WEAK SOLUTION TO PRESSURELESS TYPE SYSTEM

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ABSTRACT. In this paper, we study the Cauchy problem for a pressureless type system. The Riemann solutions only include two elementary waves: delta-wave and contact discontinuity. The existence of entropy solution is established by studying the interaction of the elementary waves and the generalized characteristics introduced in Dafermos [7].

Key Words: Pressureless type system, Generalized potential, Generalized characteristics, Delta wave

1. INTRODUCTION

We are concerned with the following pressureless type system

$$\begin{cases} \rho_t + (\rho f(u))_x = 0, \\ (\rho u)_t + (\rho u f(u))_x = 0, \end{cases} \tag{1.1}$$

with the initial data

$$(\rho, u)(x, 0) = (\rho_0(x), u_0(x)), \quad \rho_0(x) \geq 0, u_0(x) \in L^\infty(R), \tag{1.2}$$

where $\rho \geq 0$ denotes the mass density, ρu denotes the momentum and $f(\cdot)$ is a strictly increasing function. When $f(u) = u$, (1.1) is known as the model of pressureless gases.

One of the main difficulties for the mathematical analysis of (1.1) is that the solution is in general a Radon measure. This may cause the product problem of a Radon measure and a discontinuous function. Another difficulty is that the generalized variational principle introduced in [6] fails here due to the fact that there are two kinds of speeds: the physical velocity u and the characteristic speed $f(u)$, while the two speeds always coincide for usual pressureless gases.

For usual pressureless gases, the existence of weak solution was first obtained independently by Brenier and Grenier [5] and E, Rykov and Sinai [6]. Wang, Huang and Ding [18] extended their results. Recently, Boudin [4] showed that the weak solution can also be obtained as limits of solutions to a viscous pressureless model. As pointed out by [1,5-6], the convex entropy condition is not sufficient to guarantee the uniqueness of weak solution. Oleinik entropy condition is suitable instead (see [6]). Along this direction, Wang and Ding [17] first proved the uniqueness of weak solution under some restrictions on the initial data. Bouchut and James [3] also obtained similar results by different approach. More recently, F.Huang and Z.Wang [12] established the uniqueness of entropy solution for the general initial data. They showed that, besides Oleinik entropy condition, the weak continuity of energy at the initial time is also important for the uniqueness of weak

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solution. Furthermore, the uniqueness theorem of [12] is obtained in *the same space* as the existence one. We refer the readers to [12]. When $f(u) \neq u$, the system (1.1) is more complicated because there exist two kinds of speeds.

The main purpose of this paper is to prove the existence of entropy solution to (1.1),(1.2). When $f(u) \neq u$, the system (1.1) has two kinds of speeds. Thus the methods used in [3,5-6,18] are not applicable here because all these approaches are based on the fact that the two speeds always coincide. We have to seek a new approach to achieve our goal. The main novelty of our proof is that we construct a sequence of approximate solutions by piecing together a finite number of Riemann solutions with piecewise constant initial data. This idea is from the observation that the Riemann solutions only include two elementary waves: delta wave and contact discontinuity and only one wave is generated when interaction of waves takes place. Namely the total number of waves is time-decreasing. Thus we are able to construct a sequence of weak solutions (ρ_n, u_n) with piecewise constant initial data. We then prove there exists a subsequence of ρ_n (still denoted by ρ_n) such that ρ_n weakly converge to some ρ in $\mathfrak{M}$, the space of Radon measure. If there exists a function $u(x,t) \in L^\infty(\mathrm{BV}(R), R)$ such that $\rho_n u_n, \rho_n f(u_n)$ and $\rho_n u_n f(u_n)$ weakly converge to $\rho u, \rho f(u)$ and $\rho u f(u)$, respectively, then our proof is completed. Fortunately, we prove such function $u(x,t)$ exists by subtle analysis on the interaction of elementary waves and the generalized characteristics introduced in Dafermos [7].

When $f(u) = u$, the system (1.1) is the model of pressureless gases. Wang, Ding and myself [18] proposed a definition of weak solution by generalized potential. When $f(u) \neq u$, we use the same idea of [18] to give the definition of weak solution.

If ρ and u are bounded measurable functions, then the generalized potential

$$m(x,t) = \oint_{(0,0)}^{(x,t)} \rho dx - \rho f(u) dt, \tag{1.3}$$

is independent of integral path and satisfies $m_x = \rho$. Therefore, by the new variable $m(x,t)$, the system (1.1) is changed into

$$\begin{cases} m_t + f(u) m_x = 0, \\ (m_x u)_t + (m_x u f(u))_x = 0 \end{cases} \tag{1.4}$$

with the initial data

$$(m,u)|_{t=0} = (\int_0^x \rho_0(y) dy, u_0(x)). \tag{1.5}$$

We focus our attentions on (1.4) and (1.5) instead of (1.1) and (1.2). Thus our definition of weak solution is

Definition 1.1. *Assume that $m(x,t)$ is locally of bounded variation in x and $u(x,t)$ is bounded measurable to m_x, $(\rho,u) = (m_x, u)$ is called a weak solution of (1.1) and (1.2), if*

$$\begin{cases} \int\int \varphi_t m dx dt - \int\int \varphi f(u) dm dt = 0, \\ \int\int (\psi_t u + \psi_x u f(u)) dm dt = 0, \end{cases} \tag{1.6}$$

hold for any $\varphi, \psi \in C_0^\infty(R_+^2)$, *where* $\int\int \cdots dmdt$ *denotes Lebesgue-Stieltjes integral. The initial data is understood in the following sense: for any* $\phi(x) \in C_0^\infty(R)$, *as* $t \to 0$,

$$\int_{-\infty}^{\infty} \phi(x) dm(x,t) \longrightarrow \int_{-\infty}^{\infty} \phi(x)\rho_0(x)dx, \tag{1.7}$$

$$\int_{-\infty}^{\infty} \phi(x)u(x,t)dm(x,t) \longrightarrow \int_{-\infty}^{\infty} \phi(x)\rho_0(x)u_0(x)dx. \tag{1.8}$$

Remark. This definition is a natural extension of the classical definition of weak solution. When $m(x,t)$ is absolutely continuous in x, then $(\rho = m_x, u)$ is the classical weak solution to (1.1),(1.2). However, if $m(x,t)$ is locally of bounded variation in x, then ρ becomes a Radon measure.

Definition 1.2. *Let* (ρ, u) *be a weak solution of (1.1),(1.2).* (ρ, u) *is called an entropy solution of (1.1),(1.2) if there exists a constant* $E > 0$ *such that,*

$$\frac{u(x_2,t) - u(x_1,t)}{x_2 - x_1} \le \frac{E}{t} \tag{1.9}$$

holds for any $x_1 < x_2$ *and almost everywhere* $t > 0$.

In the sense of definitions 1.1 and 1.2, we have the following existence theorem.

Theorem 1. *Suppose that* $\rho_0(x), u_0(x) \in L^\infty(R)$ *with* $\rho_0(x) \ge 0$. *Then there exists an entropy solution* $(\rho(x,t), u(x,t))$ *to (1.1),(1.2).*

The program of this paper is as follows: in Sec.2, we study the Riemann problem and the interaction of waves; in Sec.3, we construct the approximate solutions and investigate the generalized characteristics; in Sec.4, we prove the existence of entropy solution to (1.1),(1.2).

2. Riemann problem

Consider the Riemann problem

$$\begin{cases} \rho_t + (\rho f(u))_x = 0, \\ (\rho u)_t + (\rho u f(u))_x = 0, \\ (\rho_0, u_0)(x) = \begin{cases} (\rho_-, u_-), & x < 0, \\ (\rho_+, u_+), & x > 0, \end{cases} \end{cases} \tag{2.1}$$

where $\rho_\pm > 0, u_\pm$ are prescribed constants.

It is noted that the characteristic equation of (2.1) is

$$\frac{dx}{dt} = f(u), \quad \frac{du}{dt} = 0. \tag{2.2}$$

This implies the characteristic is straight line and u keeps constant along each characteristic. If we seek the self-similar solution

$$(\rho, u)(t,x) = (\rho, u)(\xi), \ \xi = \frac{x}{t},$$

then (1.1) becomes

$$\begin{cases} -\xi\rho_\xi + (\rho f(u))_\xi = 0, \\ -\xi(\rho u)_\xi + (\rho u f(u))_\xi = 0. \end{cases} \tag{2.3}$$

If $\rho(\xi)$ and $u(\xi)$ are smooth, then we have

$$\rho(-\xi u_\xi + f(u)u_\xi) = 0. \tag{2.4}$$

This yields either $(\rho, u)(\xi) =$ constant or $\rho = 0$. Hence the smooth solution of (1.1),(2.1) is either constant or vacuum.

