Kinetic Schemes for the Ultra-Relativistic Euler Equations *

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Abstract. We present a kinetic numerical scheme for the relativistic Euler equations, which describe the flow of a perfect fluid in terms of the particle density n, the spatial part of the four-velocity $\mathbf{u}$ and the pressure p. The kinetic approach is very simple in the ultra-relativistic limit, but may also be applied to more general cases. The basic ingredients of the kinetic scheme are the phase-density in equilibrium and the free flight. The phase-density generalizes the non-relativistic Maxwellian for a gas in local equilibrium. The free flight is given by solutions of a collision free kinetic transport equation. The scheme presented here is an explicit method and unconditionally stable. We establish that the conservation laws of mass, momentum and energy as well as the entropy inequality are everywhere exactly satisfied by the solution of the kinetic scheme. For that reason we obtain weak admissible Euler solutions including arbitrarily complicated shock interactions. In the numerical case studies the results obtained from the kinetic scheme are compared with the first order upwind and centered schemes.

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1 Introduction

Relativity plays an important role in areas of astrophysics, high energy particle beams, high energy nuclear collisions, and free-electron laser technonolgy. The equations that describe the relativistic gas dynamics are highly nonlinear. For the practical problems it is difficult to solve these equations analytically, therefore numerical solutions are persued. A good introduction about the recent methods applied to relativistic gas dynamics can be found in the review article of Martí and Müller [23]. It is noted that, except the *kinetic beam scheme* of Yang et. al [31], all other methods developed for the relativistic Euler equations are based on macroscopic continuum description.

Kinetic approaches in order to solve the classical Euler equations of gas dynamics were successfully applied to several initial- and boundary value problems, see for example Reitz [25], Deshpande and Raul [7], Deshpande [8, 9, 10], Xu [28, 29, 30], Dreyer and Kunik [11], Junk [21]. Some interesting links between the Euler system and the so called kinetic BGK-model, which was introduced by Bhatnagar, Gross and Krook [1], are discussed in the textbooks by Cercignani [2] as well as by Godlewski and Raviart [17].

In the kinetic theory of gases there is a fundamental quantity $f = f(t, \mathbf{x}, \mathbf{q})$ called *phase density*, which is usually a function of time t, position $\mathbf{x}$ and of the momentum $\mathbf{q}$ related to the single gas atoms. It usually results from a kinetic equation like the *Boltzmann-* or *BGK-equation*, see [2] and [1]. Then the macroscopic thermodynamic quantities like particle density, energy density, velocity, and pressure are tensor-algebraic combinations of some basic integral moments of the phase density f, where the integration is performed with respect to the momentum $\mathbf{q}$ of the gas atoms. Therefore the macroscopic quantities depend only on time and space. They are all completely determined by the phase density f.

Jüttner [20] extended the classical Maxwellian for the Boltzmann gas in equilibrium to the relativistic theory. Later this phase density and the whole relativistic kinetic theory was written in a manifest Lorentz-invariant form, see the articles of Chernikov [3], [4], and the textbook deGroot, van Leeuven and van Weert [18]. The equilibrium phase-density for the Fermi- and Bose gas and the corresponding equations of state were also given by Jüttner [22].

The hyperbolic systems that can be treated by the kinetic method are those which may be generated from kinetic transfer equations and from the *Maximum Entropy Principle*. Since these systems lead to a convex entropy function, they enable several rigorous mathematical results, see for example Friedrichs and Lax [16] as well as Dafermos [6]. In the case of *thermodynamical equilibrium* the Maximum Entropy Principle constitutes a successful method in order to obtain the Maxwellian phase density for the Boltzmann gas as well as the corresponding phase densities for the Fermi- and Bose gas in equilibrium from the corresponding kinetic entropy definitions.

A kinetic scheme consists of the following ingredients:

(i) The first ingredient is the kinetic solution of a collisionless transport equation for the phase density. This is the solution in the so called free-flight period of duration $\tau_M > 0$, where τ_M is the time-step of the numerical scheme.

(ii) At the maximization times $t_n = n\tau_M$ ($n = 0, 1, 2, \ldots$), the beginning of each of these free-flight periods, the gas particles are in local equilibrium, which is described by Jüttners relativistic generalization of the classical Maxwellian phase density.

(iii) At each new maximization time $t_n > 0$ we evaluate the so called continuity conditions, which guarantee that the kinetic scheme satisfies the conservation laws and the entropy inequality. These continuity conditions determine the new initial data at t_n. The fluid dynamic variables like mass, momentum and energy density are obtained from moments of the free-flight phase density. These variables serve as initial data for the next period of free-flight. The macroscopic quantities will solve the relativistic Euler equations in the limit $\tau_M \to 0$.

The main philosophy of the kinetic schemes results from the fact that the underlying kinetic theory models most of the important physical properties inherently. For example, the kinetic solutions preserve the positivity of the particle density and pressure. Further, they satisfy the entropy inequality as well as the L_1-stablity, see [8, 11]. The kinetic schemes presented here are explicit methods and unconditionally stable. Because of its explicit nature the method is highly vectorizable and satisfies the total variation diminishing (TVD) property, see [8]. The extension of the scheme to higher dimensions is straight forward, and we will indeed start with presenting the scheme in three space dimensions. One does not need a dimensional splitting type approach in more than one space dimension. Despite these advantages we have also to mention two short comings of these kinetic schemes, namely they are numerically expensive and secondly the numerical dissipation is of the order of the time step τ_M, see [8]. The later is due to the fact that the schemes are only of first order. A higher order extension is possible in a similar manner as in [8] and will be considered in future work.

It is important to note that the basic ideas of this study can also be applied to other hyperbolic systems and kinetic equations and can be extended to initial and boundary value problems. For example, one application area is the evolution of temperature and heat flux in a Bose gas of phonons, see Dreyer, Herrmann and Kunik [12]. There are also extensions of the kinetic approach to handle multicomponent flows and magnetohydrodynamics equations, see Xu [27, 29].

In this paper we will formulate a kinetic scheme in order to solve the initial value problem for the *ultra-relativistic Euler equations*. The kinetic scheme presented here is discrete in time but continuous in space. Euler's equations (relativistic or classic) deal with a *perfect gas*, in which mean free paths and collision free times are so short that perfect isotropy is maintained about any point moving with the gas. In this case the *local equilibrium assumption* is satisfied and the corresponding phase-densities are obtained from the Maximum Entropy Principle in equilibrium. In the textbook of Weinberg [26] one can find a short introduction to special relativity and relativistic hydrodynamics with further literature also for the imperfect fluid (gas), see for example the papers of Eckart [13], [14], [15]. The *ultra-relativistic* case is the limit either of high temperature for non-zero rest mass or for a fixed temperature at very small rest mass of the gas atoms. We find the latter point of view more natural for our purposes since this corresponds to writing the Euler equations in a rescaled dimensionless form.

Now we give a short overview of this paper:

In Section 2 we will present the basic definitions of the relativistic kinetic theory, namely the macroscopic quantities considered in thermodynamics which are obtained from a kinetic phase density. Moreover the relativistic Maxwellian studied by Jüttner [20] is introduced and two limiting cases are considered, the classical Maxwellian for a cool, nonrelativistic gas and the ultra-relativistic Jüttner phase density. The former case is not pursued further here. It will be considered in a forthcoming paper.

In Section 3 we calculate the macroscopic moments of the relativistic Maxwellian in order to formulate the ultra-relativistic Euler equations as conservation laws for the particle number, momentum, and energy. The Euler equations are written in differential form as well as in a weak integral form. An entropy inequality is given in weak integral form with an entropy function which satisfies the Gibbs equation. The Rankine-Hugoniot conditions and the entropy inequality are used in order to derive a simple parameter representation for the admissible shocks.

In Section 4 we first formulate the kinetic scheme in order to solve the three-dimensional ultra-relativistic Euler equations. In contrast to the classical three-dimensional Euler equations for a non-relativistic gas we will show that the three-fold momentum integrals for the particle-density four-vector and for the energy-momentum tensor reduce simply to surface integrals where the integration is performed with respect to the unit sphere. A similar idea was used by Dreyer and Kunik [12] in order to solve the hyperbolic moment systems for a phonon Bose-gas, resulting from the Boltzmann-Peierls equation and maximum entropy principle. Then we prove that the conservation laws and the entropy inequality are satisfied for this scheme. For this purpose the continuity conditions for the zero components of the moments play a crucial role, more precisely they constitute necessary conditions in order to guarantee that the conservation laws and the entropy inequality are also satisfied across the maximization times. The continuity conditions are also required in order to initialize the kinetic scheme for the next time step.

In Section 5 we are looking at the special case of spatially one-dimensional solutions which are nevertheless solutions to the three dimensional ultra-relativistic Euler equations. In this case the surface integrals of the three-dimensional kinetic scheme reduce again to single integrals which range from -1 to $+1$. They indicate the finite domain of dependence on the preceeding initial data, which is covered by the backward light-cones. This property does not hold for classical kinetic schemes.

In Section 6 we discuss the Eulerian limit $\tau_M \to 0$ of the kinetic scheme where weak solutions are obtained from the initial value problems including arbitrary complicated shock interactions.

In Section 7 we compute a numerical a test case with an explicitly known single shock solution from Section 3. Moreover we compute a Riemann solution which shows a shock, a contact discontinuity and a rarefaction wave.

2 The Relativistic Kinetic Phase Density and its Moments

In this section we describe a relativistic gas consisting of many microscopic structureless particles in terms of the relativistic kinetic phase density. From this fundamental phase density we calculate tensorial moments which give the local macroscopic physical quantities of the gas such as the particle density, the velocity, the pressure, the temperature and so on.

In order to formulate the theory in a Lorentz-invariant form, we make use of the notations for the tensor calculus used in the textbook of Weinberg [26], with only slight modifications:

A) **The space-time coordinates** are x^μ, $\mu = 0, 1, 2, 3$, with $x^0 := ct$ for the time, x^1, x^2, x^3 for the position.

B) **The metric-tensor** is

$$g_{\mu\nu} = g^{\mu\nu} = \begin{cases} +1, & \mu = \nu = 0\,, \\ -1, & \mu = \nu = 1, 2, 3\,, \\ 0, & \mu \neq \nu\,. \end{cases}$$

C) **The proper Lorentz-transformations** are linear transformations ${\Lambda^\alpha}_\beta$ from one system of space-time with coordinates x^α to another system x'^α. They must satisfy

$$x'^\alpha = {\Lambda^\alpha}_\beta \, x^\beta , \;\; g_{\mu\nu} = {\Lambda^\alpha}_\mu \, {\Lambda^\beta}_\nu \, g_{\alpha\beta} , \;\; {\Lambda^0}_0 \geq 1 , \;\; \det \Lambda = +1 \, .$$

The conditions ${\Lambda^0}_0 \geq 1$ and $\det \Lambda = +1$ are necessary in order to exclude inversion in time and space. Then the following quantity forms a tensor with respect to proper Lorentz-transformations, the so called Levi-Civita tensor:

$$\epsilon_{\alpha\beta\gamma\delta} = \begin{cases} +1, & \alpha\beta\gamma\delta \text{ even permutation of } 0123, \\ -1, & \alpha\beta\gamma\delta \text{ odd permuation of } 0123\,, \\ 0, & \text{otherwise.} \end{cases}$$

Note that in the textbook of Weinberg [26] this tensor as well as the metric tensor both take the sign opposite to the notation used here.

