POINTWISE GREEN'S FUNCTION BOUNDS FOR SHOCK PROFILES OF SYSTEMS WITH REAL VISCOSITY

Corrado Mascia and Kevin Zumbrun

July 11, 2001; Revised: December 17, 2002

Abstract. Following the pointwise semigroup approach of [ZH,MZ.1], we establish sharp pointwise Green's function bounds and consequent linearized stability for viscous shock profiles of general hyperbolic–parabolic systems of conservation laws of dissipative type, under the necessary assumptions ([Z.1,Z.3–4]) of spectral stability, i.e., stable point spectrum of the linearized operator about the wave; transversality of the connecting profile; and hyperbolic stability of the corresponding ideal shock of the associated inviscid system, with no additional assumptions on the structure or strength of the shock. These bounds are used in a companion paper [MZ.2] to establish nonlinear stability of small-amplitude Lax shocks of symmetrizable hyperbolic–parabolic systems.

Section 1. Introduction

In this paper, extending results obtained in [ZH] and [MZ.1] for artificial viscosity and relaxation systems, we derive detailed pointwise Green's function bounds for the linearized evolution equations about viscous shock solutions, or "shock profiles," of systems of conservation laws with "real," or partially parabolic, viscosity. These yield a necessary and sufficient *Evans function condition* for linearized $L^1 \cap L^p \to L^p$ stability, which is in turn equivalent to spectral stability (i.e., stable point spectrum of the linearized operator about the wave), plus hyperbolic stability of the corresponding ideal shock of the associated inviscid system, plus transversality of the connecting profile [Z.3–4]. The Evans function condition has been verified for small-amplitude profiles in [HuZ,PZ]. However, in general it may fail, since examples of inviscid instability are well known; for further discussion, see [Z.3]. In a companion paper [MZ.2], we use the linearized decay estimates derived here to establish a full nonlinear stability result for small-amplitude Lax profiles of systems of the symmetrizable hyperbolic–parabolic class identified by Kawashima [Kaw], including in particular the compressible Navier–Stokes equations and the equations of magnetohydrodynamics (MHD).

1.1. Equations and assumptions. Following [Z.3–4], consider a general hyperbolic–parabolic system of conservation laws

$$U_t + F(U)_{x_j} = (B(U)U_x)_x \tag{1.1}$$

The first author thanks Indiana University and Scholarship CNR n.203.01.68 (07/05/97) for their hospitality and for making possible the postdoctoral visit (January–August 1999) during which this work was initiated and partly carried out. The second author thanks the Instituto per le Applicazioni del Calcolo "M. Picone" (CNR) and European TMR Project "Hyperbolic Systems of Conservation Laws" for their hospitality and for making possible the visit (April 10–May 10, 2000) during which the main part of the work was completed. Research of the first author was supported in part by European TMR Project "Hyperbolic Systems of Conservation Laws". Research of the second author was supported in part by the National Science Foundation under Grants No. DMS-9107990 and DMS-0070765.

of the form

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \ F^0 = \begin{pmatrix} f \\ g \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 \\ b_1 & b_2 \end{pmatrix}, \tag{1.2}$$

u, $f \in \mathbb{R}^{n-r}$, v, g, $b \in \mathbb{R}^r$, and a standing wave solution, or *viscous shock profile*

$$U = \bar{U}(x), \ \lim_{y \to \pm\infty} U(y) = U_\pm = (u_\pm, v_\pm), \tag{1.3}$$

under the standard assumptions:

(H0) F, $B \in C^5$.

(H1) (i) Re $\sigma(b_2) > 0$, and (ii) $\binom{df}{b}$ full rank on $\{\bar{U}(\cdot)\}$; moreover, (iii) $\sigma(f_u - b_1(b_2)^{-1} f_v)$ real, semisimple, constant multiplicity, and nonzero.

(H2) $\sigma\left(dF(U_\pm)\right)$ real, distinct, and nonzero.

(H3) Re $\sigma\left(i\xi_j dF(U_\pm) - \xi^2 B(U_\pm)\right) \le -\frac{\theta|\xi|^2}{1+|\xi|^2}$, $\theta > 0$, for $\xi \in \mathbb{R}$.

(H4) In the vicinity of $\bar{U}$, the solutions of (1.1)–(1.3) form a smooth manifold $\{\bar{U}^\delta\}$, $\delta \in B(0,r) \subset \mathbb{R}^\ell$, with $\bar{U}^0 = \bar{U}$.

Here, (H0) ensures the regularity needed for our later analysis; (H1)(i)–(iii) are standard conditions associated with local well-posedness of (1.1)–(1.2) [Kaw]; (H2) expresses strict hyperbolicity and noncharacteristicity of endstates $U_\pm$ with respect to the associated inviscid system

$$U_t + F(U)_x = 0; \tag{1.4}$$

(H3) is the dissipativity condition of Kawashima [Kaw], concerning linearized stability of the constant solutions $U \equiv U_\pm$; and (H4) is a sort of weak transversality condition, implied by but not implying transversality of the shock profile as a solution of the standing wave ODE. Condition (H0) is not optimal, being designed rather to maximize the clarity of our high-frequency analysis. At the expense of further complication, we expect that the analysis can be carried through with the same minimal regularity F, $B \in C^3$ assumed in the relaxation case [MZ.1] (minimal in the sense that it is required for our method, not necessarily in the mathematical sense).

Condition (H1)(ii) may be motivated also by consideration of the standing wave ODE

$$f^1(u,v) \equiv f^1(u_-, v_-), \tag{1.5}$$

$$b_1^{11} u' + b_2^{11} v' = g^1(u,v) - g^1(u_-, v_-). \tag{1.6}$$

For, (H1)(ii) is readily seen to be the condition that (1.6) describes a nondegenerate ODE on the r-dimensional manifold described by (1.5): well-defined, by the Implicit

Function Theorem plus full rank of df. Similarly, condition (H1)(iii) arises naturally in the analysis of the eigenvalue ODE. In the commonly occurring case $r = n - 1$, as, e.g., in gas dynamics and MHD, reality and simplicity of $\sigma(df_u - b_1(b_2)^{-1} df_v)$ are automatic, and so (H1)(iii) and (H1)(ii) are equivalent, given (H1)(i), by the determinant identity

$$\det \begin{pmatrix} f_u & f_v \\ b_1 & b_2 \end{pmatrix} = \det\,(f_u - b_1(b_2)^{-1} f_v) \det\, b_2.$$

This holds likewise in the case that dF and B are simultaneously symmetrizable, which may be reduced by coordinate transformation (see Remark 8.3, [Z.3]) to the case that dF and B are symmetric and thus $b_1 = 0$ and $f_u - b_1(b_2)^{-1} f_v = f_u$ is symmetric. (H3)(iii) always implies (H2)(ii).

We point out, finally, that (H1)(ii), and thus all of hypothesis (H1), is satisfied for gas dynamics and MHD precisely when particle and shock velocities are distinct, which is always the case along a shock; for a study of viscous profiles in these contexts, see [Gi,CS,Fre.1,FreS]. Existence of small-amplitude profiles for general systems (1.1)–(1.2) has been shown under assumptions (H0)–(H3) by Pego [P.1], using center manifold techniques; see also the result of Freistühler [Fre.2] for systems of Kawashima class. These profiles are transverse by their construction, hence satisfy (H4) as well.

1.2. Structure and classification of profiles. We begin by stating some general results regarding structure and classification of profiles, analogous to those proved in [MP] and [MZ.1,Z.4] in the strictly parabolic and relaxation case. Proofs may be found in Appendix A; see also [Z.4,Ly,LyZ].

Lemma 1.1. *Given (H1)–(H3), the endstates $U_\pm$ are hyperbolic rest points of the ODE determined by (1.6) on the r-dimensional manifold (1.5), i.e., the coefficients of the linearized equations about $U_\pm$, written in local coordinates, have no center subspace. In particular, under regularity (H0), standing wave solutions (1.3) satisfy*

$$|(d/dx)^k \big((\bar u(x), \bar v(x)) - (u_\pm, v_\pm)\big)| \le C e^{-\theta |x|}, \quad k = 0, \dots, 4, \tag{1.7}$$

as $x \to \pm\infty$.

Lemma 1.1 verifies the hypotheses of the Gap Lemma, [GZ,ZH][1], on which we shall rely to obtain the basic ODE estimates underlying our analysis in the low-frequency/large-time regime.

Let $i+$ denote the dimension of the stable subspace of $Df^1(u_+)$, i_- denote the dimension of the unstable subspace of $Df^1(u_-)$, and $i := i_+ + i_-$. Indices $i_\pm$ count the number of incoming characteristics from the right/left of the shock, while i counts the total number of incoming characteristics toward the shock. Then, the

[1]See also the version established in [KS], independently of and simultaneously to that of [GZ].

hyperbolic classification of $\bar{u}(\cdot)$, i.e., the classification of the associated hyperbolic shock (U_-, U_+), is:

$$\begin{cases} \text{Lax type} & \text{if} \quad i = n+1, \\ \text{Undercompressive (u.c.)} & \text{if} \quad i \le n, \\ \text{Overcompressive (o.c.)} & \text{if} \quad i \ge n+2. \end{cases}$$

In case all characteristics are incoming on one side, i.e. $i_+ = n$ or $i_- = n$, a shock is called *extreme*.

As in the strictly parabolic and relaxation cases, there is a close connection between the hyperbolic type of a shock and the nature of the corresponding connecting profile. Considering the standing wave ODE as an r-dimensional ODE on manifold (1.5), let us denote by $1 \le d_\pm \le r$ the dimensions of stable manifold at U_+ and unstable manifold at U_-, and $d := d_+ + d_-$. Then, we have:

Lemma 1.2 [Z.4]. *Given (H1)–(H3), there hold relations*

$$\begin{aligned} n - i_+ &= r - d_+ + \dim \mathcal{U}(A_{*+}), \\ n - i_- &= r - d_- + \dim \mathcal{S}(A_{*-}), \end{aligned} \tag{1.8}$$

where $A_* := f_u - f_v(b_2)^{-1}b_1$, and $\mathcal{U}(M)$ and $\mathcal{S}(M)$ denote unstable and stable subspaces of a matrix M. In particular, existence of a connecting profile implies $n - i = r - d$.

The final assertion implies that the "parabolic" and "hyperbolic" types of shock connections agree, i.e. Lax, under-, and overcompressive designations imply corresponding information about connections. The more detailed information (1.8) implies that "extreme" shocks have "extreme" connections:

Corollary 1.3 [Z.4]. *For (right) extreme shocks, $i_+ = n$, there holds also $d_+ = r$, i.e. the connection is also extreme, and also $\dim \mathcal{U}(A_*) \equiv 0$.*

Proof. Immediate from (1.8), using $d_+ \le r$ and $\dim \mathcal{U}(A_{*+}) \ge 0$. □

A complete description of the connection, of course, requires the further index ℓ defined in (H4) as the dimension ℓ of the connecting manifold between $(u_\pm, v_\pm)$ in the traveling wave ODE. Generically, one expects that ℓ should be equal to the surplus $d - r = i - n$. In case the connection is "dimensionally transverse" in this sense, i.e.:

$$\ell = \begin{cases} 1 & \text{undercompressive or Lax case,} \\ i - n & \text{overcompressive case,} \end{cases} \tag{1.9}$$

we call the shock "pure" type, and classify it according to its hyperbolic type; otherwise, we call it "mixed" under/overcompressive type. Throughout the paper, we shall assume that (1.9) holds, so that all viscous profiles are of *pure, hyperbolic type*. This holds in particular for the weak profiles whose existence was established in [P.1,Fre.2], which are always of pure Lax type.

Remarks. 1. The relation $0 \leq \dim \mathcal{U}(A_+) - \dim \mathcal{U}(A_{*+}) = (r - d_+) \leq r$, may be viewed as a sort of subcharacteristic relation, by analogy with the relaxation case. Indeed, as described in [Z.4], it may be used to obtain the full subcharacteristic condition

$$a_j < a_j^* < a_{j+r},$$

as a consequence of linearized stability of constant states, where a_j denote the eigenvalues of $A := dF$ and a_j^* those of A_*; this is closely related to arguments given by Yong (see, e.g., [Yo.4]) in the relaxation case. The subcharacteristic relation goes the "wrong way" in the relaxation case, and so one cannot conclude a result analogous to Corollary 1.3.

2. In the special case $r = 1$, there holds $d_+ = d_- = 1$ whenever there is a connection, and so profiles are always of Lax type $n - i = r - d = -1$. This recovers the observation of Pego [P.2] in the special case of isentropic gas dynamics that (undercompressive type) phase-transitional shocks cannot occur under the effects of viscosity alone.

3. For extreme indices $d_+ = r$ or $d_- = r$, profiles if they exist are always transverse, and likewise for indices that are "minimal" in the sense that $d_+ = \ell$ or $d_- = \ell$. In particular, *extreme Lax or overcompressive connections (if they exist) are always transverse*, hence satisfy (H4). Undercompressive shocks are never extreme.

1.3. Linearized Stability and the Evans function. Linearizing (1.1) about the stationary solution $\bar{U} = (\bar{u}, \bar{v})$, we obtain the linearized equations

$$U_t = LU := -(AU)_x + (BU_x)_x, \tag{1.10}$$

A, $B \in C^2$, where

$$\begin{pmatrix} 0 & 0 \\ b_1 & b_2 \end{pmatrix} := B(\bar{U}) \tag{1.11}$$

and

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} U := dF(\bar{U})U - dB(\bar{U})(U, \bar{U}_x) : \tag{1.12}$$

in particular,

$$(A_{11}, A_{12}) = df(\bar{U}). \tag{1.13}$$

By (H4), there exists an ℓ-parameter family of nearby stationary solutions to $\bar{U}$, $\ell \geq 1$, precluding asymptotic stability of $\bar{U}$. The appropriate notion of stability of shock profiles is, rather, *orbital stability*, or convergence to the manifold of stationary solutions $\{(\bar u^\delta, \bar v^\delta)\}$ defined in (H4). Likewise, the relevant notion of linearized stability is that of *linearized orbital stability*, defined as convergence to the *tangent stationary manifold*, Span $\{(\partial/\partial\delta_j)(\bar u^\delta, \bar v^\delta)\}$ of stationary solutions of the linearized equations (1.10) [ZH]. Note that differentiation with respect to δ_j of the standing wave ODE $F(\bar U^\delta)_x = (B(\bar U^\delta)\bar U^\delta_x)_x$ indeed yields $L\partial\bar U^\delta/\partial\delta_j = 0$, as claimed. In the Lax or undercompressive case, $\ell = 1$, the stationary manifold is just the set of translates $\bar U^\delta(x) := \bar U(x-\delta)$ of $\bar U$, and the tangent stationary manifold Span $\{\bar U_x\}$ corresponds to the usual translational zero eigenvalue arising in the study of stability of traveling waves.

An evident necessary condition for linearized orbital stability, (and thus for nonlinear orbital stability, appropriately defined) is *strong spectral stability*,

(D1) $\qquad \sigma(L) \cap \{\text{Re}\,\lambda \geq 0\} = \{0\},$

where σ refers to the spectrum of L with respect to some L^p, say L^2 for simplicity. By a standard result of Henry [He] (D1), given linearized stability of the constant solutions $U \equiv U_\pm$, as implied by (H3), is equivalent to nonnegativity of the point spectrum of L:

$$\text{Re}\,\sigma_p(L) \cap \{\text{Re}\,\lambda \geq 0\} = \{0\}; \tag{1.14}$$

for further discussion, see, e.g., [GZ,ZH,MZ.1].

In the "standard" semigroup case that L were sectorial and 0 were an isolated eigenvalue, one could conclude by standard results (see, e.g., [He,Z.1]) that necessary and sufficient conditions for *linearized and nonlinear orbital stability (in addition, with time-exponential rate)* would be (1.14) augmented with the requirement

$$\lambda = 0 \text{ is an } \ell\text{-fold eigenvalue of } L, \tag{1.15}$$

where ℓ is the dimension of the tangent stationary manifold.

However, as in the relaxation case treated in [MZ.1], neither sectoriality nor isolation of the zero eigenvalue hold here, and therein lies the interest of our analysis. In particular, decay is time-algebraic and not -exponential, and the careful description of behavior becomes important in the proof of stability. Nonetheless, following [ZH,MZ.1], we can recover a weak version of (1.14)–(1.15) based on the *Evans function* of [E.1–4,AGJ,GZ].

An analytic function playing a role for differential operators analogous to that played for finite-dimensional operators by the characteristic polynomial, the Evans function $D(\lambda)$ has the property that, on the resolvent set of L, its zeroes agree (in both location and multiplicity) with the eigenvalues of L [AGJ,GJ.1–2]. A general construction of the Evans function for operators of nondegenerate type is given in [AGJ]. An Evans function was constructed for the mixed-type operators arising in general real viscosity systems in [Z.3]; see also [SZ]. An important property of the Evans function is that it can be analytically extended to a neighborhood of $\lambda = 0$

using the Gap Lemma of [GZ,KS], that is, into the essential spectrum of L. For a detailed discussion of the Evans function and its properties, see Sections 3 and 7.

Taking the correspondence with zeroes of D as the definition of an extended, "effective" point spectrum, we can thus extend the notion of eigenvalue past the essential spectrum boundary of L, via the analytic extension of the Evans function as in [ZH,MZ.1]. The associated effective eigenprojection and eigenfunctions may then be defined in terms of the induced meromorphic extension of the resolvent kernel, via calculus of residues [ZH]; in particular, the dimension of the associated eigenspace is found as in the standard case to be equal to the order to which the Evans function vanishes. For details, see Section 7.4, below.

1.4. Result I: Linearized stability condition. The first main result of this paper, generalizing corresponding results of [ZH] and [MZ.1] for strictly parabolic and relaxation systems, is:

Theorem 1.4. *Under assumptions (H0)–(H3), shocks of "pure" hyperbolic type are $L^1 \cap L^p \to L^p$ linearly orbitally stable for $p > 1$ if and only if L has precisely ℓ effective eigenvalues in $\{\mathrm{Re}\ \lambda \geq 0\}$ (necessarily at $\lambda = 0$), or, equivalently, there holds the Evans function condition:*

($\mathcal{D}$) $\quad D(\cdot)$ has precisely ℓ zeroes in $\{\mathrm{Re}\ \lambda \geq 0\}$ (necessarily at $\lambda = 0$).

As pointed out in [ZH], the apparently simple criterion ($\mathcal{D}$) in fact hides considerable subtlety in behavior, in particular in the critical case that (D1) holds, but ($\mathcal{D}$) does not: see [ZH], section 12 for examples of possible behaviors in this case.

