GLOBAL EXISTENCE OF SMOOTH SOLUTIONS FOR PARTIALLY DISSIPATIVE HYPERBOLIC SYSTEMS WITH A CONVEX ENTROPY

B. HANOUZET* AND R. NATALINI†

ABSTRACT. We consider the Cauchy problem for a general one dimensional $n \times n$ hyperbolic symmetrizable system of balance laws. It is well known that, in many physical examples, for instance for the isentropic Euler system with damping, the dissipation due to the source term may prevent the shock formation, at least for smooth and small initial data. Our main goal is to find a set of general and realistic sufficient conditions to guarantee the global existence of smooth solutions, and possibly to investigate their asymptotic behavior. For systems which are entropy dissipative, a quite natural generalization of the Kawashima condition for hyperbolic-parabolic systems can be given. In this paper, we first propose a general framework to set this kind of problems, by using the so-called entropy variables. Therefore, we pass to prove some general statements about the global existence of smooth solutions, under different sets of conditions. In particular, the present approach is suitable for dealing with most of the physical examples of systems with a relaxation extension. Our main tools will be some refined energy estimates and the use of a suitable version of the Kawashima condition.

1. INTRODUCTION

We consider the Cauchy problem for a general one dimensional $n \times n$ hyperbolic symmetrizable system of balance laws

$$\partial_t w + \partial_x F(w) = G(w), \tag{1.1}$$

with the initial conditions

$$w(x,0) = w_0(x). \tag{1.2}$$

If the flux function F and the source term G are smooth enough, in a suitable convex open set $\Omega \subseteq \mathbb{R}^n$, it is well–known that problem (1.1)–(1.2) has a unique local (in time) smooth solution for smooth initial data. However, in the general case, and even for arbitrarily small and smooth initial data, there is no global continuation for these smooth solutions, which may develop singularities, shocks or blow–up, in finite time, see for instance [Maj78, Daf00].

On the other hand, in many physical examples, thanks to the interplay between the source term and the flux, there exist global smooth solutions for a suitable set of initial data. This is the case for the isentropic Euler system with damping

$$\begin{cases} \partial_t u - \partial_x v = 0\ , \\ \\ \partial_t v + \partial_x p(u) = -v, \end{cases} \tag{1.3}$$

1991 *Mathematics Subject Classification.* Primary: 35L65.

Key words and phrases. Dissipative hyperbolic systems, convex entropy functions, relaxation systems, Kawashima conditions.

Work partially supported by European TMR projects HCL # ERB FMRX CT96 0033

* Mathématiques Appliquées de Bordeaux, Université Bordeaux 1, 351 cours de la Libération, F-33405 Talence, France. E-mail: hanouzet@math.u-bordeaux.fr

† Istituto per le Applicazioni del Calcolo "Mauro Picone", Consiglio Nazionale delle Ricerche, Viale del Policlinico, 137, I-00161 Roma, Italy. E–mail: natalini@iac.rm.cnr.it.

with $p'(u) < 0$, which has been considered in [Nis78, HL92]. In this model, which may also be view as a nonlinear wave equation with friction, the damping may prevent the shock formation, at least for smooth and small initial data.

In the general case of system (1.1), a well–known assumption for global existence is given by the *total dissipation condition*. Suppose, for instance, that $G(0) = 0$. Since the system is symmetrizable hyperbolic, the matrix $R(0) = (R_{ij}(0))$ of the normalized right eigenvector $r_i(0)$ of the Jacobian matrix $F'(0)$ is nonsingular. System (1.1) is totally dissipative in 0 if the matrix

$$\Gamma := R^{-1}(0)G'(0)R(0) \tag{1.4}$$

is negatively diagonal dominant, which means: there exists a positive constant $\nu > 0$ such that

$$\Gamma_{ii} + \sum_{j \neq i} |\Gamma_{ji}| \leq -\nu < 0 \ ,$$

see [Li94]. Notice that, this condition is in general too strong, and for instance it is not verified for the simple example (1.3) or for general systems with relaxation, see [Daf95, Nat99].

Our main goal is to consider the following problem: to find a set of more general and realistic sufficient conditions on F and G to guarantee the global existence of smooth solutions, and possibly to investigate their asymptotic behavior.

For the systems which are endowed with a strictly convex entropy function $\mathcal{E} = \mathcal{E}(w)$, a natural assumption is the *entropy dissipation condition*, see [CLL94, MR98, Yon01], namely for every w, $\overline{w} \in \Omega$, with $G(\overline{w}) = 0$,

$$(\mathcal{E}'(w) - \mathcal{E}'(\overline{w})) \cdot G(w) \leq 0,$$

where $\mathcal{E}'(w)$ is considered as a vector in $\mathbb{R}^n$ and $\cdot$ is the scalar product in the same space. Here the condition is presented in a form which is invariant by affine perturbations. Unfortunately, this condition is too weak to prevent the formation of singularities, as shown in Section 2.

A quite natural condition can be given, most in the spirit of the papers by Kawashima [Kaw87, SK85], in the following called the condition **(K)**:

$$Ker\ G'(\overline{w}) \cap \{\text{eigenspaces of } F'(\overline{w})\} = \{0\}, \tag{1.5}$$

for every $\overline{w} \in \Omega$, with $G(\overline{w}) = 0$. This condition is satisfied in many examples, see Section 5 below. This is not the case for the equations of non–equilibrium gas dynamics. However, in this case the global existence of global solutions for small initial data has been established by Y. Zeng [Zen99], using the fact that the system can be decomposed in two blocks: for the first one we have the condition **(K)**, while the second one has essentially a linear behavior.

In this paper we first propose a general framework to set this kind of problems, by using the so–called entropy variables [God61, FL71, Boi74]. In this setting, the structure of the system and the various dissipative conditions have a clear and simple formulation. In particular the equilibrium set reduces to a linear subspace. In this regard, this framework was already considered in the special case of models for multicomponent reacting flows in [GMa98, GMb98, Mas02]. Therefore, we pass to prove a general statement about the global existence of smooth solutions. Our main tools will be some refined energy estimates and the use of a suitable version of the Kawashima condition.

The paper is organized as follows: Section 2 is devoted to the basic structures of systems with entropy dissipations. Some examples are introduced in Section 3 and Section 4 contains the main energy estimates. Next, we present a global existence result under the condition **(K)** and we reconsider the previous examples.

2. Some basic structures for dissipative hyperbolic systems

2.1. Following the general framework proposed in [CLL94], and according to the main physical examples, we shall assume in what follows that, concerning system (1.1), there exists a constant coefficient linear full–rank operator $P_1 : \mathbb{R}^n \to \mathbb{R}^k$, with $0 \le k < n$, such that

$$P_1 G(w) = 0,$$

for every $w \in \Omega$. Clearly, the operator P_1 yields k independent conserved quantities, since, setting $u = P_1 w$, we obtain

$$\partial_t u + \partial_x P_1 F(w) = 0 \ .$$

Therefore, it is easy to see that there exists another linear full–rank operator $P_2 : \mathbb{R}^n \to \mathbb{R}^{n-k}$, such that the operator $\mathbb{P} := \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$ is nonsingular. Hence, setting $v = P_2 w$, we can rewrite system (1.1) in the form

$$\begin{cases} \partial_t u + \partial_x P_1 F(w) = 0 \ , \\ \\ \partial_t v + \partial_x P_2 F(w) = P_2 G(w) \ . \end{cases}$$

In the following, we shall consider only the systems with this kind of structure. Namely, after a linear change of variables, we consider system (1.1) with $F := \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$ and $G := \begin{pmatrix} 0 \\ q \end{pmatrix}$, which takes the form

$$\begin{cases} \partial_t u + \partial_x F_1(w) = 0 \ , \\ \\ \partial_t v + \partial_x F_2(w) = q(w) \ , \end{cases} \tag{2.1}$$

with $w = (u, v) \in \Omega \subseteq \mathbb{R}^k \times \mathbb{R}^{n-k}$.

2.2. According to the general theory of hyperbolic systems of conservation laws, we shall assume that the system satisfies an entropy principle.

Hypothesis (H): there exists a strictly convex function $\mathcal{E} = \mathcal{E}(w)$, the entropy density, and a related entropy-flux function $\mathcal{F} = \mathcal{F}(w)$, such that for every smooth solution $w \in \Omega$ to system (2.1), there holds

$$\partial_t \mathcal{E}(w) + \partial_x \mathcal{F}(w) = \mathcal{G}(w) \ , \tag{2.2}$$

where $\mathcal{F}' = {}^T F' \mathcal{E}'$ and $\mathcal{G} = \mathcal{E}' \cdot G$.

Under this hypothesis, and following [God61, FL71, Boi74], it is possible to introduce a new variable, the *entropy variable*, which is

$$W = \mathcal{E}'(w). \tag{2.3}$$

This change of variable plays a special role, by allowing in particular a canonical symmetrization of the system. Actually, since $\mathcal{E}$ is a strictly convex function, we can inverse $\mathcal{E}'$ to recover the original variable w by the inverse map $\Phi := (\mathcal{E}')^{-1}$. We set

$$\begin{aligned} \mathcal{E}^*(W) \quad &:= W \cdot \Phi(W) - \mathcal{E}(\Phi(W)), \\ \mathcal{F}^*(W) \quad &:= W \cdot F(\Phi(W)) - \mathcal{F}(\Phi(W)). \end{aligned}$$

Therefore, we recover

$$w = \mathcal{E}^{*\prime}(W), \quad F(w) = \mathcal{F}^{*\prime}(W).$$

Let us set now $A_0(W) := \Phi'(W) = \mathcal{E}^{*''}(W)$, and $A_1(W) := F'(\Phi(W))A_0(W)$. It is easy to see that the matrix $A_0(W)$ is symmetric positive definite and $A_1(W)$ is symmetric. Then, selecting W as the new variable, our system reads

$$A_0(W)\partial_t W + A_1(W)\partial_x W = G(\Phi(W)). \tag{2.4}$$

To investigate the stability of system (1.1), it is very important to consider the *equilibrium points*, namely the values $\overline{w} \in \Omega$, such that $G(\overline{w}) = 0$. Set

$$\gamma = \{w \in \Omega; G(w) = 0\},$$

and

$$\Gamma = \mathcal{E}'(\gamma) = \{W \in \mathcal{E}'(\Omega); G(\Phi(W)) = 0\}.$$

Definition 2.1. *The system (1.1) is* entropy dissipative*, if, for every $\overline{w} \in \gamma$ and $w \in \Omega$, its entropy production*

$$\mathcal{R}(w, \overline{w}) := (\mathcal{E}'(w) - \mathcal{E}'(\overline{w})) \cdot G(w),$$

verifies

$$\mathcal{R}(w, \overline{w}) \leq 0. \tag{2.5}$$

In the entropy variables, the entropy dissipation condition reads

$$(W - \overline{W}) \cdot G(\Phi(W)) \leq 0,$$

for every $\overline{W} \in \Gamma$ and $W \in \mathcal{E}'(\Omega)$. For system (2.1), we can split W in two sets of variables, $W = (U, V) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, and then, setting $Q(W) := q(\Phi(W))$, the condition becomes

$$(V - \overline{V}) \cdot Q(W) \leq 0, \tag{2.6}$$

for every $\overline{V} \in \mathbb{R}^{n-k}$ such that there exists $\overline{U} \in \mathbb{R}^n$ with $(\overline{U}, \overline{V}) \in \Gamma$, and every $W \in \mathcal{E}'(\Omega)$.

