A reduced stability condition for nonlinear relaxation to conservation laws

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Abstract

We consider multidimensional hyperbolic systems of conservation laws with relaxation, together with their associated limit systems. A strong stability condition for such asymptotics has been introduced by Chen, Levermore, Liu in Comm. Pure Appl. Math. 47, 787-830, namely the existence of an entropy extension. We propose here a new stability condition, the reduced stability condition, which is weaker than the previous one, but still has the property to imply the subcharacteristic or interlacing conditions, and the dissipativity of the leading term in the Chapman-Enskog expansion. This reduced stability condition has the advantage to involve only the submanifold of equilibria, or maxwellians, so that it is much easier to check than the entropy extension condition. Our condition generalizes the one introduced by the author in the case of kinetic, i.e. diagonal semilinear relaxation. We provide an adapted stability analysis in the context of approximate Riemann solvers obtained via relaxation systems.

Key-words: Systems of conservation laws – relaxation – entropy compatibility – reduced stability condition – subcharacteristic condition – Chapman-Enskog expansion – approximate Riemann solver

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1 Introduction

We consider multidimensional systems of conservation laws

$$\partial_t u + \sum_{j=1}^{N} \partial_j F_j(u) = 0, \qquad t \in \mathbb{R},\ x \in \mathbb{R}^N, \tag{1.1}$$

where $u(t,x) \in \mathcal{U} \subset \mathbb{R}^p$, $\mathcal{U}$ convex with nonempty interior, and $F_j(u) \in \mathbb{R}^p$. The by now classical relaxation framework for getting approximate solutions to (1.1) consists in solving another system of conservation laws with right-hand side, in higher dimension $q > p$,

$$\partial_t f + \sum_{j=1}^{N} \partial_j A_j(f) = \frac{Q(f)}{\varepsilon}, \qquad t \in \mathbb{R},\ x \in \mathbb{R}^N, \tag{1.2}$$

where $f(t,x) \in \mathcal{V} \subset \mathbb{R}^q$, $\mathcal{V}$ convex with nonempty interior, and $A_j(f) \in \mathbb{R}^q$. This system is called the relaxation system, the right-hand side is the relaxation or collision term, and (1.1) is the relaxed system. In order that the solutions to (1.2) give (at least formally) approximate solutions to (1.1), we assume that we have a linear operator $L : \mathbb{R}^q \to \mathbb{R}^p$, and for any $u \in \mathcal{U}$, an equilibrium $M(u) \in \mathcal{V}$, the maxwellian equilibrium, such that for any $u \in \mathcal{U}$

$$L\,M(u) = u + k, \tag{1.3}$$

$$L\,A_j(M(u)) = F_j(u) + k'_j, \tag{1.4}$$

for some constants k and k'_j. The assumptions on Q are that

$$L\,Q(f) = 0, \tag{1.5}$$

and that

$$Q(f) = 0 \quad \text{iff} \quad f = M(u) \text{ for some } u. \tag{1.6}$$

We can notice that according to (1.3), u in (1.6) is necessarily given by $u = Lf - k$. An important example of such collision operator is the BGK operator,

$$Q(f) = M(Lf - k) - f. \tag{1.7}$$

With the above assumptions, one can check easily that starting from (1.2) and defining $u_\varepsilon = Lf_\varepsilon - k$, we obtain formally that $u_\varepsilon \to u$, solution to (1.1) as $\varepsilon \to 0$. The problem of justifying this limit in general is of course out of reach, and we refer to [10], [14] for a review of rigorous results.

The question we address here is rather the determination of coherent structures and stability conditions that enable to understand the relaxation mechanism. The most famous stability criterion is the so called subcharacteristic condition that says that the eigenvalues of (1.1) should lie "between" the eigenvalues of (1.2), in a sense that will be precised further on. But as is now known, this condition is a bit too weak, and a preferable condition, that is introduced in [4], is the existence of an entropy for (1.2) that is somehow compatible with the equilibria. However, in practice, this condition can be very difficult to check, because the entropies of (1.2) can be highly non trivial because of the large dimension q of the system. In order to face this difficulty, we introduce here a reduced stability condition that only involves the equilibria $M(u)$, $u \in \mathcal{U}$, and is therefore much easier to check. Other structural conditions are also considered in [12] and [15].

Our reduced condition generalizes the one introduced by the author in [2] in the case of kinetic, i.e. semilinear diagonal relaxation systems. Kinetic relaxation systems, as described in [2], enter the above formalism by assuming that

$$\mathbb{R}^q = (\mathbb{R}^p)^\Xi, \tag{1.8}$$

where Ξ is a measure space with positive measure $d\xi$. Thus $f \in \mathbb{R}^q$ is identified with a function $f(\xi)$, $\xi \in \Xi$. Indeed Ξ is often infinite, thus $q = \infty$ somehow. The system is assumed semilinear diagonal,

$$A_j(f)(\xi) = a_j(\xi) f(\xi), \tag{1.9}$$

for some functions $a_j : \Xi \to \mathbb{R}$. The maxwellians are now functions of ξ, $M = M(u, \xi)$, and the operator L is simply

$$Lf = \int_\Xi f(\xi)\, d\xi. \tag{1.10}$$

Thus the consistency relations (1.3)-(1.4) become moment relations. A prototype example of such system is of course the Boltzmann equation.

The plan of the paper is as follows. In Section 2, we introduce our reduced stability condition and compare it to other known stability criterion which are the entropy extension condition, the subcharacteristic condition, and the dissipativity in the Chapman-Enskog expansion. In Section 3, we analyze our reduced stability condition in the context of discrete approximations, especially for approximate Riemann solvers obtained via relaxation systems.

2 Stability criterion

This section is devoted to the comparison of different stability conditions for the relaxation of (1.2) to (1.1), namely the entropy extension condition, our reduced stability condition, the interlacing subcharacteristic condition, and the positiveness in the Chapman-Enskog expansion.

We shall always assume that all nonlinearities are sufficiently smooth, and for further reference, we state the differential form of (1.3)-(1.4),

$$L\, M'(u) = \mathrm{Id}, \tag{2.1}$$

$$L\, A_j'(M(u))M'(u) = F_j'(u). \tag{2.2}$$

We assume that the system (1.1) has a convex entropy η, which means that $\eta : \mathcal{U} \to \mathbb{R}$ is convex, and there exist entropy fluxes $G_j : \mathcal{U} \to \mathbb{R}$ such that

$$G_j' = \eta' F_j'. \tag{2.3}$$

As usual, weak entropy solutions to (1.1) are those that satisfy

$$\partial_t\, \eta(u) + \sum_{j=1}^{N} \partial_j\, G_j(u) \le 0. \tag{2.4}$$

2.1 Entropy extension

The entropy extension criterion of [4] is the following.

Definition 2.1 *We say that (EEC) holds if there exists some convex function $\mathcal{H}(f)$, $f \in \mathcal{V}$, which is an entropy for (1.2), which means that there exist some functions $\mathcal{G}_j(f)$ such that*

$$\mathcal{G}_j' = \mathcal{H}' A_j', \tag{2.5}$$

that these entropy and entropy-flux are extensions of the ones of the relaxed system,

$$\mathcal{H}(M(u)) = \eta(u) + cst, \tag{2.6}$$

$$\mathcal{G}_j(M(u)) = G_j(u) + cst, \tag{2.7}$$

and that the minimization principle holds,

$$\mathcal{H}(M(u)) \le \mathcal{H}(f) \qquad \text{whenever } u = Lf - k \in \mathcal{U}. \tag{2.8}$$

We need also to assume that the collision term Q is dissipative,

$$\mathcal{H}'(f) Q(f) \le 0. \tag{2.9}$$

Obviously, when (2.5)-(2.9) hold, we have

$$\partial_t\, \mathcal{H}(f) + \sum_{j=1}^{N} \partial_j\, \mathcal{G}_j(f) \le 0, \tag{2.10}$$

and as $\varepsilon \to 0$, we recover (2.4). We can notice that the BGK operator (1.7) automatically satisfies (2.9) because by convexity of $\mathcal{H}$ and according to (2.8) we have $\mathcal{H}'(f)(M(Lf-k)-f) \leq \mathcal{H}(M(Lf-k)) - \mathcal{H}(f) \leq 0$.
As developed in [2], in the kinetic case (1.8)-(1.10), $\mathcal{H}$ and $\mathcal{G}_j$ take the form

$$\mathcal{H}(f) = \int_\Xi H(f(\xi),\xi)\,d\xi, \quad \mathcal{G}_j(f) = \int_\Xi a_j(\xi) H(f(\xi),\xi)\,d\xi. \tag{2.11}$$

An important feature about entropy extensions is the following.