Evidently, ($\mathcal{D}$) is equivalent to the pair of conditions:

($\mathcal{D}$1) $\quad D(\cdot)$ has no zeroes in $\{\mathrm{Re}\ \lambda \geq 0\} \setminus \{0\}$.
($\mathcal{D}$2) $\quad$ (ii) $(d/d\lambda)^\ell D(0) \neq 0$.

The first condition may be recognized as the strong spectral stability condition (D1), above. The second has been shown in [Z.4], Appendix A1 to be equivalent to

(D2) $\quad \bar\Delta := \gamma\Delta \neq 0,$

where γ is a coefficient measuring transversality of the connecting manifold between $(u_\pm, v_\pm)$ (i.e., $\gamma \neq 0$ equivalent to transversality of the intersection of unstable/stable manifolds at $(u_-, v_-)/(u_+, v_+)$), and Δ is the one-dimensional "hyperbolic stability coefficient" defined in [ZS,Z.3], appropriate to the type (Lax, undercompressive, or overcompressive) of the wave: that is, ($\mathcal{D}$2) is equivalent to transversality of the connecting profile plus hyperbolic stability of the corresponding ideal shock of inviscid system (1.4).

For example, in the Lax case, Δ is just the Liu-Majda determinant

$$\Delta := \det\ (r_1^-, \dots, r_{n-i_-}^-, r_{i_++1}^+, \dots, r_n^+, [U]), \tag{1.16}$$

for which $\Delta \neq 0$ is equivalent to hyperbolic stability [M.1–3]; here $r_j^\pm$ are the eigenvectors of $dF(u_\pm)$, in order of increasing eigenvalue, and $i_\pm$ are as defined above (Section 1.2). For the generalizations of (1.16) in the undercompressive and overcompressive cases, and their interpretations with respect to hyperbolic theory, see [ZS] or [Z.4]. (Note: the overcompressive version is cited also in (7.54)–(7.55) below).

To summarize, linearized stability, $(\mathcal{D})$, is equivalent to strong spectral stability, plus transversality of the connecting profile, plus hyperbolic stability of the associated ideal shock. In the physical literature, transversality, $\gamma \neq 0$, is referred to as *structural stability* (i.e., stability of inner shock structure), and evolutionary (hyperbolic) stability $\Delta \neq 0$ as *dynamical stability* [BE]. Strong spectral stability, (D1), is an aspect that has not been explored, and is an important direction for further investigation.

Corollary 1.5. *Under assumptions (H0)–(H3), linear $L^1 \cap L^p \to L^p$ orbital stability of pure shocks, for $p > 1$, is equivalent to (D1)–(D2), or equivalently: strong spectral stability plus classical structural and dynamical stability.*

Remark 1.6. In [Z.3–Z.4], there was derived the *necessary* condition for stability of $\gamma\Delta \geq 0$, with appropriate normalization of γ, Δ. For $\gamma\Delta \neq 0$, the sign of $\gamma\Delta$ is associated with a *stability index* determining the parity of the number of unstable eigenvalues $\operatorname{Re} \lambda > 0$ of L, hence yields information intermediate to that given in (D1) and (D2). The advantage of this reduced condition is that in the extreme, Lax shock case, the sign of the quantity on the lefthand side is explicitly computable as the product of (1.16) and a Wronskian determinant involving only linear algebraic quantities associated with the limiting coefficient matrices at $x = \pm\infty$; see see [Z.4,Ly,LyZ]. (Indeed, it satisfies a formula identical to that of the strictly parabolic case, though this is far from obvious in the general case.) For history/applications/further discussion of the stability index, see e.g. [GZ,BSZ,ZS,Z.4]. Note, with the result of Theorem 1.4, that this condition is strengthened to $\gamma\Delta > 0$.

Remark 1.7. In the case discussed in [P.1], of weak shock profiles arising in regions where the inviscid system (1.4) is strictly hyperbolic, profiles are necessarily of Lax type, and transversal, $\gamma \neq 0$; moreover, the Liu-Majda condition $\Delta \neq 0$ is automatically satisfied, from standard hyperbolic considerations [M.1–3]. Thus, there holds (D2). Likewise, strong spectral stability, (D1), has been shown for Kawashima class systems using energy methods [HuZ] and in the general case by direct, Evans function techniques [PZ]. Thus, *linearized orbital stability holds always for sufficiently small-amplitude profiles, under assumptions (H0)–(H3).*

1.5. Result II: Pointwise Green's Function Bounds. Theorem 1.4 is obtained as a consequence of detailed, pointwise bounds on the Green's function $G(x,t;y)$ of the linearized evolution equations (1.10) (more properly speaking, a distribution), which we now describe. Let $a_j^{\pm}$, $j = 1, \ldots (n)$ denote the eigenvalues of $A(\pm\infty)$, and $l_j^{\pm}$ and $r_j^{\pm}$ associated left and right eigenvectors, respectively, normalized so that $l_j^{\pm t} r_k^{\pm} = \delta_k^j$. Eigenvalues $a_j(x)$, and eigenvectors l_j, r_j correspond to large-time convection rates and modes of propagation of the linearized model (1.10).

Likewise, let $a_j^*(x)$, $j=1,\dots,J\le(n-r)$ denote the eigenvalues of

$$A_* := A_{11} - A_{12}b_2^{-1}b_1, \tag{1.17}$$

m_j^* the multiplicity of a_j^*, and L_j^*, R_j^* associated $(n-r)\times m_j^*$ blocks of left and right eigenvectors of A_*, satisfying the *dynamical normalization*

$$L_j^{*t}\partial_x R_j^* = (\partial_x L_j^{*t})R_j^* \equiv 0 \tag{1.18}$$

along with the usual static normalization $L_j^{*t}R_j^* \equiv \delta_k^j I_{m_j^*}$. As shown in Lemma 4.9, [MZ.1], (1.18) may always be achieved with bounded L_j^*, R_j^*; in the "physical" case $n-r=1$, it may be acomplished simply by taking $L_j^*\equiv$ constant $=(1,0_{n-1})^{tr}$. Associated with L_j^*, R_j^*, define extended, $n\times m_j^*$ blocks

$$\mathcal{L}_j^* := \begin{pmatrix} L_j^* \\ 0 \end{pmatrix}, \quad \mathcal{R}_j^* := \begin{pmatrix} R_j^* \\ -b_2^{-1}b_1R_j^* \end{pmatrix}. \tag{1.19}$$

Eigenvalues a_j^* and eigenmodes $\mathcal{L}_j^*$, $\mathcal{R}_j^*$ correspond, respectively, to short-time hyperbolic characteristic speeds and modes of propagation for the reduced, hyperbolic part of system (1.10).

Define time-asymptotic, *scalar diffusion rates*

$$\beta_j^\pm := \left(l_j^t B r_j\right)_\pm, \quad j=1,\dots,n, \tag{1.20}$$

and local, $m_j\times m_j$ *dissipation coefficients*

$$\eta_j^*(x) := -L_j^{*t}D_*R_j^*(x), \quad j=1,\dots,J\le n-r, \tag{1.21}$$

where

$$D_*(x) := A_{12}b_2^{-1}\Big[A_{21} - A_{22}b_2^{-1}b_1 + b_2^{-1}b_1A^* + b_2\partial_x(b_2^{-1}b_1)\Big](x) \tag{1.22}$$

is an effective dissipation analogous to the effective diffusion predicted by formal, Chapman–Enskog expansion in the (dual) relaxation case.

At $x=\pm\infty$, these reduce to the corresponding quantities identified by Zeng [Ze.1,LZe.1] in her study by Fourier transform techniques of decay to *constant solutions* $(\bar u,\bar v)\equiv(u_\pm,v_\pm)$ of hyperbolic–parabolic systems, i.e., of limiting equations

$$U_t = L_\pm U := -A_\pm U_x + B_\pm U_{xx}. \tag{1.23}$$

These arise naturally through Taylor expansion of the (frozen-coefficient) Fourier symbol $(-i\xi A - \xi^2 B)_\pm$, as described in Appendix B below; in particular, as a consequence of dissipativity, (H3), we obtain

$$\beta_j^\pm > 0, \quad \mathrm{Re}\ \sigma(\eta_j^{*\pm}) > 0 \quad \text{for all } j. \tag{1.24}$$

However, note that the dynamical dissipation coefficient $D_*(x)$ *does not* agree with its static counterpart, possessing an additional term $b_2\partial_x(b_2^{-1}b_1)$, and so we cannot conclude that (1.24) holds everywhere along the profile, but only at the endpoints.

Proposition 1.8. *Under assumptions (H0)–(H4), the Green's function $G(x,t;y)$ associated with the linearized evolution equations (1.10) may in the Lax or overcompressive case be decomposed as*

$$G(x,t;y) = H + E + S + R, \tag{1.25}$$

where, for $y \le 0$:

$$\begin{aligned} H(x,t;y) &:= \sum_{j=1}^{J} a_j^{*-1}(x) a_j^*(y) \mathcal{R}_j^*(x) \zeta_j^*(y,t) \delta_{x-\bar a_j^* t}(-y) \mathcal{L}_j^{*t}(y) \\ &= \sum_{j=1}^{J} \mathcal{R}_j^*(x) \mathcal{O}(e^{-\eta_0 t}) \delta_{x-\bar a_j^* t}(-y) \mathcal{L}_j^{*t}(y), \end{aligned} \tag{1.26}$$

$\eta_0 > 0$,
(1.27)

$$E(x,t;y) := \sum_{a_k^- > 0} [c_{k,-}^0] \bar U'(x) l_k^{-t} \left(\text{errfn} \left(\frac{y + a_k^- t}{\sqrt{4\beta_k^- t}} \right) - \text{errfn} \left(\frac{y - a_k^- t}{\sqrt{4\beta_k^- t}} \right) \right),$$

and
(1.28)

$$\begin{aligned} S(x,t;y) := {}& \chi_{\{t\ge1\}} \sum_{a_k^- < 0} r_k^- l_k^{-t} (4\pi\beta_k^- t)^{-1/2} e^{-(x-y-a_k^- t)^2/4\beta_k^- t} \\ &+ \chi_{\{t\ge1\}} \sum_{a_k^- > 0} r_k^- l_k^{-t} (4\pi\beta_k^- t)^{-1/2} e^{-(x-y-a_k^- t)^2/4\beta_k^- t} \left(\frac{e^{-x}}{e^x + e^{-x}} \right) \\ &+ \chi_{\{t\ge1\}} \sum_{a_k^- > 0,\, a_j^- < 0} [c_{k,-}^{j,-}] r_j^- l_k^{-t} (4\pi\bar\beta_{jk}^- t)^{-1/2} e^{-(x-z_{jk}^-)^2/4\bar\beta_{jk}^- t} \left(\frac{e^{-x}}{e^x + e^{-x}} \right), \\ &+ \chi_{\{t\ge1\}} \sum_{a_k^- > 0,\, a_j^+ > 0} [c_{k,-}^{j,+}] r_j^+ l_k^{-t} (4\pi\bar\beta_{jk}^+ t)^{-1/2} e^{-(x-z_{jk}^+)^2/4\bar\beta_{jk}^+ t} \left(\frac{e^{x}}{e^x + e^{-x}} \right) \end{aligned}$$

denote hyperbolic, excited, and scattering terms, respectively, and R denotes a faster decaying residual (rates given in Proposition 7.1 below). Symmetric relations hold for $y \ge 0$.

Here, the averaged convection rates $\bar a_j^*(y,t)$ in (1.26) denote the time-averages over $[0,t]$ of $a_j^*(x)$ along characteristic paths $z_j^* = z_j^*(y,t)$ defined by

$$dz_j^*/dt = a_j^*(z_j^*), \quad z_j^*(0) = y, \tag{1.29}$$

and the dissipation matrix $\zeta_j^* = \zeta_j^*(y,t) \in \mathbb{R}^{m_j \times m_j}$ is defined by the dissipative flow

$$d\zeta_j^*/dt = -\eta_j^*(z_j^*)\zeta_j^*, \quad \zeta_j^*(y) = I_{m_j}. \tag{1.30}$$

Similarly, in (1.28),

$$z^{\pm}_{jk}(y,t) := a^{\pm}_j \left(t - \frac{|y|}{|a^-_k|} \right) \tag{1.31}$$

and

$$\bar\beta^{\pm}_{jk}(x,t;y) := \frac{|x^{\pm}|}{|a^{\pm}_j t|}\beta^{\pm}_j + \frac{|y|}{|a^-_k t|}\left(\frac{a^{\pm}_j}{a^-_k}\right)^2 \beta^-_k, \tag{1.32}$$

represent, respectively, approximate scattered characteristic paths and the time-averaged diffusion rates along those paths. In all equations, a_j, $a^{*\pm}_j$, l_j, $\mathcal{L}^{*\pm}_j$, r_j, $\mathcal{R}^{*\pm}_j$, $\beta^{\pm}_j$ and η^*_j are as defined just above, and scattering coefficients $[c^{j,i}_{k,-}]$, $i = -, 0, +$, are constants, uniquely determined by

$$\sum_{a^-_j<0} [c^{j,-}_{k,-}] r^-_j + \sum_{a^+_j>0} [c^{j,+}_{k,-}] r^+_j + [c^0_{k,-}]\big(U(+\infty) - U(-\infty)\big) = r^-_k \tag{1.33}$$

for each $k = 1, \dots n$, and satisfying

$$\sum_{a^-_k>0} [c^0_{k,-}] l^-_k = \sum_{a^+_k<0} [c^0_{k,+}] l^+_k = \pi, \tag{1.34}$$

where the constant vector π is the left zero effective eigenfunction of L associated with the right eigenfunction $\bar U'$. Similar, but more complicated formulae hold in the undercompressive case (see Remark 6.9, [MZ.1], or Section 3, [Z.2], for further discussion).

Proposition 1.8, a variable-coefficient generalization of the constant-coefficient results of [Ze.1,LZe.1], will be established by Laplace transform (i.e., semigroup) techniques generalizing the Fourier transform approach of [Ze.1–2,LZe], similarly as in [ZH,MZ.1]. Just as in the constant–coefficient case, term H describes propagation along characteristics of disturbances in unsmoothed, but time-exponentially damped "hyperbolic" modes associated with the reduced, hyperbolic part of system (1.10), while E and S describe the long-time, essentially parabolic behavior of perturbations.

Terms E and S are identical to those derived for strictly parabolic systems in [ZH,Z.2–3], in this case for the associated strictly parabolic system determined by "effective viscosity matrix" LBR. As discussed in those references, they may be understood by the caricature of Gaussian signals propagating for $x \gtrless 0$ according to the limiting, constant-coefficient equations at $\pm\infty$ (of the associated effective, strictly parabolic system), interacting with an "ideal" shock layer at $x = 0$ according to the rules encoded by scattering coefficients $[c^{j,i}_{k,\pm}]$, $i = -, 0, +$: That is, a signal initiating as a delta-function at $y < 0$ first decomposes into n Gaussian signals traveling in each of the characteristic modes r_k^- with approximately characteristic speed a_k^-; each of these signals then travels until it reaches the shock layer $x = 0$, whereupon it is reflected/transmitted in each of the outgoing characteristic modes $r_j^\pm$, thereafter moving with approximate characteristic speed $a_j^\pm \gtrless 0$ along path $z_{jk}^\pm$; at the same time, the incoming signal excites stationary modes, or "bound states," $(\partial/\partial\delta_j)(\bar u^\delta, \bar v^\delta)$. Incoming and outgoing Gaussian signals comprise the scattering term S, excited stationary modes the excited term E.

Only a small part of the detailed information given in the proposition is needed for the nonlinear stability analysis carried out in [MZ.2], namely $L^p \to L^q$ estimates on the time-decaying portion $H + S + R$ of the Green's function G; see Lemma 4.1 of the reference. However, the stationary portion E of the Green's function must be estimated accurately for an efficient stability analysis.

Remark 1.9. We may relax the requirement of strict hyperbolicity of $dF(u_\pm)$ in (H2) to *symmetrizable hyperbolicity* (reality plus semisimplicity of eigenvalues). If $a_j^\pm$ has multiplicity $m_j^\pm$, then, likewise, we define $l_j^\pm$ and $r_j^\pm$ as $n \times m_j^\pm$ blocks, and $\beta_j^\pm := l_j^{\pm t} B^\pm r_j^\pm$. Then, (1.25) holds with (1.27)–(1.28) replaced by matrix versions; see [LZe] for further discussion, and a nonlinear version of this multiple-mode diffusion wave.

Remark 1.10. From the intrinsically defining property of $\mathcal{L}_j^*$ and $\mathcal{R}_j^*$ (readily seen from our formulae) that they lie, respectively, in the left and right kernel of B, we may readily find that $0 \equiv (b_1, b_2)\mathcal{R}_j$, and thus $0 \equiv (b_1, b_2)H$. This quantifies the observation that the "parabolic" variable $b_1 u + b_2 v$ experiences smoothing under the evolution of (1.10), whereas the "hyperbolic" variable u does not. Likewise, data in the v-coordinate is smoothed, while data in the u-coordinate is not.

Remark 1.11. Following [Ho], we may recover the hyperbolic evolution equations (1.26), (1.29)–(1.30) by direct calculation, considering the evolution of a jump discontinuity along curve $x = z(t)$ in the original equation $U_t + (AU)_x = (BU_x)_x$; see also the related calculation of Section 1.3, [MZ.1] in the relaxation case. Introducing the parabolic variable $\tilde v := b_1 u + b_2 v$, and denoting jump across z of a variable f by $[f]$, we may write (1.10) as the pair of equations

$$u_t + (A_* u)_x + (A_{12} b_2^{-1} \tilde v)_x = 0$$

and

$$v_t + \Big((A_{21} - A_{22} b_2^{-1} b_1 + b_2 (b_2^{-1} b_1) u\Big)_x + \Big((A_{22} b_2^{-1}) + (b_2)_x b_2^{-1})\tilde v\Big)_x - (\tilde v_x)_x = 0. \tag{1.35}$$

Applying the Rankine–Hugoniot conditions, we obtain relations

$$(A_* - \dot z)[u] = 0, \quad [\tilde v] = 0, \quad \text{and}$$

$$\begin{aligned} [\tilde v_x] = &\Big(A_{21} - A_{22} b_2^{-1} b_1 + \dot z b_2^{-1} b_1 + b_2 \partial_x (b_2^{-1} b_1)\Big)[u] = \\ &\Big(A_{21} - A_{22} b_2^{-1} b_1 + b_2^{-1} b_1 A_* + b_2 \partial_x (b_2^{-1} b_1)\Big)[u] =: d_*[u], \end{aligned} \tag{1.36}$$

yielding $\dot z_j = a_j(z_j(t))$, $[u] = R_j^*(z_j(t))\zeta_j(t)$, $[\tilde v] \equiv 0$, and $[\tilde v_x] = d_* R_j^*(z_j(t))\zeta_j(t)$. Substituting these Ansatze into the $[u]$ equation

$$[u]_t + (A_*[u])_x + A_{12}b_2^{-1}[\tilde v_x] + (A_{12}b_2^{-1})_x[\tilde v] = 0$$

yields

$$[u]_t + (A_*[u])_x + D_*[u] = 0,$$

$D_* = A_{12}b_2^{-1}d_*$ as defined in (1.22); taking inner product with $L_j^*(z_j(t))$ and rearranging, we thus obtain the characteristic evolution equations

$$\dot\zeta_j = \big((\partial_x L_j^*)R_j^* - \partial_x a_j - \eta_j\big)\zeta_j, \tag{1.37}$$

$\eta_j = L_j^* D_* R_j^*$ as defined in (1.21). Under the normalization $(\partial_x L_j^*)R_j^* \equiv 0$ (note: automatic in the gas-dynamical case considered in [Ho]), this yields in the original coordinate U precisely the solution operator $a_j^{*-1}(z_j)a_j^*(y)\mathcal{R}_j^*(z_j)\zeta_j(y,t)\mathcal{L}_j^*(y)$ described in (1.26).