For a bounded discontinuity at $\xi = \sigma$, the Rankine-Hugoniot condition gives

$$\begin{cases} -\sigma[\rho] + [\rho f(u)] = 0, \\ -\sigma[\rho u] + [\rho u f(u)] = 0, \end{cases} \tag{2.5}$$

where $[\rho] = \rho_+ - \rho_-$ denotes the jump of ρ across the discontinuity. Eliminating σ in (2.5), we have

$$\rho_+\rho_-(u_+ - u_-)(f(u_+) - f(u_-)) = 0, \tag{2.6}$$

which implies $\rho_+\rho_- = 0$ or $u_- = u_+$. This means a contact discontinuity, denoted by J simply. Therefore there is only one classical elementary wave, i.e. the contact discontinuity. By virtue of the argument above, we divide the Riemann problem (2.1) into three cases.

CASE I. If $f(u_-) < f(u_+)$, then the solution is $J + J$. That is

$$(\rho(x,t), u(x,t)) = \begin{cases} (\rho_-, u_-), & x < f(u_-)t, \\ (0, b(\frac{x}{t})), & f(u_-)t < x < f(u_+)t, \\ (\rho_+, u_+), & x > f(u_+)t, \end{cases} \tag{2.7}$$

where $b(u)$ is the inverse function of $f(u)$. We note the intermediate state is vacuum. See Fig 2.1.

CASE II. If $f(u_-) = f(u_+)$, then the solution is J. That is

$$(\rho(x,t), u(x,t)) = \begin{cases} (\rho_-, u_-), & x < f(u_-)t, \\ (\rho_+, u_+), & x > f(u_+)t. \end{cases} \tag{2.8}$$

CASE III. If $f(u_-) > f(u_+)$, it is impossible to connect (ρ_-, u_-) and (ρ_+, u_+) by the contact discontinuity. The characteristics from the initial data overlap in the domain $\Omega = \{(x,t) : f(u_+)t < x < f(u_-)t\}$. A singular wave–delta wave (denoted by δ) must develop in Ω. Analogous to the shock wave, the two side states of delta wave must satisfy an additional condition. We call it Generalized Rankine-Hugoniot (GRH) condition. We now study GRH from the definition 1.1.

Generalized Rankine-Hugoniot condition. Suppose that (m, u) is a piecewise smooth solution of (1.1) in the sense of definition 1.1. Let Γ be a smooth curve across which u and m have a jump discontinuity. Let P be any point on Γ and D be a small ball centered at P. Let D_1 and D_2 be the components of D which are determined by Γ. Assume Γ is given by

$x = x(t)$. Let $\sigma(t) = x'(t)$ and $u_\delta(t) = u(x(t), t)$. For any $\varphi,\psi \in C_0^\infty(D)$. The definition 1.1 yields

$$\begin{aligned}
&\int\int_{D_1} \varphi_t m dx dt - \int\int_{D_1} \varphi f(u) dm dt + \int\int_{D_2} \varphi_t m dx dt \\
&- \int\int_{D_2} \varphi f(u) dm dt - \int_\Gamma \varphi f(u_\delta)[m] dt = 0, \\
&\int\int_{D_1} (\psi_t u + \psi_x u f(u)) dm dt + \int\int_{D_2} (\psi_t u + \psi_x u f(u)) dm dt \\
&+ \int_\Gamma (\psi_t u_\delta + \psi_x u_\delta f(u_\delta))[m] dt = 0,
\end{aligned} \tag{2.9}$$

where $[m](t) = m(x(t) + 0, t) - m(x(t) - 0, t)$ denotes the jump.

Using the fact that (m, u) is C^1 in $D_i, i = 1, 2$, the divergence theorem gives

$$\int_\Gamma \varphi(\sigma - f(u_\delta))[m] dt = 0, \tag{2.10}$$

$$\begin{aligned}
&\int_\Gamma \psi([\rho u]\sigma - [\rho u f(u)]) dt + \int_\Gamma (\psi_t + \psi_x \sigma) u_\delta [m] dt \\
&= \int_\Gamma \psi([\rho u]\sigma - [\rho u f(u)] - \frac{d}{dt}(u_\delta[m])) dt = 0.
\end{aligned} \tag{2.11}$$

Since φ and ψ are arbitrary, we have

$$\begin{cases} \frac{dx}{dt} = \sigma(t) = f(u_\delta(t)), \\ \dfrac{d}{dt}(u_\delta[m]) = [\rho u]\sigma - [\rho u f(u)]. \end{cases} \tag{2.12}$$

We call $\sigma(t) = x'(t)$ the speed of delta wave and $[m](t)$ the strength of delta wave. Similarly, choosing φ_x instead of φ in the first equation of (2.9), we have

$$\frac{d[m]}{dt} = [\rho]\sigma - [\rho f(u)]. \tag{2.13}$$

We call (2.12) and (2.13) Generalized Rankine-Hugoniot condition. The entropy condition is

$$f(u_-) > \sigma(t) > f(u_+). \tag{2.14}$$

Now let's turn to the Riemann problem (1.1),(2.1) with $u_- > u_+$. According to the discussion above, a delta wave with constant speed $\sigma = f(u_\delta)$ is expected. See Fig 2.2, where (x_0, t_0) is a point on the line $x = f(u_\delta)t$, $\xi_0 = x_0 - f(u_-)t_0$ and $\eta_0 = x_0 - f(u_+)t_0$.

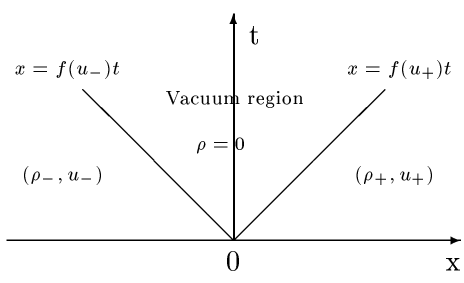

Fig 2.1

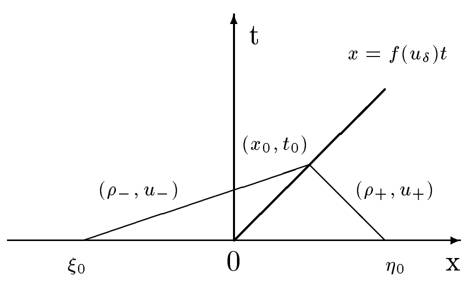

Fig 2.2

When the delta wave exists, the GRH condition yields the location of delta wave is $x = f(u_\delta)t$ and

$$\begin{aligned} [m](t) = [\rho]x - [\rho f(u)]t &= \int_{x-f(u_-)t}^{x-f(u_+)t} \rho_0(\xi)d\xi, \\ u_\delta[m](t) = [\rho u]x - [\rho u f(u)]t &= \int_{x-f(u_-)t}^{x-f(u_+)t} \rho_0(\xi)u_0(\xi)d\xi. \end{aligned} \tag{2.15}$$

Hence we have

$$[\rho]u_\delta f(u_\delta) - [\rho u]f(u_\delta) - [\rho f(u)]u_\delta + [\rho u f(u)] = 0. \tag{2.16}$$

To the contrary, there is only one constant $u_\delta \in (u_+, u_-)$ satisfying (2.16). In fact, let

$$F(s) = [\rho]sf(s) - [\rho u]f(s) - [\rho f(u)]s + [\rho u f(u)], \text{ on } u_+ < s < u_-.$$

By direct computation and the fact that $f'(s) > 0$, we have

$$\begin{aligned} F(u_-) &= \rho_+[u][f(u)] > 0, \\ F(u_+) &= -\rho_-[u][f(u)] < 0, \end{aligned}$$

and

$$\begin{aligned} F'(s) &= [\rho]f(s) + ([\rho]s - [\rho u])f'(s) - [\rho f(u)] \\ &= \rho_+(f(s) - f(u_+)) - \rho_-(f(s) - f(u_-)) \\ &\quad +\rho_+(s - u_+)f'(s) - \rho_-(s - u_-)f'(s) \\ &> 0, \text{ on } u_+ < s < u_-. \end{aligned}$$

Hence there exists one and only one point $u_\delta \in (u_+, u_-)$ satisfying $F(u_\delta) = 0$.

Therefore the solution of CASE III is δ. That is

$$(\rho(x,t), u(x,t)) = \begin{cases} (\rho_-, u_-), & x < f(u_\delta)t, \\ ([m](t)\delta(x - f(u_\delta)t), u_\delta), & x = f(u_\delta)t, \\ (\rho_+, u_+), & x > f(u_\delta)t, \end{cases} \tag{2.17}$$

where $[m](t)$ is given by (2.15) and u_δ is given by (2.16). We note that the characteristics are incoming on the delta wave.