D) **Einstein's summation convention:**
Any Greek index like α, β, that appears twice, once as a subscript and once as a superscript, is understood to be summed over 0,1,2,3 if not noted otherwise. For spatial indices, which are denoted by Latin indices like i, j, k, we will not apply this summation convention.

First we take a microscopic look at the gas and start with the kinematics of a representative gas atom with particle trajectory $\mathbf{x} = \mathbf{x}(t)$, where the time coordinate t and the space coordinate $\mathbf{x}$ are related to an arbitrary Lorentz-frame. The invariant mass of all structureless particles is assumed to be the same and is denoted by m_0. The microscopic velocity of the gas atom is $\frac{d\mathbf{x}(t)}{dt}$, and its microscopic velocity four-vector is given by $c\, q^\mu$, where the *dimensionless microscopic velocity four-vector* q^μ is defined by

$$\left(q^0, \mathbf{q}\right)^T , \qquad q^0 = q_0 = \sqrt{1+\mathbf{q}^2} \, , \qquad \mathbf{q} = \frac{\frac{1}{c}\frac{d\mathbf{x}}{dt}}{\sqrt{1-\left(\frac{1}{c}\frac{d\mathbf{x}}{dt}\right)^2}} \, . \tag{2.3}$$

The *relativistic phase density* $f(t, \mathbf{x}, \mathbf{q}) \geq 0$ is the basic quantity of the kinetic theory. A physical interpretation of the phase density will be discussed when we derive the basic quantities like particle density, velocity four-vector and pressure from its moments defined below. We make use of the fact that the so called *proper volume element* d^3q/q_0 is invariant with respect to Lorentz-transformations in order to define the

Macroscopic moments and entropy four-vector:

(i) *Particle-density four-vector*

$$N^\mu = N^\mu(t, \mathbf{x}) = \int_{\mathbb{R}^3} q^\mu \, f(t, \mathbf{x}, \mathbf{q}) \, \frac{d^3q}{q^0} \, , \tag{2.4}$$

(ii) *energy-momentum tensor*

$$T^{\mu\nu} = T^{\mu\nu}(t,\mathbf{x}) = m_0 c^2 \int_{\mathbb{R}^3} q^\mu q^\nu \, f(t,\mathbf{x},\mathbf{q}) \, \frac{d^3q}{q^0} \,, \tag{2.5}$$

(iii) *entropy four-vector*

$$S^\mu = S^\mu(t,\mathbf{x}) = -k_b \int_{\mathbb{R}^3} q^\mu \, f(t,\mathbf{x},\mathbf{q}) \, \ln\left(\frac{f(t,\mathbf{x},\mathbf{q})}{\chi}\right) \frac{d^3q}{q^0} \,. \tag{2.6}$$

Here $k_b = 1.38062 \times 10^{-23} J/K$ is Boltzmann's constant and $\chi = (\frac{m_0 c}{\hbar})^3$ with Planck's constant $\hbar = 1.05459 \times 10^{-34} Jsec$. Note that χ has the same dimension as f, namely 1/volume. We also state here that the entropy formula (2.6) can be generalized easily in such a way, that the well known case of a Fermi- or Bose gas is also included in this kinetic framework. Then formula (2.6) reads in the general case

$$S^\mu = -k_b \int_{\mathbb{R}^3} q^\mu \left[f \ln\frac{f}{\chi} - \eta\chi \left(1 + \eta\frac{f}{\chi}\right) \ln\left(1 + \eta\frac{f}{\chi}\right)\right] \frac{d^3q}{q^0} \,. \tag{2.7}$$

Here $\eta = 0$ reduces to (2.6), which is valid for the relativistic generalization of Boltzmann's statistic, whereas $\eta = +1$ is required for the Bose-Einstein statistic and $\eta = -1$ for the Fermi statistic.

Note that the spatial part $\mathbf{q} \in \mathbb{R}^3$ of the dimensionless microscopic velocity four-vector is used as an integration variable in the relativistic kinetic theory.
Now we may use the macroscopic moments N^μ, $T^{\mu\nu}$ and S^μ of the relativistic phase density f in order to calculate the other macroscopic quantities of the gas, which are

Tensor algebraic combinations of these moments:

(i) *The proper particle density*

$$n = \sqrt{N^\mu N_\mu} \,, \tag{2.8}$$

(ii) *the dimensionless velocity four-vector*

$$u^\mu = \frac{1}{n} N^\mu \,, \tag{2.9}$$

(iii) *the proper energy density*

$$e = u_\mu u_\nu T^{\mu\nu} \,, \tag{2.10}$$

(iv) *the proper pressure and temperature*

$$p = \frac{1}{3}\left(u_\mu u_\nu - g_{\mu\nu}\right) T^{\mu\nu} = k_b\, nT\,, \tag{2.11}$$

(v) *the proper entropy density*

$$\sigma = S^\mu u_\mu\,. \tag{2.12}$$

Remarks:

(i) Since $f \geq 0$, it can be shown that N^μ is a time-like vector, $N^\mu N_\mu > 0$, if f does not vanish almost everywhere for any fixed t, $\mathbf{x}$, which will be assumed in the following. It follows that the particle density n is well defined and positive. In order to see that the energy density is always positive we write it in the form

$$e = m_0 c^2 \int_{\mathbb{R}^3} (u_\mu q^\mu)^2\, f(t, \mathbf{x}, \mathbf{q})\, \frac{d^3 q}{q^0}\,. \tag{2.13}$$

(ii) The *macroscopic velocity* $\mathbf{v}$ of the gas can be obtained easily from the spatial part $\mathbf{u} = (u^1, u^2, u^3)^T$ of the dimensionless velocity four vector by

$$\mathbf{v} = c\,\frac{\mathbf{u}}{\sqrt{1+\mathbf{u}^2}}\,. \tag{2.14}$$

From this formula we can immediately read off that $|\mathbf{v}| < c$, i.e. the absolute value of the velocity is bounded by the speed of light. Note also that $u^0 = \sqrt{1+\mathbf{u}^2}$.

The attribute 'proper' denotes a Lorentz-invariant quantity, which takes its simplest form with respect to a Lorentz-frame where the gas is locally at rest. Since all quantities under consideration are written down in Lorentz-invariant form, we may omit the word 'proper' in the following.
These definitions are valid for any relativistic phase-density $f = f(t, \mathbf{x}, \mathbf{q})$, which has to be determined from a *kinetic equation* of the following form

$$q^\mu \frac{\partial f}{\partial x^\mu} = Q(f)\,. \tag{2.15}$$

As in the non-relativistic kinetic theory we have a corresponding transport part on the left-hand side and a collision part $Q(f)$ on the right-hand side. In the simplest case $Q(f)$ is determined in such a way that the following five conservation laws hold for the particle number, the energy, and the momentum

$$\frac{\partial N^\mu}{\partial x^\mu} = 0\,, \qquad \frac{\partial T^{\mu\nu}}{\partial x^\nu} = 0\,. \tag{2.16}$$

This simple case holds if the particles interact only during elastic collisions without other forces and radiation. It will nevertheless lead to an interesting and self-consistent relativistic thermodynamics, even if it is physically not realizable.

Jüttner extended the classical velocity distribution of Maxwell for a gas in equilibrium to the relativistic case. The resulting *Jüttner distribution* $f_J(n, T, \mathbf{u}, \mathbf{q})$ depends on five constant parameters, which describe the state of the gas in equilibrium, namely the particle density n, the absolute temperature T and the spatial part $\mathbf{u} \in \mathbb{R}^3$ of the dimensionless four-velocity. It is given by

$$\begin{aligned} f_J(n, T, \mathbf{u}, \mathbf{q}) &= \frac{n}{M(\beta)} \exp\left(-\beta\, u_\mu q^\mu\right) \\ &= \frac{n}{M(\beta)} \exp\left(-\beta \left(\sqrt{(1+\mathbf{u}^2)(1+\mathbf{q}^2)} - \mathbf{u}\cdot\mathbf{q}\right)\right), \end{aligned} \tag{2.17}$$

where

$$\beta = \frac{m_0 c^2}{k_b T}, \quad M(\beta) = 4\pi \int_0^\infty \vartheta^2 \exp(-\beta\sqrt{1+\vartheta^2})\, d\vartheta\,. \tag{2.18}$$

The function $M(\beta)$ is choosen in such a way that

$$n\, u^\mu = \int_{\mathbb{R}^3} q^\mu f_J(n, T, \mathbf{u}, \mathbf{q}) \frac{d^3 q}{q^0} \tag{2.19}$$

holds for the spatial part $\mathbf{u} = (u^1, u^2, u^3)^T$ of the dimensionless macroscopic velocity four-vector. This is equation (2.9), where $\mathbf{u}$ and n are in addition parameters of Jüttner's relativistic phase density. Using the Bessel functions

$$K_n(\beta) = \int_0^\infty \cosh(ns)\, \exp(-\beta\, \cosh(s))\, ds \tag{2.20}$$

and applying the substitution $\vartheta = \sinh(s)$ we may also write $M(\beta)$ in the form

$$M(\beta) = \frac{4\pi}{\beta} K_2(\beta)\,. \tag{2.21}$$

Now we shall discuss two important special cases for this phase density, namely the non-relativistic limit for a cool gas and the ultra-relativistic limit $m_0 \to 0$.

The non-relativistic limit (small temperatures, small velocities):

For the first case we rewrite (2.17) in the form

$$f_J(n, T, \mathbf{u}, \mathbf{q}) = \frac{n}{M_1(\beta)} \exp\left(-\beta \frac{(\mathbf{q}-\mathbf{u})^2 + \mathbf{u}^2\mathbf{q}^2 - (\mathbf{u}\cdot\mathbf{q})^2}{1 + \sqrt{(1+\mathbf{u}^2)(1+\mathbf{q}^2)} + \mathbf{u}\cdot\mathbf{q}}\right), \tag{2.22}$$

where

$$M_1(\beta) = M(\beta)\, \exp(\beta)\,. \tag{2.23}$$

If we apply for $\vartheta > 0$ the integral substitution

$$\xi = \sqrt{2\beta(\sqrt{1+\vartheta^2}-1)}\,, \tag{2.24}$$

then we can rewrite $M_1(\beta)$ in the form

$$M_1(\beta) = \left(\frac{2\pi}{\beta}\right)^{\frac{3}{2}} \cdot 2 \int_0^\infty \left(1+\frac{\xi^2}{2\beta}\right) \sqrt{1+\frac{\xi^2}{4\beta}} \cdot \frac{\exp(-\frac{\xi^2}{2})}{\sqrt{2\pi}}\, d\xi\,. \tag{2.25}$$

For $\beta \to \infty$, i.e. for small temperature, we can conclude from (2.25) that

$$M_1(\beta) = (\frac{2\pi}{\beta})^{\frac{3}{2}} + O(\beta^{-\frac{5}{2}}) = \left(\frac{2\pi k_b T}{m_0 c^2}\right)^{\frac{3}{2}} + O(T^{\frac{5}{2}})\,,$$

and the representation (2.22) shows that the Jüttner phase density reduces to the non-relativistic Maxwellian for $|\mathbf{u}|$, $|\mathbf{q}|$ very small, namely

$$f_c(n, T, \mathbf{u}, \mathbf{q}) = n \left(\frac{m_0 c^2}{2\pi k_b T}\right)^{\frac{3}{2}} \exp\left[-\frac{m_0 c^2 (\mathbf{q}-\mathbf{u})^2}{2K_b T}\right]\,. \tag{2.26}$$

The ultra-relativistic limit (zero rest mass of the particles):

For the ultra-relativistic limit $m_0 \to 0$ with fixed temperature we apply the substitution $\mathbf{q}' = m_0.\mathbf{q}$ in order to write (2.17) in the form

$$f_J(n, T, \mathbf{u}, \mathbf{q}) = m_0^3 \frac{n}{M_2(\tilde{\beta})} \exp\left(-\tilde{\beta}\left(\sqrt{(1+\mathbf{u}^2)(m_0^2+\mathbf{q}'^2)} - \mathbf{u}\cdot\mathbf{q}'\right)\right)\,, \tag{2.27}$$

where

$$\tilde{\beta} = \frac{\beta}{m_0} = \frac{c^2}{k_B T}\,, \quad M_2(\tilde{\beta}) = \frac{8\pi}{\tilde{\beta}^3} \int_0^\infty \frac{\xi^2}{2} \exp\left(-\sqrt{m_0^2\tilde{\beta}^2+\xi^2}\right) d\xi\,. \tag{2.28}$$

In the following we do not use primes for the new integration variable $\mathbf{q}$.