Proposition 2.2 *Assume that (2.6) holds and that $Lf - k \in \mathcal{U}$ as soon as $f \in \mathcal{V}$ (a condition that we shall assume now on). Then the minimization principle (2.8) is equivalent to the differential identity*

$$\mathcal{H}'(M(u)) = \eta'(u)L \qquad \text{for all } u \in \mathcal{U}. \tag{2.12}$$

Proof. This was proved in [2] (see also [4] for a slightly different approach), but for completeness let us give the argument. At first, if $\mathcal{H}'(M(u)) = \lambda(u)L$ for some Lagrange multiplier $\lambda(u) \in (\mathbb{R}^p)'$, then by convexity of $\mathcal{H}$, for any $f \in \mathcal{V}$ and $u \in \mathcal{U}$,

$$\begin{aligned} \mathcal{H}(f) &\geq \mathcal{H}(M(u)) + \mathcal{H}'(M(u))(f - M(u)) \\ &= \mathcal{H}(M(u)) + \lambda(u)L(f - M(u)). \end{aligned} \tag{2.13}$$

By choosing $u = Lf - k$ and by using (1.3), we obtain (2.8). Conversely, if (2.8) holds, define

$$\varphi(f) = \mathcal{H}(f) - \eta(Lf - k), \tag{2.14}$$

with η defined by $\eta(u) = \mathcal{H}(M(u))$. Then $\varphi(f) \geq 0$ for any $f \in \mathcal{V}$, and for any $u \in \mathcal{U}$, $\varphi(M(u)) = 0$. Thus $\varphi'(M(u)) = 0$, which gives (2.12), and indeed this proves that necessarily $\lambda(u) = \eta'(u)$. $\square$

Remark. In the case where $\mathcal{H}$ is convex but not differentiable as considered in [2], (2.12) needs to be replaced by the subdifferential inclusion $\mathcal{H}'(M(u)) \ni \eta'(u)L$, which means more explicitly that for any $f \in \mathcal{V}$ and $u \in \mathcal{U}$

$$\mathcal{H}(f) \geq \mathcal{H}(M(u)) + \eta'(u)L(f - M(u)). \tag{2.15}$$

This formulation is important in applications, see for example [1].

Corollary 2.3 ([4]) *If η is not known a priori but if we have a convex entropy $\mathcal{H}$ satisfying (2.5) and (2.8), then defining η and G_j by (2.6)-(2.7), we have that η is a convex entropy for (1.1) with associated entropy-fluxes G_j. In particular, if $\mathcal{H}$ is strictly convex then η also, and (1.1) is hyperbolic.*

Proof. We have that

$$\eta(u) = \min_{Lf = u+k} \mathcal{H}(f), \tag{2.16}$$

which gives easily that η is convex. Then, $\eta'(u) = \mathcal{H}'(M(u))M'(u)$ and according to Proposition 2.2 and (2.2),

$$\begin{aligned} G_j'(u) &= \mathcal{G}_j'(M(u))M'(u) \\ &= \mathcal{H}'(M(u))A_j'(M(u))M'(u) \\ &= \eta'(u)LA_j'(M(u))M'(u) \\ &= \eta'(u)F_j'(u), \end{aligned} \tag{2.17}$$

which gives the result. □

We conclude that according to (2.17), the condition (2.7) can be removed when writing (EEC), because it is a consequence of (2.5), (2.6) and (2.8).

Condition (2.8) is indeed necessary if we want to be able to find a reasonable collision term Q satisfying (2.9).

Proposition 2.4 *If we have some collision term Q satisfying (1.5)-(1.6) and for any $u \in \mathcal{U}$*

$$\operatorname{Range} Q'(M(u)) = \ker L, \tag{2.18}$$

and if $\mathcal{H}$ verifies (2.6), (2.9), then (2.8) holds.

Proof. Define $\varphi(f) = \mathcal{H}'(f)Q(f)$. Because of (1.6), we have $\varphi'(M(u)) = \mathcal{H}'(M(u))Q'(M(u))$. But by (2.9), φ is nonpositive. Since $\varphi(M(u)) = 0$ by (1.6), we deduce that $\varphi'(M(u)) = 0$, and with (2.18) that $\mathcal{H}'(M(u))$ vanishes on $\ker L$. Therefore, there exists $\lambda(u) \in (\mathbb{R}^p)'$ such that $\mathcal{H}'(M(u)) = \lambda(u)L$, and we conclude with the proof of Proposition 2.2 that (2.8) holds. □

2.2 Reduced stability condition

We here introduce our alternate stability condition for the relaxation of (1.2) to (1.1). Let us consider a direction

$$\omega = (\omega_1, \dots, \omega_N) \in \mathbb{R}^N \setminus \{0\}, \tag{2.19}$$

and the corresponding fluxes

$$F_\omega(u) = \sum_{j=1}^N \omega_j F_j(u), \qquad A_\omega(f) = \sum_{j=1}^N \omega_j A_j(f). \tag{2.20}$$

We assume that the system (1.2) is hyperbolic, i.e. that $A_\omega'(f)$ is diagonalizable over $\mathbb{R}$. In particular, for any eigenvalue λ of $A_\omega'(f)$, we denote by $P_\lambda(A_\omega'(f))$ the projector onto $\ker(A_\omega'(f) - \lambda)$, parallel to the other eigenspaces.

Here and all through the paper, the notations concerning bilinear forms are those of the appendix of [2].

Definition 2.5 *We call reduced stability condition, (RSC) for short, the positivity conditions*

$$\Big(L\, P_\lambda\big(A_\omega'(M(u))\big)M'(u)\Big)^t \eta''(u) \quad \textit{is symmetric nonnegative}, \tag{2.21}$$

for any eigenvalue λ, any $u \in \mathcal{U}$, and any direction ω.

The entropy η is fixed here, but as in [2], if η is strictly convex, (2.21) means that $L\,P_\lambda(A'_\omega(M(u)))M'(u)$ is self-adjoint nonnegative for the scalar product defined by $\eta''(u)$, thus it is necessary that this operator is diagonalizable with nonnegative eigenvalues, and this last statement is independent of η.

The main interest of (RSC) is that it involves only maxwellian states, and that it can be checked directly, unlike (EEC) that needs some unknown function $\mathcal{H}$ defined on a bigger space. We remark that (RSC) generalizes in each direction the condition introduced in [2] in the kinetic case. In particular, in this situation, (RSC) is equivalent to (EEC). In the general case, our condition is weaker.

