On kinetic formulation of measure valued and strong measure valued solutions to the Cauchy problem for hyperbolic first-order quasilinear equations. [1]

E.Yu. Panov

Novgorod State University, B.St-Petersburgskaya, 41, 173003 Novgorod, Russia
e-mail: pey@novsu.ac.ru

Abstract

We give kinetic formulation of measure valued and strong measure valued solutions to the Cauchy problem for a first-order quasilinear equation. For the corresponding kinetic equation the class of existence and uniqueness to the Cauchy problem is extracted. This class consists of so-called entropy solutions which correspond to strong measure valued solutions of the original problem. In the last section we generalized these results to the case of hyperbolic, generally nonconservative, systems and introduce the notion of a strong measure valued solution, based only on the kinetic approach under consideration.

Introduction.

In the half-space $(t,x)\in\Pi=(0,+\infty)\times\mathbb{R}^n$ consider firstly a scalar conservation law

$$u_t+\mathrm{div}_x\varphi(u)=0, \tag{1}$$

$$u=u(t,x),\ \varphi(u)=(\varphi_1(u),\dots,\varphi_n(u))\in(C^1(\mathbb{R}))^n.$$

Nonlocal theory of generalized entropy solutions (briefly - g.e.s.) to the Cauchy problem for equation (1) with an initial condition

$$u(0,x)=u_0(x)\in L^\infty(\mathbb{R}^n) \tag{2}$$

was constructed by S.N.Kruzhkov in papers [1, 2]. Remind definition of g.e.s. in Kruzhkov sense

Definition 1. A bounded measurable function $u=u(t,x)$ is called a g.e.s. to the Cauchy problem (1), (2) if

a) $\forall k\in\mathbb{R}$

$$|u-k|_t+\mathrm{div}_x[(\varphi(u)-\varphi(k))\,\mathrm{sign}(u-k)]\le 0 \tag{3}$$

in the sense of distributions on Π (in $\mathcal{D}'(\Pi)$);

b) $\operatorname*{ess\,lim}_{t\to 0}u(t,\cdot)=u_0$ in $L^1_{loc}(\mathbb{R}^n)$,

i.e. there exists a set of null Lebesgue measure $\mathcal{E}\subset(0,+\infty)$ such that for $t\notin\mathcal{E}$ $u(t,\cdot)\in L^1_{loc}(\mathbb{R}^n)$ and as $t\to 0$, $t\notin\mathcal{E}$

$$u(t,\cdot)\to u_0\quad\text{in } L^1_{loc}(\mathbb{R}^n).$$

---

[1] Supported by Russian Foundation for Basic Research (grants N 00-01-00921, N 02-01-00483)

Condition (3) means that for any nonnegative test function $g = g(t,x) \in C_0^\infty(\Pi)$

$$\int_\Pi [|u-k|g_t + \sum_{i=1}^n (\varphi_i(u) - \varphi_i(k)) \operatorname{sign}(u-k) g_{x_i}] dt dx \ge 0.$$

Taking in (3) $k = \pm\|u\|_\infty$ we derive that $\pm\{u_t + \operatorname{div}_x\varphi(u)\} \le 0$ in $\mathcal{D}'(\Pi)$, i.e. $u_t + \operatorname{div}_x\varphi(u) = 0$ and g.e.s. $u(t,x)$ satisfies the equation (1) in the distributional sense.

As is known (see [1, 2]), there always exists an unique g.e.s. to the problem (1), (2).

In [3] the following kinetic formulation of the problem (1), (2) was given. A function $u = u(t,x)$ was shown to be a g.e.s. to the problem (1), (2) if and only if the function
$f = f(t,x,v) = \chi_{u(t,x)}(v)$, where for $u, v \in \mathbb{R}$
$\chi_u(v) = \theta(u-v) - \theta(-v)$, $\theta(\lambda) = \begin{cases} 1, & \lambda > 0 \\ 0, & \lambda \le 0 \end{cases}$ is the Heaviside function, is a generalized solution (g.s.) to the corresponding problem for a kinetic equation

$$\frac{\partial}{\partial t} f + (\varphi'(v), \nabla_x f) = \frac{\partial}{\partial v} m \tag{4}$$

(here $(\cdot,\cdot)$ denotes scalar product in $\mathbb{R}^n$)
with some nonnegative locally finite measure $m = m(t,x,v)$ on $\Pi \times \mathbb{R}$ that has compact support with respect to the variable v, and with initial condition

$$f(0,x,v) = f_0(x,v) \tag{5}$$

where $f_0(x,v) = \chi_{u_0(x)}(v)$.

One should understand the initial condition (5) in a similar way to the condition $b)$ in Definition 1, that is,

$$\operatorname*{ess\,lim}_{t\to 0} f(t,\cdot,\cdot) = f_0 \text{ in } L^1_{loc}(\mathbb{R}^{n+1}). \tag{6}$$

In the papers [4, 5] the results above were extended to the more general case of measure valued solutions.

We recall (see [6, 7]) that a measure valued function on a measurable subset Ω of some Euclid space is a weakly measurable map $x \mapsto \nu_x$ of Ω into the space $\operatorname{Prob}_0(\mathbb{R})$ of probability Borel measures with compact support in $\mathbb{R}$.

The weak measurability of ν_x means that for each continuous function $p(\lambda)$ the function $x \mapsto \int p(\lambda) d\nu_x(\lambda)$ on Ω is measurable.

We say that a measure valued function ν_x is *bounded* if there exists $M > 0$ such that $\operatorname{supp}\nu_x \subset [-M, M]$ for almost all $x \in \Omega$. We shall denote by $\|\nu_x\|_\infty$ the smallest of such M.

Finally, we say that the measure valued functions of the kind $\nu_x(\lambda) = \delta(\lambda - u(x))$, where $u(x) \in L^\infty(\Omega)$ and ($\delta(\lambda - u)$ is the Dirac measure at $u \in \mathbb{R}$, are regular. We identify these measure valued functions and the corresponding functions $u(x)$, so that there is a natural embedding $L^\infty(\Omega) \subset MV(\Omega)$, where $MV(\Omega)$ is the set of bounded measure valued functions on Ω.

We now consider the measure valued solutions (briefly - m.s.) $\nu_{t,x} \in MV(\Pi)$ of the Cauchy problem for (1) with initial condition

$$\nu_{0,x} = \nu_x^0 \in MV(\mathbb{R}^n). \tag{7}$$

Definition 2. (see [7, 8]) A bounded measure valued function $\nu_{t,x}$ is called a m.s. of the problem (1), (7) if

a) $\forall k \in \mathbb{R}$

$$\frac{\partial}{\partial t}\int |\lambda - k| d\nu_{t,x}(\lambda) + \mathrm{div}_x \int (\varphi(\lambda) - \varphi(k)) \,\mathrm{sign}(\lambda - k) d\nu_{t,x}(\lambda) \le 0 \tag{8}$$

in $\mathcal{D}'(\Pi)$;

b) $\forall p(\lambda) \in C(\mathbb{R})$ $\quad \operatorname*{ess\,lim}_{t\to 0} \int p(\lambda) d\nu_{t,x}(\lambda) = \int p(\lambda) d\nu_x^0(\lambda)$ in $L^1_{loc}(\mathbb{R}^n)$.

Remark that Definition 2 is consistent with Definition 1 in that $u(t,x)$ is a g.e.s. of the problem (1), (2) if and only if this function, understood as the regular measure valued function $\nu_{t,x}(\lambda) = \delta(\lambda - u(t,x))$, is a m.s. of the problem (1), (7) with regular initial function $\nu_x^0(\lambda) = \delta(\lambda - u_0(x))$.

We can treat a measure valued function $\nu_{t,x}$ as the random field $u(t,x)$ where the value of $u(t,x)$ for fixed $(t,x) \in \Pi$ is a random variable with distribution $\nu_{t,x}$. Then condition (8) shows that $u(t,x)$ satisfies condition (3) "in the mean", that is,

$$E(|u-k|)_t + \mathrm{div}_x E((\varphi(u) - \varphi(k))\,\mathrm{sign}(u-k)) \le 0 \text{ in } \mathcal{D}'(\Pi)$$

(here E denotes expectation).

Setting in (8) $k = \pm M$, where $M = \|\nu_{t,x}\|_\infty$, we obtain

$$\frac{\partial}{\partial t}\int \lambda d\nu_{t,x}(\lambda) + \mathrm{div}_x \int \varphi(\lambda) d\nu_{t,x}(\lambda) = 0 \quad \text{in } \mathcal{D}'(\Pi), \tag{9}$$

which means the corresponding random field $u(t,x)$ satisfies (1) "in the mean".

Some results on relations between measure valued and statistic solutions of the problem (1), (7) can be found in [9].

As is known (see [10, 11]), there always exists a m.s. of the Cauchy problem (1), (7), but it is nether unique if the initial function ν_x^0 is not regular.

In [10] (see [11] for more details) the notion of a *strong measure valued solution* (briefly - s.m.s.) was introduced and existence and uniqueness of s.m.s. to the problem (1), (7) were also established.

Definition 3. (see [10, 11]) A bounded measure valued function $\nu_{t,x}$ is caled a s.m.s. of the problem (1), (7) if for all $\lambda \in (0,1)$ the function $u(t,x,\lambda) = \inf\{\, v \mid \nu_{t,x}((v,+\infty)) \le \lambda \,\}$ is a g.e.s. of the problem (1), (2) with initial function $u_0(x,\lambda) = \inf\{\, v \mid \nu_x^0((v,+\infty)) \le \lambda \,\}$.

If to be exact, in [10, 11] the functions $u(t,x,\lambda) = \inf\{\, v \mid \nu_{t,x}((-\infty,v)) > \lambda \,\}$, $u_0(x,\lambda) = \inf\{\, v \mid \nu_x^0((-\infty,v)) > \lambda \,\}$ appear that obviously leads to the equivalent definition of s.m.s.

For a regular m.s. $\nu_{t,x}(\lambda) = \delta(\lambda - u(t,x))$ we have $u(t,x,\lambda) = u(t,x)$ and $\nu_{t,x}$ is a s.m.s. Then we note (see [11]) that for fixed $(t,x) \in \Pi$ the measure $\nu_{t,x}$ is the image of the Lebesgue measure $d\lambda$ on $(0,1)$ under the map

$\lambda\mapsto u(t,x,\lambda)$: $\nu_{t,x}=u(t,x,\cdot)^*d\lambda$ (similarly, $\nu_x^0=u_0(x,\cdot)^*d\lambda$). Therefore, $\forall k\in\mathbb{R}$ $\displaystyle\int|\lambda-k|d\nu_{t,x}(\lambda)=\int_0^1|u(t,x,\lambda)-k|d\lambda,$

$$\int(\varphi(\lambda)-\varphi(k))\operatorname{sign}(\lambda-k)d\nu_{t,x}(\lambda)=$$

$$\int_0^1(\varphi(u(t,x,\lambda))-\varphi(k))\operatorname{sign}(u(t,x,\lambda)-k)d\lambda$$

and inequality (8) can be written in the form

$$\int_0^1[|u(t,x,\lambda)-k|_t+\operatorname{div}_x((\varphi(u(t,x,\lambda))-\varphi(k))\operatorname{sign}(u(t,x,\lambda)-k))]d\lambda\le 0. \tag{10}$$

Thus, Definition 3 contains strengthening of the condition (8), namely that the integrand in (10) is required to be nonpositive (in $\mathcal{D}'(\Pi)$) for all $\lambda\in(0,1)$. In particular any s.m.s. is a m.s. in the sense of Definition 2 as well. The unique solvability of the problem (1), (2) implies easily existence and uniqueness of s.m.s. of the problem (1), (7) (see details in [11]).

§ 1. Kinetic formulation of m.s.

For some $M>0$ denote by F_M the space of distribution functions $f(v)=F(\nu)(v)=\nu((v,+\infty))$ of measures ν with support in the segment $[-M,M]$. Functions $f(v)\in F_M$ don't increase and are continuous from the right with respect to $v\in\mathbb{R}$; $f(v)=0$ for $v\ge M$, $f(v)=1$ for $v<-M$. Denote also by $F_c=\bigcup\limits_{M>0}F_M$ the space of distribution functions of measures having compact supports on $\mathbb{R}$.

If $\nu_x\in MV(\Omega)$, Ω is a measurable domain in some Euclid space, then $f(x,v)=F(\nu_x)(v)\in F_M$ for a.e. $x\in\Omega$, $M=\|\nu_x\|_\infty$ and by weak measurability of the map $x\mapsto\nu_x$ we get (see for instance Lemma 3 in [12]) that the function $f(x,v)=F(\nu_x)(v)$ is measurable on the set of variables (x,v). Let $F_M(\Omega)$ be the class of functions with above properties. Obviously, inclusion $f(x,v)\in F_M(\Omega)$ is equivalent to existence of a measure valued function $\nu_x\in MV(\Omega)$ such that $\|\nu_x\|_\infty\le M$ and $f(x,v)=\nu_x((v,+\infty))$.

As was shown in [4], a distribution function $f(t,x,v)$ of a m.s. $\nu_{t,x}\in MV(\Pi)$ can be discribed as a solution of the corresponding problem (4), (5) for the kinetic equation. In [4] we also presented kinetic formulation of s.m.s. For completness we put here the main results from [4]. We shall need some technical statements.

Lemma 1. *Let $\nu_{t,x}$ be a m.s. of the problem (1), (7). Then $\forall v\in\mathbb{R}$*

$$\frac{\partial}{\partial t}\int|\lambda-v|d\nu_{t,x}(\lambda)+\operatorname{div}_x\int(\varphi(\lambda)-\varphi(v))\operatorname{sign}(\lambda-v)d\nu_{t,x}(\lambda)=-\mu^v \tag{11}$$

in $\mathcal{D}'(\Pi)$ where μ^v is a nonnegative locally finite measure on Π. Besides, if $M=\|\nu_{t,x}\|_\infty$ then $\mu^v=0$ for $v\notin[-M,M]$ and a map $v\mapsto\mu^v$ is weakly continuous, i.e. $\forall g(t,x)\in C_0(\Pi)$ the function $v\mapsto\int g(t,x)d\mu^v(t,x)$ is continuous.

Proof. Denote for $v\in\mathbb{R}$ $P=P(t,x)=\displaystyle\int|\lambda-v|d\nu_{t,x}(\lambda)$,

$$Q=Q(t,x)=\int(\varphi(\lambda)-\varphi(v))\operatorname{sign}(\lambda-v)d\nu_{t,x}(\lambda)\in\mathbb{R}^n$$

and consider a functional

$$L^v(g) = \int_\Pi [Pg_t + (Q, \nabla_x g)]dtdx, \quad g = g(t,x) \in C_0^\infty(\Pi).$$

Condition (8) implies that this functional is nonnegative: $L^v(g) \ge 0$ for $g \ge 0$. Therefore (see for instance [12, Lemma 5]), the functional L^v is continuous in $C_0(\Pi)$ and uniquely extended as a bounded nonnegative functional on $C_0(\Pi)$. Hence, by Riesz-Markov representation theorem, there exists a locally finite nonnegative measure μ^v on Π such that $L^v(g) = \int g(t,x)d\mu^v(t,x)$.

By definition of L^v we conclude that $P_t + \operatorname{div}_x Q = -\mu^v$ in $\mathcal{D}'(\Pi)$ that proves equality (11).

If $v \notin [-M,M]$ then for a.e. $(t,x) \in \Pi$ $|\lambda - v| = (v-\lambda)\operatorname{sign}(v)$, $(\varphi(\lambda) - \varphi(v))\operatorname{sign}(\lambda - v) = (\varphi(v) - \varphi(\lambda))\operatorname{sign}(v)$ on $\operatorname{supp}\nu_{t,x}$ and $\forall g(t,x) \in C_0^\infty(\Pi)$

$$L^v(g) = \pm \int_\Pi \left[\left(\int \lambda d\nu_{t,x}(\lambda) \right) g_t + \left(\int \varphi(\lambda) d\nu_{t,x}(\lambda), \nabla_x g \right) \right] dtdx = 0$$

by equality (9). Hence, $\mu^v = 0$ for $v \notin [-M,M]$.

