CONSERVATION LAWS WITH TIME DEPENDENT DISCONTINUOUS COEFFICIENTS

GIUSEPPE MARIA COCLITE AND NILS HENRIK RISEBRO

Abstract. We consider scalar conservation laws where the flux function depends discontinuously on both the spatial and temporal location. Our main results are the existence and well–posedness of an entropy solution to the Cauchy problem. The existence is established by showing that a sequence of front tracking approximations is compact in L^1, and that the limits are entropy solutions. Then, using the definition of an entropy solution taken form [11], we show that the solution operator is L^1 contractive. These results generalize the corresponding results from [16] and [11].

1. Introduction

In this paper we are concerned with the Cauchy problem for scalar conservation laws where the flux function depends on both the x and t coordinate. We study the case where this dependence takes the form $f(u,x,t) = f(u,a(x),g(t))$, through some functions a and g. Hence, we shall study the initial value problem

$$\begin{cases} u_t + f(u,a(x),g(t))_x = 0, & x \in \mathbf{R}, \quad t > 0, \\ u(x,0) = 0, & x \in \mathbf{R}, \end{cases} \tag{1.1}$$

where $f = f(u,a,g)$ is a smooth function. We regard the function $a(x)$ and $g(t)$ as coefficients, and if these are smooth, the classical results of Kružkov [16] and Oleĭnik [19] state that the above initial value problem is well posed in the class of entropy solutions.

In our case, the coefficients are allowed to be discontinuous, and we cannot apply the techniques of Kružkov and Oleĭnik directly to reach their conclusion. The main obstacle is that of the discontinuity of the spatial coefficient a. The equation where g is constant has recently received considerable attention. This started by the paper of Temple [21], in which he studied a system of non-strictly hyperbolic conservation laws. By a Lagrangian transformation, this system is equivalent to a scalar equation with discontinuous coefficients, see Wagner [24]. If one writes the scalar conservation law as a system by introducing a as a new component of the solution, we have

$$\begin{cases} u_t + f(u,a,g(t))_x = 0 \\ \qquad\qquad\qquad\quad a_t = 0. \end{cases}$$

This system has eigenvalues f_u and 0, and if $f_u(u,a,g) = 0$ for some (u,a,g), then the system is non-strictly hyperbolic, and the standard theory for systems, see Glimm [7] and (more recently) [8, 1], does not apply. In particular, one can show by a concrete example, see e.g., [21], that the total variation of the approximate solutions produced by the Glimm scheme (and also by front tracking) is not bounded in terms of the discretization parameters. Such systems are commonly called *resonant*. For resonant systems, one cannot show compactness by the usual method of establishing BV estimates on a sequence of approximate solutions.

To overcome this difficulty, in [21] Temple introduced a nonlinear mapping $\Psi = \Psi(u)$, and used this mapping to prove that the sequence of approximations produced by the Glimm scheme is compact. This approach has since been used in a number of papers for related systems, using other approximations, see Gimse and Risebro [6], Klingenberg and Risebro [14, 15] for front tracking

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approximations, Lin *et al.* [18] for Godunov type approximations, Towers [22, 23] for monotone difference schemes and Hong [9] for 2×2 Godunov schemes.

As an alternative to the use of Ψ to prove compactness, in [12, 10] Karlsen *et al.* used the Murat–Tartar compensated compactness approach to prove convergence of numerical approximations.

Regarding uniqueness of weak solutions to (1.1) in the case where a and g not smooth, this was first studied (for the constant g case) in [15] and [13]. In these papers it was shown that the solution is unique if it is the limit of solutions to equations where the coefficients are smoothed. More recently, L^1–contractivity was shown for piecewise smooth solutions in the case of convex flux functions in [23], and in a more general case by Karlsen, Risebro, and Towers [12]. Also, Seguin and Vovelle [20] proved uniqueness for L^∞ solutions for a special case of (1.1) with $g =$ const. and $a(\cdot)$ taking two values separated by a jump discontinuity. The techniques used in the present paper are heavily inspired by those used in [11], in which Karlsen, Risebro and Towers show uniqueness of solutions in the case where g is constant, and where $u \mapsto f(u, a)$ is not required to have a single local maximum. The authors of [23, 12, 20, 11] all use a Kružkov type entropy condition.

The purpose of the present paper is to extend the wellposedness theory for conservation laws with discontinuous coefficients by including a t dependent coefficient.

Conservation laws with discontinuous coefficients, both in x and t, occur in many models. The simplest such model is the hydrodynamic traffic flow model, see Ligthill and Whitham [17]. In this case the x and t dependency model the road conditions, specifically the maximal speed of any vehicle. Both of these dependencies can vary discontinuously, for instance when modeling a traffic light. Another model in which such conservation laws occur is a clarifier-thickener model of continuous sedimentation, see Bürger *et al.* [4, 2, 3]. In the papers [2, 3] the actual models were simplified so that $g(t)$ was assumed to be constant.

Now we briefly state our main result, and detail our assumptions. In order for the Riemann problem to have a bounded solution, it is convenient to assume that there is a finite interval $[\alpha, \beta]$ such that $f(\alpha, a, g) = f(\beta, a, g)$ for all a and g, and we can choose $\alpha = 0$ and $\beta = 1$. This is not necessary for the solution of the Riemann problem to be bounded, but it is certainly sufficient, see however [5] for for less restrictive assumptions that yields the same conclusions.

So therefore we assume that $f : [0, 1] \times \mathbf{R}^2 \mapsto \mathbf{R}$, $g : \mathbf{R}^+ \mapsto \mathbf{R}$ and $a : \mathbf{R} \mapsto \mathbf{R}$ are given functions which satisfy the following:

(**A**.1) a is piecewise C^1 with finitely many jump discontinuities at $x = x_1, \dots, x_M$.
(**A**.2) $\|a\|_{L^\infty} < \infty$, $\sup_{x \notin \{x_i\}_1^M} |a'(x)| < \infty$ and $a \in BV(\mathbf{R})$.
(**A**.3) $f \in C^2([0, 1] \times \mathbf{R}^2; \mathbf{R})$, $f_{uu}(u, a, g) \le -c_{uu} < 0$, for some positive constant c_{uu} for all a and g.
(**A**.4) $f(0, \cdot, \cdot) \equiv f(1, \cdot, \cdot) \equiv 0$, and there is a unique value u^* such that $f_u(u^*, \cdot, \cdot) \equiv 0$.
(**A**.5) $\partial f/\partial g \ge 0$ and $\partial f/\partial a \ge 0$.
(**A**.6) $g \in BV(\mathbf{R}^+)$.

Next, let $\Psi(u, a, g)$ be defined by

$$\Psi(u, g, a) = \operatorname{sign}(u - u^*) \frac{f(u^*, a, g) - f(u, a, g)}{f(u^*, a, g)}. \tag{1.2}$$

We demand that the initial data is such that $u_0 \in L^1(\mathbf{R}; [0, 1])$ and

$$|\Psi(u_0, a, g)|_{BV} < \infty. \tag{1.3}$$

We use the following definition of a weak entropy solution of (1.1):

Definition 1.1. *Let $T > 0$, and let $u : \Pi_T = \langle 0, T\rangle \times \mathbf{R} \mapsto [0, 1]$ be a measurable function. We call u an entropy weak solution of* (1.1) *if the following conditions hold:*

(**D**.1) *$u \in L^1(\Pi_T)$, and the map $\langle 0, T\rangle \ni t \mapsto u(\cdot, t) \in L^1(\mathbf{R})$ is Lipschitz continuous.*

(**D.2**) *The following entropy inequality holds for all constants c and all non-negative test functions φ,*

$$\begin{aligned}\iint_{\Pi_T} |u-c|\,\varphi_t + F(u,x,t,c)\varphi_x\,dtdx - \sum_{m=0}^{M}\int_{x_m}^{x_{m+1}}\int_0^T \operatorname{sign}(u-c)\,f_a(c,a(x),g(t))a'(x)\varphi\,dtdx \\ + \sum_{m=1}^{M}\int_0^T \left|f(c,a\left(x_m^+\right),g(t)) - f(c,a\left(x_m^-\right),g(t))\right|\,dt \geq 0,\end{aligned} \tag{1.4}$$

where we have set $x_0=-\infty$, $x_{M+1}=\infty$, and F is given by

$$F(u,x,t,c) = \operatorname{sign}(u-c)\left[f\left(u,a(x),g(t)\right) - f\left(c,a(x),g(t)\right)\right], \quad t>0, \quad x\in\mathbf{R}.$$

(**D.3**) $u(\cdot,t)\to u_0$ *in* $L^1(\mathbf{R})$ *as* $t\downarrow 0$.
(**D.4**) $|\Psi(u(\cdot,t),a,g(t))|_{BV}<\infty$ *for all* $t\in\langle 0,T\rangle$.

