Stability of multidimensional undercompressive shock waves

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Abstract

This paper is devoted to the study of linear and nonlinear stability of undercompressive shock waves for first order systems of hyperbolic conservation laws in a multidimensional space. We first recall the framework proposed by Freistühler to extend Majda's work on classical shock waves to undercompressive shock waves. Then we show how the so-called uniform stability condition yields a linear stability result in terms of a maximal L^2 estimate. We follow Majda's strategy on shock waves with several improvements and modifications inspired from Métivier's work. The linearized problems are solved by duality and the nonlinear equations by means of a Newton type iteration scheme. Finally, we show how this work applies to phase transitions in an isothermal van der Waals fluid.

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1 Introduction

In the area of hyperbolic conservation laws, a central problem is the natural arising of discontinuities in smooth solutions that is linked to the formation of shock waves. The first general study of this phenomenon dates back to the works of Lax [28, 29], though the interest for shock waves in gas dynamics was already an active field of both physical and mathematical research. We refer for instance to [47, 10, 11, 13, 14, 24], this list being far from exhaustive. As time was going by, the theory of hyperbolic conservation laws began to receive a rather general treatment: the scalar case was solved by Kružkov [26, 27] in the late 60's while Glimm's results [18] initiated an impressive series of works on one-dimensional problems. We refer to [6] for a review of the results in this case. For multidimensional systems, there are far fewer results than in the one-dimensional case. Using different techniques, Gårding and Kato proved the local existence of smooth solutions [17, 23]. An alternative proof that relies on Friedrichs' symmetrization can be found in [32]. However, smooth solutions to hyperbolic systems naturally develop singularities in finite time, as was shown in a remarkable case by Sideris [44]. With this blowup phenomenon in mind, it appears natural to investigate the existence of discontinuous solutions: the simplest of these consist of shock waves, that is solutions that are smooth on either side of a moving interface. Under a few technical assumptions, Majda proved in the early 80's, see [31, 30], the local existence of such multidimensional shock waves. An important step in the analysis is the study of the stability of planar shock waves with respect to small perturbations. We refer to [32, 41] for an overview of the method and its applications to gas dynamics.

In several applications such as magnetohydrodynamics or the modeling of phase transitions, it happens that certain shock waves violate the so-called Lax entropy inequalities. Depending on their nature, these discontinuous solutions bear the name of undercompressive or overcompressive shock waves (we shall recall these notions later). To select the physically relevant discontinuities, an additional admissibility criterion may be required (this is not the case for overcompressive shock waves). For isothermal subsonic phase transitions, Slemrod (for gas dynamics) and Truskinovsky (for elastodynamics) independently derived a criterion based on the theory of capillarity [45, 46], see also [42]. In this paper, we shall follow a general approach and use an abstract admissibility criterion, as suggested by Freistühler, which applies to some concrete problems (this approach was also used by Colombo and Corli in [8] for one-dimensional problems).

Our purpose is to study the linear stability of multidimensional undercompressive shock waves with respect to small perturbations: this is the extension of Majda's work on shock waves proposed by Freistühler in [15]. We follow Majda's strategy but make several technical improvements. In [31], Majda used an H^s version of pseudodifferential calculus that relied on Moser's inequalities [37]. In this paper, we follow Mokrane [36] and Métivier [33] and use the paradifferential calculus of Bony and Meyer [5, 34]. The main results of this symbolic calculus are collected in an appendix at the end of the paper. The great advantage of this technique is to use the minimal regularity in order to derive an energy estimate similar to Majda's one. In [31], Majda defined a block structure condition that enabled him to perform a symbolic construction and get satisfactory energy estimates. Métivier proved recently that the block structure condition was met by all hyperbolic systems with constant multiplicities. This result will motivate the general assumptions of section 2.

The remaining of this paper is organized as follows: in section 2, we introduce the framework of our study and state our main assumptions. We recall the definition of undercompressive shock waves. In section 3, we study the linear stability of a planar undercompressive shock and show how the so-called uniform stability condition yields a maximal L^2 estimate. The study of planar shock waves will be extended to the variable coefficients case in section 4. We recover similar results as proved by Mokrane and Métivier for classical shock waves. Section 6 is devoted to the application of our work to isothermal phase transitions in a van der Waals fluid. This problem was first addressed in [2, 3].

2 General considerations

We consider a system of n conservation laws in time-space $\mathbb{R} \times \mathbb{R}^d$:

$$\sum_{j=0}^{d} \partial_j f_j(u) = 0 \, , \tag{1}$$

where x_0 is the time variable, also denoted by t in the sequel, $(x_1, \dots, x_d)$ is the space variable and ∂_j stands for the partial derivative with respect to x_j. The fluxes $f_0, \dots, f_d$ are C^∞ functions defined on an open set U of $\mathbb{R}^n$ with values in $\mathbb{R}^n$. The jacobian matrix of f_j at point u will be denoted by $A_j(u)$. For the sake of completeness, we assume that f_0 is a C^∞ diffeomorphism on its image. One can simply think of f_0 as the identity. We assume that the space dimension d is strictly greater than 1 (that is $d \geq 2$) and make the following hyperbolicity assumption:

Assumption 1. *If we define a matrix $A(u,\xi)$ by*

$$\forall \, u \in U \, , \quad \forall \, \xi \in \mathbb{R}^d \, , \quad A(u,\xi) := \sum_{j=1}^{d} \xi_j \, A_0(u)^{-1} \, A_j(u) \, , \tag{2}$$

then $A(u,\xi)$ is diagonalizable over $\mathbb{R}$ with C^∞ eigenvalues $\lambda_1, \dots, \lambda_m$ (defined on $U \times \mathbb{R}^d \setminus \{0\}$) of fixed multiplicities $\alpha_1, \dots, \alpha_m$. Without loss of generality, we may assume that

$$\forall \, u \in U \, , \quad \forall \, \xi \in \mathbb{R}^d \setminus \{0\} \, , \quad \lambda_1(u,\xi) < \cdots < \lambda_m(u,\xi) \, .$$

We point out that assumption 1 is easy to check on the system, unlike the block structure condition. However, one could replace assumption 1 by the more abstract block structure condition, as was made in [31, 33]. We also note that the Lundquist's equations of magneto-hydrodynamics (see [12]) violate assumption 1. An open, but interesting, question is to know whether the block structure condition is satisfied by the MHD system.

We first recall a basic fact on piecewise smooth solutions to (1) (we refer for instance to [12, 40] for the proof). Let $\Gamma = \{x_d - \varphi(x_0, \dots, x_{d-1}) = 0\}$ be a smooth hypersurface in $\mathbb{R}^{d+1}$ and u be a smooth function on either side of Γ. Then u is a weak solution of (1) if and only if u satisfies (1) (in the classical sense) on either side of Γ and if the Rankine-Hugoniot conditions hold at each point of Γ:

$$\forall \, \mathbf{x} = (x_0, \dots, x_d) \in \Gamma \, , \quad \sum_{j=0}^{d-1} \partial_j \varphi \, [f_j(u)](\mathbf{x}) - [f_d(u)](\mathbf{x}) = 0 \, , \tag{3}$$

the partial derivatives of φ in the above formula being evaluated at $(x_0, \dots, x_{d-1})$. As usual we have let

$$[f_j(u)](\mathbf{x}) := \lim_{s \to 0^+} \left(f_j(\mathbf{x} + s\mathbf{n}) - f_j(\mathbf{x} - s\mathbf{n}) \right) \quad \text{with} \quad \mathbf{n} = (-\partial_0 \varphi, \dots, -\partial_{d-1}\varphi, 1)$$

a normal vector to Γ at point $\mathbf{x}$. In the sequel, we omit the dependence on $\mathbf{x}$ in the expressions $[f_j(u)]$.

In this paper, we are interested in the stability of undercompressive shock waves. To fix ideas and notations, we consider a planar shock wave

$$\mathbf{u} = \begin{cases} u_r & \text{if } x_d > \sigma t + \nu \cdot y, \\ u_l & \text{if } x_d < \sigma t + \nu \cdot y, \end{cases} \tag{4}$$

that is solution to (1). In (4), u_r and u_l are fixed vectors belonging to the open set U, $y = (x_1, \dots, x_{d-1})$ is the vector formed by the tangential space coordinates, ν is a wave vector in $\mathbb{R}^{d-1}$ and σ is the normal speed of propagation of the front. This corresponds to the equation

$$\varphi(x_0, \dots, x_{d-1}) = \sigma x_0 + \sum_{j=1}^{d-1} \nu_j x_j$$

for the shock front curve. Clearly $\mathbf{u}$ is a solution of (1) if and only if

$$\sigma \left[f_0(\mathbf{u})\right] + \sum_{j=1}^{d-1} \nu_j \left[f_j(\mathbf{u})\right] = \left[f_d(\mathbf{u})\right].$$

Performing a rotation of the axes, we may assume that $\nu = 0$. Changing the last space variable x_d into $x_d - \sigma t$ and correspondingly the flux f_d into $f_d - \sigma f_0$, we may also assume that the shock is stationary, that is $\sigma = 0$ (this will simplify the statement of assumption 2). Note that assumption 1 is still satisfied after this change of observer. We assume that the shock wave $\mathbf{u}$ is noncharacteristic. More precisely, we make the following assumption:

Assumption 2. *There exist two integers $p, q \in \{1, \dots, m\}$ such that*

$$\lambda_{q-1}(u_l, \xi) < 0 < \lambda_q(u_l, \xi) \quad \textit{and} \quad \lambda_p(u_r, \xi) < 0 < \lambda_{p+1}(u_r, \xi)$$

where $\xi := (0, \dots, 0, 1) \in \mathbb{R}^d$ is the vector of propagation of the shock wave $\mathbf{u}$.

If $q = 1$ (resp. $p = m$) the first inequality on the left (resp. the last inequality on the right) is ignored. Assumption 2 can be illustrated by figure 1 where the characteristics entering the shock front curve $\{x_d = \sigma t\}$ are depicted (here we have not assumed $\sigma = 0$).

Following Freistühler [15], we define the number

$$\kappa := (\alpha_q + \cdots + \alpha_m) + (\alpha_1 + \cdots + \alpha_p) - n\,, \tag{5}$$

representing the amount by which the number of incoming characteristics (counted with multiplicity) exceeds n. We introduce the following terminology:

Definition 1. *If $\kappa = 1$ (that is $p = q$ and λ_p is a simple eigenvalue), then* $\mathbf{u}$ *is called a p-Lax shock wave.*

If $\kappa < 1$ (that is $p < q$), then $\mathbf{u}$ *is called an undercompressive shock wave.*

If $\kappa > 1$ (that is $p > q$), then $\mathbf{u}$ *is called an overcompressive shock wave.*

As already mentioned in some previous works [22, 35, 43], in the case of an undercompressive shock wave data, the associated Riemann problem has several solutions. In other words, the Rankine-Hugoniot conditions (3) are not sufficient to single out a unique weak piecewise smooth solution. Additional constraints on the shock wave $\mathbf{u}$ are needed. As explained in the introduction, we shall follow a general approach and will not discuss the way how the additional "admissibility criterion" is derived (except in section 6 where we shall discuss an application to phase transitions in van der Waals fluids). We introduce the following definition:

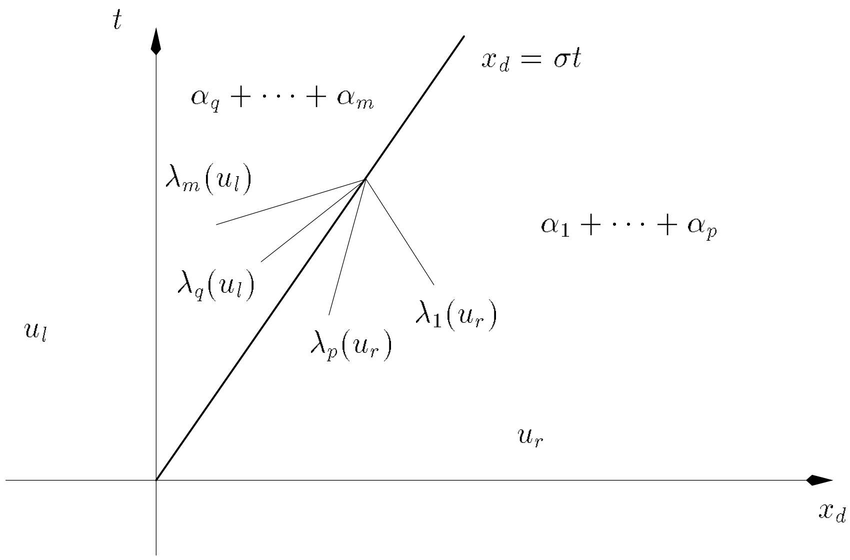

Figure 1: Characteristics entering the shock front

Definition 2. *Let* $\mathbf{u}$*, defined by* (4)*, be a planar undercompressive shock wave. We shall say that* $\mathbf{u}$ *is an* **admissible** *discontinuous solution if and only if*

$$\begin{gathered}\sigma\,[f_0(\mathbf{u})] + \sum_{j=1}^{d-1} \nu_j\,[f_j(\mathbf{u})] - [f_d(\mathbf{u})] = 0\,, \\ g(u_l, u_r, \sigma, \nu) = 0\,,\end{gathered} \tag{6}$$

where $g : U \times U \times \mathbb{R} \times \mathbb{R}^{d-1} \to \mathbb{R}^{N-n}$ *is a smooth, say* C^∞*, function. Here,* N *is an integer such that*

$$N - n = 1 - \kappa \geq 1\,,$$

that is $N - n$ *counts the number of additional jump conditions that need to be prescribed and* N *is the total amount of prescribed jump conditions including the Rankine-Hugoniot conditions.*

The local existence of admissible undercompressive shock waves is a free boundary problem (function φ defining the shock front is part of the unknown of the problem). To circumvent this difficulty, we introduce a change of variables in $\mathbb{R}^{d+1}$ with the formula:

$$\Phi(x_0, \dots, x_d) := (x_0, \dots, x_{d-1}, x_d + \varphi(x_0, \dots, x_{d-1}))\,.$$

We have chosen here the standard change of variables, as in [31, 33, 36]. It maps the hyperplane $\{x_d = 0\}$ onto the hypersurface Γ and the two half-spaces $\{\pm x_d > 0\}$ on the two sides of Γ. We now perform a change of unknown functions. If u is a smooth function on either side of Γ, then the function $u_\sharp$ defined by

$$\forall\, (x_0, \dots, x_d) \in \mathbb{R}^{d+1}, \quad u_\sharp(x_0, \dots, x_d) := u(\Phi(x_0, \dots, x_d))$$

is smooth on either side of the hyperplane $\{x_d = 0\}$. It is now a straightforward application of the chain rule to verify that u defines an admissible undercompressive shock wave if and only if

$$\begin{cases} L(u_\sharp^\pm, \varphi)\, u_\sharp^\pm = 0\,, & \pm\, x_d > 0\,, \\ B(u_\sharp^+, u_\sharp^-, \varphi) = 0\,, & x_d = 0\,, \end{cases} \tag{7}$$

where $u_\sharp^+$ (resp. $u_\sharp^-$) denotes the restriction of $u_\sharp$ to the half-space $\{x_d > 0\}$ (resp. $\{x_d < 0\}$), and operators L and B are defined by the formulas:

$$L(v,\psi)\, w := \sum_{j=0}^{d-1} A_j(v)\, \partial_j\, w + \widetilde{A_d}(v,\nabla\psi)\, \partial_d\, w \tag{8a}$$

$$\text{with} \quad \widetilde{A_d}(v,\nabla\psi) := A_d(v) - \sum_{j=0}^{d-1} \partial_j\psi\, A_j(v) \tag{8b}$$

$$B(w^+,w^-,\psi) := \begin{pmatrix} \displaystyle\sum_{j=0}^{d-1} \partial_j\psi\,[f_j(w)] - [f_d(w)] \\ \\ g(w^+,w^-,\partial_0\psi,\nabla_y\psi) \end{pmatrix}. \tag{8c}$$

As in (4), y denotes the vector of the tangential space variables. Now that we have fixed the domain, the problem reduces to the following question: given an initial datum u^0 that is smooth on either side of a hypersurface $\{x_d = \varphi^0(x_1,\dots,x_{d-1})\}$, does there exist a solution $(u_\sharp,\varphi)$ of (7) with initial value $(u_\sharp^0,\varphi^0)$, at least locally in time? We shall positively answer this question under a "uniform stability" assumption. This extends Majda's work to the framework set by Freistühler for undercompressive shock waves. As in Majda's analysis, special attention will be paid to the linear stability of the reference undercompressive shock wave $\mathbf{u}$ with respect to small perturbations.