For proof of the last assertion of the lemma remark that for a fixed function $g(t,x) \in C_0^\infty(\Pi)$ the value $L^v(g)$ continuously depends on $v \in \mathbb{R}$. Therefore, functions $v \mapsto \int g(t,x)d\mu^v(t,x)$ are continuous for any $g(t,x) \in C_0^\infty(\Pi)$. Let $C \subset \Pi$ be a compact set. Then we can choose a function $g(t,x) \in C_0^\infty(\Pi)$ such that $g(t,x) \ge 0$ and $g(t,x) = 1$ for $(t,x) \in C$. By continuity of the function $v \mapsto \int g(t,x)d\mu^v(t,x)$ the values $\mu^v(C)$ are bounded for $v \in \mathbb{R}$ (taking into account that $\mu^v = 0$ outside of $[-M,M]$). Let $g(t,x) \in C_0(\Pi)$. Choose a sequence $g_r(t,x) \in C_0^\infty(\Pi)$, $r \in \mathbb{N}$ which uniformly converges to $g(t,x)$ and such that for all $r \in \mathbb{N}$ $g_r = 0$ outside of some compact set $C \subset \Pi$. Taking into account boundedness of $\mu^v(C)$, we get that the sequence of continuous by $v \in \mathbb{R}$ functions $\int_C g_r(t,x)d\mu^v(t,x)$ uniformly converges to the function $\int g(t,x)d\mu^v(t,x)$ as $r \to \infty$. Hence, the limit function $v \mapsto \int g(t,x)d\mu^v(t,x)$ must be continuous for arbitrary $g(t,x) \in C_0(\Pi)$, that is the map $v \mapsto \mu^v$ is weakly continuous. The proof is complete.

Lemma 2. *Let* $h(v) \in C_0^1(\mathbb{R})$, $p(\lambda) = \int |\lambda - v| h'(v)dv$,

$$q_i(\lambda) = \int (\varphi_i(\lambda) - \varphi_i(v))\operatorname{sign}(\lambda - v)h'(v)dv, \quad i = 1, \dots, n.$$

Then $p(\lambda), q_i(\lambda) \in C^1(\mathbb{R})$ *and* $p'(\lambda) = 2h(\lambda)$, $q_i'(\lambda) = \varphi_i'(\lambda)p'(\lambda)$.

Proof. In the sense of distributions on $\mathbb{R}$

$$p'(\lambda) = \int |\lambda - v|'_\lambda h'(v)dv = \int \operatorname{sign}(\lambda - v)h'(v)dv =$$

$$\int_{-\infty}^{\lambda} h'(v)dv - \int_\lambda^{+\infty} h'(v)dv = 2h(\lambda);$$

$$q_i'(\lambda) = \int [(\varphi_i(\lambda) - \varphi_i(v))\operatorname{sign}(\lambda - v)]'_\lambda h'(v)dv =$$

$$\int \varphi_i'(\lambda)\operatorname{sign}(\lambda - v)h'(v)dv = \varphi_i'(\lambda)\int \operatorname{sign}(\lambda - v)h'(v)dv = \varphi_i'(\lambda)p'(\lambda).$$

It only remains to observe that $p'(\lambda) = 2h(\lambda) \in C_0^1(\mathbb{R})$, $q_i'(\lambda) = \varphi_i'(\lambda)p'(\lambda) \in C_0(\mathbb{R})$. Therefore, the functions $p(\lambda)$, $q_i(\lambda)$ are differentiable in the classical sense and $p(\lambda), q_i(\lambda) \in C^1(\mathbb{R})$ (moreover, $p(\lambda) \in C^2(\mathbb{R})$).

Lemma 3. *Let $\nu_{t,x} \in MV(\Pi)$, $\nu_x^0 \in MV(\mathbb{R}^n)$, and $f(t,x,v) = \nu_{t,x}((v,+\infty))$ and $f_0(x,v) = \nu_x^0((v,+\infty))$. Then the following conditions are equivalent:*

1) $\forall p(\lambda) \in C_0^\infty(\mathbb{R}) \quad \operatorname*{ess\,lim}_{t\to 0} \int p(\lambda)d\nu_{t,x}(\lambda) = \int p(\lambda)d\nu_x^0(\lambda)$ *in* $L^1_{loc}(\mathbb{R}^n)$;

2) $\operatorname*{ess\,lim}_{t\to 0} f(t,x,v) = f_0(x,v)$ *in* $L^1_{loc}(\mathbb{R}^{n+1})$.

Proof. First we prove the implication $2) \Rightarrow 1)$.
Let $M = \|\nu_{t,x}\|_\infty$, $p(\lambda) \in C_0^\infty(\mathbb{R})$, $h(v) = p'(v)$. By 2),

$$\operatorname*{ess\,lim}_{t\to 0} \int f(t,x,v)h(v)dv = \int f_0(x,v)h(v)dv \quad \text{in } L^1_{loc}(\mathbb{R}^n). \tag{12}$$

Integrating by parts we obtain

$$\int f(t,x,v)h(v)dv = \int f(t,x,v)dp(v) = -\int p(v)df(t,x,v) = \int p(\lambda)d\nu_{t,x}(\lambda)$$

and, in similar way, $\int f_0(x,v)h(v)dv = \int p(\lambda)d\nu_x^0(\lambda)$. By these equalities and (12) we obtain 1).

Conversely, assume that 1) holds. Then for each $p = p(\lambda) \in C_0^\infty(\mathbb{R})$ there exists a set $\mathcal{E}(p)$ of null measure such that

$$\int p(\lambda)d\nu_{t,x}(\lambda) \to \int p(\lambda)d\nu_x^0(\lambda) \quad \text{in } L^1_{loc}(\mathbb{R}^n) \tag{13}$$

as $t \to 0$, $t \notin \mathcal{E}(p)$. Now choose a dense countable set $S \subset C_0^\infty(\mathbb{R})$ and let $\mathcal{E} = \bigcup_{p\in S} \mathcal{E}(p)$. Since S is countable, $\mathcal{E}$ has measure zero and the limit relation (13) holds as $t \to 0$, $t \notin \mathcal{E}$ for all $p \in S$. Let $\rho(x) = e^{-|x|}$ (here $|x| = (x,x)^{1/2}$ is the Euclidean norm in $\mathbb{R}^n$). We claim that

$$\int_{\mathbb{R}^{n+1}} |f(t,x,v) - f_0(x,v)|\rho(x)dxdv \to 0 \tag{14}$$

as $t \to 0$, $t \notin \mathcal{E}$, which, obviously, implies 2) (remark that if $v \notin [-M,M]$ then $f(t,x,v) = f_0(x,v)$ almost everywhere in $\mathbb{R}^n$).

We carry out the proof by contradiction. Assume that (14) fails. Then there exists $\delta > 0$ and a sequence t_r, $r \in \mathbb{N}$ such that $t_r \notin \mathcal{E}$, $\lim_{r\to\infty} t_r = 0$, and

$$\liminf_{r\to\infty} \int_{\mathbb{R}^{n+1}} |f(t_r,x,v) - f_0(x,v)|\rho(x)dxdv > \delta. \tag{15}$$

By (13), we can select a subsequence of t_r (which we denote again by t_r) such that

$$\int p(\lambda)d\nu_x^{(r)}(\lambda) \underset{r\to\infty}{\to} \int p(\lambda)d\nu_x^0(\lambda)$$

a.e. in $\mathbb{R}^n$ for $\nu_x^{(r)} = \nu_{t_r,x}$. Using the standard diagonal method we can choose the same subsequence for the countable set of functions $p(\lambda) \in S$. Since S is dense in

$C_0^\infty(\mathbb{R})$ (and therefore also in $C_0(\mathbb{R})$), it follows that for almost all $x \in \mathbb{R}^n$ we have

$$\int p(\lambda)d\nu_x^{(r)}(\lambda) \underset{r\to\infty}{\to} \int p(\lambda)d\nu_x^0(\lambda)$$

for all $p(\lambda) \in C_0(\mathbb{R})$, that is, the sequence of measures $\nu_x^{(r)}$ weakly converges to ν_x^0. By Helly's theorem (see [13], Chapter 8, § 4), the corresponding sequence of distribution functions $f(t_r, x, v)$ converges for such $x \in \mathbb{R}^n$ to $f_0(x, v)$ at all continuity points of the function $v \mapsto f_0(x, v)$. The complement to the set of these values v is at most countable for any fixed $x \in \mathbb{R}^n$, therefore the (obviously, measurable) set $\left\{(x, v) \in \mathbb{R}^{n+1} \mid f(t_r, x, v) \underset{r\to\infty}{\to} f_0(x, v)\right\}$ has full Lebesgue measure. Thus, $f(t_r, x, v) \underset{r\to\infty}{\to} f_0(x, v)$ a.e. in $\mathbb{R}^{n+1}$, $0 \le f \le 1$, and by Lebesgue theorem on dominated convergence,

$$\lim_{r\to\infty} \int_{\mathbb{R}^{n+1}} |f(t_r, x, v) - f_0(x, v)|\rho(x)dxdv = 0,$$

which contradicts (15), thus proving the lemma.

Theorem 1. *A bounded measure valued function $\nu_{t,x}$ is a m.s. of the problem* (1), (7) *if and only if the corresponding distribution function $f(t, x, v) = \nu_{t,x}((v, +\infty))$ is a g.s. of the problem* (4), (5) *with initial function $f_0(x, v) = \nu_x^0((v, +\infty))$.*

Proof. Let $h(v) \in C_0^\infty(\mathbb{R})$. We multiply (11) by $h'(v)$ and integrate over $v \in \mathbb{R}$. We obtain

$$\frac{\partial}{\partial t}\int p(\lambda)d\nu_{t,x}(\lambda)+\operatorname{div}_x\int q(\lambda)d\nu_{t,x}(\lambda)=-\int_{\mathbb{R}} h'(v)\mu^v(t, x)dv \quad \text{in } \mathcal{D}'(\Pi), \tag{16}$$

where

$$p(\lambda) = \int_{\mathbb{R}} |\lambda - v|h'(v)dv,$$

$$q(\lambda) = \int_{\mathbb{R}} (\varphi(\lambda) - \varphi(v)) \operatorname{sign}(\lambda - v)h'(v)dv = (q_1(\lambda), \ldots, q_n(\lambda)) \in \mathbb{R}^n.$$

By Lemma 2, $p(\lambda) \in C^1(\mathbb{R})$, $q(\lambda) \in (C^1(\mathbb{R}))^n$ and $p'(\lambda) = 2h(\lambda)$, $q'(\lambda) = 2\varphi'(\lambda)h(\lambda)$. Hence it follows from the formula of integrating by parts that

$$\int p(\lambda)d\nu_{t,x}(\lambda) = -\int p(\lambda)d_\lambda f(t, x, \lambda) = c_0 + 2\int h(\lambda)f(t, x, \lambda)d\lambda, \tag{17}$$

$$\int q(\lambda)d\nu_{t,x}(\lambda) = -\int q(\lambda)d_\lambda f(t, x, \lambda) = \mathbf{c} + 2\int h(\lambda)\varphi'(\lambda)f(t, x, \lambda)d\lambda, \tag{18}$$

where the terms outside the integrals are

$$c_0 = -\lim_{a\to+\infty} p(\lambda)f(t, x, \lambda)\Big|_{-a}^{a}, \mathbf{c} = -\lim_{a\to+\infty} q(\lambda)f(t, x, \lambda)\Big|_{-a}^{a} \in \mathbb{R}^n.$$

Since $f(t, x, a) = 0$ and $f(t, x, -a) = 1$ for $a > 0$ sufficiently large and the functions $p(-a)$, $q_i(-a)$, $i = 1, \ldots, n$ are constant (which is a consequence of the equalities $p'(\lambda) = 2h(\lambda)$, $q'(\lambda) = 2\varphi'(\lambda)h(\lambda)$ and the fact that $h(\lambda)$ has compact support), the constant c_0 and $\mathbf{c}=(c_1,\dots,c_n)$ are well defined and independent of $(t,x)\in\Pi$. Hence, in view of (17), (18), we can transform (16) as follows

$$\frac{\partial}{\partial t}\int h(\lambda)f(t,x,\lambda)d\lambda+\operatorname{div}_x\int h(\lambda)\varphi'(\lambda)f(t,x,\lambda)d\lambda=-\frac{1}{2}\int_{\mathbb{R}}h'(v)\mu^v(t,x)dv \text{ in } \mathcal{D}'(\Pi).$$

Applying this equality to a test function $g(t,x)\in C_0^\infty(\Pi)$ we obtain

$$\int_{\Pi\times\mathbb{R}}[f(t,x,\lambda)h(\lambda)g_t(t,x)+f(t,x,\lambda)(\varphi'(\lambda),h(\lambda)\nabla_x g)]dtdxd\lambda=$$
$$\frac{1}{2}\int_{\mathbb{R}}h'(v)\Big(\int_\Pi g(t,x)d\mu^v(t,x)\Big)dv. \qquad (19)$$

Let $M=\|\nu_{t,x}\|_\infty$. We now define a measure m in $\Pi\times\mathbb{R}$ as follows:

$$\int_{\Pi\times\mathbb{R}}\xi(t,x,v)dm(t,x,v)=\frac{1}{2}\int_{\mathbb{R}}\Big(\int_\Pi \xi(t,x,v)d\mu^v(t,x)\Big)dv, \qquad (20)$$

where $\xi(t,x,v)\in C_0(\Pi\times\mathbb{R})$. In view of Lemma 1, the map $v\mapsto\mu^v$ is weakly continuous and $\mu^v=0$ for $v\notin[-M,M]$, therefore it is easy to see that the function $v\mapsto\int_\Pi\xi(t,x,v)d\mu^v(t,x)$ is continuous and has compact support, so that the right-hand side of (20) is well defined and delivers a nonnegative continuous functional in the space of functions $\xi\in C_0(\Pi\times\mathbb{R})$. By Riesz-Markov theorem there exists an unique locally finite measure $m\ge 0$ on $\Pi\times\mathbb{R}$ such that (20) holds. In addition, it is clear that $\operatorname{supp} m\subset\Pi\times[-M,M]$.

Let $\xi=\xi(t,x,v)=g(t,x)h(v)$. Then in view of (19), (20),

$$\int_{\Pi\times\mathbb{R}}[f\xi_t+f(\varphi'(v),\nabla_x\xi)]dtdxdv=\int_{\Pi\times\mathbb{R}}\xi_v dm(t,x,v)$$

for $f=f(t,x,v)$. The linear hull of functions $\xi(t,x,v)$ of the form $g(t,x)h(v)$ is dense in $C_0^\infty(\Pi\times\mathbb{R})$, therefore this equality holds by continuity for all $\xi\in C_0^\infty(\Pi\times\mathbb{R})$, that is,

$$\frac{\partial}{\partial t}f+(\varphi'(v),\nabla_x f)=\frac{\partial}{\partial v}m \quad \text{ in } \mathcal{D}'(\Pi\times\mathbb{R}).$$

Hence $f(t,x,v)$ satisfies the kinetic equation (4). By the initial condition (7), in view of Definition 2 and Lemma 3, we obtain $\operatorname*{ess\,lim}_{t\to 0}f(t,x,v)=f_0(x,v)$ in $L^1_{loc}(\mathbb{R}^{n+1})$, that is, (6) holds.

