ASYMPTOTIC STABILITY OF RIEMANN SOLUTIONS FOR A CLASS OF MULTI-D VISCOUS SYSTEMS OF CONSERVATION LAWS

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Abstract. We prove the asymptotic stability of two-states nonplanar Riemann solutions under initial and viscous perturbations for a class of multidimensional systems of conservation laws. The class of systems to which our result applies is constituted by those systems whose flux-functions in different directions share a common complete system of Riemann invariants, the level surfaces of which are hyperplanes. In particular, we obtain the uniqueness of the self-similar L^∞ entropy solution of the two-states nonplanar Riemann problem. The asymptotic stability to which the main result refers is in the sense of the convergence as $t\to\infty$ in $L^1_{\rm loc}$ of the space of directions $\boldsymbol{\xi}=\mathbf{x}/t$. That is, the solution $u(t,\mathbf{x})$ of the perturbed problem satisfies $u(t,t\boldsymbol{\xi})\to R(\boldsymbol{\xi})$ as $t\to\infty$, in $L^1_{\rm loc}(\mathbb{R}^n)$, where $R(\boldsymbol{\xi})$ is the self-similar entropy solution of the corresponding two-states nonplanar Riemann problem.

1. Introduction

We consider solutions of initial-value problems for multidimensional viscous systems of conservation laws given by

$$\partial_t u+\partial_{x_i} f^i(u)=\Delta u,\qquad (t,\mathbf{x})\in(0,\infty)\times\mathbb{R}^n, \tag{1.1}$$

where $\mathbf{x}=(x_1,\dots,x_n)$, $u\in\mathbb{R}^m$, $f^i(u)\in\mathbb{R}^m$, is a smooth vector field, $i=1,\dots,n$, $\Delta=\partial_{x_1}^2+\cdots+\partial_{x_n}^2$ is the Laplacian operator, and we adopt the convention of summation for repeated indices.

We form the initial value problem for (1.1) by prescribing an initial condition of the form

$$u(t,\mathbf{x})|_{t=0}=u_0(\mathbf{x}),\qquad \mathbf{x}\in\mathbb{R}^n, \tag{1.2}$$

where, here, we will assume $u_0\in L^\infty(\mathbb{R}^n)$.

In this paper we are concerned with the asymptotic behavior of the solution of (1.1), (1.2), $u(t,\mathbf{x})$, when the initial condition is a perturbation of a nonplanar two-states Riemann data. Namely, we assume that u_0 satisfies

$$u_0(\mathbf{x})=R_0(\mathbf{x})+\psi(\mathbf{x}), \tag{1.3}$$

where

$$R_0(\mathbf{x})=\begin{cases}u_L, & \text{if } x_i<0, \text{ for some } i\in\{1,\dots,n\}\\ u_R, & \text{if } x_i>0, \text{ for all } i\in\{1,\dots,n\}\end{cases}, \tag{1.4}$$

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and $\psi \in L^\infty(\mathbb{R}^n)$ with

$$\lim_{T\to\infty} \psi(T\mathbf{x}) = 0 \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^n). \tag{1.5}$$

We will consider systems (1.1) where the $f^i(u)$ are in the class of what we call *Temple fields*, after [27], and we further impose some additional conditions over the f^i. Namely, we will assume that the fields f^i satisfy the following conditions on a region $\mathcal{O}$ of the phase space where the solutions of (1.1)-(1.2) take values:

(H1) There exists a system of smooth functions $\{w_j(u)\}_{j=1}^m$, Riemann invariants, satisfying

$$\nabla w_j(u)\nabla f^i(u) = \lambda_j^i(u)\nabla w_i(u), \quad u \in \mathcal{O}, \qquad i = 1,\dots,n,\ j = 1,\dots,m, \tag{1.6}$$

where the λ_j^i are then the eigenvalues of ∇f^i, $i = 1,\dots,n$, $j = 1,\dots,m$;

(H2) For each $u \in \mathcal{O}$, $\{\nabla w_j(u)\}_{j=1}^m$ is a linearly independent set and the level surfaces $\{w_i = \text{constant}\}$ are hyperplanes;

(H3) For each $i \in \{1,\dots,n\}$, there exist constants $\kappa_1^i < \kappa_2^i < \cdots < \kappa_{m+1}^i$, such that

$$\kappa_1^i < \lambda_1^i(u) < \kappa_2^i < \cdots < \kappa_m^i < \lambda_m^i(u) < \kappa_{m+1}^i, \qquad \text{for all } u \in \mathcal{O}. \tag{1.7}$$

(H4) There exists a common system of unit right-eigenvectors $\{r_j(u)\}_{j=1}^m$ for $\nabla f^i(u)$, $i = 1,\dots,n$, such that

$$\nabla \lambda_j^i(u)\cdot r_j(u) > 0, \quad u \in \mathcal{O}, \qquad i = 1,\dots,n,\ j = 1,\dots,m. \tag{1.8}$$

An example of Temple field is given by $f(u) = \frac{1}{D(u)}(k_1u_1,\dots,k_mu_m)^\top$, $0 < k_1 < \cdots < k_m$, $D(u) = 1 + u_1 + \cdots + u_m$, which is found in a model system in chromatography (see [1]). One-dimensional systems in which the flux functions are Temple fields have been studied by many authors after the first analysis carried out in [27], followed by [22, 24]. The class of systems for which our result applies clearly includes the case where $f^i(u) = a^i f(u)$, $a^i \in \mathbb{R}$, $i = 1,\dots,n$, and f is any Temple field in $\mathbb{R}^m$. For $m = 2$, it was proved in [27] (see also [24]) that the Temple fields whose Riemann invariants are $w_1(u)$, $w_2(u)$, are given by

$$f_1(u) = \frac{h_1(w_1) - h_2(w_2)}{w_1 - w_2}, \qquad f_2(u) = \frac{w_2h_1(w_1) - w_1h_2(w_2)}{w_2 - w_1}, \tag{1.9}$$

where h_1 and h_2 are arbitrary smooth functions of one variable. This shows that, at least for $m = 2$, it is possible to give examples of systems (1.1) with flux functions satisfying our assumptions, beyond the simplest case when all of them are multiples of one only field.

In analyzing the asymptotic behavior of the solution $u(t,\mathbf{x})$ of (1.1), (1.2) we will prove the existence of a piecewise continuous self-similar solution of the multi-D Riemann problem for

$$\partial_t u + \partial_{x_i} f^i(u) = 0, \qquad (t,\mathbf{x}) \in (0,\infty)\times\mathbb{R}^n, \tag{1.10}$$

with Riemann data given by (1.4), $v(t,\mathbf{x}) = R(\boldsymbol{\xi})$, with $\boldsymbol{\xi} = \mathbf{x}/t$, for which we have

$$u(t,t\boldsymbol{\xi}) \longrightarrow R(\boldsymbol{\xi}), \quad \text{as } t\to\infty, \text{ in } L^1_{\text{loc}}(\mathbb{R}^n). \tag{1.11}$$

The result for planar Riemann data was announced in [16]; in that case, conditions (H3) and (H4) need only to be imposed on the first field f^1, which corresponds to the direction of propagation of the waves.

We recall that a pair $(\eta(u), \mathbf{q}(u))$ is an entropy-entropy flux pair for (1.10) if

$$\nabla \mathbf{q}(u) = \nabla \eta(u) \nabla \mathbf{f}(u), \tag{1.12}$$

where we denote $\mathbf{f} = (f^1, \dots, f^n)$.

As usual, by an entropy solution of (1.10), (1.2), we mean a bounded measurable function $u(t, \mathbf{x})$ satisfying the integral inequality

$$\int_{\mathbb{R}_+^{n+1}} \eta(u)\phi_t + \mathbf{q}(u) \cdot \nabla_{\mathbf{x}} \phi \, d\mathbf{x} \, dt + \int_{\mathbb{R}^n} \eta(u_0)\phi(0, \mathbf{x}) \, d\mathbf{x} \geq 0, \tag{1.13}$$

for all entropy-entropy flux pair for (1.10), $(\eta(u), \mathbf{q}(u))$, with $\eta(u)$ convex, and for any nonnegative $\phi \in C_0^\infty(\mathbb{R}^{n+1})$. For an account on the basic concepts and recent developments of the theory of conservation laws we refer to [9, 23].

We remark that our hypotheses (H1)-(H4) imply, in particular, that the system (1.10) is hyperbolic, in the sense that given any $\boldsymbol{\zeta} = (\zeta_1, \dots \zeta_n) \in \mathbb{R}^n$ the matrix $\zeta_1 \nabla f^1(u) + \dots + \zeta_n \nabla f^n(u)$ is diagonalizable (cf., [9, 24]). Moreover, it is well known that Temple systems admit strictly convex smooth entropies (cf., [24]), and (1.12) implies that the matrices $\nabla^2 \eta(u) \nabla f^i(u)$ are symmetric, $i = 1, \dots, n$. In particular, (1.10) is symmetrizable (cf., [18, 17, 2]). For a discussion about the relation between entropies and the stability of classical solutions of systems of conservation laws in several space variables (1.10) we refer to [10, 11].

We now state our main theorem.

Theorem 1.1. *Let f^i, $i = 1, \dots, n$, satisfy hypotheses* (H1)-(H4). *Then, there exists one and only one self-similar entropy solution $R(\boldsymbol{\xi})$ of the two-states nonplanar multi-D Riemann problem* (1.10), (1.4). *Moreover, if $u_0 \in L^\infty(\mathbb{R}^n)$ satisfies* (1.3), (1.4), (1.5), *the only solution of* (1.1), (1.2) *verifies* (1.11).

We remark that no smallness condition is needed, neither for the initial data nor for the perturbation. For the latter the only requirement is the one given by (1.5). The main feature of Theorem 1.1, of course, is not the existence of $R(\boldsymbol{\xi})$, which due to the assumptions could be easily constructed directly, but rather its uniqueness and the property (1.11).