Proposition 2.1. *Consider an entropy dissipative system in the form (2.1). Let $W^* = (U^*, V^*) \in \Gamma$. Then, every point of the form $\overline{W} = (\overline{U}, V^*) \in \mathcal{E}'(\Omega)$ is still in Γ and*

$$Q_U(\overline{W}) = 0, \tag{2.7}$$

$$D(\overline{W}) := \frac{1}{2}\left({}^TQ_V + Q_V \right)(\overline{W}) \leq 0. \tag{2.8}$$

On the other hand, if $\overline{W} = (\overline{U}, \overline{V}) \in \Gamma$ and there holds the stronger inequality

$$D(\overline{W}) < 0, \tag{2.9}$$

then for every $W^* = (U^*, V^*) \in \Gamma$ we have that $V^* = \overline{V}$.

Proof. If $W^* = (U^*, V^*) \in \Gamma$, for every $W = (U, V)$, we have that $\mathcal{R}(W, W^*) = (V - V^*) \cdot Q(W) \leq 0$. Notice that, if W is a maximum point for the function $\mathcal{R}(\cdot, W^*)$, we have

$$\mathcal{R}_U(W, W^*) = {}^TQ_U(W)(V - V^*) = 0, \tag{2.10}$$

$$\mathcal{R}_V(W, W^*) = {}^TQ_V(W)(V - V^*) + Q(W) = 0, \tag{2.11}$$

and

(2.12)

$$\mathcal{R}_{WW}(W, W^*) = \begin{pmatrix} {}^TQ_{UU}(W)(V - V^*) & {}^TQ_U(W) + {}^TQ_{UV}(W)(V - V^*) \\ Q_U(W) + {}^TQ_{UV}(W)(V - V^*) & 2D(W) + {}^TQ_{VV}(W)(V - V^*) \end{pmatrix} \leq 0.$$

Clearly the point $\overline{W} = (\overline{U}, V^*)$ is a maximum point for $\mathcal{R}(\cdot, W^*)$. Then, from (2.11), $Q(\overline{W}) = 0$, and, since

$$\mathcal{R}_{WW}(\overline{W}, W^*) = \begin{pmatrix} 0 & {}^TQ_U(\overline{W}) \\ Q_U(\overline{W}) & 2D(W) \end{pmatrix} \leq 0,$$

we obtain (2.7) and (2.8).

On the other hand, if $\overline{W}$ is an equilibrium point and (2.9) holds true, to obtain the conclusion we just multiply (2.11) by $(\overline{V} - V^*)$. □

Let us observe that, if $\overline{W} = (\overline{U}, \overline{V}) \in \Gamma$ and $W = (U, V)$ is any given point such that the segment $(U, sV + (1-s)\overline{V}) \in \mathcal{E}'(\Omega)$ for all $s \in [0,1]$, we have that the point $W^* = (U, \overline{V})$ is again in Γ. Then, we have that

$$Q(W) = \left(\int_0^1 {}^TQ_V(U, sV + (1-s)\overline{V}) ds \right) (V - \overline{V}).$$

In particular if condition (2.9) holds in $\overline{W}$, we have that there exists a matrix $B = B(W, \overline{W}) \in \mathcal{M}^{(n-k)\times(n-k)}$, positive definite, such that, for every W in a suitable neighborhood of $\overline{W}$,

$$Q(W) = -B(W, \overline{W})(V - \overline{V}). \tag{2.13}$$

It is now very natural to consider a slightly restricted class of entropy dissipative systems.

Definition 2.2. *The system (2.1) is* strictly entropy dissipative*, if there exists a matrix $B = B(W, \overline{W}) \in \mathcal{M}^{(n-k)\times(n-k)}$, positive definite, such that*

$$Q(W) = -B(W, \overline{W})(V - \overline{V}), \tag{2.14}$$

for every $W \in \mathcal{E}'(\Omega)$ and $\overline{W} = (\overline{U}, \overline{V}) \in \Gamma$.

Proposition 2.2. *If the system (2.1) is strictly entropy dissipative, there exists $\overline{V} \in \mathbb{R}^{n-k}$ such that the equilibrium manifold Γ is:*

$$\Gamma = \{W \in \mathcal{E}'(\Omega); V = \overline{V}\}.$$

Up to now we have worked with a fixed entropy function. We observe that our definition of dissipative entropy is just invariant for affine perturbations, namely of the form $\tilde{\mathcal{E}}(w) = \mathcal{E}(w) + \alpha + \beta \cdot w$, for $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}^n$. This change of entropy implies a shift of the entropy variable

$$\tilde{W} = W + \beta.$$

Fix now $\overline{w}_0 \in \gamma$. Taking the new variable $\tilde{w} = w - \overline{w}_0$, we can use the quadratic entropy function for $\tilde{w}$ given by

$$\tilde{\mathcal{E}}(\tilde{w}) = \mathcal{E}(\tilde{w} + \overline{w}_0) - \mathcal{E}(\overline{w}_0) - \mathcal{E}'(\overline{w}_0) \cdot \tilde{w}$$

and set $\tilde{W} = \tilde{\mathcal{E}}'(\tilde{w})$.

In what follows, we shall study the Cauchy problem in a neighborhood of any $\overline{w}_0 \in \gamma$, for the initial data $w_0 = \overline{w}_0 + \tilde{w}_0$. Therefore, without loss of generality, we can suppose $\overline{w}_0 = 0 \in \gamma$ and consider system (2.1) with a quadratic entropy function $\mathcal{E}$, i.e. such that

$$\mathcal{E}(0) = 0, \ \mathcal{E}'(0) = 0 \in \Gamma.$$

If system (2.1) is *strictly* entropy dissipative, then

$$Q(W) = -B(W)V \tag{2.15}$$

where $B \in \mathcal{M}^{(n-k)\times(n-k)}$ is a positive definite matrix, the entropy production is

$$\mathcal{R}(W) = -V \cdot Q(W) = -V \cdot B(W)V, \tag{2.16}$$

and

$$\Gamma = \{W \in \mathcal{E}'(\Omega); V = 0\}.$$

Remark 2.1. *It is useful to compare our definition of equilibrium point with the more thermodynamic notion proposed by Boillat and Ruggeri* [BR97, BR98]*: an equilibrium point in their sense is a state for which the entropy production vanishes and attains its maximum value. Here, we prefer to use a definition which is more commonly accepted for dynamical systems and relaxation problems. Notice that, for strictly dissipative systems the two notions coincide.*

Remark 2.2. *The dissipative entropy condition is strictly related to various stability conditions for the relaxation limit on bounded time intervals, see* [CLL94] *and the recent review* [Yon01]*. Unfortunately, this condition is not sufficient to prove the global existence of smooth solutions. To show that, we just consider the simple decoupled system*

$$\begin{cases} \partial_t u + \partial_x(\frac{1}{2}u^2) = 0 \ , \\ \\ \partial_t v + \partial_x v = -v, \end{cases}$$

which verifies the (strict) entropy dissipation condition for $\mathcal{E} = \frac{1}{2}(u^2+v^2)$ *and* $\mathcal{F} = \frac{1}{6}u^3 + \frac{1}{2}v^2$*.*

In the following sections, we shall investigate the role of coupling conditions in global existence problems.

3. Examples

This section contains some examples which show the basic structures considered in Section 2.

Example 3.1. The p-system with damping. Let us consider the system (1.3)

$$\begin{cases} \partial_t u - \partial_x v = 0 \ , \\ \\ \partial_t v + \partial_x p(u) = -v, \end{cases} \tag{3.1}$$

with $p'(u) < 0$. In this case a canonical entropy function is

$$\mathcal{E} = \frac{1}{2}v^2 + \sigma(u),$$

with $\sigma'(u) = -p(u)$. Then, the entropy variables are given by

$$W = \begin{pmatrix} U \\ \\ V \end{pmatrix} = \begin{pmatrix} -p(u) \\ \\ v \end{pmatrix},$$

and the system (3.1) can be written in the form (2.4) by means of

$$\begin{pmatrix} -(p^{-1})'(-U) & 0 \\ \\ 0 & 1 \end{pmatrix} \partial_t \begin{pmatrix} U \\ \\ V \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ \\ 1 & 0 \end{pmatrix} \partial_x \begin{pmatrix} U \\ \\ V \end{pmatrix} = - \begin{pmatrix} 0 \\ \\ V \end{pmatrix},$$

and in particular $B = 1 > 0$, and the entropy production $\mathcal{R}(W) = -V^2$.

Example 3.2. The entropy totally dissipative case. We say that the system

$$\partial_t w + \partial_x F(w) = G(w), \tag{3.2}$$

is *entropy totally dissipative* in $W = 0$, if $k = 0$, namely the operator P_2 in Section 2 is the identity in $\mathbb{R}^n$ and the system is strictly entropy dissipative for one strictly convex quadratic entropy function $\mathcal{E}$, with $\mathcal{E}(0) = 0$. It is possible to show that the condition given by Li Ta–Tsien [Li94] for proving the global existence, implies our condition.

Example 3.3. The Suliciu model. Let us consider the following semilinear system

$$
\left\{
\begin{aligned}
&\partial_t u_1 - \partial_x u_2 = 0 \ , \\
&\partial_t u_2 - \partial_x v = 0 \ , \\
&\partial_t v - \mu \partial_x u_2 = \tfrac{1}{\varepsilon}(g(u_1) - v).
\end{aligned}
\right. \tag{3.3}
$$

This model has been proposed by Suliciu [Sul90], as an approximation for the equations of isothermal viscoelasticity. From a mathematical point of view, it has been considered in some recent papers, see for instance [LP00, HP99, Tza99].

