Viscous Shock Wave and Boundary Layer Solution to an Inflow Problem for Compressible Viscous Gas

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Abstract. The inflow problem for a one-dimensional compressible viscous gas on the half line (0,+∞) is investigated. The asymptotic stability on both the viscous shock wave and a superposition of the viscous shock wave and the boundary layer solution is established under some smallness conditions. The proofs are given by an elementary energy method.

1 Introduction

The inflow problem for a one-dimensional compressible flow on the half-space $\Re_+$ is described by the following system in the *Eulerian* coordinates

$$\begin{cases} \rho_t + (\rho u)_{\tilde{x}} = 0, & \text{in } \Re_+ \times \Re_+, \\ (\rho u)_t + (\rho u^2 + p)_{\tilde{x}} = \mu u_{\tilde{x}\tilde{x}}, & \text{in } \Re_+ \times \Re_+, \\ (\rho, u)|_{\tilde{x}=0} = (\rho_-, u_-), & u_- > 0, \\ (\rho, u)|_{t=0} = (\rho_0, u_0) \to (\rho_+, u_+), & as\ \tilde{x} \to \infty. \end{cases} \tag{1.1}$$

Here $u(\tilde{x}, t)$ is the velocity, $\rho(\tilde{x}, t) > 0$ is the density, $p(\rho) = \rho^\gamma$ is the pressure, $\gamma \geq 1$ is the adiabatic constant, $\mu > 0$ is the viscosity constant, $\rho_\pm(> 0), u_\pm$ are prescribed constants.We assume the initial data satisfy the boundary condition as compatibility condition. The assumption $u_- > 0$ implies that, through the boundary $\tilde{x} = 0$ the fluid with the density ρ_- flows into the region $\Re_+$, and thus the problem (1.1) is called the inflow problem. In the cases of $u_- = 0$ and $u_- < 0$, the problems where the condition $\rho|_{\tilde{x}=0} = \rho_-$ is removed, are called the impermeable wall problem, the outflow problem respectively. For the impermeable wall problem, Matsumura and Nishihara [11] and Matsumura and Mei [7] have proved the solution to (1.1) tends to the rarefaction wave as t tends to infinity when $u_+ > u_- = 0$ without any smallness conditions, and the viscous shock wave when $u_+ < u_- = 0$ under some smallness conditions. In the setting of $u_- \neq 0$, the problems become complicated and a new wave, denoted by the boundary layer solution, or BL-solution simply, appears in the solutions due to the presence of boundary. Matsumura [6] classified all possible large time behaviors of the solutions in terms of the boundary values. In the case of $u_- < 0$, Kawashima and Nishibata [3] showed the asymptotic stability of the BL-solution. More recently, Matsumura and Nishihara [12] established the asymptotic stability of the BL-solution and the superposition of a BL-solution and a rarefaction wave for the inflow problem when $(\rho_-, u_-) \in \Omega_{sub}$ (see (1.3) and (1.6)). Shi [13] studied the rarefaction wave case when $(\rho_-, u_-), (\rho_+, u_+) \in \Omega_{super}$. However, there has been no result concerning on the viscous shock wave for both the inflow problem and the outflow one up to now. The main difficulty is to control the value $\psi(0,t)$ (see (3.1)) on the boundary, as pointed out by Matsumura and Nishihara [12], where $\psi(x,t)$ is a function corresponding to the anti-derivative of the velocity field.

In this paper, we concentrate on the viscous shock wave for the inflow problem. We establish the asymptotic stability on both the viscous shock wave and a superposition of the viscous shock wave and the BL-solution when $(\rho_-, u_-) \in \Omega_{sub}$ provided the viscous shock profile is far from the boundary initially, the strength of BL-solution and the initial perturbation are small. The main novelty of our proofs is to introduce a new variable $\bar{\psi}(x,t)$ (see (4.12)) instead of $\psi(x,t)$ in the reformulated system (see (4.12)) in order to overcome the difficulty from the term $\psi(0,t)$. When the energy method is applied to the new system, the first energy inequality does not contain the term $\psi(0,t)$, provided $|\rho_- u_-|$ is small. Namely, the estimates for the term $\psi(0,t)$ could be exactly bypassed. Thus we obtain our desired a priori estimates. It should be noted that the estimates for the term $\psi(0,t)$ are also obtained after the stability theorems are established.

We now state our main results. As in [12], we transform (1.1) to the problem in the Lagrangian coordinate

$$\begin{cases} v_t - u_x = 0, & x > s_- t, t > 0, \\ u_t + p(v)_x = \mu(\dfrac{u_x}{v})_x, & x > s_- t, t > 0, \\ (v,u)|_{x=s_- t} = (v_-, u_-), & v_- = \frac{1}{\rho_-}, u_- > 0, \\ (v,u)|_{t=0} = (v_0, u_0)(x) \to (v_+, u_+) = (\frac{1}{\rho_+}, u_+), & \text{as } x \to \infty, \end{cases} \tag{1.2}$$

where

$$v = \frac{1}{\rho}, \qquad s_- = -\frac{u_-}{v_-} < 0. \tag{1.3}$$

We now consider the inflow problem (1.2) above. The characteristic speeds of the corresponding hyperbolic system without viscosity are

$$\lambda_1 = -\sqrt{-p'(v)}, \quad \lambda_2 = \sqrt{-p'(v)}, \tag{1.4}$$

and the sound speed $c(v)$ is defined by

$$c(v) = v\sqrt{-p'(v)} = \sqrt{\gamma} v^{-\frac{\gamma-1}{2}}. \tag{1.5}$$

Comparing $|u|$ with $c(v)$, we divide the (v,u) space into three regions

$$\begin{aligned}\Omega_{\text{sub}} &= \left\{(v,u)\big||u| < c(v), v>0, u>0\right\},\\ \Gamma_{\text{trans}} &= \left\{(v,u)\big||u| = c(v), v>0, u>0\right\},\\ \Omega_{\text{super}} &= \left\{(v,u)\big||u| > c(v), v>0, u>0\right\}.\end{aligned} \tag{1.6}$$

We call them the subsonic, transonic and supersonic region respectively. When $(v_-,u_-) \in \Omega_{sub}$, since the first wave speed $\lambda_1(v_-)$ is less than the boundary speed s_-, we can expect a BL-solution which connects (v_-,u_-) and some (v_+,u_+). In fact, by the arguments in [12], such BL-solution exists if (v_-,u_-) is on the BL-solution line defined below (1.7). In the phase plane, the BL-solution line and the 2-shock wave curve through (v_-,u_-) are defined by

$$BL(v_-,u_-) = \{(v,u) \in \Omega_{sub} \cup \Gamma_{trans} | \frac{u}{v} = \frac{u_-}{v_-} = -s_-\}, \tag{1.7}$$

$$S_2(v_-,u_-) = \left\{(v,u) \in \Re_+ \times \Re_+ \big| u = u_- - s(v-v_-), v_- < v\right\}, \tag{1.8}$$

with $s = \sqrt{\frac{p(v_-)-p(v)}{v-v_-}} > 0$.

Our main results are, roughly speaking, as follows.

(I) if $(v_+,u_+) \in S_2(v_-,u_-)$, then the viscous shock wave is asymptotically stable provided that the conditions of theorem 2.1 hold.

(II) if $(v_+,u_+) \in BLS_2(v_-,u_-)$, then there exists $(\bar{v},\bar{u}) \in BL(v_-,u_-)$ such that $(v_+,u_+) \in S_2(\bar{v},\bar{u})$ and the superposition of the BL-solution connecting (v_-,u_-) with $(\bar{v},\bar{u})$ and the 2-viscous shock wave connecting $(\bar{v},\bar{u})$ with (v_+,u_+) is asymptotically stable provided that $|v_- - \bar{v}|$ is small and the conditions of theorem 2.2 hold. That is, the BL-solution is weak and the shock wave is not necessarily weak.

Our plan of this paper is as follows. In section 2, after introducing some properties of the viscous shock wave and the BL-solution, we state the main theorems. In section 3-4, we prove the asymptotic stability of viscous shock wave. Precisely speaking, in section 3, we reformulate the original problem to a new initial boundary value problem. In section 4, we establish the a priori estimates by the energy method. In section 5, the case (II) will be treated.

Notations. Throughout this paper, several positive generic constants are denoted by c, C without confusions. For function spaces, $L^p(\Omega), 1 \le p \le \infty$ denotes a usual Lebesgue space on $\Omega \subset R = (-\infty,\infty)$ with its norm

$$\|f\|_{L^p(\Omega)} = (\int_\Omega |f(x)|^p dx)^{\frac{1}{p}}, 1 \le p < \infty, \quad \|f\|_{L^\infty(\Omega)} = \sup_\Omega |f(x)|. \tag{1.9}$$

$H^l(\Omega)$ denotes the l-th order Sobolev space with its norm

$$\|f\|_l = (\sum_{j=0}^{l} \|\partial_x^j f\|^2)^{\frac{1}{2}}, \quad when\ \|\cdot\| := \|\cdot\|_{L^2(\Omega)}. \tag{1.10}$$

The domain Ω will be often abbreviated without confusions.

2 Preliminaries and Main Results

In this section, we first recall the properties of the viscous shock wave. It is well known that the travelling wave $(v,u)=(V_s,U_s)(\eta=x-st), s>0$, satisfying $(V_s,U_s)(\pm\infty)=(v_\pm,u_\pm)$ exists and is unique up to shift, under the Rankine-Hugoniot condition

$$\begin{cases} s(v_+-v_-)=u_--u_+, \\ s(u_+-u_-)=p(v_+)-p(v_-), \end{cases} \tag{2.1}$$

and the entropy condition

$$u_+<u_-. \tag{2.2}$$

Namely, (V_s,U_s) satisfies

$$\begin{cases} -sV_s'-U_s'=0, \\ -sU_s'+p(V_s)'=\mu(\dfrac{U_s'}{V_s})', \\ (V_s,U_s)(\pm\infty)=(v_\pm,u_\pm), \end{cases} \tag{2.3}$$

which yields

$$\begin{cases} U_s=-s(V_s-v_\pm)+u_\pm, \\ \dfrac{s\mu V_s'}{V_s}=-s^2V_s-p(V_s)-b\equiv: h(V_s), \\ V_s(\pm\infty)=v_\pm, \end{cases} \tag{2.4}$$

where $b=-s^2v_\pm-p(v_\pm)$. Thus, we have

Proposition 2.1. *For any $(v_+,u_+),(v_-,u_-),s>0$, satisfying $v_+>v_->0,u_->0$, the Rankine-Hugoniot condition (2.1) and the entropy condition (2.2), there exists a unique shock profile $\big(V_s,U_s\big)(\eta=x-st)$ up to a shift, which connects (v_-,u_-) and (v_+,u_+), and*

$$\begin{aligned} &0<v_-<V_s(\eta)<v_+,u_+<U_s(\eta)<u_-, \\ &h(V_s)>0,\quad V_s'=\frac{V_sh(V_s)}{s\mu}>0, \\ &|V_s(\eta)-v_\pm|=O(1)|v_+-v_-|e^{-c_\pm|\eta|}, \\ &|U_s(\eta)-u_\pm|=O(1)|v_+-v_-|e^{-c_\pm|\eta|} \end{aligned} \tag{2.5}$$

as $\eta\to\pm\infty$ where $c_\pm=\frac{v_\pm|p'(v_\pm)+s^2|}{\mu s}>0$.