Remark 1.12. In the special case of isentropic gas dynamics, written in Lagrangian coordinates, A_* is scalar and identically equal to $-s$, where s is the shock speed, and also $b_1 \equiv 0$, so that (1.21)–(1.22) simplify to $\eta_* = -D_* = c^2(x)$, where c is the sound speed. Thus, neither of the dynamical corrections $\partial_x a_j^*$ or $b_2\partial_x(b_2^{-1}b_1)$ arise in the characteristic evolution equation (1.37), and the hyperbolic behavior may be guessed by consideration of the frozen-coefficient case. However, in Eulerian coordinates, or for full gas dynamics in either Lagrangian or Eulerian coordinates, these dynamical effects are important, and must be accounted for.

Plan of the paper. In Section 2, we recall the basic spectral representation of G via inverse Laplace Transform of the resolvent kernel, and in Section 3 the standard Evans function tools that we will need. In Section 4, we make a careful construction of the resolvent kernel; using this representation, we make careful resolvent kernel estimates in Sections 5 and 6, in high- and low-frequency regimes, respectively. Combining these results, we derive in Section 7 the Green's function bounds that comprise the main results of the paper, establishing Theorem 1.8 and a partial generalization yielding necessity of $(\mathcal{D})$ for linearized stability. Finally, in Section 8, we establish sufficiency of $(\mathcal{D})$ for linearized stability, completing the proof of Theorem 1.4. For completeness, we carry out in Appendices A–C various auxiliary calculations/lemmas used throughout the paper.

Section 2. The Spectral Resolution Formula.

The starting point for our analysis, as in [ZH,MZ.1], is the spectral resolution, or inverse Laplace Transform formula for the Green's function $G(x,t;y)$ associated with the linearized evolution equations (1.10), defined by:

(i) $(\partial_t - L_x)G = 0$ in the distributional sense, for all $t > 0$; and,

(ii) $G(x,t;y) \rightharpoonup \delta(x-y)$ as $t \to 0$.

Here, G is to be interpreted in (i) as a distribution in the joint variables (x, y, t), and in (ii) as a distribution in (x, y), continuously parametrized by t. We shall see that G, uniquely so defined, by uniqueness of weak solutions of (1.10) within the class of test function initial data, in fact lies in the space of measures and not functions, so should properly be called a Green's distribution rather than a Green's function. Following the convention of [LZe,Ze.2], we use the two terms interchangeably.

The Green's distribution may be conveniently constructed using standard semigroup theory, considering the linearized operator L as a closed, densely defined operator on L^2, with domain $\mathcal{H} := \{U : U \in H^1,\ (b_1, b_2)U \in H^2\}$.

Lemma 2.1. *Given (H0)–(H4), $L : LU := -(BU_x)_x - (AU)_x$ satisfies*

$$
\begin{aligned}
|U|_{H^1} + |bU|_{H^2} &\le C(|LU|_{L^2} + |U|_{L^2}),\\
|SU|_{L^2} &\le |\lambda - \lambda_*|^{-1} |S(L-\lambda)U|_{L^2},
\end{aligned}
\tag{2.1}
$$

for all $U \in \mathcal{H}$, for some well-conditioned coordinate transformation $U \to S(x)U$, and real λ greater than some value λ_*.

Proof. By the lower triangular coordinate transformation $U \to S_1 U$,

$$
S_1 := \begin{pmatrix} I & 0 \\ -b_1 b_2^{-1} & I \end{pmatrix},
\tag{2.2}
$$

we may convert to the case that $b_1 \equiv 0$, with the slightly more general operator

$$
LU := (BU_x)_x - (AU)_x + CU,
$$

where b_2 and $A_* := A_{11} - A_{12} b_1 b_2^{-1} = A_{11}$ retain their former values, and thus their former properties of strict parabolicity and hyperbolicity with constant multiplicity, respectively. By a further, block-diagonal transformation, we may arrange also that b_2 be strictly positive definite and A_{11} symmetric; moreover, this may be chosen so that the combined transformation is uniformly well-conditioned.

Setting $\lambda = 0$ and taking the L^2 inner product of $(A_* u_x, b_2 v_{xx})$ against the resulting equation $(AU)_x - (BU_x)_x + CU = f$, we obtain after some rearrangement the bound

$$
C^{-1}(|u_x|_{L^2} + |v_{xx}|_{L^2}) \le |A_* u_x|_{L^2} + |b_2 v_{xx}|_{L^2} \le C(|u|_{L^2} + |v|_{H^1}),
$$

which in turn implies (2.1)(i). Note that we have only used nondegeneracy of the respective leading order coefficients A_* and b_2 in the u and v equations.

Taking the real part of the L^2 inner product of U against the resolvent equation

$$
\lambda U + (AU)_x - (BU_x)_x + CU = f,
\tag{2.3}
$$

integrating by parts, using the assumptions on A_{11}, b_2, and bounding terms of order $|U|_2|v_x|_2$ by Youngs inequality as $O(|U|_2^2) + \epsilon|v_x|_2^2$ with ϵ as small as needed, we obtain after some rearrangement the estimate

$$\text{Re } \lambda|U|_2^2 + O(|U|_2^2) + \theta|v_x|_2^2 \le |f|_2|U|_2, \tag{2.4}$$

whence we obtain the second claimed bound

$$|U|_2 \le |\lambda - \lambda_*|^{-1}|f|_2 = |\lambda - \lambda_*|^{-1}|(L-\lambda)U|_2 \tag{2.5}$$

for all $\lambda > \lambda_*$ on the real axis, and λ_* sufficiently large. ∎

Proposition 2.2. *Given (H0)–(H4), L is a closed, densely defined operator on L^2 with domain $\mathcal{H}$, generating a C^0 semigroup e^{Lt} satisfying $|e^{Lt}|_{L^2} \le Ce^{\omega t}$ for some real ω.*

Proof. The first assertion follows in standard fashion from bound (2.4)(i); see, e.g., [Pa] or [Z.1]. The bound (2.4)(ii) applies also to the limiting, constant-coefficient operators $L_\pm$ as $x \to \pm\infty$, whence the spectra of these operators is confined to Re $\lambda \le \lambda_*$. Because the eigenvalue equation (equation (2.3) with $f = 0$), after the coordinate transformation described in the proof of Lemma 2.1 above, may evidently be written as a nondegenerate ODE in (u, v, v'), it admits an Evans function, and so we may conclude by standard Evans function theory that the essential spectrum of L is also confined to the set Re $\lambda \le \lambda_*$; see, e.g., [GZ,ZH]. (This observation generalizes a standard result of Henry ([He], Lemma 2, pp. 138–139) in the case of a nondegenerate operator L.) Since (2.1) precludes point spectrum for real $\lambda > \lambda_*$, we thus find that all such λ belong to the resolvent set $\rho(L)$, with the resolvent bound (after the coordinate transformation described above)

$$|(L-\lambda)^{-1}|_{L^2} \le |\lambda - \lambda_*|^{-1}. \tag{2.6}$$

But, this is a standard sufficient condition that a closed, densely defined operator L generate a C^0 semigroup, with $|e^{Lt}|_{L^2} \le Ce^{\omega t}$ for all real $\omega > \lambda_*$ (see e.g. [Pa], Theorem 5.3, or [Fr,Y]). ∎

Remark 2.3. Evidently, the role played by (H1) in the proof of Lemma 2.1 is only to enforce symmetrizable hyperbolicity (i.e., real, semisimple spectrum) plus nonsingularity of matrix $A_* = A_{11} - A_{12}b_1b_2^{-1}$, and this weaker assumption is sufficient for linearized well-posedness/generation of a C^0 semigroup. An interesting Fourier transform exercise in the constant-coefficient case is to show that symmetrizable hyperbolicity of both A_* and A are necessary for linearized stability, along with (H3) with $\theta = 0$. The additional assumption of constant multiplicity will play an important technical role in the high-frequency estimates of the resolvent kernel carried out in Section 5 (see the related description of term H, Proposition 1.8); indeed, it is not clear what would be the effect on stability of dropping this assumption.

An immediate consequence of Proposition 2.2 (see, e.g. [Pa], Corollary 7.5) is:

Corollary 2.4. *For L, e^{Lt} as in Corollary 2.2, there holds for all real η greater than some η_0 the inverse Laplace Transform (spectral resolution) formula*

$$e^{Lt}\phi = P.V. \int_{\eta-i\infty}^{\eta+i\infty} e^{ikt}(L-\lambda)^{-1}\phi \, d\lambda, \tag{2.7}$$

for all $\phi \in D(L^2)$, where domain $D(L^2)$ is defined as in [Pa], p. 1.[2]

With the result of Proposition 2.2, we can express the Green's distribution simply as

$$G(x,t;y) := e^{Lt}\delta(x-y), \tag{2.8}$$

where the evolution $e^{Lt}\phi(x,y)$ of a distribution $\phi(x,y)$ with the property that, for $k \in C_0^\infty$, the distributional pairing $\langle \phi(\cdot, y), k(y)\rangle_y$ is in L^2 for each fixed x, is defined through its action on C_0^∞ test functions $h(x)k(y)$:

$$\langle e^{Lt}\phi(x,y), h(x)k(y)\rangle_{x,y} := \langle h(x), e^{Lt}\langle \phi(x,y), k(y)\rangle_y\rangle_x,$$

or

$$\langle G(x,t;y), h(x)k(y)\rangle_{x,y} := \langle h(x), e^{Lt}k(x)\rangle_x. \tag{2.9}$$

The defining properties (i)–(ii) of the Green's distribution are then consequences of the corresponding properties defining a C^0 semigroup (see, e.g., [Pa], p. 1–4).

Defining the *resolvent kernel*, similarly, through

$$G_\lambda(x,y) := (L-\lambda)^{-1}\delta(x-y), \tag{2.10}$$

where the righthand side is defined again through its action on test functions,

$$\langle (L-\lambda)^{-1}\phi(x,y), h(x)k(y)\rangle_{x,y} := \langle h(x), (L-\lambda)^{-1}\langle \phi(x,y), k(y)\rangle_y\rangle_x,$$

or

$$\langle G_\lambda(x,y), \psi(x,y)\rangle_{x,y} := \langle h(x), (L-\lambda)^{-1}k(x)\rangle_x,$$

we find through (2.7), as a tautology, the fundamental:

---

[2]Note: From the definition in [Pa], we find that the domain $D(L)$ of L satisfies $H^2 \subset \mathcal{H} \subset D(L)$. Likewise, the domain $D(L^2)$, consisting of those functions V such that $LV \in D(L)$, satisfies $H^4 \subset D(L^2)$.

Proposition 2.5. *For L, e^{Lt} as in Corollary 2.2, and G, G_λ as in (2.8)–(2.10), there holds the inverse Laplace Transform (spectral resolution) formula*

$$G(x,t;y) = P.V. \int_{\eta - i\infty}^{\eta + i\infty} e^{ikt} G_\lambda(x,y)\, d\lambda, \tag{2.11}$$

valid for all η greater than some η_0.

It is this relation that we shall use to compute the time-evolutionary Green's distribution G. Note that, on the resolvent set $\rho(L)$, the resolvent kernel G_λ likewise has an alternative, intrinsic characterization as the unique distribution satisfying

$$(L_x - \lambda) G_\lambda(x,y) = \delta(x,y) \tag{2.12}$$

and taking $f \in L^2$ to $\langle G_\lambda(x,y), f(y)\rangle_y \in L^2$. This is the characterization that we shall use in computating G_λ.

Remark 2.6. Note that a semigroup on even a still more restricted function class than L^2 would have sufficed in this construction, since we are constructing objects in the very weak class of distributions. Later, by explicit computation of G, we will verify that L in fact generates a C^0 semigroup in any L^p. Note also that (2.9) implies the expected, standard solution formula $e^{Lt}f = \langle G(x,t;y), f(y)\rangle_y$, or, formally:

$$e^{Lt} f = \int G(x,t;y) f(y) dy, \tag{2.13}$$

for f in any underlying Banach space on which e^{Lt} is defined, in this case on L^p, $1 \le p \le \infty$. ∎

Remark 2.7. As in the relaxation case [MZ.1], we point out another, more concrete route to (2.11), (2.13), generalizing the approach followed in [ZH] for the parabolic case. Namely, we may observe that, in each finite integral in the approximating sequence defined by the principal value integral (2.11), we may exchange orders of integration and distributional differentiation, using Fubini's Theorem together with the fact, established in the course of our analysis, that G_λ is uniformly bounded on the contours under consideration (in fact, we establish the much stronger result that G_λ decays exponentially in $|x-y|$, with uniform rate), to obtain

$$\begin{aligned}(\partial_t - L_x) \int_{\eta - iK}^{\eta + iK} e^{\lambda t} G_\lambda(x,y)\, d\lambda &= \delta(x-y) \int_{\eta - iK}^{\eta + iK} e^{\lambda t}\, d\lambda \\ &\rightharpoonup \delta(x-y)\delta(t)\end{aligned} \tag{2.14}$$

as $K \to \infty$, for all $t \ge 0$. So, all that we must verify is: (i') the Principal Value integral (2.11) in fact converges to some distribution $G(x,t;y)$ as $K \to \infty$; and, (ii')

$G(x,t;y)$ has a limit as $t \to 0$. For, distributional limits and derivatives commute (essentially by definition), so that (i')–(ii') together with (2.14) imply (i)–(ii) above. Facts (i')–(ii') will be established by direct calculation in the course of our analysis, thus verifying formula (2.11) at the same time that we use it to obtain estimates on G. Note that we have made no reference in this argument to the semigroup machinery cited above.

Section 3. Evans Function Framework.

Before starting our analysis, we recall, and in some cases slightly extend, the basic Evans function tools that we will need.

3.1. The gap/conjugation lemma. Consider a family of first-order systems

$$W' = \mathbb{A}(x,\lambda)W, \tag{3.1}$$

where λ varies within some domain Ω and x varies within $\mathbb{R}^1$. Equations (3.1) are to be thought of as generalized eigenvalue equations, with parameter λ representing frequency. We make the basic assumption:

(h0) Coefficient $\mathbb{A}(\cdot,\lambda)$, considered as a function from Ω into $L^\infty(x)$ is analytic in λ. Moreover, $\mathbb{A}(\cdot,\lambda)$ approaches exponentially to limits $\mathbb{A}_\pm$ as $x \to \pm\infty$, with uniform exponential decay estimates

$$|(\partial/\partial x)^k(\mathbb{A} - \mathbb{A}_\pm)| \le C_1 e^{-\theta|x|/C_2}, \quad \text{for } x \gtrless 0,\ 0 \le k \le K, \tag{3.2}$$

C_j, $\theta > 0$, on compact subsets of Ω.

Then, there holds the following "conjugation lemma" of [MéZ], a refinement of the "gap lemma" of [GZ,KS], relating solutions of the variable-coefficient ODE (3.1) to solutions of its constant-coefficient limiting equations as

$$Z' = \mathbb{A}_\pm(\lambda)Z \tag{3.3}$$

as $x \to \pm\infty$.

Lemma 3.1[MéZ]. *Under assumption (h0) alone, there exists locally to any given $\lambda_0 \in \Omega$ a pair of linear transformations $P_+(x,\lambda) = I + \Theta_+(x,\lambda)$ and $P_-(x,\lambda) = I + \Theta_-(x,\lambda)$ on $x \ge 0$ and $x \le 0$, respectively, $\Phi_\pm$ analytic in λ as functions from Ω to $L^\infty[0,\pm\infty)$, such that:*

(i) $|P_\pm|$ and their inverses are uniformly bounded, with

$$|(\partial/\partial\lambda)^j(\partial/\partial x)^k\Theta_\pm| \le C(j)C_1C_2e^{-\theta|x|/C_2} \quad \text{for } x \gtrless 0,\ 0 \le k \le K+1, \tag{3.4}$$

$j \ge 0$, where $0 < \theta < 1$ is an arbitrary fixed parameter, and $C > 0$ and the size of the neighborhood of definition depend only on θ, j, the modulus of the entries of $\mathbb{A}$ at λ_0, and the modulus of continuity of $\mathbb{A}$ on some neighborhood of $\lambda_0 \in \Omega$.

(ii) The change of coordinates $W := P_\pm Z$ reduces (3.1) on $x \geq 0$ and $x \leq 0$, respectively, to the asymptotic constant-coefficient equations (3.3).

Equivalently, solutions of (3.1) may be conveniently factorized as

$$W = (I + \Theta_\pm) Z_\pm, \tag{3.5}$$

where $Z_\pm$ are solutions of the constant-coefficient equations (3.3), and $\Theta_\pm$ satisfy bounds (3.4). In the present, one-dimensional setting, $\rho \equiv \lambda$.

Proof. As described in [MéZ], for $j = k = 0$ this is a straighforward corollary of the gap lemma as stated in [Z.3], applied to the "lifted" matrix-valued ODE for the conjugating matrices $P_\pm$; see also appendix C, below. The x-derivative bounds $0 < k \leq K+1$ then follow from the ODE and its first K derivatives. Finally, the λ-derivative bounds follow from standard interior estimates for analytic functions. □

Definition 3.2. Following [AGJ], we define the *domain of consistent splitting* for problem (3.1) as the (open) set of λ such that the limiting matrices $\mathbb{A}_+$ and $\mathbb{A}_-$ are hyperbolic (have no center subspace), and the dimensions of their stable (resp. unstable) subspaces S_+ and S_- (resp. unstable subspaces U_+ and U_-) agree.