Now we are able to pass to the ultra-relativistic limit $m_0 \to 0$. In order to do this we first have to replace the four-vector q^μ defined in (2.3) by the light vector

$$\left(q^0, \mathbf{q}\right)^T\,, \qquad q^0 = q_0 = |\mathbf{q}|\,. \tag{2.29}$$

Next we will introduce dimensionless quantities by setting $c = k_B = \hbar = 1$. Then the ultra-relativistic moments and entropy four-vector take a similar form as given in (2.4), (2.5) and (2.6)

$$N^\mu = N^\mu(t,\mathbf{x}) = \int_{\mathbb{R}^3} q^\mu\, f(t,\mathbf{x},\mathbf{q})\, \frac{d^3q}{|\mathbf{q}|}\,, \tag{2.30}$$

$$T^{\mu\nu} = T^{\mu\nu}(t,\mathbf{x}) = \int_{\mathbb{R}^3} q^\mu q^\nu\, f(t,\mathbf{x},\mathbf{q})\, \frac{d^3q}{|\mathbf{q}|}\,, \tag{2.31}$$

and the macroscopic *entropy four-vector*

$$S^\mu = S^\mu(t,\mathbf{x}) = -\int_{\mathbb{R}^3} q^\mu\, f(t,\mathbf{x},\mathbf{q})\, \ln\left(\frac{f(t,\mathbf{x},\mathbf{q})}{\chi}\right) \frac{d^3q}{|\mathbf{q}|}\,, \tag{2.32}$$

where $\chi = (\frac{m_0 c}{\hbar})^3$.
Here f is first taken as the ultra-relativistic Jüttner phase density (2.27) in its dimensionless form as

$$\begin{aligned} f_J^*(n,T,\mathbf{u},\mathbf{q}) &= \frac{n}{8\pi T^3}\, \exp\left(-\frac{u_\mu q^\mu}{T}\right) \\ &= \frac{n}{8\pi T^3}\, \exp\left(-\frac{|\mathbf{q}|}{T}\left(\sqrt{1+\mathbf{u}^2} - \mathbf{u}\cdot\frac{\mathbf{q}}{|\mathbf{q}|}\right)\right)\,. \end{aligned} \tag{2.33}$$

But we will also use equations (2.30), (2.31) and (2.32) in order to define new moments for a general phase density f. In this case we will again call (2.30), (2.31) and (2.32) the moments for the ultra-relativistic limit. For that reason we have used the general symbol f for the new moments instead of f_J^*. These more general definitions will be important for the formulation of kinetic schemes in order to solve the fluid dynamic equations in the ultra-relativistic limit.

Finally it is important to note that all the definitions given for the particle density n, velocity four-vector u^μ, energy density e and for the pressure p, which are tensor invariant algebraic combinations of the basic moments N^μ and $T^{\mu\nu}$, are still valid for an arbitrary phase density f in the ultra-relativistic limit. Neverthless we can still simplify the generally valid formula (2.11) for the pressure in the ultra-relativistic limit

$$\begin{aligned} p &= \frac{1}{3}\, u_\mu u_\nu T^{\mu\nu} - \frac{1}{3}\, g_{\mu\nu} T^{\mu\nu} \\ &= \frac{1}{3}\, u_\mu u_\nu T^{\mu\nu} - \int_{\mathbb{R}^3} g_{\mu\nu} q^\mu q^\nu\, f\, \frac{d^3q}{q^0}\,. \end{aligned} \tag{2.34}$$

Since $g_{\mu\nu} q^\mu q^\nu = q_\nu q^\nu = 0$ holds due to (2.29), we immediately conclude that

$$p = \frac{e}{3} = \frac{1}{3}\, T^{\mu\nu} u_\mu u_\nu = nT \tag{2.35}$$

in the ultra-relativistic case.

3 The Ultra-Relativistic Euler Equations

Using the ultra-relativistic Jüttner distribution (2.33) we obtain for the moments (2.30), (2.31) and (2.32)

$$N^\mu = n\, u^\mu, \qquad T^{\mu\nu} = -p\, g^{\mu\nu} \,+\, 4pu^\mu u^\nu\,, \tag{3.1}$$

$$S^\mu = -N^\mu \,\ln\frac{n^4}{p^3} + \gamma N^\mu, \qquad \sigma = -n\ln\frac{n^4}{p^3} + \gamma n \tag{3.2}$$

where γ may be any real dimensionless constant. Note that due to the mass conservation (2.16) the divergence of S^μ, which will give rise to the H-theorem formulated later, will not change when we add some multiple of N^μ to S^μ. Moreover σ obeys the Gibbs equation

$$T d\left(\frac{\sigma}{n}\right) = p\, d\left(\frac{1}{n}\right) + d\left(\frac{e}{n}\right). \tag{3.3}$$

These formulas can be easily checked for a special Lorentz frame where $u^0 = 1$, $u^1 = u^2 = u^3 = 0$, i.e. where the gas is locally at rest. Since the ultra-relativistic moments (3.1) are valid in a special Lorentz frame and since these equations are written in tensor invariant form, they are generally valid in every Lorentz frame. Using the moments (3.1) and the conservation laws (2.16), we get at regular points the three dimensional Euler equation in differential form

$$\frac{\partial}{\partial t}\,(n\,\sqrt{1+\mathbf{u}^2}) + \nabla\cdot(n\,\mathbf{u}) = 0\,, \tag{3.4}$$

$$\frac{\partial}{\partial t}(4pu^i\sqrt{1+\mathbf{u}^2}) + \sum_{k=1}^{3}\frac{\partial}{\partial x^k}\,(p\,\delta^{ik} + 4pu^iu^k) = 0\,, \tag{3.5}$$

$$\frac{\partial}{\partial t}\,(3p + 4p\mathbf{u}^2) + \sum_{k=1}^{3}\frac{\partial}{\partial x^k}\,(4pu^k\sqrt{1+\mathbf{u}^2}) = 0\,. \tag{3.6}$$

At regular points the function is continuously differentiable with repect to time and space. Note that the equations (3.5), (3.6) are a closed 4 by 4 system for p and $\mathbf{u}$. The relativistic continuity equation (3.4) decouples from the system. For given $\mathbf{u}$ it is a scalar equation for n.

Now we are looking for special solutions of the three dimensional ultra-relativistic Euler equations, which will not depend on x^2, x^3 but only on $x = x^1$. Moreover we restrict to a one-dimensional flow field $\mathbf{u} = (u(t,x),0,0)^T$

$$\begin{aligned}(n\sqrt{1+u^2})_t + (nu)_x &= 0\,,\\ (4pu\sqrt{1+u^2})_t + (p(1+4u^2))_x &= 0\,,\\ (p(3+4u^2))_t + (4pu\sqrt{1+u^2})_x &= 0\,.\end{aligned} \tag{3.7}$$

Note that these differential equations constitute a strictly hyperbolic system with the characteristic velocities

$$\lambda_1 = \frac{2u\sqrt{1+u^2}-\sqrt{3}}{3+2u^2}\,,\ \lambda_2 = \frac{u}{\sqrt{1+u^2}}\,,\ \lambda_3 = \frac{2u\sqrt{1+u^2}+\sqrt{3}}{3+2u^2}\,. \tag{3.8}$$

These eigenvalues may first be obtained in the Lorentz rest frame where $u = 0$. Then using the relativistic additivity law for the velocities, we can easily obtain (3.8) in the general Lorentz frame. In the Lorentz rest frame we obtain the positive speed of sound $\lambda = \frac{1}{\sqrt{3}}$, which is independent of the spatial direction.

The differential equations (3.7) are not sufficient if we take shock discontinuities into account. Therefore we choose a weak integral formulation which is given due to Oleinik [24] by curve integrals in time and space, namely

$$\begin{aligned}
\oint_{\partial\Omega} n\sqrt{1+u^2}dx - nudt &= 0\,,\\
\oint_{\partial\Omega} 4pu\sqrt{1+u^2}dx - p(1+4u^2)dt &= 0\,,\\
\oint_{\partial\Omega} p(3+4u^2)dx - 4pu\sqrt{1+u^2}dt &= 0\,.
\end{aligned} \tag{3.9}$$

Here $\Omega \subset \mathbb{R}_0^+ \times \mathbb{R}$ is a normal region in space-time and with a piecewise smooth, positively oriented boundary. Note that this weak formulation takes discontinuities into account, since there are no derivatives of the fields involved.The use of Oleinik's formulation enables a direct proof of conservation laws and entropy inequilty which are given in Section 4. If we apply the Gaussian divergence theorem to the weak formulation (3.9) in time-space regions where the solution is regular we come back to the differential equation form of the Euler equations (3.7).

Furthermore we require that the weak solution (3.9) must also satisfy the *entropy-inequality*

$$\oint_{\partial\Omega} S^0 dx - S^1 dt \geq 0 \tag{3.10}$$

where

$$S^0 = -n\sqrt{1+u^2}\,\ln\frac{n^4}{p^3}\,, \qquad S^1 = -nu\,\ln\frac{n^4}{p^3}\,. \tag{3.11}$$

Now we consider bounded and integrable *initial data* for a positive particle density n, transformed velocity u and absolute temperature T, which may have jumps

$$n(0,x) = n_0(x) > 0, \qquad u(0,x) = u_0(x), \qquad T(0,x) = T_0(x) > 0. \tag{3.12}$$

If $x = x(t)$ is a shock-discontinuity of the weak solution (3.9) with speed $v_s = \dot{x}(t)$, $W_- = (n_-, u_-, p_-)$ the state left to the shock and $W_+ = (n_+, u_+, p_+)$ the state to the right, then (3.9) leads to the *Rankine-Hugoniot jump conditions*

$$\begin{aligned} v_s[n_+\sqrt{1+u_+^2} - n_-\sqrt{1+u_-^2}] &= n_+u_+ - n_-u_- \,, \\ v_s[4p_+u_+\sqrt{1+u_+^2} - 4p_-u_-\sqrt{1+u_-^2}] &= (p_+ + 4p_+u_+^2) - (p_- + 4p_-u_-^2)\,, \\ v_s[(3p_+ + 4p_+u_+^2) - (3p_- + 4p_-u_-^2)] &= 4p_+u_+\sqrt{1+u_+^2} - 4p_-u_-\sqrt{1+u_-^2}\,. \end{aligned} \tag{3.13}$$

Also at the points of Rankine-Hugoniot shock curves the local form of (3.10) reads

$$-v_s(S_+^0 - S_-^0) + (S_+^1 - S_-^1) \geq 0, \tag{3.14}$$

which must be satisfied on each shock curve of (3.9). A shock that satifies (3.13) and (3.14) is called an *entropy shock*.