We shall assume that the following subcharacteristic type condition holds:

$$
\mu > g'(u_1) > 0, \tag{3.4}
$$

for all the values of u_1 under consideration. Under this assumption, we can construct a dissipative entropy function for the system (3.3). Set $\varepsilon = 1$ and

$$
\psi(\xi, \eta) := \xi\eta + \frac{\mu}{2}\xi^2 - \int_0^\eta h^{-1}(y)dy,
$$

where the function h^{-1} is the inverse function of $h(\xi) := g(\xi) - \mu\xi$.

Following [Tza99], it is easy to show that the entropy function $\mathcal{E}$ is

$$
\mathcal{E}(u_1, u_2, v) = \frac{1}{2}u_2^2 + \psi(u_1, v - \mu u_1),
$$

for the flux function

$$
\mathcal{F}(u_1, u_2, v) = -vu_2.
$$

The entropy variables are now given by $W = (U_1, U_2, V)$, with

$$
U_1 = v + \mu(h^{-1}(v - \mu u_1) - u_1), \ U_2 = u_2, \ V = u_1 - h^{-1}(v - \mu u_1).
$$

The matrices A_0 and A_1 in (2.4) are

$$
A_0 = \begin{pmatrix} (g'(g^{-1}(U_1))^{-1} & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & \mu \end{pmatrix}, \ A_1 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -\mu \\ 0 & -\mu & 0 \end{pmatrix}.
$$

The source term is

$$
\begin{aligned}
Q(W) \ &= g(g^{-1}(U_1) + V) - (U_1 + \mu V) \\
&= \left(\int_0^1 \left[g'(g^{-1}(U_1) + \tau V) - \mu \right] d\tau \right) V.
\end{aligned}
$$

Then, by the subcharacteristic condition (3.4),

$$
B = -\left(\int_0^1 \left[g'(g^{-1}(U_1) + \tau V) - \mu \right] d\tau \right) > 0,
$$

and the entropy production is $\mathcal{R}(W) = -BV^2 \leq 0$.

Example 3.4. The Kerr–Debye model. Let us consider the system

$$
\left\{
\begin{aligned}
&\partial_t d + \partial_x b = 0 \ , \\
&\partial_t b + \partial_x ((1+v)^{-1} d) = 0 \ , \\
&\partial_t v = \tfrac{1}{\varepsilon}((1+v)^{-2} d^2 - v).
\end{aligned}
\right. \tag{3.5}
$$

This model has been proposed by Ziolkowski [RWZ97], to describe the propagation of electromagnetic waves in nonlinear Kerr medium, and it has been studied in [HH00, Huy99]. Here, we consider the one dimensional model defined in $\Omega = \{v > -1\}$.

The system has the form (2.1), by setting $u = (d, b)$ and $q(u, v) = \frac{1}{\varepsilon}((1+v)^{-2}d^2 - v)$. For $\varepsilon = 1$, an entropy function is given by

$$\mathcal{E} = \frac{1}{2}(1+v)^{-1}d^2 + \frac{1}{2}b^2 + \frac{1}{4}v^2,$$

and the corresponding entropy variables are

$$U_1 = (1+v)^{-1}d, \; U_2 = b, \; V = -\frac{1}{2}((1+v)^{-2}d^2 - v).$$

The matrices A_0 and A_1 in (2.4) are given by

$$A_0 = \begin{pmatrix} 3U_1^2 + 2V + 1 & 0 & 2U_1 \\ 0 & 1 & 0 \\ 2U_1 & 0 & 2 \end{pmatrix}, \; A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and, since $Q(W) = -2V$, we have

$$\mathcal{R}(W) = -2V^2 \leq 0.$$

Example 3.5. The Jin–Xin relaxation model. The relaxation system

$$\begin{cases} \partial_t u + \partial_x v = 0 \;, \\ \\ \partial_t v + \mu^2 \partial_x u = \frac{1}{\varepsilon}(f(u) - v), \end{cases} \tag{3.6}$$

for $u, v \in \mathbb{R}^k$, $f : \mathbb{R}^k \to \mathbb{R}^k$, has been proposed by S.Jin and Z.Xin in [JX95], to give an approximation, as $\varepsilon \to 0$, of a general system of conservation laws

$$\partial_t u + \partial_x f(u) = 0. \tag{3.7}$$

Its analytical and numerical study has been continued in many recent papers [Nat96, ADN96, Nat98, ADN00, Ser00, Bia01].

In the following, we put $\varepsilon = 1$ and assume the classical stability sub-characteristic condition, see [Whi74, Liu87]

$$\mu > \sup_{u \in \Omega_1} \{|\sigma(f'(u))|\}, \tag{3.8}$$

where $\sigma(f'(u))$ is the spectrum of f', and Ω_1 is a given open convex set in $\mathbb{R}^k$.

To construct a dissipative entropy function for system (3.6), we follow [Ser00] and we assume the existence of a strictly convex entropy function η for the relaxed system (3.7). Let q be the corresponding entropy flux function. Set

$$M_{\pm}(u) := u \pm \frac{1}{\mu} f(u).$$

Thanks to condition (3.8), there exist the inverse functions $M_{\pm}^{-1}(\xi) : \mathbb{R}^k \to \mathbb{R}^k$. Therefore, we can define the auxiliary functions

$$e_{\pm}(\xi) := \frac{1}{2}\left(\eta \pm \frac{1}{\mu} q\right)(M_{\pm}^{-1}(\xi)),$$

for $\xi \in M_{\pm}^{-1}(\Omega_1)$.

The dissipative entropy function for system (3.6) is now given by

$$\mathcal{E}(u, v) = e_+(u + \frac{1}{\mu}v) + e_-(u - \frac{1}{\mu}v),$$

with the entropy flux

$$\mathcal{F}(u,v) = \mu\left(e_+(u+\frac{1}{\mu}v) - e_-(u-\frac{1}{\mu}v)\right).$$

To introduce the entropy variables, let us set

$$\Phi_\pm(\xi) = \eta'(M_\pm^{-1}(\xi)),$$

for $\xi \in M_\pm^{-1}(\Omega_1)$. We have

$$\begin{aligned} U &= \partial_u \mathcal{E}(u,v) = \tfrac{1}{2}\left(\Phi_+(u+\tfrac{1}{\mu}v) + \Phi_-(u-\tfrac{1}{\mu}v)\right), \\ V &= \partial_v \mathcal{E}(u,v) = \tfrac{1}{2\mu}\left(\Phi_+(u+\tfrac{1}{\mu}v) - \Phi_-(u-\tfrac{1}{\mu}v)\right). \end{aligned}$$

Since the matrices $\Phi'_\pm$ are nonsingular, we define

$$a_\pm(u,v) = \left(\Phi'_\pm(u \pm \frac{1}{\mu}v)\right)^{-1}.$$

Then, the matrices A_0 and A_1 in (2.4) are given by

$$A_0(U,V) = \frac{1}{2}\begin{pmatrix} a_+ + a_- & \mu(a_+ - a_-) \\ \mu(a_+ - a_-) & \mu^2(a_+ + a_-) \end{pmatrix}, \tag{3.9}$$

$$A_1(U,V) = \frac{\mu}{2}\begin{pmatrix} a_+ - a_- & \mu(a_+ + a_-) \\ \mu(a_+ + a_-) & \mu^2(a_+ - a_-) \end{pmatrix}. \tag{3.10}$$

Since in the (U,V) coordinates we have

$$f(u) - v = \pm 2\mu\left[M_\pm((\eta')^{-1}(U)) - M_\pm((\eta')^{-1}(U \pm \mu V))\right],$$

we obtain

$$\begin{aligned} f(u) - v &= -2\mu^2\left(\int_0^1 M'_+\left((\eta')^{-1}(U) + \tau((\eta')^{-1}(U+\mu V) - (\eta')^{-1}(U)\right)d\tau\right) \\ &\times \left(\int_0^1 \left((\eta')^{-1}\right)'(U + s\mu V)ds\right)V \\ &= -B(U,V)V. \end{aligned}$$

Notice that, thanks to the assumption (3.8) and to the convexity of η, the matrices $M'_\pm$ and $((\eta')^{-1})'$ are definite positive, as well as the matrix B.

Example 3.6. The BGK models. Starting from the ideas introduced for the Jin–Xin relaxation model, a more general class of kinetic systems has been recently considered, namely the BGK models with finite discrete velocities [Nat98, Bou99, Ser00, ADN00]:

$$\partial_t f_i + \lambda_i \partial_x f_i = (M_i(u) - f_i), \tag{3.11}$$

for $i = 1, \ldots, n$, where $u = \sum_{i=1}^n f_i \in \mathbb{R}$ and $M_i(u)$ are some given functions such that

$$\sum_{i=1}^n M_i(u) = u. \tag{3.12}$$

For these systems, a stability condition has been proposed in [Nat98], see also for the general case [Bou99]:

$$M'_i(u) > 0 \tag{3.13}$$

when $u \in I$, for some bounded interval $I \subseteq \mathbb{R}$.

To set these systems in our framework, we have to introduce the operator $\mathbb{P} := \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$ of subsection 2.1. Here we have $k = 1$ and we can choose

$$P_1 = (1, \dots, 1),$$

$$P_2 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & & & 1 \end{pmatrix}$$

We now rewrite system (3.11) using the transformation

$$u = \sum_{i=1}^{n} f_i, \;\; v_i = f_{i+1}, \;\; i = 1, \dots, n-1.$$

This yields

$$\begin{cases} \partial_t u + \lambda_1 \partial_x u + \sum_{i=1}^{n-1} (\lambda_{i+1} - \lambda_1) \partial_x v_i = 0 \;, \\ \\ \partial_t v_i + \lambda_{i+1} \partial_x v_i = (M_{i+1}(u) - v_i) \;, \;\; i = 1, \dots, n-1. \end{cases} \tag{3.14}$$

Following [Ser00] and [Bou99], we now pass to the construction of a family of dissipative entropy functions. Take a strictly convex function $\eta = \eta(u)$. Let $\mathcal{E}_i = \mathcal{E}_i(\xi)$, a strictly convex function such that

$$\mathcal{E}_i'(\xi) = \eta' \left(M_i^{-1}(\xi) \right) ,$$

for $i = 1, \dots, n$. We choose the entropy function as

$$\mathcal{E}(u, v) = \mathcal{E}_1(u - \sum_{i=1}^{n-1} v_i) + \sum_{i=1}^{n-1} \mathcal{E}_{i+1}(v_i).$$