From the argument above, the pressureless type system (1.1) only has two elementary waves: contact discontinuity and delta wave.

Remark. Since the system (1.1) is equivalent to

$$\begin{cases} \rho_t + (\rho f(u))_x = 0, \\ \rho(u_t + f(u)u_x) = 0, \end{cases} \tag{2.18}$$

for smooth solution, it is natural to expect that $f(u_\delta)$ is the shock speed of $u_t + f(u)u_x = 0$ if $\rho_+ = \rho_-$. For usual pressureless gases, it is true. However, by (2.16), it is in general not true when $f(u) \neq u$. This shows (1.1) is more complicated than usual pressureless gases.

Interaction of elementary waves. As we show in Riemann problem (1.1),(2.1), the system (1.1) only has two elementary waves–contact discontinuity and delta wave. It is easily seen that the two contact discontinuities never interact with each other. When the interaction of waves takes place, at least one wave is delta wave. We need to solve Riemann problem of (1.1) with the initial delta wave. Namely we consider (1.1) with the following initial data

$$(\rho, u)|_{t=0} = \begin{cases} (\rho_-, u_-), & x < 0, \\ (m_0\delta(x), v_0), & x = 0, \\ (\rho_+, u_+), & x > 0, \end{cases} \tag{2.19}$$

where $\rho_\pm \geq 0, m_0 > 0, u_\pm$ and v_0 are constants and $u_- > v_0 > u_+$. In terms of the discussion above, a delta wave is expected. If the delta wave exists, let $x = x(t)$ be the location of delta wave, $[m](t)$ the strength and $u_\delta(t) = u(x(t), t)$. Then the GRH condition yields

$$\begin{cases} x'(t) & = & f(u_\delta(t)), \\ [m](t) & = & [\rho]x - [\rho f(u)]t + m_0, \\ [m](t)u_\delta(t) & = & [\rho u]x - [\rho u f(u)]t + m_0 v_0. \end{cases} \tag{2.20}$$

Hence we have the following ordinary differential equation

$$\begin{cases} \dfrac{dx}{dt} = f(\dfrac{[\rho u]x - [\rho u f(u)]t + m_0 v_0}{[\rho] - [\rho f(u)]t + m_0}), \\ x(0) = 0, \end{cases} \tag{2.21}$$

where $[\cdot]$ denotes the jump across the curve $x = x(t)$.

To the contrary, if $f(u_+)t \leq x(t) \leq f(u_-)t$, then $[m](t) \geq m_0 > 0$,

$$\begin{aligned} & \frac{[\rho u]x(t) - [\rho u f(u)]t + m_0 v_0}{[\rho]x(t) - [\rho f(u)]t + m_0} \\ = \quad & u_+ + \frac{\rho_+(u_+ - u_-)(x(t) - f(u_-)t) + m_0(v_0 - u_+)}{[m](t)} > u_+ \end{aligned} \tag{2.22}$$

and

$$\frac{[\rho u]x(t) - [\rho u f(u)]t + m_0 v_0}{[\rho]x(t) - [\rho f(u)]t + m_0} < u_-. \tag{2.23}$$

By the standard theory of ordinary differential equation, there exists a unique solution $x = x(t)$ of (2.21) satisfying $f(u_+)t \leq x(t) \leq f(u_-)t$ and $f(u_-) > x'(t) > f(u_+)$. Therefore the solution of (1.1) and (2.19) is a delta wave. We note that

$$x''(t) = f'(u_\delta(t)) \cdot \frac{\rho_+(f(u_\delta(t)) - f(u_+))(u_+ - u_\delta(t))}{[m](t)} < 0, \tag{2.24}$$

when $\rho_- = 0$ and $x''(t) > 0$ when $\rho_+ = 0$. This means vacuum region for the Riemann solution of (1.1),(2.19) is convex.

We now consider the interaction of waves. For simplicity, we study the interaction of two waves here. The general case is similar to the two wave case. Suppose that $(\rho_-, u_-), (\rho_m, u_m)$ and (ρ_+, u_+) are constant states. Let $(\rho_-, u_-), (\rho_m, u_m)$ be connected by a wave w_1 and (ρ_m, u_m), (ρ_+, u_+) be connected by a wave w_2. Suppose that the two waves w_1 and w_2 intersect at the point (x_0, t_0). Let σ_1 and m_1 be the speed and the strength of w_1 at (x_0, t_0) respectively. Let σ_2 and m_2 be the speed and the strength of w_2 at (x_0, t_0) respectively. Then we have

$$\max\{m_1, m_2\} > 0, \quad f(u_-) > \sigma_1 > \sigma_2 > f(u_+).$$

From the time $t = t_0$, we study (1.1) with the following initial data

$$(\rho, u)|_{t=t_0} = \begin{cases} (\rho_-, u_-), & x < x_0, \\ (m_0\delta(x - x_0), v_0), & x = x_0, \\ (\rho_+, u_+), & x > x_0. \end{cases} \tag{2.25}$$

By the definition 1.1, m_0 and v_0 are uniquely given by

$$\begin{cases} m_0 = m_1 + m_2 > 0, \\ m_0 v_0 = m_1 u_1 + m_2 u_2, \end{cases} \tag{2.26}$$

if a delta wave is generated from the point (x_0, t_0), where $u_1 = b(\sigma_1)$ and $u_2 = b(\sigma_2)$. The proof is the same as in (2.9-2.13). We omit here. Since $u_- > u_1 > u_2 > u_+$, we have $u_- > v_0 > u_+$. To the contrary, by the argument of Riemann problem (1.1),(2.19), we know there exists a delta wave w from the point (x_0, t_0). Therefore the interaction of waves only generates one wave– delta wave. If the outgoing wave is regarded as a part of incoming waves, then delta wave never disappears once it is generated. See Fig 2.3.

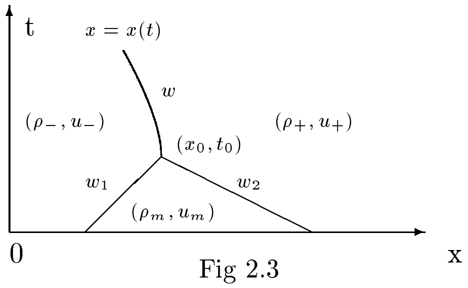

Fig 2.3

3. Construction of Approximate Solutions

This section is devoted to constructing a sequence of approximate solutions to (1.1),(1.2). Since $\rho_0 \geq 0, u_0 \in L^\infty(R)$, we choose a sequence of piecewise constant functions $\rho_n(x, 0)$ and $u_n(x, 0)$ with finitely many jump points such that

$$\begin{cases} \rho_n(x,0) > 0 \xrightarrow{\text{a.e.}} \rho_0(x) \geq 0, u_n(x,0) \xrightarrow{\text{a.e.}} u_0(x), \\ |\rho_n(x,0)|_{L^\infty}, \ |u_n(x,0)|_{L^\infty} \leq M = \max\{|u_0(x)|_{L^\infty}, \ |\rho_0(x)|_{L^\infty}\}. \end{cases} \tag{3.1}$$

For each $n \geq 1$, we construct the approximate solutions as follows: the constructive procedure starts at $t = 0$ with the initial data $(\rho_n(x, 0), u_n(x, 0))$. Suppose that $x_1 < \cdots < x_N$ be the points where $(\rho_n(x, 0), u_n(x, 0))$ is discontinuous. For each $\alpha = 1, ..., N$, we obtain the Riemann solutions (ρ_n, u_n) until a time t_1 is reached, when the first set of interaction between two or more wave-fronts takes place. After the time t_1, the solution (ρ_n, u_n) can be obtained by solving the Riemann problem with the initial delta wave which has already been investigated in (2.25). So the solution can be constructed until the time t_2 where the second set of wave interactions takes place, etc $\cdots$. According to the argument of §2, only one delta wave is generated from the point where interaction of waves takes place. The number of wave-fronts is time-decreasing. Thus the constructive procedure can be continued up to any time $t > 0$. Hence a weak solution (ρ_n, u_n) to the system (1.1) with the initial data $(\rho_n(x, 0), u_n(x, 0))$ is constructed.

In terms of §2, it is obvious that the half space $t > 0$ is divided a finite number of constant or vacuum regions by delta wave and contact discontinuity. In each constant region, $\rho_n(x,t)$ and $u_n(x,t)$ are constants. By our constructive procedure, the intersection surface of constant region and x-axis is an interval. From (2.7) and (2.24), it is not difficult to show the vacuum region is convex and has only one vertex in x-axis because our approximate initial data $\rho_n(x,0)$ is positive.

