Weakly stable multi-d shocks

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Abstract

We study the linear stability of multidimensional shock waves for systems of conservation laws in the case where Majda's uniform stability condition is violated. The linearized problem is attacked using the "good unknown" of Alinhac. We prove an energy estimate and show that the solutions to the linearized problem have singularities localized along bicharacteristic curves originating from the boundary. The application to isentropic gas dynamics is detailed.

Contents

1 Introduction

In [25] and [24], Majda proved the existence of multidimensional shock waves for hyperbolic systems of conservation laws. The analysis relied on a uniform stability assumption. However, previous works [6, 13] have exhibited some examples where the uniform stability condition breaks down. In [13], we have begun to extend Majda's linear analysis to these particular examples, namely we have proved an energy estimate on a constant coefficients linearized system. Here we adopt a general approach and prove a complete linear stability result for a class of shock waves that are not uniformly stable. The analysis is closely related to what was done in [13].

To avoid any possible confusion, we shall not include the case of non classical shock waves in this work though this field has known a significant increase of interest over the past few years, see e.g. [6, 7, 16, 17] and the references therein.

We shall focus in this paper on **multidimensional** hyperbolic **systems**: the one-dimensional case is far different from the multidimensional case since shock waves are either uniformly stable or violently unstable, see [26]. The scalar case is also known to be very different from the system case since scalar conservation laws provide us with a unified theory of existence and uniqueness of solutions in the large, see e.g. [14, 34].

We consider a system of N conservation laws in time-space $\mathbb{R} \times \mathbb{R}^d$:

$$\sum_{j=0}^{d} \partial_j f_j(u) = 0\,, \tag{1}$$

where x_0 is the time variable, also denoted by t in the sequel, $(x_1, \dots, x_d)$ is the space variable and ∂_j stands for the partial derivative with respect to x_j. The fluxes $f_0, \dots, f_d$ are C^∞ functions defined on an open set U of $\mathbb{R}^N$ with values in $\mathbb{R}^N$. The jacobian matrix of f_j at point u will be denoted by $A_j(u)$.

We assume that the system (1) does not consist of a single conservation law (in one or several space variables), that is $N \geq 2$. We also assume that the space dimension is $d \geq 2$ (see the preceeding remarks). We first assume that (1) is a symmetric hyperbolic system of conservation laws:

Assumption 1. *There exists a C^∞ mapping $\Sigma : U \longrightarrow \mathcal{M}_N(\mathbb{R})$ such that*

$$\forall\, j = 0, \dots, d \quad \forall\, u \in U \quad \Sigma(u)\, A_j(u) \text{ is symmetric,}$$

$$\forall\, K \text{ compact } \subset U \quad \exists\, c_K > 0 \text{ such that } \Sigma(u)\, A_0(u) \geq c_K\, I \text{ for all } u \in K.$$

Recall that assumption 1 is satisfied when there exists a strictly convex entropy, see [14, 34]. Assumption 1 is met by many physical examples such as Euler equations of gas dynamics, Maxwell equations or the wave equation. Moreover, assumption 1 is the key tool to solve the Cauchy problem associated to (1) for smooth initial data (namely in a Sobolev space of large index), see [26, 34].

Because the system has been assumed to be symmetric hyperbolic, the matrix $A(u, \xi)$ defined by the formula:

$$\forall\, \xi \in \mathbb{R}^d, \quad A(u, \xi) := A_0(u)^{-1} \sum_{j=1}^{d} \xi_j\, A_j(u) \tag{2}$$

is diagonalizable over $\mathbb{R}$ for all state $u \in U$ and all wave vector $\xi \in \mathbb{R}^d$ (see [34]). However, we shall need a little more than hyperbolicity to carry out the study of the linear stability of shock waves. In [22], the system was assumed to be strictly hyperbolic but it has been shown in [25] that a suitable "block structure condition" (that is met by strictly hyperbolic systems) is sufficient to carry out the study of initial boundary value problems and the study of the linear stability of shock waves, see also [12, 28, 30]. The block structure condition will be recalled further in this paper. In [27], Métivier has shown that the block structure condition was met by every hyperbolic system with constant multiplicity. We are thus naturally led to make the assumption that (1) is a system with constant multiplicity:

Assumption 2. *There exist C^∞ real valued mappings $\lambda_1, \dots, \lambda_q$ defined on $U \times \mathbb{R}^d \setminus \{0\}$, and fixed integers $m_1, \dots, m_q$ such that the λ_j's are the eigenvalues, with multiplicity m_j, of the matrix $A(u,\xi)$ defined by* (2)*. Furthermore, the λ_j's satisfy*

$$\forall\, u \in U\,, \quad \forall\, \xi \in \mathbb{R}^d \setminus \{0\}\,, \quad \lambda_1(u,\xi) < \cdots < \lambda_q(u,\xi)\,.$$

We point out that assumption 2 is easily checked on the system. However, one could replace assumption 2 by the more abstract block structure condition, as was made in [25, 28].

Example: consider Euler's equations of isentropic gas dynamics in space dimension d:

$$\begin{cases} \partial_t\, \rho + \nabla \cdot (\rho\, \mathbf{v}) = 0\,, \\ \partial_t\, (\rho\, \mathbf{v}) + \nabla \cdot (\rho\, \mathbf{v} \otimes \mathbf{v}) + \nabla\, p = 0\,, \end{cases}$$

where ρ stands for the density of the fluid, $\mathbf{v}$ for the velocity, p for the pressure. Quantities ρ and p are linked by an equation of state $p = p(\rho)$. Euler's equations form a nonlinear hyperbolic system of conservation laws. In the domain $\{\rho > 0\}$, hyperbolicity (we mean assumption 1) amounts to require that the pressure satisfies

$$c^2 := \frac{dp}{d\rho} > 0\,.$$

As usual, c denotes the sound speed in the fluid. Under this condition on the pressure law, Euler's equations are endowed with a strictly convex entropy. Moreover, the eigenvalues of the corresponding matrix $A(u,\xi)$ are given by

$$\begin{aligned} \lambda_1(u,\xi) &= \mathbf{v} \cdot \xi - c\,|\xi| && \text{with multiplicity } m_1 = 1\,, \\ \lambda_2(u,\xi) &= \mathbf{v} \cdot \xi && \text{with multiplicity } m_2 = d-1\,, \\ \lambda_3(u,\xi) &= \mathbf{v} \cdot \xi + c\,|\xi| && \text{with multiplicity } m_3 = 1\,. \end{aligned}$$

and therefore assumption 2 is met. We shall detail in section 4 how the general analysis of this paper applies in the context of isentropic gas dynamics.

Note that Lundquist's equations of magnetohydrodynamics violate assumption 2. The study of shock waves in MHD is a very intricate subject due to the appearance of many "pathologies" (nonconstant multiplicity, occurrence of under- and over-compressive shocks, etc...). We refer to [10] and to the references therein for some results on this subject.

Because of the natural development singularities in finite time [4], it appears natural to seek solutions to (1) as functions that are smooth on either side of a hypersurface of $\mathbb{R} \times \mathbb{R}^d$. Recall the following classical result:

Proposition 1.1. *Let $\Gamma = \{x_d - \varphi(x_0, \dots, x_{d-1}) = 0\}$ be a smooth hypersurface in $\mathbb{R} \times \mathbb{R}^d$, and let u be a smooth function on either side of Γ. Then u is a weak solution of* (1) *if and only if u satisfies* (1) *(in the classical sense) on either side of Γ and if the Rankine-Hugoniot conditions hold at each point of Γ:*

$$\forall\, \mathbf{x} = (x_0, \dots, x_d) \in \Gamma\,, \quad \sum_{j=0}^{d-1} \partial_j \varphi\, [f_j(u)](\mathbf{x}) - [f_d(u)](\mathbf{x}) = 0\,, \tag{3}$$

the partial derivatives of φ in the above formula being evaluated at $(x_0, \dots, x_{d-1})$. In (3)*, we have let $[f_j(u)](\mathbf{x})$ denote the jump of the quantity $f_j(u)$ across the hypersurface:*

$$[f_j(u)](\mathbf{x}) := \lim_{s \to 0^+} \left(f_j(u(\mathbf{x} + s\mathbf{n})) - f_j(u(\mathbf{x} - s\mathbf{n})) \right) \quad \textit{with} \quad \mathbf{n} = (-\partial_0 \varphi, \dots, -\partial_{d-1} \varphi, 1)\,.$$

The existence of such a solution to (1) is a free boundary problem since the function φ defining the hypersurface Γ is part of the unknown of the problem. To overcome this first difficulty, we begin by straightening the variables in order to work in a fixed domain: given a smooth function φ on $\mathbb{R}^d$, we define a change of variables in $\mathbb{R}^{d+1}$ by the formula:

$$\Phi(x_0,\dots,x_d) := (x_0,\dots,x_{d-1},x_d+\varphi(x_0,\dots,x_{d-1}))\,.$$

We have chosen here the standard change of variables (as in [25, 28, 30]): it maps the hyperplane $\{x_d = 0\}$ onto the hypersurface Γ and the two half-spaces $\{\pm x_d > 0\}$ on the two sides of Γ. Other choices for the change of variables (that may be appropriate for characteristic problems) may be found in [15]. We now perform a change of unknown functions. If u is a smooth function on either side of Γ, then the function $u_\sharp$ defined by

$$\forall\,(x_0,\dots,x_d)\in\mathbb{R}^{d+1}\,,\quad u_\sharp(x_0,\dots,x_d) := u(\Phi(x_0,\dots,x_d))$$

is smooth on either side of the hyperplane $\{x_d = 0\}$. Denoting by $u_\sharp^+$ (respectively $u_\sharp^-$) the restriction of $u_\sharp$ to the half-space $\{x_d > 0\}$ (respectively $\{x_d < 0\}$), proposition 1.1 asserts that u is a weak solution of (1) if and only if

$$\begin{cases} L(u_\sharp^\pm,\varphi)\,u_\sharp^\pm = 0 & \text{if } \pm\,x_d > 0,\\ B(u_\sharp^+,u_\sharp^-,\varphi) = 0 & \text{if } x_d = 0, \end{cases} \tag{4}$$

where operators L and B are defined by the formulas:

$$L(v,\psi)\,w := \sum_{j=0}^{d-1} A_j(v)\,\partial_j w + \widetilde{A_d}(v,\nabla\psi)\,\partial_d w \tag{5a}$$

$$\text{with}\quad \widetilde{A_d}(v,\nabla\psi) := A_d(v) - \sum_{j=0}^{d-1}\partial_j\psi\,A_j(v) \tag{5b}$$

$$B(w^+,w^-,\psi) := \sum_{j=0}^{d-1}\partial_j\psi\,[f_j(w)] - [f_d(w)]\,. \tag{5c}$$

Now that the domain is fix, the problem reduces to the following question: given an initial datum u^0 that is smooth on either side of a hypersurface $\{x_d = \varphi^0(x_1,\dots,x_{d-1})\}$, does there exist a solution $(u_\sharp,\varphi)$ of (4) with initial value $(u_\sharp^0,\varphi^0)$, at least locally in time? This question has received a positive answer in [24] under the so-called uniform stability condition (we shall recall it in section 2), see [26, 35] for a description of the method. The main idea is that equations (5a)-(5c) are satisfied for planar shocks and the linear uniform stability of these trivial solutions implies the existence of nontrivial solutions. As detailed in [6, 13, 36], the uniform stability condition breaks down in some cases and Majda's nonlinear existence result can not be applied anymore. Our purpose is therefore to derive a linear stability result under a weaker condition than Majda's one.

2 The constant coefficients analysis

We first examin the linear stability of a planar shock in order to formulate our "weak stability" assumptions. A planar shock is a solution of (1) that takes the form

$$\mathbf{u} = \begin{cases} \mathbf{u}_r & \text{if } x_d > \sigma t + \nu\cdot y,\\ \mathbf{u}_l & \text{if } x_d < \sigma t + \nu\cdot y, \end{cases} \tag{6}$$

where $\mathbf{u}_r$ and $\mathbf{u}_l$ are fixed vectors belonging to the open set U, $y = (x_1, \dots, x_{d-1})$ is the vector formed by the tangential space coordinates, ν is a wave vector in $\mathbb{R}^{d-1}$ and σ is the normal speed of propagation of the front. This corresponds to the equation

$$\varphi(x_0, \dots, x_{d-1}) = \sigma x_0 + \sum_{j=1}^{d-1} \nu_j x_j$$

for the shock front curve. We easily check that $\mathbf{u}$ is a solution of (1) if and only if the Rankine-Hugoniot relations

$$\sigma\,[f_0(\mathbf{u})] + \sum_{j=1}^{d-1} \nu_j\,[f_j(\mathbf{u})] = [f_d(\mathbf{u})] \tag{7}$$

are satisfied. Performing a rotation of the axis, we may assume $\nu = 0$. Changing last space variable x_d into $x_d - \sigma t$, we may also assume that the shock is stationary, that is $\sigma = 0$. Note that assumption 2 is still satisfied after this change of observer.

Following Lax [23], we assume that $\mathbf{u}$ is a p-shock:

Assumption 3. *There exists an integer $p \in \{1, \dots, q\}$ such that the following inequalities hold:*

$$\lambda_{p-1}(\mathbf{u}_l, e_d) < 0 < \lambda_p(\mathbf{u}_l, e_d) \quad \textit{and} \quad \lambda_p(\mathbf{u}_r, e_d) < 0 < \lambda_{p+1}(\mathbf{u}_r, e_d)$$

where $e_d := (0, \dots, 0, 1) \in \mathbb{R}^d$ is the wave vector of propagation of the shock $\mathbf{u}$. Moreover, λ_p is a simple eigenvalue, that is $m_p = 1$.

In the case $p = 1$ (respectively $p = q$), that is in the case of an extreme shock, the first inequality on the left (respectively the last on the right) is ignored. Recall that assumption 3 is made in order to avoid under- (or over-)determinacy of the boundary value problem (4). In view of the number of jump conditions, the number of characteristics (counted with their multiplicity) entering the shock front curve has to be equal to $N+1$, see figure 1. Recall also that if λ_p is a multiple eigenvalue, that is, $m_p > 1$, the p-th field is linearly degenerate by Boillat's theorem, see [34]. This is a second reason why we assume that λ_p is a simple eigenvalue.

Remark: applying the implicit functions theorem, we easily see that the set of solutions (σ, ν, u_l, u_r) of the Rankine-Hugoniot relations (7) consists, in the vicinity of $(0, 0, \mathbf{u}_l, \mathbf{u}_r)$ of all vectors of the form

$$(\sigma, \nu, u_l, g(\sigma, \nu, u_l))$$

where g is a C^∞ mapping defined on a neighbourhood of $(0, 0, \mathbf{u}_l) \in \mathbb{R} \times \mathbb{R}^{d-1} \times U$ and satisfies $g(0, 0, \mathbf{u}_l) = \mathbf{u}_r$. Moreover, shock waves that are close to our reference shock wave $\mathbf{u}$ are p-shocks, that is meet assumption 3. In all the sequel, we base most of our analysis on the reference shock wave $\mathbf{u}$ but we shall also need to deal with shock waves that are close to $\mathbf{u}$. These shock waves share the main properties of $\mathbf{u}$.

2.1 The weak stability condition

We now introduce the linearized operators around the shock $\mathbf{u}$: consider a family $u_s = \mathbf{u}_\sharp + sv$ and $\varphi_s = s\psi$. Then we define the linearized operators

$$\mathcal{L}_{\mathbf{u}}(v^\pm, \psi) := \frac{d}{ds}\, L(u_s^\pm, \varphi_s)\, u_s^\pm\,|_{s=0}\,, \tag{8a}$$

$$\mathcal{B}_{\mathbf{u}}(v, \psi) := \frac{d}{ds}\, B(u_s^+, u_s^-, \varphi_s)\,|_{s=0}\,. \tag{8b}$$

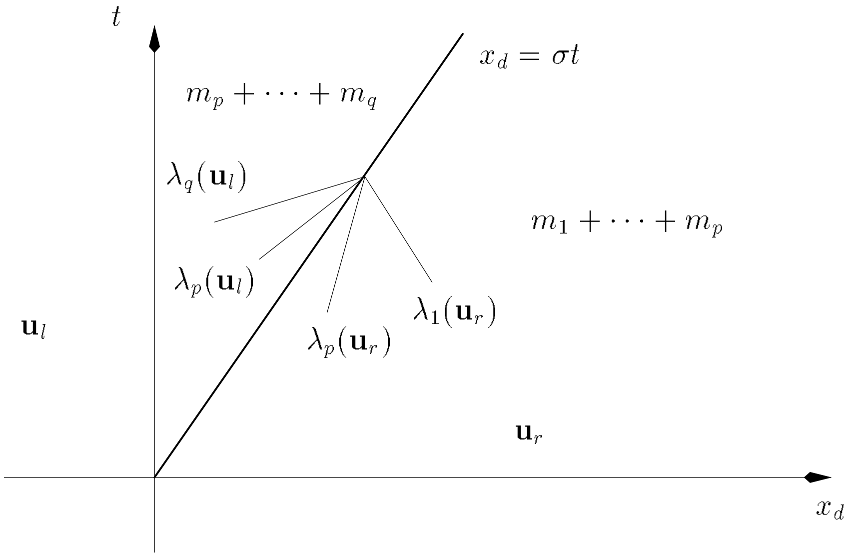

Figure 1: Characteristics entering the shock front

Since $\mathbf{u}$ is constant on either side of $\{x_d = 0\}$, the linearized operators read

$$\mathcal{L}_{\mathbf{u}}(v^{\pm}, \psi) = \mathcal{L}_{\mathbf{u}} v^{\pm} = \sum_{j=0}^{d} A_j(\mathbf{u}_{r,l})\, \partial_j\, v^{\pm}\,, \quad \pm x_d > 0\,,$$

$$\mathcal{B}_{\mathbf{u}}(v, \psi) = \sum_{j=0}^{d-1} \partial_j \psi\, [f_j(\mathbf{u})] - A_d(\mathbf{u}_r)\, v^+ + A_d(\mathbf{u}_l)\, v^-\,, \quad x_d = 0\,.$$

We let $b_j(\mathbf{u}) := [f_j(\mathbf{u})]$ (for $j = 0, \ldots, d-1$) and

$$M(\mathbf{u}) \begin{pmatrix} v^+ \\ v^- \end{pmatrix} := -A_d(\mathbf{u}_r)\, v^+ + A_d(\mathbf{u}_l)\, v^-\,.$$

Then the linearized boundary value operator $\mathcal{B}_{\mathbf{u}}$ reads

$$\mathcal{B}_{\mathbf{u}}(v, \psi) = \sum_{j=0}^{d-1} \partial_j \psi\, b_j(\mathbf{u}) + M(\mathbf{u}) \begin{pmatrix} v^+ \\ v^- \end{pmatrix}\,.$$

We are led to consider the boundary value problem for the unknown functions (v, ψ):

$$\begin{cases} \mathcal{L}_{\mathbf{u}} v^{\pm} = f_{\pm} & \text{for } \pm x_d > 0\,, \\ \mathcal{B}_{\mathbf{u}}(v, \psi) = g & \text{for } x_d = 0\,, \end{cases} \tag{9}$$

where $f_{\pm}$ and g are source terms.

Note that system (9) is a constant coefficients hyperbolic boundary value problem; moreover, the boundary $\{x_d = 0\}$ is noncharacteristic because of assumption 3: both matrices $A_d(\mathbf{u}_r)$ and $A_d(\mathbf{u}_l)$ are regular. Kreiss' theory [22] on this class of problems does not apply directly because partial derivatives of ψ are involved in the boundary conditions. Nonetheless, we attack problem (9) by the same kind of arguments: formally, we perform a Laplace transform in x_0 and a Fourier transform in the tangential space variables $(x_1, \ldots, x_{d-1})$. We also make the change of variables

$(x_d \to -x_d)$ in the evolution equation for v^-. This yields the following system of ODEs:

$$\left(\tau\, A_0(\mathbf{u}_r) + i \sum_{j=1}^{d-1} \eta_j\, A_j(\mathbf{u}_r) \right) V^+ + A_d(\mathbf{u}_r)\, \frac{dV^+}{dx_d} = \widehat{F_+}\,, \tag{10a}$$

$$\left(\tau\, A_0(\mathbf{u}_l) + i \sum_{j=1}^{d-1} \eta_j\, A_j(\mathbf{u}_l) \right) V^- - A_d(\mathbf{u}_l)\, \frac{dV^-}{dx_d} = \widehat{F_-}\,, \tag{10b}$$

in the domain $\{x_d > 0\}$, with the boundary conditions:

$$\left(\tau\, b_0(\mathbf{u}) + i \sum_{j=1}^{d-1} \eta_j\, b_j(\mathbf{u}) \right) \Psi + M(\mathbf{u}) \begin{pmatrix} V^+(0) \\ V^-(0) \end{pmatrix} = \widehat{G}\,, \tag{11}$$

on $\{x_d = 0\}$. The complex number $\tau = \gamma + i\eta_0$ has nonnegative real part and $(\eta_1, \dots, \eta_{d-1})$ is a wave vector in $\mathbb{R}^{d-1}$. In the sequel, we shall denote by η the vector $(\eta_0, \eta_1, \dots, \eta_{d-1}) \in \mathbb{R}^d$.

Because the boundary is noncharacteristic, we may rewrite (10a)-(10b) as an ODE system of the form

$$\begin{gathered} \frac{d}{dx_d} \begin{pmatrix} V^+ \\ V^- \end{pmatrix} = \mathcal{A}(\mathbf{u}, \eta, \gamma) \begin{pmatrix} V^+ \\ V^- \end{pmatrix} + \mathcal{A}_d(\mathbf{u})^{-1} \begin{pmatrix} F_+ \\ F_- \end{pmatrix} \\ \text{with } \ \mathcal{A}(\mathbf{u}, \eta, \gamma) := \begin{pmatrix} \mathcal{A}_r(\mathbf{u}, \eta, \gamma) & \mathbf{0} \\ \mathbf{0} & \mathcal{A}_l(\mathbf{u}, \eta, \gamma) \end{pmatrix} . \end{gathered} \tag{12}$$

Matrices $\mathcal{A}_{r,l}$ and $\mathcal{A}_d$ in (12) are defined by

$$\mathcal{A}_{r,l}(\mathbf{u}, \eta, \gamma) := \mp A_d(\mathbf{u}_{r,l})^{-1} \left(\gamma\, A_0(\mathbf{u}_{r,l}) + i \sum_{j=0}^{d-1} \eta_j\, A_j(\mathbf{u}_{r,l}) \right) ,$$

$$\mathcal{A}_d(\mathbf{u}) := \begin{pmatrix} A_d(\mathbf{u}_r) & \mathbf{0} \\ \mathbf{0} & -A_d(\mathbf{u}_l) \end{pmatrix} .$$

Defining

$$\mathcal{A}_j(\mathbf{u}) := \begin{pmatrix} A_j(\mathbf{u}_r) & \mathbf{0} \\ \mathbf{0} & A_j(\mathbf{u}_l) \end{pmatrix} , \quad 0 \le j \le d-1\,,$$

we find easily

$$\mathcal{A}(\mathbf{u}, \eta, \gamma) = -\mathcal{A}_d(\mathbf{u})^{-1} \left(\gamma\, \mathcal{A}_0(\mathbf{u}) + i \sum_{j=0}^{d-1} \eta_j\, \mathcal{A}_j(\mathbf{u}) \right) .$$

We also define the symbol associated to the shock front:

$$b(\mathbf{u}, \eta, \gamma) := \gamma\, b_0(\mathbf{u}) + i \sum_{j=0}^{d-1} \eta_j\, b_j(\mathbf{u})\,. \tag{13}$$

As pointed out by Hersh [18], the homogeneous part of the ODEs (12) is hyperbolic when $\gamma > 0$, that is, the matrix $\mathcal{A}$ has no purely imaginary eigenmodes when $\gamma > 0$. For $\gamma > 0$, we define $\mathcal{E}^-(\mathbf{u}, \eta, \gamma)$ as the stable subspace of $\mathcal{A}$, which is the set of initial values of solutions to (12) that are square integrable on $[0, +\infty[$ when $F_+ = F_- = 0$. Because of the decoupled nature of (12), the stable subspace $\mathcal{E}^-(\mathbf{u},\eta,\gamma)$ is nothing but the product of the stable subspaces of $\mathcal{A}_r$ and $\mathcal{A}_l$. In the case of an extreme shock, one of these two stable subspaces is trivial. The dimension of $\mathcal{E}^-(\mathbf{u},\eta,\gamma)$ is equal to the dimension of $\mathcal{E}^-(\mathbf{u},0,\gamma)$ for all η (this is just a continuity/connectedness argument). To compute the dimension of $\mathcal{E}^-(\mathbf{u},0,\gamma)$, we seek the eigenmodes of $\mathcal{A}(\mathbf{u},0,\gamma)$. These are the roots ω of the dispersion equation

$$\det\left[\omega I - \mathcal{A}(\mathbf{u},0,\gamma)\right] = 0\,.$$

The definition of $\mathcal{A}$ shows that $\omega I - \mathcal{A}(\mathbf{u},0,\gamma)$ is singular if and only if $\gamma A(\mathbf{u}_r) + \omega A_d(\mathbf{u}_r)$ or $\gamma A(\mathbf{u}_l) - \omega A_d(\mathbf{u}_l)$ are singular. As a consequence, ω satisfies the dispersion equation if and only if there exists an integer $k \in \{1,\dots,q\}$ such that

$$\lambda_k(\mathbf{u}_r,e_d)\,\omega = -\gamma \quad \text{or} \quad \lambda_k(\mathbf{u}_l,e_d)\,\omega = \gamma\,,$$

where $e_d := (0,\dots,0,1) \in \mathbb{R}^d$. Assumption 3 shows that such values of ω are negative for $k = p+1,\dots,q$ in the first case and $k = 1,\dots,p-1$ in the second case. Taking multiplicities into account, this shows that $\mathcal{E}^-(\mathbf{u},0,\gamma)$ (and therefore $\mathcal{E}^-(\mathbf{u},\eta,\gamma)$) has dimension $N-1$ as long as $\gamma > 0$.