Then, the entropy variables are given by

$$\begin{aligned} U &= \partial_u \mathcal{E} = \eta' \left(M_1^{-1}(u - \textstyle\sum_{i=1}^{n-1} v_i) \right) , \\ V_i &= \partial_{v_i} \mathcal{E} = \eta' \left(M_{i+1}^{-1}(v_i) \right) - U, \;\; i = 1, \dots, n-1. \end{aligned} \tag{3.15}$$

The inverse transformation is

$$\begin{aligned} u &= M_1 \left((\eta')^{-1}(U) \right) + \textstyle\sum_{i=1}^{n-1} M_{i+1} \left((\eta')^{-1}(U + V_i) \right) , \\ v_i &= M_{i+1} \left((\eta')^{-1}(U + V_i) \right) , \;\; i = 1, \dots, n-1. \end{aligned} \tag{3.16}$$

We show now that the equilibrium manifold

$$f_i = M_i(u), \;\; i = 1, \dots, n,$$

is given in these entropy variables by

$$V_i = 0, \;\; i = 1, \dots, n-1.$$

In fact, $f_i = M_i(u)$ is equivalent to

$$v_i = M_{i+1}(u), \;\; i = 1, \dots, n-1.$$

Then, by (3.15),

$$U = \eta' \left(M_1^{-1}(u - \sum_{i=1}^{n-1} M_{i+1}(u)) \right) = \eta'(u),$$

which implies

$$V_i = \eta'(u) - U = 0.$$

On the other hand, if $V_i = 0$, using (3.15), (3.12) and (3.16), we have that

$$\begin{aligned} u - \textstyle\sum_{i=1}^{n-1} v_i &= M_1\left((\eta')^{-1}(U)\right) \\ &= (\eta')^{-1}(U) - \textstyle\sum_{i=1}^{n-1} M_{i+1}\left((\eta')^{-1}(U)\right) \\ &= (\eta')^{-1}(U) - \textstyle\sum_{i=1}^{n-1} v_i. \end{aligned}$$

Then, for $V_i = 0$, $u = (\eta')^{-1}(U)$, and then

$$M_{i+1}(u) - v_i = M_{i+1}(u) - M_{i+1}\left((\eta')^{-1}(U)\right) = 0.$$

To prove the strict entropy dissipative condition of Definition 2.2, we have to show that the matrix B is definite positive. To this aim, we first rewrite $M_{i+1}(u) - v_i$ out of the equilibrium manifold, as

$$M_{i+1}(u) - v_i = \left(\int_0^1 M'_{i+1}\left(\alpha u + (1-\alpha)(\eta')^{-1}(U+V_i)\right) d\alpha\right)(u - (\eta')^{-1}(U+V_i)).$$

Therefore we obtain

$$\begin{aligned} B_{ii}(U,V) &= \left(\int_0^1 M'_{i+1}\left(\alpha u + (1-\alpha)(\eta')^{-1}(U+V_i)\right) d\alpha\right) \\ &\quad \times \int_0^1 [1 - M'_{i+1}((\eta')^{-1}(U+\beta V_i)]\left((\eta')^{-1}\right)'(U+\beta V_i) d\beta, \end{aligned}$$

and, for $l \neq i$,

$$\begin{aligned} B_{il}(U,V) &= -\left(\int_0^1 M'_{i+1}\left(\alpha u + (1-\alpha)(\eta')^{-1}(U+V_i)\right) d\alpha\right) \\ &\quad \times \int_0^1 [M'_{l+1}((\eta')^{-1}(U+\beta V_l)]\left((\eta')^{-1}\right)'(U+\beta V_l) d\beta. \end{aligned}$$

On the equilibrium manifold, we have

$$B_{ii}(U,0) = M'_{i+1}(u)\left(1 - M'_{i+1}(u)\right)\left((\eta')^{-1}\right)'(U),$$

and, for $l \neq i$,

$$B_{il}(U,0) = -M'_{i+1}(u) M'_{l+1}(u)\left((\eta')^{-1}\right)'(U).$$

From (3.12) and (3.13), we have that the matrix is strictly diagonal dominant, since

$$\begin{aligned} B_{ii}(U,0) &\quad - \sum_{l \neq i} |B_{il}(U,0)| \\ &= \left((\eta')^{-1}\right)'(U) M'_{i+1}(u)(1 - \textstyle\sum_{j=1}^{n-1} M'_{j+1}(u)) \\ &= \left((\eta')^{-1}\right)'(U) M'_{i+1}(u) M'_1(u) > 0. \end{aligned}$$

It is possible to extend these models to the case of equilibrium systems, when $u \in \mathbb{R}^k$ for $k > 1$, by arguing as in [ADN00, Ser00, Bou99].

Example 3.7. Gas dynamics in thermal nonequilibrium. Our last example is given by the equations of gas dynamics in vibrational nonequilibrium in Lagrangian coordinates, which with the usual physical notations read

$$\begin{cases} \partial_t v - \partial_x u = 0 \ , \\ \partial_t u + \partial_x p = 0 \ , \\ \partial_t E + \partial_x pu = 0 \ , \\ \partial_t q = \frac{1}{\tau}(Q - q), \end{cases} \tag{3.17}$$

where v, u, E, and q, are respectively the specific volume, velocity, total energy and internal vibrational energy of the gas, and τ is the relaxation time for vibrational modes, see for instance [Whi74].

To close the formulation, we have to specify the state equations for the pressure p and for the local nonequilibrium value of q. Let e_1, T_1, and T_2 be respectively the total specific translational and rotational internal energy, the related specific temperature, and the vibrational temperature. We restrict in the following our attention to the ideal gases, see [Zen99] for the general case. Then we have the following relations

$$e_1 = E - q - \frac{1}{2}u^2, \ p = p(e_1, v) = (\gamma - 1)\frac{e_1}{v}, \ T_1 = \frac{e_1}{C_v}, \ T_2 = T_2(q),$$

where γ and C_v are some positive given constants, $1 < \gamma < 3$, and T_2 is a strictly monotone function with $T_2(0) = 0$. Finally. setting $\omega = T_2^{-1}$, we can give the function Q as

$$Q = Q(T_1) = \omega(T_1).$$

Clearly the equilibrium state corresponds to the case $T_1 = T_2$.

The entropy function is now given by

$$\mathcal{E}(v, u, E, q) = -C_v \log(e_1 v^{\gamma-1}) - s_2(q),$$

where $s_2'(q) = \frac{1}{T_2(q)}$. The entropy variables $W = (U, V)$ are

$$U_1 = -\frac{p}{T_1}, \ U_2 = \frac{u}{T_1}, \ U_3 = -\frac{1}{T_1}, \ V = \frac{1}{T_1} - \frac{1}{T_2(q)},$$

in the domain $\Omega = \{U_3 < 0\} \subseteq \mathbb{R}^4$. The inverse transformation is

$$v = -(\gamma - 1)\frac{C_v}{U_1}, \ u - \frac{U_2}{U_3}, \ E = -\frac{C_v}{U_3} + \frac{1}{2}\frac{U_2^2}{U_3^2} + \omega(-\frac{1}{U_3 + V}), \ q = \omega(-\frac{1}{U_3 + V}).$$

The matrix A_0 in (2.4) is given by

$$A_0(W) = \begin{pmatrix} (\gamma-1)\frac{C_v}{U_1^2} & 0 & 0 & 0 \\ 0 & -\frac{1}{U_3} & \frac{U_2}{U_3^2} & 0 \\ 0 & \frac{U_2}{U_3^2} & \frac{C_v}{U_3^2} - \frac{U_2^2}{U_3^3} + \alpha & \alpha \\ 0 & 0 & \alpha & \alpha \end{pmatrix}, \tag{3.18}$$

where $\alpha = \alpha(U_3 + V) = \omega'(-\frac{1}{U_3+V})\left(\frac{1}{U_3+V}\right)^2$, is a strictly positive function. Since the flux function is given by $F = (-u, p, pu, 0)$, we can write

$$A_1(W) = F'(\Phi(W))A_0(W) = \begin{pmatrix} 0 & \frac{1}{U_3} & -\frac{U_2}{U_3^2} & 0 \\ \frac{1}{U_3} & 0 & -\frac{U_1}{U_3^2} & 0 \\ -\frac{U_2}{U_3^2} & -\frac{U_1}{U_3^2} & 2\frac{U_1U_2}{U_3^3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.19}$$

Finally the source term is

$$\begin{aligned} Q(W) &= \omega(-\tfrac{1}{U_3}) - \omega(-\tfrac{1}{U_3+V}) \\ &= -\left(\int_0^1 \alpha(U_3 + sV)ds\right)V = -B(W)V, \end{aligned}$$

with $B(W) > 0$.

4. Energy estimates

According with the framework introduced in Section 2, we are going to consider the Cauchy problem for a small perturbation w_0 of the zero equilibrium state, which is

$$\begin{cases} \partial_t u + \partial_x F_1(w) = 0 \ , \\ \partial_t v + \partial_x F_2(w) = q(w) \ , \end{cases} \tag{4.1}$$

with $w = (u, v) \in \Omega \subseteq \mathbb{R}^k \times \mathbb{R}^{n-k}$, for the initial condition

$$w(x, 0) = w_0(x). \tag{4.2}$$

Let $\mathcal{E}$ be a quadratic strictly convex entropy function, and assume that (4.1) is strictly dissipative with respect to $\mathcal{E}$. Setting $W = \mathcal{E}'(w)$, we have in an equivalent way

$$A_0(W)\partial_t W + A_1(W)\partial_x W = -\begin{pmatrix} 0 \\ B(W)V \end{pmatrix}, \tag{4.3}$$

where $W = (U, V) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, $A_0(W)$ is symmetric positive definite, $A_1(W)$ is symmetric, and $B \in \mathcal{M}^{(n-k)\times(n-k)}$ is positive definite. Note that

$$A := A_0^{-1}A_1 = {}^T F'(\Phi(W)).$$

For the initial conditions, we have

$$W(x, 0) = W_0(x) = \mathcal{E}'(w_0(x)). \tag{4.4}$$

Let us now introduce the classical Sobolev spaces H^s, endowed with the norm $\|\cdot\|_s$.

We consider local (in time) solutions of system (4.1) or (4.3), which are in the space $C^0([0,T]; H^2(\mathbb{R})) \cap C^1([0,T]; H^1(\mathbb{R}))$.