We now study the structure of the approximate solution (ρ_n, u_n) by the generalized characteristics. We first define the generalized characteristics $X_n(\xi,t)$. For any continuous point ξ of $(\rho_n(x,0), u_n(x,0))$ in x-axis, we introduce the genuine characteristic $L: x = \xi + tf(u_n(\xi,0))$ which may intersect a delta wave δ at $t = t_0$. We define

$$X_n(\xi,t) = \begin{cases} \xi + tf(u_n(\xi,0)), & \text{if } 0 \le t \le t_0, \\ x(t), & \text{if } t \ge t_0, \end{cases} \tag{3.2}$$

where $x(t)$ is the location of the intersecting wave δ. Note that $x(t)$ is well defined for any time $t \ge t_0$ because there is one and only one delta wave generated when wave interaction takes place. If wave interaction takes place at $t = t_1$, then $x(t)$ denotes the location of the outgoing delta wave for $t > t_1$. If ξ is of discontinuity set of $(\rho_n(x,0), u_n(x,0))$, there are two cases. When $u_n(\xi - 0, 0) > u_n(\xi + 0, 0)$, there is a delta wave δ originated from $(\xi, 0)$. We define $X_n(\xi,t), t > 0$ the location of the delta wave δ. When $u_n(\xi - 0, 0) < u_n(\xi + 0, 0)$, there are two contact discontinuities J_1 and J_2 and the intermediate state is vacuum. We define $X_n(\xi,t)$ the location of left contact discontinuity J_1 until the time t_1 when J_1 intersects other waves. When $t > t_1$, we define $X_n(\xi,t)$ the location of the outgoing delta wave. Thus the generalized characteristics $X_n(\xi,t)$ is well defined for any ξ and any time t. We have the following properties of $X_n(\xi,t)$.

Lemma 3.1. *The generalized characteristics $X_n(\xi,t)$ is increasing in ξ and Lipschitz continuous in t. In particular, $f(u_n(X_n(\xi,t),t)) = X_n'(\xi,t)$ holds except finitely many points of t, where $X_n'(\xi,t)$ denotes the upper derivative of $X_n(\xi,t)$ with respect to t. If $X_n(\xi_1,t_0) = X_n(\xi_2,t_0)$ for some t_0 and $\xi_1 \ne \xi_2$, then $X_n(\xi_1,t) = X_n(\xi_2,t)$ holds for any time $t \ge t_0$.*

Proof. By the definition of generalized characteristics, it is easy to prove Lemma 3.1. We omit the proof here.

In view of the construction procedure of the generalized characteristics, all $X_n(\xi,t)$, $\xi \in R$ may not cover the half space $t > 0$. For example, in case I of Riemann problem, all generalized characteristics never enter the vacuum region $\{(x,t) : f(u_-)t < x < f(u_+)t\}$. In fact, since the approximate solution (ρ_n, u_n) is constructed by piecing together a finite number of Riemann solutions, it is easy to show all generalized characteristics $X_n(\xi,t)$, $\xi \in R$ are away from vacuum.

Note that $X_n(\xi,t)$ is increasing in ξ. We define

$$\xi_n(x,t) = \sup\{\xi; X_n(\xi,t) < x\}, \tag{3.3}$$

$$\eta_n(x,t) = \inf\{\xi; X_n(\xi,t) > x\}. \tag{3.4}$$

We have

Lemma 3.2. *(i) $\xi_n(x,t)$ and $\eta_n(x,t)$ are increasing in x. Furthermore, for any $x_1 < x_2$, $\eta_n(x_1,t) \le \xi_n(x_2,t)$.*

(ii) If $\xi_n(x,t) < \eta_n(x,t)$, then $x = X_n(\xi,t)$, for any $\xi \in (\xi_n(x,t), \eta_n(x,t))$.

(iii) $\xi_n(x,t) = \xi_n(x-0,t), \eta_n(x,t) = \xi_n(x+0,t)$. Furthermore,

$$x - Nt \le \xi_n(x,t), \eta_n(x,t) \le x + Nt.$$

where $N = \max_{|u|\le M} |f(u)|$.

(iv) For any ξ, $\xi_n(X_n(\xi,t),t)$ is decreasing and $\eta_n(X_n(\xi,t),t)$ is increasing with respect to t.

Proof. It is easy to prove Lemma 3.2 by the definition of $X_n(\xi,t)$. We omit the proof here.

Since the half space $t > 0$ is divided a finite number of constant or vacuum regions by delta wave and contact discontinuity, it is easy to check (see Fig 2.2 for instance),

$$\xi_n(x,t) = x - f(u_n(x-0,t))t, \quad if \quad \rho_n(x-0,t) > 0, \tag{3.5}$$

$$\eta_n(x,t) = x - f(u_n(x+0,t))t, \quad if \quad \rho_n(x+0,t) > 0. \tag{3.6}$$

When (x,t) is located in a vacuum region V_a, then $\xi_n(y,\tau) = \eta_n(y,\tau) = \xi_n(x,t)$ for any $(y,\tau) \in V_a$ and the point $(\xi_n(x,t),0)$ is the vertex of V_a on x-axis because all characteristics are away from vacuum. Note that $u_n(x,t)$ can be chosen arbitrary in the vacuum region, we define

$$u_n(x,t) = b(\frac{x - \xi_n(x,t)}{t}), \quad (x,t) \in V_a$$

where $b(u)$ is the inverse function of $f(u)$. If (x,t) is located on the boundary of vacuum region, then $u_n(x-0,t) \ge u_n(x,t) \ge u_n(x+0,t)$ holds due to the fact that vacuum region is convex. We note that (3.5) and (3.6) hold for any point (x,t) by our definition of $u(x,t)$ in the vacuum region. Hence we have

Lemma 3.3. *For any point (x,t), $u_n(x-0,t) = b(\frac{x-\xi_n(x,t)}{t})$ and $u_n(x+0,t) = b(\frac{x-\eta_n(x,t)}{t})$. Furthermore, $u_n(x-0,t) \ge u_n(x,t) \ge u_n(x+0,t)$.*

Lemma 3.4. *For any $\xi_1 < \xi_2$, $X_n'(\xi_2,t) - X_n'(\xi_1,t) \le \frac{1}{t}(X_n(\xi_2,t) - X_n(\xi_1,t))$ holds for any time $t > 0$.*

Proof. Let $x_2 = X_n(\xi_2,t)$ and $x_1 = X_n(\xi_1,t)$, then $x_1 < x_2$ and $\eta_n(x_1,t) < \xi_n(x_2,t)$. Lemma 3.3 yields

$$\begin{aligned} X_n'(\xi_2,t) - X_n'(\xi_1,t) &= f(u_n(x_2,t)) - f(u_n(x_1,t)) \\ &\le f(u_n(x_2-0,t)) - f(u_n(x_1+0,t)) \\ &= \frac{x_2 - \xi(x_2,t)}{t} - \frac{x_1 - \eta(x_1,t)}{t} \le \frac{x_2 - x_1}{t}. \end{aligned}$$

Thus Lemma 3.4 is proved.

For any point (x_0,t_0), we define the left backward characteristic

$$L_{1n}(x_0,t_0) : x = x_0 + f(u_n(x_0-0,t_0))(t-t_0), \quad \text{for} \quad t < t_0, \tag{3.7}$$

and the right backward characteristic

$$L_{2n}(x_0,t_0): x = x_0 + f(u_n(x_0+0,t_0))(t-t_0), \quad \text{for} \quad t < t_0. \tag{3.8}$$

It is obvious that the line $L_{in}(x_0,t_0), i=1,2$ is contained in either constant or vacuum region. Therefore we give the following lemma without proof.

Lemma 3.5. *For any point (x_0,t_0), $\xi_n(x,t) = \eta_n(x,t) = \xi_n(x_0,t_0)$ along the line $L_{1n}(x_0,t_0)$ and $\xi_n(x,t) = \eta_n(x,t) = \eta_n(x_0,t_0)$ along the line $L_{2n}(x_0,t_0)$.*

In terms of Lemma 3.2 and 3.5, for each point (x,t), the backward characteristics $L_{1n}(x,t)$, $L_{2n}(x,t)$ and x-axis form a characteristic triangle. We denote it $\Delta_n(x,t)$. By Lemma 3.5, for any two characteristic triangles, either they disjoin with each other except x-axis or a triangle is a subset of another one.

