A gas-solid free boundary problem for compressible viscous gas

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Abstract

In this paper we propose a gas-solid free boundary problem for a one-dimensional model system of compressible viscous gas associated with the inflow problem where the solid constantly transits into the gas and produces the inflow of the gas on the free boundary. We first show the existence of the travelling wave solution and its asymptotic stability. We further discuss the case the asymptotic state of the solution is given by a combination of the travelling wave solution and rarefaction wave.

1 Introduction

We consider the one-dimensional barotropic motion of compressible viscous gas, and study the situation where the gas is adjacent to the solid by the free boundary $\tilde{x} = X(t)$. In what follows we use the *Eulerian* coordinates $(\tilde{x}, t)$ to represent the spacial and time variables. We assume that the part $\tilde{x} < X(t)$ is filled by the solid whose density $\bar{\rho}$ is a given positive constant, velocity $\bar{u}$ is zero, and the part $\tilde{x} > X(t)$ is filled by the gas with the density $\tilde{\rho}(\tilde{x}, t)$ and velocity $\tilde{u}(\tilde{x}, t)$ satisfying the conservation of mass

$$\tilde{\rho}_t + (\tilde{\rho}\tilde{u})_{\tilde{x}} = 0, \tag{1.1}$$

and the conservation of momentum

$$(\tilde{\rho}\tilde{u})_t + (\tilde{\rho}\tilde{u}^2 + \tilde{p} - \mu\tilde{u}_{\tilde{x}})_{\tilde{x}} = 0, \tag{1.2}$$

where the pressure $\tilde{p}$ is a given function of $\tilde{\rho}$, the viscosity coefficient μ is a given positive constant. Here we should note that in the solid part $\tilde{\rho} = \bar{\rho}$ and $\tilde{u} = \bar{u} = 0$ also satisfy the equations (1.1) and (1.2). On the free boundary, so that the conservation of mass holds, we assume the Rankine-Hugoniot condition

$$-\frac{dX(t)}{dt}(\tilde{\rho} - \bar{\rho}) + (\tilde{\rho}\tilde{u} - \bar{\rho}\bar{u}) = 0 \quad \text{on } \tilde{x} = X(t)$$

which implies

$$\frac{dX(t)}{dt} = \frac{\tilde{\rho}\tilde{u}}{\tilde{\rho} - \bar{\rho}}. \tag{1.3}$$

Depending on the sign of $X'(t)$ and also whether $\tilde{\rho}$ is more or less than $\bar{\rho}$, we can consider various situations. Among them we here study the situation

$$\tilde{\rho} < \bar{\rho}, \qquad \frac{dX(t)}{dt} < 0. \tag{1.4}$$

It is noted that the first condition is physically natural because the density of the gas transited in phase from the solid is usually much less than that of the solid, and the second condition makes us to focus on the phase transition process from the solid to the gas. In this situation, since the density in front of the free boundary is more than that in back of the boundary, the discontinuity of the solution at the boundary is not a classical shock, but a *detonation* type discontinuity. By (1.3) we also know the velocity $\tilde{u}$ on the boundary is positive, which means the gas constantly flows in from the free boundary, so this situation is closely related to the inflow problem on the half space for (1.1),(1.2)(cf. [3][16]). We discuss on this matter more later. Since we need another boundary condition on the density in this inflow situation, we assume

$$\tilde{\rho} = \rho_b \quad \text{on } \tilde{x} = X(t), \qquad \tilde{p}(\rho_b) = \bar{p} := \tilde{p}(\bar{\rho}), \tag{1.5}$$

where $\rho_b\,(< \bar{\rho})$ is a given positive constant, and the second assumption means the phase transition occurs along the equi-pressure level like Maxwell line in van der Waals isotherm (cf. [1,2]). We also assume

$$\tilde{p}(\tilde{\rho}) > 0, \quad \tilde{p}'(\tilde{\rho}) > 0, \quad \tilde{p}''(\tilde{\rho}) \geq 0, \qquad \text{in a neibourhood of } \tilde{\rho} = \rho_b \tag{1.6}$$

which means that once the solid changes to the gas it follows the standard property of pressure-density relation as barotropic gas. Since we need another boundary condition to determine the movement of the free boundary and the data of the density and momentum on the boundary, we also assume another Rankin-Hugoniot condition for the conservation of momentum

$$-\frac{dX}{dt}\tilde{\rho}\tilde{u} + \tilde{\rho}\tilde{u}^2 + \tilde{p} - \mu\tilde{u}_{\tilde{x}} - \bar{p} = 0 \quad \text{on } \tilde{x} = X(t). \tag{1.7}$$

By (1.3),(1.5) and (1.7), we have a nonlinear Neumann condition for the velocity

$$\mu\tilde{u}_{\tilde{x}} = \frac{\bar{\rho}\rho_b}{\bar{\rho} - \rho_b}\tilde{u}^2. \tag{1.8}$$

We know that to be more physical, especially to unify the arguments of fluid dynamical aspects and that of Stephan problems, we should further take into account the conservation of energy and its Rankin-Hugoniot condition with a jump of energy structure. However even for 2×2 systems as (1.1)(1.2) there have been no mathematical results under the free boundary condition as (1.3) because of various difficulties. In all previous works (e.g. Kazhikhov [8], Nagasawa [17]) they assume

$$\frac{dX(t)}{dt} = \tilde{u} \tag{1.9}$$

which means the particles on the free boundary always stay on the boundary. In this case, if we introduce the Lagrangian mass coordinates, we can reformulate the problem to that with the fixed boundary. On the other hand in our case (1.3) we can not do it, which gives us a main difficulty. We believe that to investigate a simple mathematical model as (1.1)(1.2) under the conditions (1.3)(1.8) is a quite meaningful one-step to overcome mathematical difficulties and proceed to more general problems. We should note that Kaliev and Kazhikhov [5] showed the existence and uniqueness of the local solution for a free boundary value problem which does not satisfy (1.9). But in their case they assumed that there is no discontinuity of the density. That is $\tilde{\rho} = \rho_b$ on the boundary. Thus we propose and concentrate on the following system in the *Eulerian* coordinates

$$\begin{cases} \tilde{\rho}_t + (\tilde{\rho}\tilde{u})_{\tilde{x}} = 0, & \text{in } \tilde{x} > X(t), \\ (\tilde{\rho}\tilde{u})_t + (\tilde{\rho}\tilde{u}^2 + \tilde{p})_{\tilde{x}} = \mu \tilde{u}_{\tilde{x}\tilde{x}}, & \text{in } \tilde{x} > X(t), \\ \tilde{\rho}(X(t),t) = \rho_b, & \\ \mu \tilde{u}_{\tilde{x}}(X(t),t) = \dfrac{\rho_b \bar{\rho}}{\bar{\rho} - \rho_b} \tilde{u}^2(X(t),t), & \\ \dfrac{dX(t)}{dt} = \dfrac{\rho_b}{\rho_b - \bar{\rho}} \tilde{u}(X(t),t), & X(0) = 0, \\ (\tilde{\rho},\tilde{u})|_{(+\infty,t)} = (\rho_+, u_+), & \\ (\tilde{\rho},\tilde{u})|_{t=0} = (\rho_0, u_0). & \end{cases} \tag{1.10}$$

In this paper we first investigate the existence of the travelling wave solution of (1.10), and establish its asymptotic stability. We further discuss the case the asymptotic state of the solution is expected to consist of the travelling wave solution and rarefaction wave.

We now state the relations of (1.10) to the inflow problem with the fixed boundary and recall the results on it. We consider the case $X(t) = \bar{s}t (\bar{s} < 0)$ is given. Let $y = \tilde{x} - \bar{s}t$, $z(y,t) = \tilde{u}(\tilde{x},t) - \bar{s}$, $\rho(y,t) = \tilde{\rho}(\tilde{x},t)$, then (1.3) and (1.5) yield the following boundary condition

$$\rho|_{y=0} = \rho_b > 0, \quad z|_{y=0} = -\frac{\bar{\rho}\bar{s}}{\rho_b} =: u_b > 0.$$

Thus (1.10) is changed into

$$\begin{cases} \rho_t + (\rho z)_y = 0, & \text{in } y > 0, \\ (\rho z)_t + (\rho z^2 + p)_y = \mu z_{yy}, & \text{in } y > 0, \\ \rho(0,t) = \rho_b > 0, & \\ z(0,t) = u_b > 0, & \\ (\rho, z)|_{(+\infty,t)} = (\rho_+, u_+ - \bar{s}), & \\ (\rho, z)|_{t=0} = (\rho_0, u_0 - \bar{s}), & \end{cases} \tag{1.11}$$

where $p = \tilde{p}(\rho(y,t))$.

The problem (1.11) is called the inflow problem with the fixed boundary. In this situation, a new wave, denoted by the boundary layer solution, or BL-solution simply, appears in the solutions due to the presence of boundary. Matsumura [10] classified all possible large time behaviors of the solutions in terms of the boundary values. Matsumura and Nishihara [16] and we [3] have proved, if the boundary values are in the subsonic region, the solution tends to the superposition of a BL-solution and a rarefaction wave as t tends to infinity, and the superposition of a BL-solution and a viscous shock wave under some smallness conditions. Shi [18] studied the rarefaction wave case when the boundary values are in the supersonic region. We refer the readers to [3,10,16,18].