The method for obtaining (1.11) is motivated by the general approach in [3]. The main point in the strategy is to prove the relation

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T |u(t, t\boldsymbol{\xi}) - R(\boldsymbol{\xi})| \, dt = 0, \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^n). \tag{1.14}$$

An important aspect of (1.14) is its equivalence to the convergence in $L^1_{\mathrm{loc}}(\mathbb{R}_+^{n+1})$ of the scaling sequence $\{u^T\}$, given by $u^T(t, \mathbf{x}) = u(Tt, T\mathbf{x})$, to $R(\mathbf{x}/t)$, when $T \to +\infty$. This fact is frequently useful when trying to prove (1.14). Once (1.14) is proved, a standard procedure established in [3] is then used to strengthen (1.14) into (1.11). This strengthening is similar to the ones encountered in [4, 21].

The remaining of this paper is organized in the following way. In Section 2 we state some results concerning more general systems (1.1). The goal of the section is to establish Lemma 2.1 which enables us to reduce the proof of (1.14) to a problem of analysing the supports of certain probability measures and showing that they consist of just one point. The latter is similar in spirit to the usual procedure in the theory of compensated compactness, since the pioneering papers of Tartar [26] and DiPerna [12], although here the probability measures have nothing to do with Young measures. In Section 3 we then specify our discussion to systems (1.1) in which the flux functions satisfy (H1)-(H4). We prove (1.14) which requires to recall some results valid for Temple fields. We also recall through a brief outline the procedures to pass from the time asymptotics given by (1.14) to the stronger one given by (1.11). Finally, we finish the proof of Theorem 1.1 by discussing the uniqueness of the self-similar Riemann solution.

2. Results for general systems (1.1)

In this section we state some results which are valid for more general systems of the form (1.1). We start with a result concerning the regularity in the large of solutions of (1.1) which are uniformly bounded in the half-space $\mathbb{R}_+^{n+1}$.

Proposition 2.1. *Suppose that $u(t,\mathbf{x})$ is a solution of (1.1), (1.2), uniformly bounded in $\mathbb{R}_+^{n+1}$, with*

$$u_0\in W^{N-1,\infty}(\mathbb{R}^n;\mathbb{R}^m),$$

for some $N\in\mathbb{Z}^+$. Then,

$$u\in L^1\left(0,T;W^{N,\infty}(\mathbb{R}^n;\mathbb{R}^m)\right),$$

for any $T>0$. More precisely, we have for $|\alpha|\le N-1$

$$\|\partial_{\mathbf{x}}^\alpha u\|_{L^\infty(\mathbf{R}_+^{n+1})}<+\infty, \tag{2.1}$$

and, if $|\alpha|=N$, for $t_0>0$ sufficiently small, there exists $C=C(t_0)>0$ such that

$$\|\partial_{\mathbf{x}}^\alpha u(t)\|_\infty\le C,\qquad \text{for } t>t_0, \tag{2.2}$$

and

$$\|\partial_{\mathbf{x}}^\alpha u(t)\|_\infty\le \frac{C}{\sqrt{t}},\qquad \text{for } 0<t\le t_0, \tag{2.3}$$

where we adopt the multi-indice notation $|\alpha|=\alpha_1+\cdots+\alpha_n$, $\partial_{\mathbf{x}}^\alpha=\partial_{x_1}^{\alpha_1}\dots\partial_{x_n}^{\alpha_n}$.

Proof. The proof is based on an integral representation valid for smooth solutions of (1.1)-(1.2), and properties of the heat kernel (cf., e.g.,[20]). Let $K(t,\mathbf{x})$ be the fundamental solution of the heat operator $\partial_t-\Delta$,

$$K(t,\mathbf{x})=(4\pi t)^{-n/2}\exp(-|\mathbf{x}|^2/4t).$$

The important fact to keep in mind about K, in the present discussion, is the following estimate for the L^1 norm of its derivatives in the space variables

$$\|\partial_{\mathbf{x}}^\alpha K(t)\|_1\le\frac{C(\alpha)}{t^{|\alpha|/2}}, \tag{2.4}$$

for any $\alpha\in\mathbb{N}^n$, where $C(\alpha)>0$ is a constant depending only on α. The solution of (1.1), (1.2) is a fixed point of the operator

$$\mathcal{L}(v)(t)=K(t)*u_0-\sum_{i=1}^n\int_0^t K_{x_i}(t-s)*f^i(v(s))\,ds. \tag{2.5}$$

We assume the operator $\mathcal{L}$ defined on

$$\mathcal{G}_T=\{\,v\colon[0,T]\to L^\infty\big(\mathbb{R}^n;\mathbb{R}^m\big)\mid \|v(t)\|_\infty\le r\,\},$$

for some $r>\|u\|_{L^\infty\left(\mathbb{R}_+^{n+1}\right)}$. It is then easy to show that $\mathcal{L}$ is a contraction in $\mathcal{G}_T$, for T sufficiently small, so that the solution of (1.1), (1.2), $u(t,\mathbf{x})$, is the only fixed point of $\mathcal{L}$ in $\mathcal{G}_T$. For $u_0 \in W^{N-1,\infty}(\mathbb{R}^n;\mathbb{R}^m)$ one can easily show that, for $T>0$ sufficiently small, there exists a constant $C>0$ such that if $v \in \mathcal{G}_T$ satisfies

$$\|\partial_{\mathbf{x}}^\alpha v(t)\|_\infty \le C\|\partial_{\mathbf{x}}^\alpha u_0\|_\infty, \tag{2.6}$$

then $\mathcal{L}(v)$ also satisfies (2.6), for $|\alpha| \le N-1$. From this and the fact that $u(t,\mathbf{x})$ is the only fixed point of $\mathcal{L}$ one easily obtains that $\partial_{\mathbf{x}}^\alpha u$ is uniformly bounded in $[0,T]\times\mathbb{R}^n$, for $|\alpha|\le N-1$, for $T>0$ sufficiently small. Since the argument for proving the above assertions is similar to the one for proving (2.2), (2.3) we content ourselves in sketching the procedure for the proof of these last estimates. We now prove the estimates (2.2), (2.3). We first consider the case $N=1$. We begin by showing that, for $T>0$ sufficiently small, there exists $C>0$ such that if $v\in\mathcal{G}_T$ satisfies

$$\|\partial_{x_i} v(t)\|_\infty \le \frac{C}{\sqrt{t}}\|u_0\|_\infty, \qquad 0<t<T, \tag{2.7}$$

for any $i\in\{1,\dots,n\}$, then $\mathcal{L}(v)$ also satisfies (2.7). So, assume $v\in\mathcal{G}_T$ satisfies (2.7). Differentiating (2.5) with respect to x_i, for any $i\in\{1,\dots,n\}$, and using (2.4) one obtains

$$\begin{aligned}\|\mathcal{L}(v)_{x_i}(t)\|_\infty &\le \frac{C_0}{\sqrt{t}}\|u_0\|_\infty + C_1 C\|u_0\|_\infty \int_0^t \frac{1}{\sqrt{t-s}}\frac{1}{\sqrt{s}}\,ds\\ &\le (C_0 + C_2 C\sqrt{T})\frac{\|u_0\|_\infty}{\sqrt{t}},\end{aligned}$$

where C_0, C_1, C_2 are constants depending only on r, $\mathbf{f}$ and K. In order to have that $\mathcal{L}(v)$ satisfies the same estimate for v in (2.7), it suffices then to choose

$$C \ge \frac{C_0}{1-C_2\sqrt{T}},$$

which is possible if T is small enough. Now, the solution u is the limit in $\mathcal{G}_T$, with the L^∞ topology, of the sequence u^k given by $u^0=0$, $u^k=\mathcal{L}(u^{k-1})$, $k=1,\dots$. We conclude by standard arguments in the theory of distributions that u satisfies (2.7), and this gives (1.2), for $N=1$. To get the uniform bound given in (1.1), for $\partial_{x_i}u(t)$, for $t>t_0$, and any $i\in\{1,\dots,n\}$, we argue as follows. For a given $t_*>0$, we consider the operator

$$\mathcal{L}(v)(t) = K(t-t_*) * u(t_*) - \sum_{i=1}^n \int_{t_*}^t K_{x_i}(t-s) * f^i(u(s))\,ds,$$

defined on

$$\mathcal{G}_{t_*,T} = \{\, v\colon [t_*, t_*+T] \to L^\infty(\mathbb{R}^n;\mathbb{R}^m) \mid \|v(t)\| \le r\,\}.$$

Then, arguing exactly as above, we obtain that

$$\|\partial_{x_i} u(t)\|_\infty \le \frac{C}{\sqrt{t-t_*}}\|u(t_*)\|_\infty, \qquad t_*<t<t_*+T, \tag{2.8}$$

with the same C and T as in (2.7). Thus, making successively $t_*=T/3,\ 2T/3,\ \dots,\ kT/3,\ \dots$, and taking $t_0=2T/3$, using the fact that u is uniformly bounded by r, since the intervals $[t_*, t_*+T]$ are overlapping, given any $t\in(t_0,\infty)$ one can find t_* in that sequence such that $t_*+(T/3)\le t\le t_*+(2T/3)$ and so, from (2.8) we get

$$\|\partial_{x_i} u(t)\|_\infty \le \frac{C\sqrt{3}}{\sqrt{T}}\|u_0\|_\infty,$$

for all $t \in (t_0, \infty)$. This gives (2.2) in the case $N = 1$. The proof of (2.1), (2.2), (2.3), for an arbitrary $N \in \mathbb{Z}^+$ now follows easily by induction applying the above arguments. $\square$

For the next results we shall need to assume that (1.1) is endowed with a strictly convex entropy η with associated entropy flux $\mathbf{q} = (q_1, \dots, q_n)$.

