On the Optimization of a Conservation Law

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Abstract

Consider the Cauchy problem for a conservation law and assume that an integral functional on its solution is defined. In this note we obtain an Euler-Lagrange equation for the stationary points of this functional. An application to the optimal management of traffic flows is considered.

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1 Introduction

Aim of this note is the study of optimization problems related to conservation laws. More precisely, we consider the Cauchy problem

$$\left\{\begin{array}{rcl}\partial_t u+\partial_x\left[f(u)\right] & = & 0\\ u(0,x) & = & u_o(x)\end{array}\right. \qquad (t,x)\in[0,+\infty[\,\times\mathbb{R}\,, \tag{1.1}$$

where the flow $f:\mathbb{R}\mapsto\mathbb{R}$ is smooth while u_o is in $\mathbf{L^1}(\mathbb{R})\cap\mathbf{BV}(\mathbb{R})$. Denote by $u(t,x)=\mathcal{S}_t(u_o)(x)$ the Kružkhov [6] solution to (1.1), $\mathcal{S}$ being the $\mathbf{L^1}$ contractive solution operator, see also [3, 5] as general references. For a fixed Lipschitzean $\psi:\mathbb{R}\mapsto\mathbb{R}$, consider an integral functional $J:\mathbf{L^1}(\mathbb{R})\cap\mathbf{BV}(\mathbb{R})\to\mathbb{R}$ of the form

$$J(u_o)=\int_a^b\int_{-\infty}^{+\infty}\psi\left(\mathcal{S}_t u_o(x)\right)\,dx\,dt \qquad\text{or}\qquad J_\varphi(u_o)=\int_{\mathbb{R}}\int_{-\infty}^{+\infty}\varphi(t,x)\,\psi\big(\mathcal{S}_t u_o(x)\big)\,dx\,dt\,,$$

with $0\leqslant a<b<+\infty$ and $\varphi\in\mathbf{C}^1_c([a,b]\times\mathbb{R})$ being a suitable weight. In this note we obtain an Euler-Lagrange equation satisfied from the stationary points of this functional.

A major difficulty in the optimization of J lies in the fact that $\mathcal{S}$ is generally *not* differentiable with respect to the usual linear differential structure of $\mathbf{L^1}$, see [1, 4] for explicit examples. Therefore, a new differential structure on $\mathbf{L^1}\cap\mathbf{BV}$ was recently introduced and studied in [1, 4]: *shift differentiability.* Within this framework, $\mathcal{S}$ turns out to be differentiable at all times at which neither interactions between shocks nor centers of compression waves occur. According to this differential structure, the tangent space at u to $\mathbf{L^1}\cap\mathbf{BV}$ is identified with $\mathbf{L^1}(Du)$, i.e. the space of absolutely integrable functions with respect to the Radon measure Du, Du being the weak derivative of u. Below we exploit this structure to obtain an equation satisfied by the points of minima and maxima of J.

As a first step, we slightly extend the theory of shift differentiability to comprehend the time dependent case and apply it to $\mathcal{S}$ to obtain the first order term in the *"shift expansion"* of J.

In the case of the management of traffic lights, shift differentiability is particularly suitable. Indeed, *shifting* the initial data corresponds to modifying the time that the light shows green or red. Below, we consider the problem of optimizing the flow of cars on a segment of a road. This application is motivated by a real problem in the Lyon (France) area [7], where the inflow on a mountain road needs to be regulated to prevent high car densities in areas dangerous due to landslides.

The Euler-Lagrange equation is derived in the next section. Then, Section 3 deals with the application to traffic flow management.

2 Main Result

Let $X=\mathbf{L^1}(\mathbb{R})\cap\mathbf{BV}(\mathbb{R})$ with the Lebesgue measure (wherever the Lebesgue measure is used, it is not explicitly mentioned). An abstract approach to shift differentiability referred to an operator $\Phi:[0,T]\times X\mapsto X$ (for some $T>0$) is possible. Nevertheless, we prefer here to avoid formal complications and stick to the most interesting case $\Phi=\mathcal{S}$ related to (1.1), although the results below can be stated in more general frameworks.

The next lemmas are slight extensions of analogous results in [4], to which we refer for further details. Fix an initial datum $u\in X$ and set $T_uX=\mathbf{L^1}(|Du|)$. Here, Du denotes the Radon measure defined as the distributional space derivative of u and $|Du|$ stands for the total variation of Du, see [10, Chapter 2]. A vector field along the curve $t\mapsto\mathcal{S}_t u$, for $t\in[0,T]$, is a map $v\colon [0,T] \times \mathbb{R} \mapsto \mathbb{R}$ such that for all $t \in [0,T]$, $v(t,\cdot) \in T_{\mathcal{S}_t u} X$. Aiming at a differential structure on X, we start introducing a relation between families of curves and vector fields along the orbits of $\mathcal{S}$.