We now concentrate on the free boundary problem (1.10). We here transform (1.10) to the problem in the Lagrangian coordinate:

$$\begin{cases} v_t - u_x = 0, & x > x(t), t > 0, \\ u_t + p(v)_x = \mu(\dfrac{u_x}{v})_x, & x > x(t), t > 0, \\ v(x(t),t) = v_b, & \\ \mu u_x(x(t),t) = \dfrac{v_b}{v_b - \bar{v}} u^2(x(t),t), & \\ \dfrac{dx(t)}{dt} = \dfrac{1}{\bar{v} - v_b} u(x(t),t), \quad x(0) = 0, & \\ (v,u)|_{(+\infty,t)} = (v_+, u_+) = (\dfrac{1}{\rho_+}, u_+), & \\ (v,u)|_{t=0} = (v_0, u_0), \quad v_0(0) = v_b, & \end{cases} \tag{1.12}$$

where $v = 1/\rho$. The transformation $(\tilde{x},t) \to (x,t)$ is given by

$$\begin{cases} \dfrac{\partial \tilde{x}(x,t)}{\partial t} = \tilde{u}(\tilde{x}(x,t),t), t > 0, x > 0 \\ \tilde{x}(x,0) = \tilde{x}_0(x), \end{cases}$$

with

$$\int_0^{\tilde{x}_0(x)} \tilde{\rho}(r,0) dr = x,$$

where $(\tilde{x},t) \in \sum_1 = \{(\tilde{x},t); \tilde{x} > \tilde{x}(0,t)\}$, and by

$$\begin{cases} \dfrac{\partial \tilde{x}(x,t)}{\partial t} = \tilde{u}(\tilde{x}(x,t),t), t > t_0(x), x > 0 \\ \tilde{x}(x,t_0(x)) = X(t_0(x)), \end{cases}$$

with

$$x = \int_0^{X(t_0(x))} \rho_b dr - \int_0^{t_0(x)} \rho_b \tilde{u}(X(t),t) dt = \bar{\rho} X(t_0(x)),$$

where $(\tilde{x},t) \in \sum_2 = \{(\tilde{x},t); X(t) < \tilde{x} < \tilde{x}(0,t)\}$. From the definition, we have

$$x \geq x(t) = \bar{\rho} X(t),$$

and

$$\int_{\tilde{x}(0,t)}^{\tilde{x}(x,t)} \tilde{\rho}(r,t)dr = x,$$

for $(\tilde{x},t) \in \sum_i (i=1,2)$. Hence, for $f(x,t) = \tilde{f}(\tilde{x}(x,t),t)$,

$$\frac{\partial}{\partial t} f(x,t) = (\frac{\partial}{\partial t} + u\frac{\partial}{\partial \tilde{x}})\tilde{f}(\tilde{x}(x,t),t), \quad \frac{\partial}{\partial x} f(x,t) = v\frac{\partial}{\partial \tilde{x}}\tilde{f}(\tilde{x}(x,t),t),$$

which gives (1.12).

We now seek a travelling wave solution of (1.12). The argument of Section 2 shows that for any given $0 < \bar{v} < v_b < v_+$, there exists a travelling wave solution $(V,U)(\xi)$, $\xi = x - \bar{s}t$, satisfying

$$\begin{cases} -\bar{s}V' - U' = 0, \\ -\bar{s}U' + p(V)' = \mu(\dfrac{U'}{V})', \end{cases} \tag{1.13}$$

with

$$\begin{cases} V(0) = v_b, \quad U(0) = u_b, \quad \mu U'(0) = v_b(v_b - \bar{v})\bar{s}^2, \\ V(+\infty) = v_+, \quad U(+\infty) = u_+, \end{cases} \tag{1.14}$$

provided that $u_+^2 < (v_+ - \bar{v})^2|p'(v_+)|$, where

$$\bar{s} = \frac{u_b}{\bar{v} - v_b}, \quad u_b = \frac{\bar{v} - v_b}{\bar{v} - v_+}u_+, \quad u_+ = (v_+ - \bar{v})^{\frac{1}{2}}(p(v_b) - p(v_+))^{\frac{1}{2}}. \tag{1.15}$$

We call (V,U) the travelling wave solution. For any $0 < \bar{v} < v_b < +\infty$, the travelling wave solution curve with the parameter $\bar{v}$ through the point $(v_b, 0)$ in the phase plane, is defined by

$$\begin{aligned} & TW_{\bar{v}}(v_b,0) \\ = \quad & \big\{(v,u); u = (v-\bar{v})^{\frac{1}{2}}(p(v_b) - p(v))^{\frac{1}{2}}, u^2 < (v-\bar{v})^2|p'(v)|, v > v_b\big\}. \end{aligned} \tag{1.16}$$

When $(v_+, u_+) \in TW_{\bar{v}}(v_b, 0)$, the solution of (1.12) is expected to the travelling wave solution satisfying (1.13)-(1.15) as t tends to infinity.

On the other hand, it is known that the 2-rarefaction curve $R_2(v_b, 0)$ through the point $(v_b, 0)$ is

$$R_2(v_b,0) = \big\{(v,u); u = -\int_{v_b}^{v} \lambda_2(s)ds, v < v_b\big\} \tag{1.17}$$

and $R_2(v_p, u_p)$ through the point (v_p, u_p) is

$$R_2(v_p,u_p) = \big\{(v,u); u = u_p - \int_{v_p}^{v} \lambda_2(s)ds, v < v_p\big\}, \tag{1.18}$$

where $\lambda_2 = \sqrt{-p'(v)}$ is the second characteristic speed of the corresponding hyperbolic system without viscosity and

$$u_p = (v_p - \bar{v})^{\frac{1}{2}}(p(v_b) - p(v_p))^{\frac{1}{2}}, u_p^2 = (v_p - \bar{v})^2|p'(v_p)|.$$

Then let us define $R_2TW_{\bar{v}}(v_b,0)$ the domain surrounded by curves $TW_{\bar{v}}(v_b,0)$, $R_2(v_b,0)$, $R_2(v_p,u_p)$ and u-axis. When (v_+,u_+) is in $R_2TW_{\bar{v}}(v_b,0)$, the solution of (1.12) is expected to tend to the superposition of a travelling wave solution and a 2-rarefaction wave as t tends to infinity.

Our aim is to investigate the stability of the travelling wave solution or a superposition of the travelling wave solution and the 2-rarefaction wave. Our results are, roughly speaking, as follows.

I) If $(v_+,u_+)\in TW_{\bar{v}}(v_b,0)$, then the travelling wave solution is stable provided that $|v_+-v_b|$ is small.

II) If $(v_+,u_+)\in R_2TW_{\bar{v}}(v_b,0)$, then there exists $(v_*,u_*)\in TW_{\bar{v}}(v_b,0)$ such that $(v_+,u_+)\in R_2(v_*,u_*)$, and the superposition of the travelling wave solution connecting (v_b,u_b) with (v_*,u_*) and the 2-rarefaction wave connecting (v_*,u_*) with (v_+,u_+) is stable provided that $|v_*-v_b|$ is small. That is, the travelling wave solution is necessary to be weak, however, the 2-rarefaction wave is not necessarily weak.

Our method is based on the energy estimates and a new approximation to the rarefaction wave. It is noted that the new approximation could be applied to extend the results of [16], where a stability theorem for the superposition of a BL-solution and a weak rarefaction wave with the fixed boundary was obtained by Matsumura and Nishihara, to the strong rarefaction wave.

Our plan of this paper is as follows. In Sections 2-4, we focus on the case I). Precisely speaking, in Section 2, we study the existence of the travelling wave solution. In Section 3, we reformulate the original problem to a new initial boundary value problem and prove the local existence of the solution. In Section 4, we establish the a priori estimates and then prove the stability of the travelling wave solution. In Section 5, the case II) is treated.

Notations. Throughout this paper, several positive generic constants are denoted by c, C without confusions. For function spaces, $L^p(\Omega), 1\le p\le\infty$ denotes a usual Lebesgue space on $\Omega\subset R=(-\infty,\infty)$ with its norm

$$\|f\|_{L^p(\Omega)}=(\int_\Omega |f(x)|^p dx)^{\frac{1}{p}}, 1\le p<\infty, \quad \|f\|_{L^\infty(\Omega)}=\sup_\Omega |f(x)|. \tag{1.19}$$

$H^l(\Omega)$ denotes the l-th order Sobolev space with its norm

$$\|f\|_l=(\sum_{j=0}^{l}\|\partial_x^j f\|^2)^{\frac{1}{2}}, \quad when\ \|\cdot\|:=\|\cdot\|_{L^2(\Omega)}. \tag{1.20}$$

The domain Ω will be often abbreviated without confusions.

2 Existence of the travelling wave solution

In this section we investigate the existence of the travelling wave solution $(V, U)(\xi)$, $\xi = x - \bar{s}t$ satisfying

$$\begin{cases} -\bar{s}V' - U' = 0, \qquad ' = d/d\xi,\ \xi = x - \bar{s}t, \\ -\bar{s}U' + p(V)' = \mu \left(\dfrac{U'}{V}\right)', \\ V(0) = v_b, U(0) = u_b, \\ \mu U'(0) = v_b(v_b - \bar{v})\bar{s}^2, \\ V(+\infty) = v_+, U(+\infty) = u_+, \end{cases} \tag{2.1}$$

where

$$\bar{s} = \frac{u_b}{\bar{v} - v_b} < 0. \tag{2.2}$$

We have the following existence result.