On the other hand, there exists a boundary layer solution of the form $(v,u)=(V_b,U_b)(x-s_-t)$ with $(V_b,U_b)(0)=(v_-,u_-),(V_b,U_b)(+\infty)=(v_+,u_+)$, if $(v_-,u_-)\in\Omega_{sub}$ and $(v_+,u_+)\in BL(v_-,u_-)$ due to Matsumura and Nishihara [12]. The BL-solution (V_b,U_b) satisfies

$$\begin{cases} -s_-V_b'-U_b'=0,\\ -s_-U_b'+p(V_b)'=\mu(\frac{U_b'}{V_b})',\\ (V_b,U_b)(0)=(v_-,u_-), \quad (V_b,U_b)(+\infty)=(v_+,u_+) \end{cases} \tag{2.6}$$

Furthermore, we have

Proposition 2.2. *Let $(v_-,u_-)\in\Omega_{sub}$, $(v_+,u_+)\in BL(v_-,u_-)\bigcap\Omega_{sub}$, then there exists a unique solution $\big(V_b,U_b\big)(\eta=x-s_-t)$ to (2.6), which satisfies*

$$|V_b(x-s_-t)-v_+,U_b(x-s_-t)-u_+|\le C|v_+-v_-|e^{-c|x-s_-t|}, \tag{2.7}$$

with some $c>0$.

We now make a coordinate transformation, in which we can make the problem (1.2) easier to handle, by

$$t=t,\quad \xi=x-s_-t. \tag{2.8}$$

Thus, the problem (1.2) becomes

$$\begin{cases} v_t-s_-v_\xi-u_\xi=0, & \xi>0,t>0,\\ u_t-s_-u_\xi+p(v)_\xi=\mu(\frac{u_\xi}{v})_\xi, & \xi>0,t>0,\\ (v,u)|_{\xi=0}=(v_-,u_-), & \\ (v,u)|_{t=0}=(v_0,u_0)\to(v_+,u_+), & as\ \xi\to+\infty. \end{cases} \tag{2.9}$$

We consider the case

$$(v_-,u_-)\in\Omega_{sub},\qquad (v_+,u_+)\in BLS_2(v_-,u_-). \tag{2.10}$$

Obviously, the large time behavior of the solutions to (2.9) should be expected to the superposition of a 2-viscous shock wave and a BL-solution. In this case, there is $(\bar v,\bar u)\in BL(v_-,u_-)$ such that $(v_+,u_+)\in S_2(\bar v,\bar u)$. We consider the situation where the initial data $(v_0(x),u_0(x))$ are given in a neighborhood of $(V_b(\xi)+V_s(\xi-\beta)-\bar v,U_b(\xi)+U_s(\xi-\beta)-\bar u)$ for some large constant $\beta>0$. Namely, we ask the viscous shock wave is far from the boundary initially. The next question is how to determine the shift α such that the solution (v,u) to (2.9) is expected to tend to $(V_b(\xi)+V_s(\xi-(s-s_-)t+\alpha-\beta)-\bar v,U_b(\xi)+U_s(\xi-(s-s_-)t+\alpha-\beta)-\bar u)$. It is known that determining the shift α is difficult even for the scalar viscous conservation laws. Fortunately, Matsumura and Nishihara [12] have shown how to determine the shift α for the system (2.9). Their results are

$$\begin{aligned}\alpha=\ &\frac{1}{v_+-\bar v}\Big\{\int_0^\infty[v_0(\xi)-V_b(\xi)-V_s(\xi-\beta)+\bar v]d\xi\\ &-(s-s_-)\int_0^\infty[V_s((s_--s)t-\beta)-\bar v]dt\Big\}.\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}
&\int_0^\infty [v(\xi,t)-V(\xi,t;\alpha,\beta)]d\xi\\
&=(s-s_-)\int_t^\infty \big(V_s((s_--s)\tau+\alpha-\beta)-\bar{v}\big)d\tau,\\
&\longrightarrow 0 \ \text{ as } t\to\infty,
\end{aligned}\tag{2.12}$$

where

$$V(\xi,t;\alpha,\beta)=V_b(\xi)+V_s(\xi-(s-s_-)t+\alpha-\beta)-\bar{v}.\tag{2.13}$$

It is noted that if $(v_+,u_+)\in S_2(v_-,u_-)$,

$$\begin{aligned}
\alpha=\ &\frac{1}{v_+-v_-}\Big\{\int_0^\infty [v_0(\xi)-V_s(\xi-\beta)]d\xi\\
&-(s-s_-)\int_0^\infty [V_s((s_--s)t-\beta)-v_-]dt\Big\}.
\end{aligned}\tag{2.14}$$

Let

$$U(\xi,t;\alpha,\beta)=U_b(\xi)+U_s(\xi-(s-s_-)t+\alpha-\beta)-\bar{u}.\tag{2.15}$$

To state our main theorems, we suppose that for some $\beta>0$,

$$v_0(\xi)-V(\xi,0;0,\beta)\in H^1\cap L^1,\ \ u_0(\xi)-U(\xi,0;0,\beta)\in H^1\cap L^1,\tag{2.16}$$

and suppose the compatibility condition

$$v_0(0)=v_-,\quad u_0(0)=u_-,\tag{2.17}$$

holds. Setting

$$\big(\Phi_0,\Psi_0\big)(\xi)=-\int_\xi^\infty \big(v_0(y)-V(y,0;0,\beta),u_0(y)-U(y,0;0,\beta)\big)dy.\tag{2.18}$$

Assume that

$$(\Phi_0,\Psi_0)\in L^2.\tag{2.19}$$

We now give our main results.

Theorem 2.1. *Suppose that* $1\le\gamma\le 3$, $(v_-,u_-)\in\Omega_{sub},(v_+,u_+)\in S_2(v_-,u_-)$, *with* $u_->0,s>0$. *Assume that (2.16),(2.17) and (2.19) hold and*

$$(\gamma-1)^2(v_+-v_-)<2\gamma v_-.\tag{2.20}$$

Then there exists a positive constant δ_0 *depending on* v_- *and* v_+. *For any given* $0<u_-=\delta<\delta_0$, *there is a positive constant* $\varepsilon_0(\delta)$, *such that if*

$$||\Phi_0,\Psi_0||_2+e^{-c_-\beta}<\varepsilon_0(\delta),\tag{2.21}$$

then (2.9) has a unique global solution $(v,u)(\xi,t)$ *satisfying*

$$v(\xi,t)-V(\xi,t;\alpha,\beta)\in C^0([0,\infty),H^1)\cap L^2(0,\infty;H^1),\tag{2.22}$$

$$u(\xi,t)-U(\xi,t;\alpha,\beta)\in C^0([0,\infty),H^1)\cap L^2(0,\infty;H^2), \tag{2.23}$$

and

$$\sup_{\xi\in\Re_+}\big|(v,u)(\xi,t)-(V,U)(\xi,t;\alpha,\beta)\big|\longrightarrow 0,\ \text{as } t\to+\infty, \tag{2.24}$$

where $\alpha=\alpha(\beta)$ is determined by (2.14).

Theorem 2.2. *Suppose that $1\le\gamma\le 3$, $(v_-,u_-)\in\Omega_{sub}$, $(v_+,u_+)\in BLS_2(v_-,u_-)$ with $u_->0$. Then there exists $(\bar v,\bar u)$ such that $(\bar v,\bar u)\in BL(v_-,u_-)$ and $(v_+,u_+)\in S_2(\bar v,\bar u)$. Assume that (2.16),(2.17) and (2.19) hold and*

$$(\gamma-1)^2(v_+-\bar v)<2\gamma\bar v. \tag{2.25}$$

Then there exists a positive constant δ_0 depending on v_- and v_+. For any given $0<u_-=\delta<\delta_0$, there exist positive constants $\varepsilon_0(\delta)$ and $\varepsilon_1(\delta)$, such that if

$$||\Phi_0,\Psi_0||_2+e^{-c_-\beta}<\varepsilon_0(\delta), \tag{2.26}$$

$$|v_--\bar v|<\varepsilon_1(\delta), \tag{2.27}$$

then (2.9) has a unique global solution $(v,u)(\xi,t)$ satisfying

$$v(\xi,t)-V(\xi,t;\alpha,\beta)\in C^0([0,\infty),H^1)\cap L^2(0,\infty;H^1), \tag{2.28}$$

$$u(\xi,t)-U(\xi,t;\alpha,\beta)\in C^0([0,\infty),H^1)\cap L^2(0,\infty;H^2), \tag{2.29}$$

and

$$\sup_{\xi\in\Re_+}\big|(v,u)(\xi,t)-(V,U)(\xi,t;\alpha,\beta)\big|\longrightarrow 0,\ \text{as } t\to+\infty, \tag{2.30}$$

where $\alpha=\alpha(\beta)$ is determined by (2.11).

3 Reformulation of the original Problem

In this section, we focus our attention on the case (I), i.e. $(v_-,u_-)\in S_2(v_-,u_-)$. In this case, $(V,U)(\xi,t;\alpha,\beta)=(V_s,U_s)(\xi-(s-s_-)t+\alpha-\beta)$. Let

$$\begin{aligned}\phi(\xi,t)&=-\int_\xi^\infty\big[v(y,t)-V(y,t;\alpha,\beta)\big]dy,\\ \psi(\xi,t)&=-\int_\xi^\infty\big[u(y,t)-U(y,t;\alpha,\beta)\big]dy,\end{aligned} \tag{3.1}$$

which means we seek the solution $(v,u)(\xi,t)$ in the form

$$\begin{aligned}v(\xi,t)&=\phi_\xi(\xi,t)+V(\xi,t;\alpha,\beta),\\ u(\xi,t)&=\psi_\xi(\xi,t)+U(\xi,t;\alpha,\beta).\end{aligned} \tag{3.2}$$

Substituting (3.2) into (2.9), and integrating the system on $[\xi, +\infty)$ with respect to ξ, we have

$$\begin{cases} \phi_t - s_-\phi_\xi - \psi_\xi = 0, & \text{in } \Re_+ \times \Re_+, \\ \psi_t - s_-\psi_\xi + p(V + \phi_\xi) - p(V) & \\ = \mu[\dfrac{U' + \psi_{\xi\xi}}{V + \phi_\xi} - \dfrac{U'}{V}], & \text{in } \Re_+ \times \Re_+. \end{cases} \tag{3.3}$$