Theorem 2.6 *If (EEC) holds for some strictly convex $\mathcal{H}$, then (RSC) holds.*

Proof. By differentiating condition (2.12) in Proposition 2.2, we get

$$\mathcal{H}''(M(u))M'(u) = L^t\eta''(u). \tag{2.22}$$

Then, since $\mathcal{H}$ is a strictly convex entropy of (1.2), $A'_\omega(M(u))$ is self-adjoint for the scalar product defined by $\mathcal{H}''(M(u))$, and thus $P_\lambda(A'_\omega(M(u)))$ also. Thus

$$\begin{aligned} &\mathcal{H}''(M(u))\cdot P_\lambda()M'(u)du\cdot P_\lambda()M'(u)dv \\ &= \mathcal{H}''(M(u))\cdot M'(u)du\cdot P_\lambda()^2M'(u)dv \\ &= L^t\eta''(u)\cdot du\cdot P_\lambda()M'(u)dv \\ &= \eta''(u)\cdot du\cdot L\,P_\lambda()M'(u)dv \\ &= \Big(L\,P_\lambda()M'(u)\Big)^t\eta''(u)\cdot du\cdot dv, \end{aligned} \tag{2.23}$$

which yields the result. □

We remark that condition (2.21) contains the information that η is an entropy, because if we multiply (2.21) by λ and take the sum over λ, we obtain with (2.2) that $(F'_\omega)^t\eta''$ is symmetric, which characterizes the existence of G_j in (2.3).

2.3 Subcharacteristic condition

The subcharacteristic condition, in its precised version, has been stated in [4]. We consider again a direction ω as in (2.19), and the fluxes (2.20). We denote

$$\lambda_1(F'_\omega(u)) \le \ldots \le \lambda_p(F'_\omega(u)) \tag{2.24}$$

the eigenvalues of $F'_\omega(u)$, repeated with multiplicities, and the eigenvalues at equilibrium repeated with multiplicities

$$\lambda_1(A'_\omega(M(u))) \le \ldots \le \lambda_q(A'_\omega(M(u))). \tag{2.25}$$

Definition 2.7 *We say that the interlacing subcharacteristic condition (ISC) holds if*

$$\lambda_k(A'_\omega(M(u))) \le \lambda_k(F'_\omega(u)) \le \lambda_{q-p+k}(A'_\omega(M(u))), \quad \text{for any } 1\le k\le p. \tag{2.26}$$

For this condition to make sense, we need of course to assume the hyperbolicity of both (1.1) and (1.2).

Proposition 2.8 ([4]) *If (EEC) holds for some strictly convex $\mathcal{H}$, then (ISC) holds.*

This result is indeed a direct consequence of (2.22) because with (2.1)-(2.2) it gives

$$\begin{aligned} M'(u)^t\,\mathcal{H}''(M(u))M'(u) &= \eta''(u), \\ M'(u)^t A'_\omega(M(u))^t\,\mathcal{H}''(M(u))M'(u) &= F'_\omega(u)^t\eta''(u), \end{aligned} \tag{2.27}$$

and the result follows from the min-max characterization of the eigenvalues. Namely, since $F'_\omega(u)$ is self-adjoint for $\eta''(u)$,

$$\lambda_k(F'_\omega(u)) = \min_{W\subset\mathbb{R}^p,\ \dim W=k}\left\{\max_{w\in W\setminus\{0\}}\frac{\eta''(u)\cdot w\cdot F'_\omega(u)w}{\eta''(u)\cdot w\cdot w}\right\}, \tag{2.28}$$

$$\lambda_k(F'_\omega(u)) = \max_{W\subset\mathbb{R}^p,\ \operatorname{codim} W=k-1}\left\{\min_{w\in W\setminus\{0\}}\frac{\eta''(u)\cdot w\cdot F'_\omega(u)w}{\eta''(u)\cdot w\cdot w}\right\}, \tag{2.29}$$

and similar formulas hold for $A'_\omega(M(u))$ with the scalar product $\mathcal{H}''(M(u))$.

Theorem 2.9 *If (RSC) holds for some strictly convex η, then (ISC) holds.*

Proof. Denote by $\mu_1 < \ldots < \mu_s$ the distinct eigenvalues of $A'_\omega(M(u))$, by $P_1,\ldots,P_s$ the associated eigenprojectors and by $n_1,\ldots,n_s$ the multiplicities $n_\ell = \dim\operatorname{Range} P_\ell$. From (2.1)-(2.2) we have

$$\mathrm{Id} = L\,M'(u) = \sum_{\ell=1}^{s} L\,P_\ell\,M'(u), \tag{2.30}$$

$$F'_\omega(u) = L\,A'_\omega(M(u))M'(u) = \sum_{\ell=1}^{s}\mu_\ell\,L\,P_\ell\,M'(u), \tag{2.31}$$

thus

$$\eta''(u) = \sum_{\ell=1}^{s}(L\,P_\ell\,M'(u))^t\eta''(u), \tag{2.32}$$

$$(F'_\omega(u))^t\eta''(u) = \sum_{\ell=1}^{s}\mu_\ell\,(L\,P_\ell\,M'(u))^t\eta''(u). \tag{2.33}$$

Let now W be a subspace of $\mathbb{R}^p$, $\dim W = k$, and denote by $\overline{m}(W)$ the maximum between braces in (2.28). There exists a unique r, $1\le r\le s$, such that $n_1+\ldots+n_{r-1} < k \le n_1+\ldots+n_r$, and since according to (2.1), $M'(u)$ is injective, $\dim(M'(u)W) = k$, and therefore we can find some $w\in W\setminus\{0\}$ such that $M'(u)w \in \operatorname{Range} P_r\oplus\ldots\oplus\operatorname{Range} P_s$. Then $P_\ell M'(u)w = 0$ for any $\ell < r$, thus with (2.32)-(2.33) and (2.21),

$$\overline{m}(W) \ge \frac{\sum_{\ell=r}^{s}\mu_\ell\,(L\,P_\ell\,M'(u))^t\eta''(u)\cdot w\cdot w}{\sum_{\ell=r}^{s}(L\,P_\ell\,M'(u))^t\eta''(u)\cdot w\cdot w} \ge \mu_r = \lambda_k(A'_\omega(M(u))), \tag{2.34}$$

which yields by (2.28) that $\lambda_k(F'_\omega(u)) \geq \lambda_k(A'_\omega(M(u)))$.
Next, consider similarly a subspace W of $\mathbb{R}^p$, $\operatorname{codim} W = k-1$, and denote by $\underline{m}(W)$ the minimum between braces in (2.29). Then $\dim(M'(u)W) = p-k+1$, thus if $1 \leq r \leq s$ is such that $n_{r+1} + \ldots + n_s < p-k+1 \leq n_r + \ldots + n_s$, i.e. $n_1 + \ldots + n_{r-1} < q-p+k \leq n_1 + \ldots + n_r$, there exists some $w \in W \backslash \{0\}$ such that $M'(u)w \in \operatorname{Range} P_1 \oplus \ldots \oplus \operatorname{Range} P_r$. Then $P_\ell M'(u)w = 0$ for any $\ell > r$, thus with (2.32)-(2.33) and (2.21),

$$\underline{m}(W) \leq \frac{\sum_{\ell=1}^r \mu_\ell \, (L\, P_\ell\, M'(u))^t \eta''(u) \cdot w \cdot w}{\sum_{\ell=1}^r (L\, P_\ell\, M'(u))^t \eta''(u) \cdot w \cdot w} \leq \mu_r = \lambda_{q-p+k}(A'_\omega(M(u))), \tag{2.35}$$

which yields by (2.29) that $\lambda_k(F'_\omega(u)) \leq \lambda_{q-p+k}(A'_\omega(M(u)))$, finishing the proof of (2.26). $\square$

2.4 Chapman-Enskog expansion

We only consider here the BGK relaxation term (1.7). For general collision terms Q, one should make further assumptions on the linearized operator around a maxwellian, that as in [4], for any δg such that $L\,\delta g = 0$, the equation $Q'(M(u))\,\delta f = \delta g$ has a unique solution δf such that $L\,\delta f = 0$, and a dissipation assumption on $Q'(M(u))$ replacing (2.9).

In order to simplify the presentation, we denote by f the solution to (1.2), (1.7) (instead of f_ε), and we set $u = Lf - k$.