Zero order estimates. Let w be a local solution to (4.1)–(4.2). Thanks to the entropy equality (2.2), we have

$$\frac{d}{dt}\int \mathcal{E}(w(x,t))dx = \int \mathcal{E}'(w(x,t)) \cdot G(w(x,t))dx, \tag{4.5}$$

and therefore, for $t \in (0,T)$,

$$\int \mathcal{E}(w(x,t))dx = \int \mathcal{E}(w_0(x))dx + \int_0^t \int \mathcal{E}'(w(x,s)) \cdot G(w(x,s))dxds. \tag{4.6}$$

From the quadratic growth of $\mathcal{E}$, there exists $\delta_0 > 0$ and two constants $c_1,\ c_2 > 0$, such that, for $|W| \leq \delta_0$

$$\frac{1}{c_1}|W|^2 \leq \mathcal{E}(\Phi(W)) \leq c_1|W|^2 \tag{4.7}$$

and

$$V \cdot (B(W)V) \geq c_2|V|^2, \tag{4.8}$$

for every $V \in \mathbb{R}^{n-k}$.

Let W be a local solution to problem (4.3)–(4.4), such that, for $t \in (0,T)$ and $x \in \mathbb{R}$

$$|W(x,t)| \leq \delta_0.$$

Then, using (4.6), (4.7), and (4.8),

$$\|W(t)\|_0^2 + c_2c_1 \int_0^t \|V(s)\|_0^2 ds \leq c_1^2\|W_0\|_0^2. \tag{4.9}$$

Lemma 4.1. *There exist $\delta_0 > 0$ and $C > 0$, such that, for every local solution W to problem (4.3)–(4.4), with*

$$\|W(t)\|_{L^\infty([0,T]\times\mathbb{R})} \leq \delta_0,$$

we have

$$\|W(t)\|_0^2 + \int_0^t \|V(s)\|_0^2 ds \leq C\|W_0\|_0^2, \tag{4.10}$$

for every $t \in (0,T)$.

Next, we estimate the L^2-norms of the first and second derivatives of the local solution W.

First order estimates. Apply the derivative ∂_x to (4.3) and take the inner product with $\partial_x W$, which yields

$$\partial_x \left(A_0(W)\partial_t W + A_1(W)\partial_x W\right) \cdot \partial_x W = -(\partial_x BV + B\partial_x V) \cdot \partial_x V. \tag{4.11}$$

We have the following identities

$$\begin{aligned} \partial_x \left(A_0(W)\partial_t W\right) \cdot \partial_x W &= \tfrac{1}{2}\partial_t \left((A_0(W)\partial_x W) \cdot \partial_x W\right) \\ &\quad -\tfrac{1}{2}\left(\partial_t A_0 \partial_x W\right) \cdot \partial_x W + \left(\partial_x A_0 \partial_t W\right) \cdot \partial_x W; \end{aligned} \tag{4.12}$$

$$\partial_x \left(A_1(W)\partial_x W\right) \cdot \partial_x W = \frac{1}{2}\partial_x \left((A_1(W)\partial_x W) \cdot \partial_x W\right) + \frac{1}{2}\left(\partial_x A_1 \partial_x W\right) \cdot \partial_x W. \tag{4.13}$$

Therefore, if we integrate equality (4.11) over $\mathbb{R}$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int (A_0(W)\partial_x W) \cdot \partial_x W dx + \int (B\partial_x V) \cdot \partial_x V dx \\ &= \int \left[\frac{1}{2}\partial_t A_0 \partial_x W - \partial_x A_0 \partial_t W - \frac{1}{2}\partial_x A_1 \partial_x W\right] \cdot \partial_x W - (\partial_x BV) \cdot \partial_x V dx. \end{aligned} \tag{4.14}$$

To estimate the right hand side of (4.14), we use the equation (4.3) in the form

$$\partial_t W = -A\partial_x W - A_0^{-1} \begin{pmatrix} 0 \\ B(W)V \end{pmatrix}. \tag{4.15}$$

Then

$$\partial_t A_0 = A_0' \partial_t W = -A_0' \left(A\partial_x W + A_0^{-1} \begin{pmatrix} 0 \\ B(W)V \end{pmatrix} \right).$$

We consider the following terms which will be useful in the estimate of (4.14).

$$\begin{aligned}
I_1 : \quad &= \tfrac{1}{2} \left[\partial_t A_0 \partial_x W + \partial_x A_1 \partial_x W \right] \cdot \partial_x W \\
&= \left\{ \frac{1}{2} \left[A_1' \partial_x W - A_0'(A\partial_x W) \right] - \frac{1}{2} A_0' (A_0^{-1} \begin{pmatrix} 0 \\ B(W)V \end{pmatrix}) \right\} \partial_x W \cdot \partial_x W \\
&= \left\{ \frac{1}{2} \left[(A_0' \partial_x W) A - A_0'(A\partial_x W) + A_0(A' \partial_x W) \right] \right. \\
&\left. -\tfrac{1}{2} A_0' (A_0^{-1} \begin{pmatrix} 0 \\ B(W)V \end{pmatrix}) \right\} \partial_x W \cdot \partial_x W \\
&= \left\{ \tfrac{1}{2} \left[A_0 (A' \partial_x W) \right] - \tfrac{1}{2} A_0' (A_0^{-1} \begin{pmatrix} 0 \\ B(W)V \end{pmatrix}) \right\} \partial_x W \cdot \partial_x W,
\end{aligned}$$

since $A_0' = \Phi''$ is symmetric.

$$\begin{aligned}
I_2 : \quad &= \left[\partial_x A_0 \partial_t W + \partial_x A_1 \partial_x W \right] \cdot \partial_x W \\
&= \left\{ \left[(A_0 A)' \partial_x W - (A_0' \partial_x W) A \right] \partial_x W - (A_0' \partial_x W)(A_0^{-1} \begin{pmatrix} 0 \\ B(W)V \end{pmatrix}) \right\} \cdot \partial_x W \\
&= \left\{ A_0 (A' \partial_x W) \partial_x W - (A_0' \partial_x W)(A_0^{-1} \begin{pmatrix} 0 \\ B(W)V \end{pmatrix}) \right\} \cdot \partial_x W.
\end{aligned}$$

Reporting $I_1 - I_2$ in (4.14) yields

$$\begin{aligned}
&\frac{1}{2} \quad \frac{d}{dt} \int (A_0 \partial_x W) \cdot \partial_x W dx + \int (B \partial_x V) \cdot \partial_x V dx \\
&= -\frac{1}{2} \int \left[A_0 (A' \partial_x W) \partial_x W \right] \cdot \partial_x W dx + \int (A_0' \partial_x W) A_0^{-1} \begin{pmatrix} 0 \\ B(W)V \end{pmatrix} \cdot \partial_x W dx \\
&- \int \left\{ \frac{1}{2} A_0' (A_0^{-1} \begin{pmatrix} 0 \\ B(W)V \end{pmatrix}) \partial_x W \cdot \partial_x W + ((B' \partial_x W) V) \cdot \partial_x V \right\} dx.
\end{aligned} \tag{4.16}$$

Remark 4.1. In (4.16) the first term on the right hand side is cubic in $\partial_x W$. It vanishes when the matrix

$$A_0(W) A'(W)(\cdot)$$

is skew–symmetric. The other terms are quadratic in $\partial_x W$ and linear in V, namely, they can be written in the form

$$\int \phi(W) \left(\partial_x W, \partial_x W \right) V dx.$$

Second order estimates. In the same way, we perform the second space derivative of (4.3) and take the inner product with $\partial_{xx} W$, which gives

$$\partial_{xx} \left(A_0(W) \partial_t W + A_1(W) \partial_x W \right) \cdot \partial_{xx} W = -\partial_{xx} \begin{pmatrix} 0 \\ B(W)V \end{pmatrix} \cdot \partial_{xx} W. \tag{4.17}$$

We have the following useful identities

$$
\begin{aligned}
\partial_{xx}\left(A_0(W)\partial_t W\right)\cdot\partial_{xx}W &= \tfrac{1}{2}\partial_t\left((A_0(W)\partial_{xx}W)\cdot\partial_{xx}W\right) - \tfrac{1}{2}(\partial_t A_0\partial_{xx}W)\cdot\partial_{xx}W \\
&\quad +2(\partial_x A_0\partial_{xt}W)\cdot\partial_{xx}W + (\partial_{xx}A_0\partial_t W)\cdot\partial_{xx}W;
\end{aligned}
\tag{4.18}
$$

$$
\begin{aligned}
\partial_{xx}\left(A_1(W)\partial_x W\right)\cdot\partial_{xx}W &= \tfrac{1}{2}\partial_x\left((A_1(W)\partial_{xx}W)\cdot\partial_{xx}W\right) \\
&\quad +\tfrac{3}{2}(\partial_x A_1\partial_{xx}W)\cdot\partial_{xx}W + (\partial_{xx}A_1\partial_x W)\cdot\partial_{xx}W.
\end{aligned}
\tag{4.19}
$$

Therefore, if we integrate equality (4.17) over $\mathbb{R}$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int (A_0(W)\partial_{xx}W)\cdot\partial_{xx}W\,dx + \int (B\partial_{xx}V)\cdot\partial_{xx}V\,dx \\
&= \int\left[\frac{1}{2}\partial_t A_0\partial_{xx}W - 2\partial_x A_0\partial_{xt}W - \partial_{xx}A_0\partial_t W\right]\cdot\partial_{xx}W\,dx \\
&- \int\left[\frac{3}{2}\partial_x A_1\partial_{xx}W + \partial_{xx}A_1\partial_x W\right]\cdot\partial_{xx}W\,dx \\
&- \int\left[2\partial_x B\partial_x V + \partial_{xx}BV\right]\cdot\partial_{xx}V\,dx.
\end{aligned}
\tag{4.20}
$$

As previously, we are going to develop some terms which will be useful for estimating (4.20). By using again (4.15), we have

$$
\begin{aligned}
J_1 :\ &= (\partial_t A_0 + \partial_x A_1)\,\partial_{xx}W\cdot\partial_{xx}W \\
&= \left[A_1'\partial_x W - A_0'A\partial_x W - A_0'A_0^{-1}\begin{pmatrix}0\\ B(W)V\end{pmatrix}\right]\partial_{xx}W\cdot\partial_{xx}W \\
&= \left[A_0'\partial_x WA - A_0'A\partial_x W + A_0A'\partial_x W - A_0'A_0^{-1}\begin{pmatrix}0\\ B(W)V\end{pmatrix}\right]\partial_{xx}W\cdot\partial_{xx}W
\end{aligned}
$$