By (3.1), the initial data satisfy

$$\begin{aligned} \phi(\xi, 0) &= -\int_\xi^{+\infty} [v_0(y) - V(y, 0; \alpha, \beta)]dy \\ &= \Phi_0(\xi) + \int_\xi^\infty [V(y, 0; \alpha, \beta) - V(y, 0; 0, \beta)]dy \\ &= \Phi_0(\xi) + \int_\xi^\infty \int_0^\alpha V'(y + \theta - \beta)d\theta dy \\ &= \Phi_0(\xi) + \int_0^\alpha [v_+ - V(\xi + \theta - \beta)]d\theta \\ &=: \phi_0(\xi), \end{aligned} \tag{3.4}$$

$$\begin{aligned} \psi(\xi, 0) &= -\int_\xi^{+\infty} [u_0(y) - U(y, 0; \alpha, \beta)]dy \\ &= \Psi_0(\xi) + \int_\xi^\infty [U(y, 0; \alpha, \beta) - U(y, 0; 0, \beta)]dy \\ &= \Psi_0(\xi) + \int_0^\alpha [u_+ - U(\xi + \theta - \beta)]d\theta \\ &=: \psi_0(\xi). \end{aligned} \tag{3.5}$$

Furthermore, we have

Proposition 3.1. *Under the assumptions (2.16), (2.17) and (2.19), the initial perturbations* $(\phi_0, \psi_0) \in H^2$ *and satisfies*

$$\|(\phi_0, \psi_0)\|_2 \to 0 \quad as \quad \|(\Phi_0, \Psi_0)\|_2 \le o(\frac{1}{\sqrt{\beta}}) \quad and \quad \beta \to +\infty. \tag{3.6}$$

Although the proof of proposition 3.1 is similar to the lemma 3.1 of [7], we give the proof here.

Proof. Let $\chi_1(x) = \int_0^\alpha [v_+ - V(x - \beta + \theta)]\, d\theta$. Then it follows from (2.5) that, if $x \ge \beta - \theta$,

$$|v_+ - V(x - \beta + \theta)| \le Ce^{-c_+(x-\beta+\theta)}. \tag{3.7}$$

Thus, we have

$$\|\chi_1\|^2 \le C\int_0^{\beta-\theta} \alpha^2\, dx + C\int_{\beta-\theta}^\infty \alpha^2 e^{-2c_+(x-\beta+\theta)}\, dx \le C\alpha^2(\beta + 1),$$

where C is independent of α and β. Similarly, we can prove that $\|\chi_1'\|^2, \|\chi_1''\|^2 \leq C\alpha^2(\beta+1)$.

On the other hand, (2.14) gives

$$|\alpha| \leq C(|\Phi_0| + e^{-c_-\beta}) \leq C(\|\Phi_0\|_2 + e^{-c_-\beta}).$$

Thus, we have

$$\|\chi_1\|_2 \to 0 \quad as \quad \|\Phi_0\|_2 \leq o(\frac{1}{\sqrt{\beta}}) \quad and \quad \beta \to +\infty.$$

In the same way, we have that $\chi_2(x) = \int_0^\alpha [u_+ - U(x-\beta+\theta)]\,d\theta$ satisfies $\|\chi_2\|_2^2 \leq C\alpha^2(\beta+1)$. It is noted that

$$\|(\phi_0,\psi_0)\|_2 \leq \|(\Phi_0,\Psi_0)\|_2 + \|(\chi_1,\chi_2)\|_2.$$

Hence, proposition 3.1 is proved.

By (2.14), the boundary data satisfy

$$\begin{aligned}\phi(0,t) &= -\int_t^{+\infty}[v(y,t)-V(y,t;\alpha,\beta)]dy\\ &= -(s-s_-)\int_t^\infty \big(V((s_- - s)\tau+\alpha-\beta)-v_-\big)d\tau,\\ &=: A(t),\end{aligned} \tag{3.8}$$

$$\begin{aligned}\psi_\xi(0,t) &= u(0,t)-U\big(0,t;\alpha,\beta\big)\\ &= u_- - U\big((s_- - s)t+\alpha-\beta\big)\\ &= A'(t)+s_-(V\big((s_- - s)t+\alpha-\beta\big)-v_-).\end{aligned} \tag{3.9}$$

It is noted that if (3.8) and (3.9) hold, then $\phi_\xi(0,t) = v_- - V(0,t;\alpha,\beta)$ automatically holds by the equation(3.3). Hence we regard (3.9) as a *Neumann* boundary condition. We now rewrite the system (3.3) in the form

$$\begin{cases}\phi_t - s_-\phi_\xi - \psi_\xi = 0, & \text{in } \Re_+\times\Re_+,\\ \psi_t - s_-\psi_\xi - f(V)\phi_\xi - \dfrac{\mu}{V}\psi_{\xi\xi} = F, & \text{in } \Re_+\times\Re_+,\\ (\phi,\psi)\big|_{t=0} = (\phi_0,\psi_0), & \\ \phi|_{\xi=0} = A(t), & \\ \psi_\xi\big|_{\xi=0} = A'(t)+s_-(V(-(s-s_-)t+\alpha-\beta)-v_-), & \end{cases} \tag{3.10}$$

here

$$f(V) = -p'(V)+\frac{\mu s V'}{V^2} = \frac{h(V)-p'(V)V}{V} \equiv \frac{K(V)}{V}, \tag{3.11}$$

$$\begin{aligned}F = &-\{p(V+\phi_\xi)-p(V)-p'(V)\phi_\xi\}\\ &+(\mu\psi_{\xi\xi}+h(V)\phi_\xi)\big(\tfrac{1}{V+\phi_\xi}-\tfrac{1}{V}\big).\end{aligned} \tag{3.12}$$

For any interval $I \subset \Re_+$, we define the solution space $X(I)$ by

$$\begin{aligned}X(I) = \{ \quad &(\phi,\psi) \in C^0(I;H^2); \phi_\xi \in L^2(I;H^1),\\ &\psi_\xi \in L^2(I;H^2), \sup_{t\in I}\|(\phi,\psi)(t)\|_2 \leq \varepsilon_1\},\end{aligned} \tag{3.13}$$

where $\varepsilon_1 = \frac{1}{2}v_-$. Let

$$N(t) = \sup_{0\le\tau\le t}(\|\phi(\tau)\|_2 + \|\psi(\tau)\|_2), \quad N_0 = \|\phi_0\|_2 + \|\psi_0\|_2. \tag{3.14}$$

By the Sobolev embedding theorem, for $(\phi,\psi) \in X([0,T])$, one obtains

$$(V+\phi_\xi)(\xi,t) \ge v_- - \|\phi_\xi\|_1 \ge \frac{1}{2}v_-, \quad (\xi,t) \in \Re_+ \times [0,T],$$

which ensures that the system (3.10) is uniformly nonsingular on $[0,T]$, and

$$|F| = O(|\phi_\xi|^2 + |\phi_\xi|\cdot|\psi_{\xi\xi}|). \tag{3.15}$$

Proposition 3.2. *(Local Existence). For any $\tau \ge 0$, consider the problem*

$$\begin{cases} \phi_t - s_-\phi_\xi - \psi_\xi = 0, & \text{in } \Re_+ \times [\tau,\infty), \\ \psi_t - s_-\psi_\xi - f(V)\phi_\xi - \dfrac{\mu}{V}\psi_{\xi\xi} = F, & \text{in } \Re_+ \times [\tau,\infty), \\ (\phi,\psi)\big|_{t=\tau} = (\phi_\tau,\psi_\tau) \in H^2, & \\ \phi|_{\xi=0} = A(t), & t \ge \tau, \\ \psi_\xi\big|_{\xi=0} = f(t) = A'(t) + s_-(V(0,t;\alpha,\beta) - v_-), & t \ge \tau, \end{cases} \tag{3.16}$$

subject to the compatibility condition $\psi_\xi(0,\tau) = f(\tau)$. Then there exists a positive constant C_0 independent of τ such that: For any $\varepsilon \in (0, \frac{\varepsilon_1}{C_0}]$ and $\beta > 1$, there exists a positive constant T_0 depending on ε and β but not on τ such that, if $\|(\phi_\tau,\psi_\tau)\|_2 \le \varepsilon$, and $\sup_{t\ge 0}(|f(t)| + |f'(t)|) \le \varepsilon$, then the problem (3.16) has a unique solution $(\phi,\psi) \in X([\tau,\tau+T_0])$ satisfying $\|(\phi,\psi)(t)\|_2 \le C_0\varepsilon$ for $t \in [\tau,\tau+T_0]$.

By using the standard way, such as Leray-Schauder's fixed-point theorem, Proposition 3.1 can be easily verified, we omit the proof here.

4 Proof of the a Priori Estimates

Let $(\phi,\psi) \in X([0,T])$ be a solution of (3.10) for a positive constant T. Without loss of generality, we may restrict $N(T) < \varepsilon_1, \beta > 1$ and $|\alpha| < 1$. Throughout this section, we use C to denote the positive constant which is independent of T, β and α.

Proposition 4.1. *(A Priori Estimates). There exists a positive constant δ_0 such that, for any given $0 < u_- = \delta < \delta_0$, there exist positive constants $\delta_1(\delta)$ ($\delta_1 \le \varepsilon_1$) and $C(\delta)$ such that if $(\phi,\psi) \in X([0,T])$ is a solution of (3.10) for some positive T and $N(T) < \delta_1$, then (ϕ,ψ) satisfies the a priori estimates*

$$\|(\phi,\psi)(t)\|_2^2 + \int_0^t \{\|\phi_\xi(\tau)\|_1^2 + \|\psi_\xi(\tau)\|_2^2\}d\tau \le C(\delta)(\|(\phi_0,\psi_0)\|_2^2 + e^{-c_-\beta}), \tag{4.1}$$

$$\int_0^t |\frac{d}{dt}\|\phi_\xi(\tau)\|^2| + |\frac{d}{dt}\|\psi_\xi(\tau)\|^2| d\tau \le C(\delta)(\|(\phi_0,\psi_0)\|_2^2 + e^{-c_-\beta}). \tag{4.2}$$

Before proving the Proposition 4.1, we first give some Lemmas.

Lemma 4.1. *For $0 \le t \le T$, the following inequalities hold:*

$$\begin{array}{c}
\left|\int_0^t (\phi\psi)\big|_{\xi=0}\right|, \quad \left|\int_0^t (\psi\psi_\xi)\big|_{\xi=0}\right|, \quad \left|\int_0^t (\phi^2)\big|_{\xi=0}\right| \le Ce^{-c_-\beta}, \\
\left|\int_0^t (\psi_\xi\psi_t)\big|_{\xi=0}\right|, \quad \left|\int_0^t (\phi\psi_\xi)\big|_{\xi=0}\right|, \quad \left|\int_0^t (\phi_\xi^2)\big|_{\xi=0}\right| \le Ce^{-c_-\beta}, \\
\left|\int_0^t (\psi_\xi^2)\big|_{\xi=0}\right|, \quad \left|\int_0^t (\phi_\xi\phi)\big|_{\xi=0}\right|, \quad \left|\int_0^t (\phi_{\xi t}^2)\big|_{\xi=0}\right| \le Ce^{-c_-\beta}, \\
\left|\int_0^t (\phi_\xi\psi)\big|_{\xi=0}\right|, \quad \left|\int_0^t (\psi_{\xi t}^2)\big|_{\xi=0}\right|, \quad \left|\int_0^t (\phi_{\xi t}\psi_\xi)\big|_{\xi=0}\right| \le Ce^{-c_-\beta},
\end{array} \tag{4.3}$$

where $c_- = \frac{v_-|p'(v_-)+s^2|}{\mu s} > 0$.