Lemma 2.10 *Up to terms in ε^2, we have*

$$\partial_t u + \sum_{j=1}^N \partial_j\, F_j(u) = \varepsilon \sum_{1 \leq i,j \leq N} \partial_j \Big(D_{ji}(u) \partial_i u \Big), \tag{2.36}$$

with

$$D_{ji}(u) = L\, A'_j(M(u)) A'_i(M(u)) M'(u) - F'_j(u) F'_i(u). \tag{2.37}$$

Proof. We have

$$\partial_t f + \sum_{j=1}^N \partial_j\, A_j(f) = \frac{M(u) - f}{\varepsilon}, \tag{2.38}$$

thus by applying L,

$$\partial_t u + \sum_{j=1}^N \partial_j\, L A_j(f) = 0. \tag{2.39}$$

From (2.38) we get that $f = M(u) + O(\varepsilon)$, and then

$$\begin{aligned} f &= M(u) - \varepsilon \Big[\partial_t f + \sum_j \partial_j\, A_j(f) \Big] \\ &= M(u) - \varepsilon \Big[\partial_t\, M(u) + \sum_j \partial_j\, A_j(M(u)) \Big] + O(\varepsilon^2). \end{aligned} \tag{2.40}$$

Therefore we can write the expansion of the nonlinearities,

$$A_j(f) = A_j(M(u)) - \varepsilon A_j'(M(u))\Big[\partial_t M(u) + \sum_i \partial_i A_i(M(u))\Big] + O(\varepsilon^2), \quad (2.41)$$

and then

$$\begin{aligned} & LA_j(f) \\ &= LA_j(M(u)) - \varepsilon LA_j'(M(u))\Big[M'(u)\partial_t u + \sum_i A_i'(M(u))M'(u)\partial_i u\Big] + O(\varepsilon^2) \\ &= F_j(u) + k_j' - \varepsilon\Big[F_j'(u)\partial_t u + \sum_i LA_j'(M(u))A_i'(M(u))M'(u)\partial_i u\Big] + O(\varepsilon^2) \\ &= F_j(u) + k_j' - \varepsilon\sum_i\Big[-F_j'(u)F_i'(u) + LA_j'(M(u))A_i'(M(u))M'(u)\Big]\partial_i u + O(\varepsilon^2), \end{aligned} \quad (2.42)$$

which yields (2.36) by substitution in (2.39). □

Definition 2.11 *We shall say that the second-order operator in (2.36) is symmetrically entropy dissipative if the bilinear form σ on $(\mathbb{R}^p)^N \times (\mathbb{R}^p)^N$ defined for $v = (v_1, \dots, v_N) \in (\mathbb{R}^p)^N$, $w = (w_1, \dots, w_N) \in (\mathbb{R}^p)^N$ by*

$$\sigma \cdot v \cdot w = \sum_{1 \le i,j \le N} \sigma_{ij} \cdot v_j \cdot w_i, \quad (2.43)$$

$$\sigma_{ij} = D_{ji}(u)^t \eta''(u), \quad 1 \le i, j \le N, \quad (2.44)$$

is symmetric (or equivalently $\sigma_{ij}^t = \sigma_{ji}$), and nonnegative, for any $u \in \mathcal{U}$.
We shall say that the second-order operator in (2.36) is directionally symmetrically entropy dissipative if the bilinear form σ satisfies only

$$\sum_{1 \le i,j \le N} \omega_j\, \omega_i\, \sigma_{ij} \quad \textit{is symmetric nonnegative}, \quad (2.45)$$

for any $(\omega_1, \dots, \omega_N) \in \mathbb{R}^N \backslash \{0\}$ and $u \in \mathcal{U}$.

This definition is very natural, because if we multiply (2.36) by $\eta'(u)$, we have formally

$$\begin{aligned} & \partial_t\, \eta(u) + \sum_j \partial_j\, G_j(u) \\ &= \varepsilon \sum_{ij} \partial_j\Big(\eta'(u)D_{ji}(u)\partial_i u\Big) - \varepsilon\sigma \cdot (\partial_1 u, \dots, \partial_N u) \cdot (\partial_1 u, \dots, \partial_N u). \end{aligned} \quad (2.46)$$

The directional dissipativity is weaker than the full dissipativity, and means that we only consider in (2.43) vectors of the form $v = (\omega_1 \underline{v}, \dots, \omega_N \underline{v})$ and $w = (\omega_1 \underline{w}, \dots, \omega_N \underline{w})$ with $(\omega_1, \dots, \omega_N) \in \mathbb{R}^N \backslash \{0\}$ and $\underline{v}, \underline{w} \in \mathbb{R}^p$.

Proposition 2.12 ([4]) *If (EEC) holds, then the system (2.36), with (2.37), is symmetrically entropy dissipative.*

Proof. According to (2.2), we have

$$D_{ji} = LA'_jA'_iM' - LA'_jM'LA'_iM' = LA'_j(\mathrm{Id} - M'L)A'_iM'. \tag{2.47}$$

We observe that thanks to (2.1), $P = M'L$ is a projector, i.e. $P^2 = P$, thus $\ker P = \ker L$ and $\operatorname{Range} P = \operatorname{Range} M'$ are supplementary. But by (2.22), these two spaces are orthogonal for $\mathcal{H}''(M(u))$. In other words $\mathcal{H}''P = P^t\mathcal{H}''$. Since $\mathcal{H}$ is an entropy, we have also $\mathcal{H}''A'_j = (A'_j)^t\mathcal{H}''$, therefore by using (2.22) again,

$$\begin{aligned}
&\sigma \cdot v \cdot w \\
&= \sum_{ij} D^t_{ji}\,\eta'' \cdot v_j \cdot w_i \\
&= \sum_{ij} \mathcal{H}'' \cdot M'v_j \cdot A'_j(\mathrm{Id} - M'L)A'_iM'w_i \\
&= \sum_{ij} \mathcal{H}'' \cdot (\mathrm{Id} - M'L)A'_jM'v_j \cdot (\mathrm{Id} - M'L)A'_iM'w_i \\
&= \mathcal{H}'' \cdot (\mathrm{Id} - M'L)\sum_j A'_jM'v_j \cdot (\mathrm{Id} - M'L)\sum_i A'_iM'w_i,
\end{aligned} \tag{2.48}$$

which is symmetric nonnegative. □

Theorem 2.13 *If (RSC) holds, then the system (2.36), with (2.37), is directionally symmetrically entropy dissipative.*

Proof. Let $\omega = (\omega_1, \ldots, \omega_N) \in \mathbb{R}^N\backslash\{0\}$ and denote by $\mu_1 < \ldots < \mu_s$ the distinct eigenvalues of $A'_\omega(M(u))$ and by $P_1, \ldots, P_s$ the associated eigenprojectors. We have

$$\begin{aligned}
&\sigma \cdot v \cdot w \\
&= \sum_{ij} \eta''(u) \cdot v_j \cdot D_{ji}(u)w_i \\
&= \sum_{ij} \eta''(u) \cdot v_j \cdot (LA'_jA'_iM' - F'_jF'_i)w_i.
\end{aligned} \tag{2.49}$$

Then, choosing $v = (\omega_1\underline{v}, \ldots, \omega_N\underline{v})$ and $w = (\omega_1\underline{w}, \ldots, \omega_N\underline{w})$ for some $\underline{v}, \underline{w} \in \mathbb{R}^p$, we get according to (2.32)-(2.33)

$$\begin{aligned}
&\sigma \cdot v \cdot w \\
&= \eta''(u) \cdot \underline{v} \cdot (LA'_\omega A'_\omega M' - F'_\omega F'_\omega)\underline{w} \\
&= \sum_{\ell=1}^{s} \mu_\ell^2\,\eta''(u) \cdot \underline{v} \cdot LP_\ell M'\underline{w} - \eta''(u) \cdot F'_\omega\underline{v} \cdot F'_\omega\underline{w} \\
&= \sum_{\ell=1}^{s} (LP_\ell M')^t\eta''(u) \cdot (\mu_\ell\,\underline{v} - F'_\omega\underline{v}) \cdot (\mu_\ell\,\underline{w} - F'_\omega\underline{w}),
\end{aligned} \tag{2.50}$$

which is symmetric nonnegative in $(\underline{v}, \underline{w})$ by (2.21). □

In one dimension ($N = 1$), of course, directional dissipativity coincides with full dissipativity. But in several dimensions, if we really want to have the full dissipativity in (2.36), this is possible, but we need to strengthen a bit the condition (RSC), by asking that the operators A'_j are simultaneously diagonalizable, as follows.

