'_k(f^\epsilon_k)$, which is possible since we suppose $H_k \in C^1$. We get

$$\partial_t H_k(f^\epsilon_k) + \partial_x \gamma^\epsilon_k H_k(f^\epsilon_k) = H'_k(f^\epsilon_k)\frac{M^\epsilon_k(f^\epsilon_k) - f^\epsilon_k}{\epsilon} \tag{3.11}$$

If we choose η such that $\eta(0) = 0$, $\eta'(0) = 0$, then $H_k(M^\epsilon_k(0)) = 0, H'_k(M^\epsilon_k(0)) = 0$, and thus $H_k \geq 0$. Integrating over a semi cone and letting it tend to infinity after summing up with respect to k we have

$$\begin{aligned}&\sum_k \int_0^L H_k(f^\epsilon_k)(x,T)dx + \sum_k \int_Q [H'_k(M^\epsilon_k(u^\epsilon)) - H'_k(f^\epsilon_k)]\tfrac{M^\epsilon_k(f^\epsilon_k) - f^\epsilon_k}{\epsilon} dQ \\ &\leq \sum_k \int_{\mathbb{R}^+} H_k(u^0)dx + \sum_k \int_0^T \gamma^\epsilon_k H_k(f^\epsilon_k(0,t))dt = \int_{\mathbb{R}^+} \eta(u^0)dx + \sum_k \int_0^T \gamma^\epsilon_k H_k(f^\epsilon_k(0,t))dt\end{aligned} \tag{3.12}$$

In the following lemma we find an upper bound for the last boundary term.

Lemma 3.1 *Assume hypothesis (3.8). For any C^2 convex entropy $\eta(\cdot)$, on the boundary we have the following estimate,*

$$\int_0^T \sum_k \gamma^\epsilon_k H_{\eta,k}(f^\epsilon_k)(0,t)dt \leq C$$

where C is independant of ϵ.

Proof : Observe that it is not restrictive to assume that the data and the solution of (1.4) are regular since using the stability results of Proposition 2.1 we can treat general data. Imposing boundary condition on the entering characteristics we have for any entropy functions

$$\begin{aligned}\sum_k \gamma^\epsilon_k H_{\eta,k}(f^\epsilon_k) &= \sum_{\gamma^\epsilon_k > 0} \gamma^\epsilon_k H_{\eta,k}(M^\epsilon_k(a_0)) + \sum_{\gamma^\epsilon_k < 0} \gamma^\epsilon_k H_{\eta,k}(f^\epsilon_k) \\ &= G_\eta(a_0) + \sum_{\gamma^\epsilon_k < 0} \gamma^\epsilon_k (H_{\eta,k}(f^\epsilon_k) - H_{\eta,k}(M^\epsilon_k(a_0))) \\ &\leq G_\eta(a_0) + \sum_{\gamma^\epsilon_k < 0} \gamma^\epsilon_k H'_{\eta,k}(M^\epsilon_k(a_0))(f^\epsilon_k - M^\epsilon_k(a_0)) \\ &= G_\eta(a_0) + \eta'(a_0) \sum_{\gamma^\epsilon_k < 0} \gamma^\epsilon_k (f^\epsilon_k - M^\epsilon_k(a_0)) \\ &= G_\eta(a_0) + \eta'(a_0)[\sum_k \gamma^\epsilon_k f^\epsilon_k - A(a_0)]\end{aligned} \tag{3.13}$$

The boundary term that we want to control can then be written

$$\int_0^T \eta'(a_0(t))v^\epsilon(t,0)dt \tag{3.14}$$

where

$$v^\epsilon = \sum_k \gamma_k^\epsilon f_k^\epsilon$$

and v^ϵ satisfies in the interior of the domain

$$\partial_t u^\epsilon + \partial_x v^\epsilon = 0. \tag{3.15}$$

Let us introduce a function ρ such that

$$\rho \in \mathcal{C}^2(\mathbb{R}^+) \quad \rho(x) = 0, \forall x > 1$$

$$\rho(0) = 1 \qquad \text{and} \quad |\rho'(\cdot)| \le c$$

We want to control

$$\begin{aligned}\int_0^T \eta'(a_0) v^\epsilon dt \quad &= -\int_0^T \int_0^1 \partial_x(\eta'(a_0)v^\epsilon)\rho dx dt - \int_0^T \int_0^1 \eta'(a_0) v^\epsilon \rho' dt \\ &= I_1 + I_2\end{aligned}$$