3 The constant coefficients case

3.1 The linearized equations

We first examin the linear stability of an admissible planar undercompressive shock wave $\mathbf{u}$ (with $\nu = 0$ and $\sigma = 0$). We introduce the linearized operators about the shock $\mathbf{u}$. Considering a family $u_s = \mathbf{u} + sv$ and $\varphi_s = s\psi$, we define the linear operators

$$\mathcal{L}_{\mathbf{u}}(v^\pm,\psi) := \frac{d}{ds}\, L(u_s^\pm,\varphi_s)\, u_s^\pm\,|_{s=0}\,, \tag{9a}$$

$$\mathcal{B}_{\mathbf{u}}(v,\psi) := \frac{d}{ds}\, B(u_s^+,u_s^-,\varphi_s)\,|_{s=0}\,. \tag{9b}$$

Since $\mathbf{u}$ is constant on either side of $\{x_d = 0\}$, the linearized operators read

$$\mathcal{L}_{\mathbf{u}}(v^\pm,\psi) = \mathcal{L}_{\mathbf{u}} v^\pm = \sum_{j=0}^{d} A_j(u_{r,l})\, \partial_j\, v^\pm\,, \quad \pm x_d > 0\,,$$

$$\mathcal{B}_{\mathbf{u}}(v,\psi) = \begin{pmatrix} \displaystyle\sum_{j=0}^{d-1} \partial_j\psi\,[f_j(\mathbf{u})] - A_d(u_r)\, v^+ + A_d(u_l)\, v^- \\ \\ \dfrac{\partial g}{\partial\sigma}\cdot\partial_0\psi + d_\nu g\cdot\nabla_y\psi + d_{u_r} g\cdot v^+ + d_{u_l} g\cdot v^- \end{pmatrix} \quad x_d = 0\,.$$

We let

$$b_0(\mathbf{u}) := \begin{pmatrix} [f_0(\mathbf{u})] \\ \partial g/\partial\sigma \end{pmatrix} \quad , \quad b_j(\mathbf{u}) := \begin{pmatrix} [f_j(\mathbf{u})] \\ \partial g/\partial\nu_j \end{pmatrix} \quad 1 \le j \le d-1\,,$$

and

$$M(\mathbf{u}) \begin{pmatrix} v^+ \\ v^- \end{pmatrix} := \begin{pmatrix} -A_d(u_r) \\ d_{u_r} g \end{pmatrix} v^+ + \begin{pmatrix} A_d(u_l) \\ d_{u_l} g \end{pmatrix} v^- .$$

Then the linearized boundary value operator $\mathcal{B}_\mathbf{u}$ reads

$$\mathcal{B}_\mathbf{u}(v, \psi) = \sum_{j=0}^{d-1} \partial_j \psi \, b_j(\mathbf{u}) + M(\mathbf{u}) \begin{pmatrix} v^+ \\ v^- \end{pmatrix} .$$

We are now led to consider the boundary value problem for the unknown functions (v, ψ):

$$\begin{cases} \mathcal{L}_\mathbf{u} v^\pm = f_\pm & \pm x_d > 0 \, , \\ \mathcal{B}_\mathbf{u}(v, \psi) = g & x_d = 0 \, , \end{cases} \tag{10}$$

where the source terms $f_\pm$ and g belong to appropriate functional spaces.

Note that system (10) is a constant coefficients hyperbolic boundary value problem; moreover, the boundary $\{x_d = 0\}$ is noncharacteristic because of assumption 2: both matrices $A_d(u_r)$ and $A_d(u_l)$ are nonsingular. But Kreiss' theory [25] on this class of problems does not apply directly because the boundary conditions involve partial derivatives of ψ. Nonetheless, we attack problem (10) by the same kind of arguments. Formally, we perform a Laplace transform in x_0 and a Fourier transform in the tangential space variables $(x_1, \dots, x_{d-1})$. We also make the change of variables $(x_d \to -x_d)$ in the evolution equation for v^-. This yields the following system of ODEs:

$$\left(\tau A_0(u_r) + i \sum_{j=1}^{d-1} \eta_j A_j(u_r) \right) V^+ + A_d(u_r) \frac{dV^+}{dx_d} = F_+ \, , \tag{11a}$$

$$\left(\tau A_0(u_l) + i \sum_{j=1}^{d-1} \eta_j A_j(u_l) \right) V^- - A_d(u_l) \frac{dV^-}{dx_d} = F_- \, , \tag{11b}$$

in the domain $\{x_d > 0\}$, with the boundary conditions:

$$\left(\tau b_0(\mathbf{u}) + i \sum_{j=1}^{d-1} \eta_j b_j(\mathbf{u}) \right) \Psi + M(\mathbf{u}) \begin{pmatrix} V^+(0) \\ V^-(0) \end{pmatrix} = G \, , \tag{12}$$

on $\{x_d = 0\}$. The complex number $\tau = \gamma + i\eta_0$ has nonnegative real part. In the sequel, we shall denote by η the vector $(\eta_0, \eta_1, \dots, \eta_{d-1}) \in \mathbb{R}^d$.

Because the boundary is noncharacteristic, we may rewrite systems (11a)-(11b) as an ODE system of the form

$$\begin{gathered} \frac{d}{dx_d} \begin{pmatrix} V^+ \\ V^- \end{pmatrix} = \mathcal{A}(\mathbf{u}, \eta, \gamma) \begin{pmatrix} V^+ \\ V^- \end{pmatrix} + \mathcal{A}_d(\mathbf{u})^{-1} \begin{pmatrix} F_+ \\ F_- \end{pmatrix} \\ \text{with } \mathcal{A}(\mathbf{u}, \eta, \gamma) := \begin{pmatrix} \mathcal{A}_r(\mathbf{u}, \eta, \gamma) & \mathbf{0} \\ \mathbf{0} & \mathcal{A}_l(\mathbf{u}, \eta, \gamma) \end{pmatrix} . \end{gathered} \tag{13}$$

Matrices $\mathcal{A}_{r,l}$ and $\mathcal{A}_d$ in (13) are defined by

$$\mathcal{A}_{r,l}(\mathbf{u}, \eta, \gamma) := \mp A_d(u_{r,l})^{-1} \left(\gamma A_0(u_{r,l}) + i \sum_{j=0}^{d-1} \eta_j A_j(u_{r,l}) \right) ,$$

$$\mathcal{A}_d(\mathbf{u}) := \begin{pmatrix} A_d(u_r) & \mathbf{0} \\ \mathbf{0} & -A_d(u_l) \end{pmatrix} .$$

Defining

$$\mathcal{A}_j(\mathbf{u}) := \begin{pmatrix} A_j(u_r) & \mathbf{0} \\ \mathbf{0} & A_j(u_l) \end{pmatrix} ,$$

we also have

$$\mathcal{A}(\mathbf{u},\eta,\gamma) := -\mathcal{A}_d^{-1} \left(\gamma\, \mathcal{A}_0(\mathbf{u}) + i \sum_{j=0}^{d-1} \eta_j\, \mathcal{A}_j(\mathbf{u}) \right) .$$

Finally, we define the symbol associated to the shock front:

$$b(\mathbf{u},\eta,\gamma) := \gamma\, b_0(\mathbf{u}) + i \sum_{j=0}^{d-1} \eta_j\, b_j(\mathbf{u}) . \tag{14}$$

As pointed out by Hersh [19], the homogeneous part of the ODEs (13) is hyperbolic when $\gamma > 0$, that is the matrix $\mathcal{A}$ does not admit purely imaginary eigenmodes when $\gamma > 0$. For $\gamma > 0$, we define $\mathcal{E}^-(\eta,\gamma)$ as the stable subspace of $\mathcal{A}$, that is the set of boundary values at $x_d = 0$ of solutions to (13) that are square integrable on $[0,+\infty[$ when $F_+ = F_- = 0$. Because of the decoupled nature of (13), the stable subspace $\mathcal{E}^-(\eta,\gamma)$ is nothing but the product of the stable subspaces of $\mathcal{A}_r$ and $\mathcal{A}_l$. The dimension of $\mathcal{E}^-(\eta,\gamma)$ is equal to the dimension of $\mathcal{E}^-(0,\gamma)$ for all η (this is just a continuity/connectedness argument). To compute the dimension of $\mathcal{E}^-(0,\gamma)$, we seek the eigenmodes of $\mathcal{A}$ when $\eta = 0$. These are the roots ω of the dispersion equation

$$\det\left[\omega I - \mathcal{A}(\mathbf{u},0,\gamma)\right] = 0 .$$

The definition of $\mathcal{A}$ shows that $\omega I - \mathcal{A}(\mathbf{u},0,\gamma)$ is singular if and only if $\gamma A(u_r) + \omega A_d(u_r)$ or $\gamma A(u_l) - \omega A_d(u_l)$ is singular. As a consequence, ω satisfies the dispersion equation if and only if there exists an integer $k \in \{1,\dots,m\}$ such that

$$\lambda_k(u_r,\xi)\,\omega = -\gamma \quad \text{or} \quad \lambda_k(u_l,\xi)\,\omega = \gamma ,$$

where $\xi = (0,\dots,0,1) \in \mathbb{R}^d$. Assumption 2 shows that such values of ω are negative for $k = p+1,\dots,m$ in the first case and $k = 1,\dots,q-1$ in the second case. Taking multiplicities into account, this shows that $\mathcal{E}^-(0,\gamma)$ has dimension $n-\kappa$ as long as $\gamma > 0$.

For fixed $\underline{\eta} \neq 0$, the stable subspace $\mathcal{E}^-(\eta,\gamma)$ admits a continuous extension to $(\underline{\eta},0)$, see [4] (the argument makes use of the compactness of Grassmanian manifolds); we still denote this extension by $\mathcal{E}^-(\underline{\eta},0)$. Note that for $\gamma = 0$, vectors in the extended stable subspace do not always correspond to boundary values of square integrable functions because of the possible occurence of purely imaginary eigenmodes. This is widely detailed in [2, 9, 31] for Euler equations of isentropic gas dynamics.

We define the hemisphere of $\mathbb{R}^{d+1}$ as

$$\Sigma_+ := \{(\eta,\gamma) \in \mathbb{R}^{d+1} \text{ s.t. } \gamma^2 + |\eta|^2 = 1 \text{ and } \gamma \geq 0\} .$$

The boundary of Σ_+ will be denoted by Σ_0, that is

$$\Sigma_0 := \{(\eta,0) \in \mathbb{R}^{d+1} \text{ s.t. } |\eta|^2 = 1\} .$$

Following [31], we make the following assumption:

Assumption 3 (Uniform stability). *There exists a positive constant $c > 0$ such that for all $(\eta,\gamma) \in \Sigma_+$, one has*

$$\forall\,(\chi,Z) \in \mathbb{C} \times \mathcal{E}^-(\eta,\gamma) , \quad |\chi\, b(\mathbf{u},\eta,\gamma) + M(\mathbf{u})\, Z| \geq c\,(|\chi| + |Z|) . \tag{15}$$

By compactness of Σ_+, assumption 3 is equivalent to the requirement that for all $(\eta,\gamma)\in\Sigma_+$, the subspace

$$\big\{(\chi,Z)\in\mathbb{C}\times\mathcal{E}^-(\eta,\gamma)\ \text{ s.t. }\ \chi\,b(\mathbf{u},\eta,\gamma)+M(\mathbf{u})\,Z=0\big\}$$

is reduced to $\{0\}$. This is true because

$$(\chi,Z)\in\mathbb{C}\times\mathbb{C}^{2n}\longmapsto\chi\,b(\mathbf{u},\eta,\gamma)+M(\mathbf{u})\,Z$$

has values in $\mathbb{C}^N$, and $N=n-\kappa+1$ is the dimension of $\mathbb{C}\times\mathcal{E}^-$. Assumption 3 is therefore the direct extension of Majda's uniform stability condition to undercompressive shock waves, once additional jump conditions have been prescribed.

In the next paragraph, we show how assumption 3 yields a maximal L^2 energy estimate for the linearized problem (10).

3.2 Stability of a planar undercompressive shock wave

Recall that b is homogeneous of degree 1 in (η,γ). So using (15) we have

$$\forall\,(\eta,\gamma)\in\mathbb{R}^d\times\mathbb{R}_+,\quad c\,(\gamma^2+|\eta|^2)\le|b(\mathbf{u},\eta,\gamma)|^2\,. \tag{16}$$

Furthermore, assumption 3 enables us to reformulate the boundary conditions in (12) by isolating the unknown shock front Ψ. Because of (16), we can define for all $(\eta,\gamma)\in\mathbb{R}^d\times\mathbb{R}_+$, $(\eta,\gamma)\neq(0,0)$, the orthogonal projector

$$\Pi(\mathbf{u},\eta,\gamma)\,h:=h-\frac{\langle h,b(\mathbf{u},\eta,\gamma)\rangle}{|b(\mathbf{u},\eta,\gamma)|^2}\,b(\mathbf{u},\eta,\gamma)\,.$$

Multiplying (12) by Π and b^* yields the new boundary conditions

$$\Pi(\mathbf{u},\eta,\gamma)M(\mathbf{u})\,V(0)=\Pi(\mathbf{u},\eta,\gamma)\,G\,, \tag{17a}$$

$$|b(\mathbf{u},\eta,\gamma)|^2\,\Psi=\langle G-M(\mathbf{u})\,V(0),b(\mathbf{u},\eta,\gamma)\rangle\,. \tag{17b}$$

Using Schwarz' inequality and (16), we deduce from (17b) the estimate

$$\forall\,(\eta,\gamma)\in\mathbb{R}^d\times\mathbb{R}_+,\quad(\gamma^2+|\eta|^2)\,|\Psi|^2\le C\,(|G|^2+|V(0)|^2) \tag{18}$$

for a suitable constant C depending only on the shock $\mathbf{u}$. It is thus sufficient to get an estimate on $V(0)$ and we shall get from (18) an estimate on Ψ. In order to obtain the desired estimate on $V(0)$, we now attack the following boundary value problem:

$$\begin{cases}\dfrac{dV}{dx_d}=\mathcal{A}(\mathbf{u},\eta,\gamma)\,V+\mathcal{A}_d(\mathbf{u})^{-1}\,F & x_d>0\,,\\ \Pi(\mathbf{u},\eta,\gamma)M(\mathbf{u})\,V(0)=\Pi(\mathbf{u},\eta,\gamma)\,G\,.\end{cases} \tag{19}$$

From assumption 3, we deduce that the boundary conditions ΠM satisfy the uniform stability condition. In other words, for all $(\eta,\gamma)\in\Sigma_+$, one has

$$\forall\,Z\in\mathcal{E}^-(\eta,\gamma)\,,\quad|\Pi(\mathbf{u},\eta,\gamma)M(\mathbf{u})\,Z|\ge c\,|Z|\,.$$

In order to state our first result, we need to introduce a few more notations. Denote by Ω and ω the domains

$$\Omega:=\mathbb{R}^{d+1}_+=\{(x_0,\dots,x_d)\in\mathbb{R}^{d+1}\text{ s.t. }x_d>0\}\quad\text{ and }\quad\omega:=\mathbb{R}^d=\partial\Omega\,.$$

For $\gamma > 0$ and $s \in \mathbb{R}$ we define the following symbols

$$\forall \, \xi \in \mathbb{R}^d, \quad \lambda^{s,\gamma}(\xi) := (\gamma^2 + |\xi|^2)^{s/2} \, .$$

The usual Sobolev spaces $H^s(\omega)$ are equipped with the weighted norms (depending on the positive parameter γ):

$$\|f\|^2_{s,\gamma} := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \lambda^{2s,\gamma}(\xi) \, |\widehat{f}(\xi)|^2 \, d\xi \, .$$

These weighted norms enable to construct a parameter version of the classical pseudodifferential calculus that is of constant use in the study of mixed initial boundary value problems for hyperbolic systems, see [1, 25, 31].