Lemma 2.1. *For any given $\bar{v}, v_b$ and v_+ with $0 < \bar{v} < v_b < v_+$, when*

$$u_b = \frac{\bar{v} - v_b}{\bar{v} - v_+} u_+, \quad u_+ = (v_+ - \bar{v})^{\frac{1}{2}} (p(v_b) - p(v_+))^{\frac{1}{2}}, \tag{2.3}$$

there exists a unique solution $(V, U)(\xi)$ to (2.1) and (2.2) satisfying $0 < v_b < V(\xi) < v_+, V' > 0$, provided that $u_+^2 < (v_+ - \bar{v})^2 |p'(v_+)|$. Furthermore, fix $\bar{v}$ and v_b, let $v_+ - v_b = \delta$, then there exists a positive constant $\delta_0 > 0$ such that, for any $\delta \leq \delta_0$,

$$|V(\xi) - v_+| = O(\delta) e^{-\frac{c}{\sqrt{\delta}}\xi}, \tag{2.4}$$

and

$$u_+, \bar{s}, u_b = O(\delta^{\frac{1}{2}}), \quad V' = O(\delta^{\frac{1}{2}}) e^{-\frac{c}{\sqrt{\delta}}\xi}. \tag{2.5}$$

Proof. When (V, U) exists, the integration of (2.1) with respect to ξ yields

$$\begin{cases} \bar{s}V + U = \bar{s}v_b + u_b = \bar{s}v_+ + u_+, \\ \mu \dfrac{U'}{V} = p(V) + \bar{s}^2(V - v_+) - p(v_+). \end{cases} \tag{2.6}$$

Let $\xi = 0$, then (2.3) holds. By (2.6) we have the ordinary equation

$$\begin{cases} -\bar{s}\mu \dfrac{V'}{V} = \mathcal{F}(V) := p(V) + \bar{s}^2(V - v_+) - p(v_+), \\ V(0) = v_b, \quad V(+\infty) = v_+. \end{cases} \tag{2.7}$$

To the contrary, since

$$u_+^2 < (v_+ - \bar{v})^2 |p'(v_+)| \tag{2.8}$$

infers $\mathcal{F}(V) > 0$ for $v_b < V < v_+$, it is easy to see there exists a unique solution (V, U) of (2.1). Furthermore, (2.3) and (2.7) infer the solution satisfies (2.4) and (2.5) if $v_+ - v_b$ is suitably small. Thus Lemma 2.1 is proved.

3 Local Existence of the solution

In this section, we first reformulate (1.12) to a new initial-boundary value problem and then prove the local existence of the solution.

Assume that $(v_+, u_+) \in TW_{\bar{v}}(v_b, 0)$, then Lemma 2.1 gives a unique solution $(V, U)(\xi), \xi = x - \bar{s}t \geq 0$ satisfying

$$\begin{cases} -\bar{s}V' - U' = 0, \\ -\bar{s}U' + p(V)' = \mu(\dfrac{U'}{V})', \\ V(0) = v_b, U(0) = u_b, \\ \mu U'(0) = v_b(v_b - \bar{v})\bar{s}^2, \\ (V, U)|_{(+\infty)} = (v_+, u_+), \end{cases} \tag{3.1}$$

where $\bar{s} = \frac{u_b}{\bar{v} - v_b} < 0$ and u_b, u_+ satisfies (2.3). To investigate the free boundary problem (1.12), we consider the coordinate transformation

$$t = t, \quad y = x - x(t), \tag{3.2}$$

where $x(t) = \bar{s}t + \gamma(t)$. By (3.2), we rewrite (1.12) as

$$\begin{cases} v_t - (\bar{s} + \gamma'(t))v_y - u_y = 0, & y > 0, t > 0, \\ u_t - (\bar{s} + \gamma'(t))u_y + p(v)_y = \mu(\dfrac{u_y}{v})_y, & y > 0, t > 0, \\ v(0, t) = v_b, \\ \mu u_y(0, t) = \dfrac{v_b}{v_b - \bar{v}} u^2(0, t), \\ \dfrac{d\gamma(t)}{dt} = \dfrac{1}{\bar{v} - v_b} u(0, t) - \bar{s}, \quad \gamma(0) = 0, \\ (v, u)|_{(+\infty, t)} = (v_+, u_+), \\ (v, u)(y, 0) = (v_0, u_0)(y), \quad y > 0. \end{cases} \tag{3.3}$$

On the other hand, we take the travelling wave solution as the form $(V, U)(y)$. Thus, by (3.1) (V, U) satisfies

$$\begin{cases} -\bar{s}V_y - U_y = 0, & y > 0, \\ -\bar{s}U_y + p(V)_y = \mu(\dfrac{U_y}{V})_y, & y > 0, \\ V(0) = v_b, U(0) = u_b, \\ \mu U_y(0) = v_b(v_b - \bar{v})\bar{s}^2, \\ (V, U)|_{(+\infty)} = (v_+, u_+). \end{cases} \tag{3.4}$$

We now put the perturbation $(\phi, \psi)(y, t)$ by

$$(v, u)(y, t) = (V, U)(y) + (\phi, \psi)(y, t), \tag{3.5}$$

so that the reformulated problem is

$$\begin{cases} \phi_t - (\bar{s} + \gamma'(t))\phi_y - \psi_y = \gamma'(t)V'(y), \;\; y > 0, t > 0, \\ \psi_t - (\bar{s} + \gamma'(t))\psi_y + (p(V+\phi) - p(V))_y \\ -\mu\Big(\dfrac{\psi_y}{V+\phi} + \dfrac{\bar{s}V'(y)\phi}{V(V+\phi)}\Big)_y = -\bar{s}\gamma'(t)V'(y), \;\; y > 0, t > 0, \\ \phi(0,t) = 0, \\ \mu\psi_y(0,t) = \dfrac{2v_b u_b}{v_b - \bar{v}}\psi(0,t) + \dfrac{v_b}{v_b - \bar{v}}\psi^2(0,t), \\ \gamma'(t) = \dfrac{1}{\bar{v} - v_b}\psi(0,t), \quad \gamma(0) = 0, \\ (\phi,\psi)|_{(+\infty,t)} = (0,0), \\ (\phi,\psi)(y,0) = (\phi_0,\psi_0) = (v_0 - V, u_0 - U)(y). \end{cases} \tag{3.6}$$

We shall combine the local existence and the a priori estimates to investigate the stability of the travelling wave solution. The a priori estimates will be treated in the next section. Here we only study the problem of the local existence.

The solution space is

$$\begin{aligned} X_{m,M}(0,T) \;\; &= \big\{\phi \in C(0,T;H_0^1), \psi \in C(0,T;H^1) | \phi_y \in L^2(0,T;L^2), \\ &\quad \psi_y \in L^2(0,T;H^1), \text{with } \sup_{[0,T]} \|(\phi,\psi)(t)\|_1 \le M, \\ &\quad \inf_{R_+ \times [0,T]} (V+\phi)(y,t) \ge m\big\}, \end{aligned} \tag{3.7}$$

for some positive constants m, M.

Proposition 3.1.(Local Existence) *For any given $0 < \bar{v} < v_b < +\infty$, there exist positive constants δ_0 and M_0 such that if $v_+ - v_b = \delta \le \delta_0$, $\phi_0 \in H_0^1$, $\psi_0 \in H^1$, $\|(\phi_0,\psi_0)\|_1 \le M (\le \frac{M_0}{b})$ and $\inf_{\mathbb{R}_+}(V+\phi_0) \ge m$, then there exists a positive constant $T_0 = T(m, M_0)$ such that there exists a unique solution $(\phi(y,t), \psi(y,t), \gamma(t))$ to (3.6) satisfying*

$$\phi(y,t), \psi(y,t) \in X_{\frac{m}{2}, bM}(0,T_0), \quad \gamma(t) \in C^1([0,T_0])$$

where $b = 3(1 + 2\sqrt{\frac{v_b}{\mu}} + \sqrt{\frac{2v_b}{v_b - \bar{v}}})$.

Proof. We first consider the characteristic equation of $(3.6)_1$. When $y \ge -x(t)(x(t) = \bar{s}t + \gamma(t))$, the characteristic starts from the y-axis. That is, for any $\bar{x}_0 \ge 0$, the characteristic $y(\bar{x}_0,t)$ from $(\bar{x}_0,0)$ satisfies

$$\begin{cases} \dfrac{dy(\bar{x}_0,t)}{dt} = -\bar{s} - \gamma'(t), \\ y(\bar{x}_0,0) = \bar{x}_0, \end{cases} \tag{3.8}$$

which yields

$$y(\bar{x}_0,t) = -x(t) + \bar{x}_0. \tag{3.9}$$

Thus, ϕ has the explicit form

$$\begin{aligned}\phi(y,t) = \quad & \phi_0(\bar{x}_0) + \int_0^t \psi_y(\bar{x}_0 - x(\tau), \tau) d\tau \\ & + \int_0^t \gamma'(\tau) V'(\bar{x}_0 - x(\tau)) d\tau, \quad \bar{x}_0 = y + x(t).\end{aligned} \tag{3.10}$$

In the same way, when $0 \leq y \leq -x(t)$, the characteristic starts from the t-axis. It is noted that the inverse function of $x(t)$ exists if $|\gamma'(t)| = |\frac{1}{\bar{v}-v_b}\psi(0,t)| \leq c\|\psi(t)\|_1$ is small. Thus, by $(3.6)_1$ and the boundary condition $\phi(0,t) = 0$, we have

$$\begin{aligned}\phi(y,t) = \quad & \int_{\bar{t}_0}^t \psi_y(x(\bar{t}_0) - x(\tau), \tau) d\tau \\ & + \int_{\bar{t}_0}^t \gamma'(\tau) V'(x(\bar{t}_0) - x(\tau)) d\tau, \quad \bar{t}_0 = x^{-1}(y + x(t)).\end{aligned} \tag{3.11}$$

On the other hand, the equation $(3.6)_2$ is regarded as the initial-boundary value problem for the parabolic equation of ψ:

$$\begin{cases} \psi_t - \mu(\dfrac{\psi_y}{V+\phi})_y = g := g(\phi, \phi_y, \psi(0,t), \psi_y) \\ \mu\psi_y(0,t) = \dfrac{2v_b u_b}{v_b - \bar{v}}\psi(0,t) + \dfrac{v_b}{v_b - \bar{v}}\psi^2(0,t), \\ \psi|_{t=0} = \psi_0, \end{cases} \tag{3.12}$$

where

$$\begin{aligned} & g(\phi, \phi_y, \psi(0,t), \psi_y) \\ = \quad & -(p(V+\phi) - p(V))_y + \left(\bar{s} + \frac{1}{\bar{v} - v_b}\psi(0,t)\right)\psi_y \\ & + \mu\left(\frac{U_y}{V+\phi} - \frac{U_y}{V}\right)_y - \frac{\bar{s}}{\bar{v} - v_b}\psi(0,t)V'(y). \end{aligned}$$

By virtue of the boundary condition of (3.6), $\gamma(t)$ has the explicit form

$$\gamma(t) = \int_0^t \frac{1}{\bar{v} - v_b}\psi(0,\tau) d\tau. \tag{3.13}$$

We now approximate $\phi_0 \in H_0^1, \psi_0 \in H^1$ by $\phi_{0k} \in H^3 \cap H_0^1, \psi_{0k} \in H^3$ such that

$$(\phi_{0k}, \psi_{0k}) \to (\phi_0, \psi_0), \quad \textit{strongly} \quad \textit{in} \quad H^1, \tag{3.14}$$

as $k \to \infty$ and $\|(\phi_{0k}, \psi_{0k}\|_1 \leq \frac{3}{2}M, \inf_{R_+}(V + \phi_{0k}) \geq \frac{2}{3}m$ hold for any k.