Let's examin the first term using (3.15) we have

$$\begin{aligned}I_1 \quad &= \int_0^T \int_0^1 \eta'(a_0)\partial_t u^\epsilon \rho dx dt \\ &= \int_0^T \int_0^1 (\partial_t(\eta'(a_0)u^\epsilon) - (\partial_t \eta'(a_0))u^\epsilon)\rho dx dt \\ &= \int_0^1 (\eta'(a_0(T))u^\epsilon(x,T) - \eta'(a_0(T))u^\epsilon(x,0))\rho dx - \int_0^T \int_0^1 (\partial_t \eta'(a_0))u^\epsilon \rho dx dt\end{aligned}$$

So that we can bound I_1 by

$$|I_1| \le C(||a_0||_{L^\infty(0,T)}||u^\epsilon||_{L^\infty((0,T)\times\mathbb{R}^+)} + ||u^\epsilon||_{L^\infty} BV(a_0))$$

independently of ϵ

For the last term it is enough to write

$$\begin{aligned}|I_2| \quad &\le |\int_0^T \int_0^1 \eta'(a_0)A(u^\epsilon)\rho' dx dt| + |\int_0^T \int_0^1 \eta'(a_0)(v^\epsilon - A(u^\epsilon))\rho' dx dt| \\ &\le C||a_0||_{L^\infty(0,T)}(||A(u^\epsilon)||_{L^1(0,T\times(0,1))} + \tfrac{\max_i\{|\theta_i|\}}{\sqrt{\epsilon}} \int_0^T \int_0^1 \textstyle\sum_i |f_i^\epsilon(t,x) - M_i(u^\epsilon,\epsilon)|dx) \\ &\le C\end{aligned}$$

So that again this is independant of ϵ, and C only depends of the data (u_0, a_0) and Lipshitz constant of $\eta'(\cdot)$. $\diamondsuit$

Now we can derive the desired L^2 estimate.

Lemma 3.2 *There exists a positive constant C such that, for all $\epsilon > 0$ and for all $k = 1, \ldots, N$*

$$\int_0^T \int_{\mathbb{R}^+} [M_k^\epsilon(f_k^\epsilon) - f_k^\epsilon]^2 dx \le \epsilon C,$$

Proof :

Using the previous lemma, one has for any C^2 convex entropy that

$$\sum_k \int_0^T \int_{\mathbb{R}+} [H^{\epsilon}_{k,\eta}{}'(M^{\epsilon}_k(w)) - H^{\epsilon}_{k,\eta}{}'(f^{\epsilon}_k)] \frac{M^{\epsilon}_k(f^{\epsilon}_k) - f^{\epsilon}_k}{\epsilon} dxdt \leq C$$

Now we choose $\eta(u) = \frac{u^2}{2}$. Again using the method of [11], one has that

$$[f^{\epsilon}_k - M^{\epsilon}_k(u^{\epsilon})]^2 \leq [H'_k(f^{\epsilon}_k) - H'_k(M^{\epsilon}_k(u^{\epsilon}))](f^{\epsilon}_k - M^{\epsilon}_k(f^{\epsilon}_k)) \tag{3.16}$$

which ends the proof. $\diamondsuit$

Remark 1. In [11] is shown that assumption (3.8) can be removed, in this case $H^{\epsilon}_{k,\eta}$ is not C^1, the main point is to replace $H'^{\epsilon}_{k,\eta}(M^{\epsilon}_k(w))$ with $\eta'(w)$ since in this case equality (3.9) does not make sense for some values of w. Then inequality (3.12) and Lemmas 3.1– 3.2 are still true.

The previous lemma is crucial for establishing convergence when ϵ goes to zero, for interior regions as for boundary terms.

An immediate consequence is that the first part of condition $i)$ in Definition (1.1) is verified by the limit function u, as the following lemma shows.