By taking the x derivative of (4.15), we obtain also

$$
\begin{aligned}
J_2 :\ &= (\partial_x A_0\partial_{xt}W + \partial_x A_1\partial_{xx}W)\cdot\partial_{xx}W \\
&= -\left\{(A_0'\partial_x W)\left[(A'\partial_x W)\partial_x W + A\partial_{xx}W + \partial_x\left(A_0^{-1}\begin{pmatrix}0\\ B(W)V\end{pmatrix}\right)\right]\right\}\cdot\partial_{xx}W \\
&+ (\partial_x A_1\partial_{xx}W)\cdot\partial_{xx}W \\
&= -\left[(A_0'\partial_x W)\,(A'\partial_x W)\,\partial_x W - A_0\,(A'\partial_x W)\,\partial_{xx}W + \partial_x\left(A_0^{-1}\begin{pmatrix}0\\ B(W)V\end{pmatrix}\right)\right]\cdot\partial_{xx}W.
\end{aligned}
$$

In a straightforward way we have also

$$
\begin{aligned}
J_3 :\ &= [\partial_{xx}A_0\partial_t W + \partial_{xx}A_1\partial_x W]\cdot\partial_{xx}W \\
&= [2\,(A_0'\partial_x W)\,(A'\partial_x W)\,\partial_x W + A_0A''(\partial_x W,\partial_x W)\partial_x W + A_0\,(A'\partial_{xx}W)\,\partial_x W]\cdot\partial_{xx}W \\
&- \left[(A_0''(\partial_x W,\partial_x W) + A_0'\partial_{xx}W)\,A_0^{-1}\begin{pmatrix}0\\ B(W)V\end{pmatrix}\right]\cdot\partial_{xx}W.
\end{aligned}
$$

Reporting $\frac{1}{2}J_1 - 2J_2 - J_3$ in (4.20) yields
(4.21)

$$\begin{aligned}
\tfrac{1}{2}\ \frac{d}{dt}&\int (A_0\partial_{xx}W)\cdot\partial_{xx}W\,dx + \int (B\partial_{xx}V)\cdot\partial_{xx}V\,dx\\
&= \frac{1}{2}\int \left[A_0'\partial_x W A - A_0'(A\partial_x W)\right]\partial_{xx}W\cdot\partial_{xx}W\,dx\\
&-\int \left[A_0A''(\partial_x W,\partial_x W)\partial_x W + A_0\left(A'\partial_{xx}W\right)\partial_x W + \frac{3}{2}A_0(A'\partial_x W)\partial_{xx}W\right]\cdot\partial_{xx}W\,dx\\
&+\int \left[\left(A_0''(\partial_x W,\partial_x W) + A_0'\partial_{xx}W\right)A_0^{-1}\begin{pmatrix}0\\ B(W)V\end{pmatrix}\right]\cdot\partial_{xx}W\,dx\\
&-\int \left[\frac{1}{2}A_0'A_0^{-1}\begin{pmatrix}0\\ B(W)V\end{pmatrix}\partial_{xx}W - \partial_x\left(A_0^{-1}\begin{pmatrix}0\\ B(W)V\end{pmatrix}\right)\right]\cdot\partial_{xx}W\,dx\\
&-\int \left(2\partial_x B\partial_x V + \partial_{xx}BV\right)\cdot\partial_{xx}V\,dx.
\end{aligned}$$

Remark 4.2. In the equality (4.21), the first term on the right-hand side is linear in $\partial_x W$ and quadratic in $\partial_{xx}W$. It vanishes if the matrix

$$A_0'(W)(\cdot)A(W) - A_0'(W)A(W)(\cdot)$$

is skew–symmetric.

The second term is cubic of the form

$$\int \psi(W,\partial_x W)\left(\partial_x W,\partial_{xx}W\right)\partial_x W\,dx,$$

where the application $\psi(W,\partial_x W)(X,Y)$ is quadratic in (X,Y).

The last terms are linear in $(V,\partial_x V)$, of the form

$$\int \psi_1(W,\partial_x W)\left(\partial_x W,\partial_{xx}W\right)V + \psi_2(W)\left(\partial_x W,\partial_{xx}W\right)\partial_x V\,dx,$$

where the applications $\psi_1(W,\partial_x W)(\cdot,\cdot)$ and $\psi_2(W,\partial_x W)(\cdot,\cdot)$ are quadratic.

5. Global existence of smooth solutions under the Kawashima condition

In this section we establish a global existence result for smooth solutions of problem (4.3)–(4.4), when the initial data are sufficiently small in the H^2–norm. It is clear that, if $k = 0$, namely in the conditions of Example 3.2, this result can be proved without any further condition, just using the energy estimates of Section 4. Unfortunately, as shown in Remark 2.2, in the general case we need some other conditions to ensure that the dissipation can prevent the formation of singularities. In this section we present a first result under a condition introduced by Kawashima [Kaw87, SK85].

5.1. The Kawashima condition.

Consider our original system (1.1), in the form

$$\partial_t w + F'(w)\partial_x w = G(w). \tag{5.1}$$

Assume that 0 is an equilibrium point with $G(0) = 0$. For this system the Kawashima condition, which actually was introduced for hyperbolic–parabolic systems, read as follows.

Condition (K): Any eigenvector of $F'(0)$ is not in the null space of $G'(0)$.

It is easily seen that this condition is invariant under diffeomorphisms which conserve the origin. Then, if system (5.1) verifies condition **(K)**, the same holds for system (2.1). For the formulation (2.4) in the entropy variables, we have that

$$\partial_t W + A_0^{-1}(W)A_1(W)\partial_x W = A_0^{-1}(W)G(\Phi(W)).$$

Then, since $A_0^{-1}(W)A_1(W) = {}^T F'(\Phi(W))$, we have the following reformulation of condition **(K)**: any eigenvector of ${}^T F'(\Phi(0))$ is not in the null space of $G'(\Phi(0))\Phi(0)$.

Finally, we restate the condition **(K)** for a strictly entropy dissipative system.

Lemma 5.1. *Assume that system (4.3) is strictly entropy dissipative. Then condition* **(K)** *is equivalent to:*

(5.2) *for every $\lambda \in \mathbb{R}$ and every $X \in \mathbb{R}^k \backslash \{0\}$, the vector $\begin{pmatrix} X \\ 0 \end{pmatrix} \in \mathbb{R}^n$ is not in the null space of $\lambda A_0(0) + A_1(0)$.*

Next, we restate, in the case of system (4.3), a result due to Shizuta and Kawashima [SK85].

Lemma 5.2. *Assume that system (4.3) is strictly entropy dissipative. Then condition (5.2) is equivalent to:*

(5.3) *there exists a constant coefficient matrix $K \in \mathbb{R}^{n\times m}$, such that*

- *$KA_0(0)$ is skew–symmetric;*
- *the matrix*
$$\frac{1}{2}\left(KA_1(0) + {}^T(KA_1(0))\right) + \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}(B(0) + {}^T B(0)) \end{pmatrix}$$
is positive definite.

Notice that in (5.3), it is possible to replace the matrix $\frac{1}{2}(B(0)+ {}^T B(0))$ by every symmetric positive definite matrix L.

5.2. **Global existence.** We can now state our global existence result under condition **(K)**.

Theorem 5.1. *Assume that system (4.3) is strictly entropy dissipative and condition (5.2) is satisfied. Then there exists $\delta > 0$ such that, if $\|W_0\|_2 \leq \delta$, there is a unique global solution $W = (U, V)$ of (4.3)–(4.4), which verifies*

$$W \in C^0([0,\infty); H^2(\mathbb{R})) \cap C^1([0,\infty); H^1(\mathbb{R})),$$

and

$$\sup_{0\leq t<+\infty} \|W(t)\|_2^2 + \int_0^{+\infty} \left(\|\partial_x U(\tau)\|_1^2 + \|V(\tau)\|_2^2\right) d\tau \leq C(\delta)\|W_0\|_2^2, \tag{5.4}$$

where $C(\delta)$ is a positive constant.

To prove Theorem 5.1, we follow the classical approach by Nishida [Nis78] and Kawashima [Kaw87], and we introduce the functional

$$N_l^2(t) := \sup_{0\leq\tau\leq t} \|W(\tau)\|_l^2 + \int_0^t \|\partial_x U(\tau)\|_{l-1}^2 + \|V(\tau)\|_l^2 d\tau, \tag{5.5}$$

for $l = 1, 2$. Let us set also

$$N_0^2(t) := \sup_{0\leq\tau\leq t} \|W(\tau)\|_0^2 + \int_0^t \|V(\tau)\|_0^2 d\tau.$$

Proposition 5.1. *Let $T > 0$, and assume that there exists a smooth solution W of (4.3)–(4.4) in the interval $[0, T]$. Then, there exist $\varepsilon > 0$ and $C > 0$ such that, if $N_2(T) \le \varepsilon$, there holds*

$$N_2^2(T) \le C\left(N_2^2(0) + N_2^3(T)\right). \tag{5.6}$$

In a classical way, Theorem 5.1 follows directly from Proposition 5.1. First we notice that, from Section 4, we are able to partially prove inequality (5.6) without using condition (5.2). By using Lemma 4.1, we obtain the zero order estimates.

Lemma 5.3. *If $N_2(T) \le \varepsilon \le \frac{\delta_0}{\alpha}$ where α is the Sobolev constant for the embedding of H^1 in L^∞, then*

$$N_0^2(T) \le C N_0^2(0).$$

On the other hand, we can use (4.16) and (4.21) to estimate some of the first and second order terms.