Conversely, if a function $f(t,x,v)\in F_M(\Pi)$ is a g.s. of the Cauchy problem (4), (5), then it follows from (4) that $\frac{\partial}{\partial v}m=0$ for $v\notin[-M,M]$ and because the measure m has compact support with respect to v we conclude that $\operatorname{supp} m\subset\Pi\times[-M,M]$. Applying (4) to the test function $g(t,x)p'(v)$, where $g=g(t,x)\in C_0^\infty(\Pi)$, $g(t,x)\ge 0$; $p(v)\in C_0^\infty(\mathbb{R})$, $p''(v)\ge 0$ for $v\in[-M,M]$, we obtain

$$\int_\Pi\Big[\Big(\int_{\mathbb{R}}f(t,x,v)p'(v)dv\Big)g_t+\Big(\int_{\mathbb{R}}f(t,x,v)\varphi'(v)p'(v)dv,\nabla_x g\Big)\Big]dtdx=$$
$$\int_{\Pi\times\mathbb{R}}g(t,x)p''(v)dm(t,x,v)\ge 0. \qquad (21)$$

In a similar way to (17) and (18), integrating by parts we obtain

$$\int_{\mathbb{R}} f(t,x,v)p'(v)dv = \int p(\lambda)d\nu_{t,x}(\lambda), \ \int_{\mathbb{R}} f(t,x,v)\varphi'(v)p'(v)dv = \int q(\lambda)d\nu_{t,x}(\lambda),$$

where $q(\lambda)=(q_1(\lambda),\dots,q_n(\lambda))$, $q(\lambda)=\int_{-\infty}^{\lambda}\varphi'(\sigma)p'(\sigma)d\sigma$. Hence, by (21)

$$\int_{\Pi}\left[\left(\int p(\lambda)d\nu_{t,x}(\lambda)\right)g_t+\left(\int q(\lambda)d\nu_{t,x}(\lambda),\nabla_x g\right)\right]dtdx\geq 0. \tag{22}$$

Let $p(\lambda)=\int_{\mathbb{R}}|\lambda-v|h'(v)dv$, where $h(v)\in C_0^\infty(\mathbb{R})$, $h'(v)\geq 0$ on $[-M,M]$. By Lemma 2 we have $p''(\lambda)=2h'(\lambda)\geq 0$ on $[-M,M]$ and, up to a constant vector, $q(\lambda)=\int_{\mathbb{R}}(\varphi(\lambda)-\varphi(v))\operatorname{sign}(\lambda-v)h'(v)dv$. Hence it follows from (22) that $\int_{\mathbb{R}}R(v)h'(v)dv\geq 0$, where we set

$$R(v)=\int_{\Pi}\Big[\Big(\int|\lambda-v|d\nu_{t,x}(\lambda)\Big)g_t+$$
$$\Big(\int(\varphi(\lambda)-\varphi(v))\operatorname{sign}(\lambda-v)d\nu_{t,x}(\lambda),\nabla_x g\Big)\Big]dtdx.$$

It is easy to see that $R(v)=-\operatorname{sign}v\cdot c$ for $v\notin[-M,M]$, where

$$c=\text{const}=\int_{\Pi}\Big[\Big(\int\lambda d\nu_{t,x}(\lambda)\Big)g_t+\Big(\int\varphi(\lambda)d\nu_{t,x}(\lambda),\nabla_x g\Big)\Big]dtdx.$$

We have

$$0\leq\int_{\mathbb{R}}R(v)h'(v)dv=\int_{-M}^{M}R(v)h'(v)dv+$$
$$c\left(\int_{-\infty}^{-M}h'(v)dv-\int_{M}^{+\infty}h'(v)dv\right)=\int_{-M}^{M}R(v)h'(v)dv+c(h(-M)+h(M)).$$

Taking here functions $h(v)$ that are constant on $[-M,M]$ we derive that $c=0$ and therefore for all $h(v)\in C_0^\infty(\mathbb{R})$, $h'(v)\geq 0$ the inequality $\int_{-M}^{M}R(v)h'(v)dv\geq 0$ holds, which implies, taking account of continuity of $R(v)$, that $R(v)$ is nonnegative for $v\in[-M,M]$. We recall that $R(v)=\pm c=0$ for $v\notin[-M,M]$. Hence, $R(v)\geq 0$ for all $v\in\mathbb{R}$, which means that (8) holds (because $g=g(t,x)\in C_0^\infty(\Pi)$ is an arbitrary function such that $g(t,x)\geq 0$).

Further, by the initial condition (6) and Lemma 3

$$\forall p(\lambda)\in C_0^\infty(\mathbb{R})\quad \operatorname*{ess\,lim}_{t\to 0}\int p(\lambda)d\nu_{t,x}(\lambda)=\int p(\lambda)d\nu_x^0(\lambda)\ \text{ in } L^1_{loc}(\mathbb{R}^n).$$

Since $C_0^\infty(\mathbb{R})$ is dense in $C_0(\mathbb{R})$ the last relation holds for all $p(\lambda)\in C_0(\mathbb{R})$, that is, the initial condition b) in Definition 2 is satisfied. Thus, $\nu_{t,x}$ is a m.s. of the problem (1), (7). The proof is complete.

Investigate what g.s. $f(t,x,v)$ of the problem (4), (5) corresponds to strong m.s. of the problem (1), (7). The following theorem gives the answer

Theorem 2. *A bounded measure valued function* $\nu_{t,x}$ *is a s.m.s. of the problem* (1), (7) *if and only if for each nondecreasing function* $s(u) \in C^1([0,1])$ *the function* $s(f(t,x,v))$ *is a g.s. to the problem* (4), (5) *with initial data* $s(f_0(x,v))$. *Here* $f(t,x,v) = \nu_{t,x}((v,+\infty))$, $f_0(x,v) = \nu_x^0((v,+\infty))$. *(It is understood that the measure* m *in the right-hand side of* (4) *depends on the function* s *).*

Proof. Let $\nu_{t,x}$ be a s.m.s. of the problem (1), (7); $s(u) \in C^1([0,1])$, $s'(u) > 0$. We can also suppose that $s(0) = 0$, $s(1) = 1$, that could be achieved by an appropriate affine transform of the function $s(u)$. We set for $\lambda \in (0,1)$

$$u(t,x,\lambda) = \inf\{\, v \mid f(t,x,v) \le \lambda \,\}, \quad u_0(x,\lambda) = \inf\{\, v \mid f_0(x,v) \le \lambda \,\};$$
$$u^{(s)}(t,x,\lambda) = \inf\{\, v \mid s(f(t,x,v)) \le \lambda \,\} = u(t,x,s^{-1}(\lambda)),$$
$$u_0^{(s)}(x,\lambda) = \inf\{\, v \mid s(f_0(x,v)) \le \lambda \,\} = u_0(x,s^{-1}(\lambda)).$$

It is clear that $s(f(t,x,v)) \in F_M(\Pi)$, $s(f_0(x,v)) \in F_M(\mathbb{R}^n)$, where $M = \|\nu_{t,x}\|_\infty$. Thus, there exist unique measure valued functions $\nu_{t,x}^{(s)} \in MV(\Pi)$ and $\nu_x^{(s)} \in MV(\mathbb{R}^n)$ such that

$$\nu_{t,x}^{(s)}((v,+\infty)) = s(f(t,x,v)), \quad \nu_x^{(s)}((v,+\infty)) = s(f_0(x,v)).$$

By Definition 3 for all $\lambda \in (0,1)$ $u^{(s)} = u(t,x,s^{-1}(\lambda))$ is a g.e.s. of the problem (1), (2) with initial function $u_0^{(s)} = u_0(x,s^{-1}(\lambda))$. Because

$$u^{(s)}(t,x,\lambda) = \inf\{\, v \mid \nu_{t,x}^{(s)}((v,+\infty)) \le \lambda \,\}, \; u_0^{(s)}(x,\lambda) = \inf\{\, v \mid \nu_x^{(s)}((v,+\infty)) \le \lambda \,\},$$

then, again by Definition 3, $\nu_{t,x}^{(s)}$ is a s.m.s. (and therefore a m.s. in the sense of Definition 2) of the problem (1), (7) with initial data $\nu_x^{(s)}$. In view of Theorem 1 the corresponding distribution function $s(f(t,x,v))$ is a g.s. of (4), (5) with initial function $s(f_0(x,v))$. The first part of the theorem (necessity) is proved, if to take into account arbitrariness of $s(u)$ (the requirement of strict monotonicity of $s(u)$, which is used in the proof, could be droped by approximation arguments).

Conversely, assume that for any nondecreasing function $s(u) \in C^1([0,1])$ the function $s(f(t,x,v))$ is a g.s. of the problem (4), (5) with initial function $s(f_0(x,v))$.

Suppose that $s(u) \in C^1([0,1])$, $s'(u) > 0$, $s(0) = 0$, $s(1) = 1$. and define functions $u^{(s)}(t,x,\lambda)$, $u_0^{(s)}(x,\lambda)$ and measure valued functions $\nu_{t,x}^{(s)}$, $\nu_x^{(s)}$ in the same way as above, in the first part of the proof. By Theorem 1 $\nu_{t,x}^{(s)}$ is a m.s. of the problem (1), (7) with initial data $\nu_x^{(s)}$. The condition (8) in the form (10) implies that $\forall k \in \mathbb{R}$

$$\int_0^1 [|u^{(s)}(t,x,\lambda) - k|_t + \operatorname{div}_x((\varphi(u^{(s)}(t,x,\lambda)) - \varphi(k)) \operatorname{sign}(u^{(s)}(t,x,\lambda) - k))] d\lambda \le 0 \;\text{ in } \mathcal{D}'(\Pi),$$

that is, $\forall h = h(t,x) \in C_0^\infty(\Pi)$, $h \ge 0$

$$\int_0^1 \Big(\int_\Pi [|u^{(s)}(\cdot,\lambda) - k| h_t + ((\varphi(u^{(s)}(\cdot,\lambda)) - \varphi(k)) \operatorname{sign}(u^{(s)}(\cdot,\lambda) - k), \nabla_x h)] dt dx \Big) d\lambda \ge 0.$$

Taking into account that $u^{(s)}(t,x,\lambda)=u(t,x,s^{-1}(\lambda))$, we can transform the last inequality as follows:

$$\int_0^1 R(s^{-1}(\lambda))d\lambda \ge 0, \qquad \text{where we denote}$$

$$R(\mu)=\int_\Pi [|u(\cdot,\mu)-k|h_t+((\varphi(u(\cdot,\mu))-\varphi(k))\,\mathrm{sign}(u(\cdot,\mu)-k),\nabla_x h)]\,dtdx.$$

Making the change $\mu=s^{-1}(\lambda)$, we obtain

$$\int_0^1 R(\mu)s'(\mu)d\mu \ge 0. \tag{23}$$

The derivative $s'(\mu)$ in (23) could be an arbitrary positive continuous function on $(0,1)$ (the normalization condition $\int_0^1 s'(\mu)d\mu=s(1)-s(0)=1$ can be always achieved after multiplication of the function $s(\mu)$ by a constant). Then (23) implies that $R(\mu)\ge 0$ a.e. on $(0,1)$. Because the function $u(t,x,\mu)$ is right-continuous with respect to the variable $\mu\in(0,1)$, the function $R(\mu)$ is also right-continuous and, therefore, $R(\mu)\ge 0$ on $(0,1)$, already without exclusion of any set of null measure. It exactly means (in view of arbitrariness of $k\in\mathbb{R}$ and $h\in C_0^\infty(\Pi)$, $h\ge 0$) that for all $\mu\in(0,1)$ the function $u(t,x,\mu)$ satisfies the condition (3).

It only remains to verify the initial condition $u(0,x,\mu)=u_0(x,\mu)$. As was mentioned above $\nu_{t,x}^{(s)}$ is a m.s. of the problem (1), (7) with the measure valued initial function $\nu_x^{(s)}$. As it follows from the initial condition (7), $\forall p(\lambda)\in C(\mathbb{R})$

$$\operatorname*{ess\,lim}_{t\to 0}\int p(\lambda)d\nu_{t,x}^{(s)}(\lambda)=\int p(\lambda)d\nu_x^{(s)}(\lambda) \quad \text{ in } L^1_{loc}(\mathbb{R}^n). \tag{24}$$

Using the change $\mu=s^{-1}(\lambda)$ we now obtain

$$\int p(\lambda)d\nu_{t,x}^{(s)}(\lambda)=\int_0^1 p(u^{(s)}(t,x,\lambda))d\lambda=$$
$$\int_0^1 p(u(t,x,s^{-1}(\lambda)))d\lambda=\int_0^1 p(u(t,x,\mu))s'(\mu)d\mu$$

and, in similar way,

$$\int p(\lambda)d\nu_x^{(s)}(\lambda)=\int_0^1 p(u_0(x,\mu))s'(\mu)d\mu.$$

Using these equalities we can rewrite the limit relation (24) in the form

$$\int_0^1 p(u(t,x,\mu))s'(\mu)d\mu \to \int_0^1 p(u_0(x,\mu))s'(\mu)d\mu \quad \text{ in } L^1_{loc}(\mathbb{R}^n) \tag{25}$$

as $t\to 0$, where t takes values in a set of full measure $(0,+\infty)\setminus\mathcal{E}$, which obviously can be chosen the same for $p(\lambda)=\lambda,\ \lambda^2$ and for a countable family of functions $\{\,s(\mu)=\mu^m\mid m=0,1,\cdots\}$, the linear hull of which is dense in $L^1((0,1))$.

Then (25) implies that $p(u(t,x,\mu)) \to p(u_0(x,\mu))$ as $t \to 0$, $t \notin \mathcal{E}$ $*$-weakly in $L^\infty(\mathbb{R}^{n+1})$ for $p(\lambda) = \lambda,\ \lambda^2$. Using these limit relation we obtain that for $\rho(x) = e^{-|x|}$

$$\int_{R^{n+1}} (u(t,x,\mu) - u_0(x,\mu))^2 \rho(x) dx d\mu = \int_{R^{n+1}} (u^2(t,x,\mu) - u_0^2(x,\mu))\rho(x) dx d\mu -$$
$$2\int_{R^{n+1}} (u(t,x,\mu) - u_0(x,\mu)) u_0(x,\mu)\rho(x) dx d\mu \underset{t\to 0,\, t\notin\mathcal{E}}{\to} 0$$

and, therefore, $u(t,x,\mu) \to u_0(x,\mu)$ in $L^1_{loc}(\mathbb{R}^{n+1})$. Denote by I the set of continuity points of the function $P(\mu) = \int_{\mathbb{R}^n} u_0(x,\mu)\rho(x)dx$. By monotonicity of $P(\mu)$ the set I has the at most countable complement. Take $\mu \in I$ and show that

$$u(t,\cdot,\mu) \to u_0(\cdot,\mu) \quad \text{in } L^1_{loc}(\mathbb{R}^n) \tag{26}$$

as $t \to 0$, $t \notin \mathcal{E}$.

Assuming that (26) fails, we can construct a sequence $t_r \notin \mathcal{E}$ such that $t_r \underset{r\to\infty}{\to} 0$ and

$$\liminf_{r\to\infty} \int_{\mathbb{R}^n} |u(t_r,x,\mu) - u_0(x,\mu)|\rho(x)dx > \delta \tag{27}$$

where δ is a some positive constant.

As it was established above $u(t,x,\lambda) \to u_0(x,\lambda)$ in $L^1_{loc}(\mathbb{R}^{n+1})$ as $t \to 0$, $t \notin \mathcal{E}$. Therefore, extracting a subsequence (which we denote again by t_r), we can assume that $u(t_r,x,\lambda) \underset{r\to\infty}{\to} u_0(x,\lambda)$ a.e. in $\mathbb{R}^{n+1}$. Hence, for almost all $\lambda \in (0,1)$ $u(t_r,\cdot,\lambda) \underset{r\to\infty}{\to} u_0(\cdot,\lambda)$ a.e. on $\mathbb{R}^n$ and by Lebesgue theorem on dominated convergence

$$\lim_{r\to\infty} \int_{\mathbb{R}^n} |u(t_r,x,\lambda) - u_0(x,\lambda)|\rho(x)dx = 0. \tag{28}$$

Let Λ be a set of λ, for which (28) holds. Because Λ has full measure then it is dense in $(0,1)$ and we can choose (taking into account that $\mu \in I$ is a continuity point of the function P) values $\mu^+,\ \mu^- \in \Lambda$ so that $\mu^- < \mu < \mu^+$ and

$$|P(\mu^+) - P(\mu^-)| = \int_{\mathbb{R}^n} (u_0(x,\mu^-) - u_0(x,\mu^+))\rho(x)dx < \frac{\delta}{2}. \tag{29}$$

By monotonicity of the functions $\mu \mapsto u(t,x,\mu)$, $\mu \mapsto u_0(x,\mu)$ we get that $\forall r \in \mathbb{N}$

$$u(t_r,x,\mu^+) \le u(t_r,x,\mu) \le u(t_r,x,\mu^-),\ u_0(x,\mu^+) \le u_0(x,\mu) \le u_0(x,\mu^-) \text{ and}$$

$$\begin{gathered} |u(t_r,x,\mu) - u_0(x,\mu)| \le \max(u(t_r,x,\mu^-), u_0(x,\mu^-)) - \\ \min((u(t_r,x,\mu^+), u_0(x,\mu^+)) \le |u(t_r,x,\mu^-) - u_0(x,\mu^-)| + \\ +|u(t_r,x,\mu^+) - u_0(x,\mu^+)| + |u_0(x,\mu^-) - u_0(x,\mu^+)|. \end{gathered} \tag{30}$$

In the latter estimate we make use of the simple inequality: for $p < P$, $q < Q$ $\max(P,Q) - \min(p,q) \le |P-Q| + |p-q| + |Q-q|$.