For all integer k, we equip the usual Sobolev space $H^k(\Omega)$ with the following norm

$$|\!|\!| f |\!|\!|^2_{k,\gamma} := \sum_{j=0}^{k} \int_0^{+\infty} \|\partial_d^j f(\cdot, x_d)\|^2_{k-j,\gamma} \, dx_d \, .$$

The Laplace transform performed in the normal modes analysis amounts to work with the new functions $\widetilde{v} = \exp(-\gamma t) v$ and $\widetilde{\psi} = \exp(-\gamma t)\psi$, with $\gamma > 0$. This leads to the introduction of the "weighted" operators:

$$\begin{aligned} & \mathcal{L}^\gamma_{\mathbf{u}} \widetilde{v} := \mathcal{L}_{\mathbf{u}} \widetilde{v} + \gamma \mathcal{A}_0(\mathbf{u}) \, \widetilde{v} \\ \text{and} \qquad & \mathcal{B}^\gamma_{\mathbf{u}}(\widetilde{v}, \widetilde{\psi}) := \mathcal{B}_{\mathbf{u}}(\widetilde{v}, \widetilde{\psi}) + \gamma \widetilde{\psi} \, b_0(u) \, . \end{aligned}$$

One easily checks that (10) is equivalent to

$$\begin{cases} \mathcal{L}^\gamma_{\mathbf{u}} \widetilde{v} = \exp(-\gamma t) f_\pm & \text{for } \pm x_d > 0 \, , \\ \mathcal{B}^\gamma_{\mathbf{u}}(\widetilde{v}, \widetilde{\psi}) = \exp(-\gamma t) g & \text{for } x_d = 0 \, . \end{cases}$$

For convenience, we drop the tilda from v and ψ (keeping in mind that these functions have been multiplied by a decreasing exponential function). The basic estimate is the same as for Lax shocks:

Theorem 3.1. *Let* $\mathbf{u}$ *be an admissible planar undercompressive shock wave that satisfies assumption 3. Then there exists a constant* $C > 0$ *such that for all* $v \in H^1(\Omega)$*, for all* $\psi \in H^1(\omega)$ *and for all* $\gamma \geq 1$*, the following estimate holds:*

$$\gamma |\!|\!| v |\!|\!|^2_0 + \|v_{|_{x_d=0}}\|^2_0 + \|\psi\|^2_{1,\gamma} \leq C \left(\frac{1}{\gamma} |\!|\!| \mathcal{L}^\gamma_{\mathbf{u}} v |\!|\!|^2_0 + \|\mathcal{B}^\gamma_{\mathbf{u}}(v, \psi)\|^2_0 \right) . \tag{20}$$

Energy estimates in Sobolev spaces of higher order are also available, provided that v *and* ψ *are sufficiently regular.*

3.3 Proof of theorem 3.1

The proof of theorem 3.1 follows the methods of Kreiss [25] and Majda [31]. Theorem 3.1 is a consequence of the existence of a Kreiss' symmetrizer:

Proposition 3.1 (Kreiss' symmetrizer). [7, 25]. *If assumption 3 is satisfied, then there exists a* C^∞ *matrix valued mapping* R *defined on* Σ_+ *such that*

$\forall \, (\eta, \gamma) \in \Sigma_+ \, , \quad R(\eta, \gamma)$ *is hermitian,* *there exist positive constants c and C verifying*

$$\forall (\eta,\gamma) \in \Sigma_+ , \quad \mathrm{Re}\ (R(\eta,\gamma)\mathcal{A}(\mathbf{u},\eta,\gamma)) \geq c\,\gamma\, I \,, \tag{21a}$$

$$R(\eta,\gamma) + C\,\beta(\eta,\gamma)^* \beta(\eta,\gamma) \geq c\, I \quad \text{with} \quad \beta(\eta,\gamma) := \Pi(\mathbf{u},\eta,\gamma) M(\mathbf{u}) \,. \tag{21b}$$

We show how proposition 3.1 enables to prove (20). We extend the symmetrizer R as a homogeneous mapping of degree 0 with respect to (η,γ), so one checks that (21) still holds for any value of $(\eta,\gamma) \in \mathbb{R}^d \times \mathbb{R}^+$, $(\eta,\gamma) \neq (0,0)$. Let $v \in H^1(\Omega)$ and $\psi \in H^1(\omega)$. We define

$$F = \mathcal{L}^\gamma_{\mathbf{u}} v \in L^2(\Omega) \quad , \quad G = \mathcal{B}^\gamma_{\mathbf{u}}(v,\psi) \in L^2(\omega) \,,$$

and write

$$\frac{d\widehat{v}}{dx_d} = \mathcal{A}(\mathbf{u},\eta,\gamma)\,\widehat{v} + \mathcal{A}_d^{-1}(\mathbf{u})\,\widehat{F} \,,$$

$$\beta(\eta,\gamma)\,\widehat{v}(0) = \Pi(\eta,\gamma)\,\widehat{G} \,.$$

We take the scalar product of the previous ODE by $R(\eta,\gamma)\,\widehat{v}$ and integrate with respect to $\eta \in \mathbb{R}^d$. Then we integrate with respect to x_d from 0 to $+\infty$ and take the real part of this equality. We get

$$-2\ \mathrm{Re} \int_0^{+\infty} \int_{\mathbb{R}^d} \langle \widehat{v}, R(\eta,\gamma)\mathcal{A}_d^{-1}(\mathbf{u})\,\widehat{F}\rangle \,d\eta dx_d =$$

$$2\ \mathrm{Re} \int_0^{+\infty} \int_{\mathbb{R}^d} \langle \widehat{v}, R(\eta,\gamma)\mathcal{A}(\mathbf{u},\eta,\gamma)\,\widehat{v}\rangle \,d\eta dx_d + \int_{\mathbb{R}^d} \langle \widehat{v}(0), R(\eta,\gamma)\,\widehat{v}(0)\rangle d\eta \,.$$

Using (21a)-(21b) and Plancherel's theorem, we obtain

$$2\ \mathrm{Re} \int_0^{+\infty} \int_{\mathbb{R}^d} \langle \widehat{v}, R(\eta,\gamma)\mathcal{A}(\mathbf{u},\eta,\gamma)\,\widehat{v}\rangle \,d\eta dx_d \geq c\,\gamma\,\|\!| v \|\!|_0^2 \,,$$

$$\int_{\mathbb{R}^d} \langle \widehat{v}(0), R(\eta,\gamma)\,\widehat{v}(0)\rangle d\eta \geq c\, \| v_{|_{x_d=0}} \|_0^2 - C\, \|G\|_0^2 \,.$$

Using Young's inequality and a uniform bound for $R(\eta,\gamma)$, we get the estimate

$$\gamma \|\!| v \|\!|_0^2 + \| v_{|_{x_d=0}} \|_0^2 \leq C\, \left(\frac{1}{\gamma} \|\!| F \|\!|_0^2 + \|G\|_0^2 \right) \,.$$

Integrating (18) with respect to η, we also get

$$\|\psi\|_{1,\gamma}^2 \leq C\, \left(\|G\|_0^2 + \| v_{|_{x_d=0}} \|_0^2 \right) \,,$$

and the addition of the two previous inequalities yields (20).

The proof of proposition 3.1 is rather long and we shall not reproduce it here. It is widely detailed in [9] in the case of isentropic Euler's equations. In the strictly hyperbolic case, the result dates back to Kreiss [25], see also [7]. We refer to [4] for the case of a general constant multiplicity hyperbolic system.

4 The variable coefficients case

It is crucial in the proof of nonlinear stability of a planar shock wave to establish an estimate of the same type as (20) for the linearized operators about interfaces that are close to but different from hyperplanes. We shall therefore establish this estimate in this section.

4.1 The linearized equations

We keep the same $\mathbf{u}$ as in the preceeding section. There exists an open set $\mathcal{U}$ in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{d-1}$ containing $(\mathbf{0}, \mathbf{0}, 0, \mathbf{0})$ such that for all $(w_r, w_l, \sigma, \nu) \in \mathcal{U}$, one has

$$\begin{aligned} & u_r + w_r \in U\,, \quad u_l + w_l \in U\,, \\ & \lambda_{q-1}(u_l + w_l, \xi) < \sigma < \lambda_q(u_l + w_l, \xi) \\ \text{and} \quad & \lambda_p(u_r + w_r, \xi) < \sigma < \lambda_{p+1}(u_r + w_r, \xi) \end{aligned}$$

where $\xi := (-\nu, 1) \in \mathbb{R}^d$. In other words, $\mathcal{U}$ is an open set such that all planar shocks associated to elements of $\mathcal{U}$ are noncharacteristic. Furthermore, one can choose $\mathcal{U}$ such that all planar undercompressive shocks associated to elements of $\mathcal{U}$ are uniformly stable, that is satisfy assumption 3. We fix a compact subset $\mathcal{K} \subset \mathcal{U}$ and consider mappings $\dot{u}_r, \dot{u}_l, \varphi$ such that φ is defined on $\mathbb{R}^d$, $\dot{u}_r$ (resp. $\dot{u}_l$) is defined on $\{x_d \geq \varphi(x_0, \ldots, x_{d-1})\}$ (resp. $\{x_d \leq \varphi(x_0, \ldots, x_{d-1})\}$) and $(\dot{u}_r, \dot{u}_l, \nabla\varphi)$ takes its values in $\mathcal{K}$. We define a function

$$\mathbf{a} := \begin{cases} u_r + \dot{u}_r(\mathbf{x}) & \text{if } x_d > \varphi(x_0, \ldots, x_{d-1}), \\ u_l + \dot{u}_l(\mathbf{x}) & \text{if } x_d < \varphi(x_0, \ldots, x_{d-1}). \end{cases} \tag{22}$$

The regularity of $\dot{u}_r, \dot{u}_l$ and φ has not been precised yet. One can think of them as smooth perturbations of the stationary shock wave $\mathbf{u}$ (some kind of first order correction in an asymptotic expansion). We shall be more precise in the sequel.

We introduce the linearized operators about the function $\mathbf{a}$: consider a family $u_s = \mathbf{a}_\sharp + sv$ and $\varphi_s = \varphi + s\psi$. Then we define

$$\mathcal{L}_{\mathbf{a}}(v^\pm, \psi) := \frac{d}{ds}\, L(u_s^\pm, \varphi_s)\, u_s^\pm\, |_{s=0}\,, \tag{23a}$$

$$\mathcal{B}_{\mathbf{a}}(v, \psi) := \frac{d}{ds}\, B(u_s^+, u_s^-, \varphi_s)\, |_{s=0}\,. \tag{23b}$$

Recall that $\mathbf{a}_\sharp$ is the function deduced from $\mathbf{a}$ after the change of variables Φ. A direct computation shows that

$$\begin{aligned} \mathcal{L}_{\mathbf{a}}(v^\pm, \psi) = & \sum_{j=0}^{d-1} A_j(\mathbf{a}_\sharp^\pm)\, \partial_j\, v^\pm + \sum_{j=0}^{d-1} (dA_j(\mathbf{a}_\sharp^\pm) \cdot v^\pm)\, \partial_j\, \mathbf{a}_\sharp^\pm + \widetilde{A_d}(\mathbf{a}_\sharp^\pm, \nabla\varphi)\, \partial_d\, v^\pm \\ & - \sum_{j=0}^{d-1} \partial_j \psi\, A_j(\mathbf{a}_\sharp^\pm)\, \partial_d\, \mathbf{a}_\sharp^\pm + (d_u \widetilde{A_d}(\mathbf{a}_\sharp^\pm, \nabla\varphi) \cdot v^\pm)\, \partial_d\, \mathbf{a}_\sharp^\pm\,, \qquad \pm x_d > 0\,, \end{aligned}$$

and

$$\mathcal{B}_{\mathbf{a}}(v, \psi) = \begin{pmatrix} \displaystyle\sum_{j=0}^{d-1} \partial_j \psi\, [f_j(\mathbf{a}_\sharp)] - \widetilde{A_d}(\mathbf{a}_\sharp^+, \nabla\varphi)\, v^+ + \widetilde{A_d}(\mathbf{a}_\sharp^-, \nabla\varphi)\, v^- \\ \\ \displaystyle\frac{\partial g}{\partial \sigma} \cdot \partial_0 \psi + d_\nu g \cdot \nabla_y \psi + d_{u_r} g \cdot v^+ + d_{u_l} g \cdot v^- \end{pmatrix}, \qquad x_d = 0\,.$$

Here above, the partial derivatives of g are evaluated at the point $(\mathbf{a}_\sharp^+, \mathbf{a}_\sharp^-, \nabla\varphi)$. We decompose the linearized operator $\mathcal{L}_{\mathbf{a}}$ as

$$\mathcal{L}_{\mathbf{a}}(v^\pm, \psi) = L(\mathbf{a}_\sharp^\pm, \varphi)\, v^\pm - \sum_{j=0}^{d-1} \partial_j \psi\, A_j(\mathbf{a}_\sharp^\pm)\, \partial_d\, \mathbf{a}_\sharp^\pm + E(\mathbf{a}_\sharp^\pm, \varphi)\, v^\pm$$

where L is defined by (8a)-(8b) and

$$E(\mathbf{a}_\sharp^\pm,\varphi)\,v^\pm := \sum_{j=0}^{d-1}(dA_j(\mathbf{a}_\sharp^\pm)\cdot v^\pm)\,\partial_j\,\mathbf{a}_\sharp^\pm + (d_u\widetilde{A}_d(\mathbf{a}_\sharp^\pm,\nabla\varphi)\cdot v^\pm)\,\partial_d\,\mathbf{a}_\sharp^\pm$$

is the zero order part (in $v^\pm$) of $\mathcal{L}_\mathbf{a}$. The principal part of $\mathcal{L}_\mathbf{a}$ is obtained by dropping the terms in ψ and the zero order terms in $v^\pm$ in the expression of $\mathcal{L}_\mathbf{a}$, and thus coincides with the operator $L(\mathbf{a}_\sharp^\pm,\varphi)$. With little risk of confusion, we shall still denote by $\mathcal{L}_\mathbf{a}$ the principal part of this operator. With the same notations as in the preceeding section, the (principal part of the) linearized equations read

$$\begin{cases}\mathcal{L}_\mathbf{a} v = \displaystyle\sum_{j=0}^{d}\mathcal{A}_j(\mathbf{a})\,\partial_j v & x_d>0\,,\\ \mathcal{B}_\mathbf{a}(v,\psi) = \displaystyle\sum_{j=0}^{d-1}\partial_j\psi\,b_j(\mathbf{a}) + M(\mathbf{a})\,v & x_d=0\,,\end{cases}\tag{24}$$

once defined $v:=(v^+,v^-)$ and performed the change of variable $x_d\to -x_d$ in the evolution equation on v^-. Our goal is now to derive an energy estimate for the weighted operators:

$$\begin{cases}\mathcal{L}_\mathbf{a}^\gamma v = \mathcal{L}_\mathbf{a} v + \gamma\,\mathcal{A}_0(\mathbf{a})\,v & x_d>0\,,\\ \mathcal{B}_\mathbf{a}^\gamma(v,\psi) = \mathcal{B}_\mathbf{a}(v,\psi) + \gamma\,\psi\,b_0(\mathbf{a}) & x_d=0\,.\end{cases}$$

4.2 The paralinearized equations

To quantify our results, we fix a constant $K>0$ and assume that

$$\|\nabla\varphi\|_{W^{1,\infty}(\omega)}\le K \quad\text{and}\quad \|\mathbf{a}_\sharp\|_{W^{1,\infty}(\Omega)}\le K\,.$$

We refer to the appendix for the definition of the paradifferential symbols and operators as well as for the paralinearization estimates. Because the coefficients in operators $\mathcal{L}_\mathbf{a}^\gamma$ and $\mathcal{B}_\mathbf{a}^\gamma$ have limited smoothness, we are going to replace both operators $\mathcal{L}_\mathbf{a}^\gamma$ and $\mathcal{B}_\mathbf{a}^\gamma$ by their paralinearized version in order to derive the desired energy estimate.