For fixed $\underline{\eta} \neq 0$, the stable subspace $\mathcal{E}^-(\mathbf{u},\eta,\gamma)$ admits a continuous extension to $(\underline{\eta},0)$, see [9] (the argument makes use of the compactness of Grassmanian manifolds); we still denote this extension by $\mathcal{E}^-(\mathbf{u},\underline{\eta},0)$. Note that for $\gamma = 0$, vectors in the extended stable subspace are not always boundary values of square integrable functions because of the possible occurence of purely imaginary eigenmodes. This is widely detailed in [6, 13] for Euler equations of isentropic gas dynamics.

We define the hemisphere of $\mathbb{R}^{d+1}$ as

$$\Sigma_+ := \{(\eta,\gamma) \in \mathbb{R}^{d+1} \text{ s.t. } \gamma^2 + |\eta|^2 = 1 \text{ and } \gamma \geq 0\}\,.$$

The boundary of Σ_+ will be denoted by Σ_0, that is

$$\Sigma_0 := \{(\eta,0) \in \mathbb{R}^{d+1} \text{ s.t. } |\eta|^2 = 1\}\,.$$

Recall the following definition:

Definition 1 (Majda). [25]. *The planar shock* $\mathbf{u}$ *is said to satisfy the uniform stability condition if there exists a positive constant* $c > 0$ *such that for all* $(\eta,\gamma) \in \Sigma_+$, *one has*

$$\forall\,(\chi,Z) \in \mathbb{C} \times \mathcal{E}^-(\mathbf{u},\eta,\gamma)\,, \quad |\chi\, b(\mathbf{u},\eta,\gamma) + M(\mathbf{u})\,Z| \geq c\,(|\chi| + |Z|)\,.$$

By compactness of Σ_+, the uniform stability condition is equivalent to the requirement that for all $(\eta,\gamma) \in \Sigma_+$, the "critical" subspace

$$\left\{(\chi,Z) \in \mathbb{C} \times \mathcal{E}^-(\mathbf{u},\eta,\gamma) \text{ s.t. } \chi\, b(\mathbf{u},\eta,\gamma) + M(\mathbf{u})\,Z = 0\right\}$$

is trivial, that is reduced to $\{0\}$. The word "critical" is not standard but its use here intends to show the major place that is occupied by this subspace in the normal modes analysis.

Recall that a planar shock is uniformly stable if and only if solutions of (9) satisfy a maximal L^2 estimate as in the study of linear hyperbolic boundary value problems, see [25, proposition 1]. The uniform stability condition is thus the direct extension of Kreiss' uniform condition [22].

As noted by Majda [25, lemma 4.1], the uniform stability condition enables to isolate the shock front Ψ appearing in (11) in a single equation. We emphasize that this operation **can not** be achieved for scalar conservation laws in space dimension more than 1.

Our approach is slightly different: we allow some instability but this instability can only stem from the traces of solutions to the dynamical system (12) and not from the shock front symbol defined by (13). More precisely, we make the following assumption:

Assumption 4. *There exists a positive constant c such that*

$$\forall\,(\eta,\gamma)\in\Sigma_+\,,\quad |b(\mathbf{u},\eta,\gamma)|\geq c\,. \tag{14}$$

Of course, an analogue estimate is valid for all states u_r, u_l close to $\mathbf{u}_r, \mathbf{u}_l$.

Remark: assumption 4 is equivalent (see [19, chapter 4]) to the requirement that there exists a C^∞ mapping $P:\Sigma_+\to Gl_N(\mathbb{C})$ such that

$$\forall\,(\eta,\gamma)\in\Sigma_+\,,\quad P(\mathbf{u},\eta,\gamma)\,b(\mathbf{u},\eta,\gamma)=\begin{pmatrix}1\\ \mathbf{0}\end{pmatrix}\,,$$

and one can even choose P depending smoothly on the states $\mathbf{u}_r, \mathbf{u}_l$.

It is clear that (14) holds if $\mathbf{u}$ satisifes Majda's uniform stability condition. In a pseudodifferential setting, b is an overdetermined elliptic symbol (γ is seen as a parameter and we are thus dealing with pseudodifferential symbols with a parameter). It is shown in [13] that assumption 4 is met in some cases where the uniform stability condition is violated: the main example concerns shock waves in isentropic gas dynamics when the pressure law is not a convex function of the density.

Assumption 4 enables to reformulate boundary conditions (11) by isolating the unknown shock front Ψ in a single equation. Because of (14), we can define for all $(\eta,\gamma)\in\mathbb{R}^d\times\mathbb{R}^+\setminus\{0\}$ the orthogonal projector

$$\Pi(\mathbf{u},\eta,\gamma)\,h:=h-\frac{\langle h,b(\mathbf{u},\eta,\gamma)\rangle}{|b(\mathbf{u},\eta,\gamma)|^2}\,b(\mathbf{u},\eta,\gamma)\,.$$

Multiplying (11) by Π and b^* yields the new boundary conditions

$$\Pi(\mathbf{u},\eta,\gamma)M(\mathbf{u})\,V(0)=\Pi(\mathbf{u},\eta,\gamma)\,G\,, \tag{15a}$$

$$|b(\mathbf{u},\eta,\gamma)|^2\,\Psi=\langle G-M(\mathbf{u})\,V(0),b(\mathbf{u},\eta,\gamma)\rangle\,. \tag{15b}$$

Observe that b is homogeneous of degree 1 in (η,γ) so (14) is equivalent to

$$\forall\,(\eta,\gamma)\in\mathbb{R}^d\times\mathbb{R}^+,\quad c\,(\gamma^2+|\eta|^2)\leq|b(\mathbf{u},\eta,\gamma)|^2\,.$$

Using Schwarz' inequality in (15b), we obtain

$$\forall\,(\eta,\gamma)\in\mathbb{R}^d\times\mathbb{R}^+,\quad (\gamma^2+|\eta|^2)\,|\Psi|^2\leq C\,(|G|^2+|V(0)|^2) \tag{16}$$

for a suitable constant C depending only on the shock $\mathbf{u}$. It is thus sufficient to get an estimate on $V(0)$ and we shall get from (16) an estimate on Ψ. In order to obtain the desired estimate on $V(0)$, we attack the following boundary value problem:

$$\begin{cases}\dfrac{dV}{dx_d}=\mathcal{A}(\mathbf{u},\eta,\gamma)\,V+\mathcal{A}_d(\mathbf{u})^{-1}\,\widehat{F} & \text{for } x_d>0,\\ \Pi(\mathbf{u},\eta,\gamma)M(\mathbf{u})\,V(0)=\Pi(\mathbf{u},\eta,\gamma)\,\widehat{G}. & \end{cases} \tag{17}$$

Isolating the front yields a boundary value problem where the boundary operator takes the form of a Fourier multiplier of order 0. Indeed, the homogeneity property of b implies that Π is homogeneous of degree 0 with respect to (η,γ). More precisely, Π is a pseudodifferential symbol of degree 0 (γ is seen as a parameter). We refer to [2] and [12] for a detailed study of pseudodifferential calculus with a parameter; the introduction of a parameter in Bony's paradifferential calculus [11, 29] has been achieved in [30], see also [28].

Our final assumption is that the boundary conditions defined by the symbol ΠM satisfy the Kreiss-Lopatinskii condition but violate the **uniform** Kreiss-Lopatinskii condition. It is important for what follows that this assumption is met by all shock waves close to $\mathbf{u}$:

Assumption 5. *For all shock waves u close to the reference shock wave $\mathbf{u}$, the following properties hold:*

If $(\eta,\gamma) \in \Sigma_+$ and $\gamma > 0$, the reduced critical subspace

$$\big\{ Z \in \mathcal{E}^-(u,\eta,\gamma) \ \ s.t.\ \ \Pi(u,\eta,\gamma) M(u)\, Z = 0 \big\}$$

is trivial.

If $(\underline{\eta},0) \in \Sigma_0$ is a point where the reduced critical subspace

$$\big\{ Z \in \mathcal{E}^-(u,\underline{\eta},0) \ \ s.t.\ \ \Pi(u,\underline{\eta},0) M(u)\, Z = 0 \big\}$$

is nontrivial, then there exists a neighbourhood $\mathcal{V}$ of $(\underline{\eta},0)$ in Σ_+ and a constant $c > 0$ such that

$$\forall\, (\eta,\gamma) \in \mathcal{V}\,, \quad \forall\, Z \in \mathcal{E}^-(u,\eta,\gamma)\,, \quad |\Pi(u,\eta,\gamma) M(u)\, Z| \geq c\,\gamma\, |Z|\,. \tag{18}$$

Furthermore, there exists a C^∞ mapping $Q(u,\cdot) : \mathcal{V} \to Gl_{2N}(\mathbb{C})$ depending smoothly on u such that for all $z = (\eta,\gamma) \in \mathcal{V}$, one has

$$Q(u,z)\, \mathcal{A}(u,z)\, Q(u,z)^{-1} = \begin{pmatrix} a^-(z) & & & & \\ & a^+(z) & & \mathbf{0} & \\ & & \omega_1(z) & & \\ & \mathbf{0} & & \ddots & \\ & & & & \omega_J(z) \end{pmatrix},$$

where $a^+(z)$ (respectively $a^-(z)$) is a square matrix whose spectrum is contained in the half plane $\{\ \mathrm{Re}\ \zeta > 0\}$ (respectively $\{\ \mathrm{Re}\ \zeta < 0\}$) and the ω_j's are complex valued. These quantities also satisfy $\omega_j \in i\mathbb{R}$ when $\gamma = 0$ and either

$$\mathrm{Re}\ \omega_j(z) \geq c\,\gamma$$

or

$$\mathrm{Re}\ \omega_j(z) \leq -c\,\gamma\,,$$

for all $z \in \mathcal{V}$.

Before stating our main result on such weakly stable planar shocks (that is planar shock waves satisfying assumptions 3-5) let us explain in a few words the meaning of assumption 5. The first condition asserts that the shock wave is not violently unstable: it must satisfy the analogue of the Kreiss-Lopatinskii condition for linear hyperbolic boundary value problems. The second condition asserts that the uniform stability condition is violated "at order 1". Recall that when Majda's uniform stability condition is met, one has an estimate of the type

$$\forall\, (\eta,\gamma) \in \mathcal{V}\,, \quad \forall\, Z \in \mathcal{E}^-(u,\eta,\gamma)\,, \quad |\Pi(u,\eta,\gamma) M(u)\, Z| \geq c\, |Z|\,.$$

In our study, the allowed instability yields a power of γ in (18).

When (η,γ) tends to an instability point $(\underline{\eta},0)$, part of the stable and unstable subspaces of $\mathcal{A}(u,\eta,\gamma)$ form a central subspace. The last requirement of assumption 5 is that the restriction of $\mathcal{A}(u,\eta,\gamma)$ to this subspace (that is central when $\gamma = 0$) is smoothly diagonalizable. In gas dynamics, there are examples of shock waves for which the uniform stability condition breaks down at a point where the symbol is not smoothly diagonalizable, see [13, 25]. Unfortunately, we have not been able to deal with this case: one major problem is the failure of the differentiability of the eigenmodes at such points. We refer to the work of Okhubo and Shirota [31] for some aspects of these phenomena.

Remark: making assumption 5 for all shock waves close to $\mathbf{u}$ is not very restrictive (first because it is satisfied in both examples we study). In fact, if assumption 5 is satisfied by the shock wave $\mathbf{u}$, then it is automatically satisfied by all shock waves close to $\mathbf{u}$ provided the whole stable and unstable subspaces form a central space when $\gamma = 0$ in the vicinity of the instability points, see [8]. In such a case, there are no blocks a^+ and a^- in the reduction of $\mathcal{A}$ in the vicinity of the instability points. This theoretical result applies to weakly stable shock waves in isentropic gas dynamics, see [13]. In the case of phase transitions in a van der Waals fluid, instability modes belong to a region of the parameters (η, γ) where some eigenmodes have negative real part and it is not a direct consequence of [8] that assumption 5 is satisfied by all planar shock waves close to $\mathbf{u}$ (but fortunately the calculations show that it is true).

2.2 The weak stability of planar shock waves

In this paragraph, we show that if $\mathbf{u}$ satisfies the weak stability condition, then the constant coefficients boundary value problem (9) is "well-posed" in the sense that solutions to (9) satisfy an energy estimate. Since we deal with shock waves that violate Majda's uniform stability condition, the energy estimate will involve losses of derivatives with respect to the right hand-side terms f and g in (9).

We need first of all to introduce a few notations. Denote by Ω and ω the domains

$$\Omega := \mathbb{R}^{d+1}_+ = \{(x_0, \dots, x_d) \in \mathbb{R}^{d+1} \text{ s.t. } x_d > 0\} \quad \text{and} \quad \omega := \mathbb{R}^d = \partial\Omega\,.$$

For $\gamma > 0$ and $s \in \mathbb{R}$ we define the following symbols

$$\forall\, \xi \in \mathbb{R}^d, \quad \lambda^{s,\gamma}(\xi) := (\gamma^2 + |\xi|^2)^{s/2}\,.$$

The usual Sobolev spaces $H^s(\omega)$ are equipped with the weighted norms (depending on the positive parameter γ):

$$\|v\|^2_{s,\gamma} := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \lambda^{2s,\gamma}(\xi)\, |\widehat{v}(\xi)|^2\, d\xi\,.$$

We shall write $\|\cdot\|_0$ instead of $\|\cdot\|_{0,\gamma}$ since there is no dependance on the parameter γ of the L^2 norm. These weighted norms enable to construct a parameter version of the classical pseudodifferential calculus which is of constant use in the study of hyperbolic boundary value problems, see [2, 22, 25, 28].

The space $L^2(\mathbb{R}^+, H^s(\mathbb{R}^d))$ is also equipped with the weighted norm:

$$|\!|\!| v |\!|\!|^2_{s,\gamma} := \int_0^{+\infty} \|v(\cdot, x_d)\|^2_{s,\gamma}\, dx_d\,. \tag{19}$$

We shall also write $|\!|\!|\cdot|\!|\!|_0$ instead of $|\!|\!|\cdot|\!|\!|_{0,\gamma}$. Typically, we shall use the spaces $L^2(\Omega)$ and $L^2(\mathbb{R}^+, H^1(\mathbb{R}^d))$.

The Laplace transform performed in the normal modes analysis amounts to work with the new functions $\widetilde{v} = \exp(-\gamma t) v$ and $\widetilde{\psi} = \exp(-\gamma t)\psi$, with $\gamma > 0$. This leads to the introduction of the "weighted" operators:

$$\mathcal{L}^\gamma_{\mathbf{u}} \widetilde{v} := \mathcal{L}_{\mathbf{u}} \widetilde{v} + \gamma \mathcal{A}_0(\mathbf{u})\, \widetilde{v} \quad \text{and} \quad \mathcal{B}^\gamma_{\mathbf{u}}(\widetilde{v}, \widetilde{\psi}) := \mathcal{B}_{\mathbf{u}}(\widetilde{v}, \widetilde{\psi}) + \gamma \widetilde{\psi}\, b_0(\mathbf{u})\,.$$

One easily checks that (9) is equivalent to

$$\begin{cases} \mathcal{L}^\gamma_{\mathbf{u}} \widetilde{v} = \exp(-\gamma t) f & \text{for } x_d > 0\,, \\ \mathcal{B}^\gamma_{\mathbf{u}}(\widetilde{v}, \widetilde{\psi}) = \exp(-\gamma t) g & \text{for } x_d = 0\,. \end{cases}$$

For convenience, we drop the tilda from v and ψ (keeping in mind that these functions have been multiplied by a decreasing exponential function and therefore also depend on the parameter γ). Our result on weakly stable planar shock waves can be stated as follows:

Theorem 2.1. *Let $\mathbf{u}$ be a planar shock wave that satisfies assumptions 3-5. Then for all $v \in H^2(\Omega)$, for all $\psi \in H^2(\omega)$ and for all $\gamma \geq 1$, one has:*

$$\gamma\, |||v|||_0^2 + \|v_{|x_d=0}\|_0^2 + \|\psi\|_{1,\gamma}^2 \lesssim \frac{1}{\gamma^3}\, |||\mathcal{L}^\gamma_{\mathbf{u}}(v,\psi)|||_{1,\gamma}^2 + \frac{1}{\gamma^2}\, \|\mathcal{B}^\gamma_{\mathbf{u}}(v,\psi)\|_{1,\gamma}^2\,. \tag{20}$$

Energy estimates in Sobolev spaces of higher order are available, provided that v and ψ are sufficiently regular. Similar estimates also hold for shock waves u close to $\mathbf{u}$.

2.3 Proof of theorem 2.1

The proof of theorem 2.1 follows the earlier works of Kreiss [22] and Majda [25] with suitable modifications. It can be found in [13] in the case of isentropic gas dynamics.

Recall first that assumption 2 ensures that (1) satisfies the so-called block structure condition. More precisely, we have:

Proposition 2.1 (Block structure condition). [27]. *If assumption 2 is satisfied, then for all shock wave u close to $\mathbf{u}$, for all $\underline{z} \in \Sigma_+$, there exists a neighbourhood $\mathcal{V}$ of $\underline{z}$ in Σ_+ and invertible matrices $Q(u,z)$ depending smoothly on u and $z \in \mathcal{V}$ such that*

$$\forall\, z \in \mathcal{V}\,, \quad Q(u,z)\mathcal{A}(u,z)\,Q(u,z)^{-1} = \begin{pmatrix} a^-(z) & & & & \\ & a^+(z) & & \mathbf{0} & \\ & & a_1(z) & & \\ & \mathbf{0} & & \ddots & \\ & & & & a_J(z) \end{pmatrix}.$$

Furthermore, the spectrum of $a^-(z)$ (respectively $a^+(z)$) is contained in the half-plane $\{\mathrm{Re}\,\zeta < 0\}$ (respectively $\{\mathrm{Re}\,\zeta > 0\}$), and for all $j = 1,\dots,J$, the matrix $a_j(z)$ has purely imaginary coefficients when $\gamma = 0$ and $a_j(\underline{z})$ satisfies

$$a_j(\underline{z}) = \omega_j I + i N_j \quad \text{with } \omega_j \in i\mathbb{R} \text{ and } N_j = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \ddots & 0 & 1 \\ 0 & \dots & 0 & 0 \end{pmatrix}.$$

Finally, the lower left-hand corner coefficient of $\partial a_j/\partial\gamma\,(\underline{z})$ is real and does not vanish. Note that the dimensions of the blocks $a^\pm$ and a_j depend on $\underline{z}$ but not on $z \in \mathcal{V}$.

Let us go back to assumption 5: if $\underline{z}$ is a point of Σ_0 such that the critical subspace

$$\big\{ Z \in \mathcal{E}^-(u,\underline{z}) \ \text{ s.t. } \ \Pi(u,\underline{z}) M(u)\, Z = 0 \big\}$$

is nontrivial, then we have assumed that all the blocks a_j (defined by proposition 1.1) are of dimension 1. In particular, when z belongs to a suitable neighbourhood of $\underline{z}$, $a_j(z)$ is a complex number ω_j such that $\omega_j \in i\mathbb{R}$ when $\gamma = 0$ and $\partial \omega_j / \partial \gamma \, (\underline{z})$ is real and does not vanish. If Re $\omega_j < 0$ when $\gamma > 0$, we obtain

$$\text{Re } \omega_j(z) \leq -c\,\gamma\,, \tag{21}$$

whereas we obtain

$$\text{Re } \omega_j(z) \geq c\,\gamma \tag{22}$$

if Re $\omega_j > 0$ when $\gamma > 0$. In both cases, c is a positive constant depending on u and $\underline{z}$. We refer to [13] for a detailed study of the block structure condition in the case of isentropic gas dynamics.

We are going to construct a microlocal symmetrizer in order to prove (20).

• Let $\underline{z} \in \Sigma_+ \setminus \Sigma_0$. Since $\mathcal{A}(u, \underline{z})$ has no purely imaginary eigenvalue (by Hersh's result [18]), there exists a neighbourhood $\mathcal{V}$ of $\underline{z}$ in Σ_+ and invertible matrices $Q(u, z)$ depending smoothly on $z \in \mathcal{V}$ (and also smoothly on u close to $\mathbf{u}$) such that

$$\begin{gathered}
\forall\, z \in \mathcal{V}\,, \quad Q(u,z)\mathcal{A}(u,z)\,Q(u,z)^{-1} = \begin{pmatrix} a^-(z) & \mathbf{0} \\ \mathbf{0} & a^+(z) \end{pmatrix}\,, \\
\text{with Sp } a^-(\underline{z}) \subset \{\text{ Re } \zeta < 0\} \quad \text{ and Sp } a^+(\underline{z}) \subset \{\text{ Re } \zeta > 0\}\,.
\end{gathered}$$

Matrices $a^-(z)$ and $a^+(z)$ are defined by Dunford's formula, see [12, 21]. As noted in [27], this reduction explains why the block structure condition needs only to be checked in the neighbourhood of points belonging to Σ_0. By Lyapunov's theorem, see [5], there exists two positive definite hermitian matrices H^+ and H^- such that

$$\text{Re } (H^{\pm}\, a^{\pm}(\underline{z})) = \pm I\,.$$

For convenience, we define the real part of a square matrix N as $(N + N^*)/2$ and we do the same for operators on a Hilbert space. Following Kreiss [22] (see also [12]), we choose r of the form

$$r(z) := \begin{pmatrix} -H^- & \mathbf{0} \\ \mathbf{0} & \alpha H^+ \end{pmatrix}\,,$$

where α is a real number greater than 1, to be chosen large enough. Because the critical subspace is trivial when z belongs to a neighbourhood of $\underline{z}$, it is proved in [12] that for α large enough, the following estimates hold

$$\begin{gathered}
\text{Re } (r(z)\,Q(u,z)\mathcal{A}(u,z)\,Q(u,z)^{-1}) \geq 1/2\, I\,, \\
r(z) + C\,\widetilde{\beta}(u,z)^*\widetilde{\beta}(u,z) \geq c\, I\,, \qquad \text{where} \qquad \widetilde{\beta}(u,z) := \Pi(u,z)\, M(u)\, Q(u,z)^{-1}\,.
\end{gathered}$$

Constants c and C are positive and depend on $(u, \underline{z})$.