We next introduce several notations defining sets which will play a role in the forthcoming developments. So, let K, M, T_0 be given fixed positive numbers. We denote:

$$
\begin{aligned}
&\Omega^+(\boldsymbol{\xi}) = \{(t,\mathbf{x}) \,:\, x_1 > \xi_1 t, \dots, x_n > \xi_n t\},\\
&\Omega^-(\boldsymbol{\xi}) = \{(t,\mathbf{x}) \,:\, x_i < \xi_i t, \text{ for some } i \in \{1,\dots,n\}\},\\
&|\mathbf{x}|_\infty = \max\{|x_i| \,:\, i = 1,\dots,n\},\\
&\Omega(T) = \{(t,\mathbf{x}) \,:\, |\mathbf{x}|_\infty \le MT_0T + K(T_0T - t),\ 0 \le t \le T_0 T\},\\
&\Omega^\pm(\boldsymbol{\xi}, T) = \Omega^\pm(\boldsymbol{\xi}) \cap \Omega(T),\\
&\nu^i[\boldsymbol{\xi}] = (\xi_i, 0, \dots, \underbrace{-1}_{(i+1)\text{-th}}, 0, \dots, 0),\\
&\partial\Omega^+(\boldsymbol{\xi}) = \cup_{i=1}^n S_i(\boldsymbol{\xi}), \quad \text{where } \nu^i[\boldsymbol{\xi}] \perp S_i(\boldsymbol{\xi})\\
&S_i(\boldsymbol{\xi}, T) = S_i(\boldsymbol{\xi}) \cap \Omega(T),\\
&\partial^0\Omega^+(\boldsymbol{\xi}, T) = \cup_{i=1}^n S_i(\boldsymbol{\xi}, T),\\
&S_i^\tau(\boldsymbol{\xi}, T) = S_i(\boldsymbol{\xi}, T) \cap \{(t,\mathbf{x}) \,:\, t = \tau\},\\
&\Omega_\tau(T) = \Omega(T) \cap \{(t,\mathbf{x}) \,:\, t = \tau\},\\
&\Omega_\tau^\pm(\boldsymbol{\xi}, T) = \Omega^\pm(\boldsymbol{\xi}, T) \cap \{(t,\mathbf{x}) \,:\, t = \tau\}.
\end{aligned}
$$

An important feature of our method for obtaining (1.14), for the systems (1.1) satisfying the hypotheses of Theorem 1.1, is the use of the Gauss-Green formula after an integration of the entropy inequality

$$
\partial_t \eta(u) + \nabla \cdot \mathbf{q}(u) \le \Delta \eta(u), \tag{2.9}
$$

over domains laterally bounded by surfaces whose intersections with the hyperplanes $t = \tau$ are unions of hypersurfaces in $\mathbb{R}^n$ of the form $S_i^\tau(\boldsymbol{\xi}, T)$, $i = 1, \dots, n$, together with a surface of the form $\Omega_\tau^+(\boldsymbol{\xi}) \cap \partial\Omega_\tau(T)$. As usual, the entropy inequality (2.9) is obtained by multiplying (1.1) by $\nabla\eta(u)$, using (1.12) and the convexity of η. The integration procedure is intended to produce boundary terms, the interesting ones being those involving the integration of the normal trace of the entropy vector $(\eta, \mathbf{q})$ over hypersurfaces whose sections by $t = \tau$ are of the type $S_i^\tau(\boldsymbol{\xi}, T)$. The two following propositions are used to show that we may discard the uninteresting boundary terms resulting from the integration process.

Proposition 2.2. *Suppose that there is a strictly convex entropy associated with (1.1) and $u \in L^\infty(\mathbb{R}_+^{n+1})$ is solution of (1.1), (1.2). Then, for a.e. $\boldsymbol{\xi} \in \mathbb{R}^n$, we have*

$$
\lim_{T\to\infty} \frac{1}{T} \int_0^T |\nabla_{\mathbf{x}} u(t, \boldsymbol{\xi} t)|\, dt = 0. \tag{2.10}
$$

In particular, for a.e. $\boldsymbol{\xi} \in \mathbb{R}^n$, $M, K, T_0 > 0$,

$$\lim_{T\to\infty} \frac{1}{T^n} \int_0^T dt \int_{S_i^t(\boldsymbol{\xi},T)} \left|\nabla_{\mathbf{x}} u(t,\mathbf{x})\right| d\mathcal{H}^{n-1} = 0, \qquad i = 1,\dots,n. \tag{2.11}$$

Proof. For $\mathbf{x} \in \mathbb{R}^n$ and $i \in \{1,\dots,n\}$ we denote $\overline{\mathbf{x}}^i = (x_1,\dots,x_{i-1})$, if $i \geq 2$, and ${}^i\overline{\mathbf{x}} = (x_{i+1},\dots,x_n)$, if $i \leq n-1$. For $i=1$ (resp., $i=n$) we simply ignore $\overline{\mathbf{x}}^i$ (resp., ${}^i\overline{\mathbf{x}}$) wherever it appears. Let $(\eta,\mathbf{q})$ be an entropy-entropy flux pair for (1.1), with η strictly convex. Multiplying (1.1) by $\nabla\eta$ one obtains

$$\partial_t \eta(u) + \nabla\cdot\mathbf{q}(u) = \Delta\eta(u) - \nabla^2\eta(\nabla_{\mathbf{x}} u, \nabla_{\mathbf{x}} u). \tag{2.12}$$

Let θ satisfy $0<\theta<1$. Following [15], we divide (2.12) by $(1+t)^{n+\theta}$ and integrate over

$$\Omega_T = \{\, (t,\mathbf{x}) \mid |\mathbf{x}|_\infty \leq Mt, 0 \leq t \leq T \,\}.$$

One obtains

$$\begin{aligned}
&\int_{|\mathbf{x}|_\infty \leq MT} \frac{\eta(u(T,\mathbf{x}))}{(1+T)^{n+\theta}}\, d\mathbf{x} + \iint\limits_{\Omega_T} \frac{\eta(u)}{(1+t)^{n+\theta+1}}\, d\mathbf{x}dt \\
&+ \sum_{i=1}^n \int_0^T dt \int_{|(\overline{\mathbf{x}}^i,\, {}^i\overline{\mathbf{x}})|_\infty \leq Mt} \left[\frac{-\xi_i\eta(u) + q^i(u) - \partial_{x_i}\eta(u)}{(1+t)^{n+\theta}} \right]_{\xi_i=-M}^{\xi_i=M} d\overline{\mathbf{x}}^i d\,{}^i\overline{\mathbf{x}} \\
&= -\iint\limits_{\Omega_T} \nabla^2\eta(u)(\nabla_{\mathbf{x}} u, \nabla_{\mathbf{x}} u)\, d\mathbf{x}dt.
\end{aligned}$$

From Proposition 2.1 it follows that the left-hand side of the above equation is bounded independently of $T>0$. We then deduce that the right-hand side is also bounded independently of $T>0$, and, so, from the strict convexity of η we get

$$\int_0^\infty dt \int_{|\mathbf{x}|_\infty \leq Mt} \frac{|\nabla_{\mathbf{x}} u|^2}{(1+t)^{n+\theta}}\, d\mathbf{x} < \infty.$$

Therefore, changing coordinates and using (1.2), one has

$$\int_0^\infty dt \int_{|\xi|_\infty \leq M} \frac{|\nabla_{\mathbf{x}} u|^2}{(1+t)^{\theta}}\, d\boldsymbol{\xi} < \infty.$$

Hence, for a.e. $\boldsymbol{\xi} \in \mathbb{R}^n$, with $|\boldsymbol{\xi}| \leq M$, we have

$$\int_0^\infty \frac{|\nabla_{\mathbf{x}} u|^2(t,\boldsymbol{\xi}t)}{(1+t)^{\theta}}\, dt < \infty. \tag{2.13}$$

But, given $\boldsymbol{\xi}$ satisfying (2.13), one obtains, for $T>1$,

$$\frac{1}{T}\int_0^T |\nabla_{\mathbf{x}} u|^2(t,\boldsymbol{\xi}t)\, dt \leq \frac{2^\theta}{T^{1-\theta}} \int_0^\infty \frac{|\nabla_{\mathbf{x}} u|^2(t,\boldsymbol{\xi}t)}{(1+t)^{\theta}}\, dt,$$

and then, using (2.13), we get

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |\nabla_{\mathbf{x}} u|^2(t,\boldsymbol{\xi}t)\, dt = 0,$$

which by Jensen's inequality implies (2.10). To conclude the proof we observe that (2.10) and Fubini's theorem imply that for a.e. $\xi \in \mathbb{R}$, we have, for all $M > 0$,

(2.14)
$$\lim_{T\to\infty} \frac{1}{T}\int_0^T dt \int_{|(\bar{\boldsymbol{\xi}}^i,\,{}^i\bar{\boldsymbol{\xi}})|_\infty \le M} \left|\nabla_{\mathbf{x}} u(t, \bar{\boldsymbol{\xi}}^i t, \xi t,\, {}^i\bar{\boldsymbol{\xi}} t)\right| d\bar{\boldsymbol{\xi}}^i d\,{}^i\bar{\boldsymbol{\xi}} = 0, \qquad i = 1, \dots, n.$$

But, due to Proposition 2.1, given $\varepsilon > 0$, we can take $M_0 > 0$ large enough so that,

(2.15)
$$\lim_{T\to\infty} \frac{1}{T}\int_\varepsilon^T dt \int_{\{|(\bar{\mathbf{x}}^i,\,{}^i\bar{\mathbf{x}})|_\infty > M_0 t\} \cap S_i^t(\boldsymbol{\xi}, T)} \left|\nabla_{\mathbf{x}} u(t, \mathbf{x})\right| \frac{1}{t^{n-1}}\, d\mathcal{H}^{n-1} < \varepsilon,\ i = 1, \dots, n,$$

and since $\varepsilon > 0$ is arbitrary (2.11) follows.