Now we give parameter representations for single entropy shocks. For this purpose we choose the initial data as follows:

Let be $(n_*, u_*, T_*) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$ and define $p_* = n_*T_*$.

We use the pressure p as a parameter which determines the strength of an entropy shock. Equations (3.13) and (3.14) are solved by

$$n(p) = n_*\sqrt{\frac{p}{p_*}\left(\frac{3p+p_*}{p+3p_*}\right)}\,, \tag{3.15}$$

$$u(p) = \frac{u_*\sqrt{p_*+3p}\sqrt{p+3p_*} \pm \sqrt{3}(p-p_*)\sqrt{1+u_*^2}}{4\sqrt{pp_*}}\,, \tag{3.16}$$

$$T(p) = \frac{p}{n(p)}\,, \tag{3.17}$$

$$u_s(p) = \frac{u_*\sqrt{3(p+3p_*)} \pm \sqrt{p_*+3p}\sqrt{1+u_*^2}}{\sqrt{8p_*}}\,, \tag{3.18}$$

$$v_s = \frac{u_s}{\sqrt{1+u_s^2}}\,, \quad v = \frac{u}{\sqrt{1+u^2}}\,, \quad v_* = \frac{u_*}{\sqrt{1+u_*^2}} \tag{3.19}$$

in the following way:

- The "+" sign in (3.16), (3.18) and $p > p_*$ gives the so called *3-shocks* with the constant state (n_*, u_*, T_*) on the right

$$(n_-, u_-, T_-) = (n(p), u(p), T(p)), \quad (n_+, u_+, T_+) = (n_*, u_*, T_*).$$

These 3-shocks satisfy both the Rankine-Hugoniot conditions (3.13) as well as the entropy condition (3.14).

- The "-" sign in (3.16), (3.18) and $p > p_0$ gives the so called *1-shocks* with the constant state (n_*, u_*, T_*) on the left:

$$(n_-, u_-, T_-) = (n_*, u_*, T_*), \quad (n_+, u_+, T_+) = (n(p), u(p), T(p)).$$

These 1-shocks satisfy both the Rankine-Hugoniot conditions (3.13) as well as the entropy condition (3.14).

Now we define the *2-shocks*, that turn out to be *contact-discontinuities without entropy-production*:

Only for these we choose $n > 0$ instead of p as a parameter and set

$$(n_-, u_-, T_-) = (n_*, u_*, T_*), \quad (n_+, u_+, T_+) = \left(n, u_*, \frac{n_* T_*}{n}\right).$$

These shocks satisfy the Rankine-Hugoniot- and entropy conditions.

Note that velocity and pressure are constant across a 2-shock. Here the shock-speed is $v_s = v_* = \frac{u_*}{\sqrt{1+u_*^2}}$.

Remark. From the Rankine-Hugoniot jump conditions one can derive by simple algebraic calculations that the only shocks are 1-, 2- and 3-shocks analogously as in the non-relativistic case, see Courant and Friedrichs [5].

4 The Kinetic Scheme for the Euler Equations

We first formulate the scheme for the three dimensional Euler equations. After that we solve the one dimensional Euler equations, using a special integration technique. Recalling the ultra-relativistic Jüttner phase density (2.33), we start with the given initial data $n_I(\mathbf{x}) = n(0, \mathbf{x})$, $T_I(\mathbf{x}) = T(0, \mathbf{x})$, $u_I(\mathbf{x}) = u(0, \mathbf{x})$.
We prescribe a time step $\tau_M > 0$ and let $t_n = n\,\tau_M$ for $n = 0, 1, 2, 3...$ be the maximization times. Then we define the moments and the entropy four-vector in the free flight for $0 < \tau < \tau_M$ as

$$\begin{aligned} N^\mu(t_n + \tau, \mathbf{x}) &= \int_{\mathbb{R}^3} q^\mu\, f_n(\mathbf{x} - \tau\frac{\mathbf{q}}{|\mathbf{q}|}, \mathbf{q}) \frac{d^3q}{|\mathbf{q}|}\,, \\ T^{\mu\nu}(t_n + \tau, \mathbf{x}) &= \int_{\mathbb{R}^3} q^\mu q^\nu\, f_n(\mathbf{x} - \tau\frac{\mathbf{q}}{|\mathbf{q}|}, \mathbf{q}) \frac{d^3q}{|\mathbf{q}|}\,, \end{aligned} \tag{4.1}$$

$$S^\mu(t_n+\tau,\mathbf{x}) = -\int_{\mathbb{R}^3} q^\mu\,(f_n \ln f_n)(\mathbf{x}-\tau\frac{\mathbf{q}}{|\mathbf{q}|},\mathbf{q})\frac{d^3q}{|\mathbf{q}|} \tag{4.2}$$

with the ultra-relativistic initial phase density at the maximization time t_n given as

$$f_n(\mathbf{y},\mathbf{q}) = f_J^*(n(t_n,\mathbf{y}),T(t_n,\mathbf{y}),\mathbf{u}(t_n,\mathbf{y}),\mathbf{q})\,. \tag{4.3}$$

Moreover n, T, u^μ are calculated from N^μ and $T^{\mu\nu}$ for the next time step from the following generally valid definitions

$$n = \sqrt{N^\mu N_\mu}, \qquad u^\mu = \frac{1}{n}N^\mu\,, \qquad T = \frac{1}{3n}u_\mu u_\nu T^{\mu\nu}\,. \tag{4.4}$$

In order to initialize the kinetic scheme for the next time step, we first require the following continuity conditions for the zero-components of the moments across the maximization time t_n, $n \geq 1$

$$\begin{aligned} N^0(t_n^+,\mathbf{x}) &= N^0(t_n^-,\mathbf{x})\,,\\ T^{0k}(t_n^+,\mathbf{x}) &= T^{0k}(t_n^-,\mathbf{x}),\ \ k=1,2,3\,,\\ T^{00}(t_n^+,\mathbf{x}) &= T^{00}(t_n^-,\mathbf{x})\,. \end{aligned} \tag{4.5}$$

Here we have used the following abbreviations for the one-sided limits across the maximization time t_n, $n \geq 1$, where ε is a positive real number

$$\begin{aligned} N^\mu(t_n^\pm,\mathbf{x}) &= \lim_{\varepsilon\to 0} N^\mu(t_n\pm\varepsilon,\mathbf{x})\,,\\ T^{\mu\nu}(t_n^\pm,\mathbf{x}) &= \lim_{\varepsilon\to 0} T^{\mu\nu}(t_n\pm\varepsilon,\mathbf{x})\,. \end{aligned}$$

Later on we will see in Proposition 4.2 that these conditions are necessary in order to guarantee the conservation laws for mass, momentum and energy across the maximization time t_n. Moreover we start again with a ultra-relativistic Jüttner distribution for the next time step. Then we obtain, using the constitutive relations (3.1), for the three dimensional Euler equations which are valid for the t_n^+ side of the maximization time

$$\begin{aligned} N^0(t_n^+,\mathbf{x}) &= n(t_n^+,\mathbf{x})\sqrt{1+\mathbf{u}^2(t_n^+,\mathbf{x})}\,,\\ T^{0k}(t_n^+,\mathbf{x}) &= 4\,p(t_n^+,\mathbf{x})\,u^k(t_n^+,\mathbf{x})\sqrt{1+\mathbf{u}^2(t_n^+,\mathbf{x})}\,,\\ T^{00}(t_n^+,\mathbf{x}) &= p(t_n^+,\mathbf{x})\left[3+4\,\mathbf{u}^2(t_n^+,\mathbf{x})\right]\,. \end{aligned} \tag{4.6}$$

Here $k=1,2,3$ again denote spatial indices. Since these components of the moments are continuous across the maximization time t_n, we can replace them by the free-flight moments for t_n^- and solve the equations (4.6) for $p, \mathbf{u}, n$ in order to initialize the kinetic scheme for the next time step by the following relations

$$p(t_n^+,\mathbf{x}) = \frac{1}{3}\left[-T^{00} + \sqrt{4(T^{00})^2 - 3\sum_{k=1}^{3}(T^{0k})^2}\,\right], \tag{4.7}$$

$$u^k(t_n^+,\mathbf{x}) = \frac{T^{0k}}{\sqrt{4p(t_n^+,\mathbf{x})[p(t_n^+,\mathbf{x}) + T^{00}]}}, \tag{4.8}$$

$$n(t_n^+,\mathbf{x}) = \frac{N^0}{\sqrt{1 + \sum_{k=1}^{3}[u^k(t_n^+,\mathbf{x})]^2}}. \tag{4.9}$$

In these formulas N^0, T^{00} and T^{0k} are abbreviations for the free flight moments $N^0(t_n^-,\mathbf{x})$, $T^{00}(t_n^-,\mathbf{x})$ and $T^{0k}(t_n^-,\mathbf{x})$ respectively.
Note that the quantities on the left hand side have to be calculated in the prescribed order from the free flight moments N^0, T^{00} and T^{0k}. Since they initialize the scheme for the next time step they conclude the formulation of the kinetic scheme.

But, we can still apply an important simplification of the volume integrals (4.1) and (4.2) for the free flight moments. We can see in (4.3) that the fields $n(t,\mathbf{y})$, $T(t,\mathbf{y})$ and $\mathbf{u}(t,\mathbf{y})$ are not depending on $|\mathbf{q}|$ but only on the unit vector $\mathbf{w} = (w^1,w^2,w^3)^T = \frac{\mathbf{q}}{|\mathbf{q}|}$. This fact enables us to reduce the three-fold volume integrals to the two-fold surface integrals by applying polar coordinates. Using the abbreviations

$$\begin{aligned}
\Phi(\mathbf{y},\mathbf{w}) &= \frac{1}{4\pi}\frac{n(\mathbf{y})}{(\sqrt{1+\mathbf{u}^2(\mathbf{y})} - \mathbf{w}\cdot\mathbf{u}(\mathbf{y}))^3}, \\
\Psi(\mathbf{y},\mathbf{w}) &= \frac{3}{4\pi}\frac{(n\,T)(\mathbf{y})}{(\sqrt{1+\mathbf{u}^2(\mathbf{y})} - \mathbf{w}\cdot\mathbf{u}(\mathbf{y}))^4},
\end{aligned} \tag{4.10}$$

we can now carry out the integration with respect to $|\mathbf{q}|$ explicitly and obtain the following reduced surface integrals for the moments

$$\begin{aligned}
N^0(t_n+\tau,\mathbf{x}) &= \oint_{\partial B(1,0)} \Phi(\mathbf{x}-\tau\mathbf{w},\mathbf{w})\,dS(\mathbf{w}), \\
N^k(t_n+\tau,\mathbf{x}) &= \oint_{\partial B(1,0)} w^k\,\Phi(\mathbf{x}-\tau\mathbf{w},\mathbf{w})\,dS(\mathbf{w}),
\end{aligned} \tag{4.11}$$

$$
\begin{aligned}
T^{00}(t_n+\tau,\mathbf{x}) &= \oint\limits_{\partial B(1,0)} \Psi(\mathbf{x}-\tau\mathbf{w},\mathbf{w})\, dS(\mathbf{w})\,, \\
T^{0k}(t_n+\tau,\mathbf{x}) &= \oint\limits_{\partial B(1,0)} w^k\, \Psi(\mathbf{x}-\tau\mathbf{w},\mathbf{w})\, dS(\mathbf{w})\,, \\
T^{km}(t_n+\tau,\mathbf{x}) &= \oint\limits_{\partial B(1,0)} w^k\, w^m\, \Psi(\mathbf{x}-\tau\mathbf{w},\mathbf{w})\, dS(\mathbf{w})\,.
\end{aligned}
\tag{4.12}
$$

Here $\mathbf{w} = \frac{\mathbf{q}}{|\mathbf{q}|}$ is the unit vector in direction of $\mathbf{q}$ and $B(r,\mathbf{x}_0)$ is the ball with radius r and center $\mathbf{x}_0$. Its boundary is the sphere $\partial B(r,\mathbf{x}_0)$. These surface integrals reflect the fact that in the ultra-relativistic case the particles are moving on the surface of the light cone.