Proof. By using (3.8),(3.9) and Proposition 2.1, we have

$$\begin{array}{l}
|\phi(0,t)| = A(t) \le O(1)e^{-c_-\beta}e^{-c_-(s-s_-)t}, \\
|\psi(0,t)| \le \sup\limits_{\xi\in\Re_+} |\psi(\xi,t)| \le CN(T) \le C,
\end{array} \tag{4.4}$$

and

$$\begin{array}{ll}
|\frac{d^k}{dt^k}A(t)| & \le O(1)e^{-c_-\beta}e^{-c_-(s-s_-)t}, \quad k=0,1,2,3, \\
|\phi_\xi(0,t)| & = |v_- - V\big((s-s_-)t+\alpha-\beta\big)| \le O(1)e^{-c_-\beta}e^{-c_-(s-s_-)t}, \\
|\phi_{\xi t}(0,t)| & = \big|[v_- - V\big((s-s_-)t+\alpha-\beta\big)]'\big| \le O(1)e^{-c_-\beta}e^{-c_-(s-s_-)t},
\end{array} \tag{4.5}$$

$$\begin{array}{ll}
|\psi_\xi(0,t)| & = |u_- - U\big((s-s_-)t+\alpha-\beta\big)| \le O(1)e^{-c_-\beta}e^{-c_-(s-s_-)t}, \\
|\psi_{\xi t}(0,t)| & = \big|[u_- - U\big((s-s_-)t+\alpha-\beta\big)]'\big| \le O(1)e^{-c_-\beta}e^{-c_-(s-s_-)t},
\end{array} \tag{4.6}$$

By using (4.4)-(4.6),

$$\begin{aligned}
\left|\int_0^t (\phi\psi)\big|_{\xi=0} d\tau\right| & \le \int_0^t O(1)e^{-c_-\beta}e^{-c_-(s-s_-)\tau}|\psi(0,\tau)| d\tau \\
& \le CN(T)\int_0^t e^{-c_-\beta}e^{-c_-(s-s_-)\tau} d\tau \\
& \le Ce^{-c_-\beta}.
\end{aligned}$$

$$\begin{aligned}
\left|\int_0^t (\psi\psi_\xi)\big|_{\xi=0} d\tau\right| & \le \int_0^t O(1)e^{-c_-\beta}e^{-c_-(s-s_-)\tau}|\psi(0,\tau)| d\tau \\
& \le CN(T)\int_0^t e^{-c_-\beta}e^{-c_-(s-s_-)\tau} d\tau \\
& \le Ce^{-c_-\beta}.
\end{aligned}$$

$$\begin{aligned}
\left|\int_0^t (\phi^2)\big|_{\xi=0} d\tau\right| &\leq \int_0^t O(1)e^{-c_-\beta}e^{-c_-(s-s_-)\tau}|\phi_\xi(0,\tau)|d\tau \\
&\leq Ce^{-c_-\beta}.
\end{aligned}$$

$$\begin{aligned}
\left|\int_0^t (\psi_\xi\psi_t)\big|_{\xi=0} d\tau\right| &= \left|\psi_\xi(0,t)\psi(0,t) - \psi_\xi(0,0)\psi(0,0) - \int_0^t \psi_{\xi\tau}(0,\tau)\psi(0,\tau)d\tau\right| \\
&\leq CN(T)\left[e^{-c_-\beta} + \int_0^t |\psi_{\xi\tau}(0,\tau)|d\tau\right] \\
&\leq Ce^{-c_-\beta}.
\end{aligned}$$

In the same way, it is easy to get the other inequalities. Lemma 4.1 is proved.

Remark. Lemma 4.1 is formally obtained. It can be rigorously proved by mollifying the functions and then taking the limit.

Lemma 4.2. *Suppose $V_s(\xi-(s-s_-)t+\alpha-\beta)$ is the viscous shock profile. Then*

$$0 \leq \frac{h(V_s)}{V_s} \leq \frac{s^2(v_+-v_-)}{v_-}, \tag{4.7}$$

$$0 < -p'(v_+) \leq f(V_s) \leq -p'(v_-) + \frac{s^2(v_+-v_-)}{v_-}, \tag{4.8}$$

$$f(V_s) - \frac{h(V_s)}{2V_s} \geq -p'(v_+) > 0. \tag{4.9}$$

The proof can be found in [7].

We now establish the a priori estimates. The following is our key Lemma.

Lemma 4.3. *There exists a positive constant δ_0 which only depends on v_- and v_+. For any given $0 < u_- < \delta_0$, the following holds*

$$\begin{aligned}
&||(\phi,\psi)(t)||^2 + \int_0^t ||\psi_\xi||^2 d\tau \\
&\leq C\left\{||(\phi_0,\psi_0)||^2 + e^{-c_-\beta} + |s_-|\int_0^t ||\phi_\xi||^2 d\tau\right. \\
&\quad \left. +N(T)\int_0^t [||\phi_\xi||^2 + ||\psi_{\xi\xi}||^2]d\tau\right\}.
\end{aligned} \tag{4.10}$$

Remark. If we directly multiply $(3.10)_2$ by $f(V)^{-1}\psi$, then we can not get (4.10) due to the bad sign of $s_-\psi^2(0,t)$. As pointed out by [12], it is difficult to control the term $\psi(0,t)$ from the boundary. It is observed that the term $s_-\psi_\xi$ in (3.10) is caused by coordinate transformation, it may be possible to erase $\psi(0,t)$ in a new system by introducing a new variable $\bar{\psi}(x,t)$ instead of $\psi(x,t)$. This is our key point. It should be noted that

$$
\begin{aligned}
&\int_0^t \psi^2(0,\tau)d\tau \le C(\delta)(\|(\phi_0,\psi_0)\|_2^2 + e^{-c_-\beta}),\\
&\int_0^t\int_0^\infty V_\xi \phi^2 d\xi d\tau \le C(\delta)(\|(\phi_0,\psi_0)\|_2^2 + e^{-c_-\beta}),
\end{aligned}
\tag{4.11}
$$

provided that proposition 4.1 holds. We shall prove (4.11) in the end of this section.

Proof. Let $\overline{\psi} = \psi - s_-\phi$, then (3.10) becomes

$$
\begin{cases}
\phi_t - 2s_-\phi_\xi - \overline{\psi}_\xi = 0, & \text{in } \Re_+ \times \Re_+,\\
\overline{\psi}_t - (f(V) - s_-^2)\phi_\xi - \dfrac{\mu}{V}\overline{\psi}_{\xi\xi} - s_-\dfrac{\mu}{V}\phi_{\xi\xi} = F, & \text{in } \Re_+ \times \Re_+,\\
(\phi,\overline{\psi})\big|_{t=0} = (\phi_0, \psi_0 - s_-\phi_0),\\
\phi\big|_{\xi=0} = A(t),\\
\overline{\psi}_\xi\big|_{\xi=0} = A'(t).
\end{cases}
\tag{4.12}
$$

Multiplying $(4.12)_1$ by ϕ and $(4.12)_2$ by $\left(f(V) - s_-^2\right)^{-1}\overline{\psi}$, then we have

$$
\begin{aligned}
&(\frac{1}{2}\phi^2)_t - (s_-\phi^2)_\xi - (\phi\overline{\psi})_\xi + \left\{\frac{1}{2}\frac{1}{f(V) - s_-^2}\overline{\psi}^2\right\}_t\\
&-\left(\frac{1}{2(f(V) - s_-^2)}\right)' V_t\overline{\psi}^2 - \left(\frac{\mu}{V(f(V) - s_-^2)}\overline{\psi}_\xi\overline{\psi}\right)_\xi\\
&+\frac{\mu}{V(f(V) - s_-^2)}\overline{\psi}_\xi^2 + \left(\frac{\mu}{V(f(V) - s_-^2)}\right)' V_\xi\overline{\psi}_\xi\overline{\psi}\\
&-\left(s_-\frac{\mu}{V(f(V) - s_-^2)}\phi_\xi\overline{\psi}\right)_\xi + s_-\frac{\mu}{V(f(V) - s_-^2)}\phi_\xi\overline{\psi}_\xi\\
&+s_-\left(\frac{\mu}{V(f(V) - s_-^2)}\right)' V_\xi\phi_\xi\overline{\psi}\\
&= F\overline{\psi}\left(f(V) - s_-^2\right)^{-1}.
\end{aligned}
\tag{4.13}
$$

By the definition of $f(V)$, one has

$$
\begin{aligned}
&\left|\left(\frac{\mu}{V(f(V) - s_-^2)}\right)' V_\xi\overline{\psi}_\xi\overline{\psi}\right| = \left|\frac{\mu(K'(V) - s_-^2)}{(K(V) - s_-^2 V)^2}V_\xi\overline{\psi}_\xi\overline{\psi}\right|\\
&\le \varepsilon\frac{\mu}{V(f(V) - s_-^2)}\overline{\psi}_\xi^2 + \frac{\mu(K'(V) - s_-^2)^2 V_\xi^2}{4\varepsilon(K(V) - s_-^2 V)^3}\overline{\psi}^2,
\end{aligned}
\tag{4.14}
$$

for any $\varepsilon > 0$, which will be determined later. Substituting this inequality into

(4.13) yields

$$
\begin{aligned}
&\left\{\frac{1}{2}\phi^2+\frac{1}{2(f(V)-s_-^2)}\overline{\psi}^2\right\}_t\\
&-\left\{s_-\phi^2+\phi\overline{\psi}+\frac{\mu}{V(f(V)-s_-^2)}\overline{\psi}_\xi\overline{\psi}+s_-\frac{\mu}{V(f(V)-s_-^2)}\phi_\xi\overline{\psi}\right\}_\xi\\
&+(1-\varepsilon-|s_-|\varepsilon)\frac{\mu}{V(f(V)-s_-^2)}\overline{\psi}_\xi^2+[Z(V)+O(1)|s_-|]V_\xi\overline{\psi}^2\\
&\leq F\overline{\psi}\left(f(V)-s_-^2\right)^{-1}+(c+\frac{1}{4\varepsilon})\frac{\mu|s_-|}{V(f(V)-s_-^2)}\phi_\xi^2,
\end{aligned}
\tag{4.15}
$$

where

$$
Z(V)=\frac{s}{2}\frac{K(V)-VK'(V)}{K(V)^2}-\frac{\mu K'(V)^2V_\xi}{4\varepsilon K(V)^3}. \tag{4.16}
$$