We now construct an explicit formula of the approximate solution (ρ_n, u_n). We define

$$\begin{aligned}
m_n(x,t) &= \int_0^{\xi_n(x,t)} \rho_n(\xi,0)d\xi,\\
q_n(x,t) &= \int_0^{\xi_n(x,t)} \rho_n(\xi,0)u_n(\xi,0)d\xi,\\
\theta_n(x,t) &= \int_0^{\xi_n(x,t)} \rho_n(\xi,0)X_n'(\xi,t)d\xi,\\
e_n(x,t) &= \int_0^{\xi_n(x,t)} \rho_n(\xi,0)u_n(\xi,0)X_n'(\xi,t)d\xi.
\end{aligned} \tag{3.9}$$

We have

Lemma 3.6.

$$\begin{aligned}
&\rho_n = m_{nx},\\
&q_{nx}(x,t) = u_n(x,t)m_{nx}(x,t),\\
&\theta_{nx}(x,t) = f(u_n(x,t))m_{nx}(x,t),\\
&e_{nx}(x,t) = u_n(x,t)f(u_n(x,t))m_{nx}(x,t)
\end{aligned} \tag{3.10}$$

hold in the sense of Radon-Nikodym derivative in x.

Proof. We divide our proof into three parts. We first consider the case when (x,t) is located in a vacuum region V_a. Then $\xi_n(y,t)$ is constant for any $(y,t) \in V_a$ and $m_n(y,t)$ is constant in V_a. This case is trivial.

We then consider the case when (x,t) lies in a constant region V_c. In this case, there are $x_1 < x < x_2$ such that $(y,t) \in V_c$ for any $y \in (x_1,x_2)$. Thus, We have,

$$\begin{aligned}
&\xi_n(x_i,t) = \eta_n(x_i,t) = x_i - f(u_n(x,t))t, \quad i=1,2,\\
&\rho_n(\xi,0) = \rho_n(x,t), \quad \forall \quad \xi \in (\xi_n(x_1,t),\xi_n(x_2,t)),\\
&u_n(\xi,0) = u_n(x,t), \quad \forall \quad \xi \in (\xi_n(x_1,t),\xi_n(x_2,t)).
\end{aligned}$$

This infers $\rho_n = m_{nx}$ and

$$\begin{aligned}
&\lim_{x_2,x_1\longrightarrow x\pm 0} \frac{q_n(x_2,t)-q_n(x_1,t)}{m_n(x_2,t)-m_n(x_1,t)} \\
&= \lim_{x_2,x_1\longrightarrow x\pm 0} \frac{\int_{\xi_n(x_1,t)}^{\xi_n(x_2,t)} \rho_n(\xi,0)u_n(\xi,0)d\xi}{\int_{\xi_n(x_1,t)}^{\xi_n(x_2,t)} \rho_n(\xi,0)d\xi} = u_n(x,t).
\end{aligned} \tag{3.11}$$

Finally, we consider the case when (x,t) lies on a delta wave δ or a contact discontinuity J. The later one is similar to the second case. We only need to study the delta wave case. In this situation, $\xi_n(x,t) < \eta_n(x,t)$. By Lemma 3.2 (ii), we have $X_n(\xi,t) = x$ and $X_n'(\xi,t) = f(u_n(x,t))$ for any $\xi \in (\xi_n(x,t), \eta_n(x,t))$. Next we shall prove

$$u_n(x,t) = \frac{\int_{\xi_n(x,t)}^{\eta_n(x,t)} \rho_n(\xi,0)u_n(\xi,0)d\xi}{\int_{\xi_n(x,t)}^{\eta_n(x,t)} \rho_n(\xi,0)d\xi}, \quad m_\delta(x,t) = \int_{\xi_n(x,t)}^{\eta_n(x,t)} \rho_n(\xi,0)d\xi, \tag{3.12}$$

where $m_\delta(x,t)$ is the strength of delta wave.

It is observed that (3.12) holds for the case III of Riemann problem. We suppose that (3.12) holds for some $t < t_0$ when the wave interaction takes place. Assume that a wave δ_1 connecting (ρ_-, u_-) and (ρ_m, u_m) intersects a wave δ_2 connecting (ρ_m, u_m) and (ρ_+, u_+) at the point (x_0, t_0), where $u_- > u_m > u_+$. Without of loss of generality, we assume that $\rho_\pm > 0$. Then $u_\pm$ are constants. Let σ_1 and m_{δ_1} be the speed and the strength of δ_1 at (x_0,t_0) respectively. Let σ_2 and m_{δ_2} be the speed and the strength of δ_2 at (x_0,t_0) respectively. Then our assumption gives

$$m_{\delta_1} = \int_{x_0-f(u_-)t_0}^{x_0-f(u_m)t_0} \rho_n(\xi,0)d\xi, \quad m_{\delta_2} = \int_{x_0-f(u_m)t_0}^{x_0-f(u_+)t_0} \rho_n(\xi,0)d\xi,$$

$$u_{\delta_1} = \frac{\int_{x_0-f(u_-)t_0}^{x_0-f(u_m)t_0} \rho_n(\xi,0)u_n(\xi,0)d\xi}{\int_{x_0-f(u_-)t_0}^{x_0-f(u_m)t_0} \rho_n(\xi,0)d\xi}, \quad u_{\delta_2} = \frac{\int_{x_0-f(u_m)t_0}^{x_0-f(u_+)t_0} \rho_n(\xi,0)u_n(\xi,0)d\xi}{\int_{x_0-f(u_m)t_0}^{x_0-f(u_+)t_0} \rho_n(\xi,0)d\xi},$$

where $u_{\delta_i} = b(\sigma_i), i = 1,2$. From (2.26), we get (3.12) at (x_0,t_0). That is

$$u_n(x_0,t_0) = \frac{\int_{\xi_n(x_0,t_0)}^{\eta_n(x_0,t_0)} \rho_n(\xi,0)u_n(\xi,0)d\xi}{\int_{\xi_n(x_0,t_0)}^{\eta_n(x_0,t_0)} \rho_n(\xi,0)d\xi}, \quad m_\delta(x_0,t_0) = \int_{\xi_n(x_0,t_0)}^{\eta_n(x_0,t_0)} \rho_n(\xi,0)d\xi,$$

where $\xi_n(x_0,t_0) = x_0 - f(u_-)t_0$, $\eta_n(x_0,t_0) = x_0 - f(u_+)t_0$. Let $t_1 > t_0$ be the next time when interaction of waves takes place. Then there is no wave interaction between t_0 and t_1. Let $x = x(t)$, $t_0 < t < t_1$ be the location of delta wave. It is obvious that $\rho_n(\xi,0) = \rho_-, u_n(\xi,0) = u_-$ when $x - f(u_-)t \le \xi \le x_0 - f(u_-)t_0$ and $\rho_n(\xi,0) = \rho_+, u_n(\xi,0) = u_+$ when $x_0 - f(u_+)t_0 \le \xi \le x - f(u_+)t$. For any $t_0 < t < t_1$, we have, from (2.20),

$$\begin{cases}
m_\delta(x,t) &= [\rho](x-x_0) - [\rho f(u)](t-t_0) + m_\delta(x_0,t_0) \\
&= \displaystyle\int_{x-f(u_-)t}^{x-f(u_+)t} \rho_n(\xi,0)d\xi, \\
m_\delta(x,t)u_n(x,t) &= [\rho u](x-x_0) - [\rho u f(u)](t-t_0) + m_\delta(x_0,t_0)u_n(x_0,t_0) \\
&= \displaystyle\int_{x-f(u_-)t}^{x-f(u_+)t} \rho_n(\xi,0)u_n(\xi,0)d\xi,
\end{cases}$$

which yields (3.12) for $t_0 < t < t_1$. By the induction principle, (3.12) is proved.

Hence by Lemma 3.2 (iii) and (3.12), we have

$$\lim_{x_2,x_1\longrightarrow x\pm 0}\frac{q_n(x_2,t)-q_n(x_1,t)}{m_n(x_2,t)-m_n(x_1,t)}=\frac{\int_{\xi_n(x,t)}^{\eta_n(x,t)}\rho_n(\xi,0)u_n(\xi,0)d\xi}{\int_{\xi_n(x,t)}^{\eta_n(x,t)}\rho_n(\xi,0)d\xi}=u_n(x,t). \tag{3.13}$$

Thus $\frac{dq_n}{dm_n}=u_n(x,t)$ holds in the sense of the Radon-Nikodym derivative. In the same fashion, we also have

$$\frac{d\theta_n}{dm_n}=f(u_n(x,t)),\frac{de_n}{dm_n}=u_n(x,t)f(u_n(x,t)).$$

It is noted that (ρ_n,u_n) is a weak solution of (1.1) with the initial data $\rho_n(x,0)$ and $u_n(x,0)$. Hence Lemma 3.6 and definition 1.1 give

$$\begin{cases}\int\int\varphi_t m_n dxdt-\int\int\varphi d\theta_n dt\\ =\int\int\varphi_t m_n dxdt-\int\int\varphi f(u_n)dm_n dt=0,\\ \int\int\psi_t dq_n+\psi_x de_n dt=\int\int(\psi_t u_n+\psi_x u_n f(u_n))dm_n dt=0\end{cases} \tag{3.14}$$

for any φ,$\psi\in C_0^\infty(R_+\times R)$. We now give our key lemma.