Lemma 5.4. *If $N_2(T) \le \varepsilon \le \frac{\delta_0}{\alpha}$, then, for $l = 1, 2$,*

$$\sup_{0 \le \tau \le T} \|W(\tau)\|_l^2 + \int_0^T \|V(\tau)\|_l^2 d\tau \le C\left(N_2^2(0) + N_2^3(T)\right).$$

Proof. Since A_0 and B are positive definite, from equality (4.16) and Remark 4.1, we have

$$\begin{aligned}
\|\partial_x W(t)\|_0^2 + & \int_0^t \|\partial_x V(\tau)\|_0^2 d\tau \\
& \le C_1(\varepsilon)\left(\|\partial_x W(0)\|_0^2 + \int_0^t \int |\partial_x W|^2 \left(|\partial_x W| + |V|\right) dx d\tau\right) \\
& \le C_1(\varepsilon)\left(\|\partial_x W(0)\|_0^2 + \left(\|V\|_\infty + \|\partial_x W(t)\|_\infty\right) \int_0^t \|\partial_x W(\tau)\|_0^2 d\tau\right),
\end{aligned}$$

where the $\|\cdot\|_\infty$ is the L^∞–norm on $(0, t) \times \mathbb{R}$. In the same way, from equality (4.21) and Remark 4.2, we have

$$\begin{aligned}
\|\partial_{xx} & W(t)\|_0^2 + \int_0^t \|\partial_{xx} V(\tau)\|_0^2 d\tau \\
& \le C_2(\varepsilon)\left(\|\partial_{xx} W(0)\|_0^2 + \int_0^t \int \left(|\partial_x W|^2 + |\partial_{xx} W|^2\right)\left(|\partial_x W| + |V| + |\partial_x V|\right) dx d\tau\right) \\
& \le C_2(\varepsilon)\left(\|\partial_{xx} W(0)\|_0^2 + \left(\|V\|_\infty + \|\partial_x W\|_\infty\right) \int_0^t \left(\|\partial_x W(\tau)\|_0^2 + \|\partial_{xx} W(\tau)\|_0^2\right) d\tau\right).
\end{aligned}$$

□

5.3. **Proof of Proposition 5.1.** Now, we have to use condition (5.2). Thanks to Lemma 5.3 and 5.4, to prove inequality (5.6) we just need to estimate the term

$$\int_0^t \|\partial_x U(\tau)\|_{l-1}^2 d\tau,$$

for $l = 1, 2$. To this purpose, we rewrite (4.3) as

$$A_0(0)\partial_t W + A_1(0)\partial_x W = H(W, \partial_x W), \tag{5.7}$$

where

$$\begin{aligned} H(W,\partial_x W): \quad &= \left[(A_1(0)-A_1(W)) - (A_0(0)-A_0(W))A_0^{-1}(W)A_1(W)\right]\partial_x W \\ &- \left[(A_0(0)-A_0(W))A_0^{-1}(W)+I\right]\begin{pmatrix} 0 \\ B(W)V \end{pmatrix}. \end{aligned}$$

From (5.7) and Lemma 5.2, we apply the matrix K and take the scalar product with $\partial_x W$. Then, integrating over $(0,t)\times\mathbb{R}$ yields

$$\int_0^t\int K\left[A_0(0)\partial_t W + A_1(0)\partial_x W\right]\cdot\partial_x W dx d\tau = \int_0^t\int KH\cdot\partial_x W dx d\tau. \tag{5.8}$$

First we have, integrating by parts

$$\begin{aligned} \int_0^t\int (KA_0(0)\partial_t W)\cdot\partial_x W dx d\tau &= -\int_0^t\int (KA_0(0)\partial_{tx} W)\cdot W dx d\tau \\ &= -\int (KA_0(0)\partial_x W)\cdot W dx\Big|_0^t + \int_0^t\int (KA_0(0)\partial_x W)\cdot\partial_t W dx d\tau \\ &= -\int (KA_0(0)\partial_x W)\cdot W dx\Big|_0^t - \int_0^t\int (KA_0(0)\partial_t W)\cdot\partial_x W dx d\tau, \end{aligned}$$

thanks to the skew–symmetry of $KA_0(0)$. Then, there exists a positive constant C_1 such that

$$\begin{aligned} \int_0^t\int (KA_0(0)\partial_t W)\cdot\partial_x W dx d\tau \quad &= -\tfrac{1}{2}\int (KA_0(0)\partial_x W)\cdot W dx\Big|_0^t \\ &\geq -C_1\left(\|W(t)\|_1^2 + \|W(0)\|_1^2\right). \end{aligned} \tag{5.9}$$

We estimate now the next term, using Lemma 5.2. We have

$$\begin{aligned} \int_0^t\int \quad (KA_1(0)\partial_x W)\cdot\partial_x W dx d\tau &= \tfrac{1}{2}\int_0^t\int \left[(KA_1(0) + {}^T(KA_1(0)))\partial_x W\right]\cdot\partial_x W dx d\tau \\ &= \int_0^t\int \left[(\frac{1}{2}(KA_1(0) + {}^T(KA_1(0)) + \begin{pmatrix} 0 & 0 \\ 0 & L \end{pmatrix})\partial_x W\right]\cdot\partial_x W dx d\tau \\ &- \int_0^t\int (LV)\cdot\partial_x V dx d\tau. \end{aligned}$$

Then

$$\int_0^t\int (KA_1(0)\partial_x W)\cdot\partial_x W dx d\tau \geq C_2\int_0^t \|\partial_x W\|_0^2 d\tau - C_3\int_0^t \|V\|_1^2 d\tau, \tag{5.10}$$

for two positive constants C_2 and C_3.

To estimate the right-hand side of (5.8), we have

$$|KH(W,\partial_x W)\cdot\partial_x W| \leq C_1(\varepsilon)\left(|V| + |W||\partial_x W|\right)|\partial_x W|. \tag{5.11}$$

We have now

$$\int_0^t\int |V||\partial_x W| dx d\tau \leq \nu\int_0^t \|\partial_x W(\tau)\|_0^2 d\tau + \frac{1}{4\nu}\int_0^t \|V(\tau)\|_0^2 d\tau,$$

where ν is an arbitrary small constant and

$$\int_0^t\int |W||\partial_x W|^2 dx d\tau \leq \|W\|_\infty\int_0^t \|\partial_x W(\tau)\|_0^2 d\tau.$$

Hence using (5.11) we obtain

$$
\begin{aligned}
|\int_0^t \int KH \cdot \partial_x W dx d\tau| \quad & \leq \tfrac{1}{2} C_2 \textstyle\int_0^t \|\partial_x W(\tau)\|_0^2 d\tau + C_4 \int_0^t \|V(\tau)\|_0^2 d\tau \\
& + C_5 \|W\|_\infty \textstyle\int_0^t \|\partial_x W(\tau)\|_0^2 d\tau.
\end{aligned} \tag{5.12}
$$

Then, summing up (5.9), (5.10) and (5.12) yields

$$
\begin{aligned}
\tfrac{1}{2} C_2 \textstyle\int_0^t \|\partial_x W(\tau)\|_0^2 d\tau \quad & \leq C_1 \left(\|W(t)\|_1^2 + \|W(0)\|_1^2 \right) + C_3 \textstyle\int_0^t \|V(\tau)\|_1^2 d\tau \\
& C_4 \textstyle\int_0^t \|V(\tau)\|_0^2 d\tau + C_5 \|W\|_\infty \int_0^t \|\partial_x W(\tau)\|_0^2 d\tau.
\end{aligned}
$$

Finally, using Lemma 5.4, we obtain, for $0 \leq t \leq T$,

$$
\int_0^t \|\partial_x U(\tau)\|_0^2 d\tau \leq C(\varepsilon) \left(N_1^2(0) + N_1^3(t) \right). \tag{5.13}
$$

We can argue in the same way, by taking the x-derivative of equation (5.7), and then the scalar product with $\partial_{xx} W$

$$
\int_0^t \int K \left[A_0(0) \partial_{tx} W + A_1(0) \partial_{xx} W \right] \cdot \partial_{xx} W dx d\tau = \int_0^t \int K \partial_x H \cdot \partial_{xx} W dx d\tau. \tag{5.14}
$$

By arguing as in the previous case, we easily obtain

$$
\int_0^t \int (K A_0(0) \partial_{tx} W) \cdot \partial_{xx} W dx d\tau \geq -C_1 \left(\|W(t)\|_2^2 + \|W(0)\|_2^2 \right), \tag{5.15}
$$

and

$$
\int_0^t \int (K A_1(0) \partial_{xx} W) \cdot \partial_{xx} W dx d\tau \geq C_2 \int_0^t \|\partial_{xx} W(\tau)\|_0^2 d\tau - C_3 \int_0^t \|V(\tau)\|_2^2 d\tau, \tag{5.16}
$$

Moreover, there holds

$$
|K \partial_x \left(H(W, \partial_x W) \right) \cdot \partial_{xx} W| \leq C_3(\varepsilon) \left(|\partial_x W|^2 + |W| |\partial_{xx} W| + |\partial_x W| |V| + |\partial_x V| \right) |\partial_{xx} W|. \tag{5.17}
$$

Therefore, we obtain

$$
\begin{aligned}
|\int_0^t \int K \partial_x \left(H(W, \partial_x W) \right) \cdot \partial_{xx} W dx d\tau| \quad & \leq \tfrac{1}{2} C_2 \textstyle\int_0^t \|\partial_{xx} W(\tau)\|_0^2 d\tau + C_6 \int_0^t \|\partial_x V(\tau)\|_0^2 d\tau \\
& + C_7 \left(\|W\|_\infty + \|\partial_x W\|_\infty \right) \int_0^t \|\partial_x W(\tau)\|_1^2 d\tau.
\end{aligned} \tag{5.18}
$$

Finally, summing up (5.15), (5.16) and (5.18) and using Lemma 5.4, we obtain, for $0 \leq t \leq T$,

$$
\int_0^t \|\partial_{xx} U(\tau)\|_0^2 d\tau \leq C_4(\varepsilon) \left(N_2^2(0) + N_2^3(t) \right). \tag{5.19}
$$

This concludes the proof of Proposition 5.1.

5.4. **Examples.** Let us reconsider the examples of Section 3, in view of the results obtained in the last section.

Example 5.1. The p-system with damping. For system (1.3), the Kawashima condition (5.2) is always verified, since, given an equilibrium state $\begin{pmatrix} U \\ 0 \end{pmatrix} \in \mathbb{R}^2$, and for every $\lambda \in \mathbb{R}$, the vector $\begin{pmatrix} X \\ 0 \end{pmatrix} \in \mathbb{R}^2$ is in the null space of

$$\lambda \begin{pmatrix} -(p^{-1})'(-U) & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

if and only if $X = 0$.

Example 5.2. The Suliciu model. Also in the case of the Suliciu system (3.3), condition (5.2) holds true. In fact, given an equilibrium state $\begin{pmatrix} U \\ 0 \end{pmatrix} \in \mathbb{R}^2 \times \mathbb{R}$, and for every $\lambda \in \mathbb{R}$, the vector $\begin{pmatrix} X \\ 0 \end{pmatrix} \in (\mathbb{R}^2 \times \mathbb{R})\backslash\{0\}$ is in the null space of

$$\lambda \begin{pmatrix} (g'(g^{-1}(U_1))^{-1} & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & \mu \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -\mu \\ 0 & -\mu & 0 \end{pmatrix},$$

if and only if $\mu = \lambda^2$ and $\mu = g'(g^{-1}(U_1)$, which is excluded by the subcharacteristic condition (3.4).