In view of (2.3), (2.4) and (3.11), one gets

$$
\begin{aligned}
Z(V)=\quad&\frac{1}{4sK(V)^3}\{2s^2[\gamma^3p(V)^2+h(V)^2+\gamma s^2Vp(V)]\\
&+2s^2\gamma[(\gamma+1)-\frac{(\gamma-1)}{\varepsilon}]p(V)h(V)-\frac{\gamma^2(\gamma-1)^2}{\varepsilon V}h(V)p(V)^2\\
&+2s^4[1-\frac{1}{2\varepsilon}]Vh(V)\},
\end{aligned}
\tag{4.17}
$$

where $O(1)$ only depends on v_-, v_+. By using Lemma 4.2, one obtains

$$
\begin{aligned}
Z(V)\geq\quad&\frac{1}{4sK(V)^3}\{2s^2[h(V)^2+\gamma s^2Vp(V)]\\
&+2s^2\gamma^2[\gamma-\frac{(v_+-v_-)}{2v_-\varepsilon}(\gamma-1)^2]p(V)^2\\
&+2s^2\gamma[\gamma+1-(\gamma-1)\varepsilon^{-1}]p(V)h(V)+2s^4[1-\frac{1}{2\varepsilon}]Vh(V)\}.
\end{aligned}
\tag{4.18}
$$

Choosing

$$
\max\{\frac{(\gamma-1)^2(v_+-v_-)}{2\gamma v_-},\frac{1}{2}\}\leq\varepsilon<1, \tag{4.19}
$$

one has

$$
Z(V)\geq C>0, \tag{4.20}
$$

due to $1\leq\gamma\leq3$. Thus, there exists a positive constant δ_0 depending on v_-, v_+. For any given $0<u_-<\delta_0$ (or $0<|s_-|\leq v_-\delta_0$), integrating (4.15) over

$[0,+\infty)\times[0,t]$, we have

$$\begin{aligned}
&\int_0^{+\infty}\left\{\frac{\phi^2}{2}+\frac{\overline{\psi}^2}{2(f(V)-s_-^2)}\right\}d\xi+\int_0^t\int_0^{+\infty}\frac{1}{2}Z(V)V_\xi\overline{\psi}^2d\xi d\tau\\
&\quad+\int_0^t\int_0^{+\infty}\frac{(1-\varepsilon-|s_-|\varepsilon)\mu\overline{\psi}_\xi^2}{V(f(V)-s_-^2)}d\xi d\tau\\
&\le C\int_0^{+\infty}\left\{\phi_0^2+\overline{\psi}_0^2\right\}d\xi\\
&\quad+\left|\int_0^t\{s_-\phi^2+\phi\overline{\psi}+\frac{\mu\overline{\psi}_\xi\overline{\psi}}{V(f(V)-s_-^2)}+\frac{s_-\mu\phi_\xi\overline{\psi}}{V(f(V)-s_-^2)}\}|_{\xi=0}d\tau\right|\\
&\quad+\int_0^t\int_0^{+\infty}F\overline{\psi}\left(f(V)-s_-^2\right)^{-1}d\xi d\tau\\
&\quad+\int_0^t\int_0^{+\infty}\frac{c\mu|s_-|}{V(f(V)-s_-^2)}\phi_\xi^2d\xi d\tau.
\end{aligned}\tag{4.21}$$

By using Lemma 4.1, Lemma 4.2, we get the energy estimate (4.10) due to $\overline{\psi}=\psi-s_-\phi$.

Lemma 4.4. *For any given* $0<u_-<\delta_0$, $\forall t\in[0,T]$,

$$\begin{aligned}
&\|\phi_\xi\|^2+\int_0^t\|\phi_\xi(\tau)\|^2d\tau\\
&\le C\left\{\|(\phi_0,\psi_0)\|_1^2+e^{-c_-\beta}+N(T)\int_0^t[\|\phi_\xi\|^2+\|\psi_\xi\|_1^2]d\tau\right\}.
\end{aligned}\tag{4.22}$$

Proof. From the system (3.10), we have

$$\frac{\mu}{V}\phi_{\xi t}-s_-\frac{\mu}{V}\phi_{\xi\xi}+f(V)\phi_\xi+s_-\psi_\xi=\psi_t-F.\tag{4.23}$$

Multiplying (4.23) by ϕ_ξ yields

$$\begin{aligned}
&\left(\frac{\mu}{2V}\phi_\xi^2\right)_t-\frac{(s-s_-)\mu}{2V^2}V_\xi\phi_\xi^2-\left\{s_-\frac{\mu}{2V}\phi_\xi^2\right\}_\xi-\frac{s_-\mu}{2V^2}V_\xi\phi_\xi^2\\
&\qquad+f(V)\phi_\xi^2+s_-\psi_\xi\phi_\xi=\psi_t\phi_\xi-F\phi_\xi.
\end{aligned}\tag{4.24}$$

Using (2.4), one obtains

$$\begin{aligned}
&\left(\frac{\mu}{2V}\phi_\xi^2\right)_t+\left(f(V)-\frac{h(V)}{2V}\right)\phi_\xi^2+s_-\psi_\xi\phi_\xi\\
&\qquad-\left\{s_-\frac{\mu}{2V}\phi_\xi^2\right\}_\xi=\psi_t\phi_\xi-F\phi_\xi.
\end{aligned}\tag{4.25}$$

The system $(3.10)_1$ gives

$$\begin{aligned}
\psi_t\phi_\xi&=(\psi\phi_\xi)_t-\psi\phi_{\xi t}=(\psi\phi_\xi)_t-\psi(s_-\phi_{\xi\xi}+\psi_{\xi\xi})\\
&=(\psi\phi_\xi)_t-(s_-\psi\phi_\xi)_\xi-(\psi\psi_\xi)_\xi+s_-\psi_\xi\phi_\xi+\psi_\xi^2.
\end{aligned}\tag{4.26}$$

Substituting (4.26) into (4.25), and using Lemma 4.2, we get

$$
\begin{aligned}
&\left(\frac{\mu}{2V}\phi_\xi^2-\psi\phi_\xi\right)_t+\left(f(V)-\frac{h(V)}{2V}\right)\phi_\xi^2\\
&\quad+\left\{\psi\psi_\xi+s_-\psi\phi_\xi-s_-\frac{\mu}{2V}\phi_\xi^2\right\}_\xi\\
&=\psi_\xi^2-F\phi_\xi.
\end{aligned}
\tag{4.27}
$$

Integrating (4.27) over $[0,+\infty)\times[0,t]$, and using Proposition 2.1, Lemma 4.2 and the inequality

$$
\int_0^{+\infty}|\psi\phi_\xi|d\xi\le\frac{\mu}{4v_+}\|\phi_\xi(t)\|^2+\frac{v_+}{\mu}\|\psi(t)\|^2,
$$

we have

$$
\begin{aligned}
&\frac{\mu}{4v_+}\|\phi_\xi(t)\|^2+|p'(v_+)|\int_0^t\|\phi_\xi(\tau)\|d\tau\\
&\le\frac{\mu}{2v_-}\|\phi_{0,\xi}\|^2+\frac{1}{2}\|\psi_0\|^2+\frac{1}{2}\|\phi_{0,\xi}\|\\
&\quad+\left|\int_0^t\{\psi\psi_\xi+s_-\psi\phi_\xi-s_-\frac{\mu}{2V}\phi_\xi^2\}\big|_{\xi=0}d\tau\right|\\
&\quad+\frac{v_+}{\mu}\|\psi(t)\|^2+\int_0^t\|\psi_\xi(\tau)\|^2d\tau+\int_0^t\int_0^{+\infty}|F\phi_\xi|d\xi d\tau.
\end{aligned}
\tag{4.28}
$$

Applying Lemma 4.1 and Lemma 4.3 to this inequality yields the estimate (4.22). Lemma 4.4 is proved.

Lemma 4.5. *For any given* $0<u_-<\delta_0$, $\forall t\in[0,T]$,

$$
\begin{aligned}
&\|\psi_\xi(t)\|^2+\int_0^t\|\psi_{\xi\xi}\|^2d\tau\\
&\le C\left\{\|(\phi_0,\psi_0)\|_1^2+e^{-c-\beta}+N(T)\int_0^t\left[\|\phi_\xi(\tau)\|^2+\|\psi_\xi(\tau)\|_1^2\right]d\tau\right\}.
\end{aligned}
\tag{4.29}
$$

Proof. Multiplying $(3.10)_2$ by $-\psi_{\xi\xi}$, one obtains

$$
(-\psi_\xi\psi_t)_\xi+(\frac{1}{2}\psi_\xi^2)_t+(\frac{s_-}{2}\psi_\xi^2)_\xi+f(V)\phi_\xi\psi_{\xi\xi}+\frac{\mu}{V}\psi_{\xi\xi}^2=-\psi_{\xi\xi}F,\tag{4.30}
$$

Lemma 4.2 and the Cauchy inequality yield

$$
|f(V)\phi_\xi\psi_{\xi\xi}|\le\frac{\mu}{2v_+}\psi_{\xi\xi}^2+\frac{c_0^2v_+}{2\mu}\phi_\xi^2,\tag{4.31}
$$

and (3.15) and the Cauchy inequality yield

$$
|-F\psi_{\xi\xi}|\le C(|\phi_\xi|^2+|\phi_\xi|\cdot|\psi_{\xi\xi}|)|\psi_{\xi\xi}|\le C|\phi_\xi|(|\phi_\xi|^2+|\psi_{\xi\xi}|^2).\tag{4.32}
$$

Substituting (4.32), (4.31) into (4.30) and using Proposition 2.1, we have

$$\begin{aligned} &\frac{1}{2}(\psi_\xi^2)_t + \Big\{\frac{s_-}{2}\psi_\xi^2 - \psi_\xi\psi_t\Big\}_\xi + \frac{\mu}{2v_+}\psi_{\xi\xi}^2 \\ &\le \frac{c_0^2 v_+}{2\mu}\phi_\xi^2 + C|\phi_\xi|(|\phi_\xi|^2 + |\psi_{\xi\xi}|^2). \end{aligned} \tag{4.33}$$

Integrating (4.33) over $[0,+\infty)\times[0,t]$, and using Lemma 4.1, Lemma 4.3 and Lemma 4.4, we have the estimate (4.29).

From Lemma 4.3-4.4, we get the following inequality.