□

Proposition 2.3. *Suppose the hypotheses of Proposition 2.2 hold. Then $\frac{1}{T}\nabla_{\mathbf{x}} u^T \to 0$ as $T \to \infty$ in $L^1_{loc}(\mathbb{R}^{n+1}_+)$. In particular, for fixed $K, T_0 > 0$, and $M_0 > 0$ sufficiently large, given any subsequence $T_k \to \infty$, there exists a further subsequence $T_l = T_{k_l} \to \infty$, such that, for a.e. $\boldsymbol{\xi} \in \mathbb{R}^n$ and $M > M_0 > 0$, we have*

$$\lim_{T_l\to\infty} \frac{1}{T_l^n}\int_0^{T_l} dt \int_{\partial\Omega_t(\boldsymbol{\xi}, T_l)} \left|\nabla_{\mathbf{x}} u(t, \mathbf{x})\right| d\mathcal{H}^{n-1} = 0. \tag{2.16}$$

Proof. We notice that if $v \in L^1((0,T); L^\infty(\mathbb{R}^n))$, for any $T > 0$, then the self-similar scaling sequence v^T converges to 0 in $L^1_{\rm loc}(\mathbb{R}^{n+1}_+)$ if and only if

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |v(t, \boldsymbol{\xi})|\, dt = 0, \qquad \text{in } L^1_{\rm loc}(\mathbb{R}^n_{\boldsymbol{\xi}}),$$

as observed in [15]. Now, setting $v(t, \mathbf{x}) = \nabla_{\mathbf{x}} u(t, \mathbf{x})$, it is easy to see that

$$v^T(t, \mathbf{x}) = \frac{1}{T}\nabla_{\mathbf{x}} u^T(t, \mathbf{x}), \qquad (t, \mathbf{x}) \in \mathbb{R}^{n+1}_+.$$

Therefore, the fact that $\frac{1}{T}\nabla_{\mathbf{x}} u^T \to 0$ as $T \to \infty$ in $L^1_{\rm loc}(\mathbb{R}^{n+1}_+)$ follows immediately from Proposition 2.2. Hence, given any subsequence $T_k \to \infty$, we can then extract a further subsequence $T_l = T_{k_l} \to \infty$, such that $v^{T_l}(t, \mathbf{x}) \to 0$ a.e. in $\mathbb{R}^{n+1}_+$. By Fubini's theorem we then have that $v^{T_l} \to 0$ a.e. in $\partial\Omega^+(\boldsymbol{\xi})$ for a.e. $\boldsymbol{\xi} \in \mathbb{R}^n$, and also $v^{T_l} \to 0$ a.e. in $\partial\Omega(1)$ for a.e. $M > M_0$, which imply that $v^{T_l} \to 0$ a.e. in $\partial\Omega(\boldsymbol{\xi}, 1)$ for a.e. $\boldsymbol{\xi} \in \mathbb{R}^n$ and $M > M_0$. Since (2.16) may be rewriten as

$$\lim_{T_l\to\infty} \int_0^{T_0} dt \int_{\partial\Omega_t(\boldsymbol{\xi}, 1)} \left|v^{T_l}(t, \mathbf{x})\right| d\mathcal{H}^{n-1} = 0,$$

the result follows.

□

Proposition 2.4. *Assume again that the hypotheses of Proposition 2.2 are valid, and that u_0 satisfies (1.3), (1.5). Then, for $M_0 > 0$ large enough, we have $u^T \to u_L$ in $L^1_{loc}(\Omega^-(-\mathbf{M}_0))$, and $u^T \to u_R$ in $L^1_{loc}(\Omega^+(\mathbf{M}_0))$, as $T \to \infty$, where $\mathbf{M}_0 = (M_0, \dots, M_0)$.*

Proof. We prove the assertion for $\Omega^-(-\mathbf{M}_0)$; the proof for $\Omega^+(\mathbf{M}_0)$ is completely analogous. We consider the entropy formed by the quadratic part of the strictly convex entropy η around u_L. That is, we take

$$\eta_*(u) = \eta(u) - \eta(u_L) - \nabla\eta(u_L)(u - u_L),$$

for which the corresponding entropy-flux is

$$\mathbf{q}_*(u) = q(u) - q(u_L) - \nabla\eta(u_L)(\mathbf{f}(u) - \mathbf{f}(u_L)).$$

For simplicity of notation we denote $(\eta_*, \mathbf{q}_*)$ also by $(\eta, \mathbf{q})$. The strict convexity guarantees the existence of M_0 such that

$$|\mathbf{q}(u)|_\infty \le \frac{M_0}{n}\eta(u). \tag{2.17}$$

Since u^T is uniformly bounded, it suffices to prove that $u^T(t,\cdot)$, for any fixed $t > 0$, corverges to u_L as $T \to \infty$ in $L^1_{\mathrm{loc}}\big(\Omega^-_t(-\mathbf{M}_0)\big)$, where

$$\Omega^-_\tau(-\mathbf{M}_0) = \Omega^-(-\mathbf{M}_0) \cap \{(t,\mathbf{x}) \,:\, t = \tau\}.$$

Equivalently, we shall prove that, given any subsequence $T_k \to \infty$, we can find a further subsequence $T_{k_l} = T_l$ such that, for any fixed $t > 0$, $u^{T_l}(t,\cdot)$ converges to u_L in $L^1_{\mathrm{loc}}\big(\Omega^-_t(-\mathbf{M}_0)\big)$. So, let a subsequence $T_k \to \infty$ be given. By Proposition 2.3, we can choose a further subsequence $T_{k_l} = T_l$ such that $\frac{1}{T_l}\nabla_{\mathbf{x}} u^{T_l}$ converges a.e. to zero in $\mathbb{R}^{n+1}_+$. So, we take this subsequence and fix an arbitrary $\tilde t > 0$. Given any point $(\tilde t, \tilde{\mathbf{x}}) \in \Omega^-_{\tilde t}(\mathbf{M}_0)$, we take $r > 0$ such that $\{\,(\tilde t, \mathbf{x}) \mid |\mathbf{x} - \tilde{\mathbf{x}}| < r\,\} \subseteq \Omega^-_{\tilde t}(-\mathbf{M}_0)$. Actually, we can choose $r > 0$ in such a way that $\frac{1}{T_l}\nabla_{\mathbf{x}} u^{T_l}$ converges to zero $\mathcal{H}^n$-a.e. over the hypersurface

$$\mathcal{S}_r = \{\,(t,\mathbf{x}) \mid |\mathbf{x} - \tilde{\mathbf{x}}| = r + M_0(\tilde t - t),\ 0 < t < \tilde t\,\},$$

as follows easily from Fubini's Theorem. Set

$$\mathcal{T}_r = \{\,(t,\mathbf{x}) \mid |\mathbf{x} - \tilde{\mathbf{x}}| < r + M_0(\tilde t - t),\ 0 < t < \tilde t\,\}.$$

Now, for $T > 0$ fixed, u^T satisfies

$$\partial_t \eta(u^T) + \nabla_{\mathbf{x}} \cdot \mathbf{q}(u^T) = \frac{1}{T}\Delta\eta(u^T) - \frac{1}{T}\nabla^2\eta(u^T)(\nabla_{\mathbf{x}} u^T, \nabla_{\mathbf{x}} u^T). \tag{2.18}$$

Integrating (2.18) over $\mathcal{T}_r$ and applying Green's formula (where we use the fact that $u^T(t,\cdot) \to u_0^T(\cdot)$ as $t \to 0$) we obtain

$$\begin{aligned} &\int_{|\mathbf{x}-\tilde{\mathbf{x}}|\le r} \eta(u^T(\tilde t, \mathbf{x}))\,d\mathbf{x} - \int_{|\mathbf{x}-\tilde{\mathbf{x}}|\le r + M_0\tilde t} \eta(u_0^T(\mathbf{x}))\,d\mathbf{x} \\ &\qquad + \int_{\mathcal{S}_r} \{\nu_t \eta(u^T) + \nu_{\mathbf{x}} \cdot \mathbf{q}(u^T)\}\,d\mathcal{H}^n \\ &\quad \le \frac{1}{T}\int_{\mathcal{S}_r} \nu_{\mathbf{x}} \cdot \nabla_{\mathbf{x}} \eta(u^T)\,d\mathcal{H}^n - \frac{1}{T}\iint_{\mathcal{T}_r} \nabla^2\eta(u^T)(\nabla_{\mathbf{x}} u^T, \nabla_{\mathbf{x}} u^T)\,d\mathbf{x}dt, \end{aligned} \tag{2.19}$$

where $(\nu_t, \nu_{\mathbf{x}})$ represents an outward unity vector normal to $\mathcal{S}_r$. Now, to get rid of the third term in (2.19), we use the fact that $|\nu_{x_i}| \le M_0 \nu_t$, $i = 1, \dots, n$, together with (2.17). It follows that

$$\int_{|\mathbf{x}-\tilde{\mathbf{x}}|\le r} \eta(u^T(t,\mathbf{x}))\,d\mathbf{x} \le \int_{|\mathbf{x}-\tilde{\mathbf{x}}|\le r + M_0\tilde t} \eta(u_0^T(\mathbf{x}))\,d\mathbf{x} + \frac{C}{T}\int_{\mathcal{S}_r} |\nabla_{\mathbf{x}} u^T|\,d\mathcal{H}^n. \tag{2.20}$$

Now, since $\eta(u)\le c|u-u_L|^2$, for some $c>0$, $u_0^T\to u_L$ in $L^1\big(\{\mathbf{x}\mid |\mathbf{x}-\tilde{\mathbf{x}}|<r+M_0\tilde{t}\}\big)$ and $\frac{1}{T_l}\nabla_{\mathbf{x}}u^{T_l}$ converges to 0 $\mathcal{H}^n$-a.e. on $\mathcal{S}_r$, we conclude that

$$\lim_{T_l\to\infty}\int_{|\mathbf{x}-\tilde{\mathbf{x}}|\le r}|u^{T_l}(\tilde{t},\mathbf{x})-u_L|\,d\mathbf{x}=0. \tag{2.21}$$

From (2.21) one readily gets that u^{T_l} converges to u_L in $L^1_{\text{loc}}\big(\Omega^-_{\tilde{t}}(-\mathbf{M}_0)\big)$, which, by the arbitrariness of $\tilde{t}>0$, implies the convergence of u^{T_l} in $L^1_{\text{loc}}\big(\Omega^-(-\mathbf{M}_0)\big)$, and, then, by the arbitrariness of the subsequence T_k from which we departed, we arrived at the desired conclusion. □

The following lemma is an important tool which allows to reduce the study of the asymptotic behavior of the solution to a question of proving that the supports of certain probability measures are concentrated in the state corresponding to the exact value of the Riemann solution.