Integrate (30) over the measure $\rho(x)dx$ and pass to the limit in the obtained inequality as $r \to \infty$. Then, taking into account that $\mu^+, \mu^- \in \Lambda$ and, therefore, the relation (28) holds for $\lambda = \mu^+, \mu^-$, we obtain the estimate

$$\limsup_{r\to\infty} \int_{\mathbb{R}^n} |u(t_r, x, \mu) - u_0(x, \mu)|\rho(x)dx \le \int_{\mathbb{R}^n} |u_0(x, \mu^-) - u_0(x, \mu^+)|\rho(x)dx.$$

Hence, using the equality $|u_0(x, \mu^-) - u_0(x, \mu^+)| = u_0(x, \mu^-) - u_0(x, \mu^+)$ and the condition (29), we derive that

$$\limsup_{r\to\infty} \int_{\mathbb{R}^n} |u(t_r, x, \mu) - u_0(x, \mu)|\rho(x)dx \le \frac{\delta}{2},$$

that contradicts (27). Thus, for $\mu \in I$ the relation (26) holds that is the initial condition $u(0, x, \mu) = u_0(x, \mu)$ is satisfied. Therefore, $u(t, x, \mu)$ is a g.e.s. of the problem (1), (2) with initial function $u_0(x, \mu)$. Further, we show that it remains true for all $\mu \in (0, 1)$. Fix $\mu \in (0, 1)$ and denote by $\bar{u}(t, x, \mu)$ the g.e.s. of the problem (1), (2) with initial data $u_0(x, \mu)$. Remark that for $\mu \in I$ $\bar{u}(t, x, \mu) = u(t, x, \mu)$, as consequence of uniqueness of g.e.s. By right-continuity of functions $u_0(x, \mu)$, $u(t, x, \mu)$ with respect to $\mu \in (0, 1)$ and property of continuous dependence of g.e.s. on initial data (see [1, 2]), we conclude that

$$u(t, x, \lambda) \to u(t, x, \mu), \quad u(t, x, \lambda) \to \bar{u}(t, x, \mu) \quad \text{in } L^1_{loc}(\Pi)$$

as $\lambda \to \mu$, $\lambda > \mu$, $\lambda \in I$. Therefore, $u(t, x, \mu) = \bar{u}(t, x, \mu)$ a.e. on Π and by the construction $u(t, x, \mu)$ is a g.e.s. of (1), (2) with initial data $u_0(x, \mu)$. That exactly means that $\nu_{t,x}$ is a s.m.s. of the problem (1), (7). The proof is complete.

Remark that Theorem 2 together with the result on existence and uniqueness of s.m.s. implies that there exists an unique function $f(t, x, v) \in F_M(\Pi)$ (for some $M > 0$) such that for any nondecreasing function $s(u) \in C^1([0, 1])$ the function $s(f(t, x, v))$ is a g.s. of the problem (4), (5) with corresponding initial data $s(f_0(x, v))$.

§ 2. Entropy solutions of the kinetic problem.

Introduce the following important class of solutions to the kinetic problem (4), (5).

Definition 4. A function $f(t, x, v) \in F_M(\Pi)$, for some $M > 0$, is called an entropy solution (e.s. for short) of the problem (4), (5) if for any function $g(v) \in F_c$

$$\frac{\partial}{\partial t} \int (f(t, x, v) - g(v))^2 dv + \operatorname{div}_x \int \varphi'(v)(f(t, x, v) - g(v))^2 dv \le 0 \ \text{ in } \mathcal{D}'(\Pi), \quad (31)$$

and the initial condition (6) is satisfied.

Remark that Definition 4 does not include the measure m, which plays the role of a free parameter in the equation (4). In this sense, the notion of entropy solution of the kinetic problem seems to be more natural in comparison with the notion of generalized solution. The main result of this section is the following

Theorem 3. *There exists an unique e.s.* $f(t, x, v)$ *of the problem* (4), (5). *Besides,* $f(t, x, v) = \nu_{t,x}((v, +\infty))$, *where* $\nu_{t,x}$ *is the corresponding strong m.s. of the problem* (1), (7).

Proof. Let $f_0(x,v)=\nu_x^0((v,+\infty))$ and $\nu_{t,x}\in MV(\Pi)$ be the unique s.m.s. of the problem (1), (7), $f(t,x,v)=\nu_{t,x}((v,+\infty))$ be the corresponding distribution function. Suppose that $M>\|\nu_{t,x}\|_\infty$ and $g(v)\in F_M\cap C^1(\mathbb{R})$. It is clear that the integrands in (31) vanish for $|v|>M$. Then, by Theorem 2 for $s(u)=u,u^2$

$$\frac{\partial}{\partial t}f+(\varphi'(v),\nabla_x f)=\frac{\partial}{\partial v}m_1, \tag{32}$$

$$\frac{\partial}{\partial t}f^2+(\varphi'(v),\nabla_x f^2)=\frac{\partial}{\partial v}m_2 \text{ in } \mathcal{D}'(\Pi), \tag{33}$$

where m_1,m_2 are nonnegative locally finite measures on $\Pi\times(-M,M)$. Let $h(t,x)\in C_0^\infty(\Pi)$, $h\ge 0$. Then using the equalities (32), (33), we obtain

$$\begin{aligned}
&\int_{\Pi\times(-M,M)}(f(t,x,v)-g(v))^2\{h_t(t,x)+(\varphi'(v),\nabla_x h)\}dtdxdv=\\
&\int_{\Pi\times(-M,M)}(f(t,x,v))^2\{h_t(t,x)+(\varphi'(v),\nabla_x h)\}dtdxdv-\\
&2\int_{\Pi\times(-M,M)}f(t,x,v)g(v)\{h_t(t,x)+(\varphi'(v),\nabla_x h)\}dtdxdv=\\
&\int_{\Pi\times(-M,M)}h_v(t,x)dm_2(t,x,v)-2\int_{\Pi\times(-M,M)}h(t,x)g'(v)dm_1(t,x,v)=\\
&-2\int_{\Pi\times(-M,M)}h(t,x)g'(v)dm_1(t,x,v)\ge 0.
\end{aligned}$$

Using approximation of an arbitrary function $g(v)\in F_M$ by a sequence of smooth functions, we derive that $\forall g(v)\in F_M$ $\forall h\in C_0^\infty(\Pi)$, $h\ge 0$

$$\int_{\Pi\times\mathbb{R}}(f(t,x,v)-g(v))^2\{h_t(t,x)+(\varphi'(v),\nabla_x h)\}dtdxdv\ge 0.$$

Here M takes an arbitrary great enough value and we obtain that the relation (31) holds. The initial condition (6) is also fulfilled by Theorem 1. Thus, $f(t,x,v)$ is a e.s. of the problem (4), (5). By construction $f(t,x,v)$ satisfies the last assertion of the theorem.

For the proof of uniqueness we apply the method of "doubling of variables" developed by S.N.Kruzhkov in [1, 2]. Let $f=f(t,x,v)$, $\bar f=\bar f(t,x,v)$ be two e.s. of the problem (4), (5). Then, for large enough $M>0$ $f(t,x,\cdot),\bar f(t,x,\cdot)\in F_M$ and by (31) with $g(v)=\bar f(\tau,\xi,v)$, $(\tau,\xi)\in\Pi$

$$\begin{aligned}
&\frac{\partial}{\partial t}\int(f(t,x,v)-\bar f(\tau,\xi,v))^2dv+\\
&\operatorname{div}_x\int\varphi'(v)(f(t,x,v)-\bar f(\tau,\xi,v))^2dv\le 0 \text{ in } \mathcal{D}'(\Pi\times\Pi).
\end{aligned}$$

In the same way, changing places of variables (t,x) and (τ,ξ), and also solutions f and $\bar f$, we obtain that

$$\begin{aligned}
&\frac{\partial}{\partial\tau}\int(f(t,x,v)-\bar f(\tau,\xi,v))^2dv+\\
&\operatorname{div}_\xi\int\varphi'(v)(f(t,x,v)-\bar f(\tau,\xi,v))^2dv\le 0 \text{ in } \mathcal{D}'(\Pi\times\Pi).
\end{aligned}$$

Putting these relations together we derive that in $\mathcal{D}'(\Pi\times\Pi)$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial\tau}\right)\int(f(t,x,v)-\bar f(\tau,\xi,v))^2dv+$$
$$(\operatorname{div}_x+\operatorname{div}_\xi)\int\varphi'(v)(f(t,x,v)-\bar f(\tau,\xi,v))^2dv\le 0. \tag{34}$$

Choose a function $\beta(s)\in C_0^\infty(\mathbb{R})$, $\beta(s)\ge 0$, $\operatorname{supp}\beta\subset[0,1]$, $\int\beta(s)ds=1$ and set for $\nu\in\mathbb{N}$ $\beta_\nu(s)=\nu\beta(\nu s)$. It is easy to see that $\beta_\nu(s)\in C_0^\infty(\mathbb{R})$, $\beta_\nu(s)\ge 0$, $\operatorname{supp}\beta_\nu\subset[0,1/\nu]$, $\int\beta_\nu(s)ds=1$ and, in particular, the sequence $\beta_\nu(s)$ converges to the Dirac δ-function in the space of distributions $\mathcal{D}'(\mathbb{R})$ as $\nu\to\infty$. Let $h=h(t,x)\in C_0^\infty(\Pi)$, $h\ge 0$; $\rho_\nu(t-\tau,x-\xi)=\beta_\nu(\tau-t)\prod_{i=1}^n\beta_\nu(\xi_i-x_i)$. Then the function $p=p(t,x;\tau,\xi)=g(t,x)\rho_\nu(t-\tau,x-\xi)\in C_0^\infty(\Pi\times\Pi)$, $p\ge 0$. Applying the inequality (34) to the test function p we obtain, after simple transforms, that

$$\int_{\Pi\times\Pi\times\mathbb{R}}(f(t,x,v)-\bar f(\tau,\xi,v))^2\{h_t+$$
$$(\varphi'(v),\nabla_x h)\}\rho_\nu(t-\tau,x-\xi)dtdxd\tau d\xi dv\ge 0. \tag{35}$$

Let E be a set of $(t,x,v)\in\Pi\times\mathbb{R}$ such that (t,x) is the Lebesgue point of the function $(t,x)\mapsto\bar f(t,x,v)$. Then E has full Lebesgue measure in $\Pi\times\mathbb{R}$ and in view of the estimate

$$|(f(t,x,v)-\bar f(\tau,\xi,v))^2-(f(t,x,v)-\bar f(t,x,v))^2|\le 2|\bar f(\tau,\xi,v)-\bar f(t,x,v)|$$

for $(t,x,v)\in E$ we have the following limit relation

$$\int_\Pi(f(t,x,v)-\bar f(\tau,\xi,v))^2\rho_\nu(t-\tau,x-\xi)d\tau d\xi\underset{\nu\to\infty}{\to}(f(t,x,v)-\bar f(t,x,v))^2$$

and, passing in (35) to the limit as $\nu\to\infty$, we derive (using also the Lebesgue theorem on dominated convergence)

$$\int_{\Pi\times\mathbb{R}}(f(t,x,v)-\bar f(t,x,v))^2\{h_t+(\varphi'(v),\nabla_x h)\}dtdxdv\ge 0 \tag{36}$$

for all nonnegative test functions $h\in C_0^\infty(\Pi)$, that is

$$\frac{\partial}{\partial t}\int(f(t,x,v)-\bar f(t,x,v))^2dv+$$
$$\operatorname{div}_x\int\varphi'(v)(f(t,x,v)-\bar f(t,x,v))^2dv\le 0\ \text{ in }\mathcal{D}'(\Pi).$$

It is clear that the integrands in (36) vanish for $|v|>M$. We set $N=\max\limits_{|v|\le M}|\varphi'(v)|$ ($|v|$ is the Euclid norm of a vector v). Let $T>0$, $R>1$, $p(u)\in C^\infty(\mathbb{R})$, $p'(u)\ge 0$, $p(u)=0$ for $u\le 0$ and $p(u)=1$ for $u\ge 1$; $q(t)\in C_0^\infty((0,T))$, $q(t)\ge 0$. Take for $\nu\in\mathbb{N}$ a test function $h(t,x)=p(R+N(T-t)-|x|)q(t)$. Obviously, $h=h(t,x)\in C_0^\infty(\Pi)$, $h\ge 0$ and by (36)

$$\int_{\Pi\times\mathbb{R}}(f(t,x,v)-\bar f(t,x,v))^2p(R+N(T-t)-|x|)q'(t)dtdxdv+$$
$$\int_{\Pi\times\mathbb{R}}(f(t,x,v)-\bar f(t,x,v))^2\{p_t+(\varphi'(v),\nabla_x p)\}q(t)dtdxdv\ge 0. \tag{37}$$

We see that $p_t = -N|\nabla_x p|$. Hence for $|v| \le M$

$$p_t + (\varphi'(v), \nabla_x p) \le p_t + |\varphi'(v)| \cdot |\nabla_x p| \le p_t + N|\nabla_x p| = 0.$$

Therefore, the second integral in (37) is nonpositive that implies

$$\int_{\Pi\times\mathbb{R}} (f(t,x,v) - \bar{f}(t,x,v))^2 p(R + N(T-t) - |x|) q'(t) dt dx dv \ge 0.$$

This inequality can be written in the form $\int I(t)q'(t)dt \ge 0$, where we denote

$$I(t) = \int_{\mathbb{R}^{n+1}} (f(t,x,v) - \bar{f}(t,x,v))^2 p(R + N(T-t) - |x|) dx dv$$

and, because $q(t) \in C_0^\infty((0,T))$ such that $q(t) \ge 0$ is arbitrary, we find that $I'(t) \le 0$ in $\mathcal{D}'((0,T))$. Then, as it follows from the initial conditions for e.s. f and $\bar{f}$, $\operatorname*{ess\,lim}_{t\to 0} I(t) = 0$. Therefore, $I(t) \le 0$ a.e. on $(0,T)$. By nonnegativity of $I(t)$, we conclude that $I(t) = 0$ a.e. on $(0,T)$. At last, taking into account that the parameters $R > 1$, $T > 0$ are arbitrary, we obtain that $f(t,x,v) = \bar{f}(t,x,v)$ a.e. on $\Pi \times \mathbb{R}$. Thus, an e.s. $f(t,x,v)$ is unique, which completes the proof.

Remark 1. As it follows from the proofs of Theorem 3 and Theorem 2, $f(t,x,v)$ is an e.s. of the problem (4), (5) if and only if $f(t,x,v)$ and $f^2(t,x,v)$ are g.s. of this problem with the corresponding initial functions. Therefore, in addition to Theorem 2, we can conclude that *the sufficient condition for a bounded measure valued function $\nu_{t,x}$ to be a strong m.s. is that the only two functions $s(f(t,x,v))$, with $s(u) = u, u^2$, are g.s. of the problem* (4), (5). Here $f(t,x,v) = \nu_{t,x}((v,+\infty))$, as usual.

§3. Approximating problem.

We consider here the approximation scheme of relaxation type for solving the kinetic problem, which was proposed in paper [5], and extends results of [3] to the case of measure valued solutions. As it was establised in [5], some subsequence of the approximating sequence converges to g.s. $f(t,x,v)$ of the problem (4), (5), thus defining some m.s. of the original problem (1), (7). Now we are able to revise this statement. We shall prove that the approximating sequence converges to the unique e.s. of the problem (4), (5), which corresponds in view of Theorems 2,3 to the strong m.s. of the problem (1), (7).

To describe the approximating problem we need a notion of *nonincreasing rearrangement.*

Let $h(v)$ be a measurable function on some interval $(-M,M)$. We set

$$m_h(\lambda) = -M + \operatorname{meas}\{v \in (-M,M) \,|\, h(v) > \lambda\},$$

where meas is the Lebesgue measure. The nonincreasing rearrangement of the function $h(v)$ is defined as follows (see [14], § 10.12). $h^*(v) = \inf\{\lambda \in \mathbb{R} \,|\, m_h(\lambda) \le v\}$, $v \in (-M,M)$. As is well known, $h^*(v)$ is a nonincreasing right-continuous function equimeasurable with $h(v)$, that is, $m_h(\lambda) \equiv m_{h^*}(\lambda)$. Since h^* and h are equimeasurable, the function $p(h(v))$, $p(h^*(v))$ are simultaneously integrable (or not) on $(-M,M)$ for an arbitrary Borel function $p(u)$, and if these functions are integrable, then (see [14, § 10.12])

$$\int_{-M}^{M} p(h^*(v))dv = \int_{-M}^{M} p(h(v))dv. \tag{38}$$

We now consider the equation

$$f_t + (\varphi'(v), \nabla_x f) = r(g-f), \tag{39}$$

$f = f(t,x,v) \in L^\infty(\Pi\times\mathbb{R})$, $r = \text{const} > 0$, the nonlinear operator $f \mapsto Ff = g$ is defined as follows

$$g = g(t,x,v) = \begin{cases} 1 & , \ v \le -M \\ 0 & , \ v \ge M \\ l(f^*(t,x;v)) & , \ v \in (-M,M) \end{cases},$$

where $f^*(t,x;v)$ is a nonincreasing rearrangement of the function $v \mapsto f(t,x,v)$ with respect to $v \in (-M,M)$ (since for almost all $(t,x) \in \Pi$ the function $v \mapsto f(t,x,v)$ is measurable, the rearrangement $f^*(t,x;v)$ is well defined); $l(f) = \max(\min(f,1),0)$ is a cut off function.