Define the following symbols:

$$\mathbf{b}(x_0,\dots,x_{d-1},\eta,\gamma) := \gamma\,\mathbf{b}_0(x_0,\dots,x_{d-1}) + i\sum_{j=0}^{d-1}\eta_j\,\mathbf{b}_j(x_0,\dots,x_{d-1})$$
$$\text{where}\quad \mathbf{b}_j(x_0,\dots,x_{d-1}) := b_j(\mathbf{a}_\sharp(x_0,\dots,x_{d-1},0))\,.$$

Then we have $\mathbf{b}_j\in W^{1,\infty}(\mathbb{R}^d)$ and as a consequence $\mathbf{b}\in\Gamma_1^1$. By theorem A.4 (see appendix A), we get

$$\|\gamma\mathbf{b}_0\,\psi - T^\gamma_{\gamma\mathbf{b}_0}\psi\|_0 \le \mathrm{Cst}\,\|\mathbf{b}_0\|_{W^{1,\infty}}\,\|\psi\|_0 \le \frac{C(K)}{\gamma}\,\|\psi\|_{1,\gamma}\,,$$
$$\|\mathbf{b}_j\,\partial_j\psi - T^\gamma_{i\eta_j\mathbf{b}_j}\psi\|_0 = \|\mathbf{b}_j\partial_j\psi - T^\gamma_{\mathbf{b}_j}\partial_j\psi\|_0 \le C(K)\,\|\psi\|_0 \le \frac{C(K)}{\gamma}\,\|\psi\|_{1,\gamma}\,,$$

and we obtain the estimate

$$\|\gamma\,\mathbf{b}_0\,\psi + \sum_{j=0}^{d-1}\partial_j\psi\,\mathbf{b_j} - T^\gamma_\mathbf{b}\psi\|_0 \le \frac{C(K)}{\gamma}\,\|\psi\|_{1,\gamma}\,.\tag{25}$$

We also define

$$\mathbf{M}(x_0,\dots,x_{d-1}) := M(\mathbf{a}_\sharp(x_0,\dots,x_{d-1},0)) \in W^{1,\infty}(\mathbb{R}^d)\,,$$

and by theorem A.4, we obtain

$$\|\mathbf{M}\, v_{|_{x_d=0}} - T^\gamma_{\mathbf{M}} v_{|_{x_d=0}}\|_0 \leq \frac{C(K)}{\gamma}\, \|v_{|_{x_d=0}}\|_0\,. \tag{26}$$

Combining (25) and (26), we deduce that there exists a constant $C(K)$ such that

$$\|\mathcal{B}^\gamma_a(v,\psi) - T^\gamma_{\mathbf{b}}\psi - T^\gamma_{\mathbf{M}} v_{|_{x_d=0}}\|_0 \leq \frac{C(K)}{\gamma}\,(\|\psi\|_{1,\gamma} + \|v_{|_{x_d=0}}\|_0)\,. \tag{27}$$

We shall therefore replace the boundary operator $(v,\psi) \mapsto \mathcal{B}^\gamma_{\mathbf{a}}(v,\psi)$ by its paralinearized version $(v,\psi) \mapsto T^\gamma_{\mathbf{b}}\psi + T^\gamma_{\mathbf{M}} v$.

We now turn to the interior equations. Defining the symbols

$$\mathbf{A}_j(x_0,\dots,x_d) := \mathcal{A}_j(\mathbf{a}_\sharp(x_0,\dots,x_d)) \in W^{1,\infty}(\Omega)\,,$$

we see that $\mathbf{A}_d$ is uniformly invertible, that is

$$\|\mathbf{A}_d^{-1}\|_{W^{1,\infty}(\Omega)} \leq C(K)\,.$$

Using theorem A.4, we obtain the estimates

$$\begin{aligned}
\|\gamma\,\mathbf{A}_d^{-1}\mathbf{A}_0\, v - \gamma T^\gamma_{\mathbf{A}_d^{-1}\mathbf{A}_0} v\|_0 &\leq C(K)\, \|v\|_0\,,\\
\|\mathbf{A}_d^{-1}\mathbf{A}_j\, \partial_j v - T^\gamma_{i\eta_j \mathbf{A}_d^{-1}\mathbf{A}_j} v\|_0 &\leq C(K)\, \|v\|_0 \qquad \text{for } 0 \leq j \leq d-1\,,
\end{aligned}$$

so if we define a symbol $\mathbf{A}$ by

$$\mathbf{A}(\mathbf{x},\eta,\gamma) := -\mathbf{A}_d(\mathbf{x})^{-1} \left(\gamma\, \mathbf{A}_0(\mathbf{x}) + i \sum_{j=0}^{d-1} \eta_j\, \mathbf{A}_j(\mathbf{x})\right)\,, \quad \mathbf{x} \in \Omega\,,$$

we have $\mathbf{A} \in \Gamma^1_1$ and the previous inequalities yield the estimate

$$\|\mathbf{A}_d^{-1}\, \mathcal{L}^\gamma_{\mathbf{a}} v - \partial_d v + T^\gamma_{\mathbf{A}} v\|_0 \leq C(K)\, \|v\|_0\,. \tag{28}$$

We shall therefore replace $v \mapsto \mathbf{A}_d^{-1}\, \mathcal{L}^\gamma_{\mathbf{a}} v$ by its paralinearized version $v \mapsto \partial_d v - T^\gamma_{\mathbf{A}} v$.

Our problem reduces to the derivation of an energy estimate of the same type as (20) for the paralinearized system:

$$\begin{cases} \partial_d v - T^\gamma_{\mathbf{A}} v = F & x_d > 0\,,\\ T^\gamma_{\mathbf{b}}\psi + T^\gamma_{\mathbf{M}} v_{|_{x_d=0}} = G & x_d = 0\,. \end{cases}$$

As we did in the constant coefficients case, we begin by eliminating the front ψ in the boundary conditions: recall that the perturbations take values in a compact set $\mathcal{K} \subset \mathcal{U}$ and all shock waves in $\mathcal{U}$ are uniformly stable. There exists thus a constant $c(K) > 0$ such that the analogue of (16) holds:

$$\forall\,(y,\eta,\gamma)\,, \quad c(K)\,(\gamma^2 + |\eta|^2) \leq |\mathbf{b}(y,\eta,\gamma)|^2\,. \tag{29}$$

We can therefore define the orthogonal projector $\Pi(y,\eta,\gamma)$ by the formula

$$\Pi(y,\eta,\gamma)\, h := h - \frac{\langle h, \mathbf{b}(y,\eta,\gamma)\rangle}{|\mathbf{b}(y,\eta,\gamma)|^2}\, \mathbf{b}(y,\eta,\gamma)\,,$$

so $\Pi \in \Gamma_1^0$ because Π is homogeneous of degree 0 in (η,γ). If we define the boundary symbol

$$\beta(y,\eta,\gamma) := \Pi(y,\eta,\gamma)\,\mathbf{M}(y)\,,$$

then we have $\beta \in \Gamma_1^0$, and we deduce from theorem A.2 the estimates

$$\|T_\Pi^\gamma T_{\mathbf{b}}^\gamma \psi\|_0 \leq C(K)\|\psi\|_0 \leq \frac{C(K)}{\gamma}\|\psi\|_{1,\gamma}\,,$$

$$\|T_\beta^\gamma v_{|x_d=0} - T_\Pi^\gamma T_{\mathbf{M}}^\gamma v_{|x_d=0}\|_0 \leq C(K)\|v_{|x_d=0}\|_{-1,\gamma} \leq \frac{C(K)}{\gamma}\|v_{|x_d=0}\|_0\,.$$

From the decomposition

$$T_\beta^\gamma v_{|x_d=0} = T_\Pi^\gamma (T_{\mathbf{b}}^\gamma \psi + T_{\mathbf{M}}^\gamma v_{|x_d=0}) - T_\Pi^\gamma T_{\mathbf{b}}^\gamma \psi + (T_\beta^\gamma v_{|x_d=0} - T_\Pi^\gamma T_{\mathbf{M}}^\gamma v_{|x_d=0})\,,$$

we get the estimate

$$\|T_\beta^\gamma v_{|x_d=0}\|_0 \leq C(K)\, \left[\|G\|_0 + \frac{1}{\gamma}\, \left(\|\psi\|_{1,\gamma} + \|v_{|x_d=0}\|_0 \right) \right]\,, \tag{30}$$

with $G = T_{\mathbf{b}}^\gamma \psi + T_{\mathbf{M}}^\gamma v_{|x_d=0}$. We shall therefore replace $\|G\|_0$ in the energy estimate by $\|T_\beta^\gamma v_{|x_d=0}\|_0$.

Applying Gårding's inequality (theorem A.3), we obtain for $\gamma \geq \gamma_0$ (γ_0 depending only on K):

$$\|\psi\|_{1,\gamma}^2 \leq C(K)\ \text{Re}\ \langle T_{\mathbf{b}^*\mathbf{b}}^\gamma \psi, \psi\rangle_{L^2} \leq C(K)\ \left(\|T_{\mathbf{b}}^\gamma \psi\|_0^2 + \ \text{Re}\ \langle R^\gamma \psi, \psi\rangle_{L^2} \right)$$

where $\{R^\gamma\}$ is a family of order ≤ 1. We thus get

$$\|\psi\|_{1,\gamma}^2 \leq C(K)\ \left(\|T_{\mathbf{b}}^\gamma \psi\|_0^2 + C(K)\, \|\psi\|_{1,\gamma}\, \|\psi\|_0 \right) \leq \frac{C(K)}{\gamma}\, \|\psi\|_{1,\gamma}^2 + C(K)\, \|T_{\mathbf{b}}^\gamma \psi\|_0^2\,.$$

Up to a greater choice of γ_0, we finally get

$$\forall\, \gamma \geq \gamma_0\,, \quad \|\psi\|_{1,\gamma}^2 \leq C(K)\, \|T_{\mathbf{b}}^\gamma \psi\|_0^2 \leq C(K)\, \left(\|G\|_0^2 + \|v_{|x_d=0}\|_0^2 \right)\,. \tag{31}$$

So if we show that there exist two constants $C(K) > 0$ and $\gamma_0 \geq 1$ such that

$$\forall\, \gamma \geq \gamma_0\,, \quad \gamma \|v\|_0^2 + \|v_{|x_d=0}\|_0^2 \leq C(K)\, \left(\frac{1}{\gamma} \|F\|_0^2 + \|G\|_0^2 \right) \tag{32}$$

where

$$F := \partial_d v - T_{\mathbf{A}}^\gamma v \quad \text{ and } \quad G := T_\beta^\gamma v_{|x_d=0}\,,$$

then we shall obtain the main variable coefficients estimate stated in the following result:

Theorem 4.1. *Let* **a** *be defined by* (22) *be a perturbation of a planar undercompressive shock* **u** *satisfying*

$$\|\dot{u}_{r,l}\|_{W^{1,\infty}} \leq K\,, \quad \|\nabla\varphi\|_{W^{1,\infty}} \leq K\,,$$

where K is a fixed positive real number. Assume also that $(\dot{u}_{r,l}, \nabla\varphi)$ takes its values in a compact subset $\mathcal{K} \subset \mathcal{U}$ and that $(\dot{u}_{r,l}, \varphi)$ has compact support. Then there exist two constants $C(K) > 0$ and $\gamma_0 \geq 1$, depending only on K and $\mathcal{K}$ such that for all $v \in H^1(\Omega)$ and all $\psi \in H^1(\omega)$, the following estimate holds:

$$\forall \gamma \geq \gamma_0 \,, \quad \gamma |\!|\!| v |\!|\!|_0^2 + \| v_{|_{x_d=0}} \|_0^2 + \|\psi\|_{1,\gamma}^2 \leq C(K) \left(\frac{1}{\gamma} |\!|\!| \mathcal{L}_{\mathbf{a}}^\gamma v |\!|\!|_0^2 + \| \mathcal{B}_{\mathbf{a}}^\gamma (v, \psi) \|_0^2 \right) \,, \tag{33}$$

where operators $\mathcal{L}_{\mathbf{a}}^\gamma$ and $\mathcal{B}_{\mathbf{a}}^\gamma$ are defined in (24).

Indeed, if (32) holds, we use (30) and (31) to derive (up to a greater choice of γ_0) the estimate

$$\gamma |\!|\!| v |\!|\!|_0^2 + \| v_{|_{x_d=0}} \|_0^2 + \|\psi\|_{1,\gamma}^2 \leq C(K) \left(\frac{1}{\gamma} |\!|\!| F |\!|\!|_0^2 + \| G \|_0^2 \right)$$

where now G is defined by $G := T_{\mathbf{b}}^\gamma \psi + T_{\mathbf{M}}^\gamma v_{|_{x_d=0}}$. Using (27) and (28), we obtain (still up to a greater choice of γ_0) the result of theorem 4.1. We thus only need to prove (32).

4.3 Proof of the estimate (32)

As the reader can imagine, we shall derive (32) with the help of (variable coefficients) Kreiss' symmetrizers. We shall base the last step of our analysis on the following result (that is the analogue of proposition 3.1):

Proposition 4.1. *Let* $\mathbf{a}$ *satisfy the same assumptions as in theorem 4.1. Then there exists a mapping*

$$\mathcal{R} : \overline{\Omega} \times (\mathbb{R}^d \times [0, +\infty[\setminus \{0\}) \longrightarrow \mathcal{M}_{2n}(\mathbb{C})$$

satisfying the following properties:

$\forall\, z$, the matrix $\mathcal{R}(z)$ is hermitian,

$\mathcal{R}$ is homogeneous of degree 0 with respect to (η, γ) and $\mathcal{R} \in \Gamma_1^0$,

$\forall\, z \in \omega \times (\mathbb{R}^d \times [0, +\infty[\setminus \{0\})$, one has

$$\mathcal{R}(z) + C\, \beta(z)^* \beta(z) \geq c\, Id \,, \tag{34}$$

there exists a finite set of matrix valued mappings such that

$$\mathrm{Re}\ (\mathcal{R}(z) \mathbf{A}(z)) = \sum_{i=1}^{I} V_i(z)^* \begin{pmatrix} \gamma H_i(z) & 0 \\ 0 & E_i(z) \end{pmatrix} V_i(z) \,,$$

where V_i and H_i are homogeneous of degree 0 with respect to (η, γ) (and belong to Γ_1^0), E_i is homogeneous of degree 1 with respect to (η, γ) (and belongs to Γ_1^1), and the following inequalities hold:

$$\sum_{i=1}^{I} V_i(z)^* V_i(z) \geq c\, Id \,, \quad H_i(z) \geq c\, Id \,, \quad E_i(z) \geq c\, (\gamma + |\eta|)\, Id \,. \tag{35}$$

The proof of proposition 4.1 is based on the same ideas as those used in [7, 25] in the constant coefficients case. One freezes the variable $y \in \overline{\Omega}$ and performs the normal modes analysis. The compact support property is used to perform the gluing technique (one can relax this assumption by assuming that $(\dot{u}_{r,l}, \varphi)$ tends sufficiently fast to 0 at infinity, see [25]). We skip the details to avoid overloading the paper. We now show how the existence of the symbolic symmetrizer yields the desired energy estimate.