Lemma 2.1. *Suppose η is a nonnegative convex entropy for* (1.1)*, with associated entropy flux $\mathbf{q}$, and and $u\in L^\infty\big(\mathbb{R}^{n+1}_+\big)$ is solution of* (1.1), (1.2)*. Let T_l be a subsequence as in Proposition 2.3 and $\mathbf{M}_0$ as in Proposition 2.4 such that $u^{T_l}\to u_L$ a.e. in $\Omega^-(-\mathbf{M}_0)$, and $u^{T_l}\to u_R$ a.e. in $\Omega^+(\mathbf{M}_0)$. Then, for a.e. $\boldsymbol{\xi}\in\mathbb{R}^n$, and $K>0$ such that*

$$|\mathbf{q}(u)|_\infty\le K\eta(u), \tag{2.22}$$

and any $M>0$ we have:
(a) *if $\eta(u_L)=0$,*

$$\limsup_{T_l\to\infty}\frac{1}{T_l^n}\sum_{i=1}^n\int_0^{T_l}\int_{S_i^t(\boldsymbol{\xi},T_l)}(-\xi_i\eta+q_i)\,(u(t,\mathbf{x}))\;d\mathcal{H}^{n-1}dt\le 0; \tag{2.23}$$

(b) *if $\eta(u_R)=0$,*

$$\limsup_{T_l\to\infty}\sum_{i=1}^n\frac{1}{T_l^n}\int_0^{T_l}\int_{S_i^t(\boldsymbol{\xi},T_l)}(\xi_i\eta-q_i)\,(u(t,\mathbf{x}))\;d\mathcal{H}^{n-1}dt\le 0. \tag{2.24}$$

Proof. We consider first the domain $\Omega^-(\boldsymbol{\xi},T)$ where K is as in the statement of the lemma, M and $\boldsymbol{\xi}$ are choosen so that (2.11) and (2.16) hold. The choice of such $\boldsymbol{\xi}$ and M is possible due to Propositions 2.2-2.3. So, integrating (2.12) over $\Omega^-(\boldsymbol{\xi},T_l)$ one obtains

$$\begin{aligned}
&\int\limits_{\Omega^-_{T_l}(\boldsymbol{\xi},T_l)}\eta(u)\,d\mathbf{x}+\sum_{i=1}^n\int_0^{T_l}\int\limits_{S_i^t(\boldsymbol{\xi},T_l)}(-\xi_i\eta+q_i)\circ u(t,\mathbf{x})\,d\mathcal{H}^{n-1}\\
&+\int_0^{T_l}\int\limits_{\partial\Omega_t^-(\boldsymbol{\xi},T_l)\cap\partial\Omega_t(T_l)}(\nu_t\eta+\nu_{\mathbf{x}}\cdot\mathbf{q})\circ u(t,\mathbf{x})\,d\mathcal{H}^{n-1}\\
&=\int\limits_{\Omega_0^-(\boldsymbol{\xi},T_l)}\eta(u)\,d\mathbf{x}+\int_0^{T_l}\int\limits_{\partial\Omega_t^-(\boldsymbol{\xi},T_l)}\nabla\eta(u)\cdot\nu_{\mathbf{x}}\,d\mathcal{H}^{n-1}-\int\limits_{\Omega^-(\boldsymbol{\xi},T_l)}\nabla^2\eta(u)(\nabla_{\mathbf{x}}u,\nabla_{\mathbf{x}}u)\,d\mathbf{x}dt.
\end{aligned}$$

From the above equation we obtain

$$\sum_{i=1}^{n}\int_0^{T_l}\int\limits_{S_i^t(\boldsymbol{\xi},T_l)}(-\xi_i\eta+q_i)\circ u(t,\mathbf{x})\,d\mathcal{H}^{n-1}$$
$$\le \int\limits_{\Omega_0^-(\boldsymbol{\xi},T_l)}\eta(u)\,d\mathbf{x}+\int_0^{T_l}\int\limits_{\partial\Omega_t^-(\boldsymbol{\xi},T_l)}\nabla\eta(u)\cdot\nu_{\mathbf{x}}\,d\mathcal{H}^{n-1}.$$

Then, we divide the above inequality by T_l^n and take lim sup when $T_l\to\infty$. So, from our choice of $\boldsymbol{\xi}, M$ and (1.3), (1.5), we get (2.23). The relation (2.24) is proved by an entirely analogous procedure, this time integrating (2.12) over $\Omega^+(\boldsymbol{\xi},T_l)$. This concludes the proof of the lemma. □

3. Proof of Theorem 1.1

In this section we give the proof of Theorem 1.1. So, we specify our discussion to systems (1.1) whose flux functions f^i satisfy hypotheses (H1)-(H4).

A crucial step in the proof of Theorem 1.1 is as follows. Let an increasing sequence $T_k\to\infty$ be given. We consider the probability measures defined for all $g\in C(\mathcal{O})$ by

(3.1)
$$\langle\mu_{i,T}^{\boldsymbol{\xi}},g(u)\rangle\equiv c_i^{-1}(\boldsymbol{\xi},K,M,T_0)\frac{1}{T^n}\int_0^T\int_{S_i^t(\boldsymbol{\xi},T)}g\left(u(t,\mathbf{x})\right)\,d\mathcal{H}^{n-1}dt,\quad i=1,\dots,n,$$

where $c_i(\boldsymbol{\xi},K,M,T_0)$ is a normalizing positive constant such that $\langle\mu_{i,T}^{\boldsymbol{\xi}},1\rangle=1$.

We prove the existence of a piecewise continuous function $\tilde R:\mathbb{R}\to\mathbb{R}^m$ and a function $\mathbf{h}:\mathbb{R}\to\mathbb{R}^n$, $\mathbf{h}(\xi)=(h_1(\xi),\dots,h_n(\xi))$, such that each $h_i:\mathbb{R}\to\mathbb{R}$ is a strictly increasing bi-Lipschitz homeomorphism of $\mathbb{R}$, independent of K,M,T_0 and of the sequence $T_k\to\infty$, for which, we have

$$\mu_{i,T_k}^{\mathbf{h}(\xi)}\rightharpoonup\delta_{\tilde R(\xi)},\qquad\text{as } T_k\to\infty,\text{ for a.e. }\xi\in\mathbb{R},\ i=1,\dots,n. \tag{3.2}$$

Here and in what follows δ_v denotes the Dirac measure with unit mass concentrated at v. We then define

$$R(\boldsymbol{\xi})=\tilde R(\xi),\qquad\text{if }(t,t\boldsymbol{\xi})\in\partial\Omega^+(\mathbf{h}(\xi)).$$

We observe that

$$\frac{1}{T^n}\int_{\partial^0\Omega^+(\mathbf{h}(\xi),T)}g\left(u(t,\mathbf{x})\right)\,d\mathcal{H}^{n-1}=\sum_{i=1}^n c_i(\boldsymbol{\xi})\langle\mu_{i,T}^{\mathbf{h}(\xi)},g(u)\rangle, \tag{3.3}$$

where we omit the dependence of c_i on K,M,T_0, as will be frequently done henceforth. Hence, (3.2) implies, in particular, that

$$\lim_{T_k\to\infty}\frac{1}{T_k^n}\int_{\partial^0\Omega^+(\mathbf{h}(\xi),T_k)}|u(t,\mathbf{x})-R(\boldsymbol{\xi})|\,d\mathcal{H}^n=0,\quad\text{for a.e. }\xi\in\mathbb{R}. \tag{3.4}$$

The point is that (3.4) implies that $u^{T_k}(t,\mathbf{x})$ converges to $R(\boldsymbol{\xi})$ in $L^1_{loc}(\mathbb{R}^{n+1}_+)$, and since the increasing sequence T_k is arbitrary, we obtain that $u^T(t,\mathbf{x})$ converges to $R(\boldsymbol{\xi})$ in $L^1_{loc}(\mathbb{R}^{n+1}_+)$. Indeed, let $\Psi:\mathbb{R}^{n+1}_+\to\mathbb{R}$ be defined by

$$\Psi(t,\mathbf{x})=\xi,\qquad\text{if }(t,\mathbf{x})\in\partial\Omega^+(\mathbf{h}(\xi)).$$

One can easily see that

$$|\nabla_{\mathbf{x}}\Psi| = t^{-1}(h_j^{-1})'(h_j(\xi)), \qquad \text{if } (t,\mathbf{x}) \in S_j(\mathbf{h}(\xi)).$$

Now, let a compact set $\mathcal{K} \subseteq \mathbb{R}_+^{n+1}$ be given. For K, M, T_0 sufficiently large, we have $\mathcal{K} \subseteq \mathcal{Q}$, where

$$\mathcal{Q} = \Omega^+(\mathbf{h}(-K)) \cap \Omega^-(\mathbf{h}(K)) \cap \Omega(1).$$

Hence, using the coarea formula (see, eg., [14]), we have

$$\begin{aligned}
&\int_{\mathcal{K}} |u^{T_k}(t,\mathbf{x}) - R(\boldsymbol{\xi})|\, d\mathbf{x}\, dt \le \int_{\mathcal{Q}} |u^{T_k}(t,\mathbf{x}) - R(\boldsymbol{\xi})|\, d\mathbf{x}\, dt \\
&= \int_0^{T_0} dt \int_{-K}^{K} d\xi \int_{\partial^0\Omega_t^+(\mathbf{h}(\xi),1)} |u^{T_k}(t,\mathbf{x}) - R(\boldsymbol{\xi})||\nabla_{\mathbf{x}}\Psi|^{-1}\, d\mathcal{H}^{n-1} \\
&\le C \int_0^{T_0} t\, dt \int_{-K}^{K} d\xi \left(\int_{\partial^0\Omega_t^+(\mathbf{h}(\xi),1)} |u^{T_k}(t,\mathbf{x}) - R(\boldsymbol{\xi})|\, d\mathcal{H}^{n-1} \right) \\
&\le C \frac{1}{T_k^n} \int_0^{T_0 T_k} dt \int_{-K}^{K} d\xi \left(\int_{\partial^0\Omega_t^+(\mathbf{h}(\xi),T_k)} |u(t,\mathbf{x}) - R(\boldsymbol{\xi})|\, d\mathcal{H}^{n-1} \right) \\
&\le C \int_{-K}^{K} \left(\frac{1}{T_k^n} \int_{\partial^0\Omega^+(\mathbf{h}(\xi),T_k)} |u(t,\mathbf{x}) - R(\boldsymbol{\xi})|\, d\mathcal{H}^{n} \right) d\xi \to 0, \qquad \text{as } k \to \infty,
\end{aligned}$$

which proves de assertion.