• Let now $\underline{z} \in \Sigma_0$ be a point where the critical subspace is trivial. It appears from proposition 2.1 that Jordan blocks may occur in the reduction of the symbol $\mathcal{A}$. In such a case, the construction of the symmetrizer is rather technical. We refer to [22, 32] and [12, 13, 30] for the details. Following these anterior works, we conclude that there exists a neighbourhood $\mathcal{V}$ of $\underline{z}$ in Σ_+ and C^∞ matrix valued mappings r and $Q(u, \cdot)$ defined on $\mathcal{V}$ such that for all $z \in \mathcal{V}$, $r(z)$ is hermitian, $Q(u, z)$ is invertible and

$$\begin{gathered}
\text{Re } (r(z)\,Q(u,z)\mathcal{A}(u,z)\,Q(u,z)^{-1}) \geq c\,\gamma\, I\,, \\
r(z) + C\,\widetilde{\beta}(u,z)^*\widetilde{\beta}(u,z) \geq c\, I\,.
\end{gathered}$$

• Let $\underline{z} \in \Sigma_0$ be a point where the critical subspace is nontrivial. From assumption 5, we know that there exists a neighbourhood $\mathcal{V}$ of $\underline{z}$ in Σ_+ and invertible matrices $Q(u,z)$, depending smoothly on $z \in \mathcal{V}$, such that

$$Q(u,z)\,\mathcal{A}(u,z)\,Q(u,z)^{-1} = \begin{pmatrix} a^-(z) & & & & & & & \\ & \omega_1(z) & & & & & & \\ & & \ddots & & & & & \\ & & & \omega_{J'}(z) & & & & \\ & & & & a^+(z) & & & \\ & & & & & \omega_{J'+1}(z) & & \\ & & & & & & \ddots & \\ & & & & & & & \omega_J(z) \end{pmatrix},$$

where the ω_j's have negative real part when $\gamma > 0$ and $j = 1, \dots, J'$ and have positive real part when $\gamma > 0$ and $j = J'+1, \dots, J$. We have just reordered the diagonal blocks appearing in assumption 3. Let us remark that the stable subspace $\mathcal{E}^-(u,z)$ (that has dimension $N-1$ for all z) is spanned by the $N-1$ first column vectors of $Q(u,z)^{-1}$. For $z \in \mathcal{V}$, we choose $r(z)$ of the form

$$r(z) := \begin{pmatrix} -\gamma^2\, H^- & & & \mathbf{0} \\ & -\gamma^2\, I_{J'} & & \\ & & \alpha\, H^+ & \\ \mathbf{0} & & & \alpha\, I_{J-J'} \end{pmatrix},$$

where α is a real number greater than 1, to be fixed, and $H^\pm$ are chosen as in the case $\gamma > 0$. Because of the local behavior of the eigenmodes ω_j's, see (21) and (22), we have

$$\forall\, z \in \mathcal{V}, \quad \mathrm{Re}\, \left(r(z)\, Q(u,z) \mathcal{A}(u,z)\, Q(u,z)^{-1} \right) \geq c \begin{pmatrix} \gamma^2\, H^- & & & \mathbf{0} \\ & \gamma^3\, I_{J'} & & \\ & & \alpha\, H^+ & \\ \mathbf{0} & & & \alpha\, \gamma\, I_{J-J'} \end{pmatrix},$$

for a suitable constant $c > 0$. If $W \in \mathbb{C}^{2N}$, we denote by W^- the vector composed by the $N-1$ first coordinates of W and W^+ the vector composed by the $N+1$ last coordinates of W (this corresponds to a decomposition between the incoming part and the outgoing part), so we can write

$$\beta(u,z)\, Q(u,z)^{-1}\, W = \beta(u,z)\, Q(u,z)^{-1} \begin{pmatrix} W^- \\ 0 \end{pmatrix} + \beta(u,z)\, Q(u,z)^{-1} \begin{pmatrix} 0 \\ W^+ \end{pmatrix},$$
$$\beta(u,z) := \Pi(u,z)\, M(u)\,,$$

and from (18), we have

$$\left| \beta(u,z)\, Q(u,z)^{-1} \begin{pmatrix} W^- \\ 0 \end{pmatrix} \right| \geq c\, \gamma\, |W^-|\,,$$

which implies

$$\gamma^2\, |W^-|^2 \lesssim |W^+|^2 + |\widetilde{\beta}(u,z)\, W|^2\,, \quad \widetilde{\beta}(u,z) := \beta(u,z)\, Q(u,z)^{-1}\,.$$

Choosing α large enough yields

$$r(z) + C \widetilde{\beta}(u,z)^* \widetilde{\beta}(u,z) \geq c\, \gamma^2\, I\,,$$

for all $z \in \mathcal{V}$. This completes the microlocal construction of the symmetrizer.

• We now turn to the proof of (20). For all $\underline{z} \in \Sigma_+$, the previous analysis establishes the existence of a neighbourhood $\mathcal{V}$ of $\underline{z}$ in Σ_+ and of smooth mappings r and Q with suitable properties. Because Σ_+ is a smooth compact manifold, there exists a finite covering $\mathcal{V}_i$, $1 \leq i \leq I$, of Σ_+ by such neighbourhoods and a smooth partition of unity χ_i, $1 \leq i \leq I$, subordinated to this covering. Functions χ_i are C^∞, nonnegative and satisfy

$$\forall\, i = 1, \ldots, I \quad \text{Supp}\; \chi_i \subset \mathcal{V}_i \quad \text{and} \quad \sum_{i=1}^{I} \chi_i^2 \equiv 1\,.$$

Let now $v \in H^2(\Omega)$ and $\psi \in H^2(\omega)$. We denote by $V(\eta, x_d)$ the Fourier transform of v with respect to the d first variables $(x_0, \ldots, x_{d-1})$. We also define

$$F := \mathcal{L}^\gamma_{\mathbf{u}}(v, \psi) \in H^1(\Omega)\,, \quad G := \mathcal{B}^\gamma_{\mathbf{u}}(v, \psi) \in H^1(\omega)\,.$$

We extend all mappings r_i and Q_i on all Σ_+ assuming them to be constant outside of $\mathcal{V}_i$ (this is of pure convenience since only the value of these mappings on Supp χ_i will be involved in the following calculations). Then we extend χ_i and Q_i as homogeneous functions of degree 0 in (η, γ). For $z = (\eta, \gamma) \in \mathbb{R}^d \times \mathbb{R}^+$, we define

$$V_i(z, x_d) := \chi_i(z)\, Q_i(\mathbf{u}, z)\, V(\eta, x_d)\,.$$

We thus get the relation

$$\frac{dV_i}{dx_d} = Q_i(\mathbf{u}, z)\, \mathcal{A}(\mathbf{u}, z)\, Q_i(\mathbf{u}, z)^{-1}\, V_i(z, x_d) + \chi_i(z)\, Q_i(\mathbf{u}, z)\, \mathcal{A}_d(\mathbf{u})^{-1}\, \widehat{F}(\eta, x_d)\,. \tag{23}$$

If $\mathcal{V}_i$ is a neighbourhood of a point $\underline{z}$ where the critical subspace is trivial, we extend r_i as a homogeneous function of degree 0 in (η, γ). We multiply (16) by $r_i(z)\, V_i(z, x_d)$ and then integrate with respect to (η, x_d), γ being fixed. Using the inequalities

$$\begin{aligned} \text{Re}\; (r_i(z)\, Q_i(\mathbf{u}, z)\, \mathcal{A}(\mathbf{u}, z)\, Q_i(\mathbf{u}, z)^{-1}) &\geq c\, \gamma\, I\,, \\ r_i(z) + C\, \widetilde{\beta}_i(\mathbf{u}, z)^*\, \widetilde{\beta}_i(\mathbf{u}, z) &\geq c\, I\,, \end{aligned}$$

we obtain Kreiss' maximal L^2 inequality

$$\gamma \|\!|\chi_i V|\!\|_0^2 + \|\chi_i V_{|x_d=0}\|_0^2 \lesssim \frac{1}{\gamma} \|\!|\chi_i \widehat{F}|\!\|_0^2 + \|\chi_i \widehat{G}\|_0^2 \lesssim \frac{1}{\gamma^3}\, \|\!|\chi_i \widehat{F}|\!\|_{1,\gamma}^2 + \frac{1}{\gamma^2}\, \|\chi_i \widehat{G}\|_{1,\gamma}^2\,.$$

If $\mathcal{V}_i$ is a neighbourhood of a point $\underline{z}$ where the critical subspace is nontrivial, we extend r_i as a homogeneous function of degree 2 in (η, γ). We have

$$r_i(z) + C\, \lambda^{2,\gamma}(\eta)\, \widetilde{\beta}_i(\mathbf{u}, z)^*\, \widetilde{\beta}_i(\mathbf{u}, z) \geq c\, \gamma^2\, I\,,$$

and therefore, multiplying (23) by $r_i(z)\, V_i(z, x_d)$ and integrating with respect to (η, x_d) yields the inequality

$$-2\; \text{Re}\; \langle\!\langle r_i\, V_i, \chi_i\, Q_i\, \mathcal{A}_d^{-1} \widehat{F} \rangle\!\rangle_{L^2(\Omega)} \geq c\, \gamma^2\, \|\chi_i V_{|x_d=0}\|_0^2 - C\, \|\chi_i \widehat{G}\|_{1,\gamma}^2 + 2\; \text{Re}\; \langle\!\langle V_i, r_i Q_i \mathcal{A} Q_i^{-1} V_i \rangle\!\rangle_{L^2(\Omega)}\,.$$

Recall that r_i has diagonal form

$$r_i(z) = \begin{pmatrix} -\gamma^2 H^- & & & \mathbf{0} \\ & -\gamma^2 I_{N-1} & & \\ & & \alpha\,\lambda^{2,\gamma}(\eta) H^+ & \\ \mathbf{0} & & & \alpha\,\lambda^{2,\gamma}(\eta)\, I_{N+1} \end{pmatrix}$$

$$\leq \begin{pmatrix} \gamma\sqrt{H^-} & & & \mathbf{0} \\ & \gamma I_{N-1} & & \\ & & \sqrt{\alpha}\,\lambda^{1,\gamma}(\eta)\sqrt{H^+} & \\ \mathbf{0} & & & \sqrt{\alpha}\,\lambda^{1,\gamma}(\eta)\, I_{N+1} \end{pmatrix}^2 =: s(z)^2 \,.$$

Recall that H^- and H^+ are hermitian poisitive definite which justifies the introduction of their positive definite square roots $\sqrt{H^-}$ and $\sqrt{H^+}$. We have

$$2\,\mathrm{Re}\,\langle\!\langle V_i, r_i Q_i \mathcal{A} Q_i^{-1} V_i \rangle\!\rangle_{L^2(\Omega)} \geq c\,\gamma\,\|\!| s(z)\,V_i \|\!|_0^2 \,,$$

and Schwarz' inequality yields

$$-2\ \mathrm{Re}\,\langle\!\langle r_i V_i, \chi_i Q_i \mathcal{A}_d^{-1} \widehat{F} \rangle\!\rangle_{L^2(\Omega)} \leq \frac{c}{2}\,\gamma\,\|\!| s(z) V_i \|\!|_0^2 + \frac{C}{\gamma}\,\|\!| \chi_i s(z) \widehat{F} \|\!|_0^2 \leq \frac{c}{2}\,\gamma\,\|\!| s(z) V_i \|\!|_0^2 + \frac{C}{\gamma}\,\|\!| \chi_i \widehat{F} \|\!|_{1,\gamma}^2 \,.$$

Eventually, we obtain

$$\gamma^3\,\|\!| \chi_i\, V \|\!|_0^2 + \gamma^2\,\| \chi_i V_{|_{x_d=0}} \|_0^2 \lesssim \frac{1}{\gamma}\,\|\!| \chi_i\, \widehat{F} \|\!|_{1,\gamma}^2 + \| \chi_i\, \widehat{G} \|_{1,\gamma}^2 \,.$$

Since the χ_i's form a partition of unity, we get

$$\gamma\,\|\!| v \|\!|_0^2 + \| v_{|_{x_d=0}} \|_0^2 \lesssim \frac{1}{\gamma^3}\,\|\!| \mathcal{L}_{\mathbf{u}}^\gamma v \|\!|_{1,\gamma}^2 + \frac{1}{\gamma^2}\,\| \mathcal{B}_{\mathbf{u}}^\gamma(v,\psi) \|_{1,\gamma}^2 \,.$$

To conclude the proof, we integrate (16) with respect to $\eta \in \mathbb{R}^d$:

$$\|\psi\|_{1,\gamma}^2 \lesssim \| v_{|_{x_d=0}} \|_0^2 + \| \mathcal{B}_{\mathbf{u}}^\gamma(v,\psi) \|_0^2 \lesssim \| v_{|_{x_d=0}} \|_0^2 + \frac{1}{\gamma^2}\,\| \mathcal{B}_{\mathbf{u}}^\gamma(v,\psi) \|_{1,\gamma}^2 \,,$$

and this gives (20).

□

Remark: in [25, proposition 2], Majda stated an energy estimate similar to (20) for isentropic Euler equations. We point out that his result was obtained under the assumption $\mathcal{L}_{\mathbf{u}}^\gamma v \equiv 0$. Theorem 2.1 thus extends this earlier result and indicates that losses of derivatives occur both in the interior domain and on the boundary. This shows a major difference between our analysis and earlier works such as [31, 33].

3 The variable coefficients analysis

Let $\mathbf{u}$ be a planar shock as in (6) satisfying assumption 3. We know that there exists an open set $\mathcal{U}$ in $\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^{d-1}$ containing the origin such that for all $(w_r, w_l, \sigma, \nu) \in \mathcal{U}$, we have

$$\mathbf{u}_r + w_r \in U \,, \quad \mathbf{u}_l + w_l \in U \,,$$
$$\lambda_{p-1}(\mathbf{u}_l + w_l, \xi) < \sigma < \lambda_p(\mathbf{u}_l + w_l, \xi) \quad \text{and} \quad \lambda_p(\mathbf{u}_r + w_r, \xi) < \sigma < \lambda_{p+1}(\mathbf{u}_r + w_r, \xi)$$

where $\xi := (-\nu, 1) \in \mathbb{R}^d$. In other words, $\mathcal{U}$ is an open set such that all planar shocks associated to elements of $\mathcal{U}$ are p-shocks. Shrinking $\mathcal{U}$ if necessary, all shock waves in $\mathcal{U}$ meet assumption 5. We fix a compact subset $\mathcal{K} \subset \mathcal{U}$ and consider mappings $\dot{u}_r, \dot{u}_l, \varphi$ such that φ is defined on $\mathbb{R}^d$, $\dot{u}_r$ (resp. $\dot{u}_l$) is defined on $\{x_d \geq \varphi(x_0, \dots, x_{d-1})\}$ (resp. $\{x_d \leq \varphi(x_0, \dots, x_{d-1})\}$). Eventually, we assume that $(\dot{u}_r, \dot{u}_l, \nabla\varphi)$ takes its values in $\mathcal{K}$ and is compactly supported. We define a function

$$\mathbf{a} := \begin{cases} \mathbf{u}_r + \dot{u}_r(\mathbf{x}) & \text{if } x_d > \varphi(x_0, \dots, x_{d-1}), \\ \mathbf{u}_l + \dot{u}_l(\mathbf{x}) & \text{if } x_d < \varphi(x_0, \dots, x_{d-1}), \end{cases} \tag{24}$$

and make the following assumption:

Assumption 6. *For all point $\mathbf{x} = (x_0, \dots, x_d)$ such that $x_d = \varphi(x_0, \dots, x_{d-1})$, the function*

$$\mathbf{a_x} := \begin{cases} \mathbf{u}_r + \dot{u}_r(\mathbf{x}) & \textit{if } y_d > \nabla\varphi \cdot (y_0, \dots, y_{d-1}), \\ \mathbf{u}_l + \dot{u}_l(\mathbf{x}) & \textit{if } y_d < \nabla\varphi \cdot (y_0, \dots, y_{d-1}), \end{cases}$$

is a planar shock wave, that is satisfies the Rankine-Hugoniot jump conditions. In the above definition of $\mathbf{a_x}$, the gradient $\nabla\varphi$ is evaluated at $(x_0, \dots, x_{d-1})$.

The regularity of $\dot{u}_r, \dot{u}_l$ and φ has not been precised. One can think of them as mild perturbations of the stationary shock wave $\mathbf{u}$ (some kind of first order correction in an asymptotic expansion). We shall be more precise in the sequel.

3.1 The linearized equations

As in paragraph 2.1, we introduce the linearized operators around $\mathbf{a}$: consider a family $u_s = \mathbf{a}_\sharp + sv$ and $\varphi_s = \varphi + s\psi$. Then we define

$$\mathcal{L}_\mathbf{a}(v^\pm, \psi) := \frac{d}{ds} L(u_s^\pm, \varphi_s)\, u_s^\pm \,|_{s=0}\,, \tag{25a}$$

$$\mathcal{B}_\mathbf{a}(v, \psi) := \frac{d}{ds} B(u_s^+, u_s^-, \varphi_s) \,|_{s=0}\,. \tag{25b}$$

Recall that $\mathbf{a}_\sharp$ is the function deduced from $\mathbf{a}$ after the change of variables Φ, see section 1. A direct computation shows that

$$\begin{aligned} \mathcal{L}_\mathbf{a}(v^\pm, \psi) = &\sum_{j=0}^{d-1} A_j(\mathbf{a}_\sharp^\pm)\, \partial_j\, v^\pm + \sum_{j=0}^{d-1} [\nabla A_j(\mathbf{a}_\sharp^\pm) \cdot v^\pm]\, \partial_j\, \mathbf{a}_\sharp^\pm + \widetilde{A_d}(\mathbf{a}_\sharp^\pm, \nabla\varphi)\, \partial_d\, v^\pm \\ &- \sum_{j=0}^{d-1} \partial_j \psi\, A_j(\mathbf{a}_\sharp^\pm)\, \partial_d\, \mathbf{a}_\sharp^\pm + [\nabla_u \widetilde{A_d}(\mathbf{a}_\sharp^\pm, \nabla\varphi) \cdot v^\pm]\, \partial_d\, \mathbf{a}_\sharp^\pm\,, \qquad \pm x_d > 0\,, \end{aligned}$$

and

$$\mathcal{B}_\mathbf{a}(v, \psi) = \sum_{j=0}^{d-1} \partial_j \psi\, [f_j(\mathbf{a}_\sharp)] - \widetilde{A_d}(\mathbf{a}_\sharp^+, \nabla\varphi)\, v^+ + \widetilde{A_d}(\mathbf{a}_\sharp^-, \nabla\varphi)\, v^-\,, \qquad x_d = 0\,.$$

We decompose the linearized operator $\mathcal{L}_\mathbf{a}$ as

$$\mathcal{L}_\mathbf{a}(v^\pm, \psi) = L(\mathbf{a}_\sharp^\pm, \varphi)\, v^\pm - \sum_{j=0}^{d-1} \partial_j \psi\, A_j(\mathbf{a}_\sharp^\pm)\, \partial_d\, \mathbf{a}_\sharp^\pm - \mathcal{C}(\mathbf{a}_\sharp^\pm, \varphi)\, v^\pm$$

where L is defined by (5a)-(5b) and

$$\mathcal{C}(\mathbf{a}_\sharp^\pm, \varphi)\, v^\pm := -\sum_{j=0}^{d-1} [\nabla A_j(\mathbf{a}_\sharp^\pm) \cdot v^\pm]\, \partial_j \mathbf{a}_\sharp^\pm - [\nabla_u \widetilde{A_d}(\mathbf{a}_\sharp^\pm, \nabla\varphi) \cdot v^\pm]\, \partial_d \mathbf{a}_\sharp^\pm \tag{26}$$

is the zero order part (in $v^\pm$) of $\mathcal{L}_\mathbf{a}$.

As was done in section 2, we write the linearized equations as a first order system in $v := (v^+, v^-)$ and ψ in the domain $x_d > 0$. Define

$$\mathcal{A}_j(\mathbf{a}) := \begin{pmatrix} A_j(\mathbf{a}_\sharp^+) & \mathbf{0} \\ \mathbf{0} & A_j(\mathbf{a}_\sharp^-) \end{pmatrix} \text{ for } 0 \le j \le d-1, \quad \mathcal{A}_d(\mathbf{a}) := \begin{pmatrix} \widetilde{A_d}(\mathbf{a}_\sharp^+, \nabla\varphi) & \mathbf{0} \\ \mathbf{0} & -\widetilde{A_d}(\mathbf{a}_\sharp^-, \nabla\varphi) \end{pmatrix},$$

$$b_j(\mathbf{a}) := [f_j(\mathbf{a}_\sharp)] \text{ for } 0 \le j \le d-1 \quad \text{and} \quad M(\mathbf{a}) := \begin{pmatrix} -\widetilde{A_d}(\mathbf{a}_\sharp^+, \nabla\varphi) & \widetilde{A_d}(\mathbf{a}_\sharp^-, \nabla\varphi) \end{pmatrix}.$$

The linearized operators read

$$\mathcal{L}_\mathbf{a}(v, \psi) = \sum_{j=0}^{d} \mathcal{A}_j(\mathbf{a})\, \partial_j v - \sum_{j=0}^{d-1} \partial_j \psi\, \mathcal{A}_j(\mathbf{a})\, \partial_d \mathbf{a}_\sharp - \mathcal{C}(\mathbf{a})\, v \quad x_d > 0\,,$$

$$\mathcal{B}_\mathbf{a}(v, \psi) = \sum_{j=0}^{d-1} \partial_j \psi\, b_j(\mathbf{a}) + M(\mathbf{a})\, v \quad x_d = 0\,.$$

Recall that the perturbation $(\dot{u}_r, \dot{u}_l, \nabla\varphi)$ takes its values in the compact set $\mathcal{K} \subset \mathcal{U}$ so the matrix $\mathcal{A}_d(\mathbf{a})$ is regular.

We now introduce the positive weight γ, that is, we change functions v and ψ and deal with $\widetilde{v} := \exp(-\gamma t) v$ and $\widetilde{\psi} := \exp(-\gamma t)\psi$. As in section 2, we introduce the weighted operators

$$\mathcal{L}_\mathbf{a}^\gamma(\widetilde{v}, \widetilde{\psi}) := \mathcal{L}_\mathbf{a}(\widetilde{v}, \widetilde{\psi}) + \gamma \mathcal{A}_0(\mathbf{a})\, \widetilde{v} - \gamma\, \widetilde{\psi}\, \mathcal{A}_0(\mathbf{a})\, \partial_d \mathbf{a}_\sharp\,,$$

$$\text{and} \quad \mathcal{B}_\mathbf{a}^\gamma(\widetilde{v}, \widetilde{\psi}) := \mathcal{B}_\mathbf{a}(\widetilde{v}, \widetilde{\psi}) + \gamma\, \widetilde{\psi}\, b_0(\mathbf{a})\,.$$

For simplicity, we drop the tildas and the $\sharp$ index. We fix an integer $m > d/2 + 3$ and assume that $\varphi \in H^m(\mathbb{R}^d)$ and $\mathbf{a} \in H^m(\Omega)$. Using some classical properties of Sobolev spaves, see [1], we have $\nabla\varphi \in W^{2,\infty}(\mathbb{R}^d)$, $\mathbf{a} \in W^{2,\infty}(\Omega)$ and $\mathbf{a} \in L^2(\mathbb{R}^+, W^{2,\infty}(\mathbb{R}^d))$.

Because the coefficients of the linearized operators have limited smoothness, a convenient way to derive an energy estimate analogous to (20) is to use the paradifferential calculus of Bony, see [11, 28, 29]. With this strategy in mind, we are going to estimate the error between the linearized operators and their paralinearized version.

3.2 The paralinearized equations

We refer to appendix A for the definition of paradifferential symbols and for the main results of paradifferential calculus.

1) Paralinearization of the boundary conditions

Define the following symbols:

$$\mathbf{b}(x_0, \dots, x_{d-1}, \eta, \gamma) := \gamma\, \mathbf{b}_0(x_0, \dots, x_{d-1}) + i \sum_{j=0}^{d-1} \eta_j\, \mathbf{b}_j(x_0, \dots, x_{d-1})$$

$$\text{where } \mathbf{b}_j(x_0, \dots, x_{d-1}) := b_j(\mathbf{a}(x_0, \dots, x_{d-1}, 0))\,.$$

Then we have $\mathbf{b}_j \in W^{2,\infty}(\mathbb{R}^d)$ and as a consequence $\mathbf{b} \in \Gamma^1_2$. Theorem A.5 yields

$$\begin{aligned}
\|\gamma \mathbf{b}_0 \, \psi - T^\gamma_{\gamma \mathbf{b}_0} \psi\|_{1,\gamma} &\lesssim \|\mathbf{b}_0\|_{W^{1,\infty}} \, \gamma \, \|\psi\|_0 \lesssim \|\psi\|_{1,\gamma} \, , \\
\|\mathbf{b}_j \, \partial_j \, \psi - T^\gamma_{i\eta_j \mathbf{b}_j} \psi\|_{1,\gamma} = \|(\mathbf{b}_j - T^\gamma_{\mathbf{b}_j})(\partial_j \, \psi)\|_{1,\gamma} &\lesssim \|\partial_j \, \psi\|_0 \lesssim \|\psi\|_{1,\gamma} \, ,
\end{aligned}$$

and we thus obtain

$$\|\gamma \mathbf{b}_0 \, \psi + \sum_{j=0}^{d-1} \partial_j \psi \, \mathbf{b_j} - T^\gamma_{\mathbf{b}} \psi\|_{1,\gamma} \lesssim \|\psi\|_{1,\gamma} \, . \tag{27}$$

We also define

$$\mathbf{M}(x_0, \dots, x_{d-1}) := M(\mathbf{a}(x_0, \dots, x_{d-1}, 0)) \in W^{2,\infty}(\mathbb{R}^d) \, ,$$

and theorem A.5 yields

$$\|\mathbf{M} \, v_{|_{x_d=0}} - T^\gamma_{\mathbf{M}} v_{|_{x_d=0}}\|_{1,\gamma} \lesssim \|v_{|_{x_d=0}}\|_0 \, . \tag{28}$$

Combining (27) and (28), we get

$$\|\mathcal{B}^\gamma_{\mathbf{a}}(v, \psi) - T^\gamma_{\mathbf{b}} \psi - T^\gamma_{\mathbf{M}} v_{|_{x_d=0}}\|_{1,\gamma} \lesssim \|\psi\|_{1,\gamma} + \|v_{|_{x_d=0}}\|_0 \, . \tag{29}$$

We shall therefore replace the linearized boundary operator $(v, \psi) \mapsto \mathcal{B}^\gamma_{\mathbf{a}}(v, \psi)$ by its paralinearized version $(v, \psi) \mapsto T^\gamma_{\mathbf{b}} \psi + T^\gamma_{\mathbf{M}} v$.