The starting point, as in the constant coefficients case, is the equality

$$\begin{cases} \partial_d v - T^\gamma_{\mathbf{A}} v = F & x_d > 0\,, \\ T^\gamma_\beta v = G & x_d = 0\,. \end{cases}$$

Let $\{R^\gamma(x_d)\}$ be given by

$$R^\gamma(x_d) := \frac{1}{2}\left((T^\gamma_{\mathcal{R}(x_d)})^* + T^\gamma_{\mathcal{R}(x_d)} \right) .$$

Because $\mathcal{R} \in \Gamma^0_1$, $\{R^\gamma\}$ is a bounded Lipschitzean family of selfadjoint operators on $L^2(\mathbb{R}^d)$ (the bounds are uniform with respect to the parameter $\gamma \geq 1$). If we multiply the first equation above by $R^\gamma(x_d) v(x_d)$ and integrate with respect to the tangential variables $(x_0, \dots, x_{d-1}) \in \mathbb{R}^d$ and to the normal variable $x_d \geq 0$, we obtain the relation

$$\begin{aligned} &\langle R^\gamma(0) v(0), v(0) \rangle_{L^2(\omega)} + 2 \text{ Re } \langle\!\langle R^\gamma T^\gamma_{\mathbf{A}} v, v \rangle\!\rangle_{L^2(\Omega)} = \\ &\quad -2 \text{ Re } \langle\!\langle R^\gamma v, F \rangle\!\rangle_{L^2(\Omega)} - \text{ Re } \langle\!\langle \frac{dR^\gamma}{dx_d} v, v \rangle\!\rangle_{L^2(\Omega)} . \end{aligned} \tag{36}$$

Note that the bounds on $\mathcal{R}$ yield the estimates:

$$\forall\, \gamma \geq 1\,, \quad \|R^\gamma v\|_0 \leq C\, \|v\|_0 \quad \text{and} \quad \|\frac{dR^\gamma}{dx_d} v\|_0 \leq C\, \|v\|_0\,,$$

so the right hand term in (36) is bounded by

$$C\, \|v\|_0\, \|F\|_0 + C\, \|v\|_0^2 . \tag{37}$$

Relation (34) shows that for γ large enough, one has

$$\text{Re } \langle T^\gamma_{\mathcal{R}(0)} v(0), v(0) \rangle_{L^2(\omega)} + C \text{ Re } \langle T^\gamma_{\beta * \beta} v(0), v(0) \rangle_{L^2(\omega)} \geq c\, \|v(0)\|^2_{L^2(\omega)}\,,$$

by Gårding's inequality (theorem A.3). Theorem A.2 shows that the remainders

$$T^\gamma_{\mathcal{R}(0)} - R^\gamma(0) \quad \text{and} \quad T^\gamma_{\beta * \beta} - (T^\gamma_\beta)^* T^\gamma_\beta$$

are of order ≤ -1, so we finally get

$$\langle R^\gamma(0) v(0), v(0) \rangle_{L^2(\omega)} + C\, \|T^\gamma_\beta v(0)\|^2_{L^2(\omega)} \geq c\, \|v(0)\|^2_{L^2(\omega)}\,, \tag{38}$$

up to a greater choice of γ_0. The final point is to derive a lower bound for the operator $R^\gamma T^\gamma_{\mathbf{A}}$ with the help of (35). Define

$$\forall\, z \in \overline{\Omega} \times (\mathbb{R}^d \times [0, +\infty[\setminus \{0\})\,, \quad a_i(z) := \begin{pmatrix} \gamma H_i(z) & 0 \\ 0 & E_i(z) \end{pmatrix} .$$

Applying theorem A.2, we obtain that the remainders

$$\text{Re } (R^\gamma T^\gamma_{\mathbf{A}}) - \text{ Re } T^\gamma_{\text{Re } \mathcal{R}\mathbf{A}} \quad \text{and} \quad T^\gamma_{V_i^* a_i V_i} - (T^\gamma_{V_i})^* T^\gamma_{a_i} T^\gamma_{V_i}$$

are of order ≤ 0. As a consequence, we get

$$\text{Re } \langle\!\langle R^\gamma T^\gamma_{\mathbf{A}} v, v\rangle\!\rangle_{L^2(\Omega)} \geq \text{Re } \sum_{i=1}^{I} \langle\!\langle T^\gamma_{a_i} v_i, v_i\rangle\!\rangle_{L^2(\Omega)} - C \,|\!|\!| v |\!|\!|_0^2$$

with $v_i := T^\gamma_{V_i} v$. Using the block decomposition of a_i and Gårding's inequality on each block, we get

$$\text{Re } \langle\!\langle T^\gamma_{a_i} v_i, v_i\rangle\!\rangle_{L^2(\Omega)} \geq c\,\gamma \,|\!|\!| v_i |\!|\!|_0^2 \,,$$

and therefore we obtain

$$\text{Re } \langle\!\langle R^\gamma T^\gamma_{\mathbf{A}} v, v\rangle\!\rangle_{L^2(\Omega)} \geq c\,\gamma \sum_{i=1}^{I} |\!|\!| v_i |\!|\!|_0^2 - C \,|\!|\!| v |\!|\!|_0^2 \,.$$

Finally, the ellipticity of the symbol $\sum_i V_i^* V_i$ yields, for γ large enough, the estimate

$$|\!|\!| v |\!|\!|_0^2 \leq C \sum_{i=1}^{I} \text{ Re } \langle\!\langle T^\gamma_{V_i^* V_i} v, v\rangle\!\rangle_{L^2(\Omega)} \leq C \sum_{i=1}^{I} |\!|\!| v_i |\!|\!|_0^2 + \frac{C}{\gamma} |\!|\!| v |\!|\!|_0^2 \,.$$

Combining the previous inequalities, we conclude that

$$\text{Re } \langle\!\langle R^\gamma T^\gamma_{\mathbf{A}} v, v\rangle\!\rangle_{L^2(\Omega)} \geq c\,\gamma \,|\!|\!| v |\!|\!|_0^2$$

for γ sufficiently large. Using (38), we see that the left hand term in (36) admits the lower bound

$$c\,\gamma \,|\!|\!| v |\!|\!|_0^2 + c\, \| v_{|x_d=0} \|_0^2 - C\, \| T^\gamma_\beta v_{|x_d=0} \|_0^2 \,,$$

so we have proved (32) and the proof of theorem 4.1 is complete.

5 Well-posedness of the linearized problem

In this section, we are going to solve the linearized equations (24) supplemented with an initial condition. Recall that the linearized operators read

$$\begin{aligned} \mathcal{L}_{\mathbf{a}} v &= \sum_{j=0}^{d} \mathcal{A}_j(\mathbf{a})\, \partial_j v \quad x_d > 0 \,, \\ \mathcal{B}_{\mathbf{a}}(v, \psi) &= \sum_{j=0}^{d-1} \partial_j \psi\, b_j(\mathbf{a}) + M(\mathbf{a})\, v \quad x_d = 0 \,. \end{aligned} \tag{39}$$

We first solve the boundary value problem

$$\begin{cases} \mathcal{L}^\gamma_{\mathbf{a}} v = f & x_d > 0 \,, \\ \mathcal{B}^\gamma_{\mathbf{a}}(v, \psi) = g & x_d = 0 \,, \end{cases}$$

with no initial condition and data f and g in weighted spaces. This is done with the help of a suitable dual problem that we introduce in the next paragraph. In [33], Métivier uses a symmetric form of the equations. These arguments can not be used here because of the under-compressive character of our shock waves. Some modifications are required. This is detailed in the next paragraph. The advantage of our method is to deal with general systems. We do not assume that the system admits a Friedrichs symmetrizer (given by a strictly convex entropy for instance).

5.1 The dual problem. Basic properties

The symbol b defined by (14) will not be used anymore so with little risk of confusion, we define

$$b(\mathbf{a}) := \begin{pmatrix} b_0(\mathbf{a}) & \cdots & b_{d-1}(\mathbf{a}) \end{pmatrix} .$$

With this definition, we have

$$\mathcal{B}_{\mathbf{a}}(v, \psi) = b(\mathbf{a}) \, \nabla \psi + M(\mathbf{a}) \, v \, .$$

Moreover, the uniform stability condition implies that the matrix $b(\mathbf{a})$ is one-to-one for all $\mathbf{a} \in \mathcal{U}$. As a consequence, choosing $\mathcal{U}$ as a contractile open set, there exists a C^∞ mapping $P : \mathcal{U} \to Gl_N(\mathbb{R})$ such that

$$\forall \, \mathbf{a} \in \mathcal{U} \, , \quad P(\mathbf{a})^{-1} \, b(\mathbf{a}) = \begin{pmatrix} Id(\mathbb{R}^d) \\ \mathbf{0} \end{pmatrix} .$$

We still denote by M the matrix $P^{-1} \, M$ so the boundary operator now reads

$$\mathcal{B}_{\mathbf{a}}(v, \psi) = \begin{pmatrix} \nabla \psi \\ \mathbf{0} \end{pmatrix} + M(\mathbf{a}) \, v \, .$$

We make the following (reasonable) assumption:

Assumption 4. *The matrix $M(\mathbf{u}) \in \mathcal{M}_{N,2n}(\mathbb{R})$ is of maximal rank.*

When dealing with Lax shock waves, the analogue of assumption 4 is automatically satisfied because the shock wave is noncharacteristic, see [33].

Up to shrinking $\mathcal{U}$, the matrix $M(\mathbf{a})$ is still of maximal rank for all $\mathbf{a} \in \mathcal{U}$ and we have the following lemma:

Lemma 5.1. *There exists a C^∞ mapping $N : \mathcal{U} \longrightarrow \mathcal{M}_{2n-N,2n}(\mathbb{R})$ such that for all $\mathbf{a} \in \mathcal{U}$, one has*

$$\mathbb{R}^{2n} = \mathrm{Ker} \; M(\mathbf{a}) \oplus \mathrm{Ker} \; N(\mathbf{a}) \, .$$

Moreover, there exist two C^∞ mappings

$$M' : \mathcal{U} \longrightarrow \mathcal{M}_{2n,N}(\mathbb{R}) \quad \text{and} \quad N' : \mathcal{U} \longrightarrow \mathcal{M}_{2n,2n-N}(\mathbb{R})$$

such that for all $\mathbf{a} \in \mathcal{U}$, one has

$$Id(\mathbb{R}^{2n}) = M'(\mathbf{a}) \, M(\mathbf{a}) + N'(\mathbf{a}) \, N(\mathbf{a}) \, .$$

Proof. The open set $\mathcal{U}$ has been chosen contractile so there exists a basis $(e_1(\mathbf{a}), \dots, e_{2n}(\mathbf{a}))$ of $\mathbb{R}^{2n}$, with C^∞ mappings e_i, satisfying

$$\mathrm{Ker} \; M(\mathbf{a}) = \; \mathrm{Span} \; (e_1(\mathbf{a}), \dots, e_{2n-N}(\mathbf{a})) \, .$$

We refer to [20] for the proof of this assertion. In this basis, we define a matrix $N(\mathbf{a}) \in \mathcal{M}_{2n-N,2n}(\mathbb{R})$ by

$$N(\mathbf{a}) := \begin{pmatrix} Id(\mathbb{R}^{2n-N}) & \mathbf{0} \end{pmatrix} .$$

The matrix $N(\mathbf{a})$ satisfies

$$\mathbb{R}^{2n} = \; \mathrm{Ker} \; M(\mathbf{a}) \oplus \; \mathrm{Ker} \; N(\mathbf{a}) \, .$$

We obtain the second part of lemma 5.1 by observing that the two mappings

$$M(\mathbf{a}): \text{ Ker } N(\mathbf{a}) \longrightarrow \mathbb{R}^N \quad \text{and} \quad N(\mathbf{a}): \text{ Ker } M(\mathbf{a}) \longrightarrow \mathbb{R}^{2n-N}$$

are isomorphisms and admit inverses $M'(\mathbf{a})$ and $N'(\mathbf{a})$ with the same regularity with respect to $\mathbf{a}$. For all $\mathbf{a} \in \mathcal{U}$, $M'(\mathbf{a})\,M(\mathbf{a})$ (resp. $N'(\mathbf{a})\,N(\mathbf{a})$) is a projector on Ker $N(\mathbf{a})$ (resp. Ker $M(\mathbf{a})$) and we use the first point of lemma 5.1 to conclude. $\square$

We define

$$M^{\text{adj}}(\mathbf{a}) := (\mathcal{A}_d(\mathbf{a})\,M'(\mathbf{a}))^T \quad \text{and} \quad N^{\text{adj}}(\mathbf{a}) := (\mathcal{A}_d(\mathbf{a})\,N'(\mathbf{a}))^T\,, \tag{40}$$

so the following equality holds:

$$\mathcal{A}_d(\mathbf{a}) = M^{\text{adj}}(\mathbf{a})^T\,M(\mathbf{a}) + N^{\text{adj}}(\mathbf{a})^T\,N(\mathbf{a})\,.$$

Then we have the following:

Lemma 5.2. *For all $\mathbf{a} \in \mathcal{U}$, the following decomposition holds:*

$$\mathbb{R}^{2n} = \text{Ker } M^{\text{adj}}(\mathbf{a}) \oplus \text{Ker } N^{\text{adj}}(\mathbf{a})\,.$$

Proof. Both matrices $M(\mathbf{a})$ and $N(\mathbf{a})$ are of maximal rank and from (40) we deduce the equalities

$$\text{Ker } M^{\text{adj}}(\mathbf{a}) = \text{Im } (\mathcal{A}_d(\mathbf{a})\,M'(\mathbf{a}))^\perp = \text{Im } (\mathcal{A}_d(\mathbf{a})\,M'(\mathbf{a})\,M(\mathbf{a}))^\perp\,,$$
$$\text{Ker } N^{\text{adj}}(\mathbf{a}) = \text{Im } (\mathcal{A}_d(\mathbf{a})\,N'(\mathbf{a}))^\perp = \text{Im } (\mathcal{A}_d(\mathbf{a})\,N'(\mathbf{a})\,N(\mathbf{a}))^\perp\,.$$

Using lemma 5.1 and the equalities above, we easily check that

$$\dim \text{ Ker } M^{\text{adj}}(\mathbf{a}) + \dim \text{ Ker } N^{\text{adj}}(\mathbf{a}) = 2n$$

and

$$\text{Ker } M^{\text{adj}}(\mathbf{a}) \cap \text{Ker } N^{\text{adj}}(\mathbf{a}) = \{0\}\,.$$

$\square$

Integrating by parts, one shows that

$$\forall\, u, v \in C_c^\infty(\Omega)\,, \quad \langle\!\langle \mathcal{L}_{\mathbf{a}} u, v \rangle\!\rangle_{L^2(\Omega)} = \langle\!\langle u, \mathcal{L}_{\mathbf{a}}^* v \rangle\!\rangle_{L^2(\Omega)}$$
$$\text{and} \quad \langle\!\langle \mathcal{L}_{\mathbf{a}}^\gamma u, v \rangle\!\rangle_{L^2(\Omega)} = \langle\!\langle u, (\mathcal{L}_{\mathbf{a}}^\gamma)^* v \rangle\!\rangle_{L^2(\Omega)}\,,$$

where $\mathcal{L}_{\mathbf{a}}^*$ and $(\mathcal{L}_{\mathbf{a}}^\gamma)^*$ are given by

$$\mathcal{L}_{\mathbf{a}}^* := -\sum_{j=0}^{d} \mathcal{A}_j(\mathbf{a})^T \partial_j - (\sum_{j=0}^{d} \partial_j \mathcal{A}_j(\mathbf{a})^T) \quad \text{and} \quad (\mathcal{L}_{\mathbf{a}}^\gamma)^* := \mathcal{L}_{\mathbf{a}}^* + \gamma \mathcal{A}_0(\mathbf{a})^T\,.$$