The inequality (1.4) implies that any entropy solution is a weak solution, as setting $c=1$ and $c=0$ will show. The condition (**D.4**) implies that the limits

$$\lim_{x\to x_m^\pm}\Psi(u,a,g)$$

exist for almost all t. Since $u\mapsto\Psi(u,a,g)$ is invertible, and the inverse is continuous, the limits

$$\lim_{x\to x_m^\pm}u(x,t)$$

also exist for almost all t. This will be needed to show uniqueness. Our main result is

Main Theorem. *Assume that f, a and g satisfy the above assumptions, (**A.1**) – (**A.6**). If u_0 and v_0 are two functions that satisfy* (1.3)*, then there exist corresponding entropy solutions u and v taking initial values u_0 and v_0 respectively. These entropy solutions satisfy*

$$\|u(\cdot,t)-v(\cdot,t)\|_{L^1(\mathbf{R})} \leq \|u_0-v_0\|_{L^1(\mathbf{R})}.$$

The rest of this paper is organized as follows. In the next section, Section 2 we define a sequence of approximate solutions by the front tracking method. This is based on the front tracking method defined in [14]. In Section 3 we proceed to establish interaction estimates which allows us to deduce that the total variation of Ψ is bounded for the front tracking approximations. Then we can use Helly's theorem, and show that any limit is an entropy solution in the above sense. In Section 4 we use an adaptation of arguments taken from [11] to show that the entropy solution operator is L^1 contractive. In this way our main theorem is proved. Finally, we conclude with a section showing the front tracking scheme used on a concrete example.

2. The front tracking scheme

We start this section by defining a front tracking scheme for the case where $g(t)\equiv\text{Const}$. This scheme is slightly different from the front tracking scheme defined for this case in e.g. [14]. The reason for this difference is that our front tracking scheme also must work when g is not constant.

Therefore we first consider the initial value problem,

$$\begin{cases} u_t+f(u,a)_x=0 & \text{for } x\in\mathbf{R},\ t>0,\\ u(x,0)=u_0(x) & \text{for } x\in\mathbf{R},\end{cases} \tag{2.1}$$

where f and a are as described above. The Riemann problem for (2.1) is the initial value problem where

$$u_0(x)=\begin{cases}u_l & x\leq 0,\\ u_r & x>0,\end{cases} \qquad a(x)=\begin{cases}a_l & x\leq 0,\\ a_r & x>0,\end{cases}$$

and its solution is detailed in [14]. This solution consists of at most one u-wave separating the u-values u_l and u_l', followed by a so–called a-wave separating the states (u_l',a_l) and (u_r',a_r). This wave is a contact discontinuity having zero speed. The solution is then completed by a u-wave separating u_r' and u_r. The first u-wave has non-positive speed, and the second non-negative. The intermediate states can u_l' and u_r' are unique, provided (1.4) holds. Furthermore $u_{l,r}'$ can equal $u_{l,r}$.

Let

$$z(u,a) = \operatorname{sign}(u^* - u)\,(f(u,a) - f(u^*,a)) \quad \text{and} \quad \alpha(a) = f(u^*,a).$$

Since $a \mapsto f(u^*,a)$ is non-decreasing, $a \mapsto \alpha(a)$ is invertible. In the (z,α) plane, a waves are straight lines of slope ± 1. An a-wave connecting two points (z_1,α_1) and (z_2,α_2) have slope 1 if z_1 and z_2 are non-positive, and slope -1 if these values are non-negative. If z_1 and z_2 have different sign, there is no a-wave connecting these points. Since u-waves connect points with the same a values, these are horizontal lines in the (z,α) plane. Now fix a (small) number $\delta > 0$, and set $\alpha_i = i\delta$, and $z_j = j\delta$, for integers i and j. We define u_0^δ and a^δ as piecewise constant functions, with a finite number of jump discontininuities, such that

$$\left.\begin{aligned} \left\|a - a^\delta\right\|_{L^1(\mathbf{R})} &\to 0,\\ \left\|u_0 - u_0^\delta\right\|_{L^1(\mathbf{R})} &\to 0 \end{aligned}\right\} \quad \text{as } \delta \to 0. \tag{2.2}$$

Label the (finite number of) values of u^δ and a^δ $u_1,\dots,u_M$, and $a_1,\dots,a_N$ respectively. Let α_j be the jth member of the ordered set

$$\{\alpha_k\}_{k=m'}^{M'} \cup \{\alpha(a_k)\}_{k=1}^{M},$$

where m' and M' are chosen such that

$$0 < m' \le \min_x \alpha(a^\delta(x)) < \max_x \alpha(a^\delta(x)) \le M'.$$

For ease of notation, set

$$a_j = \alpha^{-1}(\alpha_j).$$

Next for each α_j, we define $z_{j,k}$ to be the kth member of the ordered set

$$\{z_i\}_{i=-N'(j)}^{N'(j)} \cup \{z(u_i,a_j)\}_{i=1}^{M},$$

where $N'(j)$ is such that

$$z^{-1}\left(z_{-N'(j)},a_j\right) = 0, \quad \text{and} \quad z^{-1}\left(z_{N'(j)},a_j\right) = 1.$$

We also set

$$u_{j,k} = z^{-1}(z_{j,k},a_j), \quad \text{and} \quad f_{j,k} = f(u_{j,k},a_j).$$

Then, for each j, let the approximate flux function $f^\delta(u,a)$ be the piecewise linear interpolant,

$$f^\delta(u,a_j) = f_{j,k} + (u - u_{j,k})\frac{f_{j,k+1} - f_{j,k}}{u_{j,k+1} - u_{j,k}}, \quad \text{for } u \in [u_{j,k}, u_{j,k+1}]. \tag{2.3}$$

We have chosen the grid so that the entropy solution to the initial value problem

$$\begin{aligned} u_t + f^\delta\left(u,a^\delta\right)_x &= 0, \quad t > 0, \quad x \in \mathbf{R}\\ u(x,0) &= u_0^\delta(x), \quad x \in \mathbf{R}, \end{aligned} \tag{2.4}$$

can be constructed by front tracking for any time t. We call this front tracking solution u^δ. Furthermore u^δ will take values that are grid points, i.e., for any point (x,t) such that u^δ and a^δ is constant at (x,t),

$$z\left(u^\delta(x,t), a^\delta(x)\right) = z_{j,k}, \quad \text{for some } j \text{ and } k.$$

In particular, this means that

$$f^\delta\left(u^\delta,a^\delta\right) = f\left(u^\delta,a^\delta\right), \quad \text{almost everywhere.}$$

For an elaboration and proof of these statements, see [14]. The construction used here differs from the construction in [14] in that we have added grid points corresponding to the discretization of the initial function u_0 and the coefficient a, instead of choosing discretization that take values on the fixed grid in the (z,α) plane.