Further, let $\nu_x^0 \in MV(\mathbb{R}^n)$, $\quad \|\nu_x^0\|_\infty \le M$,
$f_0(x,v) = \nu_x^0((v,+\infty)) \in F_M(\mathbb{R}^n)$.

Consider the Cauchy problem for (39) with initial condition

$$f(0,x,v) = f_0(x,v), \tag{40}$$

which is understood in the sense of the limit relation

$$\operatorname*{ess\,lim}_{t\to 0} f(t,\cdot) = f_0 \qquad \text{in } L^1_{loc}(\mathbb{R}^{n+1}).$$

We shall consider generalized solutions (briefly - g.s.) $f(t,x,v) \in L^\infty(\Pi\times\mathbb{R})$ of the problem (39), (40), which satisfy (39) in $\mathcal{D}'(\Pi\times\mathbb{R})$ and the initial condition (40). The following theorem was proved in [5].

Theorem 4. *There exists an unique g.s. $f(t,x,v)$ of the problem* (39), (40)*, and in addition $0 \le f(t,x,v) \le 1$ a.e. in $\Pi\times\mathbb{R}$; $f(t,x,v) = 0$ for a.e. $(t,x,v) \in \Pi\times(M,+\infty)$, $f(t,x,v) = 1$ for a.e. $(t,x,v) \in \Pi\times(-\infty,-M)$. In particular, for a.e. $(t,x)\in\Pi$ we have $0 \le f^*(t,x;v) \le 1$ for all $v \in (-M,M)$, and therefore $g(t,x,v) = l(f^*(t,x;v)) = f^*(t,x;v)$.*

Theorem 5. *Let $f_1(t,x,v)$, $f_2(t,x,v)$ be g.s. of the problem* (39), (40) *with initial functions $f_1^0(x,v)$, $f_2^0(x,v) \in F_M(\mathbb{R}^n)$ respectively. Then*

$$\frac{\partial}{\partial t}\int (f_1(t,x,v) - f_2(t,x,v))^2 dv +$$
$$\operatorname{div}_x \int \varphi'(v)(f_1(t,x,v) - f_2(t,x,v))^2 dv \le 0 \quad \text{in } \mathcal{D}'(\Pi). \tag{41}$$

Proof. Since $f_1 = f_1(t,x,v)$, $f_2 = f_2(t,x,v)$ are g.s. of the problem (39), (40), then

$$\frac{\partial}{\partial t} f_1 + (\varphi'(v), \nabla_x f_1) = r(g_1 - f_1), \quad \frac{\partial}{\partial t} f_2 + (\varphi'(v), \nabla_x f_2) = r(g_2 - f_2),$$

in $\mathcal{D}'(\Pi\times\mathbb{R})$, where $g_1=Ff_1$, $g_2=Ff_2$. Taking the difference of the first and the second equalities we obtain

$$\frac{\partial}{\partial t}(f_1-f_2)+(\varphi'(v),\nabla_x(f_1-f_2))=r((g_1-g_2)-(f_1-f_2)).$$

Hence, multiplying by $2(f_1-f_2)$ and integrating with respect to v, we derive the equality

$$\frac{\partial}{\partial t}\int(f_1-f_2)^2dv+\operatorname{div}_x\int\varphi'(v)(f_1-f_2)^2dv$$
$$=2r\int((g_1-g_2)(f_1-f_2)-(f_1-f_2)^2)dv \quad \text{in } \mathcal{D}'(\Pi). \tag{42}$$

To justify this equality rigorously we should approximate the functions f_i, g_i with $i=1,2$ by sequences of the average functions, which satisfy the equality (42) in the classical sense, and then pass to the limit by the approximation parameter.

By Lemma 5, $f_1-f_2=0$ for almost all $(t,x,v)\in\Pi\times(\mathbb{R}\setminus[-M,M])$, and by Hölder's inequality,

$$\int_{\mathbb{R}}(g_1-g_2)(f_1-f_2)dv=\int_{-M}^{M}(g_1-g_2)(f_1-f_2)dv\le$$
$$\left(\int_{-M}^{M}(g_1-g_2)^2dv\right)^{1/2}\left(\int_{-M}^{M}(f_1-f_2)^2dv\right)^{1/2}\le\int_{-M}^{M}(f_1-f_2)^2dv \tag{43}$$

for a.e. $(t,x)\in\Pi$. Indeed, by Remark 2 for a.e. $(t,x)\in\Pi$ $g_1(t,x,v)=f_1^*(t,x;v)$, $g_2(t,x,v)=f_2^*(t,x;v)$ for any $v\in[-M,M]$ and by the property of rearrangements (see, for instance, lemma 4 in [5])

$$\int_{-M}^{M}(g_1(t,x,v)-g_2(t,x,v))^2dv\le\int_{-M}^{M}(f_1(t,x,v)-f_2(t,x,v))^2dv.$$

By (43), the right-hand side of (42) is nonpositive, which implies (41). The proof is complete.

Let $f_r(t,x,v)$, $r\in\mathbb{N}$ be the sequence of g.s. of the problem (39), (40) written as follows:

$$f_t+(\varphi'(v),\nabla_x f)=r(g-f),\quad g=Ff;\quad f(0,x,v)=f_0(x,v)\in F_M(\mathbb{R}^n).$$

We claim that this sequence converges in $L^1_{loc}(\Pi\times\mathbb{R})$ to the unique e.s. $f(t,x,v)$ of the problem (4), (5).

Theorem 6. *The sequence* $f_r(t,x,v)$ *converges in* $L^1_{loc}(\Pi\times\mathbb{R})$ *to the e.s.* $f(t,x,v)\in F_M(\Pi)$ *of the problem* (4), (5).

Proof. As was shown in [5], after extracting a subsequence (we keep the notation) $f_r(t,x,v)\to f(t,x,v)$ in $L^1_{loc}(\Pi\times\mathbb{R})$ as $r\to\infty$, where $f(t,x,v)\in F_M(\Pi)$ is a g.s. of the problem (4), (5).

To verify the condition (31) we choose a function $g(v)\in F_c$. We can magnify the parameter M in order that $g(v)\in F_M$. Clearly, the function $\bar f(t,x,v)=g(v)$

is a g.s. of the problem (39), (40) for any $r>0$ and by Theorem 5, applying to pair of solutions f_r, g,

$$\frac{\partial}{\partial t}\int (f_r(t,x,v)-g(v))^2 dv + \operatorname{div}_x \int \varphi'(v)(f_r(t,x,v)-g(v))^2 dv \le 0 \ \text{ in } \mathcal{D}'(\Pi),$$

which implies, in the limit as $r\to\infty$, the relation (31). Hence $f(t,x,v)$ is an e.s. of the problem (4), (5). To complete the proof observe that in view of uniqueness of e.s. the limit function $f(t,x,v)$ doesn't depend on the choice of convergent subsequence of approximations. This implies that the original sequence $f_r(t,x,v)$ strongly converges to $f(t,x,v)$ as $r\to\infty$.

The proof is complete.

§4. Some generalizations to the case of hyperbolic systems.

We consider the following first-order quasilinear system

$$\begin{gathered} u_t + A(u)u_x = 0, \\ u=u(t,x)\in\mathbb{R}^n,\ (t,x)\in\Pi=\mathbb{R}_+\times\mathbb{R}. \end{gathered} \tag{44}$$

Suppose that the matrix $A(u)$ depends continuously on $u\in\mathbb{R}^n$. The system (44) is assumed to be (generally nonstrictly) hyperbolic. This means that $A(u)$ could be diagonalized for any fixed $u\in\mathbb{R}^n$, i.e. there exist an invertible real matrix $Q(u)$ and a diagonal matrix $\Lambda(u)=\operatorname{diag}\{\lambda_1(u),\dots,\lambda_n(u)\}$, $\lambda_i(u)\in\mathbb{R}$, $i=1,\dots,n$ such that $A(u)=Q^{-1}(u)\Lambda(u)Q(u)$. Introduce the symmetric matrices $B(u)=Q^\top(u)Q(u)$, $C(u)=Q^\top(u)\Lambda(u)Q(u)$ so that the matrix $B(u)$ is positively defined and $C(u)=B(u)A(u)$. After multiplying by $B(u)$ the system (44) reduces to the following symmetric form

$$B(u)u_t + C(u)u_x = 0.$$

Remark that we can assume that the map $u\mapsto Q(u)$ is Borel and locally bounded. Suppose also that for some constant $N>0$

$$|\lambda_i(u)|\le N\ \forall u\in\mathbb{R}^n,\ i=1,\dots,n. \tag{45}$$

Let X be a Hilbert spase of vector fields $f(u)=(f_1(u),\dots,f_n(u))$, $u\in\mathbb{R}^n$ with finite value of

$$\|f\|^2=\int_{\mathbb{R}^n}(B(u)f(u),f(u))du$$

(here $(\cdot,\cdot)$ is a scalar product in $\mathbb{R}^n$). The scalar product on X is defined by the equality

$$(f,g)=\int_{\mathbb{R}^n}(B(u)f(u),g(u))du.$$

Remark that the correspondence $f\mapsto A(u)f(u)$ yields a bounded symmetric linear operator $\tilde{A}$ on X. Indeed, boundedness of $\tilde{A}$ follows from the equalities

$$(B(u)A(u)f(u),A(u)f(u))=(Q^\top(u)\Lambda(u)Q(u)f(u),Q^{-1}(u)\Lambda(u)Q(u)f(u))=$$
$$(\Lambda(u)Q(u)f(u),\Lambda(u)Q(u)f(u))\le N^2(Q(u)f(u),Q(u)f(u))=N^2(B(u)f(u),f(u)),$$

where we use (45). Besides, we see that $\|\tilde{A}\| \le N$. Symmetry of $\tilde{A}$ is proved directly by relations

$$(B(u)A(u)f(u), g(u)) =$$
$$(C(u)f(u), g(u)) = (f(u), C(u)g(u)) = (B(u)f(u), A(u)g(u)).$$

Fix some convex closed set $H \subset X$, the bounded closed subsets of which are compact.

By analogy with the scalar case we study the Cauchy problem for the kinetic system corresponded to (44)

$$f_t + A(u)f_x = 0 \tag{46}$$

where the unknown vector $f = f(t,x,u) \in \mathbb{R}^n$ belongs to the set $L^2_{loc}(\Pi, H)$ (which is a convex closed subset of the space $L^2_{loc}(\Pi, X)$ determined by condition: $f(t,x,\cdot) \in H$ for a.e. $(t,x) \in \Pi$). In an initial condition

$$f(0,x,u) = f_0(x,u) \tag{47}$$

the function f_0 is assumed to belong $L^2_{loc}(\mathbb{R}, H)$.

Definition 5. A vector function $f = f(t,x,u) \in L^2_{loc}(\Pi, H)$ is called *an entropy solution* (briefly - e.s.) of the problem (46), (47) if $\forall g = g(u) \in H$

$$\frac{\partial}{\partial t}\|f(t,x,\cdot) - g\|^2 + \frac{\partial}{\partial x}(\tilde{A}(f(t,x,\cdot) - g), f(t,x,\cdot) - g) \le 0 \text{ in } \mathcal{D}'(\Pi); \tag{48}$$

$$\operatorname*{ess\,lim}_{t\to 0} \|f(t,x,\cdot) - f_0(x,\cdot)\| = 0 \text{ in } L^2_{loc}(\mathbb{R}). \tag{49}$$

We want to establish existence and uniqueness of an e.s. to the problem (46), (47). Uniqueness of an e.s. is proved in the same way as in the scalar case (see Theorem 3). To prove existence we make use of an approximation scheme similar to (39). Namely, let $P_H : X \mapsto H$ be the projection map so that $\|f - P_H f\| = \min\limits_{h\in H} \|f - h\|$. Because X is the Hilbert space and H is its convex closed subset the projection $P_H f$ is well-defined. Besides, the following important property holds:

$$\|P_H f - P_H g\| \le \|f - g\| \ \forall f, g \in X. \tag{50}$$

Indeed, let $\bar{f} = P_H f$, $\bar{g} = P_H g$. Then, by convexity of H, for each $\alpha \in [0,1]$ the convex combination $z_\alpha = (1-\alpha)\bar{f} + \alpha\bar{g} \in H$ and by definition of the projection $\forall \alpha \in [0,1]$

$$\|f - \bar{f}\|^2 \le \|f - z_\alpha\|^2 = \|f - \bar{f} + \alpha(\bar{f} - \bar{g})\|^2 =$$
$$\|f - \bar{f}\|^2 + 2\alpha(f - \bar{f}, \bar{f} - \bar{g}) + \alpha^2\|\bar{f} - \bar{g}\|^2,$$

which implies that $(f - \bar{f}, \bar{f} - \bar{g}) \ge 0$. Similarly, changing places of f and g, we derive the inequality $(g - \bar{g}, \bar{g} - \bar{f}) \ge 0$. Putting together the obtained inequalities we conclude that $(f - g - (\bar{f} - \bar{g}), \bar{f} - \bar{g}) \ge 0$ and by Cauchy-Bunyakovskii inequality we get

$$\|\bar{f} - \bar{g}\|^2 \le (f - g, \bar{f} - \bar{g}) \le \|f - g\| \cdot \|\bar{f} - \bar{g}\|,$$

and (50) follows.

Fix $r>0$ and consider the problem

$$f_t + A(u)f_x = -r(f - P_H f), \tag{51}$$
$$f(0,x,u) = f_0(x,u). \tag{52}$$

We shall consider generalized solutions (briefly, g.s.) $f(t,x,u) \in L^2_{loc}(\Pi, X)$ of the problem (51), (52), which satisfy the equation (51) in $\mathcal{D}'(\Pi \times \mathbb{R})$ and the initial condition (52) in the sense of relation (49). Our first result is the following

Theorem 7. *There exists an unique g.s. of the problem* (51), (52).

Proof. Fix some $R>0$ and define the triangle $K_R = \{\ (t,x) \in \Pi \mid |x| < R - Nt\ \}$ and the space $L \subset L^2_{loc}(K_R, X)$ consisting of vectors $f = f(t,x,u)$ with finite norm

$$\|f\| = \operatorname*{ess\,sup}_{t>0} \left(\int_{|x| \le R - Nt} \|f(t,x,\cdot)\|^2 dx \right)^{1/2}.$$

It is easy to verify that L with introduced norm is a Banach space. Let $p = p(t,x,u) \in L$ and $f = f(t,x,u) \in L^2_{loc}(K,X)$ be the unique g.s. to the Cauchy problem for the linear hyperbolic system

$$f_t + A(u)f_x = -r(f - p), \ (t,x) \in K_R, \tag{53}$$

with initial condition

$$f(0,x,u) = f_0(x,u). \tag{54}$$

This solution can be constructed by the method of characteristics. Observe that in view of condition (45) the triangle K_R is filled by characteristics which start from points of the base (where $t=0$). Consider a map $(\tilde{Q}f)(u) = Q(u)f(u)$. It is clear that this map is a linear isometry the space X onto the Hilbert space $L^2(\mathbb{R}^n, \mathbb{R}^n)$ with usual scalar product $(f,g)_2 = \int_{\mathbb{R}^n} (f(u), g(u))du$. We shall denote by $\|\cdot\|_2$ the norm in the latter space. We set $\tilde{f} = \tilde{f}(t,x,u) = \tilde{Q}f(t,x,\cdot)(u)$. Multiplying (53) by the matrix $Q(u)$ from the left, we obtain that

$$\tilde{f}_t + \Lambda(u)\tilde{f}_x = -r(\tilde{f} - \tilde{p}) \ \text{ in } \mathcal{D}'(K_R),$$

where $\tilde{p} = \tilde{p}(t,x,u) = \tilde{Q}p(t,x,\cdot)(u)$. Taking components of the last equality we get for $i = 1, \dots, n$ that

$$(\tilde{f}_i)_t + \lambda_i(u)(\tilde{f}_i)_x = -r(\tilde{f}_i - \tilde{p}_i)$$

and with using the initial condition (54) we derive equalities

$$\tilde{f}_i(t,x,u) = e^{-rt}\tilde{f}_{0i}(x - \lambda_i(u)t, u) + r\int_0^t e^{-r(t-\tau)}\tilde{p}_i(\tau, x - \lambda_i(u)(t-\tau), u)d\tau \tag{55}$$

where $\tilde{f}_{0i}$ are components of the vector $\tilde{f}_0 = \tilde{Q}f_0(x,\cdot)$. Equalities (55) hold for every fixed $t \in [0, R/N]$ and for almost all values (x,u), $|x| \le R - Nt$, $u \in \mathbb{R}^n$ such that the functions $\tilde{f}_{0i}(x - \lambda_i(u)t, u)$ are defined, and the functions $\tau \to \tilde{p}_i(\tau, x - \lambda_i(u)(t-\tau), u)$ are defined and integrable on $[0,t]$. Remark also that for $y = x - \lambda_i(u)(t-\tau)$ we have the estimate $|y| \le |x| + |\lambda_i(u)|(t-\tau) \le R - Nt +$

$N(t-\tau) = R - N\tau$ which implies the inclusion $(\tau, y) \in K_R$. Changing the variables $y = x - \lambda_i(u)(t-\tau)$ we derive the estimate

$$\int\limits_{\tau\in[0,t],(t,x,u)\in K_R\times\mathbb{R}^n} (\tilde{p}_i(\tau, x - \lambda_i(u)(t-\tau), u))^2 d\tau\, dx du \le$$

$$\int_{K_R\times\mathbb{R}^n} (\tilde{p}_i(\tau, y, u))^2 d\tau\, dy du < \infty.$$

In particular the functions $\tilde{p}_i(\tau, x - \lambda_i(u)(t-\tau), u)$ are integrable by variable $\tau \in [0, t]$ for almost all $(x, u) \in [-R + Nt, R - Nt] \times \mathbb{R}^n$.