Using the Cauchy-Schwarz inequality one can prove that n, p, e resulting from the moment integrals (4.11) and (4.12) are well defined and positive quantities for all times and positions.

Proposition 4.1: *Let $0<\tau<\tau_M$ and $n=0,1,2,....$ We consider the moments in the free flight between the two maximization times t_n and t_{n+1}. Within this free flight zone the moments $N^\mu(t_n+\tau,\mathbf{x})$, $T^{\mu\nu}(t_n+\tau,\mathbf{x})$ and the entropy four-vector $S^\mu(t_n+\tau,\mathbf{x})$ satisfy the following conservation laws in weak integral form*

$$
\oint\limits_{\partial\Omega} N^\nu(t_n+\tau,\mathbf{x})do_\nu = 0, \quad \oint\limits_{\partial\Omega} T^{\mu\nu}(t_n+\tau,\mathbf{x})do_\nu = 0, \quad \oint\limits_{\partial\Omega} S^\nu(t_n+\tau,\mathbf{x})do_\nu = 0\,.
$$

Here the covariant vector do_ν is a positively oriented surface element to the boundary $\partial\Omega$. It can be written in covariant form as

$$
do_\kappa = \varepsilon_{\kappa\lambda\mu\nu} \sum_{i,j,m=1}^{3} \frac{\partial x^\lambda}{\partial u^i}\,\frac{\partial x^\mu}{\partial u^j}\,\frac{\partial x^\nu}{\partial u^m}\, du^i\, du^j\, du^m\,,
$$

where $x^\alpha = x^\alpha(u^1,u^2,u^3)$ is a positively oriented parameterization of the boundary $\partial\Omega$.

Remark. Note that these weak formulations correspond to the differential equations

$$
\frac{\partial N^\nu}{\partial x^\nu}(t_n+\tau,\mathbf{x}) = 0, \quad \frac{\partial T^{\mu\nu}}{\partial x^\nu}(t_n+\tau,\mathbf{x}) = 0, \quad \frac{\partial S^\nu}{\partial x^\nu}(t_n+\tau,\mathbf{x}) = 0. \tag{4.13}
$$

Proof: For $0<\tau<\tau_M$ let be $t=t_n+\tau$. If we start with the relativistic Maxwellian (4.3) as the initial phase density at the time t_n then we obtain within the time-region $0<t_n<t<t_n+\tau_M$ the free flight density $f(t,\mathbf{x},\mathbf{q}) = f(\mathbf{x}-\tau\frac{\mathbf{q}}{|\mathbf{q}|},\mathbf{q})$, which satisfies the following weak form of the free flight equation

$$
\oint\limits_{\partial\Omega} q^\nu\, f(t,\mathbf{x},\mathbf{q})do_\nu = q^\nu \oint\limits_{\partial\Omega} f(t,\mathbf{x},\mathbf{q})do_\nu = 0\,. \tag{4.14}
$$

The equation (4.14) and its multiplication with q^μ leads after integration with respect to $\mathbf{q}$ to the following equations

$$\int\limits_{\mathbb{R}^3} \left(\oint\limits_{\partial\Omega} q^\nu\, f(t,\mathbf{x},\mathbf{q}) do_\nu \right) \frac{d^3q}{|\mathbf{q}|} = 0\,,$$

$$\int\limits_{\mathbb{R}^3} \left(\oint\limits_{\partial\Omega} q^\mu\, q^\nu\, f(t,\mathbf{x},\mathbf{q}) do_\nu \right) \frac{d^3q}{|\mathbf{q}|} = 0\,. \tag{4.15}$$

Since the volume integral with respect to $\mathbf{q}$ and the surface integral with respect to t and $\mathbf{x}$ are interchangeable, we can rewrite equations (4.1) in order to get the conservation laws

$$\oint\limits_{\partial\Omega} \left(\int\limits_{\mathbb{R}^3} q^\nu\, f(t,\mathbf{x},\mathbf{q}) \frac{d^3q}{|\mathbf{q}|} \right) do_\nu = \oint\limits_{\partial\Omega} N^\nu(t_n + \tau\mathbf{x},\mathbf{q}) do_\nu = 0\,,$$

$$\oint\limits_{\partial\Omega} \left(\int\limits_{\mathbb{R}^3} q^\mu\, q^\nu\, f(t,\mathbf{x},\mathbf{q}) \frac{d^3q}{|\mathbf{q}|} \right) do_\nu = \oint\limits_{\partial\Omega} T^{\mu\nu}(t_n + \tau\mathbf{x},\mathbf{q}) do_\nu = 0\,. \tag{4.16}$$

Next we define

$$\psi(t,\mathbf{x},\mathbf{q}) = -(f_n \ln f_n)(\mathbf{x} - \tau\frac{\mathbf{q}}{|\mathbf{q}|}, \mathbf{q})\,, \tag{4.17}$$

and we conclude due to chain rule that ψ satisfies the weak form of the free-flight equation $q^\nu \frac{\partial\psi}{\partial x^\nu} = 0$, namely

$$\oint\limits_{\partial\Omega} q^\nu\, \psi(t,\mathbf{x},\mathbf{q}) do_\nu = 0\,. \tag{4.18}$$

This is coming from the Gauss Divergence Theorem.

Integrating this equation with respect to $\mathbf{q}$ and interchanging the volume and surface integrals, we finally get, using equation (4.2)

$$\oint\limits_{\partial\Omega} \left(\int\limits_{\mathbb{R}^3} q^\nu\, \psi(t,\mathbf{x},\mathbf{q}) \frac{d^3q}{|\mathbf{q}|} \right) do_\nu = \oint\limits_{\partial\Omega} S^\nu(t_n + \tau\mathbf{x},\mathbf{q}) do_\nu = 0\,. \tag{4.19}$$

$\square$

Proposition 4.2: *Let $\Omega \subset \mathbb{R}_0^+ \times \mathbb{R}^3$ be any bounded convex region in time and space. By do_ν we denote the positively oriented surface element of $\partial\Omega$. Let $\tau_M > 0$ be a fixed time step. The moment representations* (4.1) *and* (4.2) *calculated by the iterated scheme defined above have the following properties:*

(i) *The conservation laws for the particle number, the momentum and energy hold, i.e.*

$$\oint\limits_{\partial\Omega} N^\nu do_\nu = 0, \quad \oint\limits_{\partial\Omega} T^{\mu\nu} do_\nu = 0. \tag{4.20}$$

(ii) *The following entropy inequality is satisfied*

$$\oint\limits_{\partial\Omega} S^\nu do_\nu \geq 0. \tag{4.21}$$

Proof: Let be $\tau_M > 0$. We first prove part (i) of the proposition. The time axis is divided by the maximization times $0 = t_0 < t_1 < t_2 < \cdots$, so that the convex domain Ω can be decomposed into the subdomains

$$\begin{cases} \Omega_0 &= \left\{(\delta, \mathbf{x}) \in \Omega \middle| \, 0 \leq \delta \leq \frac{t_0+t_1}{2}\right\}, \\ \Omega_n &= \left\{(\delta, \mathbf{x}) \in \Omega \middle| \, \frac{t_{n-1}+t_n}{2} \leq \delta \leq \frac{t_n+t_{n+1}}{2}\right\} \; n = 1, 2, 3, \ldots. \end{cases} \tag{4.22}$$

Now we use the additivity of the curve integrals over $\partial\Omega_n$,

$$\oint_{\partial\Omega} N^\nu do_\nu = \sum_{n\geq 0} \oint_{\partial\Omega_n} N^\nu do_\nu \,, \quad \oint_{\partial\Omega} T^{\mu\nu} do_\nu = \sum_{n\geq 0} \oint_{\partial\Omega_n} T^{\mu\nu} do_\nu \,, \tag{4.23}$$

in order to state that it is sufficient to prove the conservation laws and the entropy inequality for each domain Ω_n, which only contains the maximization time t_n.

Then for ε in the range $0 < \varepsilon < \frac{1}{2}\tau_M$ we define a further decomposition of each $\Omega_n, n \geq 1$, into three parts

$$\begin{cases} \Omega^\varepsilon_{n,L} &= \left\{(\delta, \mathbf{x}) \in \Omega_n \middle| \, \delta \leq t_n - \varepsilon\right\}, \\ \Omega^\varepsilon_{n,M} &= \left\{(\delta, \mathbf{x}) \in \Omega_n \middle| \, t_n - \varepsilon \leq \delta \leq t_n + \varepsilon\right\}, \\ \Omega^\varepsilon_{n,R} &= \left\{(\delta, \mathbf{x}) \in \Omega_n \middle| \, \delta \geq t_n + \varepsilon\right\}. \end{cases} \tag{4.24}$$

The decompositions which are illustrated in the following figure, were also applied in order to prove the conservation laws and the entropy inequality for the classical Euler equations, see [11].

We obtain

$$\oint\limits_{\partial\Omega_n} N^\nu do_\nu = \oint\limits_{\partial\Omega^\varepsilon_{n,L}} N^\nu do_\nu + \oint\limits_{\partial\Omega^\varepsilon_{n,R}} N^\nu do_\nu + \oint\limits_{\partial\Omega^\varepsilon_{n,M}} N^\nu do_\nu,$$

$$\oint\limits_{\partial\Omega_n} T^{\mu\nu} do_\nu = \oint\limits_{\partial\Omega^\varepsilon_{n,L}} T^{\mu\nu} do_\nu + \oint\limits_{\partial\Omega^\varepsilon_{n,R}} T^{\mu\nu} do_\nu + \oint\limits_{\partial\Omega^\varepsilon_{n,M}} T^{\mu\nu} do_\nu$$

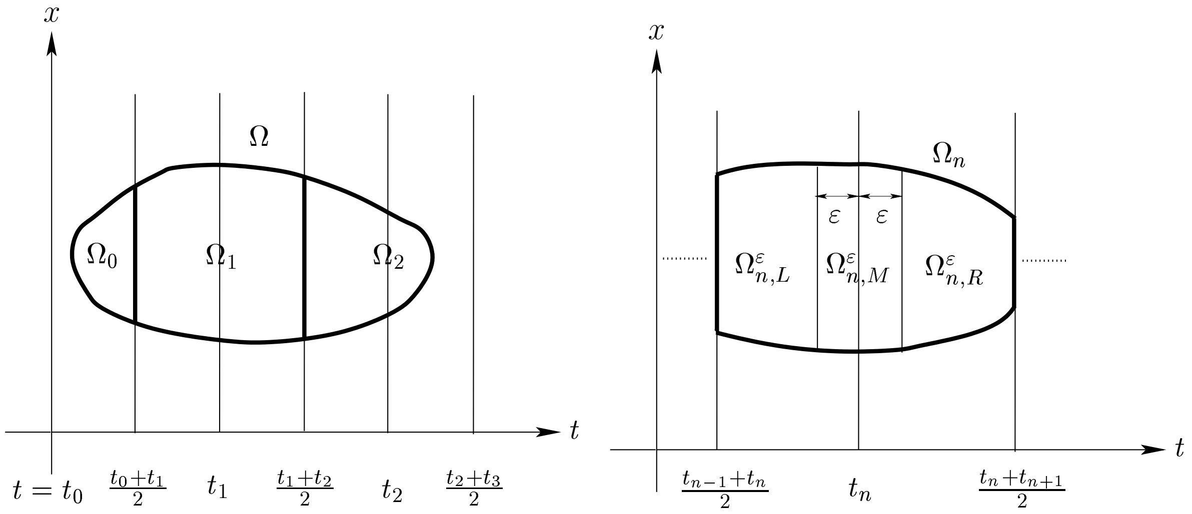

Figure 1: The decompositions of Ω and Ω_n.