Now we can define the front tracking approximation in the case where g is not constant, c.f. (1.1). Let g^δ be a piecewise constant approximation to g, such that

$$\begin{aligned} &\left\|g^\delta - g\right\|_{L^1(\mathbf{R}^+)} \to 0, \quad \text{as } \delta \to 0, \\ &\left|g^\delta\right|_{BV(\langle 0,T])} \le |g|_{BV(\langle 0,T])}\,. \end{aligned} \tag{2.5}$$

Define t^n such that g^δ is constant on each interval $I^n = \langle t^n, t^{n+1}]$. Assuming that we can define front tracking for $t < t^n$, we can then use $u^\delta(\cdot, t^n)$ as initial values for a front tracking approximation defined in $[t^n, t^{n+1}\rangle$. In order to do this we must use a "new" mapping z, since $z = z(u, a, g)$, and redefine the grid on which we operate. However, we keep the grid points corresponding to $u^\delta(\cdot, t^n)$. In this way, the grid used in the interval I^{n+1} will contain more points than the one used in I^n, but since there are only a finite number of intervals I^n such that $t^n \le T$, for a fixed δ, we use a finite number of grid points for $t \le T$. If, for $t \in I^n$, $f^\delta(\cdot, \cdot, g^\delta(t))$ denotes the approximate flux function constructed above using $f(\cdot, \cdot, g^\delta\,\big|_{I^n})$ and $u^\delta(\cdot, t^n)$, then we have that the front tracking construction u^δ will be an entropy solution of

$$\begin{aligned} u^\delta_t + f^\delta\left(u^\delta, a^\delta(x), g^\delta(t)\right)_x = 0, \quad t > 0, \quad x \in \mathbf{R} \\ u^\delta(x, 0) = u^\delta_0(x), \quad x \in \mathbf{R}. \end{aligned} \tag{2.6}$$

We call the discontinuities in u^δ *fronts*, and we have three types, u-fronts, a-fronts and g-fronts (that have infinite speed!).

3. Compactness

In this section we show that the sequence $\left\{u^\delta\right\}_{\delta>0}$ is compact in L^1, by estimating the variation of $\Psi(u^\delta, a^\delta, g^\delta)$. For each time t, such that g^δ is constant at t, we can view u^δ as consisting of a sequence of fronts, u-fronts and a-fronts.

We defined the map Ψ by (1.2), and we define the associated Temple functional of a front w by

$$T(w) = \begin{cases} |\Delta \Psi| & \text{if } w \text{ is a } u\text{-front}, \\ 2\,|\Delta a|\, g & \text{if } w \text{ is an } a\text{-front, and } \Psi_r < \Psi_l, \\ 4\,|\Delta a|\, g & \text{if } w \text{ is an } a\text{-front, and } \Psi_r > \Psi_l. \end{cases} \tag{3.1}$$

For sequence of fronts, define T additively. Next, for the front tracking approximation u^δ, we define the interaction estimate Q by

$$Q(t) = T(t)\left|g^\delta(\cdot)\right|_{BV([t,T])}, \tag{3.2}$$

where with a slight abuse of notation we write $T(t) = T(u^\delta(\cdot, t))$. With these definitions, we can state the following lemma.

Lemma 3.1. *There exist a positive constant C, depending only on f, a and g, such that for all $t > 0$, we have that the "Glimm functional"*

$$G(t) = T(t) + CQ(t), \tag{3.3}$$

is nonincreasing in time.

Proof. In each interval I^n, we know from [14] that T is non-increasing, and the lemma holds. To prove the lemma we must study interactions between u-fronts and g-fronts, and between a-fronts and g-fronts.

We start by considering the interaction between a single a-front and a single g-front. The states involved are depicted in Figure 1. We label the "incoming" a-wave (front) as a, and the outgoing a-wave as a', the left moving outgoing u-wave as u^- and the right moving outcoming u-wave as u^+. See Figure 1.

In this case we claim that

$$T(u^-) + T(a') + T(u^+) - T(a) \le C\,|\Delta g|\,|\Delta a| \tag{3.4}$$

for some constant C depending on f and its derivatives, but not on δ. To show this, we first observe that since

$$|T(a') - T(a)| \leq 2\,|\Delta a|\,|\Delta g|\,,$$

it suffices to show that

$$T(u^-) + T(u^+) \leq C\,|\Delta g|\,|\Delta a|\,. \tag{3.5}$$

First observe that since an a wave cannot cross the line $z = 0$, either both u_l and u_r are less than

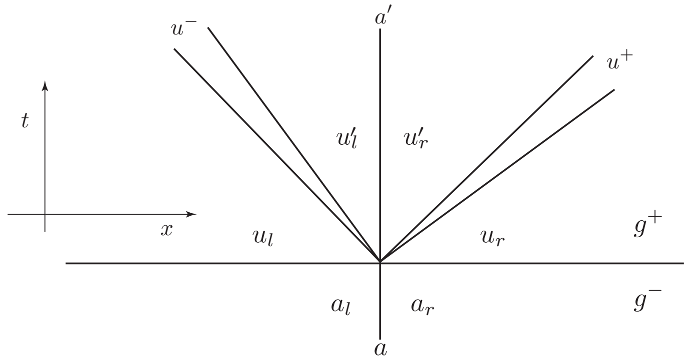

FIGURE 1. The states used in an interaction between an a-wave and a g-wave

or equal to u^*, or both are greater than or equal to u^*. If this is not so, then the "a-wave" is in fact a stationary u-wave followed by an a-wave, or vice versa. If this is so, we can perturb u^δ be an arbitrarily small amount by shifting the stationary u-wave a small distance and then treat the interaction of the g-wave and the u wave separately.

We now let

$$G\left(a_l, a_r, g_l, g_r\right) = T\left(u^-\right) + T\left(u^+\right).$$

For simplicity, we regard a_l and g^- as fixed, and the emerging waves as functions of $a = a_r$ and $g = g^+$. Trivially we have that

$$G(a_l, a_l, g^-, g) = G(a_l, a, g^-, g^-) = 0,$$

and (3.5) follows if G is continuous and

$$\frac{\partial^2 G}{\partial a \partial g}$$

is bounded, since

$$G(a_l, a_r, g^-, g^+) = \int_{a_l}^{a_r}\int_{g^-}^{g^+} \frac{\partial^2 G}{\partial a \partial g}(a_l, a, g^-, g)\,dg da.$$

First we assume that both u_l and u_r are less than or equal to u^*. In this case, if

$$f\left(u_l, a_l, g\right) \leq f\left(u^*, a, g\right), \tag{3.6}$$

then there are no left moving waves u^-, while if

$$f\left(u_l, a_l, g\right) > f\left(u^*, a, g\right), \tag{3.7}$$

there will be emerging u-waves of both positive and negative speed. These two case are depicted in Figure 2. So we find that

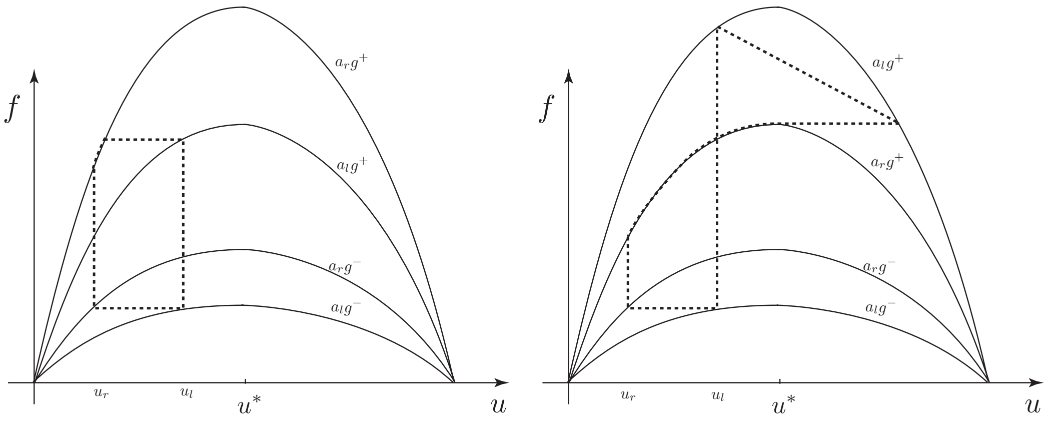

FIGURE 2. The possible results of an interaction if $u_{l,r} \leq u^*$. Left: (3.6) holds. Right: (3.7) holds.