Definition 2.14 *We shall say that the strong reduced stability condition (RSCs) holds, if for any $u \in \mathcal{U}$ the space can be decomposed as $\mathbb{R}^q = E_1 \oplus \ldots \oplus E_s$, in such a way that for each $1 \leq \ell \leq s$, E_ℓ is stable by each $A_j'(M(u))$ and the restriction of $A_j'(M(u))$ on E_ℓ is diagonal, and that denoting by P_ℓ the projector onto E_ℓ,*

$$\Big(L\, P_\ell M'(u)\Big)^t \eta''(u) \quad \textit{is symmetric nonnegative.} \tag{2.51}$$

Theorem 2.15 *If (RSCs) holds, then the system (2.36), with (2.37), is symmetrically entropy dissipative.*

Proof. Denote by μ_ℓ^j the diagonal value of $A_j'(M(u))$ on E_ℓ. We have as in (2.49)

$$\begin{aligned}
&\sigma \cdot v \cdot w \\
&= \sum_{ij} \eta''(u) \cdot v_j \cdot (LA_j'A_i'M' - F_j'F_i')w_i \\
&= \sum_{ij} \sum_{\ell=1}^{s} \mu_\ell^j \mu_\ell^i \eta''(u) \cdot v_j \cdot LP_\ell M' w_i - \sum_{ij} \eta''(u) \cdot F_j' v_j \cdot F_i' w_i \\
&= \sum_{ij} \sum_{\ell=1}^{s} (LP_\ell M')^t \eta''(u) \cdot (\mu_\ell^j\, v_j - F_j' v_j) \cdot (\mu_\ell^i\, w_i - F_i' w_i) \\
&= \sum_{\ell=1}^{s} (LP_\ell M')^t \eta''(u) \cdot \sum_j (\mu_\ell^j\, v_j - F_j' v_j) \cdot \sum_i (\mu_\ell^i\, w_i - F_i' w_i),
\end{aligned} \tag{2.52}$$

which is symmetric nonnegative by (2.51). □

Remark. In the kinetic case (1.8)-(1.10), the strong condition (RSCs) is satisfied when $M'(u,\xi)^t \eta''(u)$ is symmetric nonnegative (the condition of [2]), because according to (1.9), in this case ξ plays the role of ℓ.

3 Discrete approximations

We consider two levels of discrete approximations to the system of conservation laws (1.1) via the relaxation system (1.2). One is discrete in time only, this is the projection method, and the other is the approximate Riemann solver interpretation, which corresponds to further discretization in space. We still assume the consistency equations (1.3)-(1.4), and the existence of a convex entropy η satisfying (2.3). However, in this section we do not make use of any collision operator Q, which is replaced by projections onto maxwellians made at discrete times.

3.1 Projection method

The projection method consists in performing projections onto maxwellians at discrete times $t_n = n\Delta t$, while solving the relaxation system without right-hand side during the timestep,

$$\partial_t f + \sum_{j=1}^{N} \partial_j \, A_j(f) = 0 \quad \text{in }]t_n, t_{n+1}[\times\mathbb{R}^N, \tag{3.1}$$

$$f(t_n, x) = f^n(x) = M(u^n(x)), \tag{3.2}$$

and

$$u^n(x) = u^{n-}(x) \equiv Lf^{n-}(x) - k, \tag{3.3}$$

where we denote $f^{n-}(x) = \lim\limits_{t\to t_n, t<t_n} f(t,x)$. This can be summarized by writing

$$\partial_t f + \sum_{j=1}^{N} \partial_j \, A_j(f) = \sum_{n=1}^{\infty} \delta(t-t_n)(f^n - f^{n-}) \quad \text{in }]0,\infty[\times\mathbb{R}^N, \tag{3.4}$$

with $f^n = M(Lf^{n-} - k)$, and in this form this is very similar to the BGK collisions. We define as usual

$$u(t,x) = Lf(t,x) - k, \tag{3.5}$$

and assume as before that $u \in \mathcal{U}$.

Proposition 3.1 *As $\Delta t \to 0$, the projection method (3.1)-(3.3) is consistent with (1.1), and the equivalent equation is*

$$\partial_t u + \sum_{j=1}^{N} \partial_j \, F_j(u) = \frac{\Delta t}{2} \sum_{1\le i,j\le N} \partial_j \Big(D_{ji}(u) \partial_i u \Big), \tag{3.6}$$

with $D_{ji}(u)$ defined by (2.37).

Proof. Assuming smoothness in x, we get from (3.1)

$$f^{n+1-} = f^n - \Delta t \sum_j A_j'(f^n)\partial_j f^n + \frac{\Delta t^2}{2} \sum_{ij} \partial_j \left(A_j'(f^n) A_i'(f^n) \partial_i f^n \right) + O(\Delta t^3). \tag{3.7}$$

Therefore, by (3.2)-(3.3) and (2.37),

$$\begin{aligned} &u^{n+1-} \\ &= u^n - \Delta t \sum_j \partial_j \left[LA_j(f^n) - \frac{\Delta t}{2} \sum_i LA_j'(f^n) A_i'(f^n) \partial_i f^n \right] + O(\Delta t^3) \\ &= u^n - \Delta t \sum_j \partial_j \left[F_j(u^n) - \frac{\Delta t}{2} \sum_i \Big(D_{ji}(u^n) + F_j'(u^n) F_i'(u^n) \Big) \partial_i u^n \right] + O(\Delta t^3). \end{aligned} \tag{3.8}$$

But the exact solution u_{ex} to (1.1) satisfies

$$\begin{aligned} & u_{ex}(t_{n+1}) \\ &= u^n - \Delta t \sum_j \partial_j F_j(u^n) + \frac{\Delta t^2}{2} \sum_{i,j} \partial_j \Big[F_j'(u^n) F_i'(u^n) \partial_i u^n \Big] + O(\Delta t^3), \end{aligned} \tag{3.9}$$

thus

$$u^{n+1-} = u_{ex}(t_{n+1}) + \frac{\Delta t^2}{2} \sum_{ij} \partial_j \left(D_{ji}(u^n) \partial_i u^n \right) + O(\Delta t^3), \tag{3.10}$$

which yields (3.6). □

Since equation (3.6) is the same as the one from the Chapman-Enskog expansion (2.36) of the BGK model with ε replaced by $\Delta t/2$, Theorem 2.13 gives

Corollary 3.2 *If (RSC) holds, then the equivalent equation (3.6) of the semi-discrete projection method is directionally symmetrically entropy dissipative.*

3.2 Approximate Riemann solvers

In this section, we only consider one-dimensional problems, $N = 1$. The notion of approximate Riemann solver, introduced in [8], is a general tool that enables to generate fully discrete numerical schemes. An introduction can be found in [13]. It was noticed in [3] that kinetic schemes can be interpreted as particular approximate Riemann solvers. We here generalize this result to nonlinear relaxation schemes. Related results can be found in [9]. The relaxation approach is particularly adapted to the problem of proving entropy consistency of the approximate Riemann solvers, and each of (EEC) or (RSC) conditions gives such result, as stated below. However, other analysis are possible, as provided in [6], [5], [7].