By Minkowski inequality

$$\left(\sum_{i=1}^n \int\limits_{|x|\le R-Nt} (\tilde{f}_i(t, x, u))^2 dx du\right)^{1/2} \le$$

$$e^{-rt}\left(\sum_{i=1}^n \int\limits_{|x|\le R-Nt} (\tilde{f}_{0i}(x - \lambda_i(u)t, u))^2 dx du\right)^{1/2} +$$

$$r\int_0^t e^{-r(t-\tau)}\left(\sum_{i=1}^n \int\limits_{|x|\le R-Nt} (\tilde{p}_i(\tau, x - \lambda_i(u)(t-\tau), u))^2 dx du\right)^{1/2} d\tau. \tag{56}$$

Changing the variables $y = x - \lambda_i(u)t$, $y = x - \lambda_i(u)(t-\tau)$ in integrals from the right side of (56) we obtain that

$$\int_{|x|\le R-Nt} (\tilde{f}_{0i}(x - \lambda_i(u)t, u))^2 dx du \le \int_{|y|\le R} (\tilde{f}_{0i}(y, u))^2 dy du,$$

$$\int\limits_{|x|\le R-Nt} (\tilde{p}_i(\tau, x - \lambda_i(u)(t-\tau), u))^2 dx du \le \int_{|y|\le R-N\tau} (\tilde{p}_i(\tau, y, u))^2 dy du.$$

After substituting the estimates above to (56) we obtain

$$\left(\int_{|x|\le R-Nt} \|\tilde{f}(t, x, \cdot)\|_2^2 dx du\right)^{1/2} \le e^{-rt}\left(\int_{|y|\le R} \|\tilde{f}_0(y, \cdot)\|_2^2 dy du\right)^{1/2}$$
$$+ r\int_0^t e^{-r(t-\tau)}\left(\int_{|y|\le R-N\tau} \|\tilde{p}(\tau, y, \cdot)\|_2^2 dx du\right)^{1/2} d\tau$$

and, by the equality $\|\tilde{Q}f\|_2 = \|f\| = \|f\|_X$,

$$\left(\int_{|x|\le R-Nt} \|f(t, x, \cdot)\|^2 dx du\right)^{1/2} \le e^{-rt}\left(\int_{|y|\le R} \|f_0(y, \cdot)\|^2 dy du\right)^{1/2} +$$
$$r\int_0^t e^{-r(t-\tau)}\left(\int_{|y|\le R-N\tau} \|p(\tau, y, \cdot)\|^2 dx du\right)^{1/2} d\tau \le$$
$$e^{-rt}\|f_0\|_{L^2([-R,R],X)} + (1 - e^{-rt})\|p\|_L \le \max(\|f_0\|_{L^2([-R,R],X)}, \|p\|_L). \tag{57}$$

In particular $f(t, x, u) \in L$ and $\|f\|_L \le C = \max(\|f_0\|_{L^2([-R,R],X)}, \|p\|_L)$. By setting $f = F(p)$ we define a map $F : L \mapsto L$. Let us show that this map is contractive. Let $p, q \in L$, $f = F(p)$, $g = F(q)$. Substructing one of equalities (55), which correspond to solutions f and g, from the other we obtain equalities

$$(\tilde{f} - \tilde{g})_i(t, x, u) = r \int_0^t e^{-r(t-\tau)} (\tilde{p} - \tilde{q})_i(\tau, x - \lambda_i(u)(t - \tau), u) d\tau. \tag{58}$$

Using again the Minkowski inequality for L^2-norm of vector functions on the set $(x, u) \in [-R + Nt, R - Nt] \times \mathbb{R}^n$ and changing the variables $y = x - \lambda_i(u)(t - \tau)$ we derive from (58), in the same wave as in the proof of inequality (57), that

$$\left(\int_{|x| \le R - Nt} \|f(t, x, \cdot) - g(t, x, \cdot)\|^2 dx \right)^{1/2} \le$$

$$r \int_0^t e^{-r(t-\tau)} \left(\int_{|y| \le R - N\tau} \|p(\tau, y, \cdot) - q(\tau, y, \cdot)\|^2 dy \right)^{1/2} d\tau \le (1 - e^{-rt}) \|p - q\|_L.$$

Therefore, $\|f - g\|_L \le (1 - e^{-RT}) \|p - q\|_L$, $T = R/N$ and the map F is a contraction. All the more the composition $G(p) = F(P_H p)$ is a contraction too (we take also (50) into account). By Banach theorem there exist the unique fixed point $f \in L$ of the map G, i.e. problem (51), (52) has the unique g.s. $f = f_R$ in the triangle K_R. By uniqueness solutions f_{R_1} and f_{R_2} with $R_1 < R_2$ coincide in their common region K_{R_1}. Therefore, we can define the function $f(t, x, u) \in L^2_{loc}(\Pi, X)$ setting $f(t, x, u) = f_R(t, x, u)$ where $R > |x| + Nt$. It is clear that $f(t, x, u)$ is a g.s. of the problem (51), (52). Uniqueness of g.s. $f(t, x, u)$ follows from its iniqueness in any triangle K_R. The proof is complete.

Our next task is to prove convergence of solutions $f = f_r(t, x, u)$ to the problem (51), (52) as the parameter $r \to \infty$. To this end we need some a priory estimates for g.s. of the problem (51), (52), which are uniform with respect to the parameter $r > 0$. We have the following

Theorem 8. *Let $f(t, x, u)$, $g(t, x, u)$ be g.s. of the problem* (51), (52) *with initial data $f_0(x, u), g_0(x, u)$ respectively. Then*

1) *in* $\mathcal{D}'(\Pi)$

$$(\|f(t, x, \cdot) - g(t, x, \cdot)\|^2)_t + (\tilde{A}(f(t, x, \cdot) - g(t, x, \cdot)), f(t, x, \cdot) - g(t, x, \cdot))_x \le 0; \tag{59}$$

2) *for almost each* $t > 0$ $\forall R > 0$

$$\int_{|x| \le R} \|f(t, x, \cdot) - g(t, x, \cdot)\|^2 dx \le \int_{|x| \le R + Nt} \|f_0(x, \cdot) - g_0(x, \cdot)\|^2 dx. \tag{60}$$

Proof. Let $p = p(t, x, u) = P_H f(t, x, \cdot)(u)$, $q = q(t, x, u) = P_H g(t, x, \cdot)(u)$. Then for almost every fixed $u \in \mathbb{R}^n$

$$f_t + A(u) f_x = -r(f - p), \ g_t + A(u) g_x = -r(g - q) \ \text{ in } D'(\Pi)$$

Substracting the second equality from the first one and multiplying to the matrix $B(u)$ from the right we obtain that

$$B(u) v_t + C(u) v_x = -r B(u)(v - w) \ \text{ in } D'(\Pi), \tag{61}$$

where $v = f - g$, $w = p - q$. Let $\beta(s) \in C_0^\infty((-1,0))$, $\beta(s) \ge 0$, $\int \beta(s)ds = 1$. We consider a sequence of kernels $k_m(t,x) = m^2\beta(mt)\beta(mx)$, $m \in \mathbb{N}$ and the corresponding sequence of averaged functions (with fixed $u \in \mathbb{R}^n$) $v_m = (v * k_m)(t,x) = \int v(t-\tau, x-y, u)k_m(\tau,y)d\tau dy$, $w_m = (w * k_m)(t,x) = \int w(t-\tau, x-y, u)k_m(\tau,y)d\tau dy$. By known properties of averaged functions $v_m, w_m \in C^\infty(\Pi)$ and $v_m \to v$, $w_m \to w$ in $L^1_{loc}(\Pi)$ as $m \to \infty$. Applying the convolution operator to equality (61) we derive that

$$B(u)(v_m)_t + C(u)(v_m)_x = -rB(u)(v_m - w_m) \text{ in } \Pi$$

(in classic sense). Multiplying this equality by $2v_m$ and taking into account symmetry of matrices $B(u)$ and $C(u) = B(u)A(u)$ we obtain that

$$(B(u)v_m, v_m)_t + (C(u)v_m, v_m)_x = -2r(B(u)(v_m - w_m), v_m).$$

Multiplying this equality by the nonnegative test function $\rho = \rho(t,x) \in C_0^\infty(\Pi)$ and integrating by parts we conclude that

$$\int_\Pi [(B(u)v_m, v_m)\rho_t + (C(u)v_m, v_m)\rho_x - 2r(B(u)(v_m - w_m), v_m)\rho]dtdx = 0,$$

which implies in the limit as $m \to \infty$ that

$$\int_\Pi [(B(u)v, v)\rho_t + (C(u)v, v)\rho_x - 2r(B(u)(v - w), v)\rho]dtdx = 0.$$

Remind that the obtained equality holds for almost every $u \in \mathbb{R}^n$. Integrating it by u and taking into account the equalities $\int (B(u)v(t,x,u), v(t,x,u))du = \|v(t,x,\cdot)\|^2$, $\int (C(u)v(t,x,u), v(t,x,u))du = (\tilde{A}v(t,x,\cdot), v(t,x,\cdot))$, $\int (B(u)(v - w), v)du = (v(t,x,\cdot) - w(t,x,\cdot), v(t,x,\cdot))$ we deduce that

$$\int_\Pi [\|v(t,x,\cdot)\|^2\rho_t + (\tilde{A}v(t,x,\cdot), v(t,x,\cdot))\rho_x - 2r(v(t,x,\cdot) - w(t,x,\cdot), v(t,x,\cdot))\rho]dtdx = 0. \quad (62)$$

Remark that $\|w(t,x,\cdot)\| \le \|v(t,x,\cdot)\|$ by (50) and

$$(v(t,x,\cdot) - w(t,x,\cdot), v(t,x,\cdot)) = \|v(t,x,\cdot)\|^2 - (w(t,x,\cdot), v(t,x,\cdot)) \ge \|v(t,x,\cdot)\|^2 - \|v(t,x,\cdot)\| \cdot \|w(t,x,\cdot)\| \ge 0.$$

Therefore (62) implies that $\forall \rho \in C_0^\infty(\Pi)$, $\rho \ge 0$

$$\int_\Pi [\|v(t,x,\cdot)\|^2\rho_t + (\tilde{A}v(t,x,\cdot), v(t,x,\cdot))\rho_x]dtdx \ge 0,$$

i.e. relation (59) is fulfilled.

It is clear that the second statement of our theorem follows from the first one. We only need apply (59) to test functions which approximate the indicator functions of characteristic triangles K_R. However we shall use the other method based on equality (55).

After possible correction of the solutions f, g on the set of null measure they satisfy equalities like (55) for all $t > 0$. Substructing one equality from the other we obtain the relation

$$\tilde{v}_i(t,x,u) = e^{-rt}\tilde{v}_{0i}(x-\lambda_i(u)t,u) + r\int_0^t e^{-r(t-\tau)}\tilde{w}_i(\tau, x-\lambda_i(u)(t-\tau),u)d\tau, \quad (63)$$

where as above $v = f-g$, $w = p-q$, $v_0 = f_0 - g_0$, and the map $f \mapsto \tilde{f}$ was defined in the proof of Theorem 7. Applying the Minkowski inequality we derive from (63), that for $R > 0$

$$\left(\sum_{i=1}^n \int_{|x|\le R-Nt} \|\tilde{v}(t,x,\cdot)\|_2^2 dx\right)^{1/2} \le$$
$$e^{-rt}\left(\sum_{i=1}^n \int_{|x|\le R-Nt} (\tilde{v}_{0i}(x-\lambda_i(u)t,u))^2 dxdu\right)^{1/2} +$$
$$r\int_0^t e^{-r(t-\tau)}\left(\sum_{i=1}^n \int_{|x|\le R-Nt} (\tilde{w}_i(\tau,x-\lambda_i(u)(t-\tau),u))^2 dxdu\right)^{1/2} d\tau. \quad (64)$$

Changing the variables $y = x - \lambda_i(u)t$, $y = x - \lambda_i(u)(t-\tau)$ in integrals from the right side of (64), using the identity $\|\tilde{f}\|_2 = \|f\| = \|f\|_X$ and the inequality $\|w(t,x,\cdot)\| \le \|v(t,x,\cdot)\|$ (see (50)) we derive the estimates

$$\sum_{i=1}^n \int_{|x|\le R-Nt} (\tilde{v}_{0i}(x-\lambda_i(u)t,u))^2 dxdu \le \sum_{i=1}^n \int_{|y|\le R} (\tilde{v}_{0i}(y,u))^2 dydu =$$
$$\int_{|y|\le R} \|\tilde{v}_0(y,\cdot)\|_2^2 dy = \int_{|y|\le R} \|v_0(y,\cdot)\|^2 dy;$$
$$\sum_{i=1}^n \int_{|x|\le R-Nt} (\tilde{w}_i(\tau,x-\lambda_i(u)(t-\tau),u))^2 dxdu \le$$
$$\sum_{i=1}^n \int_{|y|\le R-N\tau} (\tilde{w}_i(\tau,y,u))^2 dydu = \int_{|y|\le R-N\tau} \|\tilde{w}(\tau,y,\cdot)\|_2^2 dy =$$
$$\int_{|y|\le R-N\tau} \|w(\tau,y,\cdot)\|^2 dy \le \int_{|y|\le R-N\tau} \|v(\tau,y,\cdot)\|^2 dy.$$

Taking them into (64) we obtain that

$$\left(\int_{|x|\le R-Nt} \|v(t,x,\cdot)\|^2 dx\right)^{1/2} \le e^{-rt}\left(\int_{|y|\le R} \|v_0(y,\cdot)\|_2^2 dydu\right)^{1/2}$$
$$+r\int_0^t e^{-r(t-\tau)}\left(\int_{|y|\le R-N\tau} \|v(\tau,y,\cdot)\|^2 dxdu\right)^{1/2} d\tau. \quad (65)$$

We set

$$\Delta(t) = e^{rt}\left(\int_{|x|\le R-Nt} \|v(t,x,\cdot)\|^2 dx\right)^{1/2}.$$

Then (65) can be rewrited in the form

$$\Delta(t) \le \Delta(0) + r\int_0^t \Delta(\tau)d\tau, \ \Delta(0) = \left(\int_{|y|\le R} \|v_0(y,\cdot)\|_2^2 dydu\right)^{1/2}$$

and by Gronwall inequality $\Delta(t) \le \Delta(0)e^{rt}$, which is equivalent to the inequality

$$\int_{|x|\le R-Nt} \|v(t,x,\cdot)\|^2 dx \le \int_{|y|\le R} \|v_0(y,\cdot)\|_2^2 dy du,$$

and (60) follows if we take $R+Nt$ instead of R. The proof is complete.