Lemma 3.3 *Let u be the limit of a subsequence of $\{u^{\epsilon}\}$. Then*

$$\partial_x B(u) \in L^2(\mathbb{R}^+ \times (0,T))$$

Proof :

Using the equations for f^{ϵ}_k we have, for all $\phi \in C^{\infty}_0(\mathbb{R}^+ \times (0,T))$

$$\int_Q \gamma_k (f^{\epsilon}_k - M^{\epsilon}_k(u^{\epsilon}))\phi dQ = \epsilon \int_Q \gamma_k f^{\epsilon}_k \phi_t + \epsilon \int_Q \gamma^2_k f^{\epsilon}_k \phi_x dQ$$

and then

$$\int_Q \gamma_k (f^{\epsilon}_k - M^{\epsilon}_k(u^{\epsilon}))\phi dQ = \epsilon \int_Q \gamma_k f^{\epsilon}_k \phi_t + \epsilon \int_Q \gamma^2_k (f^{\epsilon}_k - M^{\epsilon}_k(u^{\epsilon}))\phi_x dQ + \epsilon \int_Q \gamma^2_k M^{\epsilon}_k(u^{\epsilon})\phi_x dQ \ .$$

Summing over k and letting $\epsilon \to 0$ we have

$$\lim_{\epsilon \to 0} \sum_{k=1}^{N} \int_Q \gamma_k (f^{\epsilon}_k - M^{\epsilon}_k(u^{\epsilon}) \phi dQ = \int_Q B(u)\phi_x dQ \ ,$$

where we used Proposition 2.5, condition (M_4) and the immediate estimate

$$|\epsilon \int_Q \gamma^{\epsilon}_k f_k \phi_t dQ| \leq C ||f_k||_{L^{\infty}(Q)} ||\phi_t||_{L^1(Q)} \sqrt{\epsilon}$$

On the other hand we know that $\sum_{k=1}^{N} \gamma_k (f^{\epsilon}_k - M^{\epsilon}_k(u^{\epsilon}))$ is bounded in $L^2(Q)$; hence

$$\sum_{k=1}^{N} \gamma_k (f^{\epsilon}_k - M^{\epsilon}_k(u^{\epsilon})) \rightharpoonup \partial_x B(u) \text{ in } L^2(Q) - weak$$

and

$$\partial_x B(u) \in L^2(Q) \ .$$

$\diamondsuit$

In the following lemma, using again the L^2 estimate of Lemma 3.2, we obtain the important information that when the velocity $\theta_k \neq 0$, the trace of the solution $f^{\epsilon}_k(0,t)$ tend to the equilibrium state $M_k(a_0(t))$ when ϵ goes to zero. This is not the case for the hyperbolic equations (see [29]).

We recall that, thanks to Theorem 2.2, for $k \in Z^+ \cup Z^-$ there exists a subsequence of $\{f^{\epsilon}_k\}$ converging to some function $w_k(t)$ in $L^1((0,T))$.

Lemma 3.4 *For all $k \in Z^+ \cup Z^-$*

$$w_k(t) = M_k(u(0,t)) \qquad t \in (0,T)$$

Proof : For all $\phi \in C_0^\infty([0,+\infty) \times (0,T))$ we have

$$\int_Q \gamma_k^\epsilon (f_k^\epsilon - M_k^\epsilon(u^\epsilon))\phi dQ = \int_Q \epsilon \gamma_k^\epsilon f_k^\epsilon \phi_t dQ + \int_Q \epsilon {\gamma_k^\epsilon}^2 f_k \phi_x dQ + \int_0^T \epsilon {\gamma_k^\epsilon}^2 f_k^\epsilon(0,t)\phi(0,t)dt \ . \tag{3.17}$$

The first and the second term on the right hand side can be treated as in the previous lemma, for ϵ going to zero.

For the last term, which gives information on the boundary, it can be easily proved that

$$\epsilon \int_0^T {\gamma_k^\epsilon}^2 f_k \phi(t,0)dt \to \int_0^T {\theta_k}^2 w_k(t)\phi(t,0)dt \ .$$

For the source term part, one has again that $\gamma_k^\epsilon(f_k^\epsilon - M_k^\epsilon(u^\epsilon)) \in L^2(Q)$, then, up to extracting a subsequence, it converges in the weak topology. It follows that, letting ϵ go to zero in (3.17) we obtain

$$\int_Q h\phi dQ = \int_0^T {\theta_k}^2 w_k(t)\phi(0,t)dt + \int_Q {\theta_k}^2 M_k(u)\phi_x dQ \tag{3.18}$$

where h is a suitable function in $L^2(Q)$. Choosing as text function $\phi_\delta = \tau_\delta(x)\psi(t)$, where $\tau_\delta(x) = \max(0, 1 - \frac{x}{\delta})$ and $\psi \in C_0^\infty((0,T))$, since

$$\lim_{\delta \to 0} ||\phi_\delta||_{L^2(Q)} = 0 \ ,$$

the left hand side of (3.18) vanishes when δ goes to zero. So the claim follows letting δ go to zero in (3.18). $\diamondsuit$

The result stated in the previous lemma allows to prove that u satisfies the boundary condition in Definition 1.1.