Definition 2.1 *Fix $u \in X$ and a vector field v along $t \mapsto \mathcal{S}_t u$. A family v^ϑ, ($\vartheta \in [0,\vartheta^*]$) of curves along $t \mapsto \mathcal{S}_t u$* hat-converges *to v, $v^\vartheta \stackrel{\wedge}{\longrightarrow} v$, if for a.e. $t \in [0,T]$*

1. $\displaystyle\lim_{\vartheta\to 0} \left\| v^\vartheta(t,\cdot) - v(t,\cdot) \right\|_{\mathbf{L}^1(|D\mathcal{S}_t u|)} = 0;$
2. $\displaystyle\lim_{\vartheta\to 0} \left\| \vartheta v^\vartheta(t,\cdot) \right\|_{\mathbf{L}^\infty(\mathbb{R})} = 0;$
3. $\displaystyle\limsup_{\vartheta\to 0} Lip\left(\vartheta v^\vartheta(t,\cdot)\right) < 1.$

The next lemma is technically very useful: it ensures that in the sequel it is possible to work with families of curves v^ϑ that have more regularity than that required above.

Lemma 2.2 *Fix $u \in X$ and a vector field v along $t \mapsto \mathcal{S}_t u$ such that*

$$\left\| v(t,\cdot) \right\|_{\mathbf{L}^1(|D\mathcal{S}_t u|)} \leqslant \eta(t) \quad \text{for a.e. } t \in [0,T] \tag{2.1}$$

with $\eta \in \mathbf{L}^1([0,T])$. Then, there exists a family v^ϑ of curves along $t \mapsto \mathcal{S}_t u$ such that

(a) $v^\vartheta \stackrel{\wedge}{\longrightarrow} v$;

(b) $v^\vartheta(t,\cdot) \in \mathbf{C}_c^\infty(\mathbb{R})$ *for* $t \in [0,T]$;

(c) $\left\| v^\vartheta(t,\cdot) \right\|_{\mathbf{L}^1(|D\mathcal{S}_t u|)} \leqslant \eta(t)$ *for all* ϑ.

Proof. For any fixed $t \in [0,T]$, there exists a sequence $w_n(t,\cdot) \in \mathbf{C}_c^\infty(\mathbb{R})$ such that

$$\lim_{n\to+\infty} \left\| w_n(t,\cdot) - v(t,\cdot) \right\|_{\mathbf{L}^1(|D\mathcal{S}_t u|)} = 0, \tag{2.2}$$

$$\left\| w_n(t,\cdot) \right\|_{\mathbf{L}^1(|D\mathcal{S}_t u|)} \leqslant \left\| v(t,\cdot) \right\|_{\mathbf{L}^1(|D\mathcal{S}_t u|)} \text{ for all } n. \tag{2.3}$$

As in [4, Lemma 1], define $\zeta_n(t) = \left(n + \max_{i=1,\ldots,n} \left\{ Lip(w_i(t,\cdot)),\ \left\| w_i(t,\cdot) \right\|_{\mathbf{L}^\infty} \right\}\right)^{-2}$ and

$$v^\vartheta(t,x) = \begin{cases} \dfrac{\vartheta - \zeta_{n+1}(t)}{\zeta_n(t) - \zeta_{n+1}(t)} w_{n-1}(t,x) + \dfrac{\zeta_n(t) - \vartheta}{\zeta_n(t) - \zeta_{n+1}(t)} w_n(t,x) & \text{if } \vartheta \in \left]\zeta_{n+1}(t), \zeta_n(t)\right], \\ v(t,x) & \text{if } \vartheta = 0. \end{cases}$$

v^ϑ satisfies the requirements **(a)**, **(b)** and **(c)**. Indeed, **(a)** holds since for $\vartheta \in \left]\zeta_{n+1}(t), \zeta_n(t)\right]$, referring to Definition 2.1, we have

1. $\left\| v^\vartheta(t,\cdot) - v(t,\cdot) \right\|_{\mathbf{L}^1(|D\mathcal{S}_t u|)} \leqslant \left\| v(t,\cdot) - w_{n-1}(t,\cdot) \right\|_{\mathbf{L}^1(|D\mathcal{S}_t u|)} + \left\| v(t,\cdot) - w_n(t,\cdot) \right\|_{\mathbf{L}^1(|D\mathcal{S}_t u|)}$ and both addendum tend to 0 due to (2.2);
2. $\displaystyle\left| v^\vartheta(t,\cdot) \right| \leqslant \max_{i=n-1,n} \left| w_i(t,\cdot) \right| \leqslant \frac{1}{\sqrt{\zeta_n(t)}} \leqslant \frac{1}{\sqrt{\vartheta}};$
3. $\displaystyle\left| v^\vartheta(t,x) - v^\vartheta(t,y) \right| \leqslant \max_{i=n-1,n} \left\{ Lip(w_i(t,\cdot)) \right\} |x-y| \leqslant \frac{1}{\sqrt{\zeta_n(t)}} |x-y| \leqslant \frac{1}{\sqrt{\vartheta}} |x-y|.$

(b) is immediate, while **(c)** follows from (2.3) and (2.1).

Using the above Definition 2.1, we now introduce a special class of curves in X.