$$
\begin{aligned}
G\left(a_l, a, g^-, g\right) = {} & \frac{\operatorname{sign}\left(u_l - u_r\right)}{f\left(u^*, a, g\right)}\left[f\left(u_l, a_l, g\right) - f\left(u_r, a, g\right)\right]\chi_{\{f(u_l, a_l, g) \leq f(u^*, a, g)\}} \\
& + \left\{\frac{1}{f\left(u^*, a_l, g\right)}\left[f\left(u_l, a_l, g\right) - f\left(u^*, a, g\right)\right]\right. \\
& \qquad \left. + \frac{1}{f\left(u^*, a, g\right)}\left[f\left(u^*, a, g\right) - f\left(u_r, a, g\right)\right]\right\}\chi_{\{f(u_l, a_l, g) > f(u^*, a, g)\}}.
\end{aligned}
$$

From this expression it is straightforward to check that G is sufficiently regular, and (3.5) holds.

The case where $u_{l,r} \geq u^*$ is similar, if

$$
f\left(u_r, a, g\right) \leq f\left(u^*, a_l, g\right), \tag{3.8}
$$

there is only one outgoing u-wave, with negative speed. If

$$
f\left(u_r, a, g\right) > f\left(u^*, a_l, g\right), \tag{3.9}
$$

there are two outgoing u-waves. See Figure 3. In this case we have

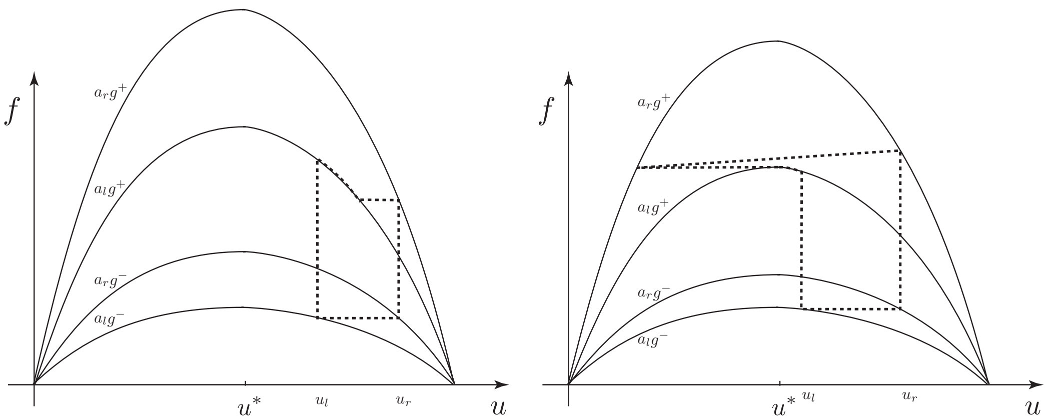

FIGURE 3. The possible results of an interaction if $u_{l,r} \geq u^*$. Left: (3.8) holds. Right: (3.9) holds.

$$G\left(a_l,a,g^-,g\right)=\frac{\operatorname{sign}\left(u_l-u_r\right)}{f\left(u^*,a_l,g\right)}\left[f\left(u_l,a_l,g\right)-f\left(u_r,a,g\right)\right]\chi_{\{f(u_r,a,g)\le f(u^*,a_l,g)\}}$$
$$+\left\{\frac{1}{f\left(u^*,a,g\right)}\left[f\left(u_r,a,g\right)-f\left(u^*,a_l,g\right)\right]\right.$$
$$\left.+\frac{1}{f\left(u^*,a_l,g\right)}\left[f\left(u^*,a_l,g\right)-f\left(u_l,a_l,g\right)\right]\right\}\chi_{\{f(u_r,a,g)>f(u^*,a_l,g)\}}.$$

Also in this case G is sufficiently regular for (3.5) to hold and thereby (3.4). This finishes the study of the interaction of a and g-fronts

Now we consider the interaction of a single u-wave and a single g-wave. The situation is depicted in Figure 4. For this interaction we claim that

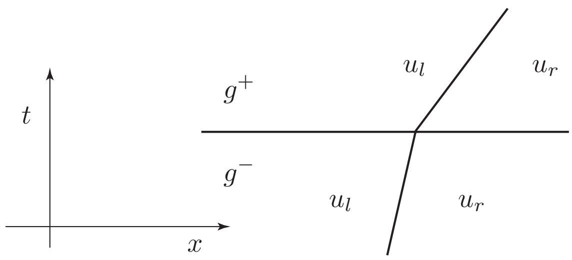

FIGURE 4. The states used in an interaction between a u-wave and a g-wave

$$\begin{aligned}\left|\Psi\left(u_r,a,g^+\right)-\Psi\left(u_l,a,g^+\right)\right|&-\left|\Psi\left(u_r,a,g^-\right)-\Psi\left(u_l,a,g^-\right)\right|\\&\le C\left|g^+-g^-\right|\left|\Psi\left(u_r,a,g^-\right)-\Psi\left(u_l,a,g^-\right)\right|.\end{aligned}\tag{3.10}$$

Since $\Psi(u^*,\cdot,\cdot)=\Psi_u(u^*,\cdot,\cdot)=0$, we can write

$$\begin{aligned}\Psi\left(u_r,a,g^+\right)-\Psi\left(u_l,a,g^+\right)&-\Psi\left(u_r,a,g^-\right)+\Psi\left(u_r,a,g^-\right)\\&=\int_{u_l}^{u_r}\left(\Psi_u\left(\sigma,a,g^+\right)-\Psi_u\left(\sigma,a,g^-\right)\right)\,d\sigma\\&=\int_{u_l}^{u_r}\int_{u^*}^{\sigma}\left(\Psi_{uu}\left(\eta,a,g^+\right)-\Psi_{uu}\left(\eta,a,g^-\right)\right)\,d\eta\,d\sigma\\&=\int_{u_l}^{u_r}\int_{u^*}^{\sigma}\int_{g^-}^{g^+}\Psi_{uug}\left(\eta,a,g\right)\,dg\,d\eta\,d\sigma.\end{aligned}\tag{3.11}$$

We also find that

$$\begin{aligned}\Psi\left(u_r,a,g^-\right)-\Psi\left(u_l,a,g^-\right)&=\int_{u_l}^{u_r}\Psi_u\left(\sigma,a,g^-\right)\,d\sigma\\&=\int_{u_l}^{u_r}\left(\Psi_u\left(\sigma,a,g^-\right)-\Psi_u\left(u^*,a,g^-\right)\right)\,d\sigma\\&=\int_{u_l}^{u_r}\int_{u^*}^{\sigma}\Psi_{uu}\left(\eta,a,g^-\right)\,d\eta\,d\sigma.\end{aligned}\tag{3.12}$$

We also have that

$$\Psi_{uu}(u,a,g) = \operatorname{sign}(u-u^*)\,\frac{-f_{uu}(u,a,g)}{f(u^*,a,g)} \geq \frac{c_{uu}}{C_{u^*}}$$

$$|\Psi_{uug}(u,a,g)| = \left|\frac{f_{uu}(u,a,g)f_g(u^*,a,g) - f_{uug}(u,a,g)f(u^*,a,g)}{f^2(u^*,a,g)}\right| \leq C_1,$$

for some constant C_1. To fix ideas assume that $u_l \leq u_r$, so that also

$$\Psi\left(u_l,a,g^\pm\right) \leq \Psi\left(u_r,a,g^\pm\right).$$

To show (3.10), we consider different cases.