Let us recall that an approximate Riemann solver for (1.1) is a function $R(x/t, u_l, u_r)$ that is an approximation of the solution to (1.1) with initial data $u^0(x) = u_l$ if $x < 0$, $u^0(x) = u_r$ if $x > 0$, in the sense that it satisfies the consistency relation

$$R(x/t, u, u) = u, \tag{3.11}$$

and the conservativity identity

$$F_l(u_l, u_r) = F_r(u_l, u_r), \tag{3.12}$$

where

$$\begin{aligned} F_l(u_l, u_r) &= F(u_l) - \int_{-\infty}^0 \Big(R(v, u_l, u_r) - u_l \Big) \, dv, \\ F_r(u_l, u_r) &= F(u_r) + \int_0^{\infty} \Big(R(v, u_l, u_r) - u_r \Big) \, dv. \end{aligned} \tag{3.13}$$

It is called dissipative with respect to the convex entropy η if

$$G_r(u_l, u_r) - G_l(u_l, u_r) \leq 0, \tag{3.14}$$

where

$$
\begin{aligned}
G_l(u_l,u_r) &= G(u_l) - \int_{-\infty}^{0} \Big(\eta(R(v,u_l,u_r)) - \eta(u_l)\Big)\,dv,\\
G_r(u_l,u_r) &= G(u_r) + \int_{0}^{\infty} \Big(\eta(R(v,u_l,u_r)) - \eta(u_r)\Big)\,dv,
\end{aligned}
\tag{3.15}
$$

and G is the entropy flux associated to η, $G' = \eta' F'$.

To the approximate Riemann solver we can associate a *Godunov-type scheme* by considering data $u^n(x)$ piecewise constant over a mesh of cells of size Δx_i and by sticking together the local approximate solvers at each interface, which is possible under a CFL condition 1/2 (in the sense that $R(x/t,u_l,u_r) = u_l$ if $x/t < -\Delta x_i/2\Delta t$, and $R(x/t,u_l,u_r) = u_r$ if $x/t > \Delta x_{i+1}/2\Delta t$). By averaging the result over the cells at time t_{n+1}, this gives

$$
\begin{aligned}
u_i^{n+1} &= \frac{1}{\Delta x_i}\int_0^{\Delta x_i/2} R(x/\Delta t, u_{i-1}^n, u_i^n)\,dx + \frac{1}{\Delta x_i}\int_{-\Delta x_i/2}^{0} R(x/\Delta t, u_i^n, u_{i+1}^n)\,dx\\
&= u_i^n - \frac{\Delta t}{\Delta x_i}[F_l(u_i^n, u_{i+1}^n) - F_r(u_{i-1}^n, u_i^n)],
\end{aligned}
\tag{3.16}
$$

which is a conservative scheme according to (3.12), with numerical flux $F(u_l,u_r) = F_l(u_l,u_r) = F_r(u_l,u_r)$. Because of Jensen's inequality, the scheme satisfies discrete entropy inequalities

$$
\eta(u_i^{n+1}) - \eta(u_i^n) + \frac{\Delta t}{\Delta x_i}[G_l(u_i^n,u_{i+1}^n) - G_r(u_{i-1}^n,u_i^n)] \le 0,
\tag{3.17}
$$

and under assumption (3.14), the inequality (3.17) becomes conservative, with any numerical flux $G(u_l,u_r)$ such that $G_r(u_l,u_r) \le G(u_l,u_r) \le G_l(u_l,u_r)$.

Approximate Riemann solvers often have the structure of a set of finitely many simple discontinuities. This means that there exists $m \ge 1$, $s_0 = -\infty < s_1 < \ldots < s_m < s_{m+1} = +\infty$ and $u_0 = u_l, u_1, \ldots, u_{m-1}, u_m = u_r$ (depending on u_l and u_r) such that $R(x/t,u_l,u_r) = u_i$ when $s_i < x/t < s_{i+1}$. Then the conservativity (3.12) becomes $\sum_{i=1}^m s_i(u_i - u_{i-1}) = F(u_r) - F(u_l)$, and the entropy inequality (3.14) becomes $\sum_{i=1}^m s_i(\eta(u_i) - \eta(u_{i-1})) \ge G(u_r) - G(u_l)$. Conservativity thus enables to define the intermediate fluxes F_i, $i = 0,\ldots,m$, by $F_0 = F(u_l)$, $F_m = F(u_r)$, $F_i - F_{i-1} = s_i(u_i - u_{i-1})$, and the numerical flux is given by $F(u_l,u_r) = F_i$ with i such that $s_i \le 0 \le s_{i+1}$.

If in the projection method (3.1)-(3.2) we start with piecewise constant data, and we modify (3.3) by introducing a further averaging over cells in the definition of u^{n+1} in terms of u^{n+1-}, then we are led to the Riemann problem for (3.1) with maxwellian initial data, and u defined in (3.5) is an approximation of the solution to (1.1). This is indeed an approximate Riemann solver, even if (3.1) is solved approximately.

Theorem 3.3 *Let $\mathcal{R}(x/t, f_l, f_r)$ be an approximate Riemann solver for (3.1) (with $N = 1$). Then*

$$
R(x/t,u_l,u_r) = L\,\mathcal{R}\Big(x/t, M(u_l), M(u_r)\Big) - k
\tag{3.18}
$$

is an approximate Riemann solver for (1.1). Moreover, if (EEC) holds and $\mathcal{R}$ is $\mathcal{H}$ entropy satisfying, then R is η entropy satisfying.

Proof. We have obviously from the consistency of $\mathcal{R}$

$$R(x/t,u,u) = L\,\mathcal{R}(x/t,M(u),M(u)) - k = L\,M(u) - k = u, \tag{3.19}$$

which gives the consistency of R (3.11). Next, denote by $A_l(f_l,f_r)$ and $A_r(f_l,f_r)$ the left and right numerical fluxes for the relaxation system (3.1). We have

$$\begin{aligned} &F_l(u_l,u_r)\\ &= F(u_l) - \int_{-\infty}^{0}\Big(R(v,u_l,u_r)-u_l\Big)\,dv\\ &= F(u_l) - L\int_{-\infty}^{0}\Big(\mathcal{R}(v,M(u_l),M(u_r))-M(u_l)\Big)\,dv\\ &= F(u_l) + L\Big[A_l(M(u_l),M(u_r)) - A(M(u_l))\Big]\\ &= L\,A_l(M(u_l),M(u_r)) - k', \end{aligned} \tag{3.20}$$

and similarly

$$\begin{aligned} &F_r(u_l,u_r)\\ &= F(u_r) + \int_{0}^{\infty}\Big(R(v,u_l,u_r)-u_r\Big)\,dv\\ &= F(u_r) + L\int_{0}^{\infty}\Big(\mathcal{R}(v,M(u_l),M(u_r))-M(u_r)\Big)\,dv\\ &= F(u_r) + L\Big[A_r(M(u_l),M(u_r)) - A(M(u_r))\Big]\\ &= L\,A_r(M(u_l),M(u_r)) - k'. \end{aligned} \tag{3.21}$$

Since $\mathcal{R}$ is conservative, we deduce the conservativity of R (3.12), with numerical flux

$$F(u_l,u_r) = L\,A(M(u_l),M(u_r)) - k'. \tag{3.22}$$

Then, assume that (EEC) holds with an entropy extension $\mathcal{H}$ and entropy flux $\mathcal{G}$, and denote by $\mathcal{G}_l(f_l,f_r)$ and $\mathcal{G}_r(f_l,f_r)$ the left and right numerical entropy fluxes associated to $\mathcal{R}$. We have according to (2.6) and to the entropy minimization principle (2.8)

$$\begin{aligned} &G_l(u_l,u_r)\\ &= G(u_l) - \int_{-\infty}^{0}\Big(\eta(R(v,u_l,u_r))-\eta(u_l)\Big)\,dv\\ &\geq G(u_l) - \int_{-\infty}^{0}\Big(\mathcal{H}(\mathcal{R}(v,M(u_l),M(u_r)))-\mathcal{H}(M(u_l))\Big)\,dv\\ &= \mathcal{G}_l(M(u_l),M(u_r)) - \mathcal{G}(M(u_l)) + G(u_l), \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} &G_r(u_l,u_r)\\ &= G(u_r) + \int_{0}^{\infty}\Big(\eta(R(v,u_l,u_r))-\eta(u_r)\Big)\,dv\\ &\leq G(u_r) + \int_{0}^{\infty}\Big(\mathcal{H}(\mathcal{R}(v,M(u_l),M(u_r)))-\mathcal{H}(M(u_r))\Big)\,dv\\ &= \mathcal{G}_r(M(u_l),M(u_r)) - \mathcal{G}(M(u_r)) + G(u_r). \end{aligned} \tag{3.24}$$

But because of (2.7), $-\mathcal{G}(M(u_l)) + G(u_l) = -\mathcal{G}(M(u_r)) + G(u_r)$, thus the entropy dissipativity of $\mathcal{R}$, i.e. $\mathcal{G}_r - \mathcal{G}_l \leq 0$, implies that of R, i.e. $G_r - G_l \leq 0$. $\square$

If the reduced stability condition (RSC) holds instead of (EEC), we have a weaker result, that says that somehow the approximate solver is entropy satisfying for data of small variation.