Definition 2.3 *Fix* $u \in X$ *and a family of curves* v^ϑ *along* $t \mapsto \mathcal{S}_t u$ *with* $Lip(\vartheta v^\vartheta(t,\cdot)) \leqslant \alpha < 1$. $v^\vartheta \star u$ *is the function implicitly defined by*

$$\left(v^\vartheta \star u\right)(t, x + \vartheta v^\vartheta(t,x)) = \mathcal{S}_t u(x)\,.$$

The curve $\vartheta \mapsto (v^\vartheta \star u)(t,\cdot)$ is indeed a continuous curve exiting $\mathcal{S}_t u$. In fact, for all $u \in X$ and all vector fields v along $t \mapsto \mathcal{S}_t u$, if v^ϑ is a family of curves along $t \mapsto \mathcal{S}_t u$ such that $v^\vartheta \stackrel{\wedge}{\longrightarrow} v$, then

$$\lim_{\vartheta \to 0} \left\| (v^\vartheta \star u)(t,\cdot) - \mathcal{S}_t u \right\|_{\mathbf{L^1}} = 0$$

as will directly follow from Proposition 2.5 below.

The next definition essentially induces an equivalence relation on the curves in X, classifying as equivalent those curves that generate the same vector field, see [2].

Definition 2.4 *Fix* $u \in X$ *and consider a family of curves* $\vartheta \mapsto u^\vartheta$ *along* $t \mapsto \mathcal{S}_t u$, *for* $\vartheta \in [0, \vartheta_*]$. *We say that* u^ϑ generates *the vector field* v *along* $t \mapsto \mathcal{S}_t u$ *if there exists* v^ϑ *hat-converging to* v *such that*

$$\lim_{\vartheta \to 0} \frac{1}{\vartheta} \left\| u^\vartheta(t) - (v^\vartheta \star u)(t) \right\|_{\mathbf{L^1}(|D\mathcal{S}_t u|)} = 0 \qquad \textit{for all } t \in [0,T].$$

We need to prove that the vector field generated by $\vartheta \mapsto u^\vartheta$, if it exists, is uniquely determined in the space $T_u X$ and that the previous definition does not depend on the choice of v^ϑ. For any $t \in [0,T]$ both these facts are ensured by [4, Theorem 1], which we state here for completeness together with a bound proved therein that will be of use below.

Proposition 2.5 *If* $u \in X$, v, w *are vector fields along* $t \mapsto \mathcal{S}_t u$ *and* $v^\vartheta \stackrel{\wedge}{\longrightarrow} v$, $w^\vartheta \stackrel{\wedge}{\longrightarrow} w$, *then for all* $t \in [0,T]$

$$\begin{aligned} \lim_{\vartheta \to 0} \frac{1}{\vartheta} \left\| (v^\vartheta \star u)(t,\cdot) - (w^\vartheta \star u)(t,\cdot) \right\|_{\mathbf{L^1}} &= \left\| v(t,\cdot) - w(t,\cdot) \right\|_{\mathbf{L^1}(|D\mathcal{S}_t u|)}, \\ \frac{1}{\vartheta} \int_{\mathbb{R}} \left| (v^\vartheta \star u)(t,x) - (w^\vartheta \star u)(t,x) \right| dx &\leqslant \int_{\mathbb{R}} \left| v^\vartheta(t,x) - w^\vartheta(t,x) \right| |D\mathcal{S}_t u|. \end{aligned} \tag{2.4}$$

Lemma 2.6 *Let* $\varphi \in \mathbf{C}^1_c([a,b] \times \mathbb{R})$. *If* $u \in X$, v *is a vector field along* $t \mapsto \mathcal{S}_t u$ *satisfying* (2.1) *and* $v^\vartheta \stackrel{\wedge}{\longrightarrow} v$, *then*

$$\lim_{\vartheta \to 0} \frac{1}{\vartheta} \int_a^b \int_{\mathbb{R}} \varphi(t,x) \left[(v^\vartheta \star u)(t,x) - \mathcal{S}_t u(x) \right] dx\, dt = - \int_a^b \int_{\mathbb{R}} \varphi(t,x)\, v(t,x)\, D\mathcal{S}_t u\, dt\,.$$

Proof. By [4, Corollary 1], for all $t \in [0,T]$

$$\lim_{\vartheta \to 0} \frac{1}{\vartheta} \int_{\mathbb{R}} \varphi(t,x) \left[(v^\vartheta \star u)(t,x) - \mathcal{S}_t u(x) \right] dx = - \int_{\mathbb{R}} \varphi(t,x)\, v(t,x)\, D\mathcal{S}_t u\,.$$

By (2.4) and Lemma 2.2, an application of the Lebesgue Dominated Convergence Theorem completes the proof.

An entirely similar procedure leads to prove the following lemma.

Lemma 2.7 *If $u \in X$, $v \in T_u X$ and $v^\vartheta \overset{\wedge}{\dashrightarrow} v$ satisfies* (2.1), *then*

$$\lim_{\vartheta \to 0} \frac{1}{\vartheta} \int_a^b \int_{\mathbb{R}} \Big[(v^\vartheta \star u)(t,x) - \mathcal{S}_t u(x) \Big] \, dx \, dt = - \int_a^b \int_{\mathbb{R}} v(t,x) \, D\mathcal{S}_t u \, dt \, .$$

Definition 2.8 *Let $\mathcal{S}$ be the solution operator generated by* (1.1). *Fix $t \in [0,T]$, the map $\mathcal{S}_t : X \mapsto X$ is* shift differentiable *at the point $u \in X$, if there exists a continuous linear map $\Lambda_{t,u} : T_u X \mapsto T_{\mathcal{S}_t u} X$ such that the following holds. For all $v \in T_u X$ there exist $v^\vartheta \overset{\wedge}{\dashrightarrow} v$ and $w^\vartheta \overset{\wedge}{\dashrightarrow} \Lambda_{t,u} v$ satisfying*

$$\lim_{\vartheta \to 0} \frac{1}{\vartheta} \Big\| (w^\vartheta \star \mathcal{S}_t u) - \mathcal{S}_t (v^\vartheta \star u) \Big\|_{\mathbf{L^1}} = 0 \, .$$