Lemma 3.7. *For any $\xi_1<\xi_2$ and $0\le t_1<t_2$, the followings hold*

$$b(\frac{X_n(\xi_2,t_2)-X_n(\xi_1,t_1)}{t_2-t_1})\ge\frac{\int_{\xi_n(X_n(\xi_1,t_1),t_1)}^{\eta_n(X_n(\xi_2,t_2),t_2)}\rho_n(\xi,0)u_n(\xi,0)d\xi}{\int_{\xi_n(X_n(\xi_1,t_1),t_1)}^{\eta_n(X_n(\xi_2,t_2),t_2)}\rho_n(\xi,0)d\xi}, \tag{3.15}$$

$$b(\frac{X_n(\xi_1,t_2)-X_n(\xi_2,t_1)}{t_2-t_1})\le\frac{\int_{\xi_n(X_n(\xi_1,t_2),t_2)}^{\eta_n(X_n(\xi_2,t_1),t_1)}\rho_n(\xi,0)u_n(\xi,0)d\xi}{\int_{\xi_n(X_n(\xi_1,t_2),t_2)}^{\eta_n(X_n(\xi_2,t_1),t_1)}\rho_n(\xi,0)d\xi}. \tag{3.16}$$

Proof. Let $X_1=X_n(\xi_1,t_1)$ and $X_2=X_n(\xi_2,t_2)$. We first draw the straight line L connecting (X_1,t_1) and (X_2,t_2),

$$L:\quad x=X_1+\frac{X_2-X_1}{t_2-t_1}(t-t_1),\quad t_1<t<t_2.$$

See Fig 3.1.

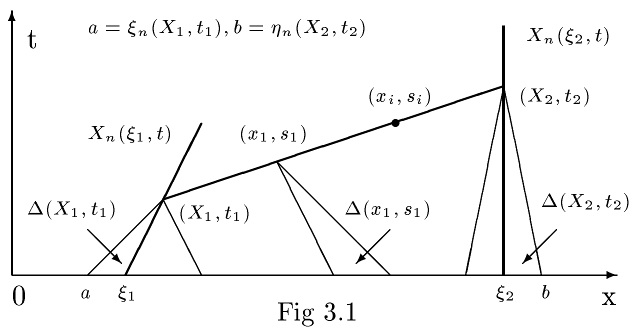

Fig 3.1

Let $S=\frac{X_2-X_1}{t_2-t_1}$ be the speed of L. We divide our proof by two steps.

Step 1: We first consider the case when there is no delta wave intersecting the line L between t_1 and t_2. By Lemma 3.5, the characteristic triangle $\Delta(X_1,t_1)$ either disjoins with $\Delta(X_2,t_2)$ except x-axis or is a subset of $\Delta(X_2,t_2)$. The first one implies that the triangle $\Delta(X_2,t_2)$ is located on the right side of $\Delta(X_1,t_1)$ due to $\xi_1<\xi_2\in\Delta(X_2,t_2)$. Note that $X_n(\xi_1,t)$ does not intersect L when $t_1<t<t_2$ because no delta wave intersects L. Thus, we have

$$\begin{aligned} &\eta_n(X_1,t_1)\le\xi_n(X_2,t_2),\\ &b(S)\ge u_n(X_1,t_1),\quad b(S)\ge u_n(X_2,t_2), \end{aligned}\tag{3.17}$$

and $b(S)\ge u_0(\xi)$ for any $\eta_n(X_1,t_1)\le\xi\le\xi_n(X_2,t_2)$. This yields (3.15). Furthermore, the following holds

$$b(S)\ge\frac{\int_{\xi_n(X_1,t_1)}^{\xi_n(X_2,t_2)}\rho_n(\xi,0)u_n(\xi,0)d\xi}{\int_{\xi_n(X_1,t_1)}^{\xi_n(X_2,t_2)}\rho_n(\xi,0)d\xi}.\tag{3.18}$$

Other case can be treated by the similar way. Therefore (3.15) holds when there is no delta wave intersecting the line L.

Step 2: We suppose that (3.15) holds for the case when there are $i(i=0,\cdots,l)$ delta waves intersecting the line L. We now prove (3.15) when there are $l+1$ waves intersecting L. We denote the intersection points as $(x_i,s_i), i=0,\cdots,l+1$ along the line L and $t_1<s_i<s_{i+1}<t_2$ holds for any i. We now consider the segment which starts from (x_1,s_1) to (X_2,t_2). As in step 1, the triangle $\Delta(x_1,s_1)$ either is located on the left side of $\Delta(X_2,t_2)$ or is a subset of $\Delta(X_2,t_2)$. Without of loss generality, we assume that $\Delta(x_1,s_1)$ lies on the left side of $\Delta(X_2,t_2)$. In this situation,

$$b(S)\ge u_n(x_1,s_1),\quad b(S)\ge u_n(X_2,t_2)$$

and for any $\xi\in(\xi_n(x_1,s_1),\eta_n(x_1,s_1))$, $X_n(\xi,s_1)=x_1$ holds. It is observed that there are $i(i=0,\cdots,l)$ waves intersecting the line L. Hence by our assumption, we have

$$b(S)\ge\frac{\int_{\xi_n(x_1,s_1)}^{\eta_n(X_2,t_2)}\rho_n(\xi,0)u_n(\xi,0)d\xi}{\int_{\xi_n(x_1,s_1)}^{\eta_n(X_2,t_2)}\rho_n(\xi,0)d\xi}.\tag{3.19}$$

On the other hand, as in step 1, the triangle $\Delta(X_1,t_1)$ either is located on the left side of $\Delta(x_1,s_1)$ or is a subset of $\Delta(x_1,s_1)$. In the first case, (3.18) gives

$$b(S)\ge\frac{\int_{\xi_n(X_1,t_1)}^{\xi_n(x_1,s_1)}\rho_n(\xi,0)u_n(\xi,0)d\xi}{\int_{\xi_n(X_1,t_1)}^{\xi_n(x_1,s_1)}\rho_n(\xi,0)d\xi}.\tag{3.20}$$

Hence we have (3.15) from (3.19) and (3.20). Other case can be treated in the similar way. Thus (3.15) holds when there are $l+1$ waves intersecting L. By the induction principle, (3.15) is proved. In the same fashion, (3.16) is also obtained. Therefore Lemma 3.7 is proved.

4. Existence of weak Solution

Proof of theorem 1. To prove theorem 1, we need to study the convergence of $X_n(\xi,t)$. Since $X_n(\xi,t)$ is increasing in ξ and $|X_n(\xi,t_2)-X_n(\xi,t_1)|\le M(t_2-t_1)$, we can choose a subsequence of $X_n(\xi,t)$ (still denoted by $X_n(\xi,t)$) such that $X_n(\xi,t)$ converge to some increasing function $X(\xi,t)$ by the diagonal method. Moreover, $X(\xi,t)$ is Lipschitz continuous in t. Similar to $X_n(\xi,t)$, we have the following properties of $X(\xi,t)$.

Lemma 4.1. *(1) $t^{-1}(X(\xi_2,t)-X(\xi_1,t))$ is time-decreasing for any $\xi_1<\xi_2$.*

(2) If $X(\xi_1,t_0)=X(\xi_2,t_0)$ for some $t_0\geq 0$, then $X(\xi_1,t)=X(\xi_2,t)$ for any $t\geq t_0$.

Proof of (1). Since $X_n(\xi,t)$ converge to $X(\xi,t)$, we only need to show that $t^{-1}(X_n(\xi_2,t)-X_n(\xi_1,t))$ is decreasing in t for any $\xi_1<\xi_2$. From Lemma 3.4, we have

$$\begin{aligned}&\frac{d}{dt}(t^{-1}(X_n(\xi_2,t)-X_n(\xi_1,t))\\&=t^{-2}(X_n(\xi_1,t)-X_n(\xi_2,t)+t(X_n'(\xi_2,t)-X_n'(\xi_1,t)))\leq 0.\end{aligned}\tag{4.1}$$

Letting $n\to\infty$ yields (1).

Proof of (2). (2) is easy from

$$t^{-1}(X(\xi_2,t)-X(\xi_1,t))\leq t_0^{-1}(X(\xi_2,t_0)-X(\xi_1,t_0))=0.\tag{4.2}$$

We define

$$\xi(x,t)=\sup\{\xi;X(\xi,t)<x\}\tag{4.3}$$

$$\eta(x,t)=\inf\{\xi;X(\xi,t)>x\}\tag{4.4}$$

Analogous to Lemma 3.2, we have

Lemma 4.2. *(i) $\xi(x,t)$ and $\eta(x,t)$ are increasing in x. Furthermore, for any $x_1<x_2$, $\eta(x_1,t)\leq\xi(x_2,t)$ holds.*

(ii) $x=X(\xi,t)$, for any $\xi\in(\xi(x,t),\eta(x,t))$.