We will use the iteration method to prove Proposition 3.1. We define the sequence $\{(\phi_k^{(n)}(y,t), \psi_k^{(n)}(y,t), \gamma_k^{(n)}(t))\}$ for each k so that

$$(\phi_k^{(0)}, \psi_k^{(0)})(y,t) = (\phi_{0k}, \psi_{0k})(y), \quad \gamma_k^{(0)}(t) = \int_0^t \frac{1}{\bar{v} - v_b}\psi_{0k}(0) d\tau \tag{3.15}$$

and for a given $((\phi_k^{(n-1)},\psi_k^{(n-1)})(y,t),\gamma_k^{(n-1)}(t))$, $\psi_k^{(n)}$ is a solution to

$$
\begin{cases}
\psi_{kt}^{(n)}-\mu(\dfrac{\psi_{ky}^{(n)}}{V+\phi_k^{(n-1)}})_y=g^{(n-1)}=g(\phi_k^{(n-1)},\phi_{ky}^{(n-1)},\psi_k^{(n-1)}(0,t),\psi_{ky}^{(n-1)})\\
\mu\psi_{ky}^{(n)}(0,t)=\dfrac{2v_bu_b}{v_b-\bar{v}}\psi_k^{(n)}(0,t)+\dfrac{v_b}{v_b-\bar{v}}[\psi_k^{(n)}(0,t)]^2,\\
\psi_k^{(n)}|_{t=0}=\psi_{0k},
\end{cases}
\tag{3.16}
$$

$$\gamma_k^{(n)}(t)=\int_0^t\frac{1}{\bar{v}-v_b}\psi_k^{(n)}(0,\tau)d\tau \tag{3.17}$$

and

$$
\phi_k^{(n)}(y,t)=\begin{cases}
\displaystyle\int_{\bar{t}_k^{(n-1)}}^t\psi_{ky}^{(n)}(x_k^{(n-1)}(\bar{t}_k^{(n-1)})-x_k^{(n-1)}(\tau),\tau)d\tau\\
\displaystyle+\int_{\bar{t}_k^{(n-1)}}^t[\gamma_k^{(n-1)}(\tau)]'V'(x_k^{(n-1)}(\bar{t}_k^{(n-1)})-x_k^{(n-1)}(\tau))d\tau,\\
if\quad 0\le y\le -x_k^{(n-1)}(t)\\
\displaystyle\phi_{0k}(\bar{x}_k^{(n-1)})+\int_0^t\psi_{ky}^{(n)}(\bar{x}_k^{(n-1)}-x_k^{(n-1)}(\tau),\tau)d\tau\\
\displaystyle+\int_0^t[\gamma_k^{(n-1)}(\tau)]'V'(\bar{x}_k^{(n-1)}-x_k^{(n-1)}(\tau))d\tau,\\
if\quad y\ge -x_k^{(n-1)}(t),
\end{cases}
\tag{3.18}
$$

where

$$
\begin{aligned}
x_k^{(n-1)}(t)&=\bar{s}t+\gamma_k^{(n-1)}(t),\\
\bar{t}_k^{(n-1)}&=(x_k^{(n-1)})^{-1}(y+x_k^{(n-1)}(t)),\\
\bar{x}_k^{(n-1)}&=y+x_k^{(n-1)}(t).
\end{aligned}
$$

We now assume M_0 is small. By the principle of contraction mapping, it is easy to prove there exists a positive time $t_0(m,M_0)<<1$ such that, if $g^{(n-1)}\in C(0,t_0;H^2)$ and $\psi_{0k}\in H^3$, there exists a unique-local solution $\psi_k^{(n)}$ to (3.16) satisfying

$$\psi_k^{(n)}\in C(0,t_0;H^3)\cap C^1(0,t_0;H^1)\cap L^2(0,t_0;H^4)$$

and $|\psi_k^{(n)}(y,t)|\le CM_0$.

Thus, if $(\phi_k^{(n-1)},\psi_k^{(n-1)})\in X_{\frac{1}{2}m,bM}$, multiplying (3.16) by $\psi_k^{(n)}$, and integrating it over R_+, we have

$$
\begin{aligned}
&\|\psi_k^{(n)}(t)\|_t^2+\frac{\mu}{v_+}\|\psi_{ky}^{(n)}\|^2+\frac{4v_bu_b}{v_b-\bar{v}}[\psi_k^{(n)}(0,t)]^2\\
\le\ &2\int_0^\infty|g^{(n-1)}\psi_k^{(n)}|dy+C|\psi_k^{(n)}(0,t)|^3\\
\le\ &C(m,M_0)(1+\|\psi_k^{(n)}(t)\|^2)+\frac{2v_bu_b}{v_b-\bar{v}}[\psi_k^{(n)}(0,t)]^2.
\end{aligned}
\tag{3.19}
$$

Thus, we have $\|\psi_k^{(n)}(t)\|^2 \leq (2M)^2$ due to $t_0 = t_0(m, M_0) << 1$ and

$$\int_0^{t_0} \|\psi_{ky}^{(n)}\|^2 dt \leq \frac{v_+}{\mu}(2M)^2. \tag{3.20}$$

Multiplying (3.16) by $-\psi_{kyy}^{(n)}$ and integrating it over $\mathbb{R}_+$, one has, if δ_0 is suitably small,

$$\begin{aligned}&(\|\psi_{ky}^{(n)}\|^2 + \frac{2v_b u_b}{v_b - \bar{v}}[\psi_k^{(n)}(0,t)]^2 + \frac{2v_b}{3(v_b - \bar{v})}[\psi_k^{(n)}(0,t)]^3)_t \\ &+\frac{\mu}{2v_+}\|\psi_{kyy}^{(n)}(t)\|^2 \leq C(m, M_0) + \|\psi_{ky}^{(n)}\|^2.\end{aligned} \tag{3.21}$$

The integration of (3.21) over $(0,t)$ gives

$$\begin{aligned}&\|\psi_{ky}^{(n)}\|^2 + \frac{2v_b u_b}{v_b - \bar{v}}[\psi_k^{(n)}(0,t)]^2 + \frac{2v_b}{3(v_b - \bar{v})}[\psi_k^{(n)}(0,t)]^3 \\ &+\frac{\mu}{2v_+}\int_0^t \|\psi_{kyy}^{(n)}(\tau)\|^2 d\tau \\ \leq\ & \|\psi_{0k}\|_1^2 + \frac{2v_b u_b}{v_b - \bar{v}}[\psi_{0k}(0)]^2 + \frac{2v_b}{3(v_b - \bar{v})}|\psi_{0k}(0)|^3 \\ &+C(m, M_0)t + \int_0^t \|\psi_{ky}^{(n)}\|^2 d\tau \\ \leq\ & (2M)^2(1 + \frac{2v_+}{\mu} + \frac{2v_b u_b}{v_b - \bar{v}}) \leq (2M)^2(1 + \frac{4v_b}{\mu} + \frac{2v_b}{v_b - \bar{v}}).\end{aligned} \tag{3.22}$$

Thus, we have

$$\|\psi_{ky}^{(n)}\|^2 \leq (3(1 + 2\sqrt{\frac{v_b}{\mu}} + \sqrt{\frac{2v_b}{v_b - \bar{v}}})M)^2 = (bM)^2, \tag{3.23}$$

and

$$\int_0^{t_0} \|\psi_{kyy}^{(n)}\|^2 dt \leq \frac{2v_+}{\mu}(bM)^2. \tag{3.24}$$

On the other hand, direct estimate on (3.18) together with (3.20),(3.23) and (3.24) gives

$$\|\phi_k^{(n)}(t)\|_1^2 \leq (3M)^2, \tag{3.25}$$

and $\inf_{R_+\times[0,T]}(V + \phi_k^{(n)})(y,t) \geq \frac{1}{2}m$.

Therefore, $(\phi_k^{(n)}, \psi_k^{(n)}) \in X_{\frac{1}{2}m, bM}(0, t_0)$. From (3.17), it is easy to see $\gamma_k^{(n)}(t) \in C^1([0, t_0])$. By a standard way, $(\phi_k^{(n)}, \psi_k^{(n)})$ is a Cauchy sequence in $C(0, t_0; H^2)$. Thus we have a solution $(\phi_k(y,t), \psi_k(y,t), \gamma_k(t))$ by letting n tends to infinity, where $\gamma_k(t) = \int_0^t \frac{1}{\bar{v} - v_b}\psi_k(0,\tau)d\tau$. In the same way, letting $k \to \infty$, we obtain the desired unique-local solution $(\phi(y,t), \psi(y,t), \gamma(t))$ to (3.6) which satisfies $(\phi, \psi) \in X_{\frac{1}{2}m, bM}(0, T_0)$ and $\gamma(t) \in C^1([0, T_0])$(taking T_0 smaller than t_0 if necessary).