The shift differentiability of $\mathcal{S}$ at any time at which neither shocks interact nor compression waves center was proved in [4]. Here we add a bound on the operator norm of the shift differential. From the standard theory of scalar conservation laws [3, 5], it is well known that $\mathcal{S}$ is Lipschitzean in $\mathbf{L^1}$ with constant 1. Hence we expect also the norm of the shift differential to be bounded by 1. This is the content of the next theorem.

Theorem 2.9 *Fix a and b so that $0 \leqslant a < b < +\infty$. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be $\mathbf{C^2}$ and strictly convex. Call $\mathcal{S}$ the semigroup generated by* (1.1). *Then, the map $\mathcal{S}_t$ is shift differentiable at all u in X such that no interaction between shocks in $\mathcal{S}_t u$ takes place at time t. Moreover, the operator norm of the shift differential $\Lambda_{t,u}$, where defined, satisfies*

$$\big\| \Lambda_{t,u} \big\| \leqslant 1 \, . \tag{2.5}$$

The bound (2.5) is obviously optimal, for equality holds in the case of a Riemann problem solved by a single shock.

Proof. The first assertion follows from [4, Theorem 4]. Consider (2.5). Let $v, w \in T_u X$ and $v^\vartheta \overset{\wedge}{\dashrightarrow} v$, $w^\vartheta \overset{\wedge}{\dashrightarrow} w$. With a slight abuse of notation, introduce two families of curves such that $(\Lambda_{t,u} v)^\vartheta \overset{\wedge}{\dashrightarrow} \Lambda_{t,u} v$ and $(\Lambda_{t,u} w)^\vartheta \overset{\wedge}{\dashrightarrow} \Lambda_{t,u} w$. Recalling that the semigroup $\mathcal{S}_t$ is contractive in the $\mathbf{L^1}(\mathbb{R})$ norm, by Proposition 2.5 and Definition 2.8,

$$\begin{aligned}
\big\| \Lambda_{t,u} v - \Lambda_{t,u} w \big\|_{\mathbf{L^1}(|D\mathcal{S}_t u|)} &= \lim_{\vartheta \to 0} \frac{1}{\vartheta} \left\| \Big((\Lambda_{t,u} v)^\vartheta \star \mathcal{S}_t u \Big) - \Big((\Lambda_{t,u} w)^\vartheta \star \mathcal{S}_t u \Big) \right\|_{\mathbf{L^1}} \\
&\leqslant \lim_{\vartheta \to 0} \frac{1}{\vartheta} \left\| \Big((\Lambda_{t,u} v)^\vartheta \star \mathcal{S}_t u \Big) - \mathcal{S}_t (v^\vartheta \star u) \right\|_{\mathbf{L^1}} \\
&\quad + \lim_{\vartheta \to 0} \frac{1}{\vartheta} \Big\| \mathcal{S}_t (v^\vartheta \star u) - \mathcal{S}_t (w^\vartheta \star u) \Big\|_{\mathbf{L^1}} \\
&\quad + \lim_{\vartheta \to 0} \frac{1}{\vartheta} \left\| \mathcal{S}_t (w^\vartheta \star u) - \Big((\Lambda_{t,u} w)^\vartheta \star \mathcal{S}_t u \Big) \right\|_{\mathbf{L^1}} \\
&\leqslant \lim_{\vartheta \to 0} \frac{1}{\vartheta} \Big\| (v^\vartheta \star u) - (w^\vartheta \star u) \Big\|_{\mathbf{L^1}} \\
&= \| v - w \|_{\mathbf{L^1}(|Du|)} \, .
\end{aligned}$$

As was to be expected, in the previous proof we used only the Lipschitzeanity of $\mathcal{S}_t$ and Definition 2.8, but not the explicit expression $\Lambda_{t,u}$, which we recall here from [4] for later use.

$$\Lambda_{t,u} v(x) = \begin{cases} v\big(\xi^+(u;x)\big) & \text{if } \mathcal{S}_t u(x-) = \mathcal{S}_t u(x+), \\ \Psi_{t,u}(x) & \text{if } \mathcal{S}_t u(x-) \neq \mathcal{S}_t u(x+), \end{cases}$$

where $\mathcal{S}_t u(x\pm) = \lim_{\xi\to x\pm} \mathcal{S}_t u(\xi)$, and

$$\begin{aligned}\Psi_{t,u}(x) = \frac{1}{\mathcal{S}_t u(x+) - \mathcal{S}_t u(x-)} \Bigg\{ & \int_{]\xi^-(u;t,x),\xi^+(u;t,x)[} v(\zeta) Du(\zeta) \\ & + v\left(\xi^+(u;t,x)\right)\left[u\left(\xi^-(u;t,x)+\right) - \mathcal{S}_t u(x-)\right] \\ & + v\left(\xi^+(u;t,x)\right)\left[\mathcal{S}_t u(x+) - u\left(\xi^+(u;t,x)-\right)\right]\Bigg\}.\end{aligned}$$

The points $\xi^+(u;t,x)$ and $\xi^-(u;t,x)$ are respectively the intersections with the x-axis of the maximal and minimal generalized backward characteristics [5, Chapter X] from the point (t,x) in the solution $u(t,x) = \mathcal{S}_t u(x)$.