Case 1: $u^* \leq u_l \leq u_r$. By (3.11) we have

$$\begin{aligned}
&\left|\Psi\left(u_r,a,g^+\right) - \Psi\left(u_l,a,g^+\right)\right| - \left|\Psi\left(u_r,a,g^-\right) - \Psi\left(u_l,a,g^-\right)\right| \\
&\qquad \leq C'\left|g^+ - g^-\right| \int_{u_l}^{u_r}\int_{u^*}^{\sigma} d\eta\, d\sigma \\
&\qquad = C'\left|g^+ - g^-\right| \left(\frac{\left|u_r^2 - u_l^2\right|}{2} - u^*\left(u_r - u_l\right)\right),
\end{aligned}$$

for some constant C' depending on the partial derivatives of f. By (3.12) we also have that

$$\Psi\left(u_r,a,g^-\right) - \Psi\left(u_l,a,g^-\right) \geq \frac{c_{uu}}{C_{u^*}} \int_{u_l}^{u_r}\int_{u^*}^{\sigma} d\eta\, d\sigma,$$

so (3.10) follows with $C = C'C_{u^*}/c_{uu}$.

Case 2: $u_l \leq u_r \leq u^*$. In this case,

$$\begin{aligned}
&\left|\Psi\left(u_r,a,g^+\right) - \Psi\left(u_l,a,g^+\right)\right| - \left|\Psi\left(u_r,a,g^-\right) - \Psi\left(u_l,a,g^-\right)\right| \\
&\qquad \leq \hat{C}\left|g^+ - g^-\right| \int_{u_l}^{u_r}\int_{\sigma}^{u^*} d\eta\, d\sigma \\
&\qquad = \hat{C}\left|g^+ - g^-\right| \left[u^*\left(u_r - u_l\right) - \frac{\left|u_r^2 - u_l^2\right|}{2}\right],
\end{aligned}$$

for some constant $\hat{C}$, and also by (3.12)

$$\Psi\left(u_r,a,g^-\right) - \Psi\left(u_l,a,g^-\right) \geq \frac{c_{uu}}{C_{u^*}} \int_{u_l}^{u_r}\int_{\sigma}^{u^*} d\eta\, d\sigma.$$

Hence (3.10) follows as in the first case.

Case 3: $u_l \leq u^* \leq u_r$. Now we write

$$\begin{aligned}
&\left|\Psi\left(u_r,a,g^+\right) - \Psi\left(u_l,a,g^+\right)\right| - \left|\Psi\left(u_r,a,g^-\right) - \Psi\left(u_l,a,g^-\right)\right| \\
&\qquad \leq C_2\left|g^+ - g^-\right| \left[\int_{u_l}^{u^*}\int_{u^*}^{\sigma} d\eta\, d\sigma + \int_{u^*}^{u_r}\int_{\sigma}^{u^*} d\eta\, d\sigma\right] \\
&\qquad = C_2\left|g^+ - g^-\right| \left(\frac{u_r^2 - 2\left(u^*\right)^2 + u_l^2}{2} - u^*\left(u_r - 2u^* + u_l\right)\right).
\end{aligned}$$

Also, by (3.12) we can estimate

$$\Psi\left(u_r,a,g^-\right) - \Psi\left(u_l,a,g^-\right) \geq \frac{c_{uu}}{C_{u^*}} \left[\int_{u_l}^{u^*}\int_{u^*}^{\sigma} d\eta\, d\sigma + \int_{u^*}^{u_r}\int_{\sigma}^{u^*} d\eta\, d\sigma\right].$$

So again (3.10) follows.

If $u_l > u_r$ we can use the same arguments as in Case 1 or 2 above to show (3.10). Since $T(t) \geq \left|a^\delta\right|_{BV}$, the lemma now follows. $\square$

Let $T^n = T \left.\right|_{I^n}$ and $g^n = g^\delta \left.\right|_{I^n}$. Since T is non-increasing in each interval I^n, from Lemma 3.1, we have that

$$T^{n+1} \leq T^n \left(1 + C\left|g^{n+1} - g^n\right|\right).$$

By the Grönwall inequality it follows that

$$\begin{aligned} T(t) &\leq T^1(0+)\exp\left(\sum_n \left|g^n - g^{n-1}\right|\right) \\ &\leq \lim_{s\downarrow 0} T(s)\exp\left(|g|_{BV}\right) \\ &\leq \left(\left|\Psi\left(u_0, a, g(0)\right)\right|_{BV} + 4\left|a\right|_{BV}\left|g(0)\right|\right)e^{|g|_{BV}}. \end{aligned} \tag{3.13}$$

where the sum in the first line above is over those n such that $t_n < t$.

From this it immediately follows that the total variation of $\Psi(u^\delta, a^\delta, g^\delta(t))$ is bounded independently of δ and t. Furthermore

$$-1 \leq \Psi\left(u^\delta(x,t), a^\delta(x), g^\delta(t)\right) \leq 1.$$

By Helly's theorem, for each fixed $t \in [0, T]$,

$$\Psi\left(u^\delta(\cdot,t), a^\delta, g^\delta(t)\right) \to \psi, \quad \text{almost everywhere as } \delta \downarrow 0,$$

and by the Lebsgues dominated convergence theorem also in $L^1(\mathbf{R})$. Furthermore, by a diagonal argument, we can achieve this convergence for a dense countable set $\{t^\gamma\} \subset [0,T]$. For t^γ in this set, define

$$u(\cdot, t^\gamma) = \Psi^{-1}\left(\psi, a, g\left(t^\gamma\right)\right).$$

Hence also $u^\delta(\cdot,t^\gamma) \to u(\cdot,t^\gamma)$. For any $t \in [0,T]$ we have that

$$\begin{aligned} \left\|u^{\delta_1}(\cdot,t) - u^{\delta_2}(\cdot,t)\right\|_{L^1(\mathbf{R})} &\leq \left\|u^{\delta_1}(\cdot,t^\gamma) - u^{\delta_1}(\cdot,t)\right\|_{L^1(\mathbf{R})} \\ &\quad + \left\|u^{\delta_1}(\cdot,t^\gamma) - u^{\delta_2}(\cdot,t^\gamma)\right\|_{L^1(\mathbf{R})} + \left\|u^{\delta_2}(\cdot,t^\gamma) - u^{\delta_2}(\cdot,t)\right\|_{L^1(\mathbf{R})}, \end{aligned}$$

where t^γ is such that $u^\delta(\cdot,t^\gamma) \to u(\cdot,t^\gamma)$. By Lemma 3.2, $t \mapsto u^\delta(\cdot,t)$ is L^1 Lipschitz continuous, so the first and third terms above can be made arbitrarily small by choosing δ_1 and δ_2 small, and the middle term can be made small by choosing t^γ close to t. Hence we have that u^δ converges to some function u in $L^1(\mathbf{R} \times [0,T])$. For the reader's convenience we show that:

Lemma 3.2. *There exists a positive constant C, independent of t, s and δ such that*

$$\left\|u^\delta(\cdot,t) - u^\delta(\cdot,s)\right\|_{L^1(\mathbf{R})} \leq C\left|t-s\right|. \tag{3.14}$$