Lemma 3.5 *For $t \in (0,T)$*

$$B(u(0,t)) = B(a_0(t)) \ .$$

Proof : From condition (M_4) we know that $B(u) = \sum_{i=1}^N \theta_k^2 M_k(u,0)$, so the result is proved if we show that

$$M_k(a_0) = M_k(u) \quad \forall k \in Z^+ \cup Z^- \ . \tag{3.19}$$

Condition (M_2) implies that, for $\epsilon = 0$ the Maxwellian functions must satisfy the following property

$$\sum_k \theta_k M_k(u) = 0 \ \ \forall u \in I \tag{3.20}$$

But due to the convergence results obtained previously and the boundary condition imposed on entering characteristics ($\theta_k > 0$), one has that on the boundary

$$M_k(a_0) = M_k(u) \quad \forall k \in Z^+ \cup Z^-$$

so that, in fact, we have the following constraint

$$\sum_{k \in Z^-} \theta_k M_k(u) + \sum_{k \in Z^+} \theta_k M_k(a_0) = 0.$$

Using again the same property (3.20) on a_0, one has

$$\sum_{\theta_k<0} \theta_k (M_k(u) - M_k(a_0)) = 0.$$

The monotonicity property of all the maxwellian functions implies that all the terms in the above sum have to be equal to zero. Therefore (3.19) holds and the claim follows.

$\diamondsuit$

If the Maxwellian functions are strictly increasing the previous lemma tells us that $u(t,0) = a_0(t)$, which corresponds to the expected regularity for non degenerate parabolic case.

Moreover if $\theta_k \neq 0$ for every $k = 1, \cdots, N$, Lemma 3.4 implies that $u(t,0) = a_0(t)$, this shows that in the strongly degenerete case one of the θ_k must vanish.

The equality (3.19) obtained in the proof of the above lemma shows that for all $k \in Z^+ \cup Z^-$ the functions $w_k(t)$ are in fact the limits of the whole sequences $\{f_k^\epsilon\}$.

In the remaining of this section we prove that the function u verifies the entropy ii) of Definition 1.1. After this proof, thanks to uniqueness result for entropy solutions of equation (1.1), we will be able to state that the whole sequence $\{u^\epsilon\}$ converges in $C([0,T]; L^1_{loc}(\mathbb{R}^+))$ to u, solution of problem (1.1)-(1.3). In order to estabilish this result we have to work with entropy that are only lipschitz continuous, more precisely we are interested in the entropies $\eta_c^\pm(s) = [s-c]_\pm$ and the associated kinetic entropies $H_{k,c}^{\pm,\epsilon}(s) = [s - M_k^\epsilon(c)]_\pm$. It a simple check to prove that the equalities (3.10) are verified for these entropies (see [11]). In such case $A_{\eta_c^\pm}(s) = H^\pm(s-c)(A(s) - A(c))$ and $B_{\eta_c^\pm}(s) = H^\pm(s-c)(B(s) - B(c))$.

Proposition 3.1 *The function $u \in C([0,T]; L^1_{\text{loc}}(\mathbb{R}^+)) \cap L^\infty((0,T) \times \mathbb{R}^+)$ verifies condition ii) of Definition 1.1.*

Proof : Let us consider the entropy $\eta_c^+(s)$. In order to simplify the notation in this proof we put $H_k(s) = H_{k,c}^{+,\epsilon}(s)$.

In [11] it is observed that it is well defined the multiplication of equation (1.4) with the multivalued function $H_k'(s) = H^+(s - M_k^\epsilon(c))$ (see Theorem 3.1 in [10]). Let us sum these equalities, multiply with a test function $\phi \in C_0^1(\mathbb{R}^+ \times [0,T))$, $\phi \geq 0$ and integrate in Q.

Let us note that

$$\sum_k H^+(f_k^\epsilon - M_k^\epsilon(c))(f_k^\epsilon - M_k^\epsilon(u^\epsilon)) = \sum_k (H^+(f_k^\epsilon - M_k^\epsilon(c)) - H^+(u^\epsilon - c))(f_k^\epsilon - M_k^\epsilon(u^\epsilon)) \geq 0 \quad (3.21)$$

In fact the equality in (3.21) is obvious, the inequlity can be proved cheking as in [11] that every term of the sum is positive.