We are now ready to state the main result of this note.

Theorem 2.10 *Let $f:\mathbb{R}\mapsto\mathbb{R}$ be $\mathbf{C^2}$ and strictly convex, $\psi:\mathbb{R}\mapsto\mathbb{R}$ be locally Lipschitzean and $\varphi:[a,b]\times\mathbb{R}\mapsto\mathbb{R}$ be $\mathbf{C}_c^{\mathbf{1}}$, with $0\leqslant a<b<+\infty$. Consider the functionals $J, J_\varphi: X\to\mathbb{R}$ defined by*

$$J(u) = \int_a^b \int_{\mathbb{R}} \psi\big(\mathcal{S}_t u(x)\big)\, dx\, dt \qquad \text{and} \qquad J_\varphi(u) = \int_a^b \int_{\mathbb{R}} \varphi(t,x)\ \psi\big(\mathcal{S}_t u(x)\big)\, dx\, dt \tag{2.6}$$

$\mathcal{S}_t$ being the semigroup generated by (1.1). Then, for all $v\in\mathbf{L^1}(|Du|)$ and for any family of curves v^ϑ such that $v^\vartheta \overset{\wedge}{\longrightarrow} v$, the following first order "shift-expansions" hold:

$$\begin{aligned}\lim_{\vartheta\to 0}\frac{1}{\vartheta}\Big[J(v^\vartheta\star u) - J(u)\Big] &= -\int_a^b\int_{\mathbb{R}} (\Lambda_{t,u}v)(x)\ D\left(\psi(\mathcal{S}_t u)\right) dt\,, \\ \lim_{\vartheta\to 0}\frac{1}{\vartheta}\Big[J_\varphi(v^\vartheta\star u) - J_\varphi(u)\Big] &= -\int_a^b\int_{\mathbb{R}} \varphi(t,x)\ (\Lambda_{t,u}v)(x)\ D\left(\psi(\mathcal{S}_t u)\right) dt\,.\end{aligned}$$

Above, by "ψ locally Lipschitzean" we mean that on every compact $K\subset\mathbb{R}$, the restriction of ψ to K is Lipschitzean. Note that the integral on the right hand side is well defined. In fact, the map $x\mapsto\psi\left(\mathcal{S}_t u(x)\right)$ is of bounded variation and, due to the local Lipschitzeanity of ψ, $\mathbf{L^1}(|D\mathcal{S}_t u|)\subseteq\mathbf{L^1}\left(\left|D\psi(\mathcal{S}_t u)\right|\right)$.

Proof. We consider the functional J_φ, since the case of J is entirely similar.

Throughout this proof, t is a time at which neither shocks interact nor compression waves center in $\mathcal{S}_t u$. Recall that the set of times at which two or more shocks collide is at most a countable, see [4, Corollary 2].

First, using Lemma 2.2, we introduce a family of curves $w^\vartheta\in\mathbf{C}_c^\infty$ such that $w^\vartheta\overset{\wedge}{\longrightarrow} v$ and $\left\|w^\vartheta\right\|_{\mathbf{L^1}(|Du|)}\leqslant\|v\|_{\mathbf{L^1}(|Du|)}$. Moreover, in view of (2.5), we have $\left\|\Lambda_{t,u}v\right\|_{\mathbf{L^1}(|D\mathcal{S}_t u|)}\leqslant\|v\|_{\mathbf{L^1}(|Du|)}$. Apply Lemma 2.2 and introduce a family of curves $(\Lambda_{t,u}w)^\vartheta$ such that $(\Lambda_{t,u}w)^\vartheta\overset{\wedge}{\longrightarrow}\Lambda_{t,u}v$ and

$$\left\|(\Lambda_{t,u}w)^\vartheta\right\|_{\mathbf{L^1}(|D\mathcal{S}_t u|)}\leqslant\|v\|_{\mathbf{L^1}(|Du|)}\,. \tag{2.7}$$

We have

$$\begin{aligned}\frac{1}{\vartheta}\big[J(v^\vartheta\star u) \quad - \quad & J(u)\big] \\ = \quad & \frac{1}{\vartheta}\int_a^b\int_{\mathbb{R}}\varphi(t,x)\left[\psi\left(\mathcal{S}_t(v^\vartheta\star u)\right) - \psi(\mathcal{S}_t u)\right] dx\, dt \\ = \quad & \frac{1}{\vartheta}\int_a^b\int_{\mathbb{R}}\varphi(t,x)\left[\psi\left(\mathcal{S}_t(v^\vartheta\star u)\right) - \psi\left(\mathcal{S}_t(w^\vartheta\star u)\right)\right] dx\, dt\end{aligned} \tag{2.8}$$

$$+\frac{1}{\vartheta}\int_a^b\int_{\mathbb{R}}\varphi(t,x)\left[\psi\left(\mathcal{S}_t(w^\vartheta\star u)\right)-\psi\left((\Lambda_{t,u}w)^\vartheta\star\mathcal{S}_t u\right)\right]\,dx\,dt \tag{2.9}$$