Lemma 3.3. *On any simply connected subset of the domain of consistent splitting, there exist analytic bases $\{v_1, \dots, v_k\}^\pm$ and $\{v_{k+1}, \dots, v_N\}^\pm$ for the subspaces $S_\pm$ and $U_\pm$ defined in Definition 3.2.*

Proof. By spectral separation of $U_\pm$, $S_\pm$, the associated (group) eigenprojections are analytic. The existence of analytic bases then follows by a standard result of Kato; see [Kat], pp. 99–102. □

By Lemma 3.1, on the domain of consistent splitting, the subspaces

$$\mathcal{S}^+ := \text{Span } \{P_+ v_1^+, \dots, P_+ v_k^+\}(\cdot) \tag{3.6}$$

and

$$\mathcal{U}^- := \text{Span } \{P_- v_{k+1}^-, \dots, P_- v_N^-\}(\cdot) \tag{3.7}$$

uniquely determine the stable manifold as $x \to +\infty$ and the unstable manifold as $x \to -\infty$ of (3.1), defined as the manifolds of solutions decaying as $x \to \pm\infty$, respectively, independent of the choice of $P_\pm$.

Definition 3.4. On any simply connected subset of the domain of consistent splitting, let $v_1^+, \dots, v_k^+$ and $v_{k+1}^-, \dots, v_N^-$ be analytic bases for S_+ and U_-, as described in Lemma 3.3. Then, the *Evans function* for (3.1) associated with this choice of limiting bases is defined as

$$D(\lambda) := \det \left(P_+ v_1^+, \dots, P_+ v_k^+, P_- v_{k+1}^-, \dots, P_- v_N^- \right)_{|x=0,\lambda}, \tag{3.8}$$

where $P_\pm$ are the transformations described in Lemma 3.1.

Remark 3.5. Note that D is independent of the choice of $P_\pm$; for, by uniqueness of stable/unstable manifolds, the exterior products (minors) $P_+ v_1^+ \wedge \cdots \wedge P_+ v_k^+$ and $P_- v_{k+1}^- \wedge \cdots \wedge P_- v_N^-$ are uniquely determined by their behavior as $x \to +\infty$, $-\infty$, respectively.

Proposition 3.6. *Both the Evans function and the stable/unstable subspaces $\mathcal{S}^+$ and $\mathcal{U}^-$ are analytic on the entire simply connected subset of the domain of consistent splitting on which they are defined. Moreover, for λ within this region, equation (3.1) admits a nontrivial solution $W \in L^2(x)$ if and only if $D(\lambda) = 0$.*

Proof. Analyticity follows by uniqueness, and local analyticity of $P_\pm$, $v_k^\pm$. Noting that the first K columns of the matrix on the righthand side of (3.8) are a basis for the stable manifold of (3.1) at $x \to +\infty$, while the final $N-K$ columns are a basis for the unstable manifold at $x \to -\infty$, we find that its determinant vanishes if and only if these manifolds have nontrivial intersection, and the second assertion follows. □

Remark 3.7. In the case that (3.1) describes an eigenvalue equation associated with an ordinary differential operator L, Proposition 3.6 implies that eigenvalues of L agree in location with zeroes of D. In [GJ.1–2], Gardner and Jones [GJ.1–2] have shown that they agree also in multiplicity; see also Lemma 6.1, [ZH], or Proposition 6.15 of this paper.

3.2. The tracking/reduction lemma. Next, consider the complementary situation of a family of equations of form

$$W' = \mathbb{A}^\epsilon(x,\lambda)W, \tag{3.9}$$

on an (ϵ,λ)-neighborhood for which the coefficient $\mathbb{A}^\epsilon$ does not exhibit uniform exponential decay to its asymptotic limits, but instead is *slowly varying*. This occurs quite generally for rescaled eigenvalue equations arising in the study of the large-frequency regime; see, e.g., [GZ,ZH,MZ.1,Z.3].

In this situation, it frequently occurs that not only $\mathbb{A}^\epsilon$ but also certain of its invariant (group) eigenspaces are slowly varying with x, i.e., there exist matrices

$$L^\epsilon = \begin{pmatrix} L_1^\epsilon \\ L_2^\epsilon \end{pmatrix}(x), \quad R^\epsilon = \begin{pmatrix} R_1^\epsilon & R_2^\epsilon \end{pmatrix}(x) \tag{3.10}$$

for which $L^\epsilon R^\epsilon(x) \equiv I$ and $|LR'| = |L'R|$ is small relative to

$$\mathbb{M}^\epsilon := L^\epsilon \mathbb{A}^\epsilon R^\epsilon(x) = \begin{pmatrix} M_1^\epsilon & 0 \\ 0 & M_2^\epsilon \end{pmatrix}(x), \tag{3.11}$$

where "$'$" as usual denotes $\partial/\partial x$. In this case, making the change of coordinates $W^\epsilon = R^\epsilon Z$, we may reduce (3.9) to the approximately block-diagonal equation

$$Z^{\epsilon\prime} = \mathbb{M}^\epsilon Z^\epsilon + \delta^\epsilon \Theta^\epsilon Z^\epsilon, \tag{3.12}$$

where $\mathbb{M}^\epsilon$ is as in (3.11), $\Theta^\epsilon(x)$ are uniformly bounded matrices, and $\delta^\epsilon(x) \le \delta(\epsilon)$ is a (relatively) small scalar.

Let us assume that such a procedure has been successfully carried out, and, moreover, that the approximate flows (i.e., solution operators) $\bar{\mathcal{F}}_j^{y\to x}$ generated by the decoupled equations

$$Z' = M_j^\epsilon Z \tag{3.13}$$

are *uniformly exponentially separated* to order $2\eta(\epsilon)$, in the sense that

$$|\bar{\mathcal{F}}_1^{y\to x}||(\bar{\mathcal{F}}_2^{y\to x})^{-1}| \le Ce^{-2\eta(\epsilon)|x-y|} \quad \text{for } x \ge y. \tag{3.14}$$

Remark 3.8. A sufficient condition for and the usual means of verification of (3.14) is existence of an approximate *uniform spectral gap*:

$$\text{Re } (M_1^\epsilon - M_2^\epsilon)(x) \le -2\eta(\epsilon) + \alpha^\epsilon(x) \tag{3.15}$$

for all x, where α^ϵ is uniformly integrable, $\int |\alpha^\epsilon(x)|\,dx \le C$, or, alternatively, existence of flows conjugate to $\bar{\mathcal{F}}_j^{y\to x}$ and satisfying this condition. Note that an exact spectral gap may be achieved from (3.15) by the coordinate change $Z_1^\epsilon \to \omega Z_1^\epsilon$, where ω is a well-conditioned (scalar) exponential weight defined by $\omega' = -\alpha^\epsilon\omega$, $\omega(0) = 1$.

Then, there holds the following *reduction lemma*, a refinement established in [MZ.1] of the "tracking lemma" given in varying degrees of generality in [GZ,ZH,Z.3].

Proposition 3.9[MZ.1]. *Consider a system (3.12) satisfying the exponential separation assumption (3.14), with Θ^ϵ uniformly bounded for ϵ sufficiently small. If, for $0 < \epsilon < \epsilon_0$, the ratio $\delta(\epsilon)/\eta(\epsilon)$ of formal error vs. spectral gap is sufficiently small relative to the bounds on Θ, in particular if $\delta(\epsilon)/\eta(\epsilon) \to 0$, then, for all $0 < \epsilon \le \epsilon_0$, there exist (unique) linear transformations $\Phi_1^\epsilon(x, \tilde{\xi}, \lambda)$ and $\Phi_2^\epsilon(x, \tilde{\xi}, \lambda)$, possessing the same regularity with respect to the parameters ϵ, $\tilde{\xi}$, λ as do coefficients $\mathbb{M}^\epsilon$ and $\delta(\epsilon)\Theta^\epsilon$, for which the graphs $\{(Z_1, \Phi_2^\epsilon Z_1)\}$ and $\{(\Phi_1^\epsilon Z_2, Z_2)\}$ are invariant under the flow of (3.12), and satisfying*

$$|\Phi_j^\epsilon|,\ |\partial_x \Phi_j^\epsilon| \le C\delta(\epsilon)/\eta(\epsilon) \text{ for all } x. \tag{3.16}$$

Proof. As described in Appendix C, [MZ.1], this may be established by a contraction mapping argument carried out for the projectivized "lifted" equations governing the flow of exterior forms; see, e.g., [Sat] for a corresponding argument in the case that M_1, M_2 are scalar. We give a more direct, "matrix-valued" version of this argument in Appendix C, below. from which one may obtain further an explicit (but rather complicated) Neumann series for Φ_j^ϵ in powers of δ/η. $\square$

Remark 3.10.

From Proposition 3.9, we obtain reduced flows

$$Z_1^{\epsilon\prime} = M_1^\epsilon Z_1^\epsilon + \delta^\epsilon(\Theta_{11} + \Theta_{12}^\epsilon \Phi_2^\epsilon) Z_1^\epsilon = M_1^\epsilon Z_1^\epsilon + \mathcal{O}(\delta^\epsilon(x)) Z_1^\epsilon \tag{3.17}$$

and

$$Z_2^{\epsilon\prime} = M_2^\epsilon Z_2^\epsilon + \delta^\epsilon(\Theta_{22} + \Theta_{21}^\epsilon \Phi_1^\epsilon) Z_2^\epsilon = M_2^\epsilon Z_2^\epsilon + \mathcal{O}(\delta^\epsilon(x)) Z_2^\epsilon \tag{3.18}$$

on the two invariant manifolds described. Let us focus on the flow (3.17), assuming without loss of generality (by a centering exponential weighting if necessary) that

$$|\bar{\mathcal{F}}_1^{y\to x}| \le C e^{-\eta(\epsilon)|x-y|} \quad \text{for } x \le y. \tag{3.19}$$

Corollary 3.11 [MZ.1]. *Assuming (3.19), the flow $\mathcal{F}_1^{y\to x}$ of the reduced equation (3.17) satisfies*

$$\mathcal{F}_1^{y\to x} = \bar{\mathcal{F}}_1^{y\to x} + \sum_{j=1}^{J} (\delta/\eta)^j \mathcal{E}_j(x, y, \epsilon) + \mathcal{O}(\delta/\eta)^{J+1} e^{-\tilde\theta\eta|x-y|}, \tag{3.20}$$

for $x \le y$, for any $0 < \tilde\theta < 1$, where $\bar{\mathcal{F}}_1^{y\to x}$ is the flow of the associated decoupled system (3.13); the iterated integrals

$$\mathcal{E}_{j+1} := \eta \int_y^x \bar{\mathcal{F}}_1^{z\to x} \big(\Theta_{11} + \Theta_{12}\Phi_2\big)(z,\epsilon) \mathcal{E}_j(z, y, \epsilon)\, dz, \qquad \mathcal{E}_0(x, y, \epsilon) := \bar{\mathcal{F}}_1^{y\to x}, \tag{3.21}$$

satisfy the uniform exponential decay estimates

$$|\mathcal{E}_j(x, y, \epsilon)| \le C_j e^{-\tilde\theta\eta|x-y|}, \tag{3.22}$$

and are as smooth in x and y and as smooth in ϵ as is $\big(\Theta_{11} + \Theta_{12}\Phi_2\big)$; and $C_j > 0$ and $\mathcal{O}(\cdot)$ are uniform in ϵ, depending only on $\tilde\theta$, M_j, and the constant C in (3.16). Symmetric bounds hold for the flow $\mathcal{F}_2^{y\to x}$ of (3.18).

Proof. This follows by a contraction mapping argument similar to (but simpler than) that used in the proof of Proposition 3.9, where expansion (3.20) is just the Jth-order Neumann expansion with remainder associated with the contraction mapping; see Appendix C of [MZ.1], or Appendix C of this paper. □

3.3. Formal block-diagonalization. To complete our discussion of ODE with slowly varying coefficients, we now supply a formal block-diagonalization procedure to be used in tandem with the rigorous error bounds of Corollary 3.11. Consider (3.1) with coefficients of the form

$$\mathbb{A}^\epsilon(x, \lambda) = A(\varepsilon x, \varepsilon), \tag{3.23}$$

ϵ small, $\lambda/|\lambda|$ (and possibly other indexing parameters) restricted to a compact set depending on ϵ, where A has formal Taylor expansion

$$A(y, \varepsilon) = \sum_{k=0}^{p} \varepsilon^k A_k(y) + O(\varepsilon^{p+1}). \tag{3.24}$$

Assume:

(h0) A decays uniformly exponentially as $y \to \pm\infty$ to limits $A_\pm$.

(h1) $A_j \in C^{p+1-j}(y)$ for $0 \le j \le p$, with derivatives uniformly bounded for all $y \in \mathbb{R}$, and $\mathcal{O}(\cdot) \in C^0$ and uniformly bounded for $y \in \mathbb{R}$, $\varepsilon \le \varepsilon_0$, some $\varepsilon_0 > 0$.

(h2) $A_0(y)$ is block-diagonalizable to form

$$D_0(y) = T_0 A_0 T_0^{-1}(y) = \text{diag}\ \{d_{0,1}, \dots, d_{0,s}\}, \quad d_j \in \mathbb{C}^{n_j \times n_j}, \tag{3.25}$$

with spectral separation between blocks (i.e., complex distance between their eigenvalues) uniformly bounded below by some $\gamma > 0$, for $\varepsilon \le \varepsilon_0$, $y \in \mathbb{R}$.

Then, we have:

Proposition 3.12[MZ.1]. *Given (h1)–(h2), there exists a uniformly well-conditioned change of coordinates $W = T\tilde{W}$ such that*

$$\tilde{W}' = D(\varepsilon x, \varepsilon)\tilde{W} + O(\varepsilon^{p+1})\tilde{W} \tag{3.26}$$

uniformly for $x \in \mathbb{R}$, $\varepsilon \le \varepsilon_0$, with

$$D(y, \varepsilon) = \sum_{k=0}^{p} \varepsilon^k D_k(y) + O(\varepsilon^{p+1}) \tag{3.27}$$

and

$$T(y, \varepsilon) = \sum_{k=0}^{p} \varepsilon^k T_k(y) \tag{3.28}$$

where D_j, $T_j \in C^{p+1-j}(y)$ with uniformly bounded derivatives for all $0 \le j \le p$, $\mathcal{O}(\epsilon^{p+1}) \in C^0(y)$ uniformly bounded, and each D_j has the same block-diagonal form

$$D_j(y) = \text{diag}\ \{d_{j,1}, \dots, d_{j,s}\}, \tag{3.29}$$

$d_{j,k} \in \mathbb{C}^{n_k \times n_k}$, as does D_0.

If there holds (h0) as well, then it is possible to choose $T_0(\cdot)$ in such a way that, also, $D_0 + \varepsilon D_1$ is determined simply by the block-diagonal part of $T_0^{-1}(A_0 + \varepsilon A_1)T_0$, or, equivalently,

$$T_0^{-1}(\partial/\partial y)T_0 \equiv 0. \tag{3.30}$$

Proof. See Propositions 4.4 and 4.8 of [MZ.1], or Appendix C of this paper. Note that our slightly weakened version of (h1) is what is actually used in the proof, and not the stronger version given in [MZ.1]. This distinction was unimportant in the relaxation case, but will be important here. □

Remarks 3.13. 1. An intuitive way to see Proposition 3.12 is to express T as a product $T_0 T_1 \cdots T_p$ and solve the resulting succession of nearby diagonalization problems; a more efficient procedure is given in [MZ.1]. From this point of view, it is clear that one derivative in y must be given up for each power of ϵ, since each transformation T_j introduces a diagonalization error of form $T_j(\partial/\partial y)T_j^{-1}$; this explains also the origin of condition (3.30). For real viscosity systems, we will need to expand to order $p=1$. Thus, we require that A_0, and therefore A and B, be C^2, which translates to the requirement F, $B \in C^3$ of (H0).

The class of perturbation series (h1) is preserved under slowly varying changes of coordinates, hence quite natural for the problem at hand. In particular, the series D obtained by diagonalization is in the same class, and so the procedure is convenient for iteration. If desired, $W^{j,\infty}(y)$ may be substituted for $C^j(y)$ everywhere.

2. The useful fact (3.30) was pointed out by Goodman [Go.1] in the diagonalizable case. Here, it is simplifying but not essential; see Remark 1.11 above.

3. We call attention to the important statement in Proposition 3.12 that any additional regularity with respect to ε in the error term $\mathcal{O}(\varepsilon^{p+1})$ in (3.24) translates to the same regularity $C^q(\varepsilon \to C^0(y))$ for the error terms $\mathcal{O}(\varepsilon^{p+1})$ in (3.27)-(3.28). In particular, if the remainder term is $C^\omega(\varepsilon \to C^0(y))$, as often happens (for example, for both relaxation and real viscous profiles), then the error term has regularity $C^\omega(\varepsilon \to C^0(y))$. This observation is used in conjunction with Proposition 3.9 and Corollary 3.11 above; specifically, it gives regularity with respect to ϵ in error terms Θ_{jk}, and the resulting graphs Φ_j, and thus in terms $\mathcal{F}^{y\to x}$, $\mathcal{E}_j$, and $\mathcal{O}(\delta/\eta)^{J+1}e^{-\tilde\theta|x-y|}$ of expansion (3.20). Likewise, we point out, in the typical case that (3.23) is obtained by rescaling, that analyticity in λ in the *original* (unrescaled) coordinates, if it holds, may also preserved by appropriate choice of T_0, even though we have expressed A in terms of the nonanalytic parameter $\lambda/|\lambda|$, and this property also is inherited in the fixed-point constructions of Proposition 3.9 and Corollary 3.11; for details, see Remarks 4.11–4.12, [MZ.1].

3.4. Splitting of a block-Jordan block. Augmenting the above discussion, we briefly consider the complementary case of an expansion (3.24) for which (h0)–(h2) hold, but (h3) is replaced by

(h3')(i) A_0 is a standard block-Jordan block of order s,

$$A_0(y) = J_{s,q} + \mu(y) I_{sq}, \quad J_{s,q} := \begin{pmatrix} 0 & I_q & \cdots & 0 \\ 0 & 0 & I_q \cdots & \\ \vdots & \vdots & \ldots & \vdots \\ 0 & \cdots & 0 & I_q \\ 0 & \cdots & \cdots & 0 \end{pmatrix}; \tag{3.31}$$

and (ii) for each y, the spectrum of the lower lefthand block $M := (A_1)_{s1}$ of A_1 lies in a compact set on which $z \to z^{1/s}$ is analytic; in particular, det $(M) \neq 0$.