2) Paralinearization of the evolution equations

We are now going to paralinearize the evolution equations after multiplying by $\mathcal{A}_d^{-1}$. Define

$$\mathbf{A}_j(x_0, \dots, x_d) := \mathcal{A}_j(\mathbf{a}(x_0, \dots, x_d)) \, , \quad \mathbf{C}(x_0, \dots, x_d) := \mathcal{C}(\mathbf{a}(x_0, \dots, x_d)) \, .$$

Because $\nabla \varphi \in W^{2,\infty}(\mathbb{R}^d)$ and $\mathbf{a} \in W^{2,\infty}(\Omega)$, we have $\mathbf{A}_j \in W^{2,\infty}(\Omega)$ and $\mathbf{C} \in W^{1,\infty}(\Omega)$. Recall that first order derivatives of $\mathbf{a}$ appear in the definition of $\mathcal{C}$, see (26), so we do not have $\mathbf{C} \in W^{2,\infty}(\Omega)$.

The matrix valued mapping $\mathbf{A}_d$ is uniformly invertible, namely

$$\|\mathbf{A}_d^{-1}\|_{W^{2,\infty}(\Omega)} \le C \, .$$

Using the definition (19) and theorem A.5, we obtain the following estimates

$$\begin{aligned}
\|\gamma \mathbf{A}_d^{-1} \mathbf{A}_0 \, v - \gamma T^\gamma_{\mathbf{A}_d^{-1} \mathbf{A}_0} v\|_{1,\gamma} &\lesssim \|v\|_0 \, , \\
\|\mathbf{A}_d^{-1} \mathbf{A}_j \, \partial_j \, v - T^\gamma_{i\eta_j \mathbf{A}_d^{-1} \mathbf{A}_j} v\|_{1,\gamma} &\lesssim \|v\|_0 \qquad \text{for } 0 \le j \le d-1 \, , \\
\|\mathbf{A}_d^{-1} \mathbf{C} \, v - T^\gamma_{\mathbf{A}_d^{-1} \mathbf{C}} v\|_{1,\gamma} &\lesssim \|v\|_0 \, .
\end{aligned}$$

Those estimates are obtained by a simple integration of the paraproduct estimates in $\mathbb{R}^d$ and from the definition of the paradifferential operators in a half-space, see appendix A.

As in (12), for $\mathbf{x} \in \Omega$, $\eta \in \mathbb{R}^d$ and $\gamma \ge 0$, we define the symbol

$$\mathbf{A}(\mathbf{x}, \eta, \gamma) := -\mathbf{A}_d(\mathbf{x})^{-1} \left(\gamma \, \mathbf{A}_0(\mathbf{x}) + i \sum_{j=0}^{d-1} \eta_j \, \mathbf{A}_j(\mathbf{x}) \right) \, .$$

It is clear that $\mathbf{A} \in \Gamma_2^1$ and the previous inequalities yield

$$\left\|\left\|\gamma \mathbf{A}_d^{-1} \mathbf{A}_0 \, v + \sum_{j=0}^{d-1} \mathbf{A}_d^{-1} \mathbf{A}_j \, \partial_j \, v - \mathbf{A}_d^{-1} \mathbf{C} \, v + T_{\mathbf{A}}^\gamma v + T_{\mathbf{A}_d^{-1}\mathbf{C}}^\gamma v\right\|\right\|_{1,\gamma} \lesssim \|\!|v|\!\|_0 \,. \tag{30}$$

We have thus estimated the error terms in v when paralinearizing $\mathcal{L}_a^\gamma(v, \psi)$. We now turn to the error terms in ψ. There are two such terms that are

$$\begin{aligned} &\gamma \psi \mathbf{A}_d^{-1} \mathbf{A}_0 \, \partial_d \, \mathbf{a} - \gamma \, T_{\mathbf{A}_d^{-1}\mathbf{A}_0 \partial_d \mathbf{a}}^\gamma \psi \\ \text{and} \quad &\mathbf{A}_d^{-1} \mathbf{A}_j \, \partial_d \, \mathbf{a} \, \partial_j \, \psi - T_{\mathbf{A}_d^{-1}\mathbf{A}_0 \partial_d \mathbf{a}}^\gamma \partial_j \, \psi \,, \quad 0 \le j \le d-1 \,. \end{aligned}$$

Using theorem A.5 and the property $\mathbf{a} \in L^2(\mathbb{R}^+, W^{2,\infty}(\mathbb{R}^d))$, we obtain

$$\begin{aligned} \|\!|\gamma \, \psi \, \mathbf{A}_d^{-1} \mathbf{A}_0 \, \partial_d \, \mathbf{a} - \gamma \, T_{\mathbf{A}_d^{-1}\mathbf{A}_0 \partial_d \mathbf{a}}^\gamma \psi|\!\|_{1,\gamma} &\lesssim \gamma \, \|\psi\|_0 \lesssim \|\psi\|_{1,\gamma} \,, \\ \|\!|\mathbf{A}_d^{-1} \mathbf{A}_j \partial_d \, \mathbf{a} \, \partial_j \, \psi - T_{\mathbf{A}_d^{-1}\mathbf{A}_j \partial_d \mathbf{a}}^\gamma \partial_j \, \psi|\!\|_{1,\gamma} &\lesssim \|\partial_j \, \psi\|_0 \lesssim \|\psi\|_{1,\gamma} \,. \end{aligned}$$

We thus get the estimate

$$\left\|\left\|\gamma \psi \mathbf{A}_d^{-1} \mathbf{A}_0 \, \partial_d \, \mathbf{a} + \sum_{j=0}^{d-1} \mathbf{A}_d^{-1} \mathbf{A}_j \, \partial_d \, \mathbf{a} \, \partial_j \, \psi + T_{\mathbf{A}\partial_d \mathbf{a}}^\gamma \psi\right\|\right\|_{1,\gamma} \lesssim \|\psi\|_{1,\gamma} \,. \tag{31}$$

Combining (30) and (31), we have proved

$$\|\!|\mathbf{A}_d^{-1} \mathcal{L}_{\mathbf{a}}^\gamma(v, \psi) - \partial_d \, v + T_{\mathbf{A}}^\gamma v + T_{\mathbf{A}_d^{-1}\mathbf{C}}^\gamma v - T_{\mathbf{A}\partial_d \mathbf{a}}^\gamma \psi|\!\|_{1,\gamma} \lesssim \|\!|v|\!\|_0 + \|\psi\|_{1,\gamma} \,. \tag{32}$$

As for the boundary conditions, we shall therefore replace the linearized operator $(v, \psi) \mapsto \mathbf{A}_d^{-1} \mathcal{L}_{\mathbf{a}}^\gamma(v, \psi)$ by its paralinearized version $(v, \psi) \mapsto \partial_d \, v - T_{\mathbf{A}}^\gamma v - T_{\mathbf{A}_d^{-1}\mathbf{C}}^\gamma v + T_{\mathbf{A}\partial_d \mathbf{a}}^\gamma \psi$.

3) Change of unknown function

Unlike in the uniformly stable case, the linearized operator involves in our case a zero order operator in v and a first order operator in ψ. We shall use a change of unknown functions, that is due to Alinhac [3] and that simplifies the expression of $\mathcal{L}_{\mathbf{a}}$. If we let $v = \dot{v} + \psi \, \partial_d \mathbf{a}$, we have

$$\mathcal{L}_{\mathbf{a}}(v, \psi) = L(\mathbf{a}, \varphi) \, \dot{v} - \mathcal{C}(\mathbf{a}, \varphi) \, \dot{v} + \psi \, \partial_d \left[L(\mathbf{a}, \varphi) \mathbf{a} \right] \,.$$

The function $\dot{v}$ is referred to as the "good unknown" of the problem. We emphasize that this change of unknown functions yields an operator in $(\dot{v}, \psi)$ with only zero order term in ψ: roughly speaking, the operator $\mathcal{L}_{\mathbf{a}}$ reduces to an operator with first and zero order term in $\dot{v}$ only since the zero order terms in ψ will be easily estimated.

As regards the paralinearized equations, the previous result suggests to make the change of unknown functions $v := \dot{v} + T_{\partial_d \mathbf{a}}^\gamma \psi$. We are going to show that the paralinearized operator is equal to $\partial_d \, \dot{v} - T_{\mathbf{A}}^\gamma \dot{v} - T_{\mathbf{A}_d^{-1}\mathbf{C}}^\gamma \dot{v}$ plus some error terms whose norm can be controlled.

A straightforward computation shows that

$$\partial_d \, v - T_{\mathbf{A}}^\gamma v - T_{\mathbf{A}_d^{-1}\mathbf{C}}^\gamma v + T_{\mathbf{A}\partial_d \mathbf{a}}^\gamma \psi = \partial_d \, \dot{v} - T_{\mathbf{A}}^\gamma \dot{v} - T_{\mathbf{A}_d^{-1}\mathbf{C}}^\gamma \dot{v} + e_1 - e_2 + e_3$$

where

$$e_1 := T_{\partial_d^2 \mathbf{a}}^\gamma \psi \,, \quad e_2 := T_{\mathbf{A}_d^{-1}\mathbf{C}}^\gamma T_{\partial_d \mathbf{a}}^\gamma \psi \,, \quad e_3 := (T_{\mathbf{A}\partial_d \mathbf{a}}^\gamma - T_{\mathbf{A}}^\gamma T_{\partial_d \mathbf{a}}^\gamma) \psi \,.$$

Because $\mathbf{a} \in H^m(\Omega)$ and $m > d/2+3$, we have

$$\|\!|e_i\|\!|_{1,\gamma} \lesssim \|\psi\|_{1,\gamma} \quad 1 \leq i \leq 3\,,$$

and therefore

$$\left\|\!\left|\left(\partial_d\, v - T^\gamma_{\mathbf{A}} v - T^\gamma_{\mathbf{A}_d^{-1}\mathbf{C}} v + T^\gamma_{\mathbf{A}\partial_d \mathbf{a}} \psi\right) - \left(\partial_d\, \dot{v} - T^\gamma_{\mathbf{A}} \dot{v} - T^\gamma_{\mathbf{A}_d^{-1}\mathbf{C}} \dot{v}\right)\right|\!\right\|_{1,\gamma} \lesssim \|\psi\|_{1,\gamma}\,.$$

As a consequence, we shall focus our attention on the operator $\partial_d\, \dot{v} - T^\gamma_{\mathbf{A}} \dot{v} - T^\gamma_{\mathbf{A}_d^{-1}\mathbf{C}} \dot{v}$ and try to derive an energy estimate for this operator.

Remark: after changing unknown functions, the paralinearized boundary operator reads $T^\gamma_{\mathbf{b}} \psi + T^\gamma_{\mathbf{M}} \dot{v}_{|x_d=0} + T^\gamma_{\mathbf{M}} T^\gamma_{\partial_d \mathbf{a}} \psi$ and we see that the last term satisifies

$$\|T^\gamma_{\mathbf{M}} T^\gamma_{\partial_d \mathbf{a}} \psi\|_{1,\gamma} \lesssim \|\psi\|_{1,\gamma}\,.$$

To summarize, we have proved

$$\begin{aligned} \left\|\!\left|\mathbf{A}_d^{-1} \mathcal{L}^\gamma_{\mathbf{a}}(v,\psi) - \left(\partial_d\, \dot{v} - T^\gamma_{\mathbf{A}} \dot{v} - T^\gamma_{\mathbf{A}_d^{-1}\mathbf{C}} \dot{v}\right)\right|\!\right\|_{1,\gamma} &\lesssim \|\!|v\|\!|_0 + \|\psi\|_{1,\gamma}\,, \\ \left\|\mathcal{B}^\gamma_{\mathbf{a}}(v,\psi) - \left(T^\gamma_{\mathbf{b}} \psi + T^\gamma_{\mathbf{M}} \dot{v}_{|x_d=0}\right)\right\|_{1,\gamma} &\lesssim \|v_{|x_d=0}\|_0 + \|\psi\|_{1,\gamma}\,. \end{aligned} \tag{33}$$

Furthermore, the definition of $\dot{v}$ gives

$$\begin{aligned} \|v_{|x_d=0}\|_0^2 &\leq 2\left(\|\dot{v}_{|x_d=0}\|_0^2 + C\|\psi\|_0^2\right) \leq 2\left(\|\dot{v}_{|x_d=0}\|_0^2 + \frac{C}{\gamma^2}\,\|\psi\|_{1,\gamma}^2\right)\,, \\ \|\!|v\|\!|_0^2 &\leq 2\left(\|\!|\dot{v}\|\!|_0^2 + C\|\psi\|_0^2\right) \leq 2\left(\|\!|\dot{v}\|\!|_0^2 + \frac{C}{\gamma^2}\,\|\psi\|_{1,\gamma}^2\right)\,, \end{aligned}$$

and we thus get

$$\gamma\|\!|\dot{v}\|\!|_0^2 + \|\dot{v}_{|x_d=0}\|_0^2 \geq \frac{1}{2}\left(\gamma\|\!|v\|\!|_0^2 + \|v_{|x_d=0}\|_0^2\right) - \frac{C}{\gamma}\|\psi\|_{1,\gamma}^2\,. \tag{34}$$

It is now clear that the change of unknown function is appropriate because an energy estimate of the same type as (20) for $(\dot{v},\psi)$ will yield a similar energy estimate for (v,ψ).

4) Eliminating the shock front

Using assumption 4, we know that there exists a positive constant $c > 0$ such that

$$\mathbf{b}^*(x_0,\dots,x_{d-1},\eta,\gamma)\,\mathbf{b}(x_0,\dots,x_{d-1},\eta,\gamma) \geq c\,(\gamma^2 + |\eta|^2)\,.$$

The constant c only depends on the compact set $\mathcal{K}$. The symbol $\mathbf{b}^*\mathbf{b}$ is of degree 2 and elliptic so applying Gårding's inequality (theorem A.3), we obtain

$$\forall\,\gamma \geq \gamma_0\,,\quad \|\psi\|_{1,\gamma}^2 \lesssim \text{ Re } \langle T^\gamma_{\mathbf{b}^*\mathbf{b}} \psi, \psi\rangle_{L^2} \lesssim \|T^\gamma_{\mathbf{b}} \psi\|_0^2 + \text{ Re } \langle R^\gamma \psi, \psi\rangle_{L^2}$$

where $\{R^\gamma\}$ is a family of order ≤ 1. We thus get

$$\|\psi\|_{1,\gamma}^2 \lesssim \|T^\gamma_{\mathbf{b}} \psi\|_0^2 + \|\psi\|_{1,\gamma}\,\|\psi\|_0 \lesssim \frac{1}{\gamma}\,\|\psi\|_{1,\gamma}^2 + \|T^\gamma_{\mathbf{b}} \psi\|_0^2\,.$$

Up to a greater choice of γ_0, we finally get

$$\begin{aligned}\forall\,\gamma \geq \gamma_0\,, \quad \|\psi\|^2_{1,\gamma} \lesssim \|T^\gamma_{\mathbf{b}}\psi\|^2_0 &\lesssim \|T^\gamma_{\mathbf{b}}\psi + T^\gamma_{\mathbf{M}}\dot{v}_{|x_d=0}\|^2_0 + \|\dot{v}_{|x_d=0}\|^2_0 \\ &\lesssim \frac{1}{\gamma^2}\,\|T^\gamma_{\mathbf{b}}\psi + T^\gamma_{\mathbf{M}}\dot{v}_{|x_d=0}\|^2_{1,\gamma} + \|\dot{v}_{|x_d=0}\|^2_0\,,\end{aligned} \tag{35}$$

which is the variable coefficients version of (16).

Introducing the orthogonal projector $\mathbf{\Pi}(y,\eta,\gamma)$ on $\mathbf{b}(y,\eta,\gamma)^\perp$, we have $\mathbf{\Pi} \in \Gamma^0_2$, $\mathbf{\Pi b} \equiv 0$ so theorem A.2 gives

$$\|T^\gamma_{\mathbf{\Pi}}T^\gamma_{\mathbf{b}}\psi\|_{1,\gamma} \lesssim \|\psi\|_{1,\gamma}\,, \quad \|T^\gamma_{\mathbf{\Pi}}T^\gamma_{\mathbf{M}}\dot{v}_{|x_d=0} - T^\gamma_{\mathbf{\Pi M}}\dot{v}_{|x_d=0}\|_{1,\gamma} \lesssim \|\dot{v}_{|x_d=0}\|_0\,.$$

Using the decomposition

$$T^\gamma_{\mathbf{\Pi M}}\dot{v}_{|x_d=0} = (T^\gamma_{\mathbf{\Pi M}} - T^\gamma_{\mathbf{\Pi}}T^\gamma_{\mathbf{M}})\dot{v}_{|x_d=0} + T^\gamma_{\mathbf{\Pi}}(T^\gamma_{\mathbf{M}}\dot{v}_{|x_d=0} + T^\gamma_{\mathbf{b}}\psi) - T^\gamma_{\mathbf{\Pi}}T^\gamma_{\mathbf{b}}\psi\,,$$

we end up with

$$\|T^\gamma_{\mathbf{\Pi M}}\dot{v}_{|x_d=0}\|_{1,\gamma} \lesssim \|\dot{v}_{|x_d=0}\|_0 + \|T^\gamma_{\mathbf{b}}\psi + T^\gamma_{\mathbf{M}}\dot{v}_{|x_d=0}\|_{1,\gamma} + \|\psi\|_{1,\gamma}\,. \tag{36}$$

From now on, we focus on the system

$$\begin{cases} \partial_d\,\dot{v} - T^\gamma_{\mathbf{A}}\dot{v} - T^\gamma_{\mathbf{A}_d^{-1}\mathbf{C}}\dot{v} = F & x_d > 0, \\ T^\gamma_{\mathbf{\Pi M}}\dot{v} = G & x_d = 0, \end{cases}$$

and try to derive an energy estimate of the type

$$\gamma\,|\!|\!|\dot{v}|\!|\!|^2_0 + \|\dot{v}_{|x_d=0}\|^2_0 \lesssim \frac{1}{\gamma^3}\,|\!|\!|F|\!|\!|^2_{1,\gamma} + \frac{1}{\gamma^2}\,\|G\|^2_{1,\gamma}\,,$$

for all $\gamma \geq \gamma_0$. Using (33),(34), (35),(36) and a greater choice of the constant γ_0, we shall obtain the variable coefficients analogue of our basic estimate (20). A precise result will be stated in paragraph 3.6.

3.3 Geometrical assumptions

Let us go back to assumption 5. For all shock waves u close to $\mathbf{u}$, we define the set of critical frequencies $\Sigma_{cr}(u)$ as the set of those $(\eta,\gamma) \in \Sigma_+$ such that the critical subspace

$$\big\{Z \in \mathcal{E}^-(u,\eta,\gamma) \text{ s.t. } \Pi(u,\eta,\gamma)M(u)\,Z = 0\big\}$$

is not reduced to $\{0\}$. From assumption 5, we know that frequencies $(\eta,\gamma) \in \Sigma_{cr}(u)$ verify $\gamma = 0$. A reasonable assumption is that $\Sigma_{cr}(u)$ has a finite number, say K, of connected components where K is independent of u. In the sequel, we shall make as if there were only one connected component. Another requirement of assumption 5 is that there exists a neighbourhood $\mathcal{V}_{cr}(u)$ of $\Sigma_{cr}(u)$ in Σ_+ and a smooth mapping Q_0 defined on $\mathcal{V}_{cr}(u)$ with values in $Gl_{2N}(\mathbb{C})$ such that

$$Q_0\,\mathcal{A}\,Q_0^{-1} = \begin{pmatrix} \omega_1 I_{n_1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \omega_J I_{n_J} \end{pmatrix}.$$

To simplify the subsequent calculations, we shall assume that the whole symbol $\mathcal{A}$ is diagonalizable and not only its central part. However, the proof would yield the same result if one took into account the blocks a^- and a^+ that may appear when reducing $\mathcal{A}$.

To deal with the variable coefficients case, we need to assume that $\Sigma_{cr}(u)$ is endowed with an "equation". More precisely, we assume that there exists a smooth **real** valued function $\widetilde{\sigma}$ (defined for all shock waves u close to $\mathbf{u}$) such that

$$\begin{aligned}\Sigma_{cr}(u) &= \{(\eta,\gamma)\in\Sigma_+ \text{ s.t. } \gamma+i\widetilde{\sigma}(u,\eta,\gamma)=0\}\\ &= \{(\eta,0)\in\Sigma_0 \text{ s.t. } \widetilde{\sigma}(u,\eta,0)=0\}\,.\end{aligned}$$

We refer to section 4 for an example. The function $\widetilde{\sigma}$ is extended as an homogeneous mapping of degree 1 with respect to (η,γ).

Given the function $\mathbf{a}$ defined by (24), we define the set of the so-called "critical points" as

$$\Sigma_{\mathbf{c}} := \{(x,\eta,\gamma)\in\partial\Omega\times\Sigma_+ \text{ s.t. } (\eta,\gamma)\in\Sigma_{cr}(\mathbf{a}(x))\}\,. \tag{37}$$

We already know from assumption 5 that there exists an open set $\mathcal{V}_{\mathbf{c}}\subset\overline{\Omega}\times\Sigma_+$ of the form

$$\mathcal{V}_{\mathbf{c}} = \bigcup_{x\in\overline{\Omega}}\{x\}\times\mathcal{V}_x\,,\quad \mathcal{V}_x\subset\Sigma_+\,,$$

and a mapping $Q_0:\mathcal{V}_{\mathbf{c}}\to Gl_{2N}(\mathbb{C})$ such that

$\Sigma_{\mathbf{c}}\subset\mathcal{V}_{\mathbf{c}}\cap\{x_d=0\}$,

Q_0 is a symbol of degree 0 and regularity 2 (because $\mathbf{A}$ is of regularity 2 and Q_0 has same regularitry as $\mathbf{A}$),

for all $z\in\mathcal{V}_{\mathbf{c}}$, one has

$$Q_0(z)\,\mathbf{A}(z)\,Q_0(z)^{-1} = \begin{pmatrix}\omega_1(z)I_{n_1} & & \mathbf{0}\\ & \ddots & \\ \mathbf{0} & & \omega_J(z)I_{n_J}\end{pmatrix} =: D_1(z)\,.$$

The set of space-frequency variables $\overline{\Omega}\times\Sigma_+$ is thus decomposed as the union of a set $\mathcal{V}_{\mathbf{c}}$ that contains all the unstable points and a set that contains only uniformly stable points. In the constant coefficients case, one can choose $\mathcal{V}_{\mathbf{c}}$ as $\overline{\Omega}\times\mathcal{V}_{cr}(\mathbf{u})$ (the critical frequencies are the same at any point in the space domain).