Theorem 3.4 *Let $\mathcal{R}(x/t, f_l, f_r)$ be the exact Riemann solver for (3.1) (with $N = 1$), and define the approximate solver R by (3.18). If (RSC) holds and η is strictly convex, then R is weakly entropy satisfying, in the sense that the entropy dissipation term $G_l(u_l, u_r) - G_r(u_l, u_r)$ has a nonnegative second order expansion as $u_l \to u$, $u_r \to u$.*

Proof. Let us define

$$\begin{aligned}\varphi(u_l, u_r) &= G_l(u_l, u_r) - G_r(u_l, u_r)\\ &= G(u_l) - G(u_r) - \int_{-\infty}^{0} \Big(\eta(R(v, u_l, u_r)) - \eta(u_l)\Big)\, dv\\ &\qquad - \int_{0}^{\infty} \Big(\eta(R(v, u_l, u_r)) - \eta(u_r)\Big)\, dv.\end{aligned} \tag{3.25}$$

Then according to the consistency (3.11), $\varphi(u, u) = 0$. We need to prove that $d\varphi(u, u) = 0$ and $d^2\varphi(u, u) \geq 0$. Let us first write the identities on the derivatives dR and d^2R of R with respect to u_l and u_r, obtained by differentiating the conservativity (3.12),

$$\begin{aligned}F'(u_l)du_l - F'(u_r)du_r &- \int_{-\infty}^{0} \Big(dR(v, u_l, u_r) \begin{pmatrix} du_l \\ du_r \end{pmatrix} - du_l\Big) dv\\ &- \int_{0}^{\infty} \Big(dR(v, u_l, u_r) \begin{pmatrix} du_l \\ du_r \end{pmatrix} - du_r\Big) dv = 0,\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}F''(u_l) \cdot du_l' \cdot du_l - F''(u_r) \cdot du_r' \cdot du_r &- \int_{-\infty}^{0} d^2R(v, u_l, u_r) \cdot \begin{pmatrix} du_l' \\ du_r' \end{pmatrix} \cdot \begin{pmatrix} du_l \\ du_r \end{pmatrix} dv\\ &- \int_{0}^{\infty} d^2R(v, u_l, u_r) \cdot \begin{pmatrix} du_l' \\ du_r' \end{pmatrix} \cdot \begin{pmatrix} du_l \\ du_r \end{pmatrix} dv = 0.\end{aligned} \tag{3.27}$$

We differentiate (3.25) to get

$$\begin{aligned}&d\varphi(u_l, u_r) \begin{pmatrix} du_l \\ du_r \end{pmatrix}\\ &= \eta'(u_l)F'(u_l)du_l - \eta'(u_r)F'(u_r)du_r\\ &\quad - \int_{-\infty}^{0} \Big(\eta'(R(v, u_l, u_r))dR(v, u_l, u_r) \begin{pmatrix} du_l \\ du_r \end{pmatrix} - \eta'(u_l)du_l\Big) dv\\ &\quad - \int_{0}^{\infty} \Big(\eta'(R(v, u_l, u_r))dR(v, u_l, u_r) \begin{pmatrix} du_l \\ du_r \end{pmatrix} - \eta'(u_r)du_r\Big) dv,\end{aligned} \tag{3.28}$$

and by taking $u_l = u_r = u$, $\eta'(u)$ factorizes and with (3.26) we obtain that $d\varphi(u, u) = 0$. Then, we differentiate (3.28) again, we take $u_l = u_r = u$, we replace the terms containing d^2R according to (3.27), and obtain

$$
\begin{aligned}
&d^2\varphi(u,u)\cdot\binom{du_l'}{du_r'}\cdot\binom{du_l}{du_r}\\
=&\Big(F'(u)^t\eta''(u)+\eta'(u)F''(u)\Big)\cdot du_l'\cdot du_l\\
&-\Big(F'(u)^t\eta''(u)+\eta'(u)F''(u)\Big)\cdot du_r'\cdot du_r\\
&-\int_{-\infty}^{0}\Big(\eta''(u)\cdot dR(v,u,u)\binom{du_l'}{du_r'}\cdot dR(v,u,u)\binom{du_l}{du_r}-\eta''(u)\cdot du_l'\cdot du_l\Big)dv\\
&-\int_{0}^{\infty}\Big(\eta''(u)\cdot dR(v,u,u)\binom{du_l'}{du_r'}\cdot dR(v,u,u)\binom{du_l}{du_r}-\eta''(u)\cdot du_r'\cdot du_r\Big)dv\\
&-\eta'(u)\Big(F''(u)\cdot du_l'\cdot du_l-F''(u)\cdot du_r'\cdot du_r\Big).
\end{aligned}
\tag{3.29}
$$

Now, according to the definition (3.18) of R,

$$dR(x/t,u_l,u_r)\binom{du_l}{du_r}=L\,d\mathcal{R}(x/t,M(u_l),M(u_r))\binom{M'(u_l)du_l}{M'(u_r)du_r}. \tag{3.30}$$

Since $\mathcal{R}$ is the exact Riemann solver for (3.1), the function

$$\delta f(t,x)=d\mathcal{R}(x/t,M(u),M(u))\binom{df_l}{df_r} \tag{3.31}$$

solves

$$
\begin{aligned}
&\partial_t\,\delta f+\partial_x\,A'(M(u))\delta f=0,\\
&\delta f(0,x)=\begin{cases}\delta f_l \text{ if } x<0,\\ \delta f_r \text{ if } x>0.\end{cases}
\end{aligned}
\tag{3.32}
$$

Denoting by P_λ the eigenprojectors of $A'(M(u))$, the solution is given by

$$\delta f(t,x)=\sum_\lambda P_\lambda(\mathbb{1}_{x-\lambda t<0}\,\delta f_l+\mathbb{1}_{x-\lambda t>0}\,\delta f_r). \tag{3.33}$$

Therefore, (3.30) gives by taking (2.1) into account

$$
\begin{aligned}
&dR(v,u,u)\binom{du_l}{du_r}\\
&=\sum_\lambda L\,P_\lambda M'(u)(\mathbb{1}_{v<\lambda}\,du_l+\mathbb{1}_{v>\lambda}\,du_r)\\
&=du_l-\sum_\lambda L\,P_\lambda M'(u)\mathbb{1}_{v>\lambda}\,(du_l-du_r)\\
&=du_r+\sum_\lambda L\,P_\lambda M'(u)\mathbb{1}_{v<\lambda}\,(du_l-du_r)\\
&=\frac{du_l+du_r}{2}-\sum_\lambda \operatorname{sgn}(v-\lambda)\,L\,P_\lambda M'(u)\frac{du_l-du_r}{2}.
\end{aligned}
\tag{3.34}
$$