since the first two integrals on the right hand side are in the free flight zone, so we conclude from equation (4.16) that these integrals vanish, i.e.,

$$\oint\limits_{\partial\Omega^\varepsilon_{n,L}} N^\nu do_\nu = \oint\limits_{\partial\Omega^\varepsilon_{n,R}} N^\nu do_\nu = 0, \qquad \oint\limits_{\partial\Omega^\varepsilon_{n,L}} T^{\mu\nu} do_\nu = \oint\limits_{\partial\Omega^\varepsilon_{n,R}} T^{\mu\nu} do_\nu = 0.$$

This implies, using $\Omega^*_n = \{\mathbf{x} \in \mathbb{R}^3 | (t_n, \mathbf{x}) \in \Omega\}$

$$\begin{aligned}
\oint\limits_{\partial\Omega_n} N^\nu do_\nu &= \oint\limits_{\partial\Omega^\varepsilon_{n,M}} N^\nu do_\nu = \lim_{\varepsilon\to 0} \oint\limits_{\partial\Omega^\varepsilon_{n,M}} N^\nu do_\nu \\
&= \int_{\Omega^*_n} \left\{ \int q^0 \left[f_n(\mathbf{x},\mathbf{q}) - f_{n-1}(\mathbf{x} - \tau_M \frac{\mathbf{q}}{|\mathbf{q}|}) \right] \frac{d^3q}{|\mathbf{q}|} \right\} d^3\mathbf{x}
\end{aligned}$$

and

$$\begin{aligned}
\oint\limits_{\partial\Omega_n} T^{\mu\nu} do_\nu &= \oint\limits_{\partial\Omega^\varepsilon_{n,M}} T^{\mu\nu} do_\nu = \lim_{\varepsilon\to 0} \oint\limits_{\partial\Omega^\varepsilon_{n,M}} T^{\mu\nu} do_\nu \\
&= \int_{\Omega^*_n} \left\{ \int q^0 q^\mu \left[f_n(\mathbf{x},\mathbf{q}) - f_{n-1}(\mathbf{x} - \tau_M \frac{\mathbf{q}}{|\mathbf{q}|}) \right] \frac{d^3q}{|\mathbf{q}|} \right\} d^3\mathbf{x},
\end{aligned}$$

where t_{n-1} is the maximization time that preceedes the maximization time t_n. The phase density f_n has to be taken to be the ultra-relativistic Jüttner phase density (2.33).

The last integral expression in these equations vanishes due to the continuity conditions

(4.5) across the maximization time t_n, which yields

$$\int_{\mathbb{R}^3} q^0 f_n(\mathbf{x},\mathbf{q}) \frac{d^3q}{|\mathbf{q}|} = \int_{\mathbb{R}^3} q^0 f_{n-1}(\mathbf{x} - \tau_M \frac{\mathbf{q}}{|\mathbf{q}|}, \mathbf{q}) \frac{d^3q}{|\mathbf{q}|}\,,$$
$$\int_{\mathbb{R}^3} q^0 q^\mu f_n(\mathbf{x},\mathbf{q}) \frac{d^3q}{|\mathbf{q}|} = \int_{\mathbb{R}^3} q^0 q^\mu f_{n-1}(\mathbf{x} - \tau_M \frac{\mathbf{q}}{|\mathbf{q}|}, \mathbf{q}) \frac{d^3q}{|\mathbf{q}|}\,. \tag{4.25}$$

This expresses the constraints that were used for the maximization procedure. We have thus established that the weak form (4.20) for an arbitrary convex domain Ω is implied by the representations (4.1).

Regarding the second part (ii) which states the existence of the entropy inequality (4.21), we start the proof again with the decompositions (4.22) and (4.24) of Ω. Since $\int_{\partial\Omega} S^\nu do_\nu = \sum_{n\geq 0}\int_{\partial\Omega_n} S^\nu do_\nu$, it is sufficient to prove $\int_{\partial\Omega_n} S^\nu do_\nu \geq 0$ for each n. We obtain

$$\oint_{\partial\Omega_n} S^\nu do_\nu = \oint_{\partial\Omega^\varepsilon_{n,L}} S^\nu do_\nu + \oint_{\partial\Omega^\varepsilon_{n,R}} S^\nu do_\nu + \oint_{\partial\Omega^\varepsilon_{n,M}} S^\nu do_\nu\,. \tag{4.26}$$

Again the first two integrals lie in the free flight zone. We can see from equation (4.19) that these integrals vanishes i.e., $\oint_{\partial\Omega^\varepsilon_{n,R}} S^\nu do_\nu = 0$, and $\oint_{\partial\Omega^\varepsilon_{n,L}} S^\nu do_\nu = 0$.
For every sufficiently small $\varepsilon > 0$ the following holds

$$\begin{aligned}\oint_{\partial\Omega_n} S^\nu do_\nu &= \lim_{\varepsilon\to 0} \oint_{\partial\Omega^\varepsilon_{n,M}} S^\nu do_\nu \\ &= \int_{\Omega^*_n} \left\{ \int q^0 \left[-(f_n \ln f_n)(\mathbf{x},\mathbf{q}) + (f_{n-1}\ln f_{n-1})(\mathbf{x} - \tau_M \frac{\mathbf{q}}{|\mathbf{q}|}, \mathbf{q}) \right] \frac{d^3q}{|\mathbf{q}|} \right\} d^3\mathbf{x},\end{aligned} \tag{4.27}$$

where $\Omega^*_n = \{\mathbf{x} \in \mathbb{R}^3 | (t_n, \mathbf{x}) \in \Omega\}$, and $t_{n-1} < t_n$ is the maximization time that preceeds t_n. □

Next we shall show that the integral (4.27) is non-negative. To this end we need the following

Lemma 4.3: *For $u, v > 0$ we have*

$$v \ln v - u \ln u = [\ln u + 1](v - u) + R(u, v) \tag{4.28}$$

with a function $R(u, v) \geq 0$.

Proof of the Lemma: Due to Taylors formula there is a $\xi > 0$ between $u, v > 0$ such that

$$v \ln v = u \ln u + (\ln u + 1)(v - u) + \frac{1}{2\xi}(v - u)^2. \tag{4.29}$$

We conclude $R(u, v) = \frac{1}{2\xi}(v - u)^2 \geq 0$. □

Continuation of proof of proposition 4.2: Now we apply Lemma 4.3 to $u = f_n(\mathbf{x}, \mathbf{q})$ and $v = f_{n-1}(\mathbf{x} - \tau_M \frac{\mathbf{q}}{|\mathbf{q}|}, \mathbf{q})$

$$
\begin{aligned}
&\int_{\mathbb{R}^3} q^0 \left[-(f_n \ln f_n)(\mathbf{x}, \mathbf{q}) + (f_{n-1} \ln f_{n-1})(\mathbf{x} - \tau_M \frac{\mathbf{q}}{|\mathbf{q}|}, \mathbf{q}) \right] \frac{d^3 q}{|\mathbf{q}|} \\
&\qquad = - \int_{\mathbb{R}^3} q^0 \left[1 + \ln f_n(\mathbf{x}, \mathbf{q}) \right] \left[f_n(\mathbf{x}, \mathbf{q}) - f_{n-1}(\mathbf{x} - \tau_M \frac{\mathbf{q}}{|\mathbf{q}|}, \mathbf{q}) \right] \frac{d^3 q}{|\mathbf{q}|} \\
&\qquad\quad + \int_{\mathbb{R}^3} R \left(f_n(\mathbf{x}, \mathbf{q}), f_{n-1}(\mathbf{x} - \tau_M \frac{\mathbf{q}}{|\mathbf{q}|}, \mathbf{q}) \right) \frac{d^3 q}{|\mathbf{q}|}.
\end{aligned} \tag{4.30}
$$

The second integral is non-negative and the first one vanishes due to the following reasons. Using Jüttner's phase density for $f_n(\mathbf{x}, \mathbf{q})$ we have

$$
\ln f_n(\mathbf{x}, \mathbf{q}) = \ln \left[\frac{n(\mathbf{x})}{8\pi T^3(\mathbf{x})} \exp \left(\frac{-u_\nu q^\nu}{T(\mathbf{x})} \right) \right] \qquad = A(x) - B(x) u_\nu q^\nu \,, \tag{4.31}
$$

where $A(\mathbf{x}) = \ln \frac{n(\mathbf{x})}{8\pi T^3(\mathbf{x})}$ and $B(\mathbf{x}) = \frac{1}{T(\mathbf{x})}$. We use the value (4.31) of $\ln f_n(\mathbf{x}, \mathbf{q})$ in (4.30). Using the definitions (4.1) for N^μ, $T^{\mu\nu}$ and the continuity conditions (4.5) for the zero components N^0, $T^{0\nu}$, we can see that the first integral in (4.30) is zero.

We have thus established the entropy inequality (4.21). □

Comparison of relativistic and classical kinetic schemes.

Finally we compare the kinetic scheme for the ultra-relativistic Euler equations with the corresponding kinetic scheme for the classical Euler equations, which was studied in detail by Deshpande and Raul [7], Deshpande [8, 9], as well as Dreyer and Kunik [11]. The basic ingredients of the kinetic schemes are the same in both cases, as given below.

a) There are given initial data for the mass-density, denoted by ρ in the classical case, the velocity $\mathbf{v}$ and the pressure p, which can be choosen as the five basic variables in the Euler equations.

b) There is a phase density f_M, usually called the **Maxwellian**, which describes the velocity distribution for the atoms of a gas in local equilibrium in terms of the five basic variables.

c) There is a time step $\tau_M > 0$ and a corresponding sequence of equidistant time steps $t_n = n\,\tau_M$ with $n \geq 0$, also called the maximization times, see [11].

d) At each maximization time $t = t_n$ we take the Maxwellian phase density, starting initially from the given initial data for $n = 0$. This Maxwellian is used as an initial phase density in order to solve the **collision free** kinetic phase density within the time range $t_n < t_n + \tau < t_{n+1}$, which will be called a free-flight interval. All thermodynamic quantities are algebraic combinations of moment integrals from this free-flight phase density f, and therefore they are defined everywhere in the free-flight interval under consideration.

It is very important to note here that the free-flight phase density is in general not a Maxwellian. Therefore in the free-flight intervals of the kinetic scheme the gas is usually neither in equilibrium nor satisfies the isotropic constitutive relations satisfied for the Euler equations !

e) In the free-flight intervals it can be seen very easily that the conservation laws for mass, momentum, energy and even for the entropy hold. The requirements that the kinetic schemes must satisfy the conservation laws and the entropy inequality across each maximization time t_{n+1} turns out to be equivalent to the five **continuity conditions**, which state that the "densities" written under the time-derivatives in the conservation laws must be continuous across t_{n+1}.