We define the dual boundary conditions of (39) in the following way:

$$N^{\text{adj}}(\mathbf{a})\,v = 0 \quad \text{and} \quad \text{div}\,(M_1^{\text{adj}}(\mathbf{a})\,v) = 0 \quad x_d = 0\,, \tag{41}$$

where $M_1^{\text{adj}}(\mathbf{a})$ is the matrix formed by d first lines of $M^{\text{adj}}(\mathbf{a})$. In (41), "div" stands for the divergence operator in $\omega = \mathbb{R}^d$. The main result of this paragraph can be stated as follows:

Proposition 5.1. *Let $\gamma \geq 1$, $f \in L^2(\Omega)$ and $g \in L^2(\omega)$. If $(v, \psi) \in L^2(\Omega) \times L^2(\omega)$ is a weak solution of*

$$\begin{cases} \mathcal{L}_{\mathbf{a}}^{\gamma} v = f & x_d > 0\,, \\ \mathcal{B}_{\mathbf{a}}^{\gamma}(v, \psi) = g & x_d = 0\,, \end{cases}$$

then for all $w \in C_c^{\infty}(\overline{\Omega})$ such that

$$N^{\mathrm{adj}}(\mathbf{a})\, w_{|_{x_d=0}} = 0 \quad \textit{and} \quad \mathrm{div}_{-\gamma}\,(M_1^{\mathrm{adj}}(\mathbf{a})\, w_{|_{x_d=0}}) = 0$$

we have

$$\langle\!\langle f, w \rangle\!\rangle_{L^2(\Omega)} + \langle g, M^{\mathrm{adj}} w \rangle_{L^2(\omega)} = \langle\!\langle v, (\mathcal{L}_{\mathbf{a}}^{\gamma})^* w \rangle\!\rangle_{L^2(\Omega)}\,.$$

The operator $\mathrm{div}_{-\gamma}$ *is defined on $H^1(\omega; \mathbb{R}^d)$ by*

$$\mathrm{div}_{-\gamma}\, u := -\gamma u_0 + \sum_{j=0}^{d-1} \partial_j u_j\,.$$

Furthermore, the linear problem

$$\begin{cases} \mathcal{L}_{\mathbf{a}}^{*} w = 0 & x_d > 0\,, \\ N^{\mathrm{adj}}(\mathbf{a})\, w = 0\,, \quad \mathrm{div}\,(M_1^{\mathrm{adj}}(\mathbf{a})\, w) = 0 & x_d = 0\,, \end{cases}$$

satisfies the backward uniform Kreiss-Lopatinskii condition, that is the analogue of the uniform stability condition (assumption 3) when γ is changed into $-\gamma$.

Proof. Because $v \in L^2(\Omega)$ and $\mathcal{L}_{\mathbf{a}}^{\gamma} v \in L^2(\Omega)$, a classical result of Friedrichs ensures that the trace of v on ω is well defined and belongs to $H^{-1/2}(\omega)$, see [16]. Moreover, the following equality holds for all $w \in C_c^{\infty}(\overline{\Omega})$:

$$\langle\!\langle f, w \rangle\!\rangle_{L^2(\Omega)} = \langle\!\langle v, (\mathcal{L}_{\mathbf{a}}^{\gamma})^* w \rangle\!\rangle_{L^2(\Omega)} - \langle \mathcal{A}_d(\mathbf{a})\, v, w \rangle_{H^{-1/2}(\omega), H^{1/2}(\omega)}\,.$$

Using the equality

$$\begin{pmatrix} \nabla^{\gamma} \psi \\ \mathbf{0} \end{pmatrix} + M(\mathbf{a})\, v = g \quad \text{with} \quad \nabla^{\gamma} \psi := \begin{pmatrix} \gamma \psi + \partial_0 \psi \\ \partial_1 \psi \\ \vdots \\ \partial_{d-1} \psi \end{pmatrix}\,,$$

we obtain

$$\begin{aligned} \langle \mathcal{A}_d(\mathbf{a})\, v, w \rangle_{H^{-1/2}(\omega), H^{1/2}(\omega)} = \langle g, w \rangle_{L^2(\omega)} + \langle \psi, \mathrm{div}_{-\gamma}\, M_1^{\mathrm{adj}}(\mathbf{a})\, w \rangle_{L^2(\omega)} \\ + \langle N(\mathbf{a})\, v, N^{\mathrm{adj}}(\mathbf{a})\, w \rangle_{H^{-1/2}(\omega), H^{1/2}(\omega)}\,. \end{aligned}$$

The first part of the proposition follows. We now turn to the study of the boundary value problem:

$$\begin{cases} \mathcal{L}_{\mathbf{a}}^{*} w = 0 & x_d > 0\,, \\ N^{\mathrm{adj}}(\mathbf{a})\, w = 0\,, \quad \mathrm{div}\,(M_1^{\mathrm{adj}}(\mathbf{a})\, w) = 0 & x_d = 0\,. \end{cases}$$

We freeze the coefficients and check the backward uniform Kreiss-Lopatinskii condition. Once the (backward) Laplace transform in x_0 and the Fourier transform in $x_1, \ldots, x_{d-1}$ performed, the interior equations read

$$\mathcal{A}_d^T \frac{d\widehat{w}}{dx_d} - \left(\gamma \mathcal{A}_0^T - i \sum_{j=0}^{d-1} \eta_j \, \mathcal{A}_j^T \right) \widehat{w} = 0 \,, \quad x_d > 0 \,,$$

and the boundary conditions read

$$N^{\mathrm{adj}} \, \widehat{w}(0) = 0 \,, \quad (-\gamma + i\eta_0, i\eta_1, \ldots, i\eta_{d-1}) \cdot M_1^{\mathrm{adj}} \, \widehat{w}(0) = 0 \,.$$

Here and in what follows, we consider the quadratic form on the spaces $\mathbb{C}^m$:

$$X \cdot Y := \sum_{i=1}^{m} X_i Y_i$$

and the sign $\perp$ denotes the orthognal with respect to this nondegenerate quadratic form. It is proved in [4, 7] that the stable subspace of the matrix

$$-(\mathcal{A}_d^T)^{-1} \left(\gamma \mathcal{A}_0^T - i \sum_{j=0}^{d-1} \eta_j \, \mathcal{A}_j^T \right)$$

is equal to $(\mathcal{A}_d \, \mathcal{E}^-(-\eta, \gamma))^\perp$. We thus need to check that for all $(\eta, \gamma) \in \Sigma_+$, the subspace

$$\Big\{ Z \in (\mathcal{A}_d \, \mathcal{E}^-(-\eta, \gamma))^\perp \text{ s.t. } N^{\mathrm{adj}} \, Z = 0 \text{ and } \\ (-\gamma + i\eta_0, i\eta_1, \ldots, i\eta_{d-1}) \cdot M_1^{\mathrm{adj}} \, Z = 0 \Big\}$$

is trivial. Let $Z \in (\mathcal{A}_d \, \mathcal{E}^-(-\eta, \gamma))^\perp$ satisfy $Z \in \operatorname{Ker} N^{\mathrm{adj}}$ and

$$(-\gamma + i\eta_0, i\eta_1, \ldots, i\eta_{d-1})^T \cdot M_1^{\mathrm{adj}} \, Z = 0 \,.$$

Let $Y \in \mathbb{C}^N$. Because of assumption 3, there exists $\chi \in \mathbb{C}$ and $X \in \mathcal{E}^-(-\eta, \gamma)$ such that

$$Y = \chi \begin{pmatrix} e \\ \mathbf{0} \end{pmatrix} + M \, X \,,$$

with $e = (-\gamma + i\eta_0, i\eta_1, \ldots, i\eta_{d-1})^T$. We now compute

$$\begin{aligned} M^{\mathrm{adj}} \, Z \cdot Y &= \chi \, (M_1^{\mathrm{adj}} \, Z \cdot e) + (M^{\mathrm{adj}} \, Z \cdot M \, X) = Z \cdot (M^{\mathrm{adj}})^T \, M \, X \\ &= (Z \cdot \mathcal{A}_d \, X) - (Z \cdot (N^{\mathrm{adj}})^T \, N \, X) = 0 \,. \end{aligned}$$

Because this equality holds for all $Y \in \mathbb{C}^N$, we have $Z \in \operatorname{Ker} M^{\mathrm{adj}}$ but we also have $Z \in \operatorname{Ker} N^{\mathrm{adj}}$, and thus $Z = 0$. This completes the proof of the proposition. $\square$

If $w \in C_c^\infty(\overline{\Omega})$ satisfies

$$N^{\mathrm{adj}}(\mathbf{a}) \, w_{|x_d=0} = 0 \quad \text{and} \quad \mathrm{div}_{-\gamma} \, (M_1^{\mathrm{adj}}(\mathbf{a}) \, w_{|x_d=0}) = 0 \,,$$

we have the estimate

$$\gamma \, \| w \|_0^2 + \| w_{|x_d=0} \|_0^2 \leq \frac{C(K)}{\gamma} \, \| (\mathcal{L}_{\mathbf{a}}^\gamma)^* w \|_0^2 \,. \tag{42}$$

This estimate is proved in a completely similar way to what we have done in the proof of theorem 4.1 (section 4). We are now able to solve the linearized equations in weighted spaces.

5.2 Existence of solutions in weighted spaces

For $\gamma \geq 1$, the space $\exp(\gamma x_0)\, L^2(\Omega)$ denotes the set of functions f defined on Ω such that $\exp(-\gamma x_0)\, f \in L^2(\Omega)$. The space $\exp(\gamma x_0)\, H^s_\gamma(\omega)$ is defined in a similar way when $s \in \mathbb{R}$. This paragraph is devoted to the proof of the following result:

Proposition 5.2. *Under assumptions 1-4 and the assumptions of theorem 4.1, there exists $\gamma_0 \geq 1$ such that for all $\gamma \geq \gamma_0$, for all $f \in \exp(\gamma x_0)\, L^2(\Omega)$ and for all $g \in \exp(\gamma x_0)\, L^2(\omega)$, there exists a weak solution $(v, \psi) \in \exp(\gamma x_0)\, L^2(\Omega) \times \exp(\gamma x_0)\, H^{1/2}(\omega)$ of the linearized equations*

$$\begin{cases} \mathcal{L}_{\mathbf{a}} v = f & x_d > 0\,, \\ \mathcal{B}_{\mathbf{a}}(v, \psi) = g & x_d = 0\,. \end{cases}$$

Proof. Let F denote the subspace

$$F := \{ w \in C^\infty_c(\overline{\Omega}) \text{ s.t. } N^{\mathrm{adj}}(\mathbf{a})\, w_{|_{x_d=0}} = 0 \quad \text{and} \quad \mathrm{div}_{-\gamma}\, (M_1^{\mathrm{adj}}(\mathbf{a})\, w_{|_{x_d=0}}) = 0 \}\,.$$

We define a linear form on $(\mathcal{L}^\gamma_{\mathbf{a}})^* F$ by the following formula

$$\ell[(\mathcal{L}^\gamma_{\mathbf{a}})^* w] := \langle\!\langle \exp(-\gamma x_0) f, w \rangle\!\rangle_{L^2(\Omega)} + \langle \exp(-\gamma x_0) g, M^{\mathrm{adj}}(\mathbf{a})\, w \rangle_{L^2(\omega)}\,.$$

The linear form ℓ is well-defined because of (42) and there exists a constant $C > 0$ such that

$$|\ell[(\mathcal{L}^\gamma_{\mathbf{a}})^* w]| \leq C\, \|\!|(\mathcal{L}^\gamma_{\mathbf{a}})^* w\|\!|_0\,.$$

Applying Hahn-Banach theorem and then Riesz theorem, we can conclude that there exists a function $\widetilde{v} \in L^2(\Omega)$ such that for all $w \in F$, one has

$$\ell[(\mathcal{L}^\gamma_{\mathbf{a}})^* w] = \langle\!\langle \widetilde{v}, (\mathcal{L}^\gamma_{\mathbf{a}})^* w \rangle\!\rangle_{L^2(\Omega)}\,. \tag{43}$$

If we first choose test functions $w \in C^\infty_c(\Omega)$, we deduce that $\widetilde{v}$ is a weak solution of

$$\mathcal{L}^\gamma_{\mathbf{a}} \widetilde{v} = \exp(-\gamma x_0) f\,,$$

or equivalently, the function $v = \exp(\gamma x_0)\widetilde{v}$ is a weak solution in $\exp(\gamma x_0) L^2(\Omega)$ of

$$\mathcal{L}_{\mathbf{a}} v = f\,.$$

Using once again Friedrichs' result [16], we claim that the trace of $\widetilde{v}$ on ω is well-defined and belongs to $H^{-1/2}_\gamma(\omega)$. Furthermore, the following equality holds for all $w \in F$:

$$\langle\!\langle \widetilde{v}, (\mathcal{L}^\gamma_{\mathbf{a}})^* w \rangle\!\rangle_{L^2(\Omega)} = \langle\!\langle \exp(-\gamma x_0) f, w \rangle\!\rangle_{L^2(\Omega)} + \langle M(\mathbf{a})\, \widetilde{v}, M^{\mathrm{adj}}(\mathbf{a})\, w \rangle_{H^{-1/2}(\omega), H^{1/2}(\omega)}\,,$$

and combining this equality with (43) yields

$$\forall\, w \in F\,, \quad \langle M(\mathbf{a})\, \widetilde{v}, M^{\mathrm{adj}}(\mathbf{a})\, w \rangle_{H^{-1/2}(\omega), H^{1/2}(\omega)} = \langle \exp(-\gamma x_0) g, M^{\mathrm{adj}}(\mathbf{a})\, w \rangle_{L^2(\omega)}\,. \tag{44}$$

Using a continuity argument, this equality still holds for all $w \in H^1(\Omega)$ such that $N^{\mathrm{adj}}(\mathbf{a})\, w = 0$ and $\mathrm{div}_{-\gamma}\, (M_1^{\mathrm{adj}}(\mathbf{a})\, w) = 0$ on ω. The previous equalities make sense because the trace of functions in $H^1(\Omega)$ is well-defined and belongs to $H^{1/2}(\omega)$.