Such an expansion typically arises as a single block $d_j(y,\epsilon) := \sum_{k=0}^{p} \epsilon^k d_{k,j}(y)$ of the matrix $D(y,\epsilon)$ obtained by the block-diagonalization procedure just described. Indeed, the situation may be expected to occur in general for operators of degenerate (mixed) type, indicating the simultaneous presence of multiple scales. (Clearly, still more degenerate situations may occur, and could be treated on a case-by-case basis.)

Remark 3.14. We have assumed that A_0 has already been put in form (3.31) because (h3')(ii) is not preserved under general coordinate changes, due to dynamical effects $T_0^{-1}(\partial/\partial y)T_0$. On the other hand a calculation similar to that of Proposition 4.8, [MZ.1] shows that $(T_0^{-1}(\partial/\partial y)T_0)_{s1} = 0$ for transformations preserving the block form (3.31) (namely, block upper-triangular matrices with diagonal blocks equal to some common invertible α), and so (h3')(ii) *is* independendent of the manner in which (3.31) is achieved from (3.24).

Performing the "balancing" transformation $A \to \mathcal{B}^{-1}A\mathcal{B}$,

$$\mathcal{B} := \text{diag}\,\{1, \epsilon^{1/s}, \dots, \epsilon^{(s-1)/s}\}, \tag{3.32}$$

we may convert this to an expansion

$$A(y,\epsilon) = \mu I + \sum_{k=1}^{p} \epsilon^{k/s} A_k(y) + \mathcal{O}(\epsilon^{(p+1)/s}) \tag{3.33}$$

in powers of $\epsilon^{1/s}$, $A_j \in C^{p+1-j}$, $\mathcal{O}(\epsilon^{(p+1)/s}) \in C^0$, where

$$A_1 = \begin{pmatrix} 0 & I_q & \cdots & 0 \\ 0 & 0 & I_q \cdots & \\ \vdots & \vdots & \dots & \vdots \\ 0 & \cdots & 0 & I_q \\ M & \cdots & \cdots & 0 \end{pmatrix}, \tag{3.34}$$

and M satisfies the spectral criterion (h3')(ii).

Then, the spectrum $(\sigma M)^{1/s}$ of A_1 separates into s groups of q, corresponding to the s different analytic representatives of $z^{1/s}$, and so A_1 is $(q \times q)$ block-diagonalizable, reducing to the previous case with ϵ replaced by $\epsilon^{1/s}$ and A_0 shifted to A_1. This corresponds to the identification of a new, slow scale $\epsilon^{1/s}$ in the problem. Note that differentiation still brings down a full factor of $\epsilon << \epsilon^{1/s}$, so that the first step in the diagonalization process can still be carried out, despite the shift in the series. Alternatively, we could remove the μI term by an exponential weighting, and rescale y to remove the initial factor $\epsilon^{1/s}$, thus converting to the standard situation described in Section 3.3.

3.5. Approximation of stable/unstable manifolds. Proposition 3.12 and Corollary 3.11 together give a recipe for the estimation of the stable/unstable manifolds of ODE with slowly varying coefficients, given (h0)–(h2): namely, expand formally to order $\delta = \epsilon^p$ such that $\delta/\eta \to 0$, then apply Corollary 3.11 to obtain a rigorous approximation in terms of the resulting block-decoupled system. In the case relevant to traveling waves that all A_j (not just A_0) converge exponentially in y to constants $A_j^\pm$ as $y \to \pm\infty$, we have, provided that (i) $d_{0,j}$ are scalar multiples of the identity; and (ii) there holds the necessary condition of neutral separation to zeroth order, $\text{Re } \sigma(d_{0,j}) \le 0 \le \text{Re } \sigma(d_{0,k})$, $j \le S < k$, for all y; that this procedure is possible if and only if the corresponding eigenspaces of the original matrix $\mathbb{A}^\epsilon$, or equivalently of $\epsilon^p D_p$, have a uniform spectral gap of order ϵ^p at the limits $x = \pm\infty$, i.e., there holds the easily checkable condition

$$\text{Re } \sigma(d_{p,j}\pm) \le -\theta < 0 < \theta \le \text{Re } \sigma(d_{p,k})^\pm, \quad j \le S < k, \tag{3.35}$$

for some uniform $\theta > 0$. For, this condition is clearly necessary by Lemma 3.1. On the other hand, suppose that the condition holds. The diagonalizing construction described in Proposition 3.12 preserves the property of exponential decay to constant states. Thus, making a change of coordinates such that

$$\text{Re } d_{p,j}^\pm \le -\theta/2 < 0 < \theta/2 \le \text{Re } d_{p,k}^\pm, \quad j \le S < k,$$

we obtain by zeroth order neutrality (preserved, by assumption (i)) plus exponential decay, that (3.15) holds with integrable error $\alpha(x) = \mathcal{O}(\epsilon e^{-\theta\epsilon|x|})$, verifying (3.14).

We remark that $d_{0,j}$ scalar is equivalent to constant multiplicity of the associated eigenvalue of A_0. For neutrally zeroth-order stable (resp. unstable) blocks, this is a standard structural assumption, corresponding in the critical regime $\lambda/|\lambda|$ pure imaginary to local well-posedness of the underlying evolution equations. For strictly zeroth-order stable (resp. unstable) blocks, diagonalizability is not necessary either for our construction or for local well-posedness.

Likewise, uniform spectral gap at $\pm\infty$ for a set of problems ϵ, $(\lambda/|\lambda|)(\epsilon)$ is roughly equivalent to asymptotic direction $\lim(\lambda/|\lambda|)$ pointing into the domain of consistent splitting as $|\lambda| \to \infty$, or equivalently $\epsilon \to 0$. *Thus, the requirement of exponential separation is in practice no restriction for applications to stability of traveling waves,* the domain of feasibility lying in general asymptotic to the domain of consistent splitting. The only real requirement is sufficient regularity in the coefficients of the linearized equations to carry out the formal diagonalization procedure to the necessary order p. Whether or not our regularity requirement is sharp is an interesting technical question, to which we do not know the answer. We point out only that the regularity required in the sectorial case is just $C^{0+\alpha}$, as is correct; see [ZH], or Remark 5.4 below.

Section 4. Construction of the Resolvent kernel.

We next derive explicit representation formulae for the resolvent kernel $G_\lambda(x,y)$ using the classical construction (see e.g. [CH], pp. 371–376) for the Green's function of an ordinary differential operator in terms of decaying solutions of the homogeneous eigenvalue equation

$$0 = (L-\lambda)U = (BU')' - (AU)' - \lambda)U, \tag{4.1}$$

matched across the singularity $x = y$ by appropriate jump conditions, in the process obtaining standard decay estimates on the resolvent kernel (see Proposition 3.4, below) suitable for analysis of intermediate frequencies λ. Here, A and B are as defined as in (1.10), $U \in \mathbb{R}^N := \mathbb{R}^n$, and $'$ as usual denotes d/dx. Our treatment here is a streamlined version of the ones in [ZH,Z.3,MZ.1]; for related analyses involving solution of the resolvent equation by variation of constants, see [AGJ,K.1–2].

The main novelty in the present case is the extension of the method to ordinary differential operators, such as L above, that are of degenerate type, i.e., the coefficient of the highest-order derivative is singular, but for which the eigenvalue equation can nonetheless be written as a nondegenerate first-order ODE (3.1) in an appropriate reduced phase space: in this case, $W = (u, v, b_1u' + b_2v') \in \mathbb{C}^{n+r}$. We do not need to explicitly calculate the matrix $\mathbb{A} \in \mathbb{C}^{(n+r)\times(n+r)}$ that results, but only to point out that its existence follows by invertibility of A_* and b_2, (H1), once we rewrite (4.1) as the lower triangular system of equations

$$\begin{aligned}
u' &= A_*^{-1}(-A_{12}b_2^{-1}z - (A_{11}' + \lambda)u - A_{12}'v),\\
v' &= b_2^{-1}z - b_2^{-1}b_1u',\\
z' &= (A_{21} - A_{22}b_2^{-1}b_1)u' + A_{22}b_2^{-1}z + A_{21}'u + (A_{22}' + \lambda)v,
\end{aligned} \tag{4.2}$$

with $z := b_1u' + b_2v'$. Note, as follows from the second equation, the important fact that every solution of (4.2) indeed corresponds to a solution of (4.1), with $z = b_1u' + b_2v'$.

4.1. Domain of consistent splitting. Define

$$\Lambda := \cap \Lambda_j^\pm, \quad j = 1, \dots, n, \tag{4.3}$$

where $\Lambda_j^\pm$ denote the open sets bounded on the left by the algebraic curves $\lambda_j^\pm(\xi)$ determined by the eigenvalues of the symbols $-\xi^2B_\pm - i\xi A_\pm$ of the limiting constant-coefficient operators

$$L_\pm w := B_\pm w'' - A_\pm w' \tag{4.4}$$

as ξ is varied along the real axis. The curves $\lambda_j^\pm(\cdot)$ comprise the essential spectrum of operators $L_\pm$.

Lemma 4.1. *The set Λ is equal to the component containing real $+\infty$ of the domain of consistent splitting for (3.1), (4.2). Moreover, assuming (H3), there holds*

$$\Lambda \subset \{\lambda : \text{Re } \lambda > -\eta|\text{Im } \lambda|/(1+|\text{Im } \lambda|), \qquad \eta > 0. \tag{4.5}$$

Proof. The second assertion, bound (4.5), follows easily from the bound (H3) on the dispersion curves $\lambda_j^{\pm}$. To establish the first assertion, we must show that, on the set Λ, the limiting eigenvalue equations

$$(L_{\pm} - \lambda)w = 0 \tag{4.6}$$

have no center manifold and the stable manifold at $+\infty$ and the unstable manifold at $-\infty$ have dimensions summing to the full dimension $N = n + r$; moreover, that these properties do not hold together on the boundary of Λ.

The fundamental modes of (4.6) are of form $e^{\mu x}V$, where μ, V satisfy the *characteristic equation*

$$(\mu^2 B - \mu A_{\pm} - \lambda)V = 0. \tag{4.7}$$

The existence of a center manifold thus corresponds with existence of solutions $\mu = i\xi$, V of (4.7), ξ real, i.e., solutions of the *dispersion relation*

$$(-\xi^2 B - i\xi A_{\pm} - \lambda)V = 0. \tag{4.8}$$

But, $\lambda \in \sigma(-\xi^2 B - i\xi A_{\pm})$ implies, by definition (4.3), that λ lies outside of Λ, establishing nonexistence of a center manifold. Moreover, it is clear from the same argument that a center manifold does exist on the boundary of Λ, since this corresponds to existence of pure imaginary eigenmodes.

Finally, nonexistence of a center manifold, together with connectivity of Λ, implies that the dimensions of stable/unstable manifolds at $+\infty/-\infty$ are constant on Λ. Taking $\lambda \to +\infty$ along the real axis, we find that these dimensions sum to the full dimension $n + r$ as claimed. For, Fourier expansion about $\xi = \infty$ of the dispersion relation (see Appendix B, below) yields $n - r$ "hyperbolic" modes

$$\lambda_j = -i\xi a_j^* + \dots, \quad j = 1, \dots, n-r,$$

where a_j^* denote the eigenvalues of b_2 and r "parabolic" modes

$$\lambda_{n-r+j} = -b_j\xi^2 + \dots, \quad j = 1, \dots r,$$

where b_j denote the eigenvalues of b_2; here, we have suppressed the $\pm$ indices for readability. Inverting these relationships to solve for $\mu := i\xi$, we find, for $\lambda \to \infty$, that there are $n - r$ hyperbolic roots $\mu_j \sim -\lambda/a_j^*$, and r parabolic roots $\mu_{n-r+j}^{\pm} \sim \sqrt{\lambda/b_j}$. By assumption $(\tilde{H}1)$(iii), the former yield a fixed number $k/(n-r-k)$ of stable/unstable roots, independent of x_1, and thus of $\pm$. Likewise, ($\tilde{H}1$(i) implies that the latter yields r stable, r unstable roots. Combining, we find the desired consistent splitting, with $(k+r)/\ (n-k)$ stable/unstable roots at both $\pm\infty$. $\square$

A standard fact, for asymptotically constant-coefficient ordinary differential operators (see, e.g. [He], Lemma 2, pp. 138–139) is that components of the domain of consistent splitting agree with components of the complement of the essential spectrum of the variable-coefficient operator L. Thus, Λ is a maximal domain in the essential spectrum complement, consisting of the component containing real, plus infinity. Moreover, provided that the coeffients of L approach their limits at integrable rate, it can be shown (see, e.g. [AGJ,GZ,ZH,MZ.1]) that each connected component consists either entirely of eigenvalues, or else entirely of *normal points*, defined as resolvent points or isolated eigenvalues of finite multiplicity. The latter fact will be seen directly in the course of our construction.

4.2. Basic construction. We now carry out the classical resolvent construction at points λ in Λ: more generally, at any point in the domain of consistent splitting. However, we will do it in a nonclassical way that reduces calculations and emphasizes the universality of the formulae obtained. The discussion is focused on the particular operator L at hand; the extension to general operators is clear.

Our starting point is a duality relation between solutions U of the eigenvalue equation and solutions $\tilde{U}$ of its adjoint, which may be obtained by the observation that

$$\begin{aligned} E(x) &:= \int \Big(\langle \tilde{U}, (L-\lambda)U\rangle - \langle (L^*-\lambda^*)\tilde{U}, U\rangle\Big)\,dx \\ &= \int \Big(\langle \tilde{U}, LU\rangle - \langle L^*\tilde{U}, U\rangle\Big)\,dx \end{aligned}$$

is a conserved quantity whenever U and $\tilde{U}$ are solutions of the eigenvalue and adjoint eigenvalue equations, respectively.

Since $\langle \tilde{U}, LU\rangle - \langle L^*\tilde{U}, U\rangle$ is a perfect derivative involving derivatives of U, $\tilde{U}$ of order strictly less than the maximum order of L, the quantity E may always be expressed as a quadratic form in phase space, in this case

$$E(x) = \langle \tilde{U}, BU'\rangle - \langle \tilde{U}, AU\rangle - \langle \tilde{U}', BU\rangle, \tag{4.9}$$

or

$$E(x) = \Big\langle \begin{pmatrix} \tilde{U} \\ \tilde{U}' \end{pmatrix}, \mathcal{S}\begin{pmatrix} U \\ U' \end{pmatrix} \Big\rangle(x), \tag{4.10}$$

where

$$\mathcal{S} := \begin{pmatrix} -A & B \\ -B & 0 \end{pmatrix}. \tag{4.11}$$

In general, we obtain a cross-banded matrix S, with elements of the jth band consisting of alternating $\pm$ multiples of the jth coefficient of the operator L, the anti-diagonal being filled with multiples of the principal coefficient, and the bands below the anti-diagonal being filled with zero blocks. Thus, for nondegenerate operators, it is immediate that $\mathcal{S}$ is invertible, and we obtain the characterization [ZH,MZ.1] of solutions of the eigenvalue equation as those functions U with the property that $E(x)$ is preserved for any solutions $\tilde{U}$ of the adjoint equation.

For degenerate operators, $\mathcal{S}$ is of course not invertible, and we must work instead in a reduced phase space. In the present case, we may rewrite (4.9) as

$$E(x) = \left\langle \begin{pmatrix} \tilde{U} \\ \tilde{Z} \end{pmatrix}, \bar{\mathcal{S}} \begin{pmatrix} U \\ Z \end{pmatrix} \right\rangle, \tag{4.12}$$

where

$$\bar{\mathcal{S}} := \begin{pmatrix} -A_{11} & -A_{12} & 0 \\ -A_{21} & -A_{22} & I_r \\ -b_2^{-1} b_1 & -I_r & 0 \end{pmatrix} \tag{4.13}$$

and

$$Z := (b_1, b_2) U', \quad \tilde{Z} := (0, b_2^*) \tilde{U}'. \tag{4.14}$$

Applying an elementary column operation (subtracting from the first column the second column times $b_2^{-1} b_1$), we find that

$$\det \bar{\mathcal{S}} = \det A_* \neq 0, \tag{4.15}$$

by (H1)(iii). Thus, we may conclude, similarly as in the nondegenerate case:

Lemma 4.2. *$W = (U, Z)$ satisfies eigenvalue equation (3.1), (4.2), if and only if*

$$\tilde{W}^* \bar{\mathcal{S}} W \equiv \text{constant} \tag{4.16}$$

for all $\tilde{W} = (\tilde{U}, \tilde{Z})$ satisfying the adjoint eigenvalue equation, and vice versa.

For future reference, we note the representation

$$\bar{\mathcal{S}}^{-1} = \begin{pmatrix} -A_*^{-1} & 0 & A_*^{-1} A_{12} \\ b_2^{-1} b_1 A_*^{-1} & 0 & -b_2^{-1} b_1 A_*^{-1} A_{12} - I \\ -\tilde{A} A_*^{-1} & I & -A_{22} + \tilde{A} A_*^{-1} A_{12} \end{pmatrix}, \tag{4.17}$$

$\tilde{A} := A_{21} - A_{22} b_2^{-1} b_1$, $A_* := A_{11} - A_{12} b_2^{-1} b_1$, which may be obtained by the above-mentioned column operation, followed by straightforward row reduction.

Observation 4.3. By the construction of $\bar{\mathcal{S}}$, we have in general the fact that $(I_n, 0) \bar{\mathcal{S}} W = \int LU \, dx$, or, in the present case,

$$(I_n, 0) \bar{\mathcal{S}} W = BU' - AU. \tag{4.18}$$

From Observation 4.3, we may immediately obtain the desired solution formula, with no further calculation. Let

$$\phi_j^+ := P_+ v_j^+, \quad j = 1\dots, k, \tag{4.19}$$

and

$$\phi_j^-, \quad j = k+1, \dots, n+r \tag{4.20}$$

denote the locally analytic bases of the stable manifold at $+\infty$ and the unstable manifold at $-\infty$ of solutions of the variable-coefficient equation (4.1), that were found in Section 3.1, and set

$$\Phi^+ := (\phi_1^+, \dots, \phi_k^+), \quad \Phi^- := (\phi_{k+1}^-, \dots, \phi_{n+r}^-), \tag{4.21}$$

and

$$\Phi := (\Phi^+, \Phi^-). \tag{4.22}$$

Define the solution operator from y to x of (4.1), denoted by $\mathcal{F}^{y\to x}$, as

$$\mathcal{F}^{y\to x} = \Phi(x;\lambda)\,\Phi^{-1}(y;\lambda) \tag{4.23}$$

and the projections $\Pi_y^\pm$ on the stable manifolds at $\pm\infty$ as

$$\Pi_y^+ = (\,\Phi^+(y;\lambda) \quad 0\,)\,\Phi^{-1}(y;\lambda) \quad \text{and} \quad \Pi_y^- = (\,0 \quad \Phi^-(y;\lambda)\,)\,\Phi^{-1}(y;\lambda). \tag{4.24}$$

Then, we have the following universal result (see, e.g. [ZH,MZ.1] in the nondegenerate case).