In turn, the continuity conditions must be used in order to **initialize** the kinetic schemes for the next time step t_{n+1}, after the free flight was performed in the time interval $t_n < t < t_{n+1}$.

Thus we have given a list of the common properties of the kinetic schemes for the classical and the relativistic Euler equations. But now there comes an interesting difference.

A) In the **classical case** the densities for mass, momentum and energy are given by ρ, $\rho\,\mathbf{v}$ and $\frac{1}{2}\,\rho\,\mathbf{v}^2 + \frac{3}{2}\,p$, and hence they are algebraic functions of the primitive variables ρ, $\mathbf{v}$, p. Here it is crucial to note that this is even true in the free-flight phase, where the gas is not in local equilibrium. Due to the continuity conditions these densities and hence the five basic variables are continuous across the maximization time.

In contrast, the fluxes of momentum and energy which are moment integrals of the classical phase densities, are in general discontinuous across t_{n+1}, since the gas is not isotropic in the free-flight domain immediately left to t_{n+1}.

B) In the **relativistic case** the densities under the time derivatives in the conservation laws are given by N^0 and $T^{0\mu}$, and in the free-flight intervals they cannot be written as functions of the basic variables n, $\mathbf{v} = \frac{\mathbf{u}}{\sqrt{1+\mathbf{u}^2}}$ and p. This is a consequence of the generally valid equations $n = \sqrt{N^\mu N_\mu}$, $u^\mu = \frac{1}{n}\,N^\mu$ and and $p = \frac{1}{3}\,(u_\mu u_\nu - g_{\mu\nu})\,T^{\mu\nu}$, which depend on **all** components of the tensors N^μ, $T^{\mu\nu}$.

Immediately before the maximization time t_{n+1} we have a non-isotropic tensor $T_-^{\mu\nu}$ in the free-flight regime, where as at t_{n+1} we have the isotropic tensor $T_+^{\mu\nu}$ which satisfies the constitutive Euler-relations (3.1) in local equilibrium. Combining this with the argumentation above and comparing with the evaluation (4.7) of the continuity conditions we finally conclude that the basic fields calculated from the relativistic kinetic scheme may in general lead to jumps across the maximization times.

Finally we note that this would not be the case if we choose the zero-components N^0 and $T^{0\mu}$ as the basic variables of the relativistic Euler system, which is mathematically possibl but not usual.

5 A Kinetic Scheme for the One Dimensional Case

In the following we are looking for spatially one dimensional solutions, which are nevertheless solutions to the full three dimensional equations. We only consider solutions which depend on t and $x = x^1$ and satisfy $n = n(t,x)$, $\mathbf{u} = (u(t,x),0,0)$, $p = p(t,x)$. We will use the generally valid equation $p = nT$ and go back to the full three dimensional scheme.

In order to calculate the surface integrals (4.11) and (4.12) we introduce instead of the unit vector $\mathbf{w}$ the new variables $-1 \leq \xi \leq 1$ and $0 \leqq \varphi \leqq 2\pi$ by

$$w^1 = \xi\,, \qquad w^2 = \sqrt{1-\xi^2}\sin\varphi\,, \qquad w^3 = \sqrt{1-\xi^2}\cos\varphi \tag{5.1}$$

with the surface element $dS(\mathbf{w}) = d\xi d\varphi$.

Note that the quantities n, T, u in the integrals (4.11) and (4.12) do not depend on the variable φ. This fact enables us to carry out the integration with respect to φ directly. Thus the two-fold surface integral reduces to a simple $\xi-$integral. For abbreviation we introduce

$$\Phi(y,\xi) = \frac{1}{2}\frac{n(y)}{(\sqrt{1+u^2(y)}-\xi u(y))^3}\,, \quad \Psi(y,\xi) = \frac{3}{2}\frac{(nT)(y)}{(\sqrt{1+u^2(y)}-\xi u(y))^4}\,, \tag{5.2}$$

then the reduced integrals for the moments can be written as

$$N^0(t_n+\tau,x) = \int_{-1}^{1}\Phi(x-\tau\xi,\xi)\,d\xi\,, \quad N^1(t_n+\tau,x) = \int_{-1}^{1}\xi\Phi(x-\tau\xi,\xi)\,d\xi\,, \tag{5.3}$$

$$\begin{aligned}
T^{00}(t_n+\tau,x) &= \int_{-1}^{1}\Psi(x-\tau\xi,\xi)\,d\xi\,,\\
T^{01}(t_n+\tau,x) &= \int_{-1}^{1}\xi\Psi(x-\tau\xi)\,d\xi\,,\\
T^{11}(t_n+\tau,x) &= \int_{-1}^{1}\xi^2\Psi(x-\tau\xi)\,d\xi\,.
\end{aligned} \tag{5.4}$$

Again the integrals reflect the fact that in the ultra-relativistic case the particles are moving on the surface of the light cone, see (4.11) and (4.12). Moreover we obtain

$$\begin{aligned}
N^2(t_n+\tau,x) &= N^3(t_n+\tau,x) = 0\,,\\
T^{10}(t_n+\tau,x) &= T^{01}(t_n+\tau,x)\,,\\
T^{22}(t_n+\tau,x) &= T^{33}(t_n+\tau,x) = \frac{1}{2}[T^{00}(t_n+\tau,x) - T^{11}(t_n+\tau,x)]\,,
\end{aligned}$$

where all the other components of $T^{\mu\nu}$ are zero. So in the one dimensional case n, u and T can be found from the generally valid relations given in (4.4) as follows

$$n(t_n+\tau,x) = \sqrt{(N^0(t_n+\tau,x))^2-(N^1(t_n+\tau,x))^2}\,, \tag{5.5}$$

$$u(t_n+\tau,x) = \frac{1}{n}N^1(t_n+\tau,x)\,, \tag{5.6}$$

$$\begin{aligned} p(t_n+\tau,x) = \frac{1}{3}[&\{1+u^2(t_n+\tau,x)\}T^{00}(t_n+\tau,x) - 2u\sqrt{1+u^2(t_n+\tau,x)} \\ &\cdot T^{01}(t_n+\tau,x) + u^2(t_n+\tau,x)T^{11}(t_n+\tau,x)]\,. \end{aligned} \tag{5.7}$$

We can now simplify the equations (4.7), which are used in order to initialize the general three-dimensional scheme, and obtain for the one-dimensional case

$$p(t_n^+,\mathbf{x}) = \frac{1}{3}\left[-T^{00}+\sqrt{4(T^{00})^2-3(T^{01})^2}\,\right]\,, \tag{5.8}$$

$$u(t_n^+,\mathbf{x}) = \frac{T^{01}}{\sqrt{4p(t_n^+,\mathbf{x})[p(t_n^+,\mathbf{x})+T^{00}]}}\,, \tag{5.9}$$

$$n(t_n^+,\mathbf{x}) = \frac{N^0}{\sqrt{1+u(t_n^+,\mathbf{x})^2}}\,. \tag{5.10}$$

Here again $N^0 = N^0(t_n^-,\mathbf{x})$, $T^{00} = T^{00}(t_n^-,\mathbf{x})$ and $T^{01} = T^{01}(t_n^-,\mathbf{x})$ are given by the free-flight moments.

6 From the Kinetic Scheme to the Eulerian Limit ($\tau_M \to 0$)

In the previous sections we have shown how to calculate the solution of the kinetic scheme in the three- and one-dimensional case, respectively. This was done for the prescribed initial data of n, u and p for a given free-flight time step $\tau_M > 0$. If we calculate these solutions for $\tau_M \to 0$ then we get the Eulerian limit

$$N^\mu \to n\,u^\mu, \qquad T^{\mu\nu} \to -p\,g^{\mu\nu} + 4pu^\mu u^\nu, \qquad S^\mu \to n\,u^\mu \ln\frac{n^4}{p^3}\,. \tag{6.1}$$

First we pass to the Eulerian limit (6.1) at the points of smoothness in the following way using (2.15) with $Q(f)=0$

$$
\begin{aligned}
\lim_{\tau\to 0}\frac{\partial}{\partial\tau}N^0(t_n+\tau,\mathbf{x}) &= \lim_{\tau\to 0}\frac{\partial}{\partial\tau}(n(t_n+\tau,\mathbf{x})\sqrt{1+u^2(t_n+\tau,\mathbf{x})}\,) \\
&= \lim_{\tau\to 0}\frac{\partial}{\partial\tau}\int_{\mathbb{R}^3}|\mathbf{q}|f_n(\mathbf{x}-\tau\frac{\mathbf{q}}{|\mathbf{q}|},\mathbf{q})\frac{d^3q}{|\mathbf{q}|} \\
&= -\lim_{\tau\to 0}\int_{\mathbb{R}^3}|\mathbf{q}|\sum_{k=1}^{3}\frac{q^k}{|\mathbf{q}|}\frac{\partial}{\partial x^k}f_n(\mathbf{x}-\tau\frac{\mathbf{q}}{|\mathbf{q}|},\mathbf{q})\frac{d^3q}{|\mathbf{q}|} \\
&= -\int_{\mathbb{R}^3}\sum_{k=1}^{3}q^k\frac{\partial}{\partial x^k}f_n(\mathbf{x},\mathbf{q})\frac{d^3q}{|\mathbf{q}|} \\
&= -\sum_{k=1}^{3}\frac{\partial}{\partial x^k}(u^k(t_n^+,\mathbf{x})n(t_n^+,\mathbf{x})) \\
&= -\nabla\cdot(\mathbf{u}(t_n^+,\mathbf{x})n(t_n^+,\mathbf{x}))\,.
\end{aligned}
$$

This implies

$$
\frac{\partial}{\partial t}(n(t_n^+,\mathbf{x})\sqrt{1+u^2(t_n^+,\mathbf{x})}\,)+\nabla\cdot(\mathbf{u}(t_n^+,\mathbf{x})n(t_n^+,\mathbf{x}))=0\,, \tag{6.2}
$$

which is the first Euler equation (3.4). Similarly we get the other two Euler equations (2.5) and (2.6) if we differentiate $T^{00}(t_n+\tau,\mathbf{x})$ and $T^{0k}(t_n+\tau,\mathbf{x})$ with respect to τ and pass to the limit $\tau\to 0$.

Secondly, on the left hand sides of (6.1) there are the moments N^μ, $T^{\mu\nu}$ and S^μ as calculated by the kinetic scheme see (4.1) and (4.2). Since we have already established the conservation laws and the entropy inequality for the solution of the kinetic scheme in Proposition 4.2, we conclude from (6.1) that this also results for the weak entropy solution in the Eulerian limit $\tau_M\to 0$. The weak entropy solution in the Eulerian limit in one space dimension is given by (3.9), (3.10) and (3.11).

7 Numerical Implementation of the Scheme

In this section we explain the numerical implementation of the one-dimensional kinetic scheme. The procedure is similar for the two-dimensional case.

- We start with the values of initial data $n(x,0)$, $u(x,0)$ and $T(x,0)$ at equidistant grid points.
- We specify the length L of the spatial domain, the number N_x of elements (intervals) in the spatial domain $0\le x\le L$, the final time t_f of output, the number E_m of maximization times and the number F_f of free flights. For$i=0,...,N_x$ we introduce the nodes $x_i=i\cdot\frac{L}{N_x}$.