Lemma 4.6. *For any given* $0 < u_- < \delta_0$, $\forall t \in [0,T]$,

$$\begin{aligned} &\|(\phi,\psi)(t)\|_1^2 + \int_0^t \{\|\phi_\xi\|^2 + \|\psi_\xi\|_1^2\}\, d\tau \\ &\le C\{\|(\phi_0,\psi_0)\|_1^2 + e^{-c_-\beta} + N(T)\int_0^t \big[\|\phi_\xi(\tau)\|^2 + \|\psi_\xi(\tau)\|_1^2\big] d\tau\}. \end{aligned} \tag{4.34}$$

Lemma 4.7. *For any given* $0 < u_- < \delta_0$, $\forall t \in [0,T]$, *then*

$$\begin{aligned} &\|\phi_{\xi\xi}(t)\|^2 + \int_0^t \|\phi_{\xi\xi}\|^2 d\tau \\ &\le \frac{C}{|s_-|^{2+\sigma}}\{\|(\phi_0,\psi_0)\|_2^2 + e^{-c_-\beta} + N(T)\int_0^t \big[\|\phi_\xi(\tau)\|^2 + \|\psi_\xi(\tau)\|_1^2\big] d\tau\} \\ &+ C\{\int_0^t \|F_\xi(\tau)\|^2 d\tau + |s_-|^\sigma \int_0^t \|\psi_{\xi\xi\xi}\|^2 d\tau\}, \end{aligned} \tag{4.35}$$

where σ is an arbitrary positive constant.

Proof. Differentiating (4.23) with respect to ξ, one gets

$$\begin{aligned} &\frac{\mu}{V}\phi_{\xi\xi t} - \frac{\mu V_\xi}{V^2}\phi_{\xi t} - s_-\frac{\mu}{V}\phi_{\xi\xi\xi} + s_-\frac{\mu V_\xi}{V^2}\phi_{\xi\xi} + f(V)\phi_{\xi\xi} \\ &+ f'(V)V_\xi\phi_\xi + s_-\psi_{\xi\xi} = \psi_{\xi t} - F_\xi. \end{aligned} \tag{4.36}$$

Multiplying (4.36) by $\phi_{\xi\xi}$, we have

$$\begin{aligned} &\Big(\frac{\mu}{2V}\phi_{\xi\xi}^2\Big)_t + \Big(f(V) - \frac{h(V)}{2V}\Big)\phi_{\xi\xi}^2 - \Big(\frac{s_-\mu}{2V}\phi_{\xi\xi}^2\Big)_\xi - \frac{\mu V_\xi}{V^2}\phi_{\xi t}\phi_{\xi\xi} \\ &+ \frac{s_-\mu V_\xi}{V^2}\phi_{\xi\xi}^2 + f'(V)V_\xi\phi_{\xi\xi}\phi_\xi + s_-\psi_{\xi\xi}\phi_{\xi\xi} = \psi_{\xi t}\phi_{\xi\xi} - F_\xi\phi_{\xi\xi}. \end{aligned} \tag{4.37}$$

From systems $(3.10)_1$, one has

$$\begin{aligned}
&\left| s_- \int_0^t \phi_{\xi\xi}^2\big|_{\xi=0} d\tau \right| = \frac{1}{|s_-|}\int_0^t (\phi_{\xi t} - \psi_{\xi\xi})^2\big|_{\xi=0} d\tau \\
&\le \frac{2}{|s_-|}\int_0^t \left[\phi_{\xi t}^2\big|_{\xi=0} + \psi_{\xi\xi}^2\big|_{\xi=0}\right] d\tau \\
&\le \frac{C}{|s_-|} e^{-c_-\beta} + \left| \frac{1}{s_-}\int_0^t \int_0^{+\infty} \frac{d}{d\xi}\psi_{\xi\xi}^2 d\xi d\tau \right| \\
&\le \frac{C}{|s_-|} e^{-c_-\beta} + \frac{1}{2|s_-|}\int_0^t \left[|s_-|^{1+\sigma}\|\psi_{\xi\xi\xi}\|^2 + \frac{1}{|s_-|^{1+\sigma}}\|\psi_{\xi\xi}\|^2 \right] d\tau \\
&\le \frac{C}{|s_-|} e^{-c_-\beta} + \frac{1}{2}|s_-|^{\sigma}\int_0^t \|\psi_{\xi\xi\xi}\|^2 d\tau + \frac{1}{|s_-|^{2+\sigma}}\int_0^t \|\psi_{\xi\xi}\|^2 d\tau,
\end{aligned} \tag{4.38}$$

where $\sigma > 0$ is an arbitrary positive constant. Using Lemma 4.2 and (2.4), we obtain

$$\left| \frac{\mu V_\xi}{V^2}\phi_{\xi t}\phi_{\xi\xi} \right| = \left| \frac{\mu V_\xi}{V^2}(s_-\phi_{\xi\xi} + \psi_{\xi\xi})\phi_{\xi\xi} \right| \le \frac{1}{8}|p'(v_+)|\phi_{\xi\xi}^2 + C\psi_{\xi\xi}^2, \tag{4.39}$$

$$|s_-\psi_{\xi\xi}\phi_{\xi\xi}| \le O(1)|s_-|\phi_{\xi\xi}^2 + C\psi_{\xi\xi}^2, \tag{4.40}$$

$$|f'(V)V_\xi\phi_{\xi\xi}\phi_\xi| \le \frac{1}{8}|p'(v_+)|\phi_{\xi\xi}^2 + C\phi_\xi^2, \tag{4.41}$$

$$|F_\xi\phi_{\xi\xi}| \le \frac{1}{8}|p'(v_+)|\phi_{\xi\xi}^2 + C|F_\xi|^2, \tag{4.42}$$

$$\begin{aligned}
\psi_{\xi t}\phi_{\xi\xi} &= (\psi_\xi\phi_{\xi\xi})_t - \psi_\xi\phi_{\xi\xi t} = (\psi_\xi\phi_{\xi\xi})_t - \psi_\xi(s_-\phi_{\xi\xi\xi} + \psi_{\xi\xi\xi}) \\
&= (\psi_\xi\phi_{\xi\xi})_t - \big(\psi_\xi(s_-\phi_{\xi\xi} + \psi_{\xi\xi})\big)_\xi + s_-\phi_{\xi\xi}\psi_{\xi\xi} + \psi_{\xi\xi}^2 \\
&= (\psi_\xi\phi_{\xi\xi})_t - (\psi_\xi\phi_{\xi t})_\xi + s_-\phi_{\xi\xi}\psi_{\xi\xi} + \psi_{\xi\xi}^2.
\end{aligned} \tag{4.43}$$

Substituting (4.38)-(4.43) into (4.37), integrating it over $[0,+\infty)\times[0,t]$, and making use of Lemma 4.2 and Lemma 4.6, we obtain the inequality (4.35). Lemma 4.7 is proved.

Lemma 4.8. *For any given* $0 < u_- < \delta_0$, $\forall t \in [0,T]$,

$$\begin{aligned}
&\|\psi_{\xi\xi}(t)\|^2 + \int_0^t \|\psi_{\xi\xi\xi}\|^2 d\tau \\
&\le \frac{C}{|s_-|^{2+\sigma}}\{\|(\phi_0,\psi_0)\|_2^2 + e^{-c_-\beta} + N(T)\int_0^t [\|\phi_\xi(\tau)\|^2 \\
&\quad + \|\psi_\xi(\tau)\|_1^2] d\tau\} + C\int_0^t \|F_\xi(\tau)\|^2 d\tau.
\end{aligned} \tag{4.44}$$

Proof. Differentiating $(3.10)_2$ with respect to ξ, multiplying the derivative by $-\psi_{\xi\xi\xi}$, we have

$$\begin{aligned}
&\left(\frac{1}{2}\psi_{\xi\xi}^2\right)_t + \left(\frac{s_-}{2}\psi_{\xi\xi}^2 - \psi_{\xi t}\psi_{\xi\xi}\right)_\xi + f(V)\phi_{\xi\xi}\psi_{\xi\xi\xi} \\
&\quad + f'(V)V_\xi\phi_\xi\psi_{\xi\xi\xi} + \frac{\mu}{V}\psi_{\xi\xi\xi}^2 - \frac{\mu}{V^2}V_\xi\psi_{\xi\xi}\psi_{\xi\xi\xi} = -F_\xi\psi_{\xi\xi\xi}
\end{aligned} \tag{4.45}$$

From systems $(3.10)_1$, one has

$$\begin{aligned}&\left|\frac{s_-}{2}\int_0^t \psi_{\xi\xi}^2\big|_{\xi=0}d\tau\right| = \left|\frac{s_-}{2}\int_0^t\int_0^{+\infty}\frac{d}{d\xi}\psi_{\xi\xi}^2 d\xi d\tau\right|\\ &\leq \frac{|s_-|}{4}\left(\int_0^t \|\psi_{\xi\xi\xi}\|^2 d\tau + \int_0^t \|\psi_{\xi\xi}\|^2 d\tau\right),\end{aligned} \tag{4.46}$$

$$\psi_{\xi t}\psi_{\xi\xi}|_{\xi=0} \leq \frac{\mu}{8v_+}\|\psi_{\xi\xi\xi}\|^2 + C\|\psi_{\xi\xi}\|^2 + C\psi_{\xi t}^2|_{\xi=0}, \tag{4.47}$$

$$|f(V)\phi_{\xi\xi}\psi_{\xi\xi\xi}| \leq C|\phi_{\xi\xi}|^2 + \frac{\mu}{8v_+}|\psi_{\xi\xi\xi}|^2, \tag{4.48}$$

$$|f'(V)V_\xi\phi_\xi\psi_{\xi\xi\xi}| \leq C|\phi_\xi|^2 + \frac{\mu}{8v_+}|\psi_{\xi\xi\xi}|^2, \tag{4.49}$$

$$|F_\xi\psi_{\xi\xi\xi}| \leq C|F_\xi|^2 + \frac{\mu}{8v_+}|\psi_{\xi\xi\xi}|^2, \tag{4.50}$$

Substituting (4.46)-(4.50) into (4.45), integrating it over $[0,+\infty)\times[0,t]$, and making use of Lemma 4.2 and Lemma 4.7, we obtain the inequality (4.44). Lemma 4.8 is proved.