Let θ be a function in $H^{1/2}(\omega)$ that takes its values in $\mathbb{R}^N$. We write $\theta = (\theta_1, \theta_2)$ where θ_1 (resp. θ_2) takes its values in $\mathbb{R}^d$ (resp. $\mathbb{R}^{N-d}$). Then there exists a function $w \in H^1(\Omega)$ such that

$$N^{\mathrm{adj}}(\mathbf{a})\, w_{|_{x_d=0}} = 0 \quad \text{and} \quad M^{\mathrm{adj}}(\mathbf{a})\, w_{|_{x_d=0}} = \theta\,.$$

This classical lifting argument works because bith matrices $N^{\mathrm{adj}}(\mathbf{a})$ and $M^{\mathrm{adj}}(\mathbf{a})$ belong to $W^{1,\infty}(\omega)$. Consequently, for all $\theta_2 \in H^{1/2}(\omega)$ we have

$$\langle M(\mathbf{a})\,\widetilde{v}, (0,\theta_2)\rangle_{H^{-1/2}(\omega),H^{1/2}(\omega)} = \langle \exp(-\gamma x_0) g, (0,\theta_2)\rangle_{L^2(\omega)}\,,$$

and for all θ_1 satisfying $\mathrm{div}_{-\gamma}\,\theta_1 = 0$, we have

$$\langle M(\mathbf{a})\,\widetilde{v}, (\theta_1,0)\rangle_{H^{-1/2}(\omega),H^{1/2}(\omega)} = \langle \exp(-\gamma x_0) g, (\theta_1,0)\rangle_{L^2(\omega)}\,.$$

The result of the proposition follows from the following lemma:

Lemma 5.3. *Let $\gamma \geq 1$ and u belong to $H_\gamma^{-1/2}(\omega;\mathbb{R}^d)$ such that for all $w \in C_c^\infty(\omega;\mathbb{R}^d)$, one has*

$$div_{-\gamma}\, w = 0 \Longrightarrow \langle u, w\rangle_{H^{-1/2}(\omega),H^{1/2}(\omega)} = 0\,.$$

Then there exists $\psi \in H_\gamma^{1/2}(\omega;\mathbb{R})$ such that

$$u = \nabla^\gamma \psi := \begin{pmatrix} \gamma\psi + \partial_0\psi \\ \partial_1\psi \\ \vdots \\ \partial_{d-1}\psi \end{pmatrix}\,.$$

Furthermore, the solution ψ satisfies $\|\psi\|_{1/2,\gamma} \leq \|u\|_{-1/2,\gamma}$.

Applying the lemma, we conclude that there exists $\widetilde{\psi} \in H_\gamma^{1/2}(\omega)$ such that

$$\begin{pmatrix} \nabla^\gamma \widetilde{\psi} \\ \mathbf{0} \end{pmatrix} + M\,\widetilde{v} = \exp(-\gamma x_0) g\,,$$

and the proposition follows by setting $\psi := \exp(\gamma x_0)\widetilde{\psi}$. □

We now give the proof of lemma 5.3.

Proof. We first consider the case where u belongs to the Schwartz' class $\mathcal{S}(\omega)$ and define

$$\psi := (\gamma^2 - \Delta)^{-1}(\mathrm{div}_{-\gamma}\, u) \in \mathcal{S}(\omega)\,.$$

Note that the operator $(\gamma^2 - \Delta)$ is invertible because $\gamma \geq 1$. Then $\mathrm{div}_{-\gamma}\,(u - \nabla^\gamma\psi) = 0$ and for all $w \in C_c^\infty(\omega;\mathbb{R}^d)$ such that $\mathrm{div}_{-\gamma}\, w = 0$, one has

$$\langle \nabla^\gamma\psi, w\rangle_{L^2(\omega)} = 0 \quad \text{and} \quad \langle u - \nabla^\gamma\psi, w\rangle_{L^2(\omega)} = 0\,.$$

Moreover, a continuity argument shows that the previous equalities still hold when w belongs to $\mathcal{S}(\omega)$. We are going to show that $u = \nabla^\gamma\psi$. If $W \in \mathcal{S}(\omega)$, we have

$$W = \nabla^\gamma\theta + w \quad \text{with} \quad \mathrm{div}_{-\gamma}\, w = 0\,.$$

Indeed, the previous decomposition is achieved by

$$\theta := (\gamma^2 - \Delta)^{-1}(\mathrm{div}_{-\gamma}\, W) \quad \text{and} \quad w = W - \nabla^\gamma\theta\,.$$

One then easily checks that

$$\langle u - \nabla^\gamma\psi, W\rangle_{L^2(\omega)} = 0$$

and therefore $u = \nabla^\gamma\psi$. The definition of ψ implies easily the estimate on the norms and the lemma follows from a density argument. □

5.3 Final steps in the analysis

Proposition 5.2 asserts the existence of weak solutions as long as the data belong to weighted spaces, but we have said nothing yet about the uniqueness of weak solutions and continuity with respect to the data. The lack of regularity prevents from using our main energy estimate (theorem 4.1). The next step in the analysis consists in showing that weak solutions exhibited in proposition 5.2 are "strong solutions" in the following sense: there exist sequences (v^ν) in $\exp(\gamma x_0)\, H^1(\Omega)$ and (ψ^ν) in $\exp(\gamma x_0)\, H^1(\omega)$ such that

$$
\begin{aligned}
\exp(-\gamma x_0)\, v^\nu \longrightarrow \exp(-\gamma x_0)\, v \quad & \text{in } L^2(\Omega)\,,\\
\mathcal{L}^\gamma_{\mathbf{a}}(\exp(-\gamma x_0)\, v^\nu) \longrightarrow f \quad & \text{in } L^2(\Omega)\,,\\
\exp(-\gamma x_0)\, v^\nu_{|x_d=0} \longrightarrow \exp(-\gamma x_0)\, v_{|x_d=0} \quad & \text{in } L^2(\omega)\,,\\
\exp(-\gamma x_0)\, \psi^\nu \longrightarrow \exp(-\gamma x_0)\, \psi \quad & \text{in } H^{1/2}_\gamma(\omega)\,,\\
\mathcal{B}^\gamma_{\mathbf{a}}(\exp(-\gamma x_0)\, v^\nu, \exp(-\gamma x_0)\, \psi^\nu) \longrightarrow g \quad & \text{in } L^2(\omega)\,.
\end{aligned}
$$

Furthermore $(\exp(-\gamma x_0)\, v^\nu, \exp(-\gamma x_0)\, \psi^\nu)$ satisfy the energy estimate (33). Using the convergences stated above, we conclude that $\exp(-\gamma x_0)\, \psi \in H^1_\gamma(\omega)$ and $\exp(-\gamma x_0)\, v$, $\exp(-\gamma x_0)\, \psi$ satisfy the energy estimate. In particular, we have a uniqueness result in weighted spaces. We refer to [33] for the proof (the regularized sequence is obtained via tangential mollifiers).

Going on in the analysis of the linearized equations, one shows that for data vanishing in the past ($f = 0$ and $g = 0$ for $x_0 < 0$), solutions also vanish in the past and one can then solve the boundary value problem (39) supplemented with an intial condition. The arguments are similar to what is done in [7, 33]. It extends the previous work of Rauch [38] that dealt with C^∞ coefficients. We can summarize the main result on the linear problem in the following way:

Theorem 5.1. *Under the assumptions of theorem 4.1, we consider data*

$$
f \in L^2(]0,T[\times\mathbb{R}^d_+)\,,\; g \in L^2(]0,T[\times\mathbb{R}^{d-1})\,,\; v_0 \in L^2(\mathbb{R}^d_+) \text{ and } \psi_0 \in H^{1/2}(\mathbb{R}^{d-1})\,,
$$

where T is a fixed positive number. Then the linear problem

$$
\begin{aligned}
\sum_{j=0}^{d} \mathcal{A}_j(\mathbf{a})\, \partial_j v = f \quad & x_d > 0\,,\\
\sum_{j=0}^{d-1} \partial_j \psi\, b_j(\mathbf{a}) + M(\mathbf{a})\, v = g \quad & x_d = 0\,,
\end{aligned}
$$

supplemented with the initial condition

$$
v_{|t=0} = v_0\,, \quad \psi_{|t=0} = \psi_0
$$

has a unique solution $(v,\psi) \in L^2(]0,T[\times\mathbb{R}^d_+) \times H^1(]0,T[\times\mathbb{R}^{d-1})$ such that $v \in C([0,T], L^2(\mathbb{R}^d_+))$ and for all $\gamma \geq \gamma_0$, for all $t \in]0,T[$, the following estimate holds:

$$
\begin{aligned}
& e^{-2\gamma t}\, \|v(t)\|^2_{L^2(\mathbb{R}^d_+)} + \gamma\, \|e^{-\gamma s} v\|^2_{L^2(]0,t[\times\mathbb{R}^d_+)} + \|e^{-\gamma s} v_{|x_d=0}\|^2_{L^2(]0,t[\times\mathbb{R}^{d-1})} + \|e^{-\gamma s}\psi\|^2_{H^1_\gamma(]0,t[\times\mathbb{R}^{d-1})}\\
& \leq C \left(\frac{1}{\gamma}\, \|e^{-\gamma s} f\|^2_{L^2(]0,t[\times\mathbb{R}^d_+)} + \|e^{-\gamma s} g\|^2_{L^2(]0,t[\times\mathbb{R}^{d-1})} + \|v_0\|^2_{L^2(\mathbb{R}^d_+)} + \|\psi_0\|^2_{H^{1/2}(\mathbb{R}^{d-1})} \right) .
\end{aligned}
$$

The constants C and γ_0 depend on the linearization state $\mathbf{a}$ as in theorem 4.1.

In order to construct an iteration scheme to solve the nonlinear equations, we need to know more about the regularity of the solutions to the linear problem. The main question is: given data that belong to higher order Sobolev spaces, does the solution also belong to a higher order Sobolev space? The question was addressed in [39] and the answer is yes provided that a certain amount of compatibility conditions hold. These compatibility conditions prevent from the formation of other discontinuities than the initial shock front curve (as in a Riemann problem). Roughly speaking, one can then replace the spaces L^2 in the preceeding theorem by spaces H^k. Of course, the corresponding initial condition ψ_0 for the shock front curve would belong to a space $H^{k+1/2}(\mathbb{R}^{d-1})$.

We now describe the iteration scheme used to solve the nonlinear equations. In [30] (see also [32]), Majda used a classical scheme

$$\begin{aligned} L(u_+^\nu, \varphi^\nu)(u_+^{\nu+1} - u_+^0) &= -L(u_+^\nu, \varphi^\nu)u_+^0 \,, \\ L(u_-^\nu, \varphi^\nu)(u_-^{\nu+1} - u_-^0) &= -L(u_-^\nu, \varphi^\nu)u_-^0 \,, \end{aligned} \tag{45}$$

for the interior equations, together with a Newton scheme for the boundary conditions. In (45), functions (u_+^0, u_-^0) are the initial conditions and the initial state for the iteration scheme. In [33], Métivier first reduces the nonlinear boundary conditions to linear boundary conditions. This is made possible by the uniform stability condition but it heavily uses the particular form of the Rankine-Hugoniot conditions. It can therefore hardly apply in our framework because of the additional jump conditions. We thus go back to Majda's method, that was also used in [36]. The scheme used for the interior equations is defined by (45) and the solution $(u_+^{\nu+1} - u_+^0, u_-^{\nu+1} - u_-^0)$ is sought as a function vanishing for $t < 0$. This is made possible by the arguments developed when solving the linearized problems. As said previously, the nonlinear boundary conditions are solved by a Newton method. Recall that we look for u_+, u_- and φ that satisfy $B(u_+, u_-, \varphi) = 0$ on $x_d = 0$, so the Newton method reads

$$dB(u_+^\nu, u_-^\nu, \varphi^\nu) \cdot (u_+^{\nu+1} - u_+^\nu, u_-^{\nu+1} - u_-^\nu, \varphi^{\nu+1} - \varphi^\nu) = -B(u_+^\nu, u_-^\nu, \varphi^\nu) \,. \tag{46}$$

Using (8c) and the definition of the linearized operator $\mathcal{B}$, see (23b), we may rewrite (46) as

$$\mathcal{B}_{\mathbf{u}^\nu}(u_+^{\nu+1}, u_-^{\nu+1}, \varphi^{\nu+1}) = \mathcal{B}_{\mathbf{u}^\nu}(u_+^\nu, u_-^\nu, \varphi^\nu) - B(u_+^\nu, u_-^\nu, \varphi^\nu) \,,$$

with $\mathbf{u}^\nu := (u_+^\nu, u_-^\nu, \varphi^\nu)$. At each step of the iteration scheme, the initial conditions are

$$(u_+^{\nu+1}, u_-^{\nu+1}, \varphi^{\nu+1})\,(t=0) = (u_+^0, u_-^0, \varphi^0) \,.$$

Using this iteration scheme, we end up with the main result of this paper:

Theorem 5.2. *Fix an integer $s > d/2 + 1$, and initial data $(u_+^0, u_-^0, \varphi^0)$ satisfying*

$$u_+^0 - u_r \in H^s(\mathbb{R}_+^d) \,, \quad u_-^0 - u_l \in H^s(\mathbb{R}_+^d) \,, \quad \varphi^0 \in H^{s+1/2}(\mathbb{R}^{d-1})$$

such that $u_+^0 - u_r$, $u_-^0 - u_l$ and φ^0 have compact support and satisfy compatibility conditions up to order $s-1$. Assume also that $(u_+^0 - u_r, u_-^0 - u_l, \varphi^0)$ takes its values in a compact subset of $\mathcal{U}$. Then there exists a unique smooth solution (u_+, u_-, φ) of (7) defined on a time interval $[0, T]$. Furthermore, the solution (u_+, u_-, φ) satisfies

$$u_+ - u_r \in H^s(]0,T[\times\mathbb{R}_+^d) \,, \; u_- - u_l \in H^s(]0,T[\times\mathbb{R}_+^d) \,, \; \varphi \in H^{s+1}(]0,T[\times\mathbb{R}^{d-1}) \,.$$

6 An example

In this section, we show how our analysis applies to the case of isothermal phase transitions in a van der Waals fluid. We refer to [3] for the proofs of the results. We are interested in the isothermal Euler equations

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho\, \mathbf{u}) = 0\,, \\ \partial_t (\rho\, \mathbf{u}) + \nabla \cdot (\rho\, \mathbf{u} \otimes \mathbf{u}) + \nabla\, p(\rho) = 0\,. \end{cases}$$

Here ρ stands for the density of the fluid, $\mathbf{u}$ for the velocity, p for the pressure. It is a well-known fact that this system meets assumption 1 in the region where it is hyperbolic, that is in the region $\{p'(\rho) > 0\}$. In the sequel, v stands for the specific volume of the fluid $v = 1/\rho$. We assume that the fluid obeys a van der Waals pressure law

$$p(\rho) = P(v) = \frac{RT}{v-b} - \frac{a}{v^2}\,.$$

The parameters a and b are positive and R stands for the perfect gas constant. When T is below the critical temperature $T_c := 8a/(27bR)$, the isothermal curve $(v \mapsto P(v))$ is nonmonotone and we thus expect phase boundaries to propagate. More precisely, P decreases but on an interval $]v_*, v^*[$. The interval $]b, v_*[$ (resp. $]v^*, +\infty[$) is referred to as the liquid (resp. vapor) phase. Another important feature of the pressure law P is the existence and uniqueness of the so-called Maxwell's equilibrium points (v_m, v^m). These are defined by

$$v_m < v_* < v^* < v^m\,, \quad P(v_m) = P(v^m) =: \overline{P} \quad \text{and} \quad \int_{v_m}^{v^m} (\overline{P} - P(v))\, dv = 0\,.$$

We let

$$U := \begin{cases} (\rho_r, \mathbf{u}_r) & \text{if } \mathbf{x} \cdot \mathbf{n} > \sigma t\,, \\ (\rho_l, \mathbf{u}_l) & \text{if } \mathbf{x} \cdot \mathbf{n} < \sigma t\,, \end{cases}$$

be a weak solution of the Euler equations with $1/\rho_r$ and $1/\rho_l$ close to Maxwell's equilibrium points (v_m, v^m) (for instance $1/\rho_l$ close to v_m and $1/\rho_r$ close to v^m). Without loss of generality, we may assume that the following inequalities hold:

$$\mathbf{u}_r \cdot \mathbf{n} - c_r < \sigma < \mathbf{u}_r \quad \text{and} \quad \mathbf{u}_l \cdot \mathbf{n} - c_l < \sigma < \mathbf{u}_l\,,$$

where c is the sound speed in the fluid (recall that c is given by $c(\rho) := p'(\rho)^{1/2}$). This corresponds to an undercompressive shock wave with $\kappa = 0$ (see (5) and definition 1). Following Slemrod and Truskinovsky [45, 46], we shall say that U is an admissible phase transition if there exists a traveling wave

$$\rho(t, \mathbf{x}) = \rho((\mathbf{x} \cdot \mathbf{n} - \sigma t)/\epsilon)\,, \quad \mathbf{u}(t, \mathbf{x}) = \mathbf{u}((\mathbf{x} \cdot \mathbf{n} - \sigma t)/\epsilon)$$

with

$$\rho(+\infty) = \rho_r\,, \rho(-\infty) = \rho_l\,, \quad \mathbf{u}(+\infty) = \mathbf{u}_r\,, \mathbf{u}(-\infty) = \mathbf{u}_l\,,$$

that is a solution to the Euler equations supplemented with a capillarity coefficient ϵ^2 and a viscocity coefficient $\nu\epsilon$ ($\nu > 0$):

$$\begin{aligned} &\partial_t \rho + \nabla \cdot (\rho\, \mathbf{u}) = 0\,, \\ &\partial_t (\rho\, \mathbf{u}) + \nabla \cdot (\rho\, \mathbf{u} \otimes \mathbf{u}) + \nabla\, p(\rho) = \nu\epsilon\Delta\mathbf{u} - \epsilon^2 \nabla(\Delta v)\,. \end{aligned}$$