Proposition 4.4. *With respect to any L^p, $1 \le p \le \infty$, the domain of consistent splitting consists entirely of normal points of L, i.e., resolvent points, or isolated eigenvalues of constant multiplicity. On this domain, the resolvent kernel G_λ is meromorphic, with representation*

$$G_\lambda(x,y) = \begin{cases} (I_n, 0)\mathcal{F}^{y\to x}\,\Pi_y^+\,\bar{\mathcal{S}}^{-1}(y)(I_n,0)^{tr} & x > y, \\ -(I_n, 0)\mathcal{F}^{y\to x}\,\Pi_y^-\,\bar{\mathcal{S}}^{-1}(y)(I_n,0)^{tr} & x < y, \end{cases} \tag{4.25}$$

$\bar{\mathcal{S}}^{-1}$ as described in (4.17). Moreover, on any compact subset K of $\rho(L)\cap\Lambda$ ($\rho(L)$ denoting resolvent set), there holds the uniform decay estimate

$$|G_\lambda(x,y)| \le Ce^{-\eta|x-y|}, \tag{4.26}$$

where $C > 0$ and $\eta > 0$ depend only on K, L.

Proof. As discussed further in [Z.1,MZ.1], to show that λ is in the resolvent set of L, with respect to any L^p, $1 \le p \le \infty$, it is enough to: (i) construct a resolvent kernel $G_\lambda(x,y)$ satisfying (2.10)

$$(L-\lambda)G_\lambda = \delta_y(x) \tag{4.27}$$

and obeying a uniform decay estimate

$$|G_\lambda(x,y)| \le Ce^{-\eta|x-y|}; \tag{4.28}$$

and, (ii) show that there are no L^p solutions of $(L-\lambda)W = 0$, i.e., λ is not an eigenvalue of L, or, equivalently, a zero of the Evans function $D(\cdot)$ (For $p < \infty$, these are necessary as well as sufficient, [Z.1]). For, then, the Hausdorff–Young inequality yields that the distributional solution formula $(L-\lambda)^{-1}f := \int G_\lambda(x,y)f(y)dy$ yields a bounded right inverse of $(L-\lambda)$ taking L^p to L^p, while nonexistence of eigenvalues implies that this is also a left inverse.

On the domain of consistent splitting, we shall show that (ii) implies (i) (they are in fact equivalent [Z.1,MZ.1]), which immediately implies the first assertion by the properties of analytic functions and the correlation between eigenvalues and zeroes of D, Remark 3.7. It remains, then, to establish existence of a solution of (4.27) satisfying the bound (4.28), under the assumption that $D(\lambda) \ne 0$.

Note, by Lemma 3.1, that the stable manifold Φ^+ decays $x \ge 0$ with uniform rate $Ce^{\eta|x|}$, $\eta > 0$. Moreover, if $D \ne 0$, then Φ^+ does not decay at $-\infty$, whereupon we may conclude from Lemma 3.1 that it grows exponentially as $x \to -\infty$, so that it in fact decays uniformly exponentially in x on the whole line. Likewise, Φ^- decays exponentially as x decreases, uniformly on the whole line. Thus, we find that (4.25) satisfies (4.26) as claimed. Clearly, it also satisfies (4.27) away from the singular point $x = y$. Finally, we note (Observation 4.3) that

$$\begin{aligned}\int_{y^-}^{y^+} LG_\lambda &= (I_n,0)\bar{\mathcal{S}}\begin{pmatrix}[G_\lambda]\\ [(b_1,b_2)G'_\lambda]\end{pmatrix}\\ &= (I_n,0)\bar{\mathcal{S}}\mathcal{F}^{y\to y}\,\bar{\mathcal{S}}^{-1}(I_n,0)^{tr} = I_n,\end{aligned}$$

validating the jump condition across $x = y$, and completing the proof. □

Proposition 4.4 includes a satisfactory intermediate-frequency bound (4.26) on the resolvent kernel. More careful analyses will be required in the large- and small-freqency limits.

Remark 4.5. Similarly as in the strictly parabolic case [ZH], formula (4.25) extends to the full phase-variable representation

(4.29)

$$\begin{pmatrix} G_\lambda(x,y) & (\partial/\partial y)G_\lambda(x,y)(0,b_2^t)^t \\ (b_1,b_2)(\partial/\partial x)G_\lambda(x,y) & (b_1,b_2)(\partial/\partial x)(\partial/\partial y)G_\lambda(x,y)(0,b_2^t)^t \end{pmatrix} = \begin{cases} \mathcal{F}^{y\to x}\,\Pi_y^+\,\bar{\mathcal{S}}^{-1}(y) & x > y, \\ -\mathcal{F}^{y\to x}\,\Pi_y^-\,\bar{\mathcal{S}}^{-1}(y) & x < y. \end{cases}$$

This formula (though not the specific phase variable) is also universal, and follows from (4.25), which states that the upper lefthand corner is correct, together with the fact that the columns of the righthand side satisfy the forward eigenvalue equation (written as a first-order system) with respect to x, by inspection, while the rows satisfy the adjoint eigenvalue equation (again, written as a first-order system) with respect to y, by duality relation (4.16), a simple consequence of which is that the columns and rows do respect the phase-variable formulation and so we may deduce from correctness of the upper lefthand corner that all entries are correct. We shall make important use of this observation in our development of effective spectral theory in Section 6.4, below.

4.3. Generalized spectral decomposition. For the treatment of low-frequency behavior, we develop a modified representation of the resolvent kernel consisting of a *scattering decomposition* in solutions of the forward and adjoint eigenvalue equations.

From (4.16), it follows that if there are k independent solutions $\phi_1^+,\dots,\phi_k^+$ of $(L-\lambda I)W=0$ decaying at $+\infty$, and $N-k$ independent solutions $\phi_{k+1}^-,\dots,\phi_N^-$ of the same equations decaying at $-\infty$, then there exist $N-k$ independent solutions $\tilde\psi_{k+1}^+,\dots,\tilde\psi_N^+$ of $(L^*-\lambda^* I)\tilde W=0$ decaying at $+\infty$, and k independent solutions $\tilde\psi_1^-,\dots,\tilde\psi_k^-$ decaying at $-\infty$. More precisely, setting

$$\Psi^+(x;\lambda) = (\,\psi_{k+1}^+(x;\lambda) \quad \cdots \quad \psi_N^+(x;\lambda)\,) \in \mathbb{R}^{N\times(N-k)}, \tag{4.30}$$

$$\Psi^-(x;\lambda) = (\,\psi_1^-(x;\lambda) \quad \cdots \quad \psi_k^-(x;\lambda)\,) \in \mathbb{R}^{N\times k}, \tag{4.31}$$

and

$$\Psi(x;\lambda) = (\,\Psi^-(x;\lambda) \quad \Psi^+(x;\lambda)\,) \in \mathbb{R}^{N\times N}, \tag{4.32}$$

where $\psi_j^\pm$ are exponentially growing solutions obtained through Lemma 3.1, we may define dual exponentially decaying and growing solutions $\tilde\psi_j^\pm$ and $\tilde\phi_j^\pm$ via

$$(\,\tilde\Psi \quad \tilde\Phi\,)_\pm^* \,\bar{\mathcal{S}}\,(\,\Psi \quad \Phi\,)_\pm \equiv I. \tag{4.33}$$

Then, we have:

Proposition 4.6. *The resolvent kernel may alternatively be expressed as*

$$G_\lambda(x,y) = \begin{cases} (I_n,0)\Phi^+(x;\lambda)M^+(\lambda)\tilde\Psi^{-*}(y;\lambda)(I_n,0)^{tr} & x>y,\\ -(I_n,0)\Phi^-(x;\lambda)M^-(\lambda)\tilde\Psi^{+*}(y;\lambda)(I_n,0)^{tr} & x<y, \end{cases} \tag{4.34}$$

where

$$M(\lambda) := \mathrm{diag}(M^+(\lambda),M^-(\lambda)) = \Phi^{-1}(z;\lambda)\bar{\mathcal{S}}^{-1}(z)\tilde\Psi^{-1*}(z;\lambda), \tag{4.35}$$

$\tilde\Psi := \begin{pmatrix} \tilde\Psi^- & \tilde\Psi^+ \end{pmatrix}$. *(Note: the righthand side of (4.35) is independent with respect to z, as a consequence of Lemma 4.2.)*

Proof. Immediate, by rearrangement of (4.27). □

Remarks. 1. Representation (4.25) reflects the classical duality principle (see, e.g. [ZH], Lemma 4.2) that the transposition $G^*_\lambda(y,x)$ of the Green's function $G_\lambda(x,y)$ associated with operator $(L-\lambda)$ should be the Green's function for the adjoint operator $(L^*-\lambda^*)$.

2. As before, it is possible to represent the matrix G_λ by means of intrinsic objects such as solution operators and projections on stable manifolds

$$G_\lambda(x,y) = \begin{cases} (I_n,0)\mathcal{F}^{z\to x}\,\Pi_z^+\,\bar{\mathcal{S}}^{-1}(z)\,\tilde\Pi_z^-\,\tilde{\mathcal{F}}^{z\to y}\,(I_n,0)^{tr} & x>y,\\ -(I_n,0)\mathcal{F}^{z\to x}\,\Pi_z^-\,\bar{\mathcal{S}}^{-1}(z)\,\tilde\Pi_z^+\,\tilde{\mathcal{F}}^{z\to y}\,(I_n,0)^{tr} & x<y, \end{cases} \tag{4.36}$$

$$\tilde{\mathcal{F}}^{z\to y} := \tilde\Psi^{-1}(z;\lambda)\,\tilde\Psi(y;\lambda),\quad \tilde\Pi_z^+ := \tilde\Psi^{-1}\begin{pmatrix}0\\ \tilde\Psi^+\end{pmatrix}(z;\lambda)\quad \tilde\Pi_z^- := \tilde\Psi^{-1}\begin{pmatrix}\tilde\Psi^-\\ 0\end{pmatrix}(z;\lambda).$$

The exponential decay asserted in Proposition 4.4 is somewhat more straightforward to see in the this dual formulation, by judiciously choosing z in (4.36) and using the exponential decay of forward and adjoint flows on $x \gtrless 0$ alone.

From Proposition 4.6, we obtain the following scattering decomposition, generalizing the Fourier transform representation in the constant–coefficient case.

Corollary 4.7 [Z.3]. *On $\Lambda\cap\rho(L)$, there hold*

$$G_\lambda(x,y) = \sum_{j,k} M^+_{jk}(\lambda)\phi^+_j(x;\lambda)\tilde\psi^-_k(y;\lambda)^* \tag{4.37}$$

for $y\le 0\le x$,

$$G_\lambda(x,y) = \sum_{j,k} d^+_{jk}(\lambda)\phi^-_j(x;\lambda)\tilde\psi^-_k(y;\lambda)^* - \sum_k \psi^-_k(x;\lambda)\tilde\psi^-_k(y;\lambda)^* \tag{4.38}$$

for $y\le x\le 0$, and

$$G_\lambda(x,y) = \sum_{j,k} d^-_{jk}(\lambda)\phi^-_j(x;\lambda)\tilde\psi^-_k(y;\lambda)^* + \sum_k \phi^-_k(x;\lambda)\tilde\phi^-_k(y;\lambda)^* \tag{4.39}$$

for $x\le y\le 0$, with

$$M^+ = (-I,0)\begin{pmatrix}\Phi^+ & \Phi^-\end{pmatrix}^{-1}\Psi^- \tag{4.40}$$

and

$$d^\pm = \big(0,I\big)\begin{pmatrix}\Phi^+ & \Phi^-\end{pmatrix}^{-1}\Psi^-. \tag{4.41}$$

Symmetric representations hold for $y \geq 0$.

Proof. The matrix M^+ in (4.35) may be expanded using duality relation (4.16) as

$$\begin{aligned} M^+ &= (-I, 0)\,(\,\Phi^+ \quad \Phi^-\,)^{-1} A^{-1}\,(\,\tilde\Psi^- \quad \tilde\Phi^-\,)^{*\,-1} \begin{pmatrix} I \\ 0 \end{pmatrix}_{|z} \\ &= (-I, 0)\,(\,\Phi^+ \quad \Phi^-\,)^{-1}\,(\,\Psi^- \quad \Phi^-\,) \begin{pmatrix} I \\ 0 \end{pmatrix}_{|z} \\ &= (-I, 0)\,(\,\Phi^+ \quad \Phi^-\,)^{-1}\,\Psi^-_{|z}, \end{aligned} \tag{4.42}$$

yielding (4.40) for $x \geq y$, in particular for $y \leq 0 \leq x$.

Next, expressing $\phi_j^\pm(x;\lambda)$ as a linear combination of basis elements at $-\infty$, we obtain the preliminary representation

$$G_\lambda(x,y) = \sum_{j,k} d^+_{jk}(\lambda)\phi_j^-(x;\lambda)\tilde\psi_k^-(y;\lambda)^* + \sum_{j,k} e^+_{jk}\psi_j^-(x;\lambda)\tilde\psi_k^-(y;\lambda)^*, \tag{4.43}$$

valid for $y \leq x \leq 0$. Duality, (4.16), with (4.25), and the fact that $\Pi_+ = I - \Pi_-$, gives

$$\begin{aligned} -\begin{pmatrix} d^+ \\ e^+ \end{pmatrix} &= (\,\tilde\Phi^- \quad \tilde\Psi^-\,)^*\, A\Pi_+\Psi^-{}_{|x} \\ &= -\,(\,\Phi^- \quad \Psi^-\,)^{-1}\,[I - (\,0 \quad \Phi^-\,)\,(\,\Phi^+ \quad \Phi^-\,)^{-1}]\Psi^- \\ &= \begin{pmatrix} 0 \\ -I_k \end{pmatrix} - \begin{pmatrix} 0 & I_{N-k} \\ 0 & 0 \end{pmatrix} (\,\Phi^+ \quad \Phi^-\,)^{-1}\,\Psi^-, \end{aligned} \tag{4.44}$$

yielding (4.38) and (4.41) for $y \leq x \leq 0$. Relations (4.39) and (4.41) follow for $x \leq y \leq 0$ in similar, but more straightforward fashion from (4.16) and (4.25). $\square$

Remark 4.8. In the constant-coefficient case, with a choice of common bases $\Psi^\pm = \Phi^\mp$ at $\pm\infty$, (4.37)–(4.41) reduce to the simple formula

$$G_\lambda(x,y) = \begin{cases} -\sum_{j=k+1}^N \phi_j^+(x;\lambda)\tilde\phi_j^{+*}(y;\lambda) & x > y, \\ \sum_{j=1}^k \phi_j^-(x;\lambda)\tilde\phi_j^{-*}(y;\lambda) & x < y, \end{cases} \tag{4.45}$$

where, generically, $\phi_j^\pm$, $\tilde\phi_j^\pm$ may be taken as pure exponentials

$$\phi_j^\pm(x)\tilde\phi_j^{\pm*}(y) = e^{\mu_j^\pm(\lambda)(x-y)}V_j^\pm(\lambda)\tilde V_j^{\pm*}(\lambda). \tag{4.46}$$

Note that, moving individual modes $\phi_j^\pm\tilde\phi_j^\pm$ in the spectral resolution formula (2.11), using Cauchy's Theorem, to contours $\mu_j(\lambda) \equiv i\xi$ lying along corresponding dispersion curves $\lambda = \lambda_j(\xi)$, we obtain the standard decomposition of e^{Lt} into eigenmodes of continuous spectrum: in this (constant-coefficient) case, just the usual representation obtained by Fourier transform solution.

Section 5. High frequency expansion.

We now turn to the crucial estimation of the resolvent kernel in the high frequency regime $|\lambda| \to +\infty$. According to the usual duality between spatial and frequency variables, large frequency λ corresponds to small time t, or *local behavior* in space and time. Therefore, we begin by examining the frozen-coefficient equations at each value x_0 of x:

$$U_t = L_{x_0} U := B(x_0)U_{xx} - A(x_0)U_x. \tag{5.1}$$

We shall for clarity state first the result in the case that A_* is strictly hyperbolic, indicating afterward the extension to the general case.

Lemma 5.1. *Let $A_*(U(x_0))$ have distinct, real eigenvalues. Then, for $|\lambda|$ sufficiently large, the eigenvalue equation $(L_{x_0} - \lambda)W = 0$ associated with the frozen-coefficient operator L_{x_0} at x_0 has a basis of $N = n + r$ solutions*

$$\{\bar\phi_1^+, \dots, \bar\phi_k^+, \bar\phi_{k+1}^-, \dots, \bar\phi_N^-\}(x; \lambda, x_0), \quad \bar\phi_j^\pm = e^{\mu_j(\lambda, x_0)x} V_j(\lambda, x_0),$$

of which $n - r$ are "hyperbolic modes," analytic in $1/\lambda$ and satisfying

$$\begin{aligned} \mu_j^\pm(\lambda, x_0) &= -\lambda/a_j^*(x_0) - \eta_j^*(x_0)/a_j^*(x_0) + \mathcal{O}(1/\lambda), \\ V_j(\lambda, x_0) &= R_j^*(x_0)/a_j^*(x_0) + \mathcal{O}(1/\lambda), \end{aligned} \tag{5.2}$$

$a_j^*(\cdot)$, $\eta_j^*(\cdot)$, and $R_j^*(\cdot)$ as defined in Section 1.5 (now with $\partial_x(b_2^{-1}b_1) \equiv 0$), and $2r$ are "parabolic modes," analytic in $1/\sqrt{\lambda}$ and satisfying

$$\begin{aligned} \mu_j^\pm(\lambda, x_0) &= -\sqrt{\lambda/\gamma_j(x_0)} + \mathcal{O}(1/\lambda), \\ V_j(\lambda, x_0) &= \begin{pmatrix} 0_{n-r} \\ s_j \\ \mu_j s_j \end{pmatrix} (x_0) + \mathcal{O}(1/\lambda), \end{aligned} \tag{5.3}$$

$\gamma_j(\cdot)$, $s_j(\cdot)$, the eigenvalues and right eigenvectors of $b_2(U(x_0))$. Likewise, the adjoint eigenvalue equation $(L_{x_0} - \lambda)^* \tilde W = 0$ has a basis of solutions

$$\{\tilde{\bar\phi}_1^-, \dots, \tilde{\bar\phi}_k^-, \tilde{\bar\phi}_{k+1}^+, \dots, \tilde{\bar\phi}_N^+\}(x; \lambda, x_0), \quad \tilde{\bar\phi}_j^\pm = e^{-\mu_j(\lambda, x_0)x} \tilde V_j(\lambda, x_0),$$

with

$$\tilde V_j(\lambda, x_0) = L_j^*(x_0) + \mathcal{O}(1/\lambda), \quad \tilde V_j(\lambda, x_0) = \begin{pmatrix} \gamma_j^{-1} b_1^{tr} t_j \\ t_j \\ -\mu_j t_j \end{pmatrix} (x_0) + \mathcal{O}(1/\lambda), \tag{5.4}$$

respectively, for hyperbolic and parabolic modes, $L_j^*(\cdot)$ as defined in Section 1.3 and $t_j(\cdot)$ the left eigenvectors of $b_2(U(x_0))$.