(iii) $\xi(x,t)=\xi(x-0,t),\eta(x,t)=\xi(x+0,t)$.

(iv) $x-Nt\leq\xi(x,t)\leq x+Nt$.

(v) For any ξ, $\xi(X(\xi,t),t)$ is time-decreasing and $\eta(X(\xi,t),t)$ is time-increasing.

Now we define

$$\begin{aligned}m(x,t)&=\int_0^{\xi(x,t)}\rho_0(\xi)d\xi,\\q(x,t)&=\int_0^{\xi(x,t)}\rho_0(\xi)u_0(\xi)d\xi,\\\theta(x,t)&=\int_0^{\xi(x,t)}\rho_0(\xi)X'(\xi,t)d\xi,\\e(x,t)&=\int_0^{\xi(x,t)}\rho_0(\xi)u_0(\xi)X'(\xi,t)d\xi.\end{aligned}\tag{4.5}$$

It is obvious that $m(x,t),q(x,t),\theta(x,t)$ and $e(x,t)$ are locally of bounded variation in x. Let $D=\{(x,t);\xi(x,t)=\eta(x,t)\}$ and $D_t=\{x;\xi(x,t)=\eta(x,t)\}$. By virtue of Lemma 4.2, R/D_t consists of at most countable points.

Lemma 4.3. *(i) For any point (x,t),*

$$\liminf_{n\to\infty}\xi_n(x,t)\geq\xi(x,t),\quad\limsup_{n\to\infty}\eta_n(x,t)\leq\eta(x,t).$$

In particular, if $(x,t)\in D$*, then* $\lim_{n\to\infty}\xi_n(x,t)=\lim_{n\to\infty}\eta_n(x,t)=\xi(x,t)$.

(ii) For any ξ *and* $t>0$,

$$\liminf_{n\to\infty}\xi_n(X_n(\xi,t),t)\geq\xi(X(\xi,t),t),\quad \limsup_{n\to\infty}\eta_n(X_n(\xi,t),t)\leq\eta(X(\xi,t),t).$$

Proof. We first prove $\liminf_{n\to\infty}\xi_n(x,t)\geq\xi(x,t)$. By way of contradiction, suppose that the conclusion does not hold, then there exists a positive constant δ_0 and a subsequence n_i of n such that $\xi_{n_i}(x,t)\leq\xi(x,t)-\delta_0$. For any $\xi_0\in(\xi(x,t)-\delta_0,\xi(x,t))$, $X_{n_i}(\xi_0,t)$ must be larger than x by the definition of $\xi_{n_i}(x,t)$. Let $n_i\to\infty$, we have $X(\xi_0,t)\geq x$. This contradicts with the definition of $\xi(x,t)$. Hence $\liminf_{n\to\infty}\xi_n(x,t)\geq\xi(x,t)$ holds. By the same way, we have $\limsup_{n\to\infty}\eta_n(x,t)\leq\eta(x,t)$. It is clear that $\lim_{n\to\infty}\xi_n(x,t)=\lim_{n\to\infty}\eta_n(x,t)=\xi(x,t)$ if $(x,t)\in D$. (ii) is also obtained in the same way. Hence Lemma 4.3 is proved.

Lemma 4.4. *For any point* $(x,t)\in D$*,* $u_n(x,t)$ *converge to* $b(\frac{x-\xi(x,t)}{t})$ *as* $n\to\infty$.

Proof. Due to Lemma 3.3, we have $u_n(x+0,t)=b(\frac{x-\eta_n(x,t)}{t})\leq u_n(x,t)\leq u_n(x-0,t)=b(\frac{x-\xi_n(x,t)}{t})$. From Lemma 4.3, we have $\lim_{n\to\infty}u_n(x,t)=b(\frac{x-\xi(x,t)}{t})$.

Lemma 4.5. *For any* $\varphi,\psi\in C_0^\infty(R_+\times R)$*, the followings hold,*

$$\begin{cases}\displaystyle\int\!\!\int\varphi_t m dxdt-\int\!\!\int\varphi d\theta dt=0,\\ \displaystyle\int\!\!\int\psi_t dqdt+\int\!\!\int\psi_x dedt=0.\end{cases}\tag{4.6}$$

Proof. Since $\xi_n(x,t)\xrightarrow{\text{a.e.}}\xi(x,t)$ due to Lemma 4.3 and $X_n'(\xi,t)$ weakly-star converge to $X'(\xi,t)$ due to the fact that $\lim_{n\to\infty}X_n(\xi,t)=X(\xi,t)$ for any ξ, (4.6) is easy from (3.14) and (4.5).

Now we define

$$u(x,t)=\begin{cases}b(\frac{x-\xi(x,t)}{t}), & \text{if } \xi(x,t)=\eta(x,t),\\ \dfrac{\int_{\xi(x,t)}^{\eta(x,t)}\rho_0(\xi)u_0(\xi)d\xi}{\int_{\xi(x,t)}^{\eta(x,t)}\rho_0(\xi)d\xi}, & \text{if } \xi(x,t)\neq\eta(x,t).\end{cases}\tag{4.7}$$

we have

Lemma 4.6. *If* $(x,t)\in R/D_t$*, then* $u(x,t)=b(X'(\xi,t))$ *for any* $\xi\in(\xi(x,t),\eta(x,t))$*, where* $X'(\xi,t)$ *denotes the upper derivative of* $X(\xi,t)$ *with respect to* t.

Proof. If $(x,t)\in R/D_t$, then $\xi(x,t)<\eta(x,t)$. For any $\xi_1<\xi_2$, $\xi_1,\xi_2\in(\xi(x,t),\eta(x,t))$ and $0<h<1$, Lemma 3.7 gives

$$b(\frac{X_n(\xi_2,t+h)-X_n(\xi_1,t)}{t_2-t_1})\geq\frac{\int_{\xi_n(X_n(\xi_1,t),t)}^{\eta_n(X_n(\xi_2,t+h),t+h)}\rho_n(\xi,0)u_n(\xi,0)d\xi}{\int_{\xi_n(X_n(\xi_1,t),t)}^{\eta_n(X_n(\xi_2,t+h),t+h)}\rho_n(\xi,0)d\xi}.\tag{4.8}$$

Since $\eta_n(X_n(\xi_2,t+h),t+h)$ and $\xi_n(X_n(\xi_1,t),t)$ are uniformly bounded with respect to n, there exists a subsequence n_i of n such that $\eta_{n_i}(X_{n_i}(\xi_2,t+h),t+h)$ and $\xi_{n_i}(X_{n_i}(\xi_1,t),t)$ converge to some constants $H(\xi_2,h)$ and $G(\xi_1)$ respectively. From Lemma 4.3(ii), we have

$$\xi_2 \le H(\xi_2,h) \le \eta(X(\xi_2,t+h),t+h), \quad \xi(x,t) \le G(\xi_1) \le \xi_1. \tag{4.9}$$

Letting $n_i \to \infty$ in (4.8) yields, for any $\xi \in (\xi(x,t),\eta(x,t))$,

$$b(\frac{X(\xi,t+h)-X(\xi,t)}{t_2-t_1}) = b(\frac{X(\xi_2,t+h)-X(\xi_1,t)}{t_2-t_1}) \ge \frac{\int_{G(\xi_1)}^{H(\xi_2,h)} \rho_0(\xi)u_0(\xi)d\xi}{\int_{G(\xi_1)}^{H(\xi_2,h)} \rho_0(\xi)d\xi} \tag{4.10}$$

due to the fact that $X(\xi_1,\tau) = X(\xi_2,\tau) = X(\xi,\tau)$ for any $\tau \ge t$.