Corolary 1. *If $f(t,x,u)$ is a g.s. of the problem (51), (52) and $g(u) \in H$ then*

$$(\|f(t,x,\cdot)-g\|^2)_t + (\tilde{A}(f(t,x,\cdot)-g), f(t,x,\cdot)-g)_x \le 0 \quad in\ \mathcal{D}'(\Pi);$$

for a.e. $t>0 \ \ \forall R>0$

$$\int_{|x|\le R} \|f(t,x,\cdot)-g\|^2 dx \le \int_{|x|\le R+Nt} \|f_0(x,\cdot)-g\|^2 dx. \tag{66}$$

Proof. Observe that $g(t,x,u) \equiv g(u)$ is g.s. of the problem (51), (52) for every $r>0$ and it only remains to apply Theorem 8 for the pair of g.s. $f(t,x,u)$ and $g(u)$.

Corollary 2. *If $f(t,x,u)$ is a g.s. of the problem (51), (52) and $\Delta x \in \mathbb{R}$ then for a.e.* $t>0 \ \ \forall R>0$

$$\int_{|x|\le R} \|f(t,x+\Delta x,\cdot)-f(t,x,\cdot)\|^2 dx \le$$

$$\int_{|x|\le R+Nt} \|f_0(x+\Delta x,\cdot)-f_0(x,\cdot)\|^2 dx \le \omega^x_{R+Nt}(|\Delta x|)$$

where

$$\omega^x_R(h) = \sup_{|\Delta x|\le h} \int_{|x|\le R} \|f_0(x+\Delta x,\cdot)-f_0(x,\cdot)\|^2 dx$$

is a square of the continuity modulus of the initial function f_0 in $L^2([-R,R],X)$.

Proof. We only need apply the second assertion of Theorem 8 to the pair of g.s. $f(t,x,u)$ and $g(t,x,u) = f(t,x+\Delta x,u)$.

Next we establish an estimate of convergence rate for g.s. of the problem (51), (52) to the initial data $f_0(x,u)$ as $t \to 0$. We have the following

Theorem 9. *Let $f(t,x,u)$ be a g.s. of the problem (51), (52). Then for a.e.* $t>0 \ \ \forall R>0$

$$\int_{|x|\le R} \|f(t,x,\cdot)-f_0(x,\cdot)\|^2 dx \le \omega^t_R(t) = (4+2N)\omega^x_{R+Nt}(t+Nt) \tag{67}$$

Proof. For any fixed $y \in \mathbb{R}$ the function $g(u) = f_0(y,u) \in H$ and by Corollary 1 for a.e. $t>0$

$$\int_{|x-y|\le t} \|f(t,x,\cdot)-f_0(y,\cdot)\|^2 dx \le \int_{|x-y|\le t+Nt} \|f_0(x,\cdot)-f_0(y,\cdot)\|^2 dx. \tag{68}$$

We also take into account that, by invariance of the set of g.s. with respect to a shift of the spatial variable, we can replace the integration region in (66) (with $R=t$)

shifting it by y. Changing the variables $z = x - y$ in the integrals from (68) and integrating over $y \in [-R, R]$ where $R > 0$, we obtain that

$$\int_{|z|\le t}\int_{|y|\le R} \|f(t, y+z, \cdot) - f_0(y, \cdot)\|^2 dx dz \le$$

$$\int_{|z|\le t+Nt}\int_{|y|\le R} \|f_0(y+z, \cdot) - f_0(y, \cdot)\|^2 dx dz \le 2(t+Nt)\omega_R^x(t+Nt). \tag{69}$$

Further, by the elementary inequality $(a+b)^2 \le 2a^2 + 2b^2$ and Corollary 2 $\forall |z| \le t$

$$\int_{|y|\le R} \|f(t, y, \cdot) - f_0(y, \cdot)\|^2 dx \le 2\int_{|y|\le R} \|f(t, y+z, \cdot) - f_0(y, \cdot)\|^2 dx +$$

$$2\int_{|y|\le R} \|f(t, y+z, \cdot) - f(t, y, \cdot)\|^2 dx \le$$

$$2\int_{|y|\le R} \|f(t, y+z, \cdot) - f_0(y, \cdot)\|^2 dx + 2\omega_{R+Nt}^x(t).$$

Integrating the obtaibed inequality over z, $|z| \le t$ and using (69), we derive an estimate

$$2t\int_{|y|\le R} \|f(t, y, \cdot) - f_0(y, \cdot)\|^2 dx \le$$

$$4(t+Nt)\omega_R^x(t+Nt) + 4t\omega_{R+Nt}^x(t) \le (8t+4Nt)\omega_{R+Nt}^x(t+Nt),$$

which implies, after division by $2t$, relation (67).

Corollary 3. *If $f(t, x, u)$ is a g.s. of the problem* (51), (52) *then for a.e.* $t, \Delta t > 0$ $\forall R > 0$

$$\int_{|x|\le R} \|f(t+\Delta t, x, \cdot) - f(t, x, \cdot)\|^2 dx \le \omega_{R+Nt}^t(\Delta t). \tag{70}$$

Proof. It is easy to see that for a.e. $\Delta t > 0$ the vector $g(t, x, u) = f(t+\Delta t, x, u)$ is a g.s. of the problem (51), (52) with the initial function $g_0(x, u) = f(\Delta t, x, u)$ (the initial condition is fulfilled by the equality like (55)). Then, by the second assertion in Corollary 1

$$\int_{|x|\le R} \|f(t+\Delta t, x, \cdot) - f(t, x, \cdot)\|^2 dx \le \int_{|x|\le R+Nt} \|f(\Delta t, x, \cdot) - f_0(x, \cdot)\|^2 dx$$

and to complete the proof it only remains to apply Theorem 9 to estimate the right-hand side of the obtained inequality.

Now we are ready to prove convergence of the approximated sequence $f_r(t, x, u)$. We have the following

Theorem 10. *Let $f_r = f_r(t, x, u)$ be a g.s. of the problem* (51), (52), $r \in \mathbb{N}$. *Then the sequence f_r converges in $L^2_{loc}(\Pi, X)$ to the unique entropy solution $f(t, x, u)$ of the problem* (46), (47).

Proof. Let $p_r(t, x, u) = P_H f_r(t, x, \cdot)(u) \in L^2_{loc}(\Pi, H)$, and $g = g(u) \in H$. Then g is a g.s. of (51), (52) for any $r > 0$ and by the equality (62) with $v = f_r - g$, $w = p_r - g$ we have the relation $\forall \rho = \rho(t, x) \in C_0^\infty(\Pi)$, $\rho \ge 0$

$$2r\int_\Pi (f_r(t, x, \cdot) - p_r(t, x, \cdot), f_r(t, x, \cdot) - g)\rho(t, x) dt dx =$$

$$\int_\Pi [\|f_r(t, x, \cdot) - g\|^2 \rho_t + (\tilde{A}(f_r(t, x, \cdot) - g), f_r(t, x, \cdot) - g)\rho_x] dt dx,$$

which implies an estimate

$$2r\int_{\Pi}(f_r(t,x,\cdot)-p_r(t,x,\cdot),f_r(t,x,\cdot)-g)\rho(t,x)dtdx\le$$
$$\int_{\Pi}\|f_r(t,x,\cdot)-g\|^2\{|\rho_t|+N|\rho_x|\}dtdx\le C, \qquad (71)$$

where C is a constant, which does not depend on r. We use also boundedness of $f_r(t,x,u)$ in $L^2_{loc}(\Pi,X)$ that easy follows from estimate (66) in Corollary 1. Then for fixed $(t,x)\in\Pi$

$$(f_r-p_r,f_r-g)=(f_r-p_r,f_r-p_r+p_r-g)=\|f_r-p_r\|^2+(f_r-p_r,p_r-g). \quad (72)$$

Because $p_r=P_Hf_r$ and H is a convex set then $\forall\alpha\in[0,1]$

$$\|f_r-p_r+\alpha(p_r-g)\|^2=\|f_r-((1-\alpha)p_r+\alpha g)\|^2\ge\|f_r-p_r\|^2.$$

Revealing the square from the left we deduce the inequality $(f_r-p_r,p_r-g)\ge 0$, which together with (72) implies that $(f_r-p_r,f_r-g)\ge\|f_r-p_r\|^2$. Then, by (71),

$$\int_{\Pi}\|f_r(t,x,\cdot)-p_r(t,x,\cdot)\|^2\rho(t,x)dtdx\le C/(2r)\underset{r\to\infty}{\to}0$$

and, using arbitrariness of the test function $\rho(t,x)$, we obtain the relation

$$f_r(t,x,u)-p_r(t,x,u)\underset{r\to\infty}{\to}0 \text{ in } L^2_{loc}(\Pi,X). \qquad (73)$$

Consider the sequence $p_r(t,x,u)$. Remind that $p_r(t,x,\cdot)=P_Hf_r(t,x,\cdot)$ and by (50)

$$\|p_r(t,x+\Delta x,\cdot)-p_r(t,x,\cdot)\|\le\|f_r(t,x+\Delta x,\cdot)-f_r(t,x,\cdot)\|,$$
$$\|p_r(t+\Delta t,x,\cdot)-p_r(t,x,\cdot)\|\le\|f_r(t+\Delta t,x,\cdot)-f_r(t,x,\cdot)\|,$$
$$\|p_r(t,x,\cdot)-g\|\le\|f_r(t,x,\cdot)-g\|\ \forall g\in H$$

and making use of Corollaries 1-3 we derive the iniform with respect to $r\in\mathbb{N}$ estimates: for a.e. $t>0$, $\forall R>0$

$$\int_{|x|\le R}\|p_r(t,x,\cdot)-g\|^2dx\le\int_{|x|\le R+Nt}\|f_0(x,\cdot)-g\|^2dx\ \forall g\in H; \qquad (74)$$
$$\int_{|x|\le R}\|p_r(t,x+\Delta x,\cdot)-p_r(t,x,\cdot)\|^2dx\le\omega^x_{R+Nt}(|\Delta x|),\ \Delta x\in\mathbb{R}; \qquad (75)$$
$$\int_{|x|\le R}\|p_r(t+\Delta t,x,\cdot)-p_r(t,x,\cdot)\|^2dx\le\omega^t_{R+Nt}(\Delta t) \text{ for a.e. } \Delta t>0. \qquad (76)$$

Here $\omega^x_{R+Nt}(h),\omega^t_{R+Nt}(h)\to 0$ as $h\to 0$. Since the norm $\|\cdot\|$ is Lipshitz continuous on X the estimates (74)-(76) imply that the scalar sequence $\|p_r(t,x,\cdot)\|$ is uniformly bounded and equicontinuous in $L^2(K)$ for any compact $K\subset\Pi$. Therefore this sequence is precompact in $L^2(K)$ and, in particular, for any $\varepsilon>0$ there exists such $M>0$ that $\forall r\in\mathbb{N}$

$$\int_K\|p_r(t,x,\cdot)\|^2\theta(\|p_r(t,x,\cdot)\|-M)dtdx<\varepsilon^2/9,$$

where $\theta(s)$ is the Heaviside function. It follows that $\forall r \in \mathbb{N}$

$$\|p_r(t,x,\cdot) - \bar{p}_r(t,x,\cdot)\|_{L^2(K,X)} < \varepsilon/3, \tag{77}$$

where

$$\bar{p}_r(t,x,u) = \begin{cases} p_r(t,x,u) & , \quad \|p_r(t,x,u)\| \le M \\ 0 & , \quad \|p_r(t,x,u)\| > M \end{cases}.$$

By our hypothesis the set $H_M = \{ h \in H \mid \|h\| \le M \} \cup \{0\}$ is compact. Thus by the Hausdorff criteria there exists a finite $\varepsilon/3$-net $S \subset H_M$. Let $Y \subset X$ be a linear span of S. Then Y is a finite-dimensional linear subspace of X. We set $q_r(t,x,u) = P_Y \bar{p}_r(t,x,\cdot)(u)$ where $P_Y : X \mapsto Y$ is an orthogonal projection operator. Since Y is also $\varepsilon/3$-net of the set H_M then $\forall r \in \mathbb{N}$

$$\|q_r(t,x,\cdot) - \bar{p}_r(t,x,\cdot)\|_{L^2(K,X)} < \varepsilon/3. \tag{78}$$

Obviously, the finite-dimensional vectors $q_r(t,x,u)$ satisfy iniform estimates like (74)-(76) after changing p_r by q_r, therefore they form the precompact in $L^2(K,X)$ sequence. Thus, this sequence possess a finite $\varepsilon/3$-net $F \subset L^2(K,X)$. In view of (77), (78) F is a ε-net for the original sequence p_r in $L^2(K,X)$ and taking into account that $\varepsilon > 0$ is arbitrary we conclude that the sequence p_r, $r \in \mathbb{N}$ is totally bounded in $L^2(K,X)$. By Hausdorff criteria this sequence is precompact in $L^2(K,X)$ for any compact $K \subset \Pi$, therefore it is precompact in $L^2_{loc}(\Pi,X)$. Thus the convergent subsequence could be extracted (we take into account that the space $L^2_{loc}(\Pi,X)$ is metrizable, so that compactness and sequential compactness are equivalent). We keep the notation p_r for this subsequence. Thus, $p_r \to f = f(t,x,u) \in L^2_{loc}(\Pi,X)$ as $r \to \infty$. By construction $p_r(t,x,\cdot) \in H$, and since H is closed $f(t,x,\cdot) \in H$ for a.e. $(t,x) \in \Pi$, i.e. $f \in L^2_{loc}(\Pi,H)$. By (73) we also have that $f_r \to f$ in $L^2_{loc}(\Pi,X)$ (strongly) as $r \to \infty$. Show that f is an e.s. of the problem (46), (47). By Corollary 1 $\forall g \in H$

$$(\|f_r(t,x,\cdot) - g\|^2)_t + (\tilde{A}(f_r(t,x,\cdot) - g), f_r(t,x,\cdot) - g)_x \le 0 \ \text{ in } \mathcal{D}'(\Pi),$$

which, taking account of strong convergence $f_r \to f$, implies in the limit as $r \to \infty$ that

$$(\|f(t,x,\cdot) - g\|^2)_t + (\tilde{A}(f(t,x,\cdot) - g), f(t,x,\cdot) - g)_x \le 0 \ \text{ in } \mathcal{D}'(\Pi),$$

i.e. condition (48) is satisfied. Further, by Theorem 9 for a.e. $t > 0$, $\forall R > 0, r \in \mathbb{N}$

$$\int_{|x| \le R} \|f_r(t,x,\cdot) - f_0(x,\cdot)\|^2 dx \le \omega_R^t(t).$$

Passing to the limit as $r \to \infty$ in the estimate above we derive that for a.e. $t > 0$, $\forall R > 0$

$$\int_{|x| \le R} \|f(t,x,\cdot) - f_0(x,\cdot)\|^2 dx \le \omega_R^t(t) \underset{t \to 0}{\to} 0$$

and the initial condition (47) is also satisfied. Thus, $f(t,x,u)$ is e.s. of (46), (47). At last, in view of uniqueness of e.s., the limit function $f(t,x,u)$ doesn't depend on the choice of a convergent subsequence. Therefore, the original sequence $f_r(t,x,u)$ converges to $f(t,x,u)$ in $L^2_{loc}(\Pi,X)$. The proof is complete.