Proof. We start by noting that since

$$\begin{aligned} &\left|\Psi\left(u^\delta(x,t), a^\delta(x), g^\delta(t)\right) - \Psi\left(u^\delta(y,t), a^\delta(y), g^\delta(t)\right)\right| \\ &\qquad \geq \left|f\left(u^\delta(x,t), a^\delta(x), g^\delta(t)\right) - f\left(u^\delta(y,t), a^\delta(y), g^\delta(t)\right)\right|, \end{aligned}$$

it follows that the total variation of f is bounded by some constant C, and C is independent of t and δ. Next, assume that $0 \leq s < t \leq T$, and let α_h be a smooth approximation to the characteristic function of the interval $[s,t]$, so that

$$\alpha_h \to \chi_{[s,t]}, \quad \text{and} \quad \alpha_h' \to \delta_s - \delta_t,$$

as $h \downarrow 0$, where δ_s denotes the Dirac delta function centered at s. Choose a test function $\varphi(x)$ such that $|\varphi| \leq 1$, and set $\varphi_h(x,t) = \varphi(x)\alpha_h(t)$. Since u^δ is a weak solution we have that

$$\iint_{\Pi_T} u^\delta \partial_t \varphi_h + f\left(u^\delta, a^\delta, g^\delta\right)\partial_x \varphi_h \, dtdx = 0,$$

and sending $h \downarrow 0$ we find that

$$\int_{\mathbf{R}} \varphi(x) \left(u^\delta(x,t) - u^\delta(x,s)\right) \, dx = \int_s^t \int_{\mathbf{R}} \varphi_x(x) f\left(u^\delta, a^\delta, g^\delta\right) \, dt dx.$$

Now

$$\begin{aligned}
\left\|u^\delta(\cdot,t) - u^\delta(\cdot,s)\right\|_{L^1(\mathbf{R})} &= \sup_{|\varphi|\le 1} \int \varphi(x) \left(u^\delta(x,t) - u^\delta(x,s)\right) \, dx \\
&= \sup_{|\varphi|\le 1} \int_s^t \int_{\mathbf{R}} \varphi_x(x) f\left(u^\delta, a^\delta, g^\delta\right) \, dx d\sigma \\
&\le \int_s^t \left|f\left(u^\delta(\cdot,\sigma), a^\delta, g^\delta(\sigma)\right|_{BV} \, d\sigma \\
&\le (t-s)C.
\end{aligned}$$

□

Next, we shall show that the limit u is an entropy solution, first we study how u^δ differs from an entropy solution in each interval $\langle x_m, x_{m+1}\rangle$. Assume that y_k are the discontinuity points of a^δ inside this interval, such that

$$x_m = y_0 < y_1 < \cdots < y_K = x_{m+1},$$

and we have that $a = a_k$ for $x \in \langle y_k, y_{k+1}\rangle$. Since u^δ is an entropy solution inside each interval $\langle y_k, y_{k+1}\rangle$,

$$\int_{y_k}^{y_{k+1}} \int_0^T \left|u^\delta - c\right| \varphi_t + F^\delta(u^\delta, x, t, c) \, dt dx - \int_0^T F^\delta(u^\delta, x, t, c) \Big|_{x=y_k^+}^{x=y_{k+1}^-} \, dt \ge 0, \tag{3.15}$$

where

$$F^\delta(u, x, t, c) = \operatorname{sign}(u-c) \left(f^\delta(u, a^\delta(x), g^\delta(t)) - f^\delta(c, a^\delta(x), g^\delta(t))\right).$$

If we set $y_k^{l,r} = y_k^\mp$, and observe that since u^δ is a weak solution,

$$f\left(u^\delta\left(t, y_k^l\right), a_k, g^\delta\right) = f\left(u^\delta\left(t, y_k^r\right), a_{k+1}, g^\delta\right) =: f_k,$$

for almost all t. Summing (3.15) for $k = 0, \ldots, K-1$, we obtain

$$\int_{x_m}^{x_{m+1}} \int_0^T \left|u^\delta - c\right| \varphi_x + F^\delta\left(u^\delta, x, t, c\right) \varphi_x \, dt dx - \int_0^T \varphi F^\delta(u^\delta, x, t, c) \Big|_{x=x_m^+}^{x=x_{m+1}^-} \, dt \tag{3.16}$$

$$- \int_0^T \sum_{k=1}^{K-1} \varphi(x_k, t) \left[\operatorname{sign}\left(u_k^r - c\right)\left(f_k - f_k^r(c)\right) - \operatorname{sign}\left(u_k^l - c\right)\left(f_k - f_k^l(c)\right)\right] \, dt \tag{3.17}$$

$$\ge 0,$$

where we have used the notation

$$u_k^{l,r} = u^\delta(y_k^\mp, t), \quad f_k^{l,r}(c) = f^\delta(c, a^\delta(y_k^\mp), g^\delta).$$

Since at each discontinuity y_k, u^δ is the solution of a Riemann problem, either *both* u_k^l and u_k^r are less than or equal to u^* or both are greater than or equal to u^*. Using this we can label those discontinuities where both u values are less than or equal to u^* as L, and the remaining ones as

G. Hence we can write the integrand in (3.17) as (for brevity we use a notation where $\varphi(x_k,t)$ is invisible, it will reappear later)

$$-\sum_L \operatorname{sign}\left(u_k^l - c\right)\left(f_k^l(c) - f_k^r(c)\right) + \left[\operatorname{sign}\left(u_k^r - c\right) - \operatorname{sign}\left(u_k^l - c\right)\right]\left(f_k - f_k^r(c)\right) \tag{3.18}$$

$$-\sum_G \operatorname{sign}\left(u_k^r - c\right)\left(f_k^l(c) - f_k^r(c)\right) + \left[\operatorname{sign}\left(u_k^l - c\right) - \operatorname{sign}\left(u_k^r - c\right)\right]\left(f_k^l(c) - f_k\right). \tag{3.19}$$

Since $f_u^\delta(u,\cdot,\cdot) > 0$ for $u < u^*$, if $u_k^l \le c \le u_k^r$, the second term in (3.18) equals

$$2\left(f_k^r(u_k^r) - f_k^r(c)\right) \ge 0,$$

and if $u_r \le c \le u_l$, the second term equals

$$-2\left(f_k^r(u_k^r) - f_k^r(c)\right) \ge 0.$$

Similarly we find that the second term in (3.19) is always non-negative. Hence

$$\begin{aligned}
(3.17) &\le -\int_0^T \sum_{k=1}^{K-1} \operatorname{sign}\left(u_k^{l,r} - c\right)\left(f^\delta(c, a_{k+1}, g^\delta) - f^\delta(c, a_k, g^\delta)\right) dt \\
&= -\int_0^T \sum_{k=1}^{K-1} \operatorname{sign}\left(u_k^{l,r} - c\right) \frac{f^\delta(c, a_{k+1}, g^\delta) - f^\delta(c, a_k, g^\delta)}{\Delta y_k}\, dt\Delta y_k,
\end{aligned}$$

where $\Delta y_k = y_{k+1} - y_k$, and we use u_k^l for discontinuities in L and u_k^r for discontinuities in G. Since a is continuously differentiable in $\langle x_m, x_{m+1}\rangle$ and $a^\delta \to a$, $g^\delta \to g$ and $u^\delta \to u$ as $\delta \downarrow 0$, we find that

$$\lim_{\delta\downarrow 0} (3.17) \ \le -\int_0^T \int_{x_m}^{x_{m+1}} \operatorname{sign}(u - c)\, f(c, a, g)_x \varphi \, dxdt. \tag{3.20}$$

By the same arguments, we find that for each discontinuity x_m

$$\begin{aligned}
F^\delta\left(u^\delta\left(x_m^+, t\right), x_m^+, t, c\right) - F^\delta\left(u^\delta\left(x_m^-, t\right), x_m^-, t, c\right)& \\
= \operatorname{sign}\left(u_m^r - c\right)\left(f_m - f_m^r(c)\right) &- \operatorname{sign}\left(u_m^l - c\right)\left(f_m - f_m^l(c)\right) \\
\ge \left|f_m^l(c) - f_m^r(c)\right|&,
\end{aligned}$$

Finally, adding (3.16) – (3.17) for m and using the above, we find that

$$\begin{aligned}
&\iint_{\Pi_T} |u - c|\,\varphi_t + F(u, x, t, c)\varphi_x \, dtdx - \sum_m \int_{x_m}^{x_{m+1}} \int_0^T \operatorname{sign}(u - c)\, f_a(u, a, g) a'(x) \varphi \, dtdx \\
&\qquad + \sum_m \int_0^T \left|f(c, a(x_m^+, g(t)) - f(c, a(x_m^-), g(t))\right| \varphi(x_m, t)\, dt \\
&\ge \lim_{\delta\downarrow 0} \sum_m \Big[\, (3.16) + (3.17)\, \Big] \\
&\ge 0.
\end{aligned}$$

Hence the limit u is an entropy solution of (1.1), since by the Lemma 3.2 also $u(\cdot,t) \to u_0$ as $t \downarrow 0$. Summing up, we have shown

Theorem 3.1. *Assume that the assumptions **(A.1)**, **(A.2)**, **(A.3)**, **(A.4)**, **(A.5)** and **(A.6)** hold. Then the sequence of front tracking solutions defined in Section 2 converges in $L^1(\Pi_T)$ to an entropy weak solution of* (1.1).