For $(x,\eta,\gamma)\in\partial\Omega\times\mathbb{R}^d\times\mathbb{R}^+$, define

$$\sigma(x,\eta,\gamma) := \widetilde{\sigma}(\mathbf{a}(x),\eta,\gamma)\,,$$

so we have $\sigma\in\Gamma^1_2(\mathbb{R}^d)$. In the subsequent analysis, we shall show that the instabilities originating from the critical points propagate in the interior domain along bicharacteristic curves. In order to control where these instabilities propagate, we are led to make the following important assumption:

Assumption 7. *Let ω_j be an eigenvalue of $\mathbf{A}$ that has negative real part when $\gamma>0$, and decompose ω_j as $\omega_j=\gamma e_j+ih_j$ (all mappings are defined on $\mathcal{V}_{\mathbf{c}}$). Then the solutions of the hamiltonian ODEs system*

$$\begin{aligned}\frac{dx_k}{dx_d} &= \frac{\partial h_j}{\partial\eta_k}(x,\eta,\gamma)\,,\quad k=0,\dots,d-1\,,\\ \frac{d\eta_k}{dx_d} &= -\frac{\partial h_j}{\partial x_k}(x,\eta,\gamma)\,,\quad k=0,\dots,d-1\,,\\ (x_0,\dots,x_{d-1},&\eta_0,\dots,\eta_{d-1},\gamma)_{|x_d=0}\in\mathcal{V}_{\mathbf{c}}\cap\{x_d=0\}\end{aligned} \tag{38}$$

are defined for all $x_d\geq 0$ (that is, stay in $\mathcal{V}_{\mathbf{c}}$ for all $x_d\geq 0$). These solutions are referred to as bicharacteristic curves.

Note that assumption 7 is met in the constant coefficients case because (38) then reduces to

$$
\begin{aligned}
\frac{dx_k}{dx_d} = \frac{\partial h_j}{\partial \eta_k}(x,\eta,\gamma)\,, \quad k=0,\dots,d-1\,,\\
\frac{d\eta_k}{dx_d} = 0\,, \quad k=0,\dots,d-1\,,\\
(x_0,\dots,x_{d-1},\eta_0,\dots,\eta_{d-1},\gamma)_{|x_d=0} \in \mathcal{V}_{\mathbf{c}} \cap \{x_d=0\}
\end{aligned}
$$

and $\mathcal{V}_{\mathbf{c}}$ is nothing but the cartesian product $\overline{\Omega} \times \mathcal{V}_{cr}(\mathbf{u})$ so the bicharacteristic curves stay in $\mathcal{V}_{\mathbf{c}}$. Furthermore, we recall that the perturbation $(\dot{u}_r, \dot{u}_l)$ has compact support. So if (x_k, η_k) is a solution of (38), η_k is constant for x_d large enough. From standard ODEs arguments, we claim that assumption 7 is satisfied when $\mathbf{a}$ is a sufficiently small perturbation of a planar shock wave. We refer to figure 2 for a schematic picture of the situation.

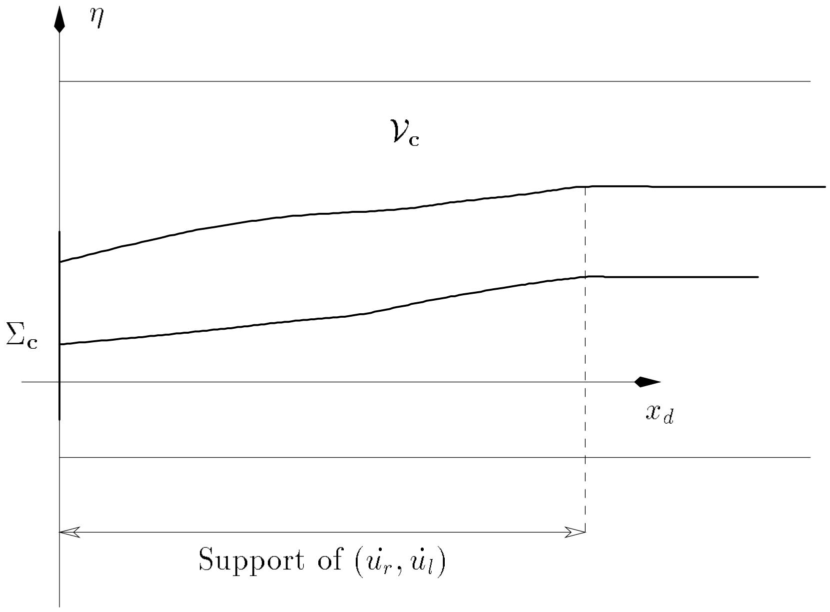

Figure 2: Bicharacteristic curves originating from the boundary

With the help of assumption 7, it is possible to construct a solution of the transport equation

$$
\begin{aligned}
\partial_{x_d}\sigma_j + \sum_{k=0}^{d-1} \partial_{x_k}\sigma_j\,\partial_{\eta_k}h_j - \partial_{\eta_k}\sigma_j\,\partial_{x_k}h_j = 0\,, \quad (x,\eta,\gamma)\in\mathcal{V}_{\mathbf{c}}\,,\\
\sigma_{j|x_d=0} = \sigma\,, \quad (x,\eta,\gamma)\in\mathcal{V}_{\mathbf{c}}\cap\{x_d=0\}\,,
\end{aligned} \tag{39}
$$

and the solution σ_j of this equation is homogeneous of degree 1 with respect to (η,γ). Note that (39) reads

$$
\partial_{x_d}\sigma_j + \{\sigma_j, h_j\} = 0\,,
$$

where $\{\sigma_j, h_j\}$ is the Poisson bracket of σ_j and h_j.

For convenience, we extend all mappings ω_j, $1\le j\le J$ (and thus D_1) as symbols of degree 1 and regularity 2 defined for all (x,η,γ). We therefore have $\omega_j \in \Gamma_2^1$. We choose these extensions such that one has either

$$
e_j \ge c > 0 \quad \text{or} \quad e_j \le -c < 0
$$

where $\omega_j = \gamma e_j + i h_j$. Extending the ω_j's allows to define a global solution σ_j of (39) that also belongs to Γ_2^1. What is important is that we have **not** changed the value of σ_j on the set $\mathcal{V}_\mathbf{c}$ since σ_j is constant along the bicharacteristic curves. The functions σ_j are weights that vanish only on the curves originating from the critical points. We shall see in the sequel that they are appropriate in the derivation of an energy estimate.

Finally, we need to precise the behavior of the restriction of the boundary symbol β to the stable subspace. Recall that the first $N-1$ column vectors of the matrix $Q_0(z)^{-1}$ span the stable subspace $\mathcal{E}^-(z)$. We write:

$$Q_0(z)^{-1} = \begin{pmatrix} Q_{in}(z) & Q_{out}(z) \end{pmatrix} , \quad Q_{in}(z) \in \mathcal{M}_{2N,N-1}(\mathbb{C}) ,$$

and make the following assumption:

Assumption 8. *There exist two mappings P_1 and P_2 defined on $\mathcal{V}_\mathbf{c} \cap \{x_d = 0\}$ such that*

for all $z \in \mathcal{V}_\mathbf{c} \cap \{x_d = 0\}$, $P_1(z) \in Gl_{2N}(\mathbb{C})$ and P_1 is a symbol of degree 0 and regularity 1,

for all $z \in \mathcal{V}_\mathbf{c} \cap \{x_d = 0\}$, $P_2(z) \in Gl_{N-1}(\mathbb{C})$ and P_2 is a symbol of degree 0 and regularity 1,

for all $z \in \mathcal{V}_\mathbf{c} \cap \{x_d = 0\}$, one has

$$P_1(z)\,\beta(z)\,Q_{in}(z)\,P_2(z) = \begin{pmatrix} \lambda^{-1,\gamma}(\eta)(\gamma + i\sigma(z)) & 0 \\ 0 & E(z) \end{pmatrix} =: \beta^-(z) \tag{40}$$

and the reactangular matrix $E(z)$ satisfies

$$\forall\, z \in \mathcal{V}_\mathbf{c} \cap \{x_d = 0\} , \quad E(z)^* E(z) \geq c\, I . \tag{41}$$

In the constant coefficients case, assumption 8 implies (18). The meaning of assumption 8 is that the restriction of β to the stable subspace has a kernel of dimension 1 (this was not part of assumption 5) and "β vanishes at order 1 on this kernel".

With these preliminary reductions in mind, we can turn to the derivation of our energy estimates. We fix a nonnegative cut-off function χ verifying

χ is a smooth (that is, C^∞) symbol of degree 0 and Supp $\chi \subset \mathcal{V}_\mathbf{c}$,

$\chi \equiv 1$ in a neighbourhood of the bicharacteristic curves starting from $\Sigma_\mathbf{c}$. In other words, out of the region where $\chi \equiv 1$, one has $|\sigma_j| \geq c > 0$ for all j.

Define $\chi_0 := (1 - \chi)$ and observe that χ_0 has its support in the set of uniformly stable points.

3.4 Energy estimates near instability points

The aim of this paragraph is to derive three different energy estimates in the neighbourhood of instability points. We show how to control the H^1 norm of the outgoing modes. For the incoming modes, we show how to control the L^2 norm and the H^1 norm far from the bicharacteristic curves starting from the critical set $\Sigma_\mathbf{c}$.

Before establishing our main energy estimates, we state a reduction result that will help us to deal with the zero order terms in the equations. In terms of symbolic calculus, we look for a symbol Q_{-1} of degree -1 and regularity 1 such that

$$(Q_0 + Q_{-1}) \# (\partial_d + \mathbf{A} + \mathbf{A}_d^{-1}\mathbf{C}) = (\partial_d + D_1 + D_0) \# (Q_0 + Q_{-1})$$

where the composition of symbols is to be understood as the expansion to first or second order, see theorem A.2. Recall that our symbols have finite smoothness in the space variable so the expansions of adjoints or composed symbols are only finite and not asymptotic. The existence of Q_{-1} is given by the following lemma:

Lemma 1. *Let Q_0 and D_1 be defined as in the preceeding paragraph. There exists a symbol Q_{-1} of degree -1 and regularity 1, defined on $\mathcal{V}_{\mathbf{c}}$, such that*

$$(Q_0+Q_{-1})(\mathbf{A}+\mathbf{A}_d^{-1}\mathbf{C})+\partial_d Q_0+\frac{1}{i}\sum_{k=0}^{d-1}(\partial_{\eta_k}Q_0\partial_{x_k}\mathbf{A}-\partial_{\eta_k}D_1\partial_{x_k}Q_0)-(D_1+D_0)(Q_0+Q_{-1})$$

is a symbol of degree -1 and D_0 is a block diagonal symbol (of degree 0 and regularity 1) whose blocks have dimension $n_1,\dots,n_J$ as those of D_1.

Proof. Using the equality $Q_0\mathbf{A}=D_1Q_0$, the problem reduces to finding a symbol Q_{-1} of degree -1 such that

$$[Q_{-1}Q_0^{-1},D_1]+Q_0\mathbf{A}_d^{-1}\mathbf{C}Q_0^{-1}+\partial_d Q_0+\frac{1}{i}\sum_{k=0}^{d-1}\partial_{\eta_k}Q_0\partial_{x_k}\mathbf{A}-\partial_{\eta_k}D_1\partial_{x_k}Q_0$$

is block diagonal (it will automatically be a symbol of degree 0). Here above $[M,N]$ denotes the commutator of two matrices M and N. Using that D_1 is diagonal, a simple calculation shows that one can choose $Q_{-1}Q_0^{-1}$ (and therefore Q_{-1}) such that the extra diagonal blocks of $[Q_{-1}Q_0^{-1},D_1]$ cancel those of

$$Q_0\mathbf{A}_d^{-1}\mathbf{C}Q_0^{-1}+\partial_d Q_0+\frac{1}{i}\sum_{k=0}^{d-1}\partial_{\eta_k}Q_0\partial_{x_k}\mathbf{A}-\partial_{\eta_k}D_1\partial_{x_k}Q_0\,.$$

Because the diagonal blocks of $[Q_{-1}Q_0^{-1},D_1]$ are identically zero, one can only cancel the extra diagonal blocks. It stems from this simple calculation that $Q_{-1}Q_0^{-1}$ is of degree -1 and regularity 1, and therefore so is Q_{-1}. $\square$

Note that Q_{-1} and D_0 are only defined for space-frequency variables belonging to $\mathcal{V}_{\mathbf{c}}$ but as was done for D_1, we extend D_0 as a global symbol of degree 0 and regularity 1.

In all the sequel, we denote by Q the sum Q_0+Q_{-1}. The following calculations heavily use the fact that Q defines a "good diagonalization basis" of the paralinearized operator

$$\dot v\longmapsto \partial_d\dot v-T^\gamma_{\mathbf{A}}\dot v-T^\gamma_{\mathbf{A}_d^{-1}\mathbf{C}}\dot v\,.$$

Let $\dot v\in H^2(\Omega)$ and define

$$F:=\partial_d\dot v-T^\gamma_{\mathbf{A}}\dot v-T^\gamma_{\mathbf{A}_d^{-1}\mathbf{C}}\dot v\in H^1(\Omega)\,.$$

We also define

$$w:=T^\gamma_{\chi Q}\dot v$$

and we first show that w satisfies a paradifferential equation whose first and zero order symbol are block diagonal. The paradifferential equation involves error terms that will be absorbed at the very end of the analysis. In all this paragraph, R^γ always denotes an operator of order ≤ -1 that represents the current error terms in the computations. We have

$$\begin{aligned}\partial_d\, w&=T^\gamma_{(\partial_d\chi)Q}\dot v+T^\gamma_{\chi\partial_d Q}\dot v+T^\gamma_{\chi Q}(\partial_d\dot v)=T^\gamma_{(\partial_d\chi)Q}\dot v+T^\gamma_{\chi\partial_d Q}\dot v+T^\gamma_{\chi Q}(T^\gamma_{\mathbf{A}}\dot v+T^\gamma_{\mathbf{A}_d^{-1}\mathbf{C}}\dot v+F)\\&=T^\gamma_{(\partial_d\chi)Q_0}\dot v+T^\gamma_{\chi\partial_d Q_0}\dot v+T^\gamma_{\chi Q(\mathbf{A}+\mathbf{A}_d^{-1}\mathbf{C})}\dot v+T^\gamma_{r^1}\dot v+R^\gamma\dot v+T^\gamma_{\chi Q}F\,,\end{aligned}$$

where

$$r^1 := \frac{1}{i} \sum_{k=0}^{d-1} \partial_{\eta_k}(\chi Q_0)\, \partial_{x_k} \mathbf{A} \,.$$

Lemma 1 implies that

$$\chi \partial_d Q_0 + \chi Q(\mathbf{A} + \mathbf{A}_d^{-1}\mathbf{C}) + r^1 - \left((D_1 + D_0)\chi Q + \frac{1}{i} \sum_{k=0}^{d-1} (\partial_{\eta_k}\chi) Q_0\, \partial_{x_k}\mathbf{A} + \chi(\partial_{\eta_k} D_1)\, \partial_{x_k} Q_0 \right)$$

is of degree -1, and we thus get

$$\partial_d\, w = T^\gamma_{(\partial_d \chi)Q_0} \dot{v} + (T^\gamma_{D_1} + T^\gamma_{D_0})\, w + T^\gamma_{r^2} \dot{v} + T^\gamma_{r^3} \dot{v} + R^\gamma \dot{v} + T^\gamma_{\chi Q} F \,,$$

where

$$r^2 := \frac{1}{i} \sum_{k=0}^{d-1} (\partial_{\eta_k}\chi) Q_0\, \partial_{x_k}\mathbf{A} + \chi(\partial_{\eta_k} D_1)\, \partial_{x_k} Q_0 \,,$$

$$r^3 := -\frac{1}{i} \sum_{k=0}^{d-1} (\partial_{\eta_k} D_1)\, \partial_{x_k}(\chi Q_0) = -\frac{1}{i} \sum_{k=0}^{d-1} (\partial_{\eta_k} D_1)\, \partial_{x_k}(\chi Q_0) \,.$$

After simplifying $r^2 + r^3$, we get the relation

$$\partial_d\, w = T^\gamma_{D_1} w + T^\gamma_{D_0} w + T^\gamma_r \dot{v} + R^\gamma \dot{v} + T^\gamma_{\chi Q} F \,, \tag{42}$$

with

$$r := (\partial_d \chi)\, Q_0 + \frac{1}{i} \sum_{k=0}^{d-1} (\partial_{\eta_k}\chi)\, Q_0\, \partial_{x_k}\mathbf{A} - (\partial_{x_k}\chi)\, (\partial_{\eta_k} D_1)\, Q_0 \,. \tag{43}$$

As a consequence, r is of order 0 and is identically zero in the domain where $\chi \equiv 1$, therefore r has its support far from the "unstable" points.

Recall that D_1 is diagonal and D_0 is block diagonal:

$$D_1 = \begin{pmatrix} \omega_1\, I_{n_1} & & \\ & \ddots & \\ & & \omega_J\, I_{n_J} \end{pmatrix} , \quad D_0 = \begin{pmatrix} C_1 & & \\ & \ddots & \\ & & C_J \end{pmatrix} ,$$

so (42) can be written as a collection of J equations

$$\partial_d\, w_j = T^\gamma_{\omega_j} w_j + T^\gamma_{C_j} w_j + T^\gamma_{r_j} \dot{v} + R^\gamma \dot{v} + T^\gamma_{\chi Q_j} F \,, \tag{44}$$

with Re $\omega_j < 0$ when $\gamma > 0$ and $1 \le j \le J'$ and Re $\omega_j > 0$ when $\gamma > 0$ and $J' + 1 \le j \le J$.

1) Estimate for the outgoing modes

We first deal with the case where Re $\omega_j > 0$ when $\gamma > 0$. Recall that ω_j is defined for all (η, γ) and satisfies

$$\forall\, (\eta, \gamma) \in \mathbb{R}^d \times \mathbb{R}^+ , \qquad \text{Re } \omega_j \ge c\, \gamma \,.$$

We choose $\Lambda^{2,\gamma}$ as a symmetrizer for (44), where $\Lambda^{2,\gamma}$ is the Fourier multiplier of symbol $\lambda^{2,\gamma}(\eta)$. Taking the scalar product in $L^2(\Omega)$ of (44) with $\Lambda^{2,\gamma} w_j$, we get

$$\begin{aligned}-\|w_j(0)\|^2_{1,\gamma} = 2 \text{ Re } \langle\!\langle \Lambda^{1,\gamma} T^\gamma_{\omega_j} w_j, \Lambda^{1,\gamma} w_j \rangle\!\rangle_{L^2(\Omega)} + 2 \text{ Re } \langle\!\langle \Lambda^{1,\gamma} T^\gamma_{C_j} w_j, \Lambda^{1,\gamma} w_j \rangle\!\rangle_{L^2(\Omega)} \\ +2 \text{ Re } \langle\!\langle \Lambda^{1,\gamma} T^\gamma_{r_j} \dot{v}, \Lambda^{1,\gamma} w_j \rangle\!\rangle_{L^2(\Omega)} + 2 \text{ Re } \langle\!\langle \Lambda^{1,\gamma} R^\gamma \dot{v}, \Lambda^{1,\gamma} w_j \rangle\!\rangle_{L^2(\Omega)} \\ +2 \text{ Re } \langle\!\langle \Lambda^{1,\gamma} T^\gamma_{\chi Q_j} F, \Lambda^{1,\gamma} w_j \rangle\!\rangle_{L^2(\Omega)} \,.\end{aligned}$$

Taking the order of the different operators into account and using Young's inequality, we obtain

$$\begin{aligned}&-2 \text{ Re } \langle\!\langle \Lambda^{1,\gamma} T^\gamma_{C_j} w_j, \Lambda^{1,\gamma} w_j \rangle\!\rangle_{L^2(\Omega)} \le C\, \| w_j \|^2_{1,\gamma} \,, \\ &-2 \text{ Re } \langle\!\langle \Lambda^{1,\gamma} T^\gamma_{r_j} \dot{v}, \Lambda^{1,\gamma} w_j \rangle\!\rangle_{L^2(\Omega)} \le \frac{C}{\gamma}\, \| T^\gamma_{r_j} \dot{v} \|^2_{1,\gamma} + \varepsilon\, \gamma\, \| w_j \|^2_{1,\gamma} \,, \\ &-2 \text{ Re } \langle\!\langle \Lambda^{1,\gamma} R^\gamma \dot{v}, \Lambda^{1,\gamma} w_j \rangle\!\rangle_{L^2(\Omega)} \le \frac{C}{\gamma}\, \| \dot{v} \|^2_0 + \varepsilon\, \gamma\, \| w_j \|^2_{1,\gamma} \,, \\ &-2 \text{ Re } \langle\!\langle \Lambda^{1,\gamma} T^\gamma_{\chi Q_j} F, \Lambda^{1,\gamma} w_j \rangle\!\rangle_{L^2(\Omega)} \le \frac{C}{\gamma}\, \| T^\gamma_{\chi Q} F \|^2_{1,\gamma} + \varepsilon\, \gamma\, \| w_j \|^2_{1,\gamma} \,.\end{aligned}$$

Using theorem A.2, the difference

$$\Lambda^{1,\gamma}\, T^\gamma_{\omega_j} - T^\gamma_{\omega_j}\, \Lambda^{1,\gamma}$$

is of order ≤ 1 so we have

$$2 \text{ Re } \langle\!\langle \Lambda^{1,\gamma} T^\gamma_{\omega_j} w_j, \Lambda^{1,\gamma} w_j \rangle\!\rangle_{L^2(\Omega)} \ge 2 \text{ Re } \langle\!\langle T^\gamma_{\omega_j} \Lambda^{1,\gamma} w_j, \Lambda^{1,\gamma} w_j \rangle\!\rangle_{L^2(\Omega)} - C\, \| w_j \|^2_{1,\gamma} \,.$$

Applying Gårding's inequality (theorem A.3), we finally get

$$2 \text{ Re } \langle\!\langle \Lambda^{1,\gamma} T^\gamma_{\omega_j} w_j, \Lambda^{1,\gamma} w_j \rangle\!\rangle_{L^2(\Omega)} \ge (c\,\gamma - C)\, \| w_j \|^2_{1,\gamma} \,.$$

It is now clear that an appropriate choice of ε yields the H^1 estimate

$$\gamma\, \| w_j \|^2_{1,\gamma} + \| w_j(0) \|^2_{1,\gamma} \lesssim \frac{1}{\gamma}\, \| T^\gamma_{\chi Q} F \|^2_{1,\gamma} + \frac{1}{\gamma}\, \left(\| \dot{v} \|^2_0 + \| T^\gamma_r \dot{v} \|^2_{1,\gamma} \right) \,. \tag{45}$$

2) Estimate for the incoming modes

We now deal with the case where Re $\omega_j < 0$ when $\gamma > 0$ (and therefore Re $\omega_j \le -c\gamma$). We first choose the identity as a symmetrizer and perform the same computation as above. One can indeed proceed in a similar way because the symmetrizer is a constant coefficients operator (that is, a Fourier multiplier). Because of the sign of Re ω_j, we have

$$\gamma\, \| w_j \|^2_0 \lesssim \| w_j(0) \|^2_0 + \frac{1}{\gamma^3}\, \| T^\gamma_{\chi Q} F \|^2_{1,\gamma} + \frac{1}{\gamma^3}\, \left(\| \dot{v} \|^2_0 + \| T^\gamma_{r_j} \dot{v} \|^2_{1,\gamma} \right) \,,$$

and we rewrite this estimate as

$$\gamma^3\, \| w_j \|^2_0 \lesssim \gamma^2\, \| w_j(0) \|^2_0 + \frac{1}{\gamma}\, \| T^\gamma_{\chi Q} F \|^2_{1,\gamma} + \frac{1}{\gamma}\, \left(\| \dot{v} \|^2_0 + \| T^\gamma_r \dot{v} \|^2_{1,\gamma} \right) \,. \tag{46}$$

The right-hand terms in (45) and (46) have similar expressions. The only difference is that the boundary value of w_j is on the right-hand side of the inequality when we deal with an incoming mode.