4 Stability of the travelling wave solution

This section is devoted to the stability of the travelling wave solution. Our stability theorem is

Theorem 4.1.(Stability theorem) *For any given* $0 < \bar{v} < v_b < +\infty$, *there exist positive constants* δ_1 *and* C_1. *If* $(v_+, u_+) \in TW_{\bar{v}}(v_b, 0)$, $v_+ - v_b = \delta \leq \delta_1$, $v_0 - V \in H_0^1$, $u_0 - U \in H^1$ *and* $\|v_0 - V, u_0 - U\|_1 \leq C_1\delta$, *then there exists a global solution* $(v(x,t), u(x,t), x(t))$ *to (1.12) satisfying*

$$(v - V, u - U)(x,t) \in C(0, +\infty; H^1(x(t), +\infty)),$$

$$(v - V)_x(x,t) \in L^2(0, +\infty; L^2(x(t), +\infty)),$$

$$(u - U)_x(x,t) \in L^2(0, +\infty; H^1(x(t), +\infty)), x(t) \in C^1([0, +\infty)),$$

$$x'(t) - \bar{s} \in L^2(0, +\infty),$$

and

$$\sup_{x \geq x(t)} |(v,u)(x,t) - (V,U)(x - x(t))| \to 0, \quad as \quad t \to +\infty.$$

Theorem 4.1 is derived directly from the local existence (Proposition 3.1) and the following a priori estimates.

Proposition 4.2.(A priori estimates) *For any given* $\bar{v}$ *and* v_b, *there exist positive constants* δ_1 *and* C_0. *If* $v_+ - v_b = \delta \leq \delta_1 (\leq \delta_0)$, $(\phi, \psi) \in X_{\frac{1}{4}v_b, bM}(0,T)$ *with* $M \leq \frac{C_0}{b}\delta$ *is a solution to (3.6) for some positive constant* T, *then*

$$\begin{aligned} &\|(\phi,\psi)(t)\|_1^2 + \delta^{\frac{1}{2}} \int_0^t \psi^2(0,\tau)d\tau + \delta^{\frac{1}{2}} \int_0^t \int_0^\infty \phi^2 V'(y)dyd\tau \\ &+ \int_0^t \|\phi_y(\tau)\|^2 d\tau + \int_0^t \|\psi_y(\tau)\|_1^2 d\tau \leq C\|(\phi_0,\psi_0)\|_1^2 \end{aligned} \tag{4.1}$$

where $C > 1$ *is a constant.*

Proof. By virtue of the previous section, (ϕ, ψ) satisfies

$$\begin{cases} \phi_t - (\bar{s} + \gamma'(t))\phi_y - \psi_y = \gamma'(t)V'(y), \;\; y > 0, t > 0, \\ \psi_t - (\bar{s} + \gamma'(t))\psi_y + (p(V+\phi) - p(V))_y \\ -\mu\left(\dfrac{\psi_y}{V+\phi} + \dfrac{\bar{s}V'(y)\phi}{V(V+\phi)}\right)_y = -\bar{s}\gamma'(t)V'(y), \;\; y > 0, t > 0, \\ \phi(0,t) = 0, \\ \mu\psi_y(0,t) = \dfrac{2v_b u_b}{v_b - \bar{v}}\psi(0,t) + \dfrac{v_b}{v_b - \bar{v}}\psi^2(0,t), \\ \gamma'(t) = \dfrac{1}{\bar{v} - v_b}\psi(0,t), \quad \gamma(0) = 0, \\ (\phi,\psi)|_{(+\infty,t)} = (0,0), \\ (\phi,\psi)(y,0) = (\phi_0,\psi_0) = (v_0 - V, u_0 - U)(y). \end{cases} \tag{4.2}$$

Let

$$\Phi(v,V)=p(V)\phi-\int_V^{V+\phi}p(s)ds.$$

Multiplying the first equation of (4.2) by $(p(V)-p(V+\phi))$ and the second one by ψ, we have

$$\begin{aligned}
&\{[\frac{1}{2}\psi^2+\Phi(v,V)]\}_t-(\bar{s}+\gamma')\Delta\phi V_y\\
&+\{(p(V+\phi)-p(V))\psi-(\bar{s}+\gamma')(\Phi(v,V)+\frac{1}{2}\psi^2)\\
&-\mu(\frac{\psi_y}{V+\phi}+\frac{\bar{s}V_y\phi}{V(V+\phi)})\psi\}_y+\mu\frac{\psi_y^2}{V+\phi}\\
&+\frac{\mu\bar{s}V_y\psi_y\phi}{V(V+\phi)}+(p(V+\phi)-p(V))\gamma'V_y+\bar{s}\gamma'V_y\psi=0,
\end{aligned}\tag{4.3}$$

where $\Delta\phi=p(V+\phi)-p(V)-p'(V)\phi$ satisfies $c\phi^2\le\Delta\phi\le C\phi^2$.

Let $E=\int_0^\infty\frac{1}{2}\psi^2+\Phi(v,V)dy$. The integration of (4.3) over $R_+\times[0,T]$ gives

$$\begin{aligned}
&E(t)-E(0)+\mu\int_0^t\int_0^\infty\frac{\psi_y^2}{V+\phi}dyd\tau-\int_0^t\int_0^\infty(\bar{s}+\gamma')\Delta\phi V_ydyd\tau\\
&+\int_0^t[\frac{1}{2}(\bar{s}+\gamma')\psi(0,\tau)+\mu\frac{\psi_y(0,\tau)}{v_b}]\psi(0,\tau)d\tau+\int_0^t\int_0^\infty\frac{\mu\bar{s}V_y\psi_y\phi}{V(V+\phi)}dyd\tau\\
&+\int_0^t\int_0^\infty(p(V+\phi)-p(V))\gamma'V_ydyd\tau+\int_0^t\int_0^\infty\bar{s}\gamma'V_y\psi dyd\tau=0.
\end{aligned}\tag{4.4}$$

Since $\bar{s}=O(\delta^{\frac{1}{2}})$ is negative and $|\gamma'|=O(\delta)$, we have

$$-(\bar{s}+\gamma')\Delta\phi V_y\ge c\delta^{\frac{1}{2}}\phi^2V_y.\tag{4.5}$$

On the other hand, the boundary conditions of (4.2) give

$$\mu\frac{\psi_y(0,\tau)}{v_b}\psi(0,\tau)=\frac{2u_b}{v_b-\bar{v}}\psi^2(0,\tau)+\frac{1}{v_b-\bar{v}}\psi^3(0,\tau)\ge-\frac{3}{2}\bar{s}\psi^2(0,\tau),\tag{4.6}$$

which infers

$$[\frac{1}{2}(\bar{s}+\gamma')\psi(0,\tau)+\mu\frac{\psi_y(0,\tau)}{v_b}]\psi(0,\tau)\ge-\frac{1}{2}\bar{s}\psi^2(0,\tau)\ge c\delta^{\frac{1}{2}}\psi^2(0,\tau).\tag{4.7}$$

Thus, from (4.4)-(4.7), we have

$$\begin{aligned}
&E(t)-E(0)+\mu\int_0^t\int_0^\infty\frac{\psi_y^2}{V+\phi}dyd\tau+c\delta^{\frac{1}{2}}\int_0^t\int_0^\infty\phi^2V_ydyd\tau\\
&+c\delta^{\frac{1}{2}}\int_0^t\psi^2(0,\tau)d\tau=-\int_0^t\int_0^\infty\frac{\mu\bar{s}V_y\psi_y\phi}{V(V+\phi)}dyd\tau\\
&-\int_0^t\int_0^\infty(p(V+\phi)-p(V))\gamma'V_ydyd\tau-\int_0^t\int_0^\infty\bar{s}\gamma'V_y\psi dyd\tau.
\end{aligned}\tag{4.8}$$

We now estimate each term of the right hand side of (4.8). From Lemma 2.1 and

$$c\phi^2 \le \Phi(v,V) \le C\phi^2, \tag{4.9}$$

we have

$$\frac{\mu\bar{s}V_y\psi_y\phi}{V(V+\phi)} \le \lambda\psi_y^2 + C\bar{s}^2\phi^2V_y^2 \le \lambda\psi_y^2 + C\delta^{\frac{3}{2}}\phi^2V_y, \tag{4.10}$$

$$\begin{aligned} &\int_0^\infty |(p(V+\phi)-p(V))\gamma' V_y|dy \\ \le\ & C\int_0^\infty |\gamma' V_y\phi|dy \le C|\gamma'|\|\phi_y\|\int_0^\infty y^{\frac{1}{2}}V_y dy \\ \le\ & C\delta^{\frac{5}{4}}|\gamma'|\|\phi_y\| \le \lambda\delta^{\frac{1}{2}}|\gamma'|^2 + C\delta^2\|\phi_y\|^2, \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} \int_0^\infty \bar{s}\gamma' V_y\psi dy &= -\bar{s}\gamma'\int_0^\infty (V-v_+)\psi_y dy \\ &\quad -\bar{s}(v_b-v_+)\gamma'\psi(0,\tau) \\ &\le \lambda\|\psi_y\|^2 + C\delta^{\frac{3}{2}}\psi^2(0,\tau), \end{aligned} \tag{4.12}$$

where we have used the fact that

$$|\phi(y,\tau)| = |\int_0^y \phi_y dy| \le y^{\frac{1}{2}}\|\phi_y\|,$$

and λ is a suitably small positive constant.

Combining (4.8)-(4.12), we get the basic lemma.