The expansions (5.2)–(5.4) hold also in the nonstrictly hyperbolic case, with $\bar\phi_j$, $\bar{\tilde\phi}_j$ and R_j^*, L_j^* now denoting $n \times m_j^*$ blocks, and μ_j, η_j denoting $m_j^* \times m_j^*$ matrices, $j = 1, \dots, J$, where m_j^* is the multiplicity of eigenvalue a_j^* of $A_*(x_0)$.

Proof. Similarly as in the proof of Lemma 4.1, this follows by inversion of the corresponding expansions about $\xi = \infty$ of the eigenvalues $\lambda_j(\xi)$ and eigenvectors $V_j(\xi)$ of the frozen-coefficient Fourier symbol $-i\xi A(x_0) - \xi^2 B(x_0)$, using the basic relation $\mu_j = i\xi$ relating characteristic and dispersion equations (4.7) and (4.8). The expansions of the Fourier symbol are carried out in Appendix B. $\square$

By (4.45), Lemma 5.1 gives an expression for the constant-coefficient resolvent kernel $G_{\lambda;x_0}$ at x_0 of, to lowest order, a "hyperbolic" part,

(5.5)
$$H_{\lambda;x_0} \sim \begin{cases} \sum_{j=1}^k a_j^*(x_0)^{-1} e^{(-\lambda/a_j^* - \eta_j^*/a_j^*)(x_0)(x-y)} R_j^*(x_0) L_j^{*t}(x_0) & x < y, \\ -\sum_{j=k+1}^J a_j^*(x_0)^{-1} e^{(-\lambda/a_j^* - \eta_j^*/a_j^*)(x_0)(x-y)} R_j^*(x_0) L_j^{*t}(x_0) & x > y, \end{cases}$$

with next-order contribution a "parabolic" part $P_{\lambda;x_0}$, analytic in $\lambda^{-1/2}$ (i.e., for λ away from the negative real axis) and satisfying

$$P_{\lambda;x_0} = \mathcal{O}(|\lambda|^{-1/2} e^{-\theta|\lambda|^{1/2}|x-y|}), \quad \theta > 0, \tag{5.6}$$

on some sector

$$\Omega_P := \{\lambda :\ \mathrm{Re}\ \lambda \geq -\theta_1 |\mathrm{Im}\ \lambda| + \theta_2\}, \quad \theta_j > 0. \tag{5.7}$$

Our main result of this section, and the main technical challenge of the paper as a whole, will be to establish on an appropriate unbounded subset of the resolvent set $\rho(L)$ that the variable coefficient solutions $\phi_j^\pm$, $\tilde\phi_j^\pm$, and thus the resolvent kernel, satisfy analogous formulae, with static quantities replaced by the corresponding dynamical ones defined in Section 1.5. More precisely, define

$$\Omega := \{\lambda : -\eta_1 \leq \mathrm{Re}\ \lambda\}, \tag{5.8}$$

with $\eta_1 > 0$ sufficiently small that $\Omega \setminus B(0, r)$ is compactly contained in the set of consistent splitting Λ (defined in (4.3)), for some small $r > 0$ to be chosen later; this is possible, by Lemma 4.1(i).

Proposition 5.2. *Assume that there hold (H0)–(H4), plus strict hyperbolicity of $A_*(\cdot)$. Then, for any $r > 0$, and $\eta_1 = \eta_1(r) > 0$ chosen sufficiently small, there holds $\Omega \setminus B(0, r) \subset \Lambda \cap \rho(L)$. Moreover, for $R > 0$ sufficiently large, there holds on $\Omega \setminus B(0, R)$ the decomposition*

$$G_\lambda(x, y) = H_\lambda(x, y) + P_\lambda(x, y) + \Theta_\lambda(x, y), \tag{5.9}$$

(5.10)
$$H_\lambda(x,y) := \begin{cases} -\sum_{j=k+1}^{J} a_j^*(y)^{-1} e^{\int_y^x (-\lambda/a_j^* - \eta_j^*/a_j^*)(z)\,dz} R_j^*(x) L_j^{*t}(y) & x>y,\\ \sum_{j=1}^{k} a_j^*(y)^{-1} e^{\int_y^x (-\lambda/a_j^* - \eta_j^*/a_j^*)(z)\,dz} R_j^*(x) L_j^{*t}(y) & x<y, \end{cases}$$

$$\Theta_\lambda(x,y) = \lambda^{-1}B(x,y;\lambda) + \lambda^{-1}(x-y)C(x,y;\lambda) + \lambda^{-2}D(x,y;\lambda), \tag{5.11}$$

where

$$B(x,y;\lambda) = \begin{cases} \sum_{j=k+1}^{J} e^{-\int_y^x \lambda/a_j^*(z)\,dz} b_j^+(x,y) & x>y,\\ \sum_{j=1}^{k} e^{-\int_y^x \lambda/a_j^*(z)\,dz} b_j^-(x,y) & x<y, \end{cases} \tag{5.12}$$

(5.13)
$$C(x,y;\lambda) = \begin{cases} -\sum_{i,j=k+1}^{J} \mathrm{mean}_{z\in[x,y]} e^{-\int_y^z \lambda/a_i^*(s)ds - \int_z^x \lambda/a_j^*(s)ds} c_{i,j}^+(x,y;z), & x>y,\\ \sum_{i,j=1}^{k} \mathrm{mean}_{z\in[x,y]} e^{-\int_y^z \lambda/a_i^*(s)ds - \int_z^x \lambda/a_j^*(s)ds} c_{i,j}^-(x,y;z), & x<y, \end{cases}$$

with

$$|b_j^\pm|,\ |c_{i,j}^\pm| \le Ce^{-\theta|x-y|} \tag{5.14}$$

and

$$D(x,y;\lambda) = \mathcal{O}(e^{-\theta(1+\mathrm{Re}\ \lambda)|x-y|} + e^{-\theta\sqrt{|\lambda|}|x-y|}), \tag{5.15}$$

for some uniform $\theta > 0$ independent of x, y, z, each described term separately analytic in λ, and P_λ is analytic in λ on a (larger) sector Ω_P as in (5.7), with θ_1 sufficiently small, and θ_2 sufficiently large, satisfying uniform bounds

$$(\partial/\partial x)^\alpha(\partial/\partial y)^\beta P_\lambda(x,y) = \mathcal{O}(|\lambda|^{(|\alpha|+|\beta|-1)/2} e^{-\theta|\lambda|^{1/2}|x-y|}), \quad \theta>0, \tag{5.16}$$

for $|\alpha| + |\beta \le 2$ and $0 \le |\alpha|, |\beta| \le 1$.

Likewise, there hold derivative bounds

(5.17)
$$\begin{aligned}(\partial/\partial x)\Theta_\lambda(x,y) = \big(B_x^0(x,y;\lambda) + (x-y)C_x^0(x,y;\lambda)\big) + \lambda^{-1}\big(B_x^1(x,y;\lambda)\\ + (x-y)C_x^1(x,y;\lambda) + (x-y)^2 D_x^1(x,y;\lambda)\big) + \lambda^{-3/2}E_x(x,y;\lambda)\end{aligned}$$

and

(5.18)
$$\begin{aligned}(\partial/\partial y)\Theta_\lambda(x,y) = \big(B_y^0(x,y;\lambda) + (x-y)C_y^0(x,y;\lambda)\big) + \lambda^{-1}\big(B_y^1(x,y;\lambda)\\ + (x-y)C_y^1(x,y;\lambda) + (x-y)^2 D_y^1(x,y;\lambda)\big) + \lambda^{-3/2}E_y(x,y;\lambda),\end{aligned}$$

where B^α_β and C^α_β satisfy bounds of form (5.12) and (5.13), D^1_β now denotes the iterated integral

(5.19)

$$D^1_\beta(x,y;\lambda)=\begin{cases} -\sum_{h,i,j=k+1}^J \mathrm{mean}_{y\le w\le z\le x} e^{-\int_y^w \lambda/a_h^*(s)ds-\int_w^z \lambda/a_i^*(s)ds-\int_z^x \lambda/a_j^*(s)ds} \\ \qquad \times d^{\beta,+}_{h,i,j}(x,y;z), & x>y,\\ \sum_{h,i,j=1}^k \mathrm{mean}_{x\le z\le w\le y} e^{-\int_y^w \lambda/a_h^*(s)ds-\int_w^z \lambda/a_i^*(s)ds-\int_z^x \lambda/a_j^*(s)ds} \\ \qquad \times d^{\beta,-}_{h,i,j}(x,y;z), & x<y,\end{cases}$$

with $|d^{\beta,-}_{h,i,j}|\le Ce^{-\theta|x-y|}$, and E_β satisfies a bound of form (5.15).

The bounds (5.11)–(5.18) hold also in the nonstrictly hyperbolic case, with (5.10) replaced by

(5.20)

$$H_\lambda(x,y)=\begin{cases} -\sum_{j=K+1}^J a_j^*(y)^{-1}e^{-\int_y^x \lambda/a_j^*(z)\,dz}R_j^*(x)\tilde\zeta_j(x,y)L_j^{*t}(y) & x>y,\\ \sum_{j=1}^K a_j^*(y)^{-1}e^{-\int_y^x \lambda/a_j^*(z)\,dz}R_j^*(x)\tilde\zeta_j(x,y)L_j^{*t}(y) & x<y,\end{cases}$$

where $\tilde\zeta_j(x,y)\in\mathbb{R}^{m_j^*\times m_j^*}$ denotes a dissipative flow similar to ζ_j in (1.30), but with respect to variable x: i.e.,

$$d\tilde\zeta_j/dx=\eta_j^*(x)\tilde\zeta_j(x,y)/a_j^*(x),\quad \tilde\zeta_j(y,y)=I, \tag{5.21}$$

or, equivalently, $\tilde\zeta_j(x,y)=\zeta_j(y,\tau)$ for $z_j(y,\tau)=x$, where z_j as in (1.29) denotes the characteristic path associated with a_j^*; R_j^*, L_j^* now denote $n\times m_j^*$ blocks; and μ_j, η_j^* denote $m_j^*\times m_j^*$ matrices, $j=1,\dots,K$, $j=K+1,\dots,J$, where m_j^* is the multiplicity of eigenvalue a_j^* of $A_*(x_0)$, with

$$a_1^*\le\cdots\le a_K^*<0<a_{K+1}^*\le\cdots\le a_J^*.$$

(Note that we have not here assumed $(\mathcal{D})$).

Proof. Our argument is similar in spirit to the proof of Proposition 4.4 [MZ.1] in the hyperbolic relaxation case. However, the mixed hyperbolic–parabolic nature of the underlying equations, and the associated presence of multiple scales, makes the analysis more delicate. In accordance with the behavior we seek to identify, it is convenient to express the eigenvalue equation in "local" variables $\tilde u:=A_*u$, $\tilde v:=b_1u+b_2v$ similar to those used in proving Proposition 2.1, yielding

$$\begin{aligned}\tilde u_x&=-\lambda A_*^{-1}\tilde u-(A_{12}b_2^{-1}\tilde v)_x,\\ (\tilde v_x)_x&=\Big[((A_{21}-A_{22}b_2^{-1}b_1+b_2\partial_x(b_2^{-1}b_1))A_*^{-1}\tilde u)_x\\ &\quad+(A_{22}+\partial_x(b_2))b_2^{-1}\tilde v)_x+\lambda b_2^{-1}b_1A_*^{-1}\tilde u+\lambda b_2^{-1}\tilde v\Big].\end{aligned}\tag{5.22}$$

Similarly as in the relaxation case, the substitution of $A_* u$ for u converts the hyperbolic part of the equation from divergence to nondivergence form, simplifying slightly the flow along characteristics in the hyperbolic term H_λ; for related discussion, see Remark 1.11.

Following standard procedure (e.g., [AGJ,GZ,ZH,Z.3]), we perform the rescaling

$$\tilde{x} := |\lambda| x, \quad \tilde{\lambda} := \lambda/|\lambda|. \tag{5.23}$$

suggested by the leading order hyperbolic behavior of $\mu(x_0)$, (5.2), to obtain after some rearrangement the perturbation equation

$$Y' = A(\tilde{x}, |\lambda|^{-1})Y, \quad Y := (\tilde{u}, \tilde{v}, \tilde{v}')^{tr}, \tag{5.24}$$

where

$$A(\tilde{x}, |\lambda|^{-1}) = A_0(\tilde{x}) + |\lambda|^{-1} A_1(\tilde{x}) + \mathcal{O}(|\lambda|^{-2}), \tag{5.25}$$

(5.26)

$$A_0(\tilde{x}) := \begin{pmatrix} -\tilde{\lambda} A_*^{-1} & 0 & -A_{12} b_2^{-1} \\ 0 & 0 & I_r \\ 0 & 0 & 0 \end{pmatrix}, \quad A_1(\tilde{x}) := \begin{pmatrix} 0 & \partial_x (A_{12} b_2^{-1}) & 0 \\ 0 & 0 & 0 \\ -\tilde{\lambda} d_* A_*^{-2} & -\tilde{\lambda} b_2^{-1} & e_* b_2^{-1} \end{pmatrix},$$

$$\begin{aligned} d_* &:= A_{21} - A_{22} b_2^{-1} b_1 + b_2^{-1} b_1 A_* + b_2 \partial_x (b_2^{-1} b_1), \\ e_* &:= A_{22} + d_* A_*^{-1} A_{12} + \partial_x (b_2), \end{aligned} \tag{5.27}$$

and "$'$" denotes $\partial/\partial\tilde{x}$. Here, expansion (5.25) is to be considered as a continuous family of one-parameter perturbation equations, indexed by $\tilde{\lambda} \in S^1$. The form of e_* is not important, and is included only for completeness; d_*, on the other hand, may be recognized as the important quantity defined in (1.36).

We seek to develop corresponding expansions of the resolvent kernel in orders of $|\lambda|^{-1}$, or, equivalently, by representation (4.25), expansions of stable/unstable manifolds of (5.24), and the reduced flows therein. This we will accomplish by the methods described in Sections 3.1–3.2, first reducing to an approximately block-diagonal system segregating spectrally separated (in particular, stable/unstable) modes, with formal error of a suitably small order, then converting the formal error into rigorous error bounds using Proposition 3.9 together with Corollary 3.11. An interesting aspect of this calculation is the block-diagonalization of parabolic modes, which are not uniformly spectrally separated in this (hyperbolic) scaling, but rather comprise a block-Jordan block of order $s = 2$ associated with eigenvalue $\mu = 0$. Accordingly, we treat them separately, by the method sketched in Section 3.4, after the initial block-diagonalization has been carried out. This results in a slower decaying error series for these blocks, converging in powers of $|\lambda|^{-1/2}$; parabolic modes are well-behaved, however, and so the loss of accuracy is harmless.

We will carry out in detail the basic estimate (5.11), indicating the extension to derivative bounds (5.17)–(5.18) by a few brief remarks. First, observe that in the modified coordinates $Y = \mathcal{Q}W$, (4.25) just becomes

$$G_\lambda(x,y) = \begin{cases} (I_n, 0)\mathcal{Q}^{-1}(x)\mathcal{F}_Y^{y\to x}\Pi_Y^- \mathcal{Q}\bar{\mathcal{S}}^{1}(y)(I_n,0)^{tr} & x<y, \\ \\ -(I_n, 0)\mathcal{Q}^{-1}(x)\mathcal{F}_Y^{y\to x}\Pi_Y^+ \mathcal{Q}\bar{\mathcal{S}}^{-1}(y)(I_n,0)^{tr} & x>y, \end{cases} \tag{5.28}$$

where $\Pi_Y^\pm$ and $\mathcal{F}_Y^{y\to x}$ denote projections and flows in Y-coordinates, and the lower triangular matrices

$$\mathcal{Q} = \begin{pmatrix} I_{n-r} & 0 & 0 \\ 0 & I_r & 0 \\ 0 & 0 & |\lambda|^{-1}I_r \end{pmatrix} \begin{pmatrix} A_* & 0 & 0 \\ b_1 & b_2 & 0 \\ \partial_x(b_1) & \partial_x(b_2) & I_r \end{pmatrix}, \tag{5.29}$$

$$\mathcal{Q}^{-1} = \begin{pmatrix} A_*^{-1} & 0 & 0 \\ -b_2^{-1}b_1A_*^{-1} & b_2^{-1} & 0 \\ -b_2\partial_x(b_2^{-1}b_1)A_*^{-1} & -\partial_x(b_2)b_2^{-1} & I_r \end{pmatrix} \begin{pmatrix} I_{n-r} & 0 & 0 \\ 0 & I_r & 0 \\ 0 & 0 & |\lambda| I_r \end{pmatrix} \tag{5.30}$$

relate Y coordinates to the original coordinates $W = (u, v, b_1u_x + b_2v_x)^{tr}$ of (4.25):

$$Y = \mathcal{Q}W, \quad W = \mathcal{Q}^{-1}Y. \tag{5.31}$$

It is readily checked that $\mathcal{Q} \in C^{p+1}$ for f, $B \in C^{p+2}$, whence $\mathcal{Q} \in C^2$ under the regularity hypothesis (H0) ($p = 3$).