4. Uniqueness

In this section we prove the following theorem:

Theorem 4.1. *Assume that the assumptions (**A.1**), (**A.2**), (**A.3**), (**A.4**), (**A.5**) and (**A.6**) hold. Let $u = u(x,t)$ and $v = v(x,t)$ be two entropy weak solutions of*

$$u_t + f(u,a,g)_x = 0, \quad \textit{in the strip } \Pi_T,$$

for some $T > 0$, satisfying the initial conditions

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), \qquad x \in \mathbf{R}.$$

Then we have that

$$\|u(t,\cdot) - v(t,\cdot)\|_{L^1(\mathbf{R})} \le \|u(s,\cdot) - v(s,\cdot)\|_{L^1(\mathbf{R})}, \tag{4.1}$$

for each $0 \le s \le t < T$.

Proof. We use the convention that $u = u(y,s)$ and $v = v(x,t)$. Assume that φ is a non-negative test function of the variables (x,t,y,s) such that

$$\varphi(x,t,y,s) = 0, \quad \text{for} \quad (x,t,y,s) \in (\langle 0,T\rangle \times [x_m - h_m, x_m + h_m])^2, \quad \text{for } m = 1,\dots,M, \tag{4.2}$$

and for some positive numbers $h_1,\dots,h_M$. Because the support of φ is bounded away from the discontinuities in a, we can proceed (for a while) as in Kružkov's paper [16]. Since u and v are entropy solutions,

$$-\iint\limits_{\Pi_T} |v-u|\,\varphi_t + F(v,x,t,u)\varphi_x\,dtdx + \iint\limits_{\Pi_T} \operatorname{sign}(v-u)\, f_a(u,a(x),g(t))a'(x)\varphi\,dtdx \le 0, \tag{4.3}$$

$$-\iint\limits_{\Pi_T} |u-v|\,\varphi_s + F(u,y,s,v)\varphi_y\,dsdy + \iint\limits_{\Pi_T} \operatorname{sign}(u-v)\, f_a(v,a(y),g(s))a'(y)\varphi\,dsdy \le 0. \tag{4.4}$$

Next we observe that

$$\begin{aligned} F(v,x,t,u)\varphi_x - &\operatorname{sign}(v-u)\, f_a(u,a(x),g(t))\varphi \\ &= \operatorname{sign}(v-u)\,(f(v,a(x),g(t)) - f(u,a(y),g(s)))\,\varphi_x \\ &\qquad - \operatorname{sign}(v-u)\,\partial_x\left[(f(u,a(x),g(t)) - f(u,a(y),g(s)))\,\varphi\right], \end{aligned}$$

and

$$\begin{aligned} F(u,y,s,v)\varphi_y - &\operatorname{sign}(u-v)\, f_a(v,a(y),g(s))\varphi \\ &= \operatorname{sign}(u-v)\,(f(u,a(y),g(s)) - f(v,a(x),g(t)))\,\varphi_y \\ &\qquad - \operatorname{sign}(u-v)\,\partial_y\left[(f(v,a(y),g(s)) - f(v,a(x),g(t)))\,\varphi\right]. \end{aligned}$$

Using these observations in (4.3) and (4.4) we find that

$$\begin{aligned} -\iint\limits_{\Pi_T} &|v-u|\,\varphi_x + \operatorname{sign}(v-u)\left[f(v,a(x),g(t)) - f(u,a(y),g(s))\right]\varphi_x \\ &+ \operatorname{sign}(v-u)\,\partial_x\left[f(u,a(x),g(t)) - f(u,a(y),g(s))\right]\varphi\,dtdx \le 0, \end{aligned} \tag{4.5}$$

$$\begin{aligned} -\iint\limits_{\Pi_T} &|u-v|\,\varphi_y + \operatorname{sign}(u-v)\left[f(u,a(y),g(s)) - f(v,a(x),g(t))\right]\varphi_y \\ &+ \operatorname{sign}(u-v)\,\partial_y\left[f(v,a(y),g(s)) - f(v,a(x),g(t))\right]\varphi\,dsdy \le 0. \end{aligned} \tag{4.6}$$

Integrating these two with respect to $dsdy$ and $dtdx$ and adding the results, we find

$$\begin{aligned}\iiiint\limits_{\Pi_T^2} & |v-u|\,(\varphi_t+\varphi_s)+\operatorname{sign}(v-u)\,[f(v,a(x),g(t))-f(u,a(y),g(s))]\,(\varphi_x+\varphi_y) \\ & +\operatorname{sign}(v-u)\Big[\partial_x\left(\left(f(u,a(x),g(t))-f(v,a(y),g(s)\right)\varphi\right) \\ & \qquad +\partial_y\left(\left(f(v,a(x),g(t))-f(v,a(y),g(s))\right)\varphi\right)\Big]\,dtdx\,dsdy\le 0.\end{aligned} \tag{4.7}$$

Let $\psi(z,r)$ be a non-negative test function with support bounded away from the jumps in a, and let δ_ρ be a standard approximate Dirac delta function in one variable. Now choose

$$\varphi(x,t,y,s)=\psi\left(\frac{x+y}{2},\frac{t+s}{2}\right)\delta_\rho\left(\frac{x-y}{2}\right)\delta_\rho\left(\frac{t-s}{2}\right).$$

Since the discontinuities in a does influence the above inequality (4.7), we can repeat the arguments in [16] to show that

$$\iint\limits_{\Pi_T}|v(x,t)-u(x,t)|\,\psi_x+F(v(x,t),a(x),g(t),u(x,t))\psi_x\,dtdx\ge 0. \tag{4.8}$$

Next, we have to consider the discontinuity points of a. To this end let $\chi_h(x)$ denote the characteristic function of the interval $\langle -h,h\rangle$, and set

$$\mu_h=1-\delta_{h/2}*\chi_h,\qquad \Theta_h(x)=\sum_{m=1}^M\mu_h\left(x-x_m\right).$$

Now $\Theta_h\to 1$ in $L^1(\mathbf{R})$ as $h\downarrow 0$. Let φ be a non-negative test function and set

$$\psi_h=\varphi\cdot\Theta_h.$$

Then ψ_h is an admissible test function in (4.8). By the Lebesgue dominated convergence theorem, we have that

$$\lim_{h\downarrow 0}\iint\limits_{\Pi_T}|u-v|\,\partial_t\psi_h\,dtdx=\iint\limits_{\Pi_T}|u-v|\,\varphi_x\,dtdx \tag{4.9}$$

$$\begin{aligned}\lim_{h\downarrow 0}\iint\limits_{\Pi_T}F(v,x,t,u)\partial_x\psi_h\,dtdx=&\iint\limits_{\Pi_T}F(v,x,t,u)\varphi_x\,dtdx\\ &+\int_0^T\sum_{m=1}^M F(v,x,t,u)\,\Big|_{x=x_{m^-}}^{x=x_m^+}\,\varphi(x_m,t)\,dt.\end{aligned} \tag{4.10}$$