Proof of Proposition 4.1. By using Sobolev embedding theorem, (3.11) and (3.15), we have

$$\begin{aligned}&\|F_\xi\|^2 \leq C\int_0^{+\infty}(\phi_\xi^4 + \phi_\xi^2\phi_{\xi\xi}^2 + \psi_{\xi\xi}^2\phi_{\xi\xi}^2 + \psi_{\xi\xi\xi}^2\phi_\xi^2 + \phi_\xi^2\psi_{\xi\xi}^2)d\xi\\ &\leq C\sup_{\xi\in\Re_+}[\phi_\xi^2\int_0^{+\infty}(\phi_\xi^2+\phi_{\xi\xi}^2+\psi_{\xi\xi\xi}^2+\psi_{\xi\xi}^2)d\xi + \psi_{\xi\xi}^2\int_0^{+\infty}\phi_{\xi\xi}^2\,d\xi]\\ &\leq C[\|\phi_\xi\|_1^2(\|\phi_\xi\|^2+\|\psi_{\xi\xi\xi}\|^2) + \|\psi_{\xi\xi}\|_1^2\|\phi_{\xi\xi}\|^2]\\ &\leq C\|\phi_\xi\|_1^2(\|\phi_\xi\|_1^2+\|\psi_\xi\|_2^2)\\ &\leq CN(T)(\|\phi_\xi\|_1^2+\|\psi_\xi\|_2^2).\end{aligned} \tag{4.51}$$

Assume that $0<u_-<\delta$ is given. Choose $\delta_1 = O(1)\delta^{3+\sigma}$. Let $N(T)\leq\delta_1$, then Lemma 4.3-Lemma 4.8 infer the inequality (4.1). To prove inequality (4.2), we differentiate the system $(3.10)_1$ with respect to ξ, multiply it by ϕ_ξ, and integrate the resulting equality with respect to ξ, we get

$$\frac{d}{dt}\|\phi_\xi(t)\|^2 = 2\int_0^{+\infty}\psi_{\xi\xi}\phi_\xi d\xi + 2s_-\int_0^{+\infty}\phi_\xi\phi_{\xi\xi}d\xi. \tag{4.52}$$

Integrate (4.52) over $[0,+\infty)$, from (4.1), we prove (4.2) for ϕ. In the same way, we can also prove (4.2) for ψ. Proposition 4.1 is proved.

Theorem 4.1. *Suppose that the assumptions of theorem 2.1 hold. Then there exists a positive constant $\varepsilon_0(\delta)$, such that if (2.20) and (2.21) are satisfied, then the initial-boundary value problem (3.10) has a unique global solution $(\phi,\psi)\in$*

$X([0,+\infty))$ *satisfying inequalities (4.1) and (4.2) for any* $t \geq 0$. *Moreover, the solution is asymptotically stable*

$$\sup_{\xi\in\Re_+} |(\phi_\xi,\psi_\xi)(\xi,t)| \longrightarrow 0, \quad \text{as } t\to+\infty.$$

Proof. From Proposition 3.2 and Proposition 4.1, we get the existence of a unique global solution $(\phi,\psi) \in X([0,+\infty))$ satisfying inequalities (4.1) and (4.2) for any $t \geq 0$, provided that $\|(\phi_0,\psi_0)\|_2$ and β^{-1} are chosen small enough. Furthermore, $\|(\phi_{\xi\xi},\psi_{\xi\xi})(t)\|$ is uniformly bounded over $[0,+\infty)$ due to (4.1). By the *Sobolev* embedding theorem, we obtain

$$\sup_{\xi\in\Re_+} |(\phi_\xi,\psi_\xi)(\xi,t)|^2 \leq 2\{\|\phi_\xi(t)\|\|\phi_{\xi\xi}(t)\| + \|\psi_\xi(t)\|\|\psi_{\xi\xi}(t)\|\} \longrightarrow 0,$$

as $t\to+\infty$. This completes the proof of Theorem 4.1.

Proof of Theorem 2.1. From Theorem 4.1, Theorem 2.1 is obtained at once.

Remark. In terms of proposition 4.1, naturally we have

$$\int_0^t \psi^2(0,\tau)d\tau \leq \frac{C}{|s_-|^{3+\sigma}}(\|(\phi_0,\psi_0)\|_2^2 + e^{-c_-\beta}), \tag{4.53}$$

$$\int_0^t\int_0^\infty V_\xi\phi^2 d\xi d\tau \leq \frac{C}{|s_-|^{2+\sigma}}(\|(\phi_0,\psi_0)\|_2^2 + e^{-c_-\beta}). \tag{4.54}$$

In fact, multiplying $(3.10)_1$ by $f(V)\phi$ and $(3.10)_2$ by ψ, we have

$$\begin{aligned} & -\frac{s}{2}f'(V)V_\xi\phi^2 + (\frac{s_-}{2}\psi^2)_\xi \\ = \ & (\frac{1}{2}f(V)\phi^2 + \frac{1}{2}\psi^2)_t + \frac{\mu}{V}\psi_\xi^2 \\ & -(\frac{s_-}{2}f(V)\phi^2 + f(V)\phi\psi + \frac{\mu}{V}\psi_\xi\psi)_\xi \\ & +f'(V)V_\xi\phi\psi - \frac{\mu}{V^2}V_\xi\psi_\xi\psi - F\psi. \end{aligned} \tag{4.55}$$

It is observed that

$$\int_0^t\int_0^\infty V_\xi\bar{\psi}^2 d\xi d\tau \geq \frac{1}{2}\int_0^t\int_0^\infty V_\xi\psi^2 d\xi d\tau - c|s_-|^2\int_0^t\int_0^\infty V_\xi\phi^2 d\xi d\tau. \tag{4.56}$$

and

$$\begin{aligned} f'(V) & = -p''(V) + \tfrac{h'(V)V-h(V)}{V^2} \\ & \leq -p''(V) + \tfrac{h'(V)}{V} \\ & \leq -p''(V) - \tfrac{s^2+p'(V)}{V} \\ & \leq -\gamma^2 V_+^{-\gamma-2}. \end{aligned} \tag{4.57}$$

Integrating (4.55) over $[0,+\infty)\times[0,t]$, from (4.21),(4.56),(4.57) and proposition 4.1, we have

$$\int_0^t\int_0^\infty V_\xi\phi^2 d\xi d\tau+|s_-|\int_0^t\psi^2(0,\tau)d\tau\le\frac{C}{|s_-|^{2+\sigma}}(\|(\phi_0,\psi_0)\|_2^2+e^{-c_-\beta}). \quad (4.58)$$

5 Superposition of viscous shock wave and BL-solution

In this section, we investigate the case

$$(v_-,u_-)\in\Omega_{sub},\qquad (v_+,u_+)\in BLS_2(v_-,u_-), \tag{5.1}$$

where a superposition of the 2-viscous shock wave and the BL-solution is expected to be an asymptotic state of the solution to (1.2). In this case, there is $(\bar v,\bar u)\in BL(v_-,u_-)$ such that $(v_+,u_+)\in S_2(\bar v,\bar u)$ and there are the BL-solution $(V_b,U_b)(\xi)$ satisfying (2.6) and the 2-viscous shock wave $(V_s,U_s)(\xi-(s-s_-)t+\alpha-\beta)$ satisfying (2.3), where $\alpha=\alpha(\beta)$ is determined by (2.11). Let

$$\begin{aligned}V(\xi,t;\alpha,\beta)&=V_b(\xi)+V_s(\xi-(s-s_-)t+\alpha-\beta)-\bar v,\\ U(\xi,t;\alpha,\beta)&=U_b(\xi)+U_s(\xi-(s-s_-)t+\alpha-\beta)-\bar u,\end{aligned} \tag{5.2}$$

and

$$\begin{aligned}\phi(\xi,t)&=-\int_\xi^\infty\big[v(y,t)-V(y,t;\alpha,\beta)\big]dy,\\ \psi(\xi,t)&=-\int_\xi^\infty\big[u(y,t)-U(y,t;\alpha,\beta)\big]dy.\end{aligned} \tag{5.3}$$

Substituting (5.3) into (2.9), and integrating the system on $[\xi,+\infty)$ with respect to ξ, we have

$$\begin{cases}\phi_t-s_-\phi_\xi-\psi_\xi=0, & \text{in } \Re_+\times\Re_+,\\ \psi_t-s_-\psi_\xi-f_1(V)\phi_\xi-\dfrac{\mu}{V}\psi_{\xi\xi}=F_1+Q_1, & \text{in } \Re_+\times\Re_+,\\ (\phi,\psi)\big|_{t=0}=(\phi_0,\psi_0),\\ \phi|_{\xi=0}=A(t),\\ \psi_\xi\big|_{\xi=0}=A'(t)+s_-(V(-(s-s_-)t+\alpha-\beta)-v_-),\end{cases} \tag{5.4}$$

where

$$f_1(V)=f_1(V_s,V_b)=-p'(V)+\frac{h(V_s)}{V}+\frac{g(V_b)}{V}, \tag{5.5}$$

$$\begin{aligned}F_1=&-\{p(V+\phi_\xi)-p(V)-p'(V)\phi_\xi\}\\ &+(\mu\psi_{\xi\xi}+h(V_s)\phi_\xi+g(V_b)\phi_\xi)\big(\tfrac{1}{V+\phi_\xi}-\tfrac{1}{V}\big),\end{aligned} \tag{5.6}$$

$$Q_1=-\{p(V)-p(V_s)-p(V_b)+p(\bar v)\}+h(V_s)\frac{V_b-\bar v}{V+\phi_\xi}+g(V_b)\frac{V_s-\bar v}{V+\phi_\xi}, \tag{5.7}$$

$$A(t) = -(s - s_-)\int_t^\infty \big(V_s((s_- - s)\tau + \alpha - \beta) - \bar v\big) d\tau, \tag{5.8}$$

with $h(V_s) = \frac{\mu s V_s'}{V_s}$ and $g(V_b) = \frac{\mu s_- V_b'}{V_b}$. It is easy to see that (3.15) and Lemma 4.1 still hold here. Furthermore, from proposition 2.1 and 2.2, we have

$$\begin{aligned}
&\int_0^\infty \int_0^\infty |Q_1| d\xi dt \\
\le\ & \int_0^\infty \int_0^\infty |V_s(\xi - (s - s_-)t + \alpha - \beta) - \bar v||V_b(\xi) - \bar v| d\xi dt \\
\le\ & C\int_0^\infty \int_0^{(s-s_-)t+\beta-\alpha} e^{-c|\xi|} e^{-c_-|\xi-(s-s_-)t+\alpha-\beta|} d\xi dt \\
& + C\int_0^\infty \int_{(s-s_-)t+\beta-\alpha}^\infty e^{-c|\xi|} d\xi dt \\
\le\ & Ce^{-c\beta}.
\end{aligned} \tag{5.9}$$

As in Lemma 4.3, let $\overline{\psi} = \psi - s_-\phi$, then (5.4) becomes

$$\begin{cases}
\phi_t - 2s_-\phi_\xi - \overline{\psi}_\xi = 0, & \text{in } \Re_+ \times \Re_+, \\
\overline{\psi}_t - (f_1(V) - s_-^2)\phi_\xi - \frac{\mu}{V}\overline{\psi}_{\xi\xi} - s_-\frac{\mu}{V}\phi_{\xi\xi} = F_1 + Q_1, & \text{in } \Re_+ \times \Re_+, \\
(\phi, \overline{\psi})\big|_{t=0} = (\phi_0, \psi_0 - s_-\phi_0), \\
\phi\big|_{\xi=0} = A(t), \\
\overline{\psi}_\xi\big|_{\xi=0} = A'(t).
\end{cases} \tag{5.10}$$