- The time step Δt is calculated by $\Delta t = \frac{t_f}{E_m F_f}$. Here we have two time loops, one is an outer loop for the maximization times and the other one is an inner loop for the free flight phase. The step in the spatial domain is $\Delta x = L/N_x$.
- Our aim is to calculate the moments (5.3) and (5.4). These moments are then used to update the fields n, u and T.
- Since we only know the values of the fields at the nodal points, the free flight fields in the integrands of (5.3) and (5.4) must be calculated from the knowledge of the nodal values at the points x_i. Here we use linear interpolation between two subsequent nodal points x_i and x_{i+1}. We use the following interpolation formula

$$f_n(x_j - \xi\tau, \xi) = (1 - \eta)\, f_J(x_i, \xi) + \eta\, f_J(x_{i+1}, \xi)\,,$$

where $x_j - \xi\tau = x_i + \eta\,(x_{i+1} - x_i)$ for $0 \leq \eta \leq 1$. Here f_n and f_J are free-flight and Jüttner phase densities, respectively. The relation between x_i, x_j and η is shown in Figure 2.

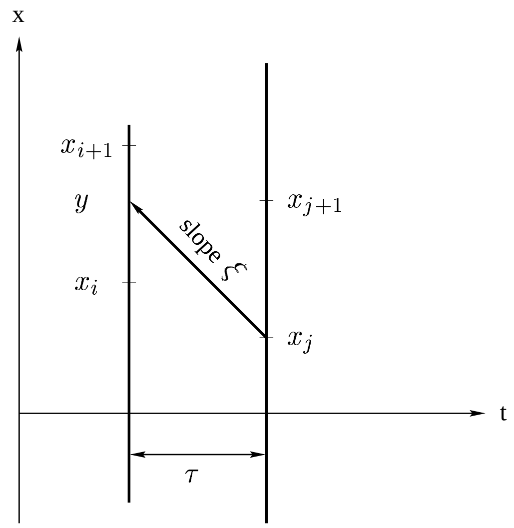

Figure 2: Interpolation of $y = x_j - \xi\tau$ at the grid points x_i and x_{i+1}.

- The $\xi-$ integration is performed with the composite trapezoidal rule.
- When we are in the free flight the values of the fields n, u and T are calculated by using the generally valid algebraic relations (5.5), while at the maximization time, i.e. after the end of free flight loop, the fields are updated by using the continuity relations (5.8) in order to initialize the scheme for next time step.

In the following we present numerical test cases for the solution of the one-dimensional ultra-relativistic Euler equations. For this purpose a computer program was developed using the same procedure discussed above.

7.1 Single shock solution of the Euler equations

In this example we test our kinetic scheme for a single shock problem. We supplied intial data to the program for which we know that a single shock solution results from the Rankine-Hugoniot jump conditions. We select the initial data and the space-time range such that the shock exactly reaches the right lower corner at the time axis. In Figure 3 the left hand figure represents the plot of the particle density n in the time range $0 \le t \le 1.271$ and in the space range $0 \le x \le L = 2$. We 1000 mesh points in the spatial domain and time step $\tau_M = 0.01271$. The figure shows that the kinetic scheme captures this shock in exactly the same way as predicted by the Rankine-Hugoniot jump conditions. The right hand plot in Figure 3 represents the particle density n at the fixed time $t = 0.635$ for the same initial data. The Riemannian initial data with a jump at $x = L/2 = 1$ are choosen as

$$(n, u, p) = \begin{cases} (1.0, 0.0, 1.0) & \text{if } x < 1.0\,, \\ (2.725, -0.6495, 4.0) & \text{if } x \ge 1.0\,. \end{cases}$$

In this example we found that our kinetic scheme gives a sharp shock resolution. This is a good test for the kinetic scheme, and its success indicates that the conservation laws for mass, momentum and energy as well as the entropy inequality are satisfied. We have already proved these properties for the solutions of the kinetic scheme.

7.2 Riemann solutions of the Euler equations

Here we test our kinetic scheme for the solution of one dimensional Riemann problems. For the comparison we use exact Riemann solution, first order upwind and Lax-Friedrichs centeral schemes. It was found that kinetic scheme has comparable accuracy with the first order upwind and Lax-Friedrichs schemes.

Test Problem 1: The inital data are

$$(n, u, p) = \begin{cases} (5.0, 0.0, 10.0) & \text{if } x < 0.5\,, \\ (1.0, 0.0, 0.5) & \text{if } x \ge 0.5\,. \end{cases}$$

The spatial domain is taken as $[0, 1]$ with 400 mesh elements and the final time is $t = 0.5$. The solution consists of a left shock, a contact and a right rarefaction wave. Figure 4 represents plots for the particle density n and pressure p. We can see that the schemes does not resolve the contact discontinuity very well.

Test Problem 2: The inital data are

$$(n, u, p) = \begin{cases} (1.0, 1.0, 3.0) & \text{if } x < 0.5\,, \\ (1.0, -0.5, 2.0) & \text{if } x \geq 0.5\,. \end{cases}$$

The spatial domain is taken as $[0, 1]$ with 400 mesh elements and the final time is $t = 0.5$. The solution consist of left shock, a contact and a right shock. Figure 5 represents a plots for the particle density n and pressure p at $t = 0.5$.

Test Problem 3: The intial dare are

$$(n, u, p) = \begin{cases} (1.0, -0.5, 2.0) & \text{if } x < 0.5\,, \\ (1.0, 0.5, 2.0) & \text{if } x \geq 0.5\,. \end{cases}$$

This problem has a solution consisting of two strong rarefactions and a trivial stationary contact discontinuity. The spatial domain is taken as $[0, 1]$ with 400 mesh elements and the final time is $t = 0.5$. Figure 6 show the solution profiles.

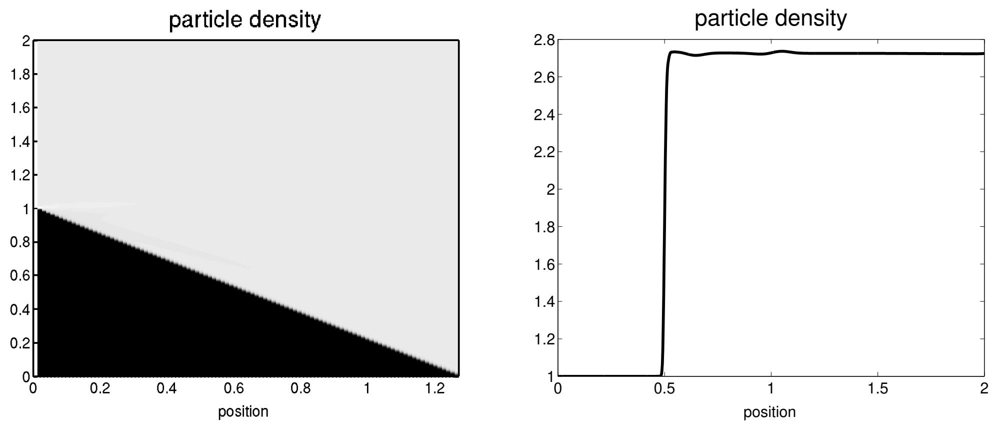

Figure 3: Left: Particle density $n(t, x)$ in the time range $0 \leq t \leq 1.271$.
Right: Particle density at t=0.635.

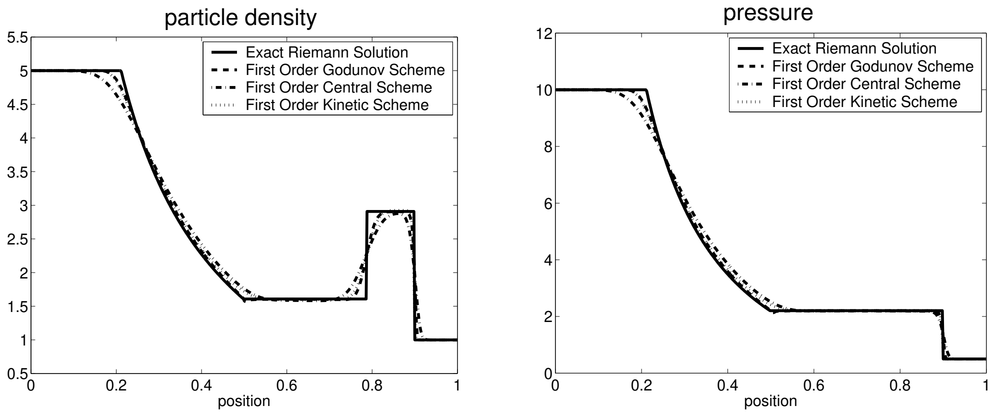

Figure 4: Comparison of the results from the test problem 1 at time $t = 0.5$.

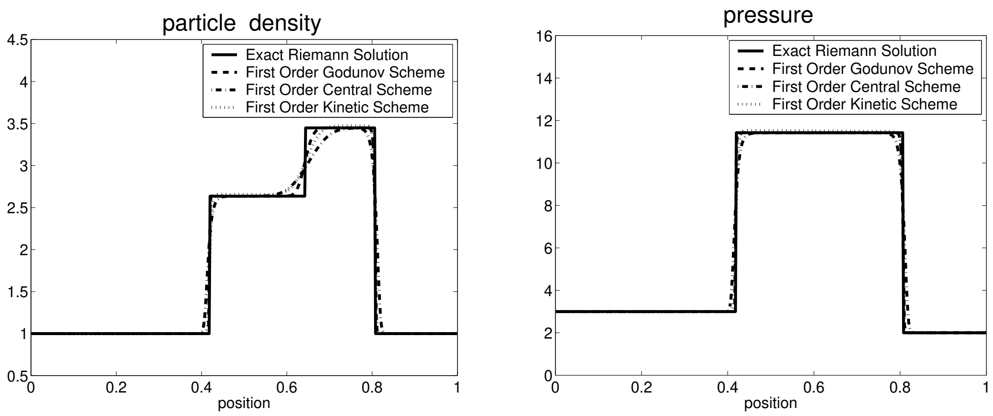

Figure 5: Comparison of the results from the test problem 2 at time $t = 0.5$.

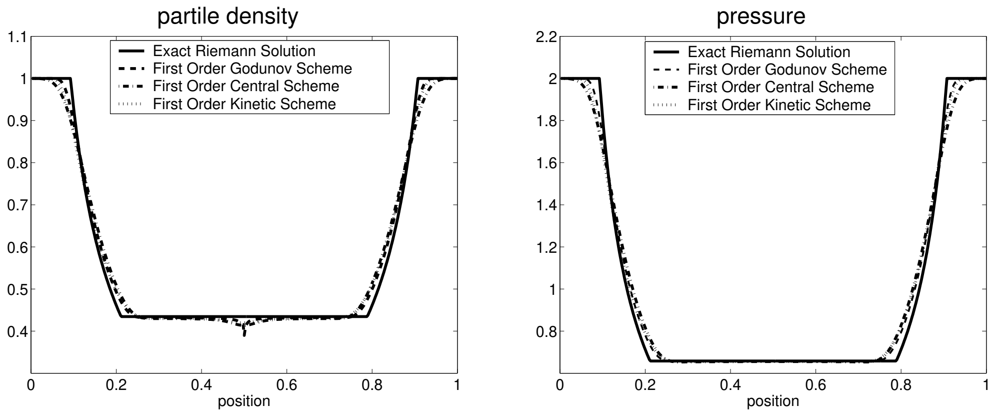

Figure 6: Comparison of the results from the test problem 3 at time $t = 0.5$.

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