Example 5.3. The Kerr–Debye model. For the Kerr–Debye model system (3.3), we have that condition (5.2) holds around an equilibrium point $(U_1, U_2, 0)$ if and only if the linear system

$$\begin{cases} \lambda(3U_1^2 + 1)x_1 + x_2 = 0 \ , \\ \lambda x_2 + x_1 = 0 \ , \\ 2\lambda U_1 x_1 = 0, \end{cases}$$

has only trivial solutions (x_1, x_2) for every value $\lambda \in \mathbb{R}$. In this case we have that condition (5.2) is verified only for $U_1 \neq 0$, otherwise we have the 1–dimensional family of solutions $x_1 = \lambda x_2$ for $\lambda = \pm 1$.

Example 5.4. The Jin–Xin relaxation and BGK models. Let us consider now the relaxation model (3.6). From the form of the matrices A_0 and A_1, given respectively by (3.9) and (3.10), we have that a vector $\begin{pmatrix} X \\ 0 \end{pmatrix} \in (\mathbb{R}^k \times \mathbb{R}^k)$ is in the kernel of $\lambda A_0 + A_1$, for some $\lambda \in \mathbb{R}$, if and only if

$$(\lambda + \mu)a_+ X = (\lambda - \mu)a_- X = 0.$$

Therefore, since the matrices $a_\pm$ are both nonsingular from the sub-characteristic condition (3.8), this implies $X = 0$ and then condition (5.2) is always verified. Similar arguments apply to the BGK systems (3.11), where the Kawashima condition is guaranteed by the monotonicity condition (3.13).

We conclude this series of examples, by a negative results, already observed in [Zen99].

Example 5.5. Gas dynamics in thermal nonequilibrium. It is easy to show that for system (3.17), the condition (5.2) is not verified. Actually, thanks to (3.18) and (3.19), we have that a vector $\begin{pmatrix} X \\ 0 \end{pmatrix} \in (\mathbb{R}^3 \times \mathbb{R})$ is in the kernel of $\lambda A_0 + A_1$, for some $\lambda \in \mathbb{R}$, if and only if

$$\begin{cases} \lambda(\gamma-1)\frac{C_v}{U_1^2}x_1 + \frac{1}{U_3}x_2 - \frac{U_2}{U_3^2}x_3 = 0 \ , \\ \frac{1}{U_3}x_1 - \frac{\lambda}{U_3}x_2 + \left(\lambda\frac{U_2}{U_3^2} - \frac{U_1}{U_3^2}\right)x_3 = 0 \ , \\ -\frac{U_2}{U_3^2}x_1 + \left(\lambda\frac{U_2}{U_3^2} - \frac{U_1}{U_3^2}\right)x_2 + \left(\lambda(\frac{C_v}{U_3^2} - \frac{U_2^2}{U_3^3} + \alpha) + 2\frac{U_1U_2}{U_3^3}\right)x_3 = 0 \ , \\ \lambda\alpha x_3 = 0, \end{cases}$$

Therefore, if for instance $U_2 = 0$, we have nontrivial solutions for $\lambda = 0$ and $X = (\frac{U_1}{U_3}x_3, 0, x_3)$.

Let us recall that, for this last example, even if the Kawashima conditions was not verified, Y. Zeng was able to obtain a global existence result, by observing that not all the components of $\partial_x U$ and ∂xxU are involved in the trilinear part of the energy estimates, and then it was possible to use a reduced functional. In this regard, it should be worth giving a weaker version of Kawashima conditions, to include this and other examples.

References

[ADN96] D. Aregba-Driollet and R. Natalini, *Convergence of relaxation schemes for conservation laws*, Appl. Anal. **61** (1996), 163–193.

[ADN00] D. Aregba-Driollet and R. Natalini, *Discrete kinetic schemes for multidimensional systems of conservation laws*, SIAM J. Numer. Anal. **37** (2000), no. 6, 1973–2004 (electronic).

[Bia01] S. Bianchini, *A Glimm type functional for a special Jin-Xin relaxation model*, Ann. Inst. H. Poincaré Anal. Non Linéaire **18** (2001), no. 1, 19–42.

[Boi74] G. Boillat, *Sur l'Existence et la Recherche d'Équations de Conservation Supplémentaires pour les Systèmes Hyperboliques*, C.R. Acad. Sc. Paris, **278** A (1974), 909–912 .

[BR97] G. Boillat and T. Ruggeri, *Hyperbolic principal subsystems: entropy convexity and subcharacteristic conditions*, Arch. Rational Mech. Anal. **137** (1997), no. 4, 305–320.

[BR98] ———, *On the shock structure problem for hyperbolic system of balance laws and convex entropy*, Contin. Mech. Thermodyn. **10** (1998), no. 5, 285–292.

[Bou99] F. Bouchut, *Construction of BGK models with a family of kinetic entropies for a given system of conservation law*, J. Statist. Phys. (1999), no. 95, 113–170.

[CLL94] G.-Q. Chen, C.D. Levermore, and T.-P. Liu, *Hyperbolic conservation laws with stiff relaxation terms and entropy*, Comm. Pure and Appl. Math. **47** (1994), 787–830.

[Daf95] C.M. Dafermos, *A system of hyperbolic conservation laws with frictional damping*, Z. Angew. Math. Phys. **46** (1995), no. Special Issue, S294–S307, Theoretical, experimental, and numerical contributions to the mechanics of fluids and solids.

[Daf00] C. M. Dafermos, *Hyperbolic conservation laws in continuum physics*, Springer-Verlag, Berlin, 2000.

[FL71] K. O. Friedrichs and P. D. Lax, *Systems of conservation equations with a convex extension*, Proc. Nat. Acad. Sci. U.S.A. **68** (1971), 1686–1688.

[GMa98] V. Giovangigli and M. Massot, *Asymptotic Stability of Equilibrium States for Multicomponent Reacting Flows*, Math. Models and Methods in Applied Sciences, **2** (1998), 251-297.

[GMb98] ———, *The Local Cauchy Problem for Multicomponent Reactive Flows in Full Vibrational Nonequilibrium*, Math. Meth. Appl. Sci. (1998), 1415-1439

[God61] S. K. Godunov, *An interesting class of quasi-linear systems*, Dokl. Akad. Nauk SSSR **139** (1961), 521–523.

[HH00] B. Hanouzet and Ph. Huynh, *Approximation par relaxation d'un système de Maxwell non linéaire*, C. R. Acad. Sci. Paris Sér. I Math. **330** (2000), no. 3, 193–198.

[HL92] L. Hsiao and T.-P. Liu, *Convergence to nonlinear diffusion waves for solutions of a system of hyperbolic conservation laws with damping*, Comm. Math. Physics **143** (1992), 599–605.

[HP99] L. Hsiao and R. Pan, *Zero relaxation limit to centered rarefaction waves for a rate-type viscoelastic system*, J. Differential Equations **157** (1999), no. 1, 20–40.

[Huy99] Ph. Huynh, *Étude théorique et numérique de modèles de Kerr*, Ph.D. thesis, Université Bordeaux I, 1999.

[JX95] S. Jin and Z. Xin, *The relaxation schemes for systems of conservation laws in arbitrary space dimensions*, Comm. Pure and Appl. Math. **48** (1995), 235–276.

[RWZ97] J.B. Judkins and R-W. Ziolkowski, *Full wave vector Maxwell equation modeling of the self-focusing of ultrashort optical pulses in a non linear Kerr medium exhibiting a finite response time*, J. Opt. Soc. Am. B **45** (1997), 375–391.

[Kaw87] S. Kawashima, *Large-time behaviour of solutions to hyperbolic-parabolic systems of conservation laws and applications*, Proc. Roy. Soc. Edinburgh Sect. A **106** (1987), no. 1-2, 169–194.

[Li94] T. T. Li, *Global classical solutions for quasilinear hyperbolic systems*, Masson, Paris, 1994.

[Liu87] T.-P. Liu, *Hyperbolic conservation laws with relaxation*, Comm. Math. Phys. **108** (1987), 153–175.

[LP00] H. Li and R. Pan, *Zero relaxation limit for piecewise smooth solutions to a rate-type viscoelastic system in the presence of shocks*, J. Math. Anal. Appl. **252** (2000), no. 1, 298–324.

[Maj78] A. Majda, *Compressible fluid flow and systems of conservation laws in several space dimensions*, Appl. Math. Sci., vol. 53, Springer–Verlag, 1978.

[Mas02] M. Massot, *Singular perturbation analysis for the reduction of complex chemistry in gaseous mixtures using the entropic structure*, Discrete and Continuous Dynamical Systems - Series B, **2** (2002), 433–456.

[MR98] I. Müller and T. Ruggeri, *Rational extended thermodynamics*, second ed., Springer-Verlag, New York, 1998, With supplementary chapters by H. Struchtrup and Wolf Weiss.

[Nat96] R. Natalini, *Convergence to equilibrium for the relaxation approximation of conservation laws*, Comm. Pure and Appl. Math. **49** (1996), 795–823.

[Nat98] ———, *A discrete kinetic approximation of entropy solutions to multidimensional scalar conservation laws*, J. Differential Equations **148** (1998), no. 2, 292–317.

[Nat99] ———, *Recent results on hyperbolic relaxation problems*, Analysis of systems of conservation laws (Aachen, 1997), Chapman & Hall/CRC, Boca Raton, FL, 1999, pp. 128–198.

[Nis78] T. Nishida, *Nonlinear hyperbolic equations and related topics in fluid dynamics*, Département de Mathématique, Université de Paris-Sud, Orsay, 1978, Publications Mathématiques d'Orsay, No. 78-02.

[Ser00] D. Serre, *Relaxations semi-linéaire et cinétique des systèmes de lois de conservation*, Ann. Inst. H. Poincaré Anal. Non Linéaire **17** (2000), no. 2, 169–192.

[SK85] Y. Shizuta and S. Kawashima, *Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation*, Hokkaido Math. J. **14** (1985), no. 2, 249–275.

[Sul90] I. Suliciu, *On modelling phase transitions by means of rate-type constitutive equations. Shock wave structure*, Internat. J. Engng. Sci. **28** (1990), no. 8, 829–841.

[Tza99] A. E. Tzavaras, *Materials with internal variables and relaxation to conservation laws*, Arch. Ration. Mech. Anal. **146** (1999), no. 2, 129–155.

[Whi74] G.B. Whitham, *Linear and nonlinear waves*, A Wiley & Sons, New York, 1974.

[Yon01] W.-A. Yong, *Basic aspects of hyperbolic relaxation systems*, Advances in the theory of shock waves, Birkhäuser Boston, Boston, MA, 2001, pp. 259–305.

[Zen99] Y. Zeng, *Gas dynamics in thermal nonequilibrium and general hyperbolic systems with relaxation*, Arch. Ration. Mech. Anal. **150** (1999), no. 3, 225–279.