Hence, with this choice of test function, (4.8) reads

$$\begin{aligned}\iint\limits_{\Pi_T}&|v(x,t)-u(x,t)|\,\varphi_x+F(v(x,t),a(x),g(t),u(x,t))\varphi_x\,dtdx\\ &\ge-\int_0^T\sum_{m=1}^M F(v,x,t,u)\,\Big|_{x=x_{m^-}}^{x=x_m^+}\,\varphi(x_m,t)\,dt.\end{aligned} \tag{4.11}$$

Now we shall prove that

$$\int_0^T\sum_{m=1}^M F(v,x,t,u)\,\Big|_{x=x_{m^-}}^{x=x_m^+}\,\varphi(x_m,t)\,dt\le 0. \tag{4.12}$$

Next choose some $t \in \langle 0, T\rangle$ such that g is continuous at t, and a discontinuity point of a x_m. Set

$$f^u = \lim_{x\to x_m^+} f(u(x,t), a(x), g(t)) = \lim_{x\to x_m^-} f(u(x,t), a(x), g(t)),$$

and

$$f^v = \lim_{x\to x_m^+} f(v(x,t), a(x), g(t)) = \lim_{x\to x_m^-} f(v(x,t), a(x), g(t)).$$

Furthermore, we use the notation $u_{l,r} = \lim_{x\to x_{m}\mp} u(x,t)$, $v_{l,r} = \lim_{x\to x_m^\mp} v(x,t)$ (these limits exist due to ((**D.4**))), and set

$$f_{l,r}(u) = f(u, a_{l,r}, g), \quad \text{and} \quad F_{l,r}(u,c) = \operatorname{sign}(u-c)\,(f_{l,r}(u) - f_{l,r}(c)).$$

Now, (4.12) will follow if we show that

$$\begin{aligned} A &= F_l\,(u_l, v_l) - F_r\,(u_r, v_r) \\ &= [\operatorname{sign}(u_l - v_l) - \operatorname{sign}(u_r - v_r)]\,(f^u - f^v) \geq 0. \end{aligned} \tag{4.13}$$

Now for any entropy solution w with left and right limits w_l and w_r we have that the following implications hold

$$a_l < a_r \Rightarrow \begin{cases} f^w \leq f_r(c) & \text{for } w_l \leq c \leq w_r, \\ f^w \geq f_l(c) & \text{for } w_r \leq c \leq w_l, \end{cases} \tag{4.14}$$

$$a_r < a_l \Rightarrow \begin{cases} f^w \leq f_l(c) & \text{for } w_l \leq c \leq w_r, \\ f^w \geq f_r(c) & \text{for } w_r \leq c \leq w_l. \end{cases} \tag{4.15}$$

From (4.13), $A = 0$ if $\operatorname{sign}(u_l - v_l) = \operatorname{sign}(u_r - v_r)$, therefore we only have to consider the cases where this is not so. If $\operatorname{sign}(u_l - v_l) \neq \operatorname{sign}(u_r - v_r)$ either $\{v_l, v_r\} \subset [\min(u_l, u_r), \max(u_l, u_r)]$, or $\{u_l, u_r\} \subset [\min(v_l, v_r), \max(v_l, v_r)]$. In all these cases we can easily use (4.14) or (4.15) to show that $A \geq 0$. As an example, assume that $v_l < u_{l,r}$ and $u_{l,r} < v_r$. Then, since v is an entropy solution

$$A = 2(f^u - f^v) = \begin{cases} f_r\,(u_r) - f^v \geq 0 & \text{if } a_l < a_r, \text{ by (4.14)}, \\ f_l\,(u_l) - f^v \geq 0 & \text{if } a_l > a_r, \text{ by (4.15)}. \end{cases}$$

The other cases are considered similarly. Now we have established that for each non-negative test function φ,

$$\iint_{\Pi_T} |u - v|\,\varphi_t + F(u(x,t), a(x), g(t), v(x,t))\varphi_x\,dtdx \geq 0,$$

and it is then standard procedure, see e.g., [16], to show that this implies L^1 contraction. □

5. An example

To demonstrate that the front tracking construction also has some potential as a practical numerical method, in this section we show an example of how the front tracking construction works on a concrete example. We use the simple flux function

$$f(u, a, g) = 4agu(1-u), \tag{5.1}$$

and

$$\begin{aligned} a(x) &= \left(1 + 8\chi_{\{x<0.25\}}\,|x|\right) + 2\cos^2(\pi x) \\ g(t) &= 0.6 + 0.55\cos\left(2\pi t\left(\chi_{\{0.15<t<0.65\}} + 1\right)\right) \\ u_0(x) &= 0.5(1 - 0.8\sin(\pi x)). \end{aligned} \tag{5.2}$$

In the example we present, we have used periodic initial data in the interval $x \in [0,1]$, $\delta = 1/20$ and $\Delta t = 0.025$. In Figure 5 we show the initial data u_0^δ and the approximate coefficients a^δ) and g^δ. In the next figure (Figure 6 we show the front tracking solution at $t = 0.5$ and the fronts for $0 \leq t \leq 0.5$. The a-fronts are depicted as broken horizontal lines, the g-fronts as vertical broken lines and the u-fronts as solid lines.

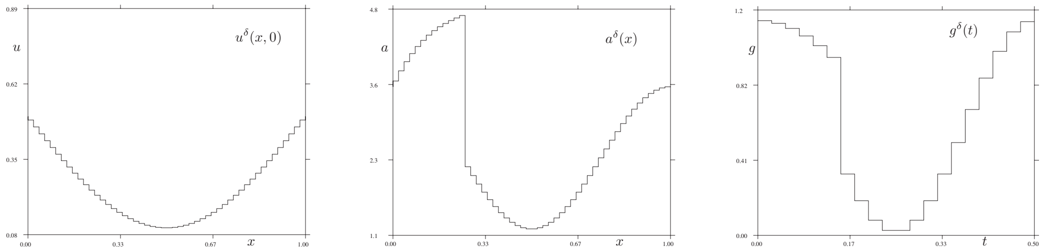

FIGURE 5. The initial function u_0^δ (left), the coefficient a^δ (middle) and g^δ (right).

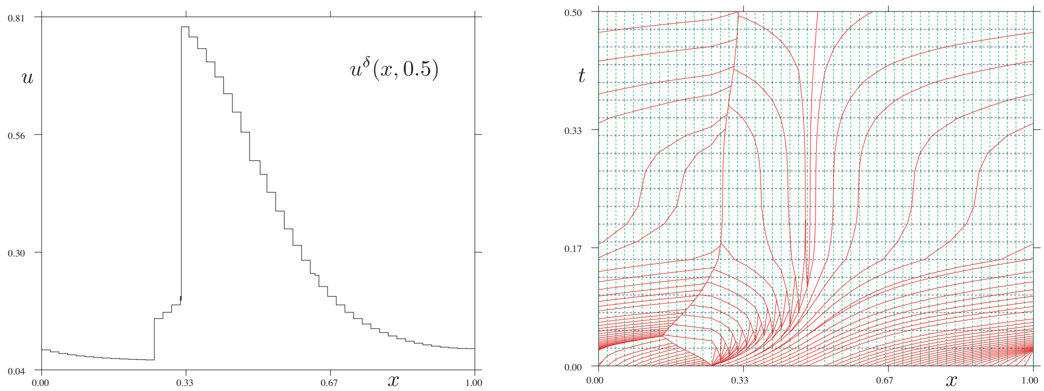

FIGURE 6. The front tracking solution $u^\delta(x,0.5)$ (left) and the fronts (right).

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(Giuseppe Maria Coclite)
SISSA–ISAS
via Beirut 2–4
34014 Trieste
Italy
E-mail address: `coclite@sissa.it`

(Nils Henrik Risebro)
Department of Mathematics
University of Oslo
P.O. Box 1053, Blindern
N–0316 Oslo, Norway
E-mail address: `nilshr@math.uio.no`