We obtain

$$-\int_Q \sum_k H_k(f_k^\epsilon)\phi_t \, dQ - \int_Q \sum_k \gamma_k^\epsilon H_k(f_k^\epsilon)\phi_x \, dQ - \int_{\mathbb{R}^+} \eta_c^+(u^0)\phi(x,0)\, dx \leq \int_{\mathbb{R}^+} \sum_k \gamma_k^\epsilon H_k(f_k^\epsilon)\phi(0,t)\, dt$$

Proceeding as in the proof of Lemma 3.1 (see (3.13)) it can be observed that

$$\sum_k \gamma_k^\epsilon H_k(f_k^\epsilon) \leq H^+(a_0 - c)(A(a_0) - A(c)) + H^+(a_0 - c)\sum_k \gamma_k^\epsilon (f_k^\epsilon - M_k^\epsilon(a_0)) \quad (3.22)$$

Let us consider test function ϕ such that $H^+(a_0 - c)\phi(t,0) = 0$, then for such test function we obtain, using inequality (3.22)

$$\int_Q \sum_k H_k(f_k^\epsilon)\phi_t \, dQ + \int_Q \sum_k \gamma_k^\epsilon H_k(f_k^\epsilon)\phi_x \, dQ + \int_{\mathbb{R}^+} [u_0 - c]_+ \phi(x,0)\, dx \geq 0 \quad (3.23)$$

The idea is to pass to the limit when ϵ goes to zero in the inequality (3.23), it is easy to verify that the first and the last term in the inequality (3.23) converge to the corresponding term of (1.8). We want to investigate the second term. To this aim we prove that for every $k = 1, \cdots, N$

$$H_k(f_k^\epsilon) = H_k(M_k^\epsilon(u^\epsilon)) - (H_k')(f_k^\epsilon)(M_k^\epsilon(u^\epsilon) - f_k^\epsilon) + o(\sqrt{\epsilon}) \quad \text{in } L^1_{loc}(Q) \tag{3.24}$$

As in [11] we have that

$$\begin{aligned} 0 \quad & \leq H_k(M_k^\epsilon(u^\epsilon)) - H_k(f_k^\epsilon) - H_k'(f_k^\epsilon)(M_k^\epsilon(u^\epsilon) - f_k^\epsilon) \\ & \leq (\eta_c'^+(u^\epsilon) - H_k'(f_k^\epsilon))(M_k^\epsilon(u^\epsilon) - f_k^\epsilon) \\ & \leq \textstyle\sum_i (\eta_c'^+(u^\epsilon) - H_i'(f_i^\epsilon))(M_i^\epsilon(u^\epsilon) - f_i^\epsilon). \end{aligned} \tag{3.25}$$

Proceeding as (3.12) and Lemma 3.1 we have

$$\begin{aligned} & \textstyle\int_Q \sum_i (\eta_c'^+(u^\epsilon) - H_i'(f_i^\epsilon))(M_i^\epsilon(u^\epsilon) - f_i^\epsilon)\, dQ \leq \\ & \epsilon(\textstyle\int_{\mathbb{R}^+} \eta_c^+(u^0)\, dx + \int_0^T H^+(a_0 - c)(A(a_0) - A(c)) + H^+(a_0 - c) \sum_i \gamma_k^\epsilon(f_i^\epsilon - M_i^\epsilon(a_0))\, dt). \end{aligned} \tag{3.26}$$

Using Lemma 3.4 and inequalities (3.25)–(3.26) we obtain (3.24). From (3.24) we deduce

$$\sum_k \gamma_k^\epsilon H_k(f_k^\epsilon) = H^+(u - c)(A(u) - A(c)) - \sum_k \gamma_k^\epsilon H_k'(f_k^\epsilon)(M_k^\epsilon(u^\epsilon) - f_k^\epsilon) + o(1) \quad \text{in } L^1_{loc}(Q). \tag{3.27}$$

To complete the proof, using the equation (1.4), we prove as in Lemma 3.3 that

$$\lim_{\epsilon \to 0} \sum_k \gamma_k^\epsilon H_k'(f_k^\epsilon)(M_k^\epsilon(u^\epsilon) - f_k^\epsilon) = (B_{\eta_c^+})_x = H^+(u - c)(B(u))_x \quad \text{in } L^2_{loc}(Q) - weak \tag{3.28}$$

Letting ϵ goes to zero in the inequality (3.23) and using (3.27)–(3.28) we obtain the result in the case of the entropy η_c^+. The proof for the entropy η_c^- is the same.

$\diamondsuit$

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