Consider the important particular case when H is a cone. We prove the following

Theorem 11. *Let H be a closed convex cone in X. Then the vector $f(t,x,u)$, satisfying initial condition (47), is an e.s. of (46), (47) if and only if $\forall h\in H$*

$$\frac{\partial}{\partial t}(f(t,x,\cdot),h)+\frac{\partial}{\partial x}(\tilde{A}f(t,x,\cdot),h)\ge 0 \text{ in } \mathcal{D}'(\Pi) \text{ and} \tag{79}$$

$$\frac{\partial}{\partial t}\|f(t,x,\cdot)\|^2+\frac{\partial}{\partial x}(\tilde{A}f(t,x,\cdot),f(t,x,\cdot))=0 \text{ in } \mathcal{D}'(\Pi). \tag{80}$$

Proof. Let $h\in H$. Since H is a cone then $rh\in H \quad \forall r\ge 0$ and by condition (48)

$$\begin{gathered}\frac{\partial}{\partial t}\|f(t,x,\cdot)\|^2+\frac{\partial}{\partial x}(\tilde{A}f(t,x,\cdot),f(t,x,\cdot))-\\ 2r\left[\frac{\partial}{\partial t}(f(t,x,\cdot),h)+\frac{\partial}{\partial x}(\tilde{A}f(t,x,\cdot),h)\right]=\frac{\partial}{\partial t}\|f(t,x,\cdot)-rh\|^2+\\ \frac{\partial}{\partial x}(\tilde{A}(f(t,x,\cdot)-rh),f(t,x,\cdot)-rh)\le 0 \text{ in } \mathcal{D}'(\Pi).\end{gathered} \tag{81}$$

Because $r\ge 0$ is arbitrary we obtain (79). Taking in (81) $r=0$ we derive the inequality

$$\frac{\partial}{\partial t}\|f(t,x,\cdot)\|^2+\frac{\partial}{\partial x}(\tilde{A}f(t,x,\cdot),f(t,x,\cdot))\le 0 \text{ in } \mathcal{D}'(\Pi). \tag{82}$$

The opposite inequality

$$\frac{\partial}{\partial t}\|f(t,x,\cdot)\|^2+\frac{\partial}{\partial x}(\tilde{A}f(t,x,\cdot),f(t,x,\cdot))\ge 0 \text{ in } \mathcal{D}'(\Pi) \tag{83}$$

easy follows from (79) with using the method of "doubling of variables". Indeed, take in (79) $h=f(\tau,y,\cdot)$, $(\tau,y)\in\Pi$. Then we obtain the inequality

$$\frac{\partial}{\partial t}(f(t,x,\cdot),f(\tau,y,\cdot))+\frac{\partial}{\partial x}(\tilde{A}f(t,x,\cdot),f(\tau,y,\cdot))\ge 0 \text{ in } \mathcal{D}'(\Pi).$$

Similarly, changing places of variables t,x and τ,y and taking account of symmetry of the operator $\tilde{A}$ we come to the relaion

$$\frac{\partial}{\partial \tau}(f(t,x,\cdot),f(\tau,y,\cdot))+\frac{\partial}{\partial y}(\tilde{A}f(t,x,\cdot),f(\tau,y,\cdot))\ge 0 \text{ in } \mathcal{D}'(\Pi).$$

Adding the inequalities above, we deduce that

$$\begin{gathered}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \tau}\right)(f(t,x,\cdot),f(\tau,y,\cdot))+\\ \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)(\tilde{A}f(t,x,\cdot),f(\tau,y,\cdot))\ge 0 \text{ in } \mathcal{D}'(\Pi).\end{gathered} \tag{84}$$

Let $h = h(t,x) \in C_0^\infty(\Pi)$, $h \ge 0$; $\rho_\nu(t-\tau, x-y) = \beta_\nu(\tau - t)\beta_\nu(y-x)$ where the sequence $\beta_\nu(s) = \nu\beta(\nu s)$, $\nu \in \mathbb{N}$ was defined above in the proof of Theorem 3. Applying (84) to a nonnegative test function $p = p(t,x;\tau,\xi) = h(t,x)\rho_\nu(t-\tau, x-y) \in C_0^\infty(\Pi \times \Pi)$ we obtain after symple transformations that

$$Q_\nu = \int_{\Pi\times\Pi} [(f(t,x,\cdot), f(\tau,y,\cdot))h_t(t,x) +$$

$$(\tilde{A}f(t,x,\cdot), f(\tau,y,\cdot))h_x(t,x)]\rho_\nu(t-\tau,x-y)dtdxd\tau dy \le 0. \tag{85}$$

We set

$$Q = \int_\Pi [\|f(t,x,\cdot)\|^2 h_t(t,x) + (\tilde{A}f(t,x,\cdot), f(t,x,\cdot))h_x(t,x)]dtdx.$$

Then, taking account of the equality $\int \rho_\nu(t-\tau, x-y)d\tau dy = 1$, we conclude that

$$|Q_\nu - Q| \le \int_{\Pi\times\Pi} \Big|(f(t,x,\cdot), f(t,x,\cdot) - f(\tau,y,\cdot))h_t(t,x) +$$

$$(\tilde{A}f(t,x,\cdot), f(t,x,\cdot) - f(\tau,y,\cdot))h_x(t,x)\Big|\rho_\nu(t-\tau,x-y)dtdxd\tau dy \le$$

$$C\int_{K\times\Pi} \|f(t,x,\cdot)\| \cdot \|f(t,x,\cdot) - f(\tau,y,\cdot)\|\rho_\nu(t-\tau,x-y)dtdxd\tau dy$$

where $K \subset \Pi$ is a support of the function $h(t,x)$, $C = \max(|h_t(t,x)| + N|h_x(t,x)|)$. Making use of Cauchy-Bunyakowskii inequality and the property $\int \rho_\nu(t-\tau, x-y)d\tau dy = 1$, we derive an estimate

$$|Q_\nu - Q|^2 \le C^2 \int_K \|f(t,x,\cdot)\|^2 dtdx \times$$

$$\int_{K\times\Pi} \|f(t,x,\cdot) - f(\tau,y,\cdot)\|^2 \rho_\nu(t-\tau,x-y)dtdxd\tau dy,$$

the right-hand side of which tends to zero as $\nu \to \infty$, by the average continuity property. Thus, $Q_\nu \to Q$ as $\nu \to \infty$ and (85) implies that $Q \le 0$ for any nonnegative test function $h(t,x)$. This means that the relation (83) holds. Inequalities (82), (83) implies in turn (80).

Conversely, suppose that conditions (79), (80) are fulfilled, and $h \in H$. Then,

$$\frac{\partial}{\partial t}\|f(t,x,\cdot) - h\|^2 + \frac{\partial}{\partial x}(\tilde{A}(f(t,x,\cdot) - h), f(t,x,\cdot) - h) =$$

$$\frac{\partial}{\partial t}\|f(t,x,\cdot)\|^2 + \frac{\partial}{\partial x}(\tilde{A}f(t,x,\cdot), f(t,x,\cdot)) -$$

$$2\left[\frac{\partial}{\partial t}(f(t,x,\cdot), h) + \frac{\partial}{\partial x}(\tilde{A}f(t,x,\cdot), h)\right] \le 0 \text{ in } \mathcal{D}'(\Pi)$$

and the defining condition (48) is satisfied. The theorem is proved.

Corollary 4. *Let H be a closed linear subspace of X. Then condition* (48) *is equivalent to the identity:* $\forall h \in H$

$$\frac{\partial}{\partial t}(f(t,x,\cdot), h) + \frac{\partial}{\partial x}(\tilde{A}f(t,x,\cdot), h) = 0 \text{ in } \mathcal{D}'(\Pi). \tag{86}$$

Proof. Assume that condition (48) is fulfilled. Since H is a cone, then (79) is satisfied. Taking in this condition functions $\pm h \in H$ we derive identity (86). Conversely, if (86) is satisfied then this condition immediately implies (79). Then, taking in (86) $h = f(\tau, y, \cdot)$ and applying method of doubling of variables we obtain (80), in the same way as in the proof of Theorem 11. By Theorem 11 $f(t, x, \cdot)$ satisfies to condition (48). The proof is complete.

Remark 2. In the case when H is a closed linear subspace existence and uniqueness of e.s. to the problem (46), (47) can be proved without the condition of strong compactness of bounded closed subsets of H (which holds only for finite-dimensional subspace). Indeed, we do not use this condition in the proof of the uniqueness, and the existence can be established on the base of weak convergence of the sequence $f_r = f_r(t, x, u)$ of g.s. to (51), (52). Obviously, the limit function $f(t, x, u)$ must satisfy condition (86), which is equivalent to (48) by Corollary 4. To verify initial condition (47) we make use the estimate in Theorem 9 (in the same way as in the proof of Theorem 10).

In the case of a scalar equation $u_t + a(u)u_x = 0$, $(t, x) \in \Pi = \mathbb{R}_+ \times \mathbb{R}$ we set

$$H = \{\ f(u) - \theta(-u) \mid f \in F_M\ \},$$

where $M > 0$, $\theta(\lambda) = \begin{cases} 1, & \lambda > 0 \\ 0, & \lambda \le 0 \end{cases}$ is the Heaviside function. We see that $H \subset L^2(\mathbb{R})$ is a compact convex set. Indeed, the set $H = -\theta(-u) + F_M$ is closed, consists of functions supported in $[-M, M]$, and the class F_M is compact in $L^2[-M, M]$ by Helly theorem. Therefore, there exists an unique e.s. $f(t, x, u) - \theta(-u)$ of the corresponding kinetic equation $f_t + a(u)f_x = 0$ with the initial data $f(0, x, u) = f_0(x, u) - \theta(-u)$, $f_0(x, \cdot) \in F_M$. It is clear that $f(t, x, u)$ coincides e.s. in the sense of Definition 4. As follows from Theorem 3 this solution is also e.s. with respect to the wider class $\tilde{H} = \{\ \lambda f \mid f \in H, \lambda \ge 0\ \}$ that is the conic envelope of H.

Also observe that for scalar equations the theorems on existence and uniqueness of e.s. proved in this section above are easy generalized to the case of many spatial variables $x \in \mathbb{R}^n$.

Consider some closed convex subset $H \subset X$ bounded closed subset of which are compact, and suppose in addition that $\mathrm{div} f(u) = l + \nu$ $\forall f(u) \in H$ where $\nu \in \mathrm{Prob}_0(\mathbb{R}^n)$ is a finite Borel probability measure on $\mathbb{R}^n$, $l \in \mathcal{D}'(\mathbb{R}^n)$ is some fixed distribution. Denote

$$M = \{\ \mathrm{div} f(u) - l \mid f(u) \in H\ \} \subset \mathrm{Prob}_0(\mathbb{R}^n).$$

Remark that for $n > 1$ Dirac measures could not belong to M.

Let $\nu_x^0 = \mathrm{div} f_0(x, u) - l$ and $\nu_{t,x} = \mathrm{div} f(t, x, u) - l$ where $f(t, x, u) \in L^2_{loc}(\Pi, H)$ is an unique e.s. of the problem (46), (47) with initial data $f_0(x, u)$. By analogy with the scalar case $n = 1$ we call the measure valued function $\nu_{t,x}$ a *strong measure valued solution* to the original problem (44) with the measure valued initial data ν_x^0. Of course, this notion depends on choice of "kinetic class" H, which is not uniquely determined by the family of measures M. We can restrict ourselfes to the class of potential vectors $f(u) = \nabla\varphi(u)$. Then (for $l = 0$) the vector $f(u)$

is uniquely defined by its divergence since the function $\varphi(u)$ must be an unique generalized solution to the Poisson equation $\Delta\varphi = \nu$ and therefore

$$f(u) = \nabla\varphi(u) = \int \nabla\mathcal{E}(u-v)d\nu(v)$$

where $\mathcal{E}(u)$ is the fundamental solution of the Laplace equation. The natural restriction to the class of measures $\nu \in M$ is the condition $f(u) \in X$.

To demonstrate connection between strong measure valued solutions and measure valued solutions in DiPerna sense (see [7, 8]) we consider the case when the matrix $A(u)$ is symmetric and the matrix $B(u)$ is chosen to be the identity matrix. By analogy with a scalar equation we consider wider conical kinetic class , which consists of potential vectors $f(u) \in X$, $f(u) = \nabla p(u)$ in $\mathcal{D}'(\mathbb{R}^n)$ such that the distribution $\mathrm{div} f = \Delta p$ is nonnegative finite Borel measure in $\mathbb{R}^n$ (not necessary - probability). Assume as usual that H is a convex closed cone, bounded closed subsets of which are compact. Then for any initial data $f_0(x,u) \in L^2_{loc}(\mathbb{R}, H)$ there exists the unique e.s. $f = f(t,x,u) \in L^2_{loc}(\Pi, H)$ of the problem (46), (47). We consider measure valued (in wider sense) functions $\nu_x^0 = \mathrm{div}_u f_0(x,u)$ and $\nu_{t,x} = \mathrm{div}_u f(t,x,u)$. Let continuous functions $p(u)$, $q(u)$ be such that $\nabla p(u) \in H$,

$$\mathrm{div}(\nabla q(u) - A(u)\nabla p(u)) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^n) \tag{87}$$

and $p, q \to 0$ as $u \to \infty$. We call the function $p(u)$ an *entropy* of the system (44) and the function $q(u)$ a corresponding *entropy flux* . Remark that $p(u)$ is a subharmonic function, which is more general than the usual convexity requirement. The condition (87) is a weaker form of the relation $\nabla q(u) = A(u)\nabla p(u)$ postulated for classical entropy pairs (p,q) in Lax sense. In contrast to the Lax relation (which is overdetermined for $n > 2$), the flux q is uniquely defined by (87) for any general enough entropy $p(u)$.

By Theorem 11 the relation (79) is fulfilled with $h = \nabla p$. Integrating by parts we conclude that

$$(f(t,x,\cdot), \nabla p) = -\int p(u)d\nu_{t,x}(u);$$

$$(\tilde{A}f(t,x,\cdot), \nabla p) = (f(t,x,\cdot), \nabla q) = -\int q(u)d\nu_{t,x}(u)$$

and by (79) we derive the entropy condition

$$\frac{\partial}{\partial t}\int p(u)d\nu_{t,x}(u) + \frac{\partial}{\partial x}\int q(u)d\nu_{t,x}(u) \le 0 \quad \text{in } \mathcal{D}'(\Pi),$$

similar to DiPerna condition.

References

[1] *S.N. Kruzhkov,* "Generalized solutions in the large for non-linear first-order equations", *Dokl. Acad. Nauk SSSR.* 1969. V. 187, no. 1. P. 29-32; English transl. in *Soviet Math. Dokl.* 1969. V. 10.

[2] *S.N. Kruzhkov,* "First order quasilinear equations in several independent variables", *Mat. Sbornik.* 1970. V. 81. No 2. P. 228-255; English transl. in *Math. USSR Sb.* 1970. V. 10. No 2. P. 217-243.

[3] *P.L. Lions, B. Perthame, E. Tadmor,* "A kinetic formulation of multidimensional scalar conservation laws and related equations", *J. Amer. Math. Soc.* 1994. V.7. No 1. P. 169-191.

[4] *E.Yu. Panov,* "On kinetic formulation of measure valued solution of a first-order quasilinear equation", *Fundamental'naya i prikladnaya matematika.* 1998. V. 4. N 1. P. 317-332. (Russian)

[5] *E.Yu. Panov,* "An approximation scheme for measure-valued solutions of a first-order quasilinear equation", *Mat. Sbornik.* 1997. V. 188, No. 1. P. 83-108; English transl. in *Sbornik: Mathematics.* 1997. V. 188, No. 1. P. 87-113.

[6] *L. Tartar,* "Compensated compactness and applications to partial differential equations", *Research notes in mathematics, nonlinear analysis, and mechanics:* Heriot-Watt Symposium. 1979. V. 4. P. 136-212.

[7] *R.J. DiPerna,* "Generalized solutions to conservation laws,in system of nonlinear partial differential equations", *NATO ASI Series,* edited by J.M.Ball,D.Reidel Pub. Co. 1983.

[8] *R.J. DiPerna,* "Measure-valued solutions to conservation laws", *Arch. Rational Mech. Anal.* 1985. V.88. P. 223-270.

[9] *E.Yu.Panov,* "On statistical solutions of the Cauchy problem for a first-order quasilinear equation", *Matemat. Modelirovanie.* 2002. V. 14, No. 3. P. 17-26. (Russian)

[10] *E.Yu. Panov,* "Strong measure-valued solutions of the Cauchy problem for a first-order quasilinear equation with a bounded measure-valued initial function", *Vestnik Mosk. Univ. Ser.1 Mat. Mekh.* 1993. No. 1. P. 20-23; English transl. in *Moscow Univ. Math. Bull.* 1993. V.48. No. 1.

[11] *E.Yu.Panov,* "On measure valued solutions of the Cauchy problem for a first order quasilinear equation", *Izvest. Ross. Akad. Nauk.* 1996. No 2. P. 107-148; English transl. in *Izvestiya: Mathematics.* 1996. V. 60. No 2. P. 335-377.

[12] *E.Yu.Panov,* "On sequences of measure-valued solutions of first-order quasilinear equations", *Matem. Sbornik.* 1994. V .185. No. 2. P. 87-106; English transl. in *Russian Acad. Sci. Sb. Math.* 1995. V. 81.

[13] *I.P. Natanson,* Theory of functions of a real variable, Gostehizdat, Moscow 1957; English transl., Vols. I, II, Ungar, New York 1955, 1961.

[14] *G.G. Hardy, G.E. Littlewood, and G. Pólya,* Inequalities, Cambridge University Press, Cambridge 1952.