$$+\frac{1}{\vartheta}\int_a^b\int_{\mathbb{R}}\varphi(t,x)\left[\psi\left((\Lambda_{t,u}w)^\vartheta\star\mathcal{S}_t u\right)-(\Lambda_{t,u}w)^\vartheta\star\psi(\mathcal{S}_t u)\right]\,dx\,dt \tag{2.10}$$

$$+\frac{1}{\vartheta}\int_a^b\int_{\mathbb{R}}\varphi(t,x)\left[(\Lambda_{t,u}w)^\vartheta\star\psi(\mathcal{S}_t u)-\psi(\mathcal{S}_t u)\right]\,dx\,dt\,. \tag{2.11}$$

First we prove that (2.8) tends to 0 as $\vartheta\to 0$. By the boundedness of φ and the local Lipschitzeanity of ψ, it is sufficient to prove that

$$\lim_{\vartheta\to 0}\frac{1}{\vartheta}\int_a^b\int_{\mathbb{R}}\left[(v^\vartheta\star u_o)-(w^\vartheta\star u_o)\right]\,dx\,dt=0\,.$$

The above follows from an application of the Lebesgue Theorem, which is possible thanks to Proposition 2.5 and (2.7).

Consider (2.9). The shift differentiability of $\mathcal{S}_t$ at t implies that

$$\lim_{\vartheta\to 0}\frac{1}{\vartheta}\int_{\mathbb{R}}\left[\left(\mathcal{S}_t(w^\vartheta\star u_o)\right)-\left((\Lambda_{t,u}w)^\vartheta\star\mathcal{S}_t u_o\right)\right]\,dx=0$$

and, reasoning as in the case above, one gets that (2.9) tends to 0 as $\vartheta\to 0$.

Finally, (2.10) is identically 0, since by Definition 2.3, $\psi\left((\Lambda_{t,u}w)^\vartheta\star\mathcal{S}_t u\right)=(\Lambda_{t,u}w)^\vartheta\star\psi(\mathcal{S}_t u)$.

As $\vartheta\to 0$, the limit of the last term (2.11) is computed by Lemma 2.6.

It is now straightforward to obtain the Euler-Lagrange equation for the functionals (2.6).

Corollary 2.11 *In the same assumptions of Theorem 2.10, if $u\in X$ is a point of extremum of the functional J_φ, resp. J, then the equality*

$$\int_a^b\int_{\mathbb{R}}\varphi(t,x)\;(\Lambda_{t,u}v)(x)\;D\left(\psi(\mathcal{S}_t u)\right)\,dt=0\quad,\ \textit{resp.}\ \int_a^b\int_{\mathbb{R}}(\Lambda_{t,u}v)(x)\;D\left(\psi(\mathcal{S}_t u)\right)\,dt=0\,, \tag{2.12}$$

holds for all $v\in\mathbf{L}^1\left(|Du|\right)$.

3 Traffic Flow Management

According to the Lighthill-Whitham [8] and Richards [9] (LWR) model the car density ρ along a road with neither entries nor exits satisfies the scalar conservation law

$$\begin{cases}\partial_t\rho+\partial_x\left[\rho\,v(\rho)\right]=0,\\ \rho(0,x)=\rho_o(x),\end{cases} \tag{3.1}$$

where $t\in[0,+\infty[$ is time, $x\in\mathbb{R}$ is the coordinate along the road and $v=v(\rho)$ is the average traffic speed at density ρ, with $\rho\in[0,R]$. R is the maximal car density sustained by the road, i.e. when $\rho=R$ cars are lined up bumper to bumper in a queue. A typical speed law $v=v(\rho)$ leads to a flow $q=\rho\cdot v(\rho)$ of the form in Figure 1, which is the one we assume here: q is smooth, strictly concave down with zeros at $\rho=0$ and $\rho=R$. $\check R$ denotes the unique point of maximum, with $q(\check R)=\check Q$.

From the point of view of traffic flow management, it is often more reasonable to consider the flow as unknown, which essentially leads to exchange the roles of t and x, yielding

$$\begin{cases}\partial_x q+\partial_t\left[\rho(q)\right]=0\,,\\ q(t,0)=q_o(t)\,,\end{cases} \tag{3.2}$$

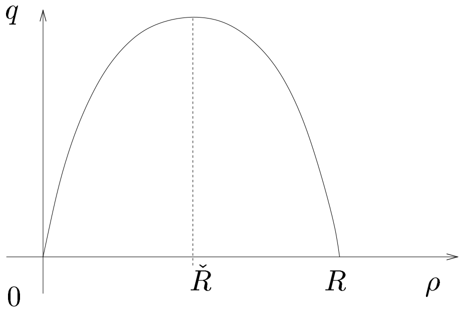

Figure 1: A typical flow for the LWR model (3.1).

where $q \in [0, \check{Q}]$ and, equivalently, $\rho \in [0, \check{R}]$.