Let us denote now η'' for $\eta''(u)$ and dR for $dR(v,u,u)$. For the term in the integral on $v<0$ in (3.29), we have

$$\begin{aligned}
&\eta'' \cdot du_l' \cdot du_l - \eta'' \cdot dR\begin{pmatrix} du_l' \\ du_r' \end{pmatrix} \cdot dR\begin{pmatrix} du_l \\ du_r \end{pmatrix} \\
= {}& \eta'' \cdot \frac{du_l' - du_r'}{2} \cdot \frac{du_l - du_r}{2} \\
& - \eta'' \cdot \left(dR\begin{pmatrix} du_l' \\ du_r' \end{pmatrix} - \frac{du_l' + du_r'}{2} \right) \cdot \left(dR\begin{pmatrix} du_l \\ du_r \end{pmatrix} - \frac{du_l + du_r}{2} \right) \\
& - \eta'' \cdot \left(dR\begin{pmatrix} du_l' \\ du_r' \end{pmatrix} - du_l' \right) \cdot \frac{du_l + du_r}{2} \\
& - \eta'' \cdot \frac{du_l' + du_r'}{2} \cdot \left(dR\begin{pmatrix} du_l \\ du_r \end{pmatrix} - du_l \right).
\end{aligned} \tag{3.35}$$

Similarly, for the term in the integral on $v>0$ in (3.29),

$$\begin{aligned}
&\eta'' \cdot du_r' \cdot du_r - \eta'' \cdot dR\begin{pmatrix} du_l' \\ du_r' \end{pmatrix} \cdot dR\begin{pmatrix} du_l \\ du_r \end{pmatrix} \\
= {}& \eta'' \cdot \frac{du_l' - du_r'}{2} \cdot \frac{du_l - du_r}{2} \\
& - \eta'' \cdot \left(dR\begin{pmatrix} du_l' \\ du_r' \end{pmatrix} - \frac{du_l' + du_r'}{2} \right) \cdot \left(dR\begin{pmatrix} du_l \\ du_r \end{pmatrix} - \frac{du_l + du_r}{2} \right) \\
& - \eta'' \cdot \left(dR\begin{pmatrix} du_l' \\ du_r' \end{pmatrix} - du_r' \right) \cdot \frac{du_l + du_r}{2} \\
& - \eta'' \cdot \frac{du_l' + du_r'}{2} \cdot \left(dR\begin{pmatrix} du_l \\ du_r \end{pmatrix} - du_r \right).
\end{aligned} \tag{3.36}$$

Now we integrate for $v<0$ the two last terms in (3.35) by using (3.34),

$$\begin{aligned}
&\int_{-\infty}^{0} \left[-\eta'' \cdot \left(dR\begin{pmatrix} du_l' \\ du_r' \end{pmatrix} - du_l' \right) \cdot \frac{du_l + du_r}{2} \right. \\
&\qquad \left. -\eta'' \cdot \frac{du_l' + du_r'}{2} \cdot \left(dR\begin{pmatrix} du_l \\ du_r \end{pmatrix} - du_l \right) \right] dv \\
= {}& \eta'' \cdot \sum_{\lambda<0} -\lambda\, L\, P_\lambda M' \left(du_l' - du_r' \right) \cdot \frac{du_l + du_r}{2} \\
& + \eta'' \cdot \frac{du_l' + du_r'}{2} \cdot \sum_{\lambda<0} -\lambda\, L\, P_\lambda M' \left(du_l - du_r \right).
\end{aligned} \tag{3.37}$$

We integrate similarly for $v>0$ the two last terms in (3.36) by using (3.34),

$$\begin{aligned}
&\int_{0}^{\infty} \left[-\eta'' \cdot \left(dR\begin{pmatrix} du_l' \\ du_r' \end{pmatrix} - du_r' \right) \cdot \frac{du_l + du_r}{2} \right. \\
&\qquad \left. -\eta'' \cdot \frac{du_l' + du_r'}{2} \cdot \left(dR\begin{pmatrix} du_l \\ du_r \end{pmatrix} - du_r \right) \right] dv \\
= {}& \eta'' \cdot \sum_{\lambda>0} -\lambda\, L\, P_\lambda M' \left(du_l' - du_r' \right) \cdot \frac{du_l + du_r}{2} \\
& + \eta'' \cdot \frac{du_l' + du_r'}{2} \cdot \sum_{\lambda>0} -\lambda\, L\, P_\lambda M' \left(du_l - du_r \right).
\end{aligned} \tag{3.38}$$

We add up the results of (3.37) and (3.38), and since $\sum \lambda L\, P_\lambda M' = LA'M' = F'$ by (2.2), we obtain

$$\begin{aligned} &-\eta'' \cdot F'(du_l' - du_r') \cdot \frac{du_l + du_r}{2} - \eta'' \cdot \frac{du_l' + du_r'}{2} \cdot F'(du_l - du_r) \\ &= -(F')^t \eta'' \cdot du_l' \cdot du_l + (F')^t \eta'' \cdot du_r' \cdot du_r. \end{aligned} \tag{3.39}$$

With the previous computations, we obtain a new expression for (3.29), noticing that the terms containing F'' simplify,

$$\begin{aligned} &d^2\varphi(u,u) \cdot \begin{pmatrix} du_l' \\ du_r' \end{pmatrix} \cdot \begin{pmatrix} du_l \\ du_r \end{pmatrix} \\ &= \int_{-\infty}^{\infty} \Bigg(\eta''(u) \cdot \frac{du_l' - du_r'}{2} \cdot \frac{du_l - du_r}{2} \\ &\quad - \eta''(u) \cdot \left(dR \begin{pmatrix} du_l' \\ du_r' \end{pmatrix} - \frac{du_l' + du_r'}{2} \right) \cdot \left(dR \begin{pmatrix} du_l \\ du_r \end{pmatrix} - \frac{du_l + du_r}{2} \right) \Bigg) dv, \end{aligned} \tag{3.40}$$

where again dR stands for $dR(v,u,u)$. In order to prove the nonnegativity of $d^2\varphi(u,u)$, it is enough to prove that for any v,

$$\begin{aligned} &\eta''(u) \cdot \left(dR \begin{pmatrix} du_l \\ du_r \end{pmatrix} - \frac{du_l + du_r}{2} \right) \cdot \left(dR \begin{pmatrix} du_l \\ du_r \end{pmatrix} - \frac{du_l + du_r}{2} \right) \\ &\leq \eta''(u) \cdot \frac{du_l - du_r}{2} \cdot \frac{du_l - du_r}{2}. \end{aligned} \tag{3.41}$$

We observe that by (3.34),

$$dR \begin{pmatrix} du_l \\ du_r \end{pmatrix} - \frac{du_l + du_r}{2} = -\sum_\lambda \operatorname{sgn}(v - \lambda)\, L\, P_\lambda M'(u) \frac{du_l - du_r}{2}, \tag{3.42}$$

and since (RSC) gives that $L\, P_\lambda M'(u)$ is selfadjoint nonnegative for $\eta''(u)$, we conclude with the following lemma.

Lemma 3.5 *Let $B_1, \ldots, B_s$ be selfadjoint nonnegative operators on an euclidean space, satisfying $\sum B_\ell = \mathrm{Id}$. Then for any scalars $\alpha_1, \ldots, \alpha_s$,*

$$\left\| \sum_{\ell=1}^{s} \alpha_\ell B_\ell \right\| \leq \sup_{1 \leq \ell \leq s} |\alpha_\ell|. \tag{3.43}$$

Proof. The operator $B = \sum \alpha_\ell B_\ell$ is selfadjoint, thus

$$\|B\| = \sup_{du \neq 0} \frac{|du \cdot B du|}{du \cdot du}, \tag{3.44}$$

and since $du \cdot B_\ell du \geq 0$,

$$\begin{aligned} |du \cdot B du| &\leq \sum |\alpha_\ell|\, du \cdot B_\ell du \\ &\leq \sup |\alpha_\ell| \times \sum du \cdot B_\ell du \\ &= \sup |\alpha_\ell| \times du \cdot du, \end{aligned} \tag{3.45}$$

which gives the result. □

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