Let $x(h) = X(\xi_2,t+h)$. Because $\eta(x(h),t+h)$ is an increasingly monotonic sequence (see Lemma 4.2(iv)), we have

$$\lim_{h\to 0} \eta(x(h),t+h) = A \ge \eta(x,t). \tag{4.11}$$

If $A > \eta(x,t)$, then for any $\xi_0 \in (\eta(x,t),A)$, $X(\xi_0,t+h) = x(h)$. Letting $h \to 0$ yields $X(\xi_0,t) = x$. This contradicts with the definition of $\eta(x,t)$. Thus we obtain

$$\lim_{h\to 0} \eta(x(h),t+h) = \eta(x,t). \tag{4.12}$$

Since $H(\xi_2,h)$ is uniformly bounded with respect to h, there exists a subsequence h_i of h (still denoted by h) such that $H(\xi_2,h)$ converge to some constant $H(\xi_2)$ as $h \to 0$. From (4.9) and (4.12), $H(\xi_2)$ satisfies $\xi_2 \le H(\xi_2) \le \eta(x,t)$. Thus we get

$$b(X'(\xi,t)) \ge \frac{\int_{G(\xi_1)}^{H(\xi_2)} \rho_0(\xi)u_0(\xi)d\xi}{\int_{G(\xi_1)}^{H(\xi_2)} \rho_0(\xi)d\xi}, \tag{4.13}$$

Letting $\xi_2 \to \eta(x,t)$ and $\xi_1 \to \xi(x,t)$ gives

$$b(X'(\xi,t)) \ge \frac{\int_{\xi(x,t)}^{\eta(x,t)} \rho_0(\xi)u_0(\xi)d\xi}{\int_{\xi(x,t)}^{\eta(x,t)} \rho_0(\xi)d\xi} = u(x,t). \tag{4.14}$$

In the same way, we also have

$$\lim_{h\to 0} \xi(x(h),t+h) = \xi(x,t), \tag{4.15}$$

and then

$$b(X'(\xi,t)) \le \frac{\int_{\xi(x,t)}^{\eta(x,t)} \rho_0(\xi)u_0(\xi)d\xi}{\int_{\xi(x,t)}^{\eta(x,t)} \rho_0(\xi)d\xi} = u(x,t). \tag{4.16}$$

Therefore $u(x,t) = b(X'(\xi,t))$ for any $\xi \in (\xi(x,t),\eta(x,t))$. Lemma 4.6 is proved.

Lemma 4.7. *For any point (x,t), $u(x-0,t) = b(\frac{x-\xi(x,t)}{t})$, $u(x+0,t) = b(\frac{x-\eta(x,t)}{t})$. Furthermore, $u(x+0,t) \le u(x,t) \le u(x-0,t)$.*

Proof. It is noted that R/D_t is a zero measure set, Hence for any point (x,t), we choose a sequence of x_{1m} satisfying $x_{1m} \in D_t$, $x_{1m} < x$ and $x_{1m} \to x-0$. In terms of (4.7) and Lemma 4.2(iii), we have

$$\lim_{m\to\infty} u(x_{1m},t) = \lim_{m\to\infty} b(\frac{x_{1m}-\xi(x_{1m},t)}{t}) = b(\frac{x-\xi(x,t)}{t}). \tag{4.17}$$

In the same way, we also have $u(x+0,t)=b(\frac{x-\eta(x,t)}{t})$.

To prove $u(x+0,t)\le u(x,t)\le u(x-0,t)$, we only need to study the case when $\xi(x,t)\ne\eta(x,t)$. In this situation, $x=X(\xi,t)$ holds for any $\xi\in(\xi(x,t),\eta(x,t))$. On the other hand, Lemma 3.2(iv) infers that $(X_n(\xi,t),t)\in\Delta(X_n(\xi,t+h),t+h)$ for any $h>0$. Let $x_n(h)=X_n(\xi,t+h),\ x(h)=X(\xi,t+h)$. Then

$$\frac{x_n(h)-X_n(\xi,t)}{h}\le\frac{x_n(h)-\xi_n(x_n(h),t+h)}{t+h}. \tag{4.18}$$

Let $n\to\infty$, we get from Lemma 4.3(ii)

$$\frac{x(h)-x}{h}\le\frac{x(h)-\xi(x(h),t+h)}{t+h}. \tag{4.19}$$

Letting $h\to 0$ gives

$$X'(\xi,t)\le\frac{x-\xi(x,t)}{t}=f(u(x-0,t)). \tag{4.20}$$

This yields $u(x,t)\le u(x-0,t)$. In the same way, we also have $u(x,t)\ge u(x+0,t)$. Therefore Lemma 4.7 is proved.

To prove theorem 1, it is sufficient to prove,

$$\begin{aligned}&\frac{dq}{dm}=u(x,t),\frac{d\theta}{dm}=f(u(x,t)),\\&\frac{de}{dm}=u(x,t)f(u(x,t))\end{aligned} \tag{4.21}$$

hold for almost everywhere x in the sense of measure m_x due to Lemma 4.5.

If there exists a neighborhood $\mathfrak{N}$ of (x,t) such that $\xi(x,t)$ is constant in $\mathfrak{N}$, then the functions $m(x,t),q(x,t),\theta(x,t)$ and $e(x,t)$ are constants in $\mathfrak{N}$. This case is trivial. So we always assume that for any $x_1<x<x_2$, $m(x_2,t)>m(x_1,t)$ holds. Let $y_1=\xi(x_1,t)$, $y_2=\xi(x_2,t)$, then $y_1<y_2$ and

$$y_1\to\xi(x,t),\ y_2\to\eta(x,t),\ \text{as } x_1,x_2\to x, \tag{4.22}$$

due to Lemma 4.2.

We first consider the case when $\xi(x,t)=\eta(x,t)$. It is noted that for any $\varepsilon>0$, $X(y_2-\varepsilon,t)<x_2$. Then we compute by Lemma 3.7,

$$\begin{aligned}&\frac{q(x_2,t)-q(x_1,t)}{m(x_2,t)-m(x_1,t)}=\lim_{\varepsilon\to+0}\lim_{n\to 0}\frac{\int_{y_1}^{y_2-\varepsilon}\rho_n(\xi,0)u_n(\xi,0)d\xi}{\int_{y_1}^{y_2-\varepsilon}\rho_n(\xi,0)d\xi}\\ \le\ &\lim_{\varepsilon\to+0}\lim_{n\to 0}b(\frac{X_n(y_2-\varepsilon,t)-y_1}{t})\le\lim_{\varepsilon\to+0}b(\frac{X(y_2-\varepsilon,t)-y_1}{t})\le b(\frac{x_2-y_1}{t}).\end{aligned} \tag{4.23}$$

In the same fashion, we also have

$$\frac{q(x_2,t)-q(x_1,t)}{m(x_2,t)-m(x_1,t)}\ge b(\frac{x_1-y_2}{t}). \tag{4.24}$$

Let $x_2,x_1\to x\pm 0$ in (4.23) and (4.24), we get

$$\lim_{x_2,x_1\to x\pm 0}\frac{q(x_2,t)-q(x_1,t)}{m(x_2,t)-m(x_1,t)}=b(\frac{x-\xi(x,t)}{t})=u(x,t). \tag{4.25}$$

When $\xi(x,t) \neq \eta(x,t)$, Lemma 4.2 yields

$$\lim_{x_2,x_1 \longrightarrow x\pm 0} \frac{q(x_2,t)-q(x_1,t)}{m(x_2,t)-m(x_1,t)} = \frac{\int_{\xi(x,t)}^{\eta(x,t)} \rho_0(\xi)u_0(\xi)d\xi}{\int_{\xi(x,t)}^{\eta(x,t)} \rho_0(\xi)d\xi} = u(x,t). \tag{4.26}$$

Thus we have $q_x = um_x$. It is noted that $u(x,t) = b(X'(\xi,t))$ for any $\xi \in (\xi(x,t),\eta(x,t))$. In the same way as in (4.22-4.26), (4.21) also holds for $\theta(x,t)$ and $e(x,t)$. Hence $(m,u)(x,t)$ defined in (4.5) and (4.7) satisfies (1.6) from Lemma 4.5 and (4.21). By the explicit formulas of $m(x,t)$ and $u(x,t)$, it is easy to see (1.7) and (1.8) hold. On the other hand, for any $x_1 < x_2$, we calculate from Lemma 4.7,

$$\begin{aligned} \frac{u(x_2,t)-u(x_1,t)}{x_2-x_1} &\leq \frac{u(x_2-0,t)-u(x_1+0,t)}{x_2-x_1} \\ &= \frac{b(\frac{x_2-\xi(x_2,t)}{t}) - b(\frac{x_1-\eta(x_1,t)}{t})}{x_2-x_1} \leq \frac{E}{t}, \end{aligned} \tag{4.27}$$

due to $\eta(x_1,t) \leq \xi(x_2,t)$, where $E = \max_{|u|\leq M}\{b'(f(u))\}$. Therefore $(\rho = m_x, u)(x,t)$ is an entropy solution of (1.1) and (1.2) and theorem 1 is proved.

Remark. From the definition of $\xi(x,t),\eta(x,t)$, the discontinuity of $m(x,t)$ only takes place on set S, where $S = \{(x,t); \xi(x,t) \neq \eta(x,t), t > 0\}$. It is obvious that S consists of at most countable Lipschitz continuous curves.

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