At $x = 0$ a traffic light regulates the inflow of cars. The aim of this control is to have a given flow. We are thus lead to consider a cost functional of the type

$$J(q_o) = \int_a^b \int_{\mathbb{R}} \psi\left(\mathcal{S}_x q_o(t)\right)\, dt\, dx \tag{3.3}$$

where $\psi \in \mathbf{Lip}\left([0, +\infty[\right)$ is a nonnegative function such that $\psi(q_*) = 0$ as in Figure 2. Here, a and b are the beginning and the end of the dangerous portion of the road. Aiming at an

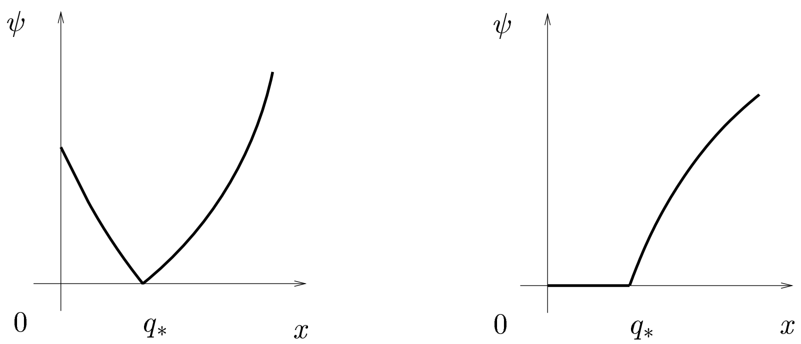

Figure 2: Two possible cost functions for (3.3). Left, a generic situation and, right, a possible choice for the threshold problem.

analytical treatment, we assume that the inflow q_o of cars is periodic and constant throughout the green light. In a numerical framework, this assumption can easily be abandoned. Thus, we set

$$q_o(t) = \begin{cases} \overline{q} & \text{if } (t - t_o)/(t_r + t_g) \in \left[0, t_g/(t_r + t_g)\right[\mod \mathbf{Z}\,, \\ 0 & \text{if } (t - t_o)/(t_r + t_g) \in \left[t_g/(t_r + t_g), 1\right[\mod \mathbf{Z}\,. \end{cases} \tag{3.4}$$

Above, t_r (resp. t_g) is the time the traffic light shows red (resp. green), so that q_o is $(t_r + t_g)$-periodic, while t_o is the first time at which the light turns green, see Figure 3. Two qualitatively different behaviors in the solution to (3.2)-(3.4) may appear, depending on whether each shock hits first the rarefaction wave in front of it (Case A) or the one behind it (Case B), see Figure 4. In both cases, at $x = x_*$ the shock starts to bend due to the interaction and at some x^* a second interaction takes place, between the curved shock and a rarefaction, see Figure 4.

The periodic structure of the solution allows us to restrict the optimization procedure to a single period.

Denote by $1/c$ the characteristic speed of (3.2), so that c is the usual characteristic speed of the LWR model (3.1). It is easy to see that Case A holds if and only if

$$\frac{t_g}{t_r} < \frac{c(0)\left(\overline{q} - \rho(\overline{q})c(\overline{q})\right)}{c(\overline{q})\left(\rho(\overline{q})c(0) - \overline{q}\right)}\,.$$

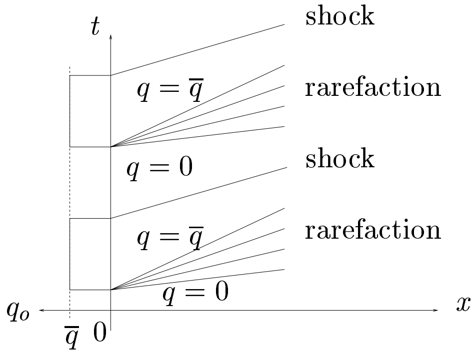

Figure 3: Left, the initial data (3.4) in the $(q_o, t)-$plane and, right, the corresponding solution to (3.2) in the (x, t) plane.

In both cases, let x_* be the point where the interaction begins and call $t = \sigma(x)$, for $x > x_*$ the support of the shock during the interaction. Note that $\sigma(x)$ is smooth and can be found by means of the conservation of q, see (3.6) below.

Due to the periodicity and the form of q_o, any shift in $\mathbf{L}^1(Dq_o)$ is essentially of the form

$$w(t) = \begin{cases} \alpha & \text{if } t = t_o\,, \\ \beta & \text{if } t = t_o + t_g\,. \end{cases}$$

This amounts to consider variations of the initial data in which the time the traffic light shows green passes from t_g to $t_g + \beta\vartheta$ and the red time from t_r to $t_r + \alpha\vartheta$. Direct computations yield, in Case A:

$$\int_a^b \int_{\mathbb{R}} (\Lambda_{x,q_o} w)(t)\; D\left(\psi(\mathcal{S}_x q_o)\right) dx =$$

$$\begin{cases} (\beta-\alpha)(b-a)\left[\psi(\overline{q}) - \psi(0)\right] & \text{if } a < b < x_* < x^*\,, \\ (\beta-\alpha)\left[(\psi(\overline{q}) - \psi(0))(x_* - a) + \int_{x_*}^b \left\{\psi\left(\mathcal{S}_x q_o(\sigma(x)-)\right) - \psi(0)\right\} dx\right] & \text{if } a < x_* < b < x^*\,, \\ (\beta-\alpha)\int_a^b \left\{\psi\left(\mathcal{S}_x q_o(\sigma(x)-)\right) - \psi(0)\right\} dx & \text{if } x_* < a < b < x^*\,. \end{cases}$$