We now prove a more subtle estimate for the incoming modes. We choose $S_j := (T^\gamma_{\sigma_j})^* T^\gamma_{\sigma_j}$ as a symmetrizer for (44). Recall that S_j is of order ≤ 2 since $\sigma_j \in \Gamma^1_2$. We take the scalar product in $L^2(\Omega)$ of (44) with $S_j w_j$. This yields

$$\begin{aligned}
-\|T^\gamma_\sigma w_j(0)\|_0^2 = \; & \mathrm{Re}\, \langle\!\langle (\partial_d S_j) w_j, w_j \rangle\!\rangle_{L^2(\Omega)} + 2\,\mathrm{Re}\, \langle\!\langle S_j T^\gamma_{\omega_j} w_j, w_j \rangle\!\rangle_{L^2(\Omega)} \\
& +2\,\mathrm{Re}\, \langle\!\langle S_j T^\gamma_{C_j} w_j, w_j \rangle\!\rangle_{L^2(\Omega)} + 2\,\mathrm{Re}\, \langle\!\langle S_j T^\gamma_{r_j} \dot v, w_j \rangle\!\rangle_{L^2(\Omega)} \\
& +2\,\mathrm{Re}\, \langle\!\langle S_j R^\gamma \dot v, w_j \rangle\!\rangle_{L^2(\Omega)} + 2\,\mathrm{Re}\, \langle\!\langle S_j T^\gamma_{\chi Q_j} F, w_j \rangle\!\rangle_{L^2(\Omega)} \,.
\end{aligned}$$

First observe that

$$\partial_d S_j = (T^\gamma_{\partial_d \sigma_j})^* \, T^\gamma_{\sigma_j} + (T^\gamma_{\sigma_j})^* \, T^\gamma_{\partial_d \sigma_j} \,,$$

so we have

$$\mathrm{Re}\, \langle\!\langle (\partial_d S_j) w_j, w_j \rangle\!\rangle_{L^2(\Omega)} = 2\,\mathrm{Re}\, \langle\!\langle T^\gamma_{\partial_d \sigma_j} w_j, T^\gamma_{\sigma_j} w_j \rangle\!\rangle_{L^2(\Omega)} \,.$$

We recall that $\sigma_j \in \mathbb{R}$, hence the difference

$$T^\gamma_{\sigma_j} T^\gamma_{C_j} - T^\gamma_{C_j} T^\gamma_{\sigma_j}$$

is of order ≤ 0, and we get

$$\begin{aligned}
2\,\mathrm{Re}\, \langle\!\langle T^\gamma_{\sigma_j} T^\gamma_{C_j} w_j, T^\gamma_{\sigma_j} w_j \rangle\!\rangle_{L^2(\Omega)} & \le C\, \|T^\gamma_{\sigma_j} w_j\|_0^2 + C\, \|w_j\|_0 \, \|T^\gamma_{\sigma_j} w_j\| \\
& \le C\, \|T^\gamma_{\sigma_j} w_j\|_0^2 + \frac{C}{\gamma}\, \|w_j\|_0^2 + \varepsilon\, \gamma\, \|T^\gamma_{\sigma_j} w_j\|_0^2 \,.
\end{aligned}$$

Because $T^\gamma_{\sigma_j}$ is of order ≤ 1, we obtain thanks to Young's inequality

$$\begin{aligned}
2\,\mathrm{Re}\, \langle\!\langle T^\gamma_{\sigma_j} T^\gamma_{r_j} \dot v, T^\gamma_{\sigma_j} w_j \rangle\!\rangle_{L^2(\Omega)} & \le \frac{C}{\gamma}\, \|T^\gamma_{r_j} \dot v\|_{1,\gamma}^2 + \varepsilon\, \gamma\, \|T^\gamma_{\sigma_j} w_j\|_0^2 \,, \\
2\,\mathrm{Re}\, \langle\!\langle T^\gamma_{\sigma_j} R^\gamma \dot v, T^\gamma_{\sigma_j} w_j \rangle\!\rangle_{L^2(\Omega)} & \le \frac{C}{\gamma}\, \|\dot v\|_0^2 + \varepsilon\, \gamma\, \|T^\gamma_{\sigma_j} w_j\|_0^2 \,, \\
2\,\mathrm{Re}\, \langle\!\langle T^\gamma_{\sigma_j} T^\gamma_{\chi Q_j} F, T^\gamma_{\sigma_j} w_j \rangle\!\rangle_{L^2(\Omega)} & \le \frac{C}{\gamma}\, \|T^\gamma_{\chi Q} F\|_{1,\gamma}^2 + \varepsilon\, \gamma\, \|T^\gamma_{\sigma_j} w_j\|_0^2 \,.
\end{aligned}$$

Collecting these first inequalities, we already have

$$\begin{aligned}
-2\,\mathrm{Re}\, \langle\!\langle T^\gamma_{\partial_d \sigma_j} w_j + T^\gamma_{\sigma_j} T^\gamma_{\omega_j} w_j, T^\gamma_{\sigma_j} w_j \rangle\!\rangle_{L^2(\Omega)} & \le \|T^\gamma_\sigma w_j(0)\|_0^2 + (C + 4\varepsilon\, \gamma)\, \|T^\gamma_{\sigma_j} w_j\|_0^2 \\
& + \frac{C}{\gamma}\, \|T^\gamma_{\chi Q} F\|_{1,\gamma}^2 + \frac{C}{\gamma}\, \|w_j\|_0^2 + \frac{C}{\gamma}\, \left(\|\dot v\|_0^2 + \|T^\gamma_r \dot v\|_{1,\gamma}^2 \right) \,.
\end{aligned} \tag{47}$$

We are now going to derive a lower bound for the left-hand term in (47). Write $\omega_j = \gamma\, e_j + i h_j$ with $e_j, h_j \in \mathbb{R}$ and $e_j \in \Gamma^0_2$, $h_j \in \Gamma^1_2$. From theorem A.2, we have

$$T^\gamma_{\sigma_j} T^\gamma_{\omega_j} = \gamma\, T^\gamma_{e_j} T^\gamma_{\sigma_j} + \gamma\, T^\gamma_{-i\{\sigma_j, e_j\}} + \gamma R^\gamma + T^\gamma_{i h_j} T^\gamma_{\sigma_j} + T^\gamma_{\{\sigma_j, h_j\}} + R^\gamma_0 \,,$$

where R^γ still denotes an operator of order ≤ -1 and R^γ_0 denotes an operator of order ≤ 0. Recall that σ_j is a solution to the transport equation

$$\begin{cases} \partial_d\, \sigma_j + \{\sigma_j, h_j\} = 0 & x_d > 0 \,, \\ \sigma_j{}_{|_{x_d = 0}} = \sigma \,, \end{cases}$$

so we get

$$
\begin{aligned}
-2 \text{ Re } \langle\!\langle T^\gamma_{\partial_d \sigma_j} w_j + T^\gamma_{\sigma_j} T^\gamma_{\omega_j} w_j , T^\gamma_{\sigma_j} w_j \rangle\!\rangle_{L^2(\Omega)} &= -2\gamma \ \text{ Re } \langle\!\langle T^\gamma_{e_j} T^\gamma_{\sigma_j} w_j , T^\gamma_{\sigma_j} w_j \rangle\!\rangle_{L^2(\Omega)} \\
&\quad -2\gamma \ \text{ Re } \langle\!\langle T^\gamma_{-i\{\sigma_j, e_j\}} w_j , T^\gamma_{\sigma_j} w_j \rangle\!\rangle_{L^2(\Omega)} - 2\gamma \ \text{ Re } \langle\!\langle R^\gamma w_j , T^\gamma_{\sigma_j} w_j \rangle\!\rangle_{L^2(\Omega)} \\
&\quad -2 \, \langle\!\langle (\text{Re } T^\gamma_{ih_j}) T^\gamma_{\sigma_j} w_j , T^\gamma_{\sigma_j} w_j \rangle\!\rangle_{L^2(\Omega)} - 2 \text{ Re } \langle\!\langle R^\gamma_0 w_j , T^\gamma_{\sigma_j} w_j \rangle\!\rangle_{L^2(\Omega)} \, .
\end{aligned}
$$

Let us first examine the last three terms of the right-hand side. Because $ih_j \in i\mathbb{R}$, the operator Re $T^\gamma_{ih_j}$ is of order ≤ 0 and we have

$$
\begin{aligned}
-2 \langle\!\langle (\text{Re } T^\gamma_{ih_j}) T^\gamma_{\sigma_j} w_j , T^\gamma_{\sigma_j} w_j \rangle\!\rangle_{L^2(\Omega)} &\geq -C \, \|T^\gamma_{\sigma_j} w_j\|_0^2 \, , \\
-2 \text{ Re } \langle\!\langle R^\gamma_0 w_j , T^\gamma_{\sigma_j} w_j \rangle\!\rangle_{L^2(\Omega)} &\geq -C \, \|w_j\|_0 \, \|T^\gamma_{\sigma_j} w_j\|_0 \, , \\
-2\gamma \ \text{ Re } \langle\!\langle R^\gamma w_j , T^\gamma_{\sigma_j} w_j \rangle\!\rangle_{L^2(\Omega)} \geq -C \, \gamma \, \|w_j\|_{-1,\gamma} \, \|T^\gamma_{\sigma_j} w_j\|_0 &\geq -C \, \|w_j\|_0 \, \|T^\gamma_{\sigma_j} w_j\|_0 \, .
\end{aligned}
\tag{48}
$$

Applying Gårding's inequality (theorem A.3), we obtain

$$
-2\gamma \ \text{ Re } \langle\!\langle T^\gamma_{e_j} T^\gamma_{\sigma_j} w_j , T^\gamma_{\sigma_j} w_j \rangle\!\rangle_{L^2(\Omega)} \geq c \, \gamma \, \|T^\gamma_{\sigma_j} w_j\|_0^2 \, , \tag{49}
$$

so it only remains to derive a lower bound for the term

$$
-2\gamma \ \text{ Re } \langle\!\langle T^\gamma_{-i\{\sigma_j, e_j\}} w_j , T^\gamma_{\sigma_j} w_j \rangle\!\rangle_{L^2(\Omega)} \, .
$$

Because $\sigma_j \in \Gamma^1_2$, the operator $T^\gamma_{-i\{\sigma_j, e_j\}}$ is of order ≤ 0 and therefore

$$
-2\gamma \ \text{ Re } \langle\!\langle T^\gamma_{-i\{\sigma_j, e_j\}} w_j , T^\gamma_{\sigma_j} w_j \rangle\!\rangle_{L^2(\Omega)} \geq -C \, \gamma \, \|w_j\|_0 \, \|T^\gamma_{\sigma_j} w_j\|_0 \geq -\varepsilon \, \gamma \, \|T^\gamma_{\sigma_j} w_j\|_0^2 - C \, \gamma \, \|w_j\|_0^2 \, . \tag{50}
$$

Choosing ε appropriately and taking the sum of (47)-(48)-(49)-(50), we get the second estimate for the incoming modes:

$$
\gamma \, \|T^\gamma_{\sigma_j} w_j\|_0^2 \lesssim \|T^\gamma_\sigma w_j(0)\|_0^2 + \frac{1}{\gamma} \, \|T^\gamma_{\chi Q} F\|_{1,\gamma}^2 + \gamma \, \|w_j\|_0^2 + \frac{1}{\gamma} \, \left(\|\dot{v}\|_0^2 + \|T^\gamma_r \dot{v}\|_{1,\gamma}^2 \right) \, . \tag{51}
$$

Take the sum of (46) and (51). Choosing γ large enough, we have

$$
\gamma^3 \, \|w_j\|_0^2 + \gamma \, \|T^\gamma_{\sigma_j} w_j\|_0^2 \lesssim \gamma^2 \, \|w_j(0)\|_0^2 + \|T^\gamma_\sigma w_j(0)\|_0^2 + \frac{1}{\gamma} \, \|T^\gamma_{\chi Q} F\|_{1,\gamma}^2 + \frac{1}{\gamma} \, \left(\|\dot{v}\|_0^2 + \|T^\gamma_r \dot{v}\|_{1,\gamma}^2 \right) \, . \tag{52}
$$

Let us decompose the vector w in

$$
w = \begin{pmatrix} w_{in} \\ w_{out} \end{pmatrix} \, , \quad w_{in} \in \mathbb{C}^{N-1} \, , \quad w_{out} \in \mathbb{C}^{N+1} \, .
$$

The vector w_{in} is the collection of the w_j's that correspond to incoming modes and w_{out} is the collection of the w_j's that correspond to outgoing modes. Taking the sum of (45) and (52), we obtain

$$
\begin{aligned}
&\gamma \, \|w_{out}\|_{1,\gamma}^2 + \gamma^3 \, \|w_{in}\|_0^2 + \sum_{incoming} \gamma \, \|T^\gamma_{\sigma_j} w_j\|_0^2 + \|w_{out}(0)\|_{1,\gamma}^2 \\
&\lesssim \gamma^2 \, \|w_{in}(0)\|_0^2 + \|T^\gamma_\sigma w_{in}(0)\|_0^2 + \frac{1}{\gamma} \, \|T^\gamma_{\chi Q} F\|_{1,\gamma}^2 + \frac{1}{\gamma} \, \left(\|\dot{v}\|_0^2 + \|T^\gamma_r \dot{v}\|_{1,\gamma}^2 \right) \, ,
\end{aligned}
\tag{53}
$$

and we want to show an estimate of the type

$$
\gamma^2 \, \|w_{in}(0)\|_0^2 + \|T^\gamma_\sigma w_{in}(0)\|_0^2 \lesssim \|G\|_{1,\gamma}^2 + \|w_{out}(0)\|_{1,\gamma}^2 \, .
$$

3) Estimate for the boundary terms

We fix four cut-off functions $\chi_1, \chi_2, \chi_3, \chi_4$ such that

$$\begin{aligned}\chi_1 \equiv 1 \text{ on a neighbourhood of Supp } \chi \cap \{x_d = 0\}\,,\\ \chi_2 \equiv 1 \text{ on a neighbourhood of Supp } \chi_1\,,\\ \chi_3 \equiv 1 \text{ on a neighbourhood of Supp } \chi_2\,,\\ \chi_4 \equiv 1 \text{ on a neighbourhood of Supp } \chi_3\,,\end{aligned}$$

and Supp $\chi_4 \subset \mathcal{V}_{\mathbf{c}} \cap \{x_d = 0\}$. These cut-off functions are introduced in order to use the localized Gårding's inequality (theorem A.4). We write

$$T^\gamma_{\chi_2\beta Q_0^{-1}} w(0) = T^\gamma_{\chi_2\beta Q_{in}} w_{in}(0) + T^\gamma_{\chi_2\beta Q_{out}} w_{out}(0)\,.$$

Using the definition $w = T^\gamma_{\chi Q}\dot{v}$, we obtain

$$T^\gamma_{\chi_2\beta Q_0^{-1}} w(0) = T^\gamma_\chi G + R^\gamma \dot{v}(0)\,,$$

where R^γ is an operator of order ≤ -1. We thus get

$$\|T^\gamma_{\chi_2\beta Q_{in}} w_{in}(0)\|_{1,\gamma} \lesssim \|G\|_{1,\gamma} + \|w_{out}(0)\|_{1,\gamma} + \|\dot{v}(0)\|_0\,. \tag{54}$$

We are now going to introduce the basis of the stable subspace in which β has a reduced expression. Because $\chi_4\chi_2 = \chi_2$, we have

$$\chi_2\beta Q_{in} = (\chi_4 P_1^{-1})\,\underbrace{(\chi_2 P_1 \beta Q_{in} P_2)}_{\chi_2\beta^-}(\chi_4 P_2^{-1})\,.$$

We therefore obtain

$$T^\gamma_{\chi_2\beta Q_{in}} w_{in}(0) = T^\gamma_{\chi_4 P_1^{-1}} T^\gamma_{\chi_2\beta^-} T^\gamma_{\chi_4 P_2^{-1}} w_{in}(0) + R^\gamma w_{in}(0)\,,$$

and this yields the inequality

$$\|T^\gamma_{\chi_4 P_1^{-1}} T^\gamma_{\chi_2\beta^-} T^\gamma_{\chi_4 P_2^{-1}} w_{in}(0)\|_{1,\gamma} \lesssim \|G\|_{1,\gamma} + \|w_{out}(0)\|_{1,\gamma} + \|\dot{v}(0)\|_0\,. \tag{55}$$

We first show that we have an estimate of the type

$$\|T^\gamma_{\chi_4 P_1^{-1}} T^\gamma_{\chi_2\beta^-} T^\gamma_{\chi_4 P_2^{-1}} w_{in}(0)\|_{1,\gamma} \geq c\,\|T^\gamma_{\chi_2\beta^-} T^\gamma_{\chi_4 P_2^{-1}} w_{in}(0)\|_{1,\gamma} - C\,\|w_{in}(0)\|_0\,.$$

Define

$$z := T^\gamma_{\chi_2\beta^-} T^\gamma_{\chi_4 P_2^{-1}} w_{in}(0)\,.$$

We first check that

$$\|T^\gamma_{\chi_4 P_1^{-1}} z\|_{1,\gamma} = \|\Lambda^{1,\gamma} T^\gamma_{\chi_4 P_1^{-1}} z\|_0 \geq \|T^\gamma_{\chi_4 P_1^{-1}} \Lambda^{1,\gamma} z\|_0 - C\,\|w_{in}(0)\|_0\,.$$

We also note that

$$T^\gamma_{\chi_3} \Lambda^{1,\gamma} z = \Lambda^{1,\gamma} z + R_0^\gamma z\,,$$

where R_0^γ is of order ≤ 0. We thus get

$$\|T^\gamma_{\chi_4 P_1^{-1}} z\|_{1,\gamma} \geq \|T^\gamma_{\chi_4 P_1^{-1}} T^\gamma_{\chi_3} \Lambda^{1,\gamma} z\|_0 - C\,\|w_{in}(0)\|_0\,.$$

We are now going to use the ellipticity of $(P_1^{-1})^* P_1^{-1}$ on the support of χ_4. We write

$$\begin{aligned}\|T^\gamma_{\chi_4 P_1^{-1}} T^\gamma_{\chi_3} \Lambda^{1,\gamma} z\|_0^2 &= \langle (T^\gamma_{\chi_4 P_1^{-1}})^* T^\gamma_{\chi_4 P_1^{-1}} T^\gamma_{\chi_3} \Lambda^{1,\gamma} z, T^\gamma_{\chi_3} \Lambda^{1,\gamma} z\rangle \\ \geq\ \mathrm{Re}\ \langle T^\gamma_{\chi_4^2 (P_1^{-1})^* P_1^{-1}} T^\gamma_{\chi_3} \Lambda^{1,\gamma} z, T^\gamma_{\chi_3} \Lambda^{1,\gamma} z\rangle &- C\,\|T^\gamma_{\chi_3} \Lambda^{1,\gamma} z\|_0\,\|T^\gamma_{\chi_3} \Lambda^{1,\gamma} z\|_{-1,\gamma}\,,\end{aligned}$$

and we apply Gårding's inequality (theorem A.4) to obtain

$$\|T^\gamma_{\chi_4 P_1^{-1}} T^\gamma_{\chi_3} \Lambda^{1,\gamma} z\|_0^2 \geq c\,\|T^\gamma_{\chi_3} \Lambda^{1,\gamma} z\|_0^2 - C\,\|w_{in}(0)\|_0^2 \geq c\,\|\Lambda^{1,\gamma} z\|_0^2 - C\,\|w_{in}(0)\|_0^2\,.$$

Plugging this inequality in (55) yields

$$\|\Lambda^{1,\gamma} T^\gamma_{\chi_2 \beta^-} T^\gamma_{\chi_4 P_2^{-1}} w_{in}(0)\|_0 \lesssim \|G\|_{1,\gamma} + \|w_{out}(0)\|_{1,\gamma} + \|\dot{v}(0)\|_0\,. \tag{56}$$

Observe that the difference

$$\Lambda^{1,\gamma} T^\gamma_{\chi_2 \beta^-} - T^\gamma_{\chi_2 \lambda^{1,\gamma} \beta^-}$$

is of order ≤ 0 so (56) also reads

$$\|T^\gamma_{\chi_2 \lambda^{1,\gamma} \beta^-} T^\gamma_{\chi_4 P_2^{-1}} w_{in}(0)\|_0 \lesssim \|G\|_{1,\gamma} + \|w_{out}(0)\|_{1,\gamma} + \|\dot{v}(0)\|_0\,. \tag{57}$$

It is time to use the particular structure of β^- to derive a lower bound for the left-hand term in (57). Recall that

$$\chi_2 \lambda^{1,\gamma} \beta^- = \begin{pmatrix} \chi_2 (\gamma + i\sigma) & 0 \\ 0 & \chi_2 \lambda^{1,\gamma} E \end{pmatrix}\,.$$

We have

$$T^\gamma_{\chi_4 P_2^{-1}} w_{in}(0) = T^\gamma_{\chi_1} T^\gamma_{\chi_4 P_2^{-1}} w_{in}(0) + R^\gamma w_{in}(0)\,,$$

so we rewrite (57) as

$$\|T^\gamma_{\chi_2 \lambda^{1,\gamma} \beta^-} T^\gamma_{\chi_1} T^\gamma_{\chi_4 P_2^{-1}} w_{in}(0)\|_0 \lesssim \|G\|_{1,\gamma} + \|w_{out}(0)\|_{1,\gamma} + \|\dot{v}(0)\|_0\,. \tag{58}$$

The decomposition

$$T^\gamma_{\chi_4 P_2^{-1}} w_{in}(0) := \begin{pmatrix} w_1 \\ w' \end{pmatrix}\,, \quad w_1 \in \mathbb{C}, \quad w' \in \mathbb{C}^{N-2}\,,$$

gives

$$\|T^\gamma_{\chi_2 \lambda^{1,\gamma} \beta^-} T^\gamma_{\chi_1} T^\gamma_{\chi_4 P_2^{-1}} w_{in}(0)\|_0^2 = \|T^\gamma_{\chi_2 (\gamma + i\sigma)} T^\gamma_{\chi_1} w_1\|_0^2 + \|T^\gamma_{\chi_2 \lambda^{1,\gamma} E} T^\gamma_{\chi_1} w'\|_0^2\,.$$

Because $E^* E$ is elliptic, we obtain

$$\|T^\gamma_{\chi_2 \lambda^{1,\gamma} E} T^\gamma_{\chi_1} w'\|_0 \geq c\,\|w'\|_{1,\gamma} - C\,\|w_{in}(0)\|_0 \geq c\,\|T^\gamma_\sigma w'\|_0 - C\,\|w_{in}(0)\|_0\,,$$

and using that $\sigma \in \mathbb{R}$, we also obtain

$$\|T^\gamma_{\chi_2(\gamma+i\sigma)} T^\gamma_{\chi_1} w_1\|_0 \geq c\,\left(\gamma\,\|w_1\|_0 + \|T^\gamma_\sigma w_1\|_0\right) - C\,\|w_{in}(0)\|_0\,.$$

Eventually, we get the lower-bound

$$\|T^\gamma_{\chi_2\lambda^{1,\gamma}\beta - T^\gamma_{\chi_1}} T^\gamma_{\chi_4 P_2^{-1}} w_{in}(0)\|_0 \geq c\,\left(\gamma\,\|T^\gamma_{\chi_4 P_2^{-1}} w_{in}(0)\|_0 + \|T^\gamma_\sigma T^\gamma_{\chi_4 P_2^{-1}} w_{in}(0)\|_0\right) - C\,\|\dot{v}(0)\|_0\,. \tag{59}$$

To conclude, we perform the same kind of calculations as those already done with $\chi_4 P_1^{-1}$ to show that

$$\|T^\gamma_{\chi_4 P_2^{-1}} w_{in}(0)\|_0 \geq c\,\|w_{in}(0)\|_0 - C\,\|\dot{v}(0)\|_{-1,\gamma}\,,$$
$$\|T^\gamma_\sigma T^\gamma_{\chi_4 P_2^{-1}} w_{in}(0)\|_0 \geq c\,\|T^\gamma_\sigma w_{in}(0)\|_0 - C\,\|\dot{v}(0)\|_0\,.$$

The boundary terms thus satisfy:

$$\gamma^2\,\|w_{in}(0)\|_0^2 + \|T^\gamma_\sigma w_{in}(0)\|_0^2 \lesssim \|G\|_{1,\gamma}^2 + \|w_{out}(0)\|_{1,\gamma}^2 + \|\dot{v}(0)\|_0^2\,. \tag{60}$$

Combining (53) and (60), we obtain our main energy estimate localized near the instability points:

$$\begin{aligned}\gamma\,\|\!|w_{out}|\!\|_{1,\gamma}^2 + \gamma^3\,\|\!|w_{in}|\!\|_0^2 + \sum_{incoming} \gamma\,\|\!|T^\gamma_{\sigma_j} w_j|\!\|_0^2 + \|w_{out}(0)\|_{1,\gamma}^2 + \gamma^2\,\|w_{in}(0)\|_0^2 + \|T^\gamma_\sigma w_{in}(0)\|_0^2 \\ \lesssim \|G\|_{1,\gamma}^2 + \|\dot{v}(0)\|_0^2 + \frac{1}{\gamma}\,\|\!|T^\gamma_{\chi Q} F|\!\|_{1,\gamma}^2 + \frac{1}{\gamma}\,\left(\|\!|\dot{v}|\!\|_0^2 + \|\!|T^\gamma_r \dot{v}|\!\|_{1,\gamma}^2\right)\,.\end{aligned} \tag{61}$$

3.5 Energy estimates far from instability points

In this paragraph, we show how to estimate the H^1 norm of $\dot{v}$ far from the instability points. We fix a smooth cut-off function $\widetilde{\chi}$ such that

$$\widetilde{\chi} \equiv 1 \text{ on Supp } \chi_0 \cap \{x_d = 0\}\,,$$

so we have $\widetilde{\chi}\chi_0|_{x_d=0} \equiv \chi_0|_{x_d=0}$, and we also ask that the support of $\widetilde{\chi}$ does not meet the set of critical points $\Sigma_{\mathbf{c}}$. This is possible because the support of $\chi_0|_{x_d=0}$ does not meet $\Sigma_{\mathbf{c}}$. With this requirement, the uniform stability condition is met at all point in the support of $\widetilde{\chi}$. The cut-off function $\widetilde{\chi}$ is introduced in order to use a localized Gårding's inequality.