Lemma 4.3. *Let the conditions of Proposition 4.2 hold, then*

$$\begin{aligned} &\|(\phi,\psi)(t)\|^2 + \int_0^t \|\psi_y(\tau)\|^2 d\tau + \delta^{\frac{1}{2}}\int_0^t \psi^2(0,\tau)d\tau \\ &+\delta^{\frac{1}{2}}\int_0^t\int_0^{+\infty} V_y\phi^2 dy d\tau \\ \le\ & C\|(\phi,\psi)(0)\|^2 + C\delta^2\int_0^t \|\phi_y(\tau)\|^2 d\tau. \end{aligned} \tag{4.13}$$

Following [14], we adapt a new variable

$$\tilde{v} = \frac{v}{V}, \tag{4.14}$$

to estimate $\|\phi_y\|$. Then the equation $(4.2)_2$ is rewritten as

$$\begin{aligned} &\big(\mu\frac{\tilde{v}_y}{\tilde{v}} - \psi\big)_t - (\bar{s}+\gamma'(t))\big(\mu\frac{\tilde{v}_y}{\tilde{v}} - \psi\big)_y - p'(v)V\tilde{v}_y + (p'(V)-p'(v)\tilde{v})V_y \\ &= \mu(\frac{V_y}{V})_y\gamma'(t) - \bar{s}\gamma'(t)V_y. \end{aligned} \tag{4.15}$$

Thus, we have

Lemma 4.4. *It holds that*

$$
\begin{aligned}
&\|\frac{\tilde v_y}{\tilde v}(t)\|^2+\int_0^t\int_0^{+\infty}\frac{\tilde v_y^2}{\tilde v^{\gamma+2}}dyd\tau\\
&\le C(\|\phi_0\|_1^2+\|\psi_0\|^2).
\end{aligned}
\tag{4.16}
$$

Proof. Multiplying (4.15) by $\frac{\tilde v_y}{\tilde v}$, one gets

$$
\begin{aligned}
&\{\frac{\mu}{2}(\frac{\tilde v_y}{\tilde v})^2-\psi(\frac{\tilde v_y}{\tilde v})\}_t-p'(v)V\frac{\tilde v_y^2}{\tilde v}\\
&+\{\psi\frac{\tilde v_t}{\tilde v}-\frac{\mu(\bar s+\gamma'(t))}{2}(\frac{\tilde v_y}{\tilde v})^2\}_y\\
&=\frac{\psi_y^2}{v}+\frac{\bar s\phi\psi_yV_y}{vV}+(p'(V)-p'(v)\tilde v)V_y\frac{\tilde v_y}{\tilde v}\\
&+(\mu(\frac{V_y}{V})_y\gamma'-\bar s\gamma'(t)V_y)\frac{\tilde v_y}{\tilde v}+\gamma'(t)\psi_y\frac{V_y}{V}.
\end{aligned}
\tag{4.17}
$$

Since

$$
|(\frac{\tilde v_y}{\tilde v})(0,t)|=\frac{\phi_y}{v_b}=\frac{\gamma'(t)V'(0)+\psi_y(0,t)}{v_b(\bar s+\gamma'(t))}\le C|\psi(0,t)|,
\tag{4.18}
$$

$$
|(p'(V)-p'(v)\tilde v)V_y\frac{\tilde v_y}{\tilde v}|\le\lambda\tilde v_y^2+CV_y^2\phi^2,
\tag{4.19}
$$

and

$$
\begin{aligned}
&\int_0^\infty|(\frac{V_y}{V})_y\gamma'(\tau)\frac{\tilde v_y}{\tilde v}|\\
&\le\int_0^\infty\lambda\tilde v_y^2dy+C\int_0^\infty|(\frac{V_y}{V})_y|^2dy|\gamma'(\tau)|^2\\
&\le\int_0^\infty\lambda\tilde v_y^2dy+C\delta^{\frac12}|\gamma'(\tau)|^2,
\end{aligned}
\tag{4.20}
$$

where we have used $V'(0)=O(\delta^{\frac12})$ and $\psi_y(0,t)=O(\delta^{\frac12})|\psi(0,t)|$. It is noted that

$$
c_1\phi_y^2-c_2V_y^2\phi^2\le|\frac{\tilde v_y}{\tilde v}|^2\le C_1\phi_y^2+C_2V_y^2\phi^2,
$$

where $c_i,C_i,i=1,2$ are positive constants which only depend on $\bar v$ and v_b. Integrating (4.17) over $(0,+\infty)\times(0,t)$ and using Lemma 4.3, we get Lemma 4.4.

Lemma 4.5. *It holds that*

$$
\|\psi_y(t)\|^2+\int_0^t\|\psi_{yy}(\tau)\|^2d\tau\le C\|\phi_0,\psi_0\|_1^2.
\tag{4.21}
$$

Proof. Multiplying $(4.2)_2$ by $-\psi_{yy}$, we have

$$\begin{aligned}
&(\frac{\psi_y}{2})^2_t + \big(-\psi_t\psi_y + \frac{\bar{s}+\gamma'(t)}{2}\psi_y^2\big)_y + \mu\frac{\psi_{yy}^2}{v} \\
&= \{-\mu\frac{\psi_y(V_y+\phi_y)}{(V+\phi)^2} + \mu(\frac{\bar{s}V_y\phi}{(V+\phi)V})_y \\
&\quad -\big(p(V+\phi)-p(V)\big)_y\}(-\psi_{yy}) \\
&\quad +\bar{s}\gamma'(t)V_y\psi_{yy}.
\end{aligned} \tag{4.22}$$

Integrating (4.22) over $(0,+\infty)\times(0,t)$, one has

$$\begin{aligned}
&\frac{1}{2}\|\psi_y(t)\|^2 + \int_0^t \psi_t(0,\tau)\psi_y(0,\tau)d\tau - \int_0^t \frac{\bar{s}+\gamma'(t)}{2}\psi_y^2(0,\tau)d\tau \\
&+\frac{\mu}{2v_+}\int_0^t \|\psi_{yy}(\tau)\|^2 d\tau \\
\leq\ & C(\|\phi_0\|_1^2 + \|\psi_0\|_1^2) + C\int_0^t \|\phi_y(\tau)\|^2 + \|\psi_y(\tau)\|^2 d\tau \\
&+C\int_0^t\int_0^\infty \bar{s}^2V_{yy}^2\phi^2 dy d\tau.
\end{aligned} \tag{4.23}$$

By the boundary conditions of (4.2), we have

$$\psi_t(0,t)\psi_y(0,t) = \frac{v_b}{v_b-\bar{v}}\{u_b\psi^2(0,t) + \frac{1}{3}\psi^3(0,t)\}_t. \tag{4.24}$$

On the other hand,

$$\int_0^t\int_0^\infty \bar{s}^2V_{yy}^2\phi^2 dy d\tau \leq C\delta\int_0^t\int_0^\infty |V_{yy}|y dy\cdot\|\phi_y\|^2 d\tau \leq C\delta^2\int_0^t \|\phi_y\|^2 d\tau.$$

Thus, combining (4.22)-(4.24) and the fact that $\bar{s}<0$, we get Lemma 4.5.

Proposition 4.2 is obtained at once from Lemma 4.3-4.5.

Proof of Theorem 4.1. Let δ_1, C_0, C and M_0 are the constants in Proposition 3.1 and 4.2. We assume $C_0\delta_1 \leq M_0$. This is possible because we can choose δ_1 smaller if necessary. We now let $v_+ - v_b = \delta \leq \delta_1$ and the initial data $\|(\phi_0,\psi_0)\|_1 \leq M \leq \frac{C_0\delta}{\sqrt{C}b} \leq M_0/b$. Then by Proposition 3.1, there exists a positive time $t_0 = t_0(v_b, M_0)$ such that there is a unique-local solution (ϕ,ψ,γ) to (3.6) satisfying

$$\phi(y,t), \psi(y,t) \in X_{\frac{v_b}{4}, bM}(0,t_0), \quad \gamma(t)\in C^1([0,t_0]).$$

It is noted that $bM \leq C_0\delta$. Thus, by virtue of Proposition 4.2, we have

$$\|(\phi,\psi)(t_0)\|_1 \leq \sqrt{C}\|(\phi_0,\psi_0)\|_1 \leq C_0\delta/b \leq M_0/b$$

Which satisfies the conditions of Proposition 3.1. Hence, from the time $t = t_0$, again using Proposition 3.1 and 4.2, we know there exists a solution (ϕ, ψ, γ) to (3.6) in the interval $[t_0, 2t_0]$ and

$$\|(\phi, \psi)(2t_0)\|_1 \leq \sqrt{C}\|\phi_0, \psi_0\|_1 \leq C_0\delta/b \leq M_0/b.$$

Repeating above procedure, we obtain the asymptotic stability theorem 4.1.

5 Superposition of a travelling wave solution and a rarefaction wave

This section is devoted to the superposition of a travelling wave solution and a rarefaction wave. Assume that

$$(v_+, u_+) \in R_2TW_{\bar{v}}(v_b, 0), \tag{5.1}$$

where $0 < \bar{v} < v_b$.

In this case, there exists $(v_*, u_*) \in TW_{\bar{v}}(v_b, 0)$ such that $(v_+, u_+) \in R_2(v_*, u_*)$, and the superposition of the travelling wave solution connecting (v_b, u_b) with (v_*, u_*) and the 2-rarefaction wave connecting (v_*, u_*) with (v_+, u_+).

We now consider the 2-rarefaction wave $(v^R, u^R)(\frac{x}{t})$ connecting (v_*, u_*) with (v_+, u_+), which is the weak solution of the *Riemann* problem

$$\begin{cases} v_t - u_x = 0, \quad (x,t) \in \Re \times (0, +\infty), \\ v_t + p(v)_x = 0, \\ (v, u)\big|_{t=0} = (v_0^R, u_0^R)(x) = \begin{cases} (v_*, u_*), & x < 0, \\ (v_+, u_+), & x > 0. \end{cases} \end{cases} \tag{5.2}$$

It is known that $(v^R, u^R)(\frac{x}{t})$ has the explicit form

$$\begin{aligned} &(v^R, u^R)(\frac{x}{t}) \\ &= \begin{cases} (v_*, u_*), & -\infty \leq \dfrac{x}{t} \leq \lambda_2(v_*), \\ \big(\lambda_2^{-1}(\dfrac{x}{t}), u_* - \displaystyle\int_{v_*}^{\lambda_2^{-1}(\frac{x}{t})} \lambda_2(s)ds\big), & \lambda_2(v_*) \leq \dfrac{x}{t} \leq \lambda_2(v_+), \\ (v_+, u_+), & \lambda_2(v_+) \leq \dfrac{x}{t} \leq +\infty, \end{cases} \end{aligned} \tag{5.3}$$

where $\lambda_2(v) = \sqrt{-p'(v)}$.