We will need the following three lemmas concerning fields in the Temple class, that is, fields satisfying hypotheses (H1) and (H2) in Section 1. The first is due to D. Serre [24], the second to A. Heibig [19] and the third to Chen-Frid [3]; we refer to the corresponding cited papers for the proofs.

Lemma 3.1. (D. Serre [24]) *For a Temple field $f\colon \mathcal{O} \subseteq \mathbb{R}^m \to \mathbb{R}^m$, defined in a in a convex domain $\mathcal{O} \subseteq \mathbb{R}^m$ convex, let $l_j(u)$ be the vector with the same direction as $\nabla w_j(u)$, satisfying $l_j(u)\cdot r_j(u) = 1$. We have*

$$l_j(v)\cdot(u-v) = 0 \Longrightarrow l_j(v)\cdot(f(u)-f(v)) = 0, \qquad j = 1,\dots,m,$$

$(u,v) \in \mathcal{O}\times\mathcal{O}$. In particular, if (1.1) satisfies the hypotheses of Theorem 1.1 for $u \in \mathcal{O}$, then, for any $v \in \mathcal{O}$, the following are entropy-entropy flux pairs for (1.1):

$$\begin{aligned}
(\eta_j^e, \mathbf{q}_j^e) \equiv \big((l_j(v)\cdot(u-v))_+\,,\\
H(l_j(v)\cdot(u-v))l_j(v)\cdot(\mathbf{f}(u)-\mathbf{f}(v))\big)\,,
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
(\eta_j^w, \mathbf{q}_j^w) \equiv \big((l_j(v)\cdot(v-u))_+\,,\\
H(l_j(v)\cdot(v-u))l_j(v)\cdot(\mathbf{f}(v)-\mathbf{f}(u))\big)\,,
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
(\eta_j^*, \mathbf{q}_j^*) \equiv \big(|l_i(v)\cdot(u-v)|\,,\\
\operatorname{sgn}(l_j(v)\cdot(u-v))l_j(v)\cdot(\mathbf{f}(u)-\mathbf{f}(v))\big)\,,
\end{aligned} \tag{3.7}$$

for $j = 1,\dots,m$. Here, $H(s)$ is the well-known Heaviside function and we use the notation $(s)_+ = sH(s)$.

Lemma 3.2. (A. Heibig [19]) *If $f\colon \mathcal{O} \subseteq \mathbb{R}^m \to \mathbb{R}^m$ is a Temple field, defined in a $\mathcal{O} \subseteq \mathbb{R}^m$ convex, then there exists an unique map $\bar{A}\colon \mathcal{O}\times\mathcal{O} \to M_{m\times m}(\mathbb{R})$ such that* **(i)** *For all $(u,v) \in \mathcal{O}\times\mathcal{O}$,*

$$f(u) - f(v) = \bar{A}(u,v)(u-v),$$

$\bar A(u,u) = A(u)$, where $A(u) = \nabla f(u)$; is the Jacobian matrix of f;
(ii) *for all $(u,v) \in \mathcal{O}\times\mathcal{O}$, $A(v)$ and $\bar A(u,v)$ have the same (left and right) eigenvectors;*
(iii) *$\bar A$ is a smooth function.*

Lemma 3.3. (Chen-Frid [3]) *Let $\bar\lambda_j(u,v)$ be the j-th eigenvalue of the matrix $\bar A(u,v)$, given by Lemma 3.2 and suppose $\mathcal{O}$ has the form*

$$\mathcal{O} = \bigcap_{j=1}^{m}\{\, u\in\mathbb{R}^m \;:\; |w_j(u) - w_j(\bar u)| \le M_j \,\}, \tag{3.8}$$

for certain $M_j > 0$, $j = 1,\dots,m$. Then,

$$\min_{u\in\mathcal{O}} \lambda_j(u) \le \bar\lambda_j(u,v) \le \max_{u\in\mathcal{O}} \lambda_j(u), \qquad (u,v)\in\mathcal{O}\times\mathcal{O}. \tag{3.9}$$

We begin now the proof of Theorem 1.1, properly said. Let $\mathcal{O}$ be as in Lemma 3.3 and assume that $u_0(\mathbf{x}) \in \mathcal{O}$ for all $\mathbf{x}\in\mathbb{R}^n$. Since, as is well known, (H1) and (H2) imply that $\mathcal{O}$ is positively invariant under the flow of (1.1) (cf. [8]), the solution of (1.1)-(1.2), $u(t,\mathbf{x})$, can be defined for all $t > 0$ and we have $u(t,\mathbf{x}) \in \mathcal{O}$, for $(t,\mathbf{x}) \in \mathbb{R}^{n+1}_+$. We then assume that f^i, $i = 1,\dots,n$, in (1.1) satisfy (H1)-(H4) of Section 1 over a bounded region containing $\mathcal{O}$.

We start the definition of $\mathbf{h}$ by setting $h_1(\xi) = \xi$, $\xi\in\mathbb{R}$, and $h_i(\xi) = \xi$, $i = 2,\dots,n$, for $|\xi| > M_0$, where M_0 is as in Proposition 2.4. We will complete the definition of h_i, $i = 2,\dots,n$, along the proof, in such a way that $h_i([-M_0,\kappa_1^1]) = [-M_0,\kappa_1^i]$, $h_i([\kappa_{m+1}^1, M_0]) = [\kappa_{m+1}^i, M_0]$, and $h_i([\kappa_j^1,\kappa_{j+1}^1]) = [\kappa_j^i,\kappa_{j+1}^i]$, $j = 1,\dots,m$. So, we next define h_i in the intervals $[-M_0,\kappa_1^1]$ and $[\kappa_{m+1}^1, M_0]$ as the only affine map preserving orientation such that $h_i([-M_0,\kappa_1^1]) = [-M_0,\kappa_1^i]$ and $h_i([\kappa_{m+1}^1, M_0]) = [\kappa_{m+1}^i, M_0]$. Observe that Proposition 2.4 implies that (3.1) holds for $|\xi| > M_0$. In what follows we split the remaining of the proof into several steps.

#1. $\xi\in(-M_0,\kappa_1^1)\cup(\kappa_{m+1}^1, M_0)$. Let us consider first $\xi\in(-M_0,\kappa_1^1)$. From the weak compactness of the probability measures with support in $\mathcal{O}$, we have that $\mu_{i,T_k}^{\mathbf{h}(\xi)} \rightharpoonup \mu_i^\xi$, for some probability measure μ_i^ξ, with support in $\mathcal{O}$, by passing to a subsequence if necessary, for $i = 1,\dots,n$. We apply **(a)** of Lemma 2.1 to the $(\eta_j^*(u,u_L), \mathbf{q}_j^*(u,u_L))$, given by (3.7). We then get

$$\sum_{i=1}^{n} c_i(\mathbf{h}(\xi))\langle \mu_i^\xi, -h_i(\xi)|l_j(u_L)\cdot(u-u_L)| \\ + \operatorname{sgn}(l_j(u_L)\cdot(u-u_L))\, l_j(u_L)\cdot(f^i(u) - f^i(u_L))\rangle \le 0, \tag{3.10}$$

for $j = 1,\dots,m$. Using Lemma 3.2, we may write (3.10) in the form

$$\sum_{i=1}^{n} c_i(\mathbf{h}(\xi))\langle \mu_i^\xi, |l_j(u_L)\cdot(u-u_L)|(-h_i(\xi) + \bar\lambda_j^i(u,u_L))\rangle \le 0, \quad j = 1,\dots,m,$$

where $\bar\lambda_j^i(u,u_L)$ is as in Lemma 3.3. Thus, since $h_i(\xi)<\kappa_1^i\le\bar\lambda_j^i(u,v)$, $i=1,\dots,n$, $j=1,\dots,m$, $u,v\in\mathcal{O}$, we get

$$\sum_{i=1}^n c_i(\mathbf{h}(\xi))\langle\mu_i^\xi,|l_j(u_L)\cdot(u-u_L)|(-h_i(\xi)+\bar\lambda_j^i(u,u_L))\rangle=0,\quad j=1,\dots,m.$$

Therefore, we must have

$$\operatorname{supp}\mu_i^\xi\subseteq\{u\in\mathcal{O}\,:\,l_j(u_L)\cdot(u-u_L)=0,j=1,\dots,m\}=\{u_L\}.$$

We then conclude $\mu_i^\xi=\delta_{u_L}$. Since this holds for all weakly convergent subsequence of $\mu_{i,T_k}^{\mathbf{h}(\xi)}$, we conclude $\mu_{i,T_k}^{\mathbf{h}(\xi)}\rightharpoonup\delta_{u_L}$, for $\xi\in(-M_0,\kappa_1^1)$. So, we define $\bar R(\xi)=u_L$, for $\xi<\kappa_1^1)$. The case $\xi\in(k_{m+1}^1,M_0)$ is analogous from **(b)** of Lemma 2.1 applied to $(\eta_i^*(u,u_R),\mathbf{q}_i^*(u,u_R))$, $i=1,\dots,n$. We then define $\bar R(\xi)=u_R$, for $\xi>\kappa_{m+1}^1$.