Initial diagonalization. For f, $B \in C^{p+2}$, we may check further that $A_j \in C^{p-j+1}$ as required in the theory of Section 3.3, suitable for approximate block-diagonalization to formal error $\mathcal{O}(|\lambda|^{-p-1})$: formal error $\mathcal{O}(|\lambda|^{-4})$ in the present case $p = 3$. Applying the formal diagonalization procedure of Proposition 3.12 to (5.24), and choosing the special initialization (3.30), at the same time taking care as described in Remark 3.13(3) to preserve analyticity with respect to λ^{-1} in original coordinates, we obtain the approximately block-diagonalized system

$$Z' = D(\tilde{x}, |\lambda|^{-1})Z, \quad TZ := Y, \quad D := TAT^{-1}, \tag{5.32}$$

$$T(\tilde{x}, |\lambda|^{-1}) = T_0(\tilde{x}) + |\lambda|^{-1}T_1(\tilde{x}) + \cdots + |\lambda|^{-3}T_3(\tilde{x}), \tag{5.33}$$

$$D(\tilde{x}, |\lambda|^{-1}) = D_0(\tilde{x}) + |\lambda|^{-1}D_1(\tilde{x}) + \cdots + D_3(\tilde{x})|\lambda|^{-3} + \mathcal{O}(|\lambda|^{-4}), \tag{5.34}$$

where without loss of generality (since T_0 is uniquely determined up to a constant linear coordinate change)

(5.35)
$$T_0^{-1} = \begin{pmatrix} L_*^{tr} & 0 & \tilde\lambda^{-1} L_*^{tr} A_* A_{12} b_2^{-1} \\ 0 & I_r & 0 \\ 0 & 0 & I_r \end{pmatrix}, \quad T_0 = \begin{pmatrix} R_* & 0 & -\tilde\lambda^{-1} A_* A_{12} b_2^{-1} \\ 0 & I_r & 0 \\ 0 & 0 & I_r \end{pmatrix},$$

$L_* := (L_1^*, \dots, L_J^*)$, $R_* := (R_1^*, \dots, R_J^*)$, L_j^*, R_j^* as defined in (1.18), and so

(5.36)
$$D_0 = \begin{pmatrix} -\operatorname{diag}\{\tilde\lambda I_{m_j^*}/a_j^*\} & 0 & 0 \\ 0 & 0 & I_r \\ 0 & 0 & 0 \end{pmatrix}, \quad D_1 = \begin{pmatrix} -\operatorname{diag}\{\eta_j^* I_{m_j^*}/a_j^*\} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\tilde\lambda b_2^{-1} & * \end{pmatrix},$$

a_j^* and m_j^* as defined in (1.17), and η_j^* as defined in (1.21)–(1.22). Here, we have used the simple block upper-triangular form of A_0 to deduce the form of (5.35), then calculated (5.36) from the relations $D_0 = T_0^{-1} A_0 T_0$, $D_1 = T_0^{-1} A_1 T_0$, the second a consequence of the special normalization (3.30).

The parabolic block. At this point, we have diagonalized into J $m_j^* \times m_j^*$ hyperbolic blocks corresponding to eigenvalues $\tilde\mu = -\tilde\lambda / a_j^*$ of A_0, and a $2r \times 2r$ parabolic block-Jordan block (the lower righthand corner)

$$N := \begin{pmatrix} 0 & I_r \\ 0 & 0 \end{pmatrix} + |\lambda|^{-1} \begin{pmatrix} 0 & 0 \\ -\tilde\lambda b_2^{-1} & * \end{pmatrix} + \mathcal{O}(|\lambda|^{-2}), \tag{5.37}$$

corresponding to eigenvalue $\tilde\mu = 0$ of A_0. We now focus on the latter block, applying the procedure of Section 3.4. "Balancing" by transformation $\mathcal{B} := \{1, |\lambda|^{1/2}\}$ yields

$$\tilde M := \mathcal{B}^{-1} N \mathcal{B} = |\lambda|^{-1/2} \tilde M_1 + \mathcal{O}(|\lambda|^{-1}), \quad \tilde M_1 := \begin{pmatrix} 0 & I_r \\ -\tilde\lambda b_2^{-1} & 0 \end{pmatrix}, \tag{5.38}$$

or $|\lambda|^{-1/2}$ times the standard rescaled expansion of the strictly parabolic case [ZH].

Proceeding as in that case, we observe that $\sigma(\tilde M_1) = \pm\sqrt{\sigma(\tilde\lambda b_2^{-1})}$ has a uniform spectral gap of order one, by (H1)(i), on the sector Ω_P defined in (5.7), for $\theta_1 > 0$ sufficiently small. Thus (see, e.g. Proposition A.9, p. 361 of [St]), there exists a well-conditioned coordinate change $S = S(\tilde M_1(y))$ depending continuously on $\tilde M_1$, such that

$$\hat M_1 = \operatorname{diag}\{\hat M_1^-, \hat M_1^+\}(\tilde x) := S^{-1} \tilde M S(\tilde x)$$

with $\hat M_1^-$ uniformly negative definite and $\hat M_1^+$ uniformly positive definite. Applying this coordinate change, and noting that the "dynamic error" $S^{-1}(\partial/\partial\tilde x) S$ is of order $(\partial/\partial\tilde x)\tilde M_1 = \mathcal{O}(|\lambda|^{-1})$, we obtain, finally, the formal expansion

$$\hat M(\tilde x, |\tilde\lambda|^{-1}) = |\lambda|^{-1/2} \operatorname{diag}\{\hat M_1^-, \hat M_1^+\} + \mathcal{O}(|\lambda|^{-1}). \tag{5.39}$$

On the sector Ω_P, blocks $\pm$ are exponentially separated to order $|\lambda|^{-1/2}$, by Remark 3.8. Thus, we may apply Proposition 3.9 to find that, on Ω_P, there is a further transformation $\hat{S} = I_{2r} + \mathcal{O}(|\lambda|^{-1/2})$ converting (5.39) to fully diagonalized form

$$M(\tilde{x}, |\tilde{\lambda}|^{-1}) = |\lambda|^{-1/2} \operatorname{diag}\{M_1^-, M_1^+\}, \tag{5.40}$$

where $M_1^\pm = \hat{M}_1^\pm + \mathcal{O}(|\lambda|^{-1/2})$ are still uniformly positive/negative definite. We will define P_λ to be the part of the Green's function associated to the flow of this diagonal block. Note that it has been obtained independent of any separation (or lack thereof) from hyperbolic modes, so is analytic on all of Ω_p

More precisely, denote by

$$\mathcal{Z} := \mathcal{T}Y := \begin{pmatrix} I_{n-r} & 0 \\ 0 & \hat{S}S\mathcal{B} \end{pmatrix} (T_0 + |\lambda|^{-1}T_1)Y \tag{5.41}$$

the concatenation of coordinate changes made in the course of the above reductions, $\mathcal{Z} = (\zeta_1, \dots, \zeta_J, \rho_-, \rho_+)^{tr}$. Then, we have, similarly as in (5.28), the representation

$$G_\lambda(x,y) = \begin{cases} (I_n, 0)\mathcal{Q}^{-1}\mathcal{T}^{-1}(x)\mathcal{F}_{\mathcal{Z}}^{y\to x}\Pi_{\mathcal{Z}}^-\mathcal{T}\mathcal{Q}\bar{\mathcal{S}}^{1}(y)(I_n,0)^{tr} & x<y, \\ \\ -(I_n, 0)\mathcal{Q}^{-1}\mathcal{T}^{-1}(x)\mathcal{F}_{\mathcal{Z}}^{y\to x}\Pi_{\mathcal{Z}}^+\mathcal{T}\mathcal{Q}\bar{\mathcal{S}}^{-1}(y)(I_n,0)^{tr} & x>y, \end{cases} \tag{5.42}$$

where $\Pi_{\mathcal{Z}}^\pm$ and $\mathcal{F}_{\mathcal{Z}}^{y\to x}$ denote projections and flows in $\mathcal{Z}$-coordinates, and $\mathcal{Z}$ satisfies the approximately decoupled system of equations

$$\begin{aligned} \zeta_j' &= -(\tilde{\lambda}/a_j^* - |\lambda|^{-1}\eta_j^*/a_j^* + c_j|\lambda|^{-2} + d_j|\lambda|^{-3})\zeta_j + \mathcal{O}(|\lambda|^{-4})(\rho,\zeta), \\ \rho_\pm' &= |\lambda|^{-1/2}M_1^\pm\rho_\pm + \mathcal{O}(|\lambda|^{-7/2})\zeta. \end{aligned} \tag{5.43}$$

(Note: the degraded error estimate in the ρ-equations is due to the ill-conditioning of the balancing transformation $\mathcal{B}$.) Approximating $\Pi_{\mathcal{Z}}^\pm$ and $\mathcal{F}_{\mathcal{Z}}^{y\to x}$ by the corresponding projections $\bar{\Pi}_{\mathcal{Z}}^\pm = \bar{\Pi}_{\mathcal{Z}}^{H,\pm} + \bar{\Pi}_{\mathcal{Z}}^{P,\pm}$ and flows $\bar{\mathcal{F}}_{\mathcal{Z}}^{y\to x}$ for the associated decoupled equations, where $\bar{\Pi}_{\mathcal{Z}}^{H,\pm}$ and $\bar{\Pi}_{\mathcal{Z}}^{P,\pm}$ denote the subprojections on hyperbolic and parabolic blocks, we obtain for $x<y$ the approximation

$$\bar{G}_\lambda(x,y) := (I_n,0)^{tr}\mathcal{Q}^{-1}\mathcal{T}^{-1}(x)\bar{\mathcal{F}}_{\mathcal{Z}}^{y\to x}(\bar{\Pi}_{\mathcal{Z}}^{H,-} + \bar{\Pi}_{\mathcal{Z}}^{H,-})\mathcal{T}\mathcal{Q}\bar{\mathcal{S}}^{1}(y)(I_n,0), \tag{5.44}$$

and similarly for $x>y$. Defining

$$\begin{aligned} P_\lambda(x,y) &:= (I_n,0)\mathcal{Q}^{-1}\mathcal{T}^{-1}(x)\bar{\mathcal{F}}_{\mathcal{Z}}^{y\to x}\bar{\Pi}_{\mathcal{Z}}^{P,-}\mathcal{T}\mathcal{Q}\bar{\mathcal{S}}^{1}(y)(I_n,0)^{tr} \\ &= (I_n,0)\mathcal{Q}^{-1}\mathcal{T}^{-1}(x)\bar{\mathcal{F}}_{\mathcal{Z}}^{y\to x}\operatorname{diag}\{0_{n-r}, I_r, 0_r\}\mathcal{T}\mathcal{Q}\bar{\mathcal{S}}^{1}(y)(I_n,0)^{tr}, \end{aligned} \tag{5.45}$$

and untangling the various coordinate changes, we find that P_λ satisfies bounds (5.16), similarly as in the strictly parabolic case [ZH]. Specifically, we have by direct calculation that

$$(I_n,0)\mathcal{Q}^{-1} = \begin{pmatrix} A_*^{-1} & 0 & 0 \\ -b_2^{-1}b_1A_*^{-1} & b_2^{-1} & 0 \end{pmatrix} = \mathcal{O}(1) \tag{5.46}$$

and

$$\mathcal{Q}\bar{\mathcal{S}}^{-1}(I_n,0)^{tr} = \begin{pmatrix} -I_{n-r} & 0 \\ 0 & 0 \\ \mathcal{O}(|\lambda|^{-1}) & |\lambda|^{-1}I_r \end{pmatrix}, \tag{5.47}$$

whence (by the special form of T_0)

$$(T_0 + |\lambda|^{-1}T_1)\mathcal{Q}\bar{\mathcal{S}}^{-1}(I_n,0)^{tr} = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) \\ 0 & 0 \\ \mathcal{O}(|\lambda|^{-1}) & |\lambda|^{-1}I_r \end{pmatrix} \tag{5.48}$$

and therefore

$$\Pi^{P,-}_{\mathcal{Z}}\mathcal{T}\mathcal{Q}\bar{\mathcal{S}}^{-1}(I_n,0)^{tr} = \mathcal{O}(|\lambda|^{-1/2}). \tag{5.49}$$

Combining with the evident bound

$$|\bar{\mathcal{F}}^{y\to x}_{\mathcal{Z}}\bar{\Pi}^{P,-}_{\mathcal{Z}}| \le Ce^{-\theta|\lambda|^{-1/2}|\tilde{x}-\tilde{y}|} = Ce^{-\theta|\lambda|^{1/2}|x-y|}, \tag{5.50}$$

for some $\theta > 0$, coming from $M_1^- \le -\theta|\lambda|^{-1/2}$, we obtain (5.16) for $|\alpha| = 0$. Derivative bounds then follow from the fact that x and y derivatives of all transformations are at worst order one, while

$$|(\partial/\partial(x,y))^\alpha \bar{\mathcal{F}}^{y\to x}_{\mathcal{Z}}\bar{\Pi}^{P,-}_{\mathcal{Z}}| = |\lambda|^{|\alpha|}|(\partial/\partial(\tilde{x},\tilde{y}))^\alpha \bar{\mathcal{F}}^{y\to x}_{\mathcal{Z}}\bar{\Pi}^{P,-}_{\mathcal{Z}}| = \mathcal{O}(|\lambda|^{-|\alpha/2|})$$

by inspection of the decoupled equation $\rho_-' = |\lambda|^{-1/2}M_1^-\rho_-$. For closely related but much simpler calculations in the strictly parabolic case, see the proof of Proposition 7.3, [ZH] (discussion just below (7.24)).

Approximation error. Let us now return to the approximately decoupled equations (5.43), or, converting back to the original spatial coordinates:

$$\begin{aligned} \zeta_j' &= -(\lambda/a_j^* - \eta_j^*/a_j^* + c_j\lambda^{-1} + d_j\lambda^{-2})\zeta_j + \mathcal{O}(|\lambda|^{-3})(\rho,\zeta), \\ \rho_\pm' &= \lambda^{1/2}M_1^\pm\rho_\pm + \mathcal{O}(|\lambda|^{-5/2})\zeta. \end{aligned} \tag{5.51}$$

By condition (3.35) together with (1.24) to the ζ_j equations and condition (3.15) to the $\rho_\pm$ equations, we find that the stable and unstable manifolds of the decoupled flow, consisting respectively of ρ_- together with ζ_j, $a_j^* > 0$ and ρ_+ together with ζ_j, $a_j^* > 0$, are exponentially separated to order $\theta\min\{|\lambda|^{1/2}, 1 + \text{Re }\lambda\}$, for some $\theta > 0$. Applying Proposition 3.9 together with Corollary 3.11, therefore, yields just the decoupled flows with error terms of order $|\lambda|^{-5/2}(e^{-\theta|\lambda|^{1/2}|x-y|}e^{-\theta(1+\text{Re }\lambda)|x-y|})$, and diagonalization error $\mathcal{O}(|\lambda|^{-5/2})$.

Recalling representation (5.42), we find after some rearrangement that G_λ is given by the decoupled approximant $\bar G_\lambda$ of (5.44) plus error terms (coming from flow plus diagonalization error) of form $|\lambda|^{-5/2}(e^{-\theta|\lambda|^{1/2}|x-y|}e^{-\theta(1+\mathrm{Re}\ \lambda)|x-y|})$, absorbable in term D. The contribution to $\bar G_\lambda$ from decoupled parabolic modes, as already discussed, is exactly P_λ. Having separated off the parabolic modes, we have now essentially reduced to the purely hyperbolic case treated in [MZ.1], and we proceed accordingly. Focusing on the decoupled hyperbolic equations

$$\zeta_j' = -(\lambda/a_j^* - \eta_j^*/a_j^*)\zeta_j + (c_j\lambda^{-1} + d_j\lambda^{-2})\zeta_j, \tag{5.52}$$

and applying Corollary 3.11 once again, considering the second term of the right-hand side as a perturbation, we find, substituting into (5.44), that the contribution to $\bar G_\lambda$ from decoupled hyperbolic modes is H_λ plus terms of form Θ_λ as claimed. We omit the details of this latter calculation, which are entirely similar to those of the relaxation case. We note in passing that exponential separation implies that the stable/unstable manifolds of (5.51) are *uniformly transverse* for $\lambda \in \Omega$, $|\lambda|$ sufficiently large, yielding $\Omega \setminus B(0,R) \subset \rho(L) \cap \Lambda$, for R sufficiently large, as claimed.

Derivative estimates. Derivative estimates (5.17)–(5.18) now follow in straighforward fashion, by differentiation of (5.42), noting from the approximately decoupled equations that differentiation of the flow brings down a factor (to absorbable error) of λ in hyperbolic modes, $\lambda^{1/2}$ in parabolic modes. $\square$

Remark 5.3. 1. Though we do not carry it out, it is clear that we could, if desired, expand the parabolic term P_λ to lowest order to (later) obtain short-time estimates on parabolic modes analogous to those obtained by parametrix methods (see, e.g. [Le,LU,LSU,Fr,ZH]) in the strictly parabolic case.

2. As pointed out in [ZH, errata], the regularity needed to treat the parabolic block is only $C^{0+\alpha}$ of standard short-time Schauder or parametrix theory. For, substituting $B^\delta := B * \eta^\delta$ for B, where η^δ is a standard smoothing kernel, and $\delta := |\lambda|^{-\frac{1}{2(1+\alpha)}}$, we obtain in the parabolic block a diagonalization series in powers of $\mathcal{O}(|\lambda|^{-\frac{\alpha}{2(1+\alpha)}}) \to 0$, which is sufficient for the argument. Indeed, a uniform modulus of continuity is enough to carry through the contraction argument for some small fixed δ, if we are willing to give up the description of G_λ to lowest order, as we do here. The regularity needed to treat the hyperbolic block alone is C^3; see [MZ.2].

Section 6. Low frequency bounds.

Our next object is the estimation of the resolvent kernel in the remaining critical regime $|\lambda| \to 0$, corresponding to large time behavior of the Green's function G, or *global behavior* in space and time. Here, the details of the system are not important, and we may follow essentially the same treatment as in the strictly parabolic or relaxation case [ZH,Z.2–3,MZ.1]. From a global perspective, the structure of the linearized equations is that of two nearly constant-coefficient regions separated by a thin shock layer near $x = 0$; accordingly, behavior is essentially governed by the two, limiting far-field equations:

$$U_t = L_\pm U, \tag{6.1}$$

coupled by an appropriate transmission relation at $x = 0$.

Similarly as in the previous section, we begin by examining these limiting systems. For simplicity of exposition, we make the provisional assumption:

(A1) Matrices $N_\pm := \begin{pmatrix} g_u & g_v \end{pmatrix} \begin{pmatrix} f_u & f_v \\ b_1 & b_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I_r \end{pmatrix}_{|(U_\pm)}$ have distinct eigenvalues.

(These $r \times r$ matrices govern behavior near $U_\pm$ in standing wave ODE (1.5)-(1.