To derive the desired energy estimate, we shall use a classical Kreiss' symmetrizer, as in [25, 28].

Proposition 3.1 (Kreiss' symmetrizers). *There exists a mapping*

$$S : \overline{\Omega} \times (\mathbb{R}^d \times \mathbb{R}^+ \setminus \{0\}) \longrightarrow \mathcal{M}_{2n}(\mathbb{C})$$

satisfying the following properties:

$\forall\, z$, *the matrix* $S(z)$ *is hermitian,*

S *is a symbol of degree* 2 *and regularity* 2,

$\forall\, z \in \partial\Omega \times (\mathbb{R}^d \times \mathbb{R}^+ \setminus \{0\})$, *one has*

$$\widetilde{\chi}(z)^2\,S(z) + C\,\widetilde{\chi}(z)^2\,\lambda^{2,\gamma}(\eta)\,\beta(z)^*\beta(z) \geq c\,\widetilde{\chi}(z)^2\,\lambda^{2,\gamma}(\eta)\,I\,, \tag{62}$$

there exists a finite set of matrix valued mappings such that

$$\mathrm{Re}\ (S(z)\mathbf{A}(z)) = \sum_l V_l(z)^* \begin{pmatrix} \gamma H_l(z) & 0 \\ 0 & E_l(z) \end{pmatrix} V_l(z)\,,$$

where V_l and E_l are homogeneous of degree 1 with respect to (η,γ) (and belong to Γ^1_2), H_l is homogeneous of degree 0 with respect to (η,γ) (and belongs to Γ^0_2), and the following inequalities hold:

$$\sum_l V_l(z)^* V_l(z) \geq c\,\lambda^{2,\gamma}(\eta)\,I\,, \quad H_l(z) \geq c\,I\,, \quad E_l(z) \geq c\,\lambda^{1,\gamma}(\eta)\,I\,. \tag{63}$$

We define $w := T^\gamma_{\chi_0}\dot{v}$ and compute the equation satisfied by w. The calculations are entirely similar to those done in the preceeding paragraph, namely

$$\partial_d\, w = T^\gamma_{\mathbf{A}}\, w + T^\gamma_{\mathbf{A}_d^{-1}\mathbf{C}}\, w + T^\gamma_{r^0}\,\dot{v} + R^\gamma\,\dot{v} + T^\gamma_{\chi Q}\,F\,, \tag{64}$$

with

$$r^0 := \partial_d\chi_0 + \frac{1}{i}\,\{\chi_0,\mathbf{A}\} = \partial_d\chi_0 + \frac{1}{i}\sum_{k=0}^{d-1}(\partial_{\eta_k}\chi_0)\partial_{x_k}\mathbf{A} - (\partial_{\eta_k}\mathbf{A})\,\partial_{x_k}\chi_0\,. \tag{65}$$

Let $\{\mathbf{S}^\gamma(x_d)\}$ be given by

$$\mathbf{S}^\gamma(x_d) := \frac{1}{2}\left((T^\gamma_{S(x_d)})^* + T^\gamma_{S(x_d)}\right)\,.$$

Because $S \in \Gamma^2_2$, $\{\mathbf{S}^\gamma\}$ is a bounded Lipschitzean family of selfadjoint operators from $H^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ (the bounds are uniform with respect to the parameter $\gamma \geq 1$). The starting point to derive the energy estimate is to take the scalar product of (64) with $\mathbf{S}^\gamma\,w$ and integrate with respect to $(x_0,\ldots,x_d)\in\Omega$. We find

$$\begin{aligned} \langle \mathbf{S}^\gamma(0)w(0), w(0)\rangle_{L^2} + 2\ \mathrm{Re}\ \langle\!\langle \mathbf{S}^\gamma T^\gamma_{\mathbf{A}} w, w\rangle\!\rangle_{L^2(\Omega)} = -\ \mathrm{Re}\ \langle\!\langle \frac{d\mathbf{S}^\gamma}{dx_d} w, w\rangle\!\rangle_{L^2(\Omega)} \\ -2\ \mathrm{Re}\ \langle\!\langle \mathbf{S}^\gamma T^\gamma_{\mathbf{A}_d^{-1}\mathbf{C}} w, w\rangle\!\rangle_{L^2(\Omega)} - 2\ \mathrm{Re}\ \langle\!\langle \mathbf{S}^\gamma T^\gamma_{r^0}\dot{v}, w\rangle\!\rangle_{L^2(\Omega)} - 2\ \mathrm{Re}\ \langle\!\langle \mathbf{S}^\gamma R^\gamma \dot{v}, w\rangle\!\rangle_{L^2(\Omega)} \\ -2\ \mathrm{Re}\ \langle\!\langle \mathbf{S}^\gamma T^\gamma_{\chi_0} F, w\rangle\!\rangle_{L^2(\Omega)}\,. \end{aligned} \tag{66}$$

The right-hand side of (66) is easily estimated. We write $\mathbf{S}^\gamma$ as

$$\mathbf{S}^\gamma = \Lambda^{1,\gamma}\,\Lambda^{-1,\gamma}\,\mathbf{S}^\gamma$$

and use that $\Lambda^{1,\gamma}$ is self-adjoint. Because $\Lambda^{-1,\gamma}\mathbf{S}^\gamma$ is of order ≤ 1, we obtain

$$\begin{aligned} -\ \mathrm{Re}\ \langle\!\langle \frac{d\mathbf{S}^\gamma}{dx_d} w, w\rangle\!\rangle_{L^2(\Omega)} &\leq C\,\|w\|^2_{1,\gamma}\,, \\ -2\ \mathrm{Re}\ \langle\!\langle \mathbf{S}^\gamma T^\gamma_{\mathbf{A}_d^{-1}\mathbf{C}} w, w\rangle\!\rangle_{L^2(\Omega)} &\leq C\,\|w\|^2_{1,\gamma}\,, \\ -2\ \mathrm{Re}\ \langle\!\langle \mathbf{S}^\gamma T^\gamma_{r^0}\dot{v}, w\rangle\!\rangle_{L^2(\Omega)} &\leq \frac{C}{\gamma}\,\|T^\gamma_{r^0}\dot{v}\|^2_{1,\gamma} + \varepsilon\,\gamma\,\|w\|^2_{1,\gamma}\,, \\ -2\ \mathrm{Re}\ \langle\!\langle \mathbf{S}^\gamma R^\gamma \dot{v}, w\rangle\!\rangle_{L^2(\Omega)} &\leq \frac{C}{\gamma}\,\|\dot{v}\|^2_0 + \varepsilon\,\gamma\,\|w\|^2_{1,\gamma}\,, \\ -2\ \mathrm{Re}\ \langle\!\langle \mathbf{S}^\gamma T^\gamma_{\chi_0} F, w\rangle\!\rangle_{L^2(\Omega)} &\leq \frac{C}{\gamma}\,\|T^\gamma_{\chi_0} F\|^2_{1,\gamma} + \varepsilon\,\gamma\,\|w\|^2_{1,\gamma}\,, \end{aligned}$$

so we get

$$\langle \mathbf{S}^\gamma(0) w(0), w(0)\rangle_{L^2} + 2\ \mathrm{Re}\ \langle\!\langle \mathbf{S}^\gamma T^\gamma_{\mathbf{A}} w, w\rangle\!\rangle_{L^2(\Omega)} \leq (C + 3\varepsilon\,\gamma)\, \|\!| w |\!\|^2_{1,\gamma} + \frac{C}{\gamma}\, \|\!| T^\gamma_{\chi_0} F |\!\|^2_{1,\gamma} + \frac{C}{\gamma}\, \left(\|\!| \dot{v} |\!\|^2_0 + \|\!| T^\gamma_{r^0} \dot{v} |\!\|^2_{1,\gamma} \right) . \tag{67}$$

We are now going to derive a lower bound for the left-hand side of (67) by means of Gårding's inequalities. We first deal with the boundary term. First note that

$$\mathbf{S}(0) - T^\gamma_{S(0)}$$

is of order ≤ 1, so we have

$$\langle \mathbf{S}^\gamma(0) w(0), w(0)\rangle_{L^2} = \ \mathrm{Re}\ \langle T^\gamma_{S(0)}(0) w(0), w(0)\rangle_{L^2} + \mathcal{O}(\|w(0)\|_{1,\gamma}\, \|w(0)\|_0) .$$

Using (62), we can apply Gårding's inequality (theorem A.4) and derive

$$\mathrm{Re}\ \langle T^\gamma_{S(0)}(0) w(0), w(0)\rangle_{L^2} + C\ \ \mathrm{Re}\ \langle T^\gamma_{\lambda^{2,\gamma}\beta^*\beta} w(0), w(0)\rangle_{L^2} \geq c\, \|w(0)\|^2_{1,\gamma} - C\, \|\dot{v}(0)\|^2_0 .$$

Observe that

$$T^\gamma_{\lambda^{2,\gamma}\beta^*\beta} - \left[\Lambda^{1,\gamma} T^\gamma_\beta\right]^* \left[\Lambda^{1,\gamma} T^\gamma_\beta\right]$$

is of order ≤ 1 so we have

$$\mathrm{Re}\ \langle T^\gamma_{\lambda^{2,\gamma}\beta^*\beta} w(0), w(0)\rangle_{L^2} = \|T^\gamma_\beta w(0)\|^2_{1,\gamma} + \mathcal{O}(\|w(0)\|_{1,\gamma}\, \|w(0)\|_0) .$$

For γ large enough, we therefore obtain

$$\langle \mathbf{S}^\gamma(0) w(0), w(0)\rangle_{L^2} \geq c\, \|w(0)\|^2_{1,\gamma} - C\, \|T^\gamma_\beta w(0)\|^2_{1,\gamma} - C\, \|\dot{v}(0)\|^2_0 , \tag{68}$$

and we can now deal with the interior term. Since

$$\mathrm{Re}\ \mathbf{S}^\gamma T^\gamma_{\mathbf{A}} - T^\gamma_{\mathrm{Re}\ (S\mathbf{A})}$$

is of order ≤ 2, we have

$$2\ \mathrm{Re}\ \langle\!\langle \mathbf{S}^\gamma T^\gamma_{\mathbf{A}} w, w\rangle\!\rangle_{L^2(\Omega)} \geq 2\ \mathrm{Re}\ \langle\!\langle T^\gamma_{\mathrm{Re}\ (S\mathbf{A})} w, w\rangle\!\rangle_{L^2(\Omega)} - C\, \|\!| w |\!\|^2_{1,\gamma} .$$

Define

$$\forall\, z \in \overline{\Omega} \times (\mathbb{R}^d \times \mathbb{R}^+ \setminus \{0\})\,, \quad a_i(z) := \begin{pmatrix} \gamma H_i(z) & 0 \\ 0 & E_i(z) \end{pmatrix} .$$

Then the remainders

$$T^\gamma_{V_l^* a_l V_l} - (T^\gamma_{V_l})^* T^\gamma_{a_l} T^\gamma_{V_l}$$

are of order ≤ 2, so we get

$$2\ \mathrm{Re}\ \langle\!\langle \mathbf{S}^\gamma T^\gamma_{\mathbf{A}} w, w\rangle\!\rangle_{L^2(\Omega)} \geq \mathrm{Re}\ \sum_l \langle\!\langle T^\gamma_{a_l} w_l, w_l\rangle\!\rangle_{L^2(\Omega)} - C\, \|\!| w |\!\|^2_{1,\gamma}$$

where $w_l := T^\gamma_{V_l} w$. Using the block decomposition of a_i and Gårding's inequality (theorem A.3) on each block, we obtain

$$\mathrm{Re}\ \langle\!\langle T^\gamma_{a_l} w_l, w_l\rangle\!\rangle_{L^2(\Omega)} \geq c\,\gamma\, \|\!| w_l |\!\|^2_0 ,$$

and the ellipticity of the symbol $\sum_l V_l^* V_l$ yields, for γ large enough, the estimate

$$\|w\|_{1,\gamma}^2 \lesssim \sum_l \operatorname{Re} \langle\!\langle T^\gamma_{V_l^* V_l} w, w \rangle\!\rangle_{L^2(\Omega)} \lesssim \sum_l \|w_l\|_0^2 + \frac{1}{\gamma} \|w\|_{1,\gamma}^2 .$$

Combining all these inequalities, we conclude that

$$\operatorname{Re} \langle\!\langle \mathbf{S}^\gamma T^\gamma_{\mathbf{A}} w, w \rangle\!\rangle_{L^2(\Omega)} \geq c\,\gamma \|w\|_{1,\gamma}^2 \tag{69}$$

for γ large enough. We now use (68) and (69) to derive a lower bound for the left-hand term of (67). Choosing ε appropriately, we end up with

$$\gamma \|w\|_{1,\gamma}^2 + \|w(0)\|_{1,\gamma}^2 - \|T^\gamma_\beta w(0)\|_{1,\gamma}^2 - \|\dot v(0)\|_0^2 \lesssim \frac{1}{\gamma} \|T^\gamma_{\chi_0} F\|_{1,\gamma}^2 + \frac{1}{\gamma} \left(\|\dot v\|_0^2 + \|T^\gamma_{r^0} \dot v\|_{1,\gamma}^2 \right) .$$

To conclude, observe that the remainder

$$T^\gamma_\beta T^\gamma_{\chi_0} - T^\gamma_{\chi_0} T^\gamma_\beta$$

is of order ≤ -1 so we have

$$\|T^\gamma_\beta w(0)\|_{1,\gamma}^2 \lesssim \|G\|_{1,\gamma}^2 + \|\dot v(0)\|_0^2 .$$

Eventually, we have proved

$$\gamma \|T^\gamma_{\chi_0} \dot v\|_{1,\gamma}^2 + \|T^\gamma_{\chi_0} \dot v(0)\|_{1,\gamma}^2 \lesssim \|G\|_{1,\gamma}^2 + \|\dot v(0)\|_0^2 + \frac{1}{\gamma} \|T^\gamma_{\chi_0} F\|_{1,\gamma}^2 + \frac{1}{\gamma} \left(\|\dot v\|_0^2 + \|T^\gamma_{r^0} \dot v\|_{1,\gamma}^2 \right) . \tag{70}$$

3.6 The main result

This paragraph is devoted to the very end of the analysis. We first prove the following:

Theorem 3.1. *For all $\dot v \in H^1(\Omega)$, we define the localized norm of $\dot v$ as*

$$\tilde{\|\dot v\|}^2 := \|T^\gamma_{\chi_0} \dot v\|_{1,\gamma}^2 + \|\dot v_{out}\|_{1,\gamma}^2 + \gamma^2 \|\dot v_{in}\|_0^2 + \sum_{1 \leq j \leq J'} \|T^\gamma_{\sigma_j} \dot v_j\|_0^2$$

where we keep the notations of paragraph 3.3:

$$T^\gamma_{\chi Q} \dot v = \begin{pmatrix} \dot v_{in} \\ \dot v_{out} \end{pmatrix} , \quad \dot v_{in} \in \mathbb{C}^{N-1} , \quad \dot v_{out} \in \mathbb{C}^{N+1} \quad \textit{and} \quad \dot v_{in} = \begin{pmatrix} \dot v_1 \\ \vdots \\ \dot v_{J'} \end{pmatrix} .$$

For $\dot v(0) \in H^1(\mathbb{R}^d)$ we define in a similar way the localized norm of $\dot v(0)$ as

$$\tilde{\|\dot v(0)\|}^2 := \|T^\gamma_{\chi_0} \dot v(0)\|_{1,\gamma}^2 + \|\dot v_{out}(0)\|_{1,\gamma}^2 + \gamma^2 \|\dot v_{in}(0)\|_0^2 + \|T^\gamma_\sigma \dot v_{in}(0)\|_0^2 .$$

Then there exist two constants $C > 0$ and $\gamma_0 \geq 1$ such that for all $\gamma \geq \gamma_0$ and for all $(v, \psi) \in H^2(\Omega) \times H^2(\mathbb{R}^d)$, the following estimate holds:

$$\gamma \tilde{\|\dot v\|}^2 + \tilde{\|\dot v(0)\|}^2 \leq C \left(\frac{1}{\gamma} \|F\|_{1,\gamma}^2 + \|G\|_{1,\gamma}^2 \right) , \tag{71}$$

where

$$\dot v := v - T^\gamma_{\partial_d \mathbf{a}} \psi , \quad F := \partial_d \dot v - T^\gamma_{\mathbf{A}} \dot v - T^\gamma_{\mathbf{A}_d^{-1} \mathbf{C}} \dot v \quad \textit{and} \quad G := T^\gamma_{\mathbf{\Pi M}} \dot v_{|x_d = 0} .$$

Proof. Using (61) and (70), we have already proved that there exists two constants C and γ_0 such that for all $\gamma \geq \gamma_0$, one has

$$\gamma \, \|\dot{v}\tilde{\|}^2 + \|\dot{v}(0)\tilde{|}|^2 \leq C \left(\frac{1}{\gamma} \, \|F\|_{1,\gamma}^2 + \|G\|_{1,\gamma}^2 \right) + C \, \|\dot{v}(0)\|_0^2 + \frac{C}{\gamma} \, \left(\|\dot{v}\|_0^2 + \|T_r^\gamma \dot{v}\|_0^2 + \|T_{r^0}^\gamma \dot{v}\|_0^2 \right) \tag{72}$$

where r is given by (43) and r^0 is given by (65). We first show that the localized norm verifies

$$\gamma \, \|\dot{v}\|_0 \leq C \, \|\dot{v}\tilde{\|} \tag{73}$$

for γ large enough and $C > 0$. Write

$$I = \chi_0 I + \chi I = \chi_0 I + (\widetilde{\chi} Q_0^{-1}) \chi Q_0 \, ,$$

where $\widetilde{\chi}$ is a suitable cut-off function such that $\widetilde{\chi}\chi = \chi$. Then we have

$$\begin{aligned} \gamma \, \|\dot{v}\|_0 &\leq \gamma \, \|T_{\chi_0}^\gamma \dot{v}\|_0 + C \, \gamma \, \|T_{\chi Q_0}^\gamma \dot{v}\|_0 \leq \|T_{\chi_0}^\gamma \dot{v}\|_{1,\gamma} + C \, \gamma \, \|T_{\chi Q}^\gamma \dot{v}\|_0 + C \, \gamma \, \|T_{\chi Q_{-1}}^\gamma \dot{v}\|_0 \\ &\leq \|T_{\chi_0}^\gamma \dot{v}\|_{1,\gamma} + \|\dot{v}_{out}\|_{1,\gamma} + \gamma \, \|\dot{v}_{in}\|_0 + C \, \|\dot{v}\|_0 \leq C \, \|\dot{v}\tilde{\|} + C \, \|\dot{v}\|_0 \end{aligned}$$

and this gives (73) when γ is large enough. In a completely similar way, we get

$$\gamma \, \|\dot{v}(0)\|_0 \leq C \, \|\dot{v}(0)\tilde{|}| \, . \tag{74}$$

Using (73) and (74), (72) reads

$$\gamma \, \|\dot{v}\tilde{\|}^2 + \|\dot{v}(0)\tilde{|}|^2 \leq C \left(\frac{1}{\gamma} \, \|F\|_{1,\gamma}^2 + \|G\|_{1,\gamma}^2 \right) + \frac{C}{\gamma} \, \left(\|T_r^\gamma \dot{v}\|_{1,\gamma}^2 + \|T_{r^0}^\gamma \dot{v}\|_{1,\gamma}^2 \right) \, , \tag{75}$$

and we need to absorb the last two terms.

To absorb the term $T_r^\gamma \dot{v}$, we decompose the symbol r as a linear combination of χ_0 and the σ_j's. Recall that r is a symbol of degree 0 and regularity 1 that is identically zero in the region $\{\chi \equiv 1\}$. In the region $\{\chi \leq 1/2\}$, we have $\chi_0 \geq 1/2$ so we can write

$$r := \alpha_0 \, (\chi_0 \, I) \, ,$$

with α a symbol of degree 0. In the region $\{\chi \geq 1/2\}$, we can write

$$r := \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{J'} \\ \alpha_{out} \end{pmatrix} \begin{pmatrix} \sigma_1 I_{n_1} & & & \\ & \ddots & & \\ & & \sigma_{J'} I_{n_{J'}} & \\ & & & I \end{pmatrix} (\chi \, Q_0) \, ,$$

because the two last matrices are regular in the region where r is not identically zero. Up to introducing new cut-off functions, we can decompose r as

$$r = \alpha_0 \, (\chi_0 \, I) + \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{J'} \\ \alpha_{out} \end{pmatrix} \begin{pmatrix} \sigma_1 I_{n_1} & & & \\ & \ddots & & \\ & & \sigma_{J'} I_{n_{J'}} & \\ & & & I \end{pmatrix} (\chi \, Q_0) \, ,$$

where α_0 and α_{out} are of degree 0 and $\alpha_1, \ldots, \alpha_{J'}$ are of degree -1 (because the σ_j's are of degree 1). At this stage, we easily derive

$$\|T_r^\gamma \dot{v}\|_{1,\gamma} \lesssim \|T_{\chi_0}^\gamma \dot{v}\|_{1,\gamma} + \|\dot{v}_{out}\|_{1,\gamma} + \sum_{1 \leq j \leq J'} \|T_{\sigma_j}^\gamma \dot{v}_j\|_0 \lesssim \|\dot{v}\tilde{\|} \, .$$

The same kind of arguments also work for the term $T^\gamma_{r^0}$ because r^0 is identically zero near the bicharacteristic curves. Eventually, (75) yields, for γ large enough:

$$\gamma \, \| \dot{v} \tilde{\|}^2 + \| \dot{v}(0) \tilde{\|}^2 \lesssim \frac{1}{\gamma} \, \| F \|^2_{1,\gamma} + \| G \|^2_{1,\gamma} \,. \tag{76}$$

$\square$

Theorem 3.1 gives a precise statement of the location of the possible singularities of $\dot{v}$, that is, where $\dot{v}$ is less regular that the source terms F and G. However, an important consequence of theorem 3.1 is that our constant coefficients energy estimate holds for variable coefficients system:

Theorem 3.2. *Let* $\mathbf{a}$*, defined by* (24)*, satisfy assumptions 6-8. Then for all* $v \in H^2(\Omega)$*,* $\psi \in H^2(\mathbb{R}^d)$ *and for all* $\gamma \geq \gamma_0$ *(where* γ_0 *only depends on* $\mathbf{a}$*), one has*

$$\gamma \, \| v \|^2_0 + \| v_{|x_d=0} \|^2_0 + \| \psi \|^2_{1,\gamma} \lesssim \frac{1}{\gamma^3} \, \| \mathcal{L}^\gamma_{\mathbf{a}}(v,\psi) \|^2_{1,\gamma} + \frac{1}{\gamma^2} \, \| \mathcal{B}^\gamma_{\mathbf{a}}(v,\psi) \|^2_{1,\gamma} \,. \tag{77}$$

Proof. The result is a direct consequence of the paralinearization estimates. Let v and ψ be given and define $\dot{v}$ as the good unknown of the problem. Let

$$F := \partial_d \dot{v} - T^\gamma_{\mathbf{A}} \dot{v} - T^\gamma_{\mathbf{A}_d^{-1}\mathbf{C}} \dot{v} \quad \text{and} \quad G := T^\gamma_{\mathbf{\Pi M}} \dot{v}_{|x_d=0} \,.$$

Using (76) and (73)-(74), we get

$$\gamma \, \| \dot{v} \|^2_0 + \| \dot{v}(0) \|^2_0 \lesssim \frac{1}{\gamma^3} \, \| F \|^2_{1,\gamma} + \frac{1}{\gamma^2} \, \| G \|^2_{1,\gamma} \,.$$

We know from (36) that G satisfies

$$\| G \|^2_{1,\gamma} \lesssim \| \dot{v}(0) \|^2_0 + \| T^\gamma_{\mathbf{b}} \psi + T^\gamma_{\mathbf{M}} \dot{v}_{|x_d=0} \|^2_{1,\gamma} + \| \psi \|^2_{1,\gamma} \,,$$

so we have

$$\gamma \, \| \dot{v} \|^2_0 + \| \dot{v}(0) \|^2_0 \lesssim \frac{1}{\gamma^3} \, \| F \|^2_{1,\gamma} + \frac{1}{\gamma^2} \, \| T^\gamma_{\mathbf{b}} \psi + T^\gamma_{\mathbf{M}} \dot{v}_{|x_d=0} \|^2_{1,\gamma} + \frac{1}{\gamma^2} \, \| \psi \|^2_{1,\gamma} \,.$$

Using (33), we obtain

$$\gamma \, \| \dot{v} \|^2_0 + \| \dot{v}(0) \|^2_0 \lesssim \frac{1}{\gamma^3} \, \| F \|^2_{1,\gamma} + \frac{1}{\gamma^2} \, \| \mathcal{B}^\gamma_{\mathbf{a}}(v,\psi) \|^2_{1,\gamma} + \frac{1}{\gamma^2} \, \| \psi \|^2_{1,\gamma} \,,$$

and using once again (33) (for the interior term $\| F \|_{1,\gamma}$) yields

$$\gamma \, \| \dot{v} \|^2_0 + \| \dot{v}(0) \|^2_0 \lesssim \frac{1}{\gamma^3} \, \| \mathcal{L}^\gamma_{\mathbf{a}}(v,\psi) \|^2_{1,\gamma} + \frac{1}{\gamma^2} \, \| \mathcal{B}^\gamma_{\mathbf{a}}(v,\psi) \|^2_{1,\gamma} + \frac{1}{\gamma^2} \, \| \psi \|^2_{1,\gamma} \,.$$

Estimate (34) yields the estimate for v (and not $\dot{v}$) while (35) enables to recover the estimate on the shock front ψ and to derive (77). $\square$

4 The example of gas dynamics

When dealing with a concrete example, it is more convenient to have distinct notations for the Laplace variable and the Fourier variable. We shall thus denote by τ the Laplace dual variable of t (τ is a complex number of positive real part) and by $\eta \in \mathbb{R}^{d-1}$ the Fourier dual variable of $(x_1, \ldots, x_{d-1})$. Consequently, the stable subspace will be denoted by $\mathcal{E}^-(\tau, \eta)$, the front symbol will be denoted by $b(\tau, \eta)$ and so on.