Now we consider the more difficult cases when $\xi\in(k_j^1,k_{j+1}^1)$, $j=1,\dots,m$. Let $w=(w_1,\dots,w_m)$, where the w_j's are the Riemann invariants of hypothesis (H2). In the Cartesian space of the w-coordinates, we set $w^{(1)}=w(u_L)$, $w^{(m+1)}=w(u_R)$, and

$$w^{(j)}=(w_1(u_L),\dots,w_{j-1}(u_L),w_j(u_R),\dots,w_m(u_R)),\qquad j=2,\dots,m.$$

We then let $u^{(j)}=u(w^{(j)})$, $j=1,\dots,m+1$.

For a function $\mathbf{h}$ with the properties described above and $\xi\in(k_j^1,k_{j+1}^1)$, with $j\in\{1,\dots,m\}$, we again consider the probability measures $\mu_{i,T_k}^{\mathbf{h}(\xi)}$ which, after a passage to a subsequence, satisfy $\mu_{i,T_k}^{\mathbf{h}(\xi)}\rightharpoonup\mu_i^\xi$, for a certain probability measure μ_i^ξ.

#2. We claim that

$$\operatorname{supp}\mu_i^\xi\subseteq L_j\equiv\{u\in\mathcal{O}\,:\,u=u^{(j)}+sr_j(u^{(j)}),\ s\in\mathbb{R}\}.\tag{3.11}$$

Indeed, we have

$$L_j=\bigcap_{\alpha=1}^{j-1}\{u\in\mathcal{O}\,:\,l_\alpha(u_R)\cdot(u-u_R)=0\}\cap\bigcap_{\alpha=j+1}^{m}\{u\in\mathcal{O}\,:\,l_\alpha(u_L)\cdot(u-u_L)=0\}.$$

Thus we apply again (2.24) with $(\eta,\mathbf{q})=(\eta_\alpha^*(u,u_R),\mathbf{q}_\alpha^*(u,u_R))$, $\alpha=1,\dots,j-1$, and (2.23) with $(\eta,\mathbf{q})=(\eta_\alpha^*(u,u_L),\mathbf{q}_\alpha^*(u,u_L))$, $\alpha=j+1,\dots,m$. As above, the application of (2.24) gives

$$\sum_{i=1}^n c_i(\mathbf{h}(\xi))\langle\mu_i^\xi,|l_\alpha(u_R)\cdot(u-u_R)|(h_i(\xi)-\bar\lambda_\alpha^i(u,u_R))\rangle\le0,\qquad\alpha=1,\dots,j-1,\tag{3.12}$$

while the application of (2.23) gives

$$\sum_{i=1}^n c_i(\mathbf{h}(\xi))\langle\mu_i^\xi,|l_\alpha(u_L)\cdot(u-u_L)|(-h_i(\xi)+\bar\lambda_\alpha^i(u,u_L))\rangle\le0,\qquad\alpha=j+1,\dots,m.\tag{3.13}$$

Since $h_i([k_j^1,k_{j+1}^1])=[k_j^i,k_{j+1}^i]$, we have $h_i(\xi)>\kappa_{\alpha+1}^i>\bar\lambda_\alpha^i(u,u_R)$, for $\alpha=1,\dots,j-1$, and $h_i(\xi)<\kappa_\alpha^i<\bar\lambda_\alpha^i(u,u_L))$, for $\alpha=j+1,\dots,m$. Hence, the equality holds in both (3.12) and (3.13), and this is possible only if (3.11) holds.

Next we give a precise definition of $\mathbf{h}$ in the intervals (k_j^1, k_{j+1}^1), $j = 1, \dots, m$, and verify (3.2) in these intervals.

#3. Let $\xi \in (\kappa_j^1, m_j^1) \cup (M_j^1, \kappa_{j+1}^1)$, where $m_j^i = \min\{\lambda_j^i(u^{(j)}), \lambda_j^i(u^{(j+1)})\}$ and $M_j^i = \max\{\lambda_j^i(u^{(j)}), \lambda_j^i(u^{(j+1)})\}$. We also denote $m_{*j}^i = \inf_{u\in\mathcal{O}} \lambda_j^i(u)$ and $M_{*j}^i = \sup_{u\in\mathcal{O}} \lambda_j^i(u)$. Let us first address the case $\xi \in (\kappa_j^1, m_j^1)$. We define h_i in the interval $[\kappa_j^1, m_{*j}^1]$ as the only affine function which applies this interval onto the interval $[\kappa_j^i, m_{*j}^i]$ preserving the orientation. For $\xi \in (\kappa_j^1, m_{*j}^1)$, again $\mu_{i,T_k}^{\mathbf{h}(\xi)} \rightharpoonup \mu_i^\xi$ for some probability measure μ_i^ξ, by passing to a subsequence if necessary. We take the entropy pair $(\eta^*(u, u^{(j)}), \mathbf{q}^*(u, u^{(j)}))$, observing that $\eta^*(u_L, u^{(j)})$ because $w_j(u_L) = w_j(u^{(j)})$. We then apply (2.23) to obtain

$$\sum_{i=1}^n c_i(\mathbf{h}(\xi))\langle \mu_i^\xi, |l_j(u^{(j)}) \cdot (u - u^{(j)})|(-h_i(\xi) + \lambda_j^i(u, u^{(j)}))\rangle \le 0, \tag{3.14}$$

where

$$\lambda_j^i(u, v) = \int_0^1 \lambda_j^i(\theta u + (1-\theta)v)\, d\theta$$

is the j-th eigenvalue of $\int_0^1 \nabla f^i(\theta u + (1-\theta)v)\, d\theta$ if $u, v \in L_j$. Now, for $u \in L_j \cap \mathcal{O}$ inequality (3.14) is possible only if equality holds, which implies $\mu_i^\xi = \delta_{u^{(j)}}$. Hence, $\mu_{i,T_k}^{\mathbf{h}(\xi)} \rightharpoonup \delta_{u^{(j)}}$ in this case.

For $\xi \in (m_{*j}^1, m_j^1)$ we proceed as follows. Let u_ξ be such that $\lambda_j^1(u_\xi) = \xi$. Define $h_i(\xi) = \lambda_j^i(u_\xi)$, $i = 2, \dots, n$. We first take the entropy pair $(\eta^w(u, u_\xi),\ \mathbf{q}^w(u, u_\xi))$ and observe that $\eta^w(u_R, u_\xi) = 0$, which holds because $l_j(u_\xi)$ and $u_R - u_\xi$ point inward the same half-space determined by the hyperplane $w_j = w_j(u_\xi)$. The latter holds since $w_j(u_R) = w_j(u^{(j+1)}) > w_j(u_\xi)$ and λ_j^1 is an increasing function of w_j over L_j. Considering again probability measures μ_i^ξ obtained as above and applying (2.24), we get

$$\sum_{i=1}^n c_i(\mathbf{h}(\xi))\langle \mu_i^\xi, (l_j(u_\xi) \cdot (u_\xi - u))_+ (h_i(\xi) - \lambda_j^i(u, u_\xi))\rangle \le 0. \tag{3.15}$$

We notice that $(l_j(u_\xi) \cdot (u_\xi - u))_+ = (s_\xi - s)_+$, where s_ξ is given by $u_\xi = u^{(j)} + s_\xi r_j(u^{(j)})$. We also observe that $\lambda_j^i(u, u_\xi) \le h_i(\xi) = \lambda_j^i(u_\xi)$ if $u = u^{(j)} + s r_j(u^{(j)})$ and $s \le s_\xi$. These observations lead to

$$\operatorname{supp} \mu_i^\xi \subseteq \{u \in L_j \,:\, u = u^{(j)} + s r_j(u^{(j)}),\ s \ge s_\xi\}. \tag{3.16}$$

To complete the analysis for $\xi \in (m_{*j}^1, m_j^1)$ we again take the pair $(\eta^*(u, u^{(j)}),$ $\mathbf{q}^*(u, u^{(j)}))$ and apply (2.23) obtaining (3.14). Since $\lambda_j^i(u, u^{(j)})) \ge \lambda_j^i(u_\xi, u^{(j)})) \ge \lambda_j^i(u_\xi) = h_i(\xi)$ if $u = u^{(j)} + s r_j(u^{(j)})$ and $s \ge s_\xi$, we conclude $\mu_i^\xi = \delta_{u^{(j)}}$, and so we get $\mu_{i,T_k}^{\mathbf{h}(\xi)} \rightharpoonup \delta_{u^{(j)}}$ for $\xi \in (m_{*j}^1, m_j^1)$. Accordingly, we define $\bar{R}(\xi) = u^{(j)}$, for $\xi \in (\kappa_j^1, m_j^1)$.

For $\xi \in (M_j^1, \kappa_{j+1}^1)$ we follow analogous procedures to define h_i in this interval, $i = 2, \dots, n$, and obtain similarly $\mu_{i,T_k}^{\mathbf{h}(\xi)} \rightharpoonup \delta_{u^{(j+1)}}$ in this case. So, we define $\bar{R}(\xi) = u^{(j+1)}$, for $\xi \in (M_j^1, \kappa_{j+1}^1)$.

#4. We now come to the most interesting case, which is when $\xi \in (m_j^1, M_j^1)$. Due to (H4) we have just two possibilities:

(i) $\lambda_j^i(u^{(j)}) < \lambda_j^i(u^{(j+1)})$, $i = 1, \dots, n$;
(ii) $\lambda_j^i(u^{(j)}) > \lambda_j^i(u^{(j+1)})$, $i = 1, \dots, n$.