To study the large time behavior of solutions to (1.12), it is necessary to construct a smooth approximate rarefaction wave $(\tilde{V}, \tilde{U})(x, t)$ of $(v^R, u^R)(\frac{x}{t})$ in $\Re \times (0, +\infty)$, and its restriction $(V^R, U^R)(\frac{x}{t}) := (\tilde{V}, \tilde{U})(x, t)\big|_{x \geq x(t)}$. For this reason, we investigate the *Riemann* problem on $\Re \times (0, +\infty)$ of *Burgers* equation

$$\begin{cases} w_t^R + w^R w_x^R = 0, \quad (x,t) \in \Re \times \Re_+, \\ w^R(x, 0) = w_0^R(x) = \begin{cases} w_- = \lambda_2(v_*), \ x < 0, \\ w_+ = \lambda_2(v_+), \ x > 0. \end{cases} \end{cases} \tag{5.4}$$

Here $0 < w_- < w_+$. The weak solution of (5.4) with the entropy condition is a rarefaction wave $w^R(\frac{x}{t})$ connecting w_- and w_+

$$w^R(\frac{x}{t}) = \begin{cases} w_-, & x \leq w_- t, \\ \dfrac{x}{t}, & w_- t < x < w_+ t, \\ w_+. & w_+ t \leq x. \end{cases} \tag{5.5}$$

We now approximate $w^R(\frac{x}{t})$ by

$$\begin{cases} w_t + w w_x = 0, \quad (x,t) \in \Re \times (0, +\infty), \\ w\big|_{t=0} = w_0(x) \\ \qquad = \begin{cases} w_-, & x < 0, \\ w_- + \tilde{w}\kappa_q \displaystyle\int_0^{\varepsilon x} z^q e^{-z} dz, & x \geq 0. \end{cases} \end{cases} \tag{5.6}$$

Here $\tilde{w} = w_+ - w_-$, κ_q is a constant such that $\kappa_q \int_0^{+\infty} z^q e^{-z} dz = 1$ for large constant $q \geq 8$ and ε is a positive constant determined later. We have

Lemma 5.1 *Let $0 < w_- < w_+$, then the problem (5.6) has a unique smooth solution $w(x,t)$ satisfying*

i. $w_- \leq w(x,t) < w_+, w_x \geq 0, for\ x \geq 0, t \geq 0.$

ii. *For any $p (1 \leq p \leq +\infty)$, there exists a constant $C_{p,q}$ such that $for\ t \geq 0$*
$\|w_x(\cdot,t)\|_{L^p} \leq C_{p,q} \min\left(\tilde{w}\varepsilon^{1-\frac{1}{p}}, \tilde{w}^{\frac{1}{p}} t^{-1+\frac{1}{p}}\right),$
$\|w_{xx}(\cdot,t)\|_{L^p} \leq C_{p,q} \min\left(\tilde{w}\varepsilon^{2-\frac{1}{p}}, \tilde{w}^{\frac{1}{q}} \varepsilon^{1-\frac{1}{p}+\frac{1}{q}} t^{-1+\frac{1}{q}}\right).$

iii. *when* $x \leq 0, w(x,t) - w_- = w_x(x,t) = w_{xx}(x,t) = 0.$

iv. $\limsup\limits_{t \to +\infty, x \in \Re} \left| w(x,t) - w^R(x,t) \right| = 0.$

Proof. Since the solution $w(x,t)$ of (5.6) has the explicit form

$$w(x,t) = w_0(x_0(x,t)), x = x_0(x,t) + w_0(x_0(x,t))t,$$

and

$$w_0'(x_0) = \begin{cases} 0, & x_0 \leq 0 \\ \tilde{w}k_q \varepsilon (\varepsilon x_0)^q e^{-\varepsilon x_0}, & x_0 > 0, \end{cases}$$

$$|w_0''(x_0)| \leq C \tilde{w}^{\frac{1}{q}} \varepsilon^{1+\frac{1}{q}} |w_0'(x_0)|^{1-\frac{1}{q}} e^{-\frac{\varepsilon x_0}{2q}},$$

it is not difficult to get Lemma 5.1 by virtue of the method of [14]. We omit the detail here.

Now we define the approximate solution $(\tilde{V}, \tilde{U})(x,t)$ as follows

$$(\tilde{V}, \tilde{U})(x,t) = \left(\lambda_2^{-1}\big(w(x,t)\big), u_* - \int_{v_*}^{\lambda_2^{-1}\left(w(x,t)\right)} \lambda_2(s) ds\right). \tag{5.7}$$

Setting

$$(V^R, U^R)(y,t) := (\tilde{V}, \tilde{U})\big|_{x \geq x(t)}, \quad y = x - x(t). \tag{5.8}$$

Since $w(x,t)$ is the smooth solution of the problem (5.6), it is easy to see

$$\begin{cases} V_t^R - (\bar{s} + \gamma'(t))V_y^R - U_y^R = 0, \\ U_t^R - (\bar{s} + \gamma'(t))U_y^R + p(V^R)_y = 0, \quad (y,t) \in R_+ \times (0, +\infty), \\ (V^R, U^R)\big|_{y=0} = (v_*, u_*), \\ (V^R, U^R)\big|_{t=0} = \left(\lambda_2^{-1}(w_0(y)), u_* - \int_{v_*}^{\lambda_2^{-1}(w_0(y))} \lambda_2(s)ds\right). \end{cases} \tag{5.9}$$

Note that $|V_{yy}^R| \leq C(|w_{xx}| + |w_x|^2)$, one has from Lemma 5.1,

Lemma 5.2. *Let $\delta_2 = |v_+ - v_*| + |u_+ - u_*|$, then $(V^R, U^R)(y,t)$ satisfies, if $q \geq p$,*

i. $U_y^R(y,t) \geq 0, |U_y^R| \leq C\varepsilon\delta_2,$ *for* $t \geq 0, y \geq 0,$

ii. *For any $p(1 \leq p \leq +\infty)$, there exist a constant $C_{p,q}$ such that*
$\|V_y^R(\cdot,t)\|_{L^p(\{y\geq 0\})} \leq C_{p,q} \min\{\delta_2 \varepsilon^{1-\frac{1}{p}}, \delta_2^{\frac{1}{p}}(1+t)^{-1+\frac{1}{p}}\},$
$\|V_{yy}^R(\cdot,t)\|_{L^p(\{y\geq 0\})} \leq C_{p,q} \min\{\delta_2 \varepsilon^{2-\frac{1}{p}}, \delta_2^{\frac{1}{q}}(1+t)^{-1+\frac{1}{q}}\}, \quad t \geq 0.$

iii. $(V^R, U^R)|_{y \leq -x(t)} = (v_*, u_*), \quad (V_y^R, U_y^R, V_{yy}^R, U_{yy}^R)|_{y \leq -x(t)} = 0.$

iv. $\displaystyle \limsup_{t \to +\infty, y \in \{y \geq 0\}} |(V^R, U^R)(y,t) - (v^R, u^R)(\frac{y + x(t)}{t})| = 0.$

On the other hand, the travelling wave solution $(V_B, U_B)(y)$ connecting (v_b, u_b) with (v_*, u_*) satisfies, from Lemma 2.1,

$$\begin{cases} -\bar{s}V_y - U_y = 0, & y > 0, \\ -\bar{s}U_y + p(V)_y = \mu(\dfrac{U_y}{V})_y, & y > 0, \\ V(0) = v_b, U(0) = u_b, \\ \mu U_y(0) = v_b(v_b - \bar{v})\bar{s}^2, \\ (V,U)|_{(+\infty)} = (v_*, u_*), \end{cases} \tag{5.10}$$

where

$$\bar{s} = \frac{u_b}{\bar{v} - v_b} < 0, u_b = \frac{\bar{v} - v_b}{\bar{v} - v_*}u_*, u_* = (v_* - \bar{v})^{\frac{1}{2}}(p(v_b) - p(v_*))^{\frac{1}{2}}. \tag{5.11}$$

Let

$$V(y,t) = V_B(y) + V^R(y,t) - v_*, U(y,t) = U_B(y) + U^R(y,t) - u_*, \tag{5.12}$$

we set the perturbation $(\phi, \psi)(y,t)$ by $(v,u)(y,t) = (V + \phi, U + \psi)(y,t)$. Then the reformulated equation is, from (1.12), (5.9) and (5.10),

$$\begin{cases} \phi_t - (\bar{s} + \gamma'(t))\phi_y - \psi_y = \gamma'(t)V_{By}, \ \ y > 0, t > 0, \\ \psi_t - (\bar{s} + \gamma'(t))\psi_y + (p(V+\phi) - p(V))_y \\ -\mu\big(\dfrac{\psi_y}{V+\phi} - \dfrac{U_y\phi}{V(V+\phi)}\big)_y = -\bar{s}\gamma'(t)V_{By} + G(y), \ \ y > 0, t > 0, \\ \phi(0,t) = 0, \\ \mu\psi_y(0,t) = \dfrac{2v_b u_b}{v_b - \bar{v}}\psi(0,t) + \dfrac{v_b}{v_b - \bar{v}}\psi^2(0,t), \\ \gamma'(t) = \dfrac{1}{\bar{v} - v_b}\psi(0,t), \ \ \gamma(0) = 0, \\ (\phi,\psi)|_{(+\infty,t)} = (0,0), \\ (\phi,\psi)(y,0) = (\phi_0,\psi_0) = (v_0 - V_0, u_0 - U_0)(y), \end{cases} \tag{5.13}$$

where

$$G = -(p(V) - p(V_B) - p(V^R) + p(v_*)) + \mu\Big(\frac{U_y}{V} - \frac{U_{By}}{V_B}\Big) = -G_1 + G_2. \tag{5.14}$$

We now derive the a priori estimates like Proposition 4.2. First we fix $\bar{v}$ and v_b. Then we choose a suitably small constant δ_0 which will be given later. We assume that $v_* - v_b = \delta \leq \delta_0$ and $(\psi,\psi) \in X_{\frac{1}{4}v_b,M}(0,T)$ is a solution to (5.13) with $M \leq C_0\delta^{\frac{3}{5}}$ for some positive constants T and C_0.