Multiplying $(5.10)_1$ by ϕ and $(5.10)_2$ by $\big(f_1(V) - s_-^2\big)^{-1}\overline{\psi}$, we have

$$\begin{aligned}
&(\frac{1}{2}\phi^2)_t - (s_-\phi^2)_\xi - (\phi\overline{\psi})_\xi + \left\{\frac{1}{2}\frac{1}{f_1(V) - s_-^2}\overline{\psi}^2\right\}_t \\
&- \left(\frac{1}{2(f_1(V) - s_-^2)}\right)_t \overline{\psi}^2 - \left(\frac{\mu}{V(f_1(V) - s_-^2)}\overline{\psi}_\xi\overline{\psi}\right)_\xi \\
&+ \frac{\mu}{V(f_1(V) - s_-^2)}\overline{\psi}_\xi^2 + \left(\frac{\mu}{V(f_1(V) - s_-^2)}\right)_\xi \overline{\psi}_\xi\overline{\psi} \\
&- \left(s_-\frac{\mu}{V(f_1(V) - s_-^2)}\phi_\xi\overline{\psi}\right)_\xi + s_-\frac{\mu}{V(f_1(V) - s_-^2)}\phi_\xi\overline{\psi}_\xi \\
&+ s_-\left(\frac{\mu}{V(f_1(V) - s_-^2)}\right)_\xi \phi_\xi\overline{\psi} \\
&= (F_1 + Q_1)\overline{\psi}\left(f_1(V) - s_-^2\right)^{-1}.
\end{aligned} \tag{5.11}$$

Let $|v_- - \bar v| = \tilde\delta$. Noting that $V_t = V_{st}$, the term $-\left(\frac{1}{2(f_1(V) - s_-^2)}\right)_t$ can only control the term which includes $V_{s\xi}$. By the discussion above, we have

$$\begin{aligned}
&\int_0^{+\infty}\left\{\frac{\phi^2}{2}+\frac{\overline{\psi}^2}{2(f_1(V)-s_-^2)}\right\}d\xi+\int_0^t\int_0^{+\infty}[\tilde{Z}(V_s)+O(1)(|s_-|+\tilde{\delta})]V_{s\xi}\overline{\psi}^2d\xi d\tau\\
&+\int_0^t\int_0^{+\infty}\frac{c\mu\overline{\psi}_\xi^2}{V(f_1(V)-s_-^2)}d\xi d\tau\\
&\le C\int_0^{+\infty}\left\{\phi_0^2+\overline{\psi}_0^2\right\}d\xi+C\int_0^t\int_0^{+\infty}V_{b\xi}^2\bar{\psi}^2d\xi d\tau\\
&+\left|\int_0^t\{s_-\phi^2+\phi\overline{\psi}+\frac{\mu\overline{\psi}_\xi\overline{\psi}}{V(f_1(V)-s_-^2)}+\frac{s_-\mu\phi_\xi\overline{\psi}}{V(f_1(V)-s_-^2)}\}|_{\xi=0}d\tau\right|\\
&+\int_0^t\int_0^{+\infty}(F_1+Q_1)\overline{\psi}\left(f_1(V)-s_-^2\right)^{-1}d\xi d\tau\\
&+\int_0^t\int_0^{+\infty}\frac{c\mu|s_-|}{V(f_1(V)-s_-^2)}\phi_\xi^2d\xi d\tau,
\end{aligned}\tag{5.12}$$

where

$$\tilde{Z}(V_s)=\frac{s}{2}(\frac{V_s}{K(V_s)})'-\frac{\mu K'(V_s)^2V_{s\xi}}{4\varepsilon K(V_s)^3}\ge C>0,\tag{5.13}$$

with

$$K(V_s)=-p'(V_s)V_s+h(V_s),\tag{5.14}$$

if (2.25) holds and

$$\max\{\frac{(\gamma-1)^2(v_+-\bar{v})}{2\gamma\bar{v}},\frac{1}{2}\}\le\varepsilon<1.\tag{5.15}$$

Now we estimate the term $\int_0^t\int_0^{+\infty}V_{b\xi}^2\bar{\psi}^2d\xi d\tau$ by the idea of Kawashima and Nikkuni [2]. Since

$$\psi(\xi,t)=\psi(0,t)+\int_0^\xi\psi_\xi(y,t)dy,\tag{5.16}$$

we have

$$|\psi(\xi,t)|\le C(|\psi(0,t)|+\xi^{\frac{1}{2}}\|\psi_\xi\|),\tag{5.17}$$

which yields, from proposition 2.2,

$$\begin{aligned}
&\int_0^t\int_0^{+\infty}V_{b\xi}^2\bar{\psi}^2d\xi d\tau\\
\le\;& C\int_0^t\int_0^{+\infty}V_{b\xi}^2[\psi(0,t)^2+|s_-|^2\phi(0,t)^2+\xi(|s_-|^2\|\phi_\xi\|^2+\|\psi_\xi\|^2)]d\xi d\tau\\
\le\;& C\tilde{\delta}^2\int_0^t(\psi(0,t)^2+\|\psi_\xi\|^2+|s_-|^2\|\phi_\xi\|^2)d\tau+Ce^{-c\beta}.
\end{aligned}\tag{5.18}$$

Hence, (5.9) and (5.12) give

$$
\begin{aligned}
&\|(\phi,\psi)(t)\|^2+\int_0^t\|\psi_\xi\|^2d\tau+\int_0^t\int_0^\infty[\tilde{Z}(V_s)+O(1)(|s_-|+\tilde{\delta})]V_{s\xi}\psi^2d\xi d\tau\\
&\le C\left\{\|(\phi_0,\psi_0)\|^2+e^{-c_-\beta}+|s_-|\int_0^t\|\phi_\xi\|^2d\tau+|s_-|^2\int_0^t\int_0^\infty V_{s\xi}\phi^2d\xi d\tau\right.\\
&\quad\left.+\tilde{\delta}^2\int_0^t\psi(0,\tau)^2d\tau+N(T)\int_0^t[\|\phi_\xi\|^2+\|\psi_{\xi\xi}\|^2]d\tau\right\},
\end{aligned}
\tag{5.19}
$$

where we have used the fact that

$$
\int_0^t\int_0^\infty V_{s\xi}\bar{\psi}^2d\xi d\tau\ge\frac{1}{2}\int_0^t\int_0^\infty V_{s\xi}\psi^2d\xi d\tau-c|s_-|^2\int_0^t\int_0^\infty V_{s\xi}\phi^2d\xi d\tau. \tag{5.20}
$$

Suppose that $\tilde{\delta}$ is small enough such that $\tilde{\delta}\le o(|s_-|)$. Multiplying $(5.4)_1$ by $f_1(V)\phi$ and $(5.4)_2$ by ψ yields

$$
\begin{aligned}
&-\frac{s-s_-}{2}\frac{\partial}{\partial V_s}f_1(V_s,V_b)V_{s\xi}\phi^2+(\frac{s_-}{2}\psi^2)_\xi-\tfrac{1}{2}s_-[f_1(V)]_\xi\phi^2\\
=\;&(\frac{1}{2}f_1(V)\phi^2+\frac{1}{2}\psi^2)_t+\frac{\mu}{V}\psi_\xi^2\\
&-(\frac{s_-}{2}f_1(V)\phi^2+f_1(V)\phi\psi+\frac{\mu}{V}\psi_\xi\psi)_\xi\\
&+[f_1(V)]_\xi\phi\psi+[\frac{\mu}{V}]_\xi\psi_\xi\psi-(F_1+Q_1)\psi.
\end{aligned}
\tag{5.21}
$$

The Cauchy inequality yields

$$
\begin{aligned}
|[f_1(V)]_\xi\phi\psi|&\le\varepsilon V_{s\xi}\phi^2+C(V_{s\xi}\psi^2+V_{b\xi}\phi^2+V_{b\xi}\psi^2)\\
&\le\varepsilon V_{s\xi}\phi^2+CV_{s\xi}\psi^2+CV_{b\xi}[\psi(0,t)^2+\phi(0,t)^2\\
&\quad+\xi(\|\phi_\xi\|^2+\|\psi_\xi\|^2)],
\end{aligned}
\tag{5.22}
$$

with small constant ε, and

$$
|[\frac{\mu}{V}]_\xi\psi_\xi\psi|\le C(\psi_\xi^2+V_{s\xi}\psi^2+V_{b\xi}\psi^2). \tag{5.23}
$$

Integrating (5.21) over $[0,+\infty)\times[0,t]$ and using (5.18) and (5.19), we have

$$
\begin{aligned}
&\int_0^t V_{s\xi}\phi^2d\xi d\tau+|s_-|\int_0^t\psi(0,\tau)^2d\tau\\
\le\;&C\left\{\|(\phi,\psi)(t)\|^2+\int_0^t\|\psi_\xi\|^2d\tau+\int_0^t\int_0^\infty V_{s\xi}\psi^2d\xi d\tau\right\}\\
&+C\left\{\|(\phi_0,\psi_0)\|^2+e^{-c_-\beta}+|s_-|\int_0^t\|\phi_\xi\|^2d\tau\right.\\
&\left.+N(T)\int_0^t[\|\phi_\xi\|^2+\|\psi_{\xi\xi}\|^2]d\tau\right\},
\end{aligned}
\tag{5.24}
$$

where we have used the fact that $\frac{\partial}{\partial V_s}f_1(V_s,V_b)=f'(V_s)+O(1)\tilde{\delta}$ with $f'(V_s)<-\gamma^2\bar{v}^{-\gamma-2}$ due to (4.57).

Substituting (5.24) into (5.19), we have

$$
\begin{aligned}
&[1 - c(|s_-|^2 + \frac{\tilde{\delta}^2}{|s_-|})](\|(\phi,\psi)(t)\|^2 + \int_0^t \|\psi_\xi\|^2 d\tau + \int_0^t \int_0^\infty V_{s\xi}\psi^2 d\xi d\tau) \\
&\le C\left\{ \|(\phi_0,\psi_0)\|^2 + e^{-c-\beta} + |s_-| \int_0^t \|\phi_\xi\|^2 d\tau \right. \\
&\quad \left. + N(T) \int_0^t [\|\phi_\xi\|^2 + \|\psi_{\xi\xi}\|^2] d\tau \right\},
\end{aligned}
\tag{5.25}
$$

which yields, if $|s_-|$ is small,

$$
\begin{aligned}
&\|(\phi,\psi)(t)\|^2 + \int_0^t \|\psi_\xi\|^2 d\tau + \int_0^t \int_0^\infty V_{s\xi}\psi^2 d\xi d\tau \\
&\le C\left\{ \|(\phi_0,\psi_0)\|^2 + e^{-c-\beta} + |s_-| \int_0^t \|\phi_\xi\|^2 d\tau \right. \\
&\quad \left. + N(T) \int_0^t [\|\phi_\xi\|^2 + \|\psi_{\xi\xi}\|^2] d\tau \right\}.
\end{aligned}
\tag{5.26}
$$

The estimates of higher order derivatives are also obtained, though the calculations are rather tedious. Therefore theorem 2.2 is proved.

Remark. Theorem 2.2 implies that the BL-solution is necessarily weak though the shock wave is not necessary to be weak.

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