Let us analyze (i) first. As above, we let u_ξ be such that $\lambda_j^1(u_\xi) = \xi$, and define $h_i(\xi) = \lambda_j^i(u_\xi)$, $i = 2, \dots, n$. We also define $\bar R(\xi) = u_\xi$. In order to prove that $\mu_{i,T_k}^\xi \rightharpoonup \delta_{\bar R(\xi)}$, we consider the entropy pairs $(\eta^w(u, u_\xi), \mathbf{q}^w(u, u_\xi))$ and $(\eta^e(u, u_\xi), \mathbf{q}^e(u, u_\xi))$. We notice that $\eta^w(u_R, u_\xi) = 0$ and $\eta^e(u_L, u_\xi) = 0$, the first because $u_R - u_\xi$ and $l_j(u_\xi)$ point inward the same half-space determined by $w_j = w_j(u_\xi)$, and the second because the same holds for $u_\xi - u_L$ and $l_j(u_\xi)$. Applying (2.24) to $(\eta^w(u, u_\xi), \mathbf{q}^w(u, u_\xi))$ and (2.23) to $(\eta^e(u, u_\xi), \mathbf{q}^e(u, u_\xi))$, we obtain, respectively, (3.15) and the analogous one

$$\sum_{i=1}^n c_i(\mathbf{h}(\xi)) \langle \mu_i^\xi, (l_j(u_\xi) \cdot (u - u_\xi))_+ (-h_i(\xi) + \lambda_j^i(u, u_\xi)) \rangle \le 0. \tag{3.17}$$

From (3.15) it follows as above that (3.16) holds. Similarly, from (3.17) it follows

$$\operatorname{supp} \mu_i^\xi \subseteq \{u \in L_j \,:\, u = u^{(j)} + s r_j(u^{(j)}),\ s \le s_\xi\}. \tag{3.18}$$

Hence, we conclude $\mu_i^\xi = \delta_{\bar R(\xi)}$ and so $\mu_{i,T_k}^\xi \rightharpoonup \delta_{\bar R(\xi)}$.

We finally consider the alternative (ii). Let $\lambda_j^i(u, v)$ be as above, and set

$$\sigma_j^i := \lambda_j^i(u^{(j)}, u^{(j+1)}) = \lambda_j^i(u^{(j+1)}, u^{(j)}), \qquad i = 1, \dots, n.$$

We then define $h_i(\xi)$ in the interval $[\lambda_j^1(u^{(j+1)}), \sigma_j^1]$ as the only affine map carrying this interval onto $[\lambda_j^i(u^{(j+1)}), \sigma_j^i]$, preserving the orientation; similarly, we define $h_i(\xi)$ in the interval $[\sigma_j^1, \lambda_j^1(u^{(j)})]$ as the only affine map carrying this interval onto $[\sigma_j^i, \lambda_j^i(u^{(j)})]$, preserving the orientation. We also define $\bar R(\xi) = u^{(j)}$, for $\lambda_j^1(u^{(j+1)}) \le \xi \le \sigma_j^1$ and $\bar R(\xi) = u^{(j+1)}$, for $\sigma_j^1 \le \xi \le \lambda_j^1(u^{(j)})$. Let μ_i^ξ be obtained as above. We first show that

$$\operatorname{supp} \mu_i^\xi \subseteq \{u \in L_j \,:\, u = u^{(j)} + s r_j(u^{(j)}),\ s(u^{(j+1)}) \le s \le 0\}. \tag{3.19}$$

where $s(u^{(j+1)})$ is such that $u^{(j+1)} = u^{(j)} + s(u^{(j+1)}) r_j(u^{(j)})$. To this, we consider the pairs $(\eta^w(u, u^{(j+1)}), \mathbf{q}^w(u, u^{(j+1)}))$ and $(\eta^e(u, u^{(j)}), \mathbf{q}^e(u, u^{(j)}))$, observing that $\eta^w(u_R, u^{(j+1)}) = 0$ and $\eta^e(u_L, u^{(j)}) = 0$. Applying (2.24) to $(\eta^w(u, u^{(j+1)}), \mathbf{q}^w(u, u^{(j+1)})$ and (2.23) to $(\eta^e(u, u^{(j)}), \mathbf{q}^e(u, u^{(j)}))$, we obtain, respectively,

$$\sum_{i=1}^n c_i(\mathbf{h}(\xi)) \langle \mu_i^\xi, (l_j(u^{(j+1)}) \cdot (u^{(j+1)} - u))_+ (h_i(\xi) - \lambda_j^i(u, u^{(j+1)})) \rangle \le 0, \tag{3.20}$$

and

$$\sum_{i=1}^n c_i(\mathbf{h}(\xi)) \langle \mu_i^\xi, (l_j(u^{(j)}) \cdot (u - u^{(j)}))_+ (-h_i(\xi) + \lambda_j^i(u, u^{(j)})) \rangle \le 0. \tag{3.21}$$

Now, $(l_j(u^{(j+1)}) \cdot (u^{(j+1)} - u))_+ = (s(u^{(j+1)}) - s)_+$ and $(l_j(u^{(j)}) \cdot (u - u^{(j)}))_+ = s_+$. Hence, (3.20) implies

$$\operatorname{supp} \mu_i^\xi \subseteq \{u \in L_j \,:\, u = u^{(j)} + s r_j(u^{(j)}),\ s \ge s(u^{(j+1)})\},$$

and (3.21) implies

$$\operatorname{supp} \mu_i^\xi \subseteq \{u \in L_j \,:\, u = u^{(j)} + s r_j(u^{(j)}),\ s \le 0\},$$

which together give (3.19).

Now, for $\lambda_j^1(u^{(j+1)}) < \xi < \sigma_j^1$, we take the entropy pair $(\eta^e(u,u^{(j)}),\mathbf{q}^e(u,u^{(j)}))$, apply (2.23) to get (3.21). Since $h_i(\xi) < \sigma_j^i \le \lambda_j^i(u,u^{(j)})$, for $u = u^{(j)} + sr_j(u^{(j)})$ and $s \ge s(u^{(j+1)})$, we must have $\mu_i^\xi = \delta_{u^{(j)}}$ and then $\mu_{i,T_k}^{\mathbf{h}(\xi)} \rightharpoonup \delta_{u^{(j)}}$. Similarly, for $\sigma_j^1 < \xi < \lambda_j^1(u^{(j)})$, we take the entropy pair $(\eta^w(u,u^{(j+1)}),\mathbf{q}^w(u,u^{(j+1)}))$, apply (2.24) to obtain (3.20). Since $h_i(\xi) > \sigma_j^i \ge \lambda_j^i(u,u^{(j)})$, for $u = u^{(j)} + sr_j(u^{(j)})$ and $s \le 0$, we must have $\mu_i^\xi = \delta_{u^{(j+1)}}$ and then $\mu_{i,T_k}^{\mathbf{h}(\xi)} \rightharpoonup \delta_{u^{(j+1)}}$. This concludes the proof of (3.2).

#5. As it was shown above, (3.2), holding for $K > 0$ sufficiently large, $T_0 > 0$, and almost all $M > M_0$, with M_0 sufficiently large, implies the convervence of $u^T(t,\mathbf{x})$ to $R(\mathbf{x}/t)$ in $L^1_{loc}(\mathbb{R}^n_+)$. The latter, in turn, implies (1.14). As it was mentioned in Section 1, (1.14) can be improved to (1.11). The latter is achieved using a method established in [3], motivated by [21]. The strategy is similar to the one for obtaining the decay of periodic solutions in L^1_{loc}, from the decay in time-average along rays (see [4]), and is based on the existence of a strictly convex entropy for (1.10). Namely, we use the entropy inequalities, the convergence in time-average, and a result connecting both resembling a continuous version of Hardy's convergence theorem (cf. [28], p.156), to get the usual convergence in time (1.11). The details of the one dimensional case are given in [3], while the the multidimensional case can be seen in [6]. The basic steps are as follows. Given a smooth strictly convex entropy $\eta(u)$ for (1.10), we consider its quadratic part around v

$$\alpha(u,v) = \eta(u) - \eta(v) - \nabla\eta(v)(u-v).$$

In the space of the variables $\boldsymbol{\xi}$, we take any ball B over which $R(\boldsymbol{\xi})$ is Lipschitz. Using the equations (1.1), (1.10), for $u(t,t\boldsymbol{\xi})$ and $R(\boldsymbol{\xi})$, respectively, we obtain, after integration over B,

$$\frac{d}{dt}\int_B \alpha(u(t,t\boldsymbol{\xi}),R(\boldsymbol{\xi}))\,d\boldsymbol{\xi} \le \frac{C}{t},$$

for some constant $C > 0$, independent of t. Now, (1.14) implies

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\left(\int_B \alpha(u(t,t\boldsymbol{\xi}),R(\boldsymbol{\xi}))\,d\boldsymbol{\xi}\right)\,dt = 0.$$

Therefore, the nonnegative function $Y(t) = \int_B \alpha(u(t,t\boldsymbol{\xi}),R(\boldsymbol{\xi}))\,d\boldsymbol{\xi}$ satisfies

$$\frac{d}{dt}Y(t) \le \frac{C}{t}, \tag{3.22}$$

and

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T Y(t)\,dt = 0, \tag{3.23}$$

and an easy argument shows that (3.22) and (3.23) imply

$$\lim_{t\to\infty} Y(t) = 0. \tag{3.24}$$

Now, by Besicovitch's covering theorem, any compact set $\mathcal{K} \subseteq \mathbb{R}^n$ can be covered by a finite number of disjoint countable families of balls B where $R(\boldsymbol{\xi})$ is Lipschitz, and so (3.24) implies (1.11).

#6. It remains only to prove the uniqueness of the self-similar L^∞ entropy solution $R(\boldsymbol{\xi})$ of the two-states Riemann problem (1.10), (1.4). To this we make the important remark that the methods developed above are also applicable to L^∞

entropy solutions of (1.10),(1.2), whose initial data satisfies (1.3)-(1.5), with flux-functions satisfying (H1)-(H4). The only nontrivial adaptation to be made is that the use of the classical Gauss-Green formula must now be justified by appealing to the theory of L^∞ divergence-measure fields developed in [7]. So, if $\tilde{R}(\boldsymbol{\xi})$ is another self-similar L^∞ entropy solution of that problem, (1.11) holds with $u(t, t\boldsymbol{\xi})$ replaced by $\tilde{R}(\boldsymbol{\xi})$. Since both $R(\boldsymbol{\xi})$ and $\tilde{R}(\boldsymbol{\xi})$ are independent of t, we conclude $\tilde{R}(\boldsymbol{\xi}) = R(\boldsymbol{\xi})$, a.e. in $\mathbb{R}^n$.

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