Consider Euler's equations of isentropic gas dynamics:

$$\begin{cases} \partial_t \, \rho + \nabla \cdot (\rho \, \mathbf{v}) = 0 \, , \\ \partial_t \, (\rho \, \mathbf{v}) + \nabla \cdot (\rho \, \mathbf{v} \otimes \mathbf{v}) + \nabla \, p = 0 \, , \end{cases} \tag{78}$$

where p is an increasing function of $\rho > 0$ but which is not convex. System (78) satisfies both assumptions 1 and 2. For a planar shock wave

$$\mathbf{u} = \begin{cases} (\rho_l, \mathbf{v}_l) & \text{if } x \cdot \nu < \sigma t, \\ (\rho_r, \mathbf{v}_r) & \text{if } x \cdot \nu > \sigma t, \end{cases}$$

where ν is a unit vector in $\mathbb{R}^d$, the Rankine-Hugoniot conditions read

$$\begin{aligned} \rho_r (\mathbf{v}_r \cdot \nu - \sigma) = \rho_l (\mathbf{v}_l \cdot \nu - \sigma) =: j \, , \\ j[\mathbf{v}] + [p]\nu = 0 \, , \end{aligned}$$

and we shall assume $j \neq 0$. Then the tangential velocity is continuous across the shock front and, up to changing observer, we may assume

$$\nu = (0, \ldots, 0, 1) \, , \quad \mathbf{v}_{r,l} = (0, \ldots, 0, v_{r,l}) \, , \quad j = \rho_r v_r = \rho_l v_l > 0 \, .$$

The Mach number on both sides of the shock front is defined by

$$M_{r,l} := \frac{v_{r,l}}{c_{r,l}} \quad \text{with} \quad c_{r,l} := \sqrt{p'(\rho_{r,l})} \, .$$

As usual, c denotes the sound speed in the fluid. One easily checks that $\mathbf{u}$ is a 1-shock if and only if

$$M_l > 1 \, , \quad M_r < 1 \, , \quad \rho_r > \rho_l \, .$$

In such a case, the following is proved in [25]:

Proposition 4.1 (Majda). [25]. *The shock* $\mathbf{u}$ *is uniformly stable if and only if*

$$M_r^2 \left(\frac{\rho_r}{\rho_l} - 1 \right) < 1 \, .$$

Otherwise, the shock is only weakly linearly stable.

When p is not a convex function of ρ, one may have

$$M_r^2 \left(\frac{\rho_r}{\rho_l} - 1 \right) > 1 \, . \tag{79}$$

In this case, the following is proved in [13]:

Proposition 4.2. [13]. *There exists $V_1 > 0$ such that for all $(\tau, \eta) \in \mathbb{C} \times \mathbb{R}^{d-1}$ satisfying* Re $\tau \geq 0$, *$(\tau, \eta) \neq (0,0)$ and $\tau \neq \pm i V_1 |\eta|$, one has*

$$\left\{(Z, \chi) \in \mathcal{E}^-(\tau, \eta) \times \mathbb{C} \text{ s.t. } \chi\, b(\tau, \eta) + M\, Z = 0\right\} = \{0\},$$

and for $\eta \neq 0$, the set

$$\left\{(Z, \chi) \in \mathcal{E}^-(\pm i V_1 |\eta|, \eta) \times \mathbb{C} \text{ s.t. } \chi\, b(\pm V_1|\eta|, \eta) + M\, Z = 0\right\}$$

is a one dimensional subspace of $\mathbb{C}^{2d+3}$.

By definition, V_1^2 is the smallest root of the polynomial

$$P_1(X) = (c_r^2 - u_r^2)(X^2 + u_r^2 u_l^2) + \left[4 u_r^2 c_r^2 - 2 u_r u_l (c_r^2 + u_r^2)\right] X\,,$$

that has two distinct positive roots (the greatest is denoted V_2^2). Furthermore we have

$$c_r^2 - u_r^2 < V_1^2 < u_r u_l \frac{c_r^2 - u_r^2}{c_r^2 + u_r^2} < V_2^2\,.$$

We recall a few results of [13]. The eigenvalues ω_j's of $\mathcal{A}(\mathbf{u}, \tau, \eta)$ are $\omega_2^r := -\tau/u_r$, $\omega_2^l := \tau/u_l$ and the roots $\omega_{1,3}^r$, $\omega_{1,3}^l$ of the polynomial equations:

$$(\tau + u_r \omega)^2 = c_r^2 (\omega^2 - |\eta|^2)\,, \tag{80a}$$

$$(\tau - u_l \omega)^2 = c_l^2 (\omega^2 - |\eta|^2)\,. \tag{80b}$$

We choose ω_3^r as the root of negative real part of (80b) when Re $\tau > 0$. We also define $a_3^r := \tau u_r - (c_r^2 - u_r^2)\omega_3^r$.

It is proved in [13] that assumptions 4 and 5 hold as long as (79) is satisfied. Moreover, in a suitable neighbourhood of the critical frequencies $\Sigma_{cr}(\mathbf{u})$, there exists a C^∞ basis (that we write under the form of a rectangle matrix Q_{in}) of the stable subspace $\mathcal{E}^-$ such that

$$\beta(\tau, \eta)\, Q_{in} = \begin{pmatrix} \rho_r (c_r^2 \tau + u_r a_3^r) & 2ij\eta^t \\ ij\eta (c_r^2 \tau + u_l a_3^r) & -\rho_r (\tau^2 I_{d-1} + u_r u_l \eta \otimes \eta) \end{pmatrix}. \tag{81}$$

Simple calculations on matrices will show that assumption 8 holds. Let us first look at the 2-dimensional case: η is a real number and βq_{in} is a 2×2 matrix. For all complex numbers $\xi_1, \xi_2, \xi_3, \xi_4, \xi_5$ such that $\xi_4 \neq 0$ and $\xi_5 \neq 0$, the identity

$$\begin{pmatrix} 1/\xi_5 & -\xi_2/(\xi_4 \xi_5) \\ 0 & 1/\xi_4 \end{pmatrix} \begin{pmatrix} \xi_1 & \xi_2 \\ \xi_3 & \xi_4 \end{pmatrix} \begin{pmatrix} \xi_4 & 0 \\ -\xi_3 & 1 \end{pmatrix} = \begin{pmatrix} (\xi_1 \xi_4 - \xi_2 \xi_3)/\xi_5 & 0 \\ 0 & 1 \end{pmatrix} \tag{82}$$

is a straightforward calculation. If we write

$$\beta(\tau, \eta)\, Q_{in} = \begin{pmatrix} \xi_1 & \xi_2 \\ \xi_3 & \xi_4 \end{pmatrix},$$

we can easily check that ξ_4 does not vanich in the neighbourhood of the critical frequencies. In the neighbourhood of $(iV_1|\eta|, \eta) \in \Sigma_+$, the determinant of $\beta(\tau, \eta)\, Q_{in}$ reads

$$\xi_1 \xi_4 - \xi_2 \xi_3 = (\tau - i V_1 |\eta|)\, h(\tau, \eta) \quad \text{or} \quad \det(\beta(\tau, \eta)\, Q_{in}) = (\tau + i V_1 |\eta|)\, h(\tau, \eta)$$

for a suitable C^∞ function h that does not vanish [13]. Setting $\xi_5 := h(\tau, \eta)$, we obtain two regular matrices $P_1(\tau, \eta)$ and $P_2(\tau, \eta)$ such that

$$P_1(\tau, \eta)\, \beta(\tau, \eta)\, Q_{in}\, P_2(\tau, \eta) = \begin{pmatrix} \tau - i V_1 |\eta| & 0 \\ 0 & 1 \end{pmatrix}.$$

This is nothing but assumption 8 since the set of critical frequencies is precisely defined by $\tau = iV_1|\eta|$ or $\tau = -iV_1|\eta|$. In this case, the set of critical frequencies $\Sigma_{cr}(\mathbf{u})$ has exactly four connected components and we have a real equation of each of them.

In space dimension 3, the computations are similar. Observe that the matrix $\tau^2 I_{d-1} + u_r u_l \eta \otimes \eta$ is regular (near the critical frequencies) according to proposition 4.2. Hence βQ_{in} reads

$$\beta\, Q_{in} := \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 \\ \xi_4 & \xi_5 & \xi_6 \\ \xi_7 & \xi_8 & \xi_9 \end{pmatrix} = \begin{pmatrix} \xi_1 & \ell \\ * & \Xi \end{pmatrix} ,$$

where $\Xi := (\tau^2 I_{d-1} + u_r u_l \eta \otimes \eta)$. Therefore, one has $\det \Xi \neq 0$ near the critical frequencies. We check the identity

$$\begin{pmatrix} 1 & -\ell\, \Xi^{-1} \\ 0 & \Xi^{-1} \end{pmatrix} \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 \\ \xi_4 & \xi_5 & \xi_6 \\ \xi_7 & \xi_8 & \xi_9 \end{pmatrix} \begin{pmatrix} \xi_5\xi_9 - \xi_6\xi_8 & 0 & 0 \\ \xi_6\xi_7 - \xi_4\xi_9 & 1 & 0 \\ \xi_4\xi_8 - \xi_5\xi_7 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \det(\beta\, Q_{in}) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

and we can conclude that assumption 8 is also satisfied in the 3-dimensional case.

A Paradifferential calculus with a parameter

In this appendix, we collect the main results of the paradifferential calculus of Bony and Meyer [11, 29] that we use in this paper. The introduction of a positive parameter was achieved by Mokrane [30], see also [28]. We refer to these papers for the proofs of the results stated below. We first recall the classification of paradifferential symbols:

Definition 2. *A paradifferential symbol of degree $m \in \mathbb{R}$ and regularity k ($k = 0$ or $k = 1$) is a function $a(x,\xi,\gamma) : \mathbb{R}^d \times \mathbb{R}^d \times [0,+\infty[\to \mathbb{C}^{N\times N}$ such that a is C^∞ smooth with respect to ξ and for all $\alpha \in \mathbb{N}^d$, there exists a constant C_α verfying*

$$\forall\,(\xi,\gamma)\,,\quad \|\partial_\xi^\alpha a(\cdot,\xi,\gamma)\|_{W^{k,\infty}} \leq C_\alpha\,\lambda^{m-|\alpha|,\gamma}(\xi) = C_\alpha\,(\gamma^2+|\xi|^2)^{(m-|\alpha|)/2}\,.$$

The set of paradifferential symbols of degree m and regularity k is denoted by Γ_k^m. We denote by Σ_k^m the set of paradifferential symbols a such that for a suitable $\epsilon \in]0,1[$ one has

$$\forall\,(\xi,\gamma)\,,\quad \mathrm{Supp}\ \mathcal{F}_x\,a(\cdot,\xi,\gamma) \subset \{\eta \in \mathbb{R}^d / |\eta| \leq \epsilon\,(\gamma^2+|\xi|^2)^{1/2}\}\,.$$

Of course, the symbols in Σ_k^m are C^∞ functions with respect to both variables x and ξ, and for all $a \in \Sigma_k^m$, we have the estimates

$$\forall\,(x,\xi,\gamma)\,,\quad |\partial_x^\beta\partial_\xi^\alpha \sigma\,(x,\xi,\gamma)| \leq C_{\alpha,\beta}\,\lambda^{m-|\alpha|+|\beta|,\gamma}(\xi)\,.$$

Thus any symbol $a \in \Sigma_k^m$ belongs to Hörmander's class $S_{1,1}^m$ [20] and defines an operator $P^\gamma(a)$ on the Schwartz' class $\mathcal{S}$ by the usual formula

$$\forall\,u \in \mathcal{S}\,,\quad P^\gamma(a)u(x) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}\,a(x,\xi,\gamma)\,\widehat{u}(\xi)\,d\xi\,.$$

We shall use the following terminology:

Definition 3. *A family of operators $\{P^\gamma\}$ defined for $\gamma \geq 1$ will be said of order $\leq m$ ($m \in \mathbb{R}$) if the operators P^γ are uniformly bounded from H_γ^{s+m} to H_γ^s:*

$$\forall\,\gamma \geq 1\,,\quad \forall\,u \in H_\gamma^{s+m}\,,\quad \|P^\gamma u\|_{s,\gamma} \leq C(s,m)\,\|u\|_{s+m,\gamma}\,.$$

The following theorem is crucial for the sequel of the analysis:

Theorem A.1. *If $a \in \Sigma_k^m$, the family $\{P^\gamma(a)\}$ is of order $\leq m$.*

The regularization of symbols in the class Γ_k^m is achieved by a convolution with admissible cut-off functions:

Definition 4. *Let $\psi : \mathbb{R}^d \times \mathbb{R}^d \times [1,+\infty[\to [0,+\infty[$ be a C^∞ function such that the following estimates hold for all $\alpha,\beta \in \mathbb{N}^d$:*

$$\forall\,(\eta,\xi,\gamma),\quad |\partial_\eta^\alpha\partial_\xi^\beta \psi\,(\eta,\xi,\gamma)| \leq C_{\alpha,\beta}\,\lambda^{-|\alpha|-|\beta|,\gamma}(\xi)\,.$$

We shall say that ψ is an admissible cut-off function if there exist real numbers $0 < \epsilon_1 < \epsilon_2 < 1$ satisfying

$$\begin{aligned}
\psi(\eta,\xi,\gamma) = 1 \quad &\text{if} \quad |\eta| \leq \epsilon_1(\gamma^2+|\xi|^2)^{1/2}\,,\\
\psi(\eta,\xi,\gamma) = 0 \quad &\text{if} \quad |\eta| \geq \epsilon_2(\gamma^2+|\xi|^2)^{1/2}\,.
\end{aligned}$$

An example of cut-off function is the following: let χ be a nonnegative C^∞ function on $\mathbb{R}^d \times \mathbb{R}$ such that

$$\gamma_1^2 + |\xi_1|^2 \geq \gamma_2^2 + |\xi_2|^2 \Longrightarrow \chi(\xi_1,\gamma_1) \leq \chi(\xi_2,\gamma_2)\,,$$
$$\begin{cases} \chi(\xi,\gamma) = 1 & \text{if } \left(\gamma^2 + |\xi|^2\right)^{1/2} \leq 1/2, \\ \chi(\xi,\gamma) = 0 & \text{if } \left(\gamma^2 + |\xi|^2\right)^{1/2} \geq 1. \end{cases}$$

We define a function $\varphi(\xi,\gamma) := \chi(\xi/2,\gamma/2) - \chi(\xi,\gamma)$. Then the function ψ_0 defined by

$$\psi_0(\eta,\xi,\gamma) := \sum_{p \geq 0} \chi(2^{2-p}\eta, 0)\, \varphi(2^{-p}\xi, 2^{-p}\gamma)$$

is an admissible cut-off function (one can take $\epsilon_1 = 1/16$ and $\epsilon_2 = 1/2$).

If ψ is an admissible cut-off function, the inverse Fourier transform K^ψ of $\psi(\cdot,\xi,\gamma)$ satisfies

$$\forall\,(\xi,\gamma)\,, \quad \|\partial_\xi^\alpha K^\psi(\cdot,\xi,\gamma)\|_{L^1} \leq C_\alpha\, \lambda^{-|\alpha|,\gamma}(\xi)\,.$$

These L^1 bounds on $\partial_\xi^\alpha K^\psi$ enable to establish the following proposition:

Proposition A.1. *Let ψ be an admissible cut-off function. The mapping*

$$a \longmapsto \sigma_a^\psi(x,\xi,\gamma) := \int_{\mathbb{R}^d} K^\psi(x-y,\xi,\gamma)\, a(y,\xi,\gamma)\, dy$$

is continuous from Γ_k^m to Σ_k^m for all m.

If $a \in \Gamma_1^m$, then $a - \sigma_a^\psi \in \Gamma_0^{m-1}$. In particular, if ψ_1 and ψ_2 are two admissible cut-off functions and $a \in \Gamma_1^m$, then $\sigma_a^{\psi_1} - \sigma_a^{\psi_2} \in \Sigma_0^{m-1}$.

Fixing an admissible cut-off function ψ, we define the paradifferential operator $T_a^{\psi,\gamma}$ by the formula

$$T_a^{\psi,\gamma} := P^\gamma(\sigma_a^\psi)\,.$$

If ψ_1 and ψ_2 are two admissible cut-off functions and $a \in \Gamma_1^m$, then proposition A.1 and theorem A.1 show that the family $\{T_a^{\psi_1,\gamma} - T_a^{\psi_2,\gamma}\}$ is of order $\leq (m-1)$.

The symbolic calculus is based on the following theorem:

Theorem A.2. *Let $a \in \Gamma_1^m$ and $b \in \Gamma_1^{m'}$. Then $ab \in \Gamma_1^{m+m'}$ and the family*

$$\{T_a^{\psi,\gamma} \circ T_b^{\psi,\gamma} - T_{ab}^{\psi,\gamma}\}_{\gamma \geq 1}$$

is of order $\leq m + m' - 1$ for all admissible cut-off function ψ.

Let $a \in \Gamma_1^m$. Then for all admissible cut-off function ψ, the family

$$\{(T_a^{\psi,\gamma})^* - T_{a^*}^{\psi,\gamma}\}_{\gamma \geq 1}$$

is of order $\leq m - 1$.

Let $a \in \Gamma_2^m$ and $b \in \Gamma_2^{m'}$. Then $ab \in \Gamma_2^{m+m'}$ and the family

$$\{T_a^{\psi,\gamma} \circ T_b^{\psi,\gamma} - T_{ab}^{\psi,\gamma} - T^{\psi,\gamma}_{-i\sum_j \partial_{\xi_j} a \partial_{x_j} b}\}_{\gamma \geq 1}$$

is of order $\leq m + m' - 2$ for all admissible cut-off function ψ.

Let $a \in \Gamma_2^m$. Then the family

$$\{(T_a^{\psi,\gamma})^* - T_{a^*}^{\psi,\gamma} - T_{-i\sum_j \partial_{\xi_j}\partial_{x_j} a^*}^{\psi,\gamma}\}_{\gamma \geq 1}$$

is of order $\leq m-2$ for all admissible cut-off function ψ.

The next theorem is the parameter version of Gårding's inequality:

Theorem A.3. *Let $a \in \Gamma_1^{2m}$ and ψ be and admissible cut-off function. Assume that there exists a constant $c > 0$ such that*

$$\forall\, (x, \xi, \gamma), \quad \mathrm{Re}\; a(x, \xi, \gamma) \geq c\, \lambda^{2m,\gamma}(\xi)\, Id\,.$$

Then there exists $\gamma_0 \geq 1$ such that

$$\forall\, \gamma \geq \gamma_0\,, \forall\, u \in H_\gamma^m\,, \quad \mathrm{Re}\; \langle T_a^{\psi,\gamma} u, u\rangle_{L^2} \geq \frac{c}{2}\, \|u\|_{m,\gamma}^2\,.$$

We also have a localized version of Gårding's inequality:

Theorem A.4. *Let $a \in \Gamma_1^{2m}$, $\chi \in \Gamma_1^0$ and ψ be and admissible cut-off function. Assume that there exists $\widetilde{\chi} \in \Gamma_1^0$ and a constant $c > 0$ such that $\widetilde{\chi} \geq 0$, $\widetilde{\chi}\,\chi = \chi$ and*

$$\forall\, (x, \xi, \gamma), \quad \widetilde{\chi}^2(x, \xi, \gamma)\, \mathrm{Re}\; a(x, \xi, \gamma) \geq c\, \widetilde{\chi}^2(x, \xi, \gamma)\, \lambda^{2m,\gamma}(\xi)\, I\,.$$

Then there exists $\gamma_0 \geq 1$ and $C > 0$ such that

$$\forall\, \gamma \geq \gamma_0\,, \forall\, u \in H_\gamma^m(\mathbb{R}^d)\,, \quad \mathrm{Re}\; \langle T_a^{\psi,\gamma} T_\chi^{\psi,\gamma} u, T_\chi^{\psi,\gamma} u\rangle_{H^{-m},H^m} \geq \frac{c}{2}\, \|T_\chi^{\psi,\gamma} u\|_{m,\gamma}^2 - C\, \|u\|_{m-1,\gamma}^2\,.$$

We now study the case of paraproducts: they are defined by the particular choice of ψ_0 as cut-off function. We shall write T_a^γ instead of $T_a^{\psi_0,\gamma}$ for the paradifferential operators obtained after smoothing by the function ψ_0. We have the important result:

Theorem A.5. *Let $a \in W^{1,\infty}(\mathbb{R}^d)$, $u \in L^2(\mathbb{R}^d)$ and $\gamma \geq 1$. Then we have*

$$\|a\, u - T_a^\gamma u\|_0 \leq \frac{C}{\gamma}\, \|a\|_{W^{1,\infty}}\, \|u\|_0\,,$$

$$\|a\, \partial_j u - T_a^\gamma(\partial_j u)\|_0 \leq C\, \|a\|_{W^{1,\infty}}\, \|u\|_0\,,$$

for a suitable constant C that is independent of (a, u, γ).

If $a \in W^{2,\infty}(\mathbb{R}^d)$, we have

$$\|a\, u - T_a^\gamma u\|_{1,\gamma} \leq \frac{C}{\gamma}\, \|a\|_{W^{2,\infty}}\, \|u\|_0\,,$$

$$\|a\, \partial_j u - T_a^\gamma(\partial_j u)\|_{1,\gamma} \leq C\, \|a\|_{W^{2,\infty}}\, \|u\|_0\,,$$

for a suitable constant C that is independent of (a, u, γ).

We can extend the paradifferential calculus to symbols defined on a half-space in the following way: we still denote by Γ_k^m the set of symbols $a(x_0, \dots, x_d, \eta, \gamma)$ defined on $\Omega \times (\mathbb{R}^d \times [0, +\infty[\setminus\{0\})$ such that the mapping $x_d \mapsto a(\cdot, x_d, \cdot)$ is bounded into Γ_k^m. We define the paradifferential operator T_a^γ by

$$\forall\, u \in C_c^\infty(\overline{\Omega})\,, \quad \forall\, x_d \geq 0\,, \quad (T_a^\gamma u)(\cdot, x_d) := T_{a(x_d)}^\gamma u(\cdot, x_d)\,.$$

Using theorem A.5 and integrating with respect to x_d, we obtain for all symbol $a \in W^{1,\infty}(\Omega)$ and all $u \in L^2(\Omega)$ the estimates:

$$|\!|\!|a\, u - T_a^\gamma u|\!|\!|_0 \leq \frac{C}{\gamma}\, \|a\|_{W^{1,\infty}(\Omega)}\, |\!|\!|u|\!|\!|_0\,,$$

$$|\!|\!|a\, \partial_j u - T_a^\gamma(\partial_j u)|\!|\!|_0 \leq C\, \|a\|_{W^{1,\infty}(\Omega)}\, |\!|\!|u|\!|\!|_0\,, \quad 0 \leq j \leq d-1\,.$$

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