Let F denote the specific free energy of the fluid, $F'(v) = -P(v)$. Then it is shown in [3] that admissible phase transitions satisfy the Rankine-Hugoniot conditions together with the additional (nonexplicit) jump condition

$$[F] + [v]\,\frac{P(v_r) + P(v_l)}{2} = -\nu\, j \int_{-\infty}^{+\infty} v'(\xi)^2\, d\xi\,,$$

where

$$j := \rho_r(\mathbf{u}_r \cdot \mathbf{n} - \sigma) = \rho_l(\mathbf{u}_l \cdot \mathbf{n} - \sigma)$$

denotes the mass transfer across the discontinuity. On account of the Rankine-Hugoniot conditions, the additional jump condition can be rewritten as

$$\left[f'(\rho) + \frac{1}{2}(\mathbf{u} \cdot \mathbf{n} - \sigma)^2\right] = -\nu\, j \int_{-\infty}^{+\infty} v'(\xi)^2\, d\xi\,,$$

where $f(\rho) := \rho F(1/\rho)$ stands for the free energy per unit volume. We have thus a (nonexplicit) form for the function g introduced in definition 2. The following theorem is proved in [3] (we refer to this paper for the precise statement):

Theorem 6.1 (Benzoni). [3]. *For sufficiently small viscosity coefficient ν, there exist admissible phase transitions and such phase transitions are linearly uniformly stable, that is meet assumption 3.*

In order to apply the analysis developed in this paper, we need to check assumption 4. Performing a change of observer, we may assume that $\sigma = 0$, $\mathbf{n} = (0, \dots, 0, 1)$ and that the velocities on either side of the shock are parallel to $\mathbf{n}$, that is $\mathbf{u}_{r,l} = (0, \dots, 0, u_{r,l})$.

For the particular system of Euler's equations supplemented with the above additional jump condition, it is proved in [3] that the matrix M involved in the linearized boundary conditions is given by

$$M = \begin{pmatrix} u_r & 0 & \rho_r & -u_l & 0 & -\rho_l \\ 0 & \rho_r u_r I_{d-1} & 0 & 0 & -\rho_l u_l I_{d-1} & 0 \\ u_r^2 + c_r^2 & 0 & 2\rho_r u_r & -(u_l^2 + c_l^2) & 0 & -2\rho_l u_l \\ f''(\rho_r) & 0 & u_r & (\tilde{\nu} u_l - f''(\rho_l)) & 0 & (\tilde{\nu}\rho_l - u_l) \end{pmatrix},$$

where $\tilde{\nu} = \alpha\nu + o(\nu)$, α being a positive constant. Recall that f is the free energy per unit volume. In order to check that M is of maximal rank, we prove that the matrix M_1 consisting of the first $(d+1)$ columns and of the last column of M is invertible. We have

$$f''(\rho) = \frac{1}{\rho}\, p'(\rho) = \frac{c^2}{\rho}$$

and therefore

$$M_1 = \begin{pmatrix} u_r & \mathbf{0} & \rho_r & -\rho_l \\ \mathbf{0} & \rho_r u_r I_{d-1} & \mathbf{0} & \mathbf{0} \\ u_r^2 + c_r^2 & \mathbf{0} & 2\rho_r u_r & -2\rho_l u_l \\ \dfrac{c_r^2}{\rho_r} & \mathbf{0} & u_r & (\tilde{\nu}\rho_l - u_l) \end{pmatrix}.$$

The determinant of M_1 is given by

$$\det M_1 = -(\rho_r u_r)^{d-1}\,(c_r^2 - u_r^2)\,([u]\,[\rho] + \tilde{\nu}\,\rho_r\,\rho_l)\,,$$

so M_1 is an invertible matrix for sufficiently small viscosity coefficient ν. Assumption 4 is therefore met and we can apply the results of section 5 to these admissible phase transitions.

A Paradifferential calculus

In this appendix, we collect the main results of the paradifferential calculus of Bony and Meyer [5, 34] that we use in this paper. The introduction of a positive parameter was achieved by Mokrane [36], see also [33]. We refer to these papers for the proofs of the results stated below. We first recall the classification of paradifferential symbols:

Definition 3. *A paradifferential symbol of degree $m \in \mathbb{R}$ and regularity k ($k = 0$ or $k = 1$) is a function $a(x, \xi, \gamma) : \mathbb{R}^d \times \mathbb{R}^d \times [0, +\infty[\to \mathbb{C}^{N \times N}$ such that a is C^∞ smooth with respect to ξ and for all $\alpha \in \mathbb{N}^d$, there exists a constant C_α verfying*

$$\forall\,(\xi,\gamma)\,,\quad \|\partial_\xi^\alpha a(\cdot,\xi,\gamma)\|_{W^{k,\infty}} \le C_\alpha\,\lambda^{m-|\alpha|,\gamma}(\xi) = C_\alpha\,(\gamma^2+|\xi|^2)^{(m-|\alpha|)/2}\,.$$

The set of paradifferential symbols of degree m and regularity k is denoted by Γ_k^m. We denote by Σ_k^m the set of paradifferential symbols a such that for a suitable $\epsilon \in]0,1[$ one has

$$\forall\,(\xi,\gamma)\,,\quad \mathrm{Supp}\;\mathcal{F}_x\,a(\cdot,\xi,\gamma) \subset \{\eta \in \mathbb{R}^d/|\eta| \le \epsilon\,(\gamma^2+|\xi|^2)^{1/2}\}\,.$$

Of course, the symbols in Σ_k^m are C^∞ functions with respect to both variables x and ξ, and for all $a \in \Sigma_k^m$, we have the estimates

$$\forall\,(x,\xi,\gamma)\,,\quad |\partial_x^\beta \partial_\xi^\alpha \sigma\,(x,\xi,\gamma)| \le C_{\alpha,\beta}\,\lambda^{m-|\alpha|+|\beta|,\gamma}(\xi)\,.$$

Thus any symbol $a \in \Sigma_k^m$ belongs to Hörmander's class $S_{1,1}^m$ [21] and defines an operator $P^\gamma(a)$ on the Schwartz' class $\mathcal{S}$ by the usual formula

$$\forall\,u \in \mathcal{S}\,,\quad P^\gamma(a)u(x) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}\,a(x,\xi,\gamma)\,\widehat{u}(\xi)\,d\xi\,.$$

We shall use the following terminology:

Definition 4. *A family of operators $\{P^\gamma\}$ defined for $\gamma \ge 1$ will be said of order $\le m$ ($m \in \mathbb{R}$) if the operators P^γ are uniformly bounded from H_γ^{s+m} to H_γ^s:*

$$\forall\,\gamma \ge 1\,,\quad \forall\,u \in H_\gamma^{s+m}\,,\quad \|P^\gamma u\|_{s,\gamma} \le C(s,m)\,\|u\|_{s+m,\gamma}\,.$$

The following theorem is crucial for the sequel of the analysis:

Theorem A.1. *If $a \in \Sigma_k^m$, the family $\{P^\gamma(a)\}$ is of order $\le m$.*

The regularization of symbols in the class Γ_k^m is achieved by a convolution with admissible cut-off functions:

Definition 5. *Let $\psi : \mathbb{R}^d \times \mathbb{R}^d \times [1, +\infty[\to [0, +\infty[$ be a C^∞ function such that the following estimates hold for all $\alpha, \beta \in \mathbb{N}^d$:*

$$\forall\,(\eta,\xi,\gamma),\quad |\partial_\eta^\alpha \partial_\xi^\beta \psi\,(\eta,\xi,\gamma)| \le C_{\alpha,\beta}\,\lambda^{-|\alpha|-|\beta|,\gamma}(\xi)\,.$$

We shall say that ψ is an admissible cut-off function if there exist real numbers $0 < \epsilon_1 < \epsilon_2 < 1$ satisfying

$$\begin{aligned}\psi(\eta,\xi,\gamma) = 1 \quad &\textit{if} \quad |\eta| \le \epsilon_1(\gamma^2+|\xi|^2)^{1/2}\,,\\ \psi(\eta,\xi,\gamma) = 0 \quad &\textit{if} \quad |\eta| \ge \epsilon_2(\gamma^2+|\xi|^2)^{1/2}\,.\end{aligned}$$

An example of cut-off function is the following: let χ be a nonnegative C^∞ function on $\mathbb{R}^d \times \mathbb{R}$ such that

$$\gamma_1^2 + |\xi_1|^2 \geq \gamma_2^2 + |\xi_2|^2 \Longrightarrow \chi(\xi_1,\gamma_1) \leq \chi(\xi_2,\gamma_2)\,,$$
$$\begin{cases} \chi(\xi,\gamma) = 1 & \text{if } \left(\gamma^2 + |\xi|^2\right)^{1/2} \leq 1/2, \\ \chi(\xi,\gamma) = 0 & \text{if } \left(\gamma^2 + |\xi|^2\right)^{1/2} \geq 1. \end{cases}$$

We define a function $\varphi(\xi,\gamma) := \chi(\xi/2,\gamma/2) - \chi(\xi,\gamma)$. Then the function ψ_0 defined by

$$\psi_0(\eta,\xi,\gamma) := \sum_{p\geq 0} \chi(2^{2-p}\eta, 0)\, \varphi(2^{-p}\xi, 2^{-p}\gamma)$$

is an admissible cut-off function (one can take $\epsilon_1 = 1/16$ and $\epsilon_2 = 1/2$).

If ψ is an admissible cut-off function, the inverse Fourier transform K^ψ of $\psi(\cdot,\xi,\gamma)$ satisfies

$$\forall\,(\xi,\gamma)\,, \quad \|\partial_\xi^\alpha K^\psi(\cdot,\xi,\gamma)\|_{L^1} \leq C_\alpha\, \lambda^{-|\alpha|,\gamma}(\xi)\,.$$

These L^1 bounds on $\partial_\xi^\alpha K^\psi$ enable to establish the following proposition:

Proposition A.1. *Let ψ be an admissible cut-off function. The mapping*

$$a \longmapsto \sigma_a^\psi(x,\xi,\gamma) := \int_{\mathbb{R}^d} K^\psi(x-y,\xi,\gamma)\, a(y,\xi,\gamma)\, dy$$

is continuous from Γ_k^m to Σ_k^m for all m.

If $a \in \Gamma_1^m$, then $a - \sigma_a^\psi \in \Gamma_0^{m-1}$. In particular, if ψ_1 and ψ_2 are two admissible cut-off functions and $a \in \Gamma_1^m$, then $\sigma_a^{\psi_1} - \sigma_a^{\psi_2} \in \Sigma_0^{m-1}$.

Fixing an admissible cut-off function ψ, we define the paradifferential operator $T_a^{\psi,\gamma}$ by the formula

$$T_a^{\psi,\gamma} := P^\gamma(\sigma_a^\psi)\,.$$

If ψ_1 and ψ_2 are two admissible cut-off functions and $a \in \Gamma_1^m$, then proposition A.1 and theorem A.1 show that the family $\{T_a^{\psi_1,\gamma} - T_a^{\psi_2,\gamma}\}$ is of order $\leq (m-1)$.

The symbolic calculus is based on the following theorem:

Theorem A.2. *Let $a \in \Gamma_1^m$ and $b \in \Gamma_1^{m'}$. Then $ab \in \Gamma_1^{m+m'}$ and the family $\{T_a^{\psi,\gamma} \circ T_b^{\psi,\gamma} - T_{ab}^{\psi,\gamma}\}_{\gamma\geq 1}$ is of order $\leq m + m' - 1$ for all admissible cut-off function ψ.*

Let $a \in \Gamma_1^m$. Then for all admissible cut-off function ψ, the family $\{(T_a^{\psi,\gamma})^* - T_{a^*}^{\psi,\gamma}\}_{\gamma\geq 1}$ is of order $\leq (m-1)$.

The next theorem is the parameter version of Gårding's inequality:

Theorem A.3. *Let $a \in \Gamma_1^{2m}$ and ψ be and admissible cut-off function. Assume that there exists a constant $c > 0$ such that*

$$\forall\,(x,\xi,\gamma), \quad \mathrm{Re}\; a(x,\xi,\gamma) \geq c\,\lambda^{2m,\gamma}(\xi)\, Id\,.$$

Then there exists $\gamma_0 \geq 1$ such that

$$\forall\,\gamma \geq \gamma_0\,, \forall\, u \in H_\gamma^m\,, \quad \mathrm{Re}\;\langle T_a^{\psi,\gamma} u, u\rangle_{L^2} \geq \frac{c}{2}\,\|u\|_{m,\gamma}^2\,.$$

We now study the case of paraproducts: they are defined by the particular choice of ψ_0 as cut-off function. We shall write T_a^γ instead of $T_a^{\psi_0,\gamma}$ for the paradifferential operators obtained after smoothing by the function ψ_0. We have the important result:

Theorem A.4. *Let $a \in W^{1,\infty}(\mathbb{R}^d)$, $u \in L^2(\mathbb{R}^d)$ and $\gamma \geq 1$. Then we have*

$$\|a\, u - T_a^\gamma u\|_0 \leq \frac{C}{\gamma} \|a\|_{W^{1,\infty}} \|u\|_0 \,,$$
$$\|a\, \partial_j u - T_a^\gamma (\partial_j u)\|_0 \leq C\, \|a\|_{W^{1,\infty}} \|u\|_0 \,,$$

for a suitable constant C that is independent of (a, u, γ).

We now extend the paradifferential calculus to symbols defined on a half-space: we still denote by Γ_k^m the set of symbols $a(x_0, \dots, x_d, \eta, \gamma)$ defined on $\Omega \times (\mathbb{R}^d \times [0, +\infty[\, \setminus \{0\})$ such that the mapping $x_d \mapsto a(\cdot, x_d, \cdot)$ is bounded into Γ_k^m. We define the paradifferential operator T_a^γ by

$$\forall\, u \in C_c^\infty(\overline{\Omega})\,, \quad \forall\, x_d \geq 0\,, \quad (T_a^\gamma u)(\cdot, x_d) := T_{a(x_d)}^\gamma u(\cdot, x_d)\,.$$

Using theorem A.4 and integrating with respect to x_d, we obtain for all symbol $a \in W^{1,\infty}(\Omega)$ and all $u \in L^2(\Omega)$ the estimates:

$$|\!|\!| a\, u - T_a^\gamma u |\!|\!|_0 \leq \frac{C}{\gamma} \|a\|_{W^{1,\infty}(\Omega)} |\!|\!| u |\!|\!|_0 \,,$$
$$|\!|\!| a\, \partial_j u - T_a^\gamma (\partial_j u) |\!|\!|_0 \leq C\, \|a\|_{W^{1,\infty}(\Omega)} |\!|\!| u |\!|\!|_0 \,, \quad 0 \leq j \leq d-1\,.$$

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