Multiplying $(5.13)_1$ by $p(V) - p(V+\phi)$ and $(5.13)_2$ by ψ, we have

$$\begin{aligned} &\{[\tfrac{1}{2}\psi^2 + \Phi(v,V)]\}_t + \Delta\phi U_y^R - (\bar{s}+\gamma')\Delta\phi V_{By} \\ &+\{(p(V+\phi) - p(V))\psi - (\bar{s}+\gamma')(\Phi(v,V) + \tfrac{1}{2}\psi^2) \\ &-\mu\big(\frac{\psi_y}{V+\phi} - \frac{U_y\phi}{V(V+\phi)}\big)\psi\}_y + \mu\frac{\psi_y^2}{V+\phi} \\ &-\frac{\mu(U_y^R + U_{By})\psi_y\phi}{V(V+\phi)} + (p(V+\phi) - p(V))\gamma' V_{By} \\ &+\bar{s}\gamma' V_{By}\psi - G_y\psi = 0. \end{aligned} \tag{5.15}$$

Since $p''(V) > 0$, one has

$$\Delta\phi = p(V+\phi) - p(V) - p'(V)\phi = f(v,V)\phi^2 \geq 0. \tag{5.16}$$

We regard

$$Q = \Delta\phi U_y^R - \frac{\mu U_y^R\psi_y\phi}{V(V+\phi)} + \mu\frac{\psi_y^2}{v} \tag{5.17}$$

as the quadratic equation:

$$\begin{aligned} &(\sqrt{\mu}\frac{\psi_y}{\sqrt{v}})^2 - \frac{\sqrt{\mu U_y^R}}{V\sqrt{vf(v,V)}} \cdot \sqrt{\mu}\frac{\psi_y}{\sqrt{v}} \cdot \sqrt{U_y^R f(v,V)}\phi \\ &+(\sqrt{U_y^R f(v,V)}\phi)^2. \end{aligned} \tag{5.18}$$

By Lemma 5.2, if ε is suitably small, the discriminate of (5.18) satisfies

$$D = \frac{\mu U_y^R}{V^2 v f(v,V)} - 4 < 0. \tag{5.19}$$

It is noted that all terms including $V_B(y)$ could be treated by the same methods of the previous section, thus if δ_0 is suitably small, then we have

$$\begin{aligned}
&\|(\phi,\psi)(t)\|^2 + \int_0^t \|\psi_y(\tau)\|^2 d\tau + \delta^{\frac{1}{2}} \int_0^t \psi^2(0,\tau) d\tau \\
&+\delta^{\frac{1}{2}} \int_0^t \int_0^{+\infty} V_{By}\phi^2 dy d\tau + \int_0^t \int_0^{+\infty} U_y^R \phi^2 dy d\tau \\
&\leq C\|(\phi,\psi)(0)\|^2 + C\delta^2 \int_0^t \|\phi_y(\tau)\|^2 d\tau + \int_0^t \int_0^{+\infty} |G_y||\psi| dy d\tau.
\end{aligned} \tag{5.20}$$

We now estimate the last term of (5.20). We compute

$$\begin{aligned}
&|G_{1y}| = |p'(V)(V_{By} + V_y^R) - p'(V_B)V_{By} - p'(V^R)V_y^R| \\
&\leq |V_{By}||V^R - v_*| + |V_y^R||V_B - v_*|.
\end{aligned} \tag{5.21}$$

It is observed that $(V^R - v_*, V_y^R)|_{y \leq -\bar{s}t - \gamma(t)} = 0$, thus we have, from Lemma 2.1,

$$\begin{aligned}
&\|G_{1y}\|^2 = \int_{-\left(\bar{s}t+\gamma(t)\right)}^{+\infty} |G_{1y}|^2(y,t) dy \\
&\leq C \sup_{y \geq -\left(\bar{s}t+\gamma(t)\right)} \{|V^R(y,t) - v_*|^2 |V_{By}|\} \int_{-\left(\bar{s}t+\gamma(t)\right)}^{+\infty} |V_{By}| dy \\
&\quad +C \sup_{y \geq -\left(\bar{s}t+\gamma(t)\right)} \{|V_y^R(y,t)|^2 |v_* - V_B|\} \int_{-\left(\bar{s}t+\gamma(t)\right)}^{+\infty} |v_* - V_B| dy \\
&\leq C\delta^{\frac{3}{2}} \delta_2^2 e^{-ct}.
\end{aligned} \tag{5.22}$$

This infers, by $\|\psi\| \leq C_0 \delta^{\frac{3}{5}}$,

$$\begin{aligned}
&\int_0^t \int_0^{\infty} |G_{1y}||\psi| dy d\tau \\
&\leq C \int_0^t \|G_{1y}\| \|\psi\| d\tau \leq C\delta^{\frac{27}{20}}.
\end{aligned} \tag{5.23}$$

On the other hand, we calculate,

$$G_2 = \mu\Big(\frac{U_y}{V} - \frac{U_{By}}{V_B}\Big) = \mu\Big(\frac{U_y^R}{V} - \frac{U_{By}(V^R - v_*)}{VV_B}\Big). \tag{5.24}$$

and

$$\begin{aligned}
&|G_{2y}| \leq C(|V_{yy}^R| + |V_y^R|^2 + |V_y^R||V_{By}| \\
&+|U_{Byy}||V^R - v_*| + |U_{By}||V^R - v_*||V_{By}|).
\end{aligned} \tag{5.25}$$

From Lemma 5.2 and (5.25), we have

$$\int_0^\infty |G_{2y}|dy \le C(\|V^R_{yy}\|_{L^1} + \delta e^{-ct} + \varepsilon^{\frac{1}{6}}(1+t)^{-\frac{5}{6}}). \tag{5.26}$$

Thus,

$$\begin{aligned}
&\int_0^t \int_0^\infty |G_{2y}||\psi|dyd\tau \\
\le\ & C\int_0^t \|\psi\|^{\frac{1}{2}}\|\psi_y\|^{\frac{1}{2}}(\|V^R_{yy}\|_{L^1} + \delta e^{-ct} + \varepsilon^{\frac{1}{6}}(1+t)^{-\frac{5}{6}})d\tau \\
\le\ & \lambda\int_0^t \|\psi_y\|^2 + C(\int_0^t \delta^{\frac{2}{5}}\|V^R_{yy}\|^{\frac{4}{3}}_{L^1} + \delta^{\frac{26}{15}}e^{-ct} + \varepsilon^{\frac{2}{9}}\delta^{\frac{2}{5}}(1+\tau)^{-\frac{10}{9}}d\tau) \\
\le\ & \lambda\int_0^t \|\psi_y\|^2 d\tau + C\varepsilon^{\frac{1}{6}}\delta^{\frac{2}{5}} + C\delta^{\frac{26}{15}} \\
\le\ & \lambda\int_0^t \|\psi_y\|^2 d\tau + C\delta^{\frac{7}{5}},
\end{aligned} \tag{5.27}$$

if $\varepsilon = O(\delta^6)$, where λ is a small positive constant. Therefore, we have the following estimate

$$\begin{aligned}
&\|(\phi,\psi)(t)\|^2 + \int_0^t \|\psi_y(\tau)\|^2 d\tau + \delta^{\frac{1}{2}}\int_0^t \psi^2(0,\tau)d\tau \\
&+\delta^{\frac{1}{2}}\int_0^t\int_0^{+\infty} V_{By}\phi^2 dyd\tau + \int_0^t\int_0^{+\infty} U^R_y\phi^2 dyd\tau \\
&\le C\|(\phi,\psi)(0)\|^2 + C\delta^2\int_0^t \|\phi_y(\tau)\|^2 d\tau + C\delta^{\frac{27}{20}}.
\end{aligned} \tag{5.28}$$

Remark 5.3. If we choose the approximate waves (V^R, U^R) defined by (5.8) instead of the waves in [16], and let ε be suitably small, then it is not difficult to extend the stability theorem of [16] to the strong rarefaction wave by the same method.

The estimates of higher order derivatives are also obtained, though the calculations are rather tedious. Thus, we have the following theorem.

Theorem 5.4.(the case $(v_+,u_+) \in R_2TW_{\bar v}(v_b,0)$) *For any given $0<\bar v<v_b<+\infty$, assume that $(v_+,u_+) \in R_2TW_{\bar v}(v_b,0)$. Define $(V,U)(y,t)$ by (5.12). Then there exist positive constants δ_0 and C_0. If $v_*-v_b=\delta\le\delta_0$, $v_0(y)-V(y,0)\in H^1_0$, $u_0(y)-U(y,0)\in H^1$ and $\|v_0-V(y,0), u_0-U(y,0)\|_1 \le C_0\delta^{\frac{3}{5}}$, then there exists a global solution $(v(x,t),u(x,t),x(t))$ to (1.12) satisfying*

$$(v-V,u-U)(x,t)\in C(0,+\infty;H^1(x(t),+\infty)),$$

$$(v-V)_x(x,t)\in L^2(0,+\infty;L^2(x(t),+\infty)),$$

$$(u-U)_x(x,t)\in L^2(0,+\infty;H^1(x(t),+\infty)), x(t)\in C^1([0,+\infty)),$$

$$x'(t) - \bar{s} \in L^2(0, +\infty),$$

and

$$\sup_{x \geq x(t)} |(v,u)(x,t) - (V,U)(x - x(t))| \to 0, \quad as \quad t \to +\infty.$$

Remark 5.5. Theorem 5.4 implies that the 2-rarefaction wave is not necessary to be weak, though the travelling wave solution is necessarily weak.

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