While in Case B we have

$$\int_a^b \int_{\mathbb{R}} (\Lambda_{x,q_o} w)(t)\; D\left(\psi(\mathcal{S}_x q_o)\right) dx =$$

$$\begin{cases} (\beta-\alpha)(b-a)\left[\psi(\overline{q}) - \psi(0)\right] & \text{if } a < b < x_* < x^*\,, \\ (\beta-\alpha)\left[(\psi(\overline{q}) - \psi(0))(x_* - a) + \int_{x_*}^b \left\{\psi(\overline{q}) - \psi\left(\mathcal{S}_x q_o(\sigma(x)+)\right)\right\} dx\right] & \text{if } a < x_* < b < x^*\,, \\ (\beta-\alpha)\int_a^b \left\{\psi\left(\mathcal{S}_x q_o(\sigma(x)+)\right) - \psi(\overline{q})\right\} dx & \text{if } x_* < a < b < x^*\,. \end{cases}$$

In both cases, for $a > x^*$, we have:

$$\int_a^b \int_{\mathbb{R}} (\Lambda_{x,q_o} w)(t)\; D\left(\psi(\mathcal{S}_x q_o)\right) dx =$$

$$= (\beta-\alpha)\int_a^b \left[\psi\left(\mathcal{S}_x q\left(\sigma(x) + (t_r + t_g)-\right)\right) - \psi\left(\mathcal{S}_x q\left(\sigma(x)+\right)\right)\right] dx$$

while the remaining possibilities are similar to the enoes considered above.

If ψ has the form displayed on the right in Figure 2, then the Euler-Lagrange equation (2.12) can be explicitly solved.

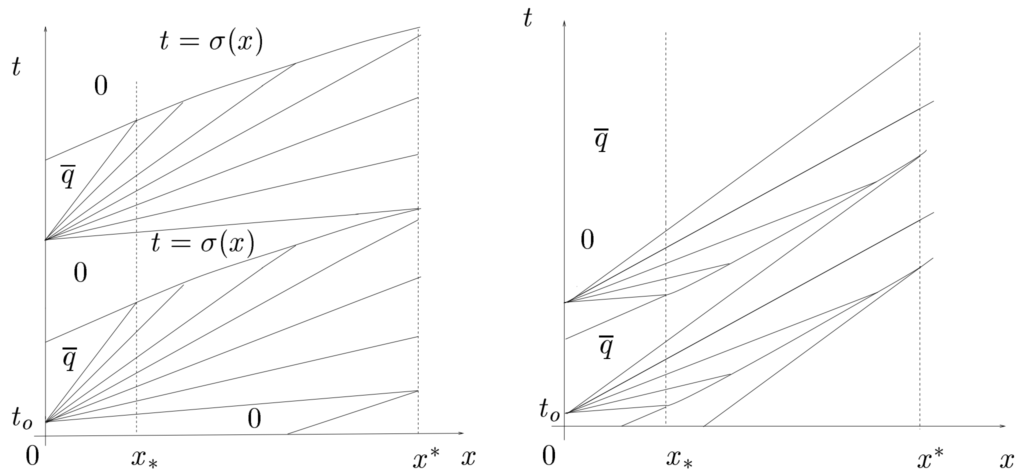

Figure 4: Behaviors of the solution to (3.2)-(3.4). Left Case A, right Case B.

Case A is straightforward if $a \leqslant x^*$, namely whenever the dangerous stretch is *near* to the road entry : as is to be expected the solution is $\overline{q} \leqslant q_*$. Indeed, as Figure 4, left, shows, on any space interval there exist times at which the flow attains the inflow value $\overline{q}$.

If the dangerous stretch is *far* from the road entry, i.e. $a > x_*$, then it is possible to allow an inflow $\overline{q}$ higher than the threshold q_*. Due to the interaction between shocks and rarefactions, J attains an absolute minimum as soon as

$$\mathcal{S}_a q_o \left(\sigma(a)+\right) \leqslant q_* \,. \tag{3.5}$$

Case B is similar. If $a \leqslant x^*$ the solution is $\overline{q} \leqslant q_*$, while, if $a > x^*$ is possible to allow an inflow $\overline{q}$ higher than the threshold q_* as in the previous case.

Consider the standard assumption of an affine speed law

$$v(\rho) = V \cdot \left(1 - \frac{\rho}{R}\right) ,$$

where V is the maximal possible speed. In this case $\check{R} = R/2$ and $\check{Q} = RV/4$. The function $\sigma(x)$, for $x \geqslant x_*$, is given by

$$\sigma(x) = \frac{x}{V} + 2\frac{\overline{q}t_g}{RV} + 2\sqrt{\frac{\overline{q}t_g x}{RV^2} + \left(\frac{\overline{q}t_g}{RV}\right)^2} \,. \tag{3.6}$$

while the above optimality condition (3.5) reads

$$\overline{q} \leqslant \frac{1}{4} \cdot \frac{R}{Vt_g} \cdot \frac{\left[(RV - 4q_*)Vt_o + 4aq_*\right]^2}{(RV - 4q_*) \cdot \left[(RV - 4q_*)t_o + Ra\right]} \,.$$

Note that the longer the time t_g of the green light, the lower becomes the right hand side above.

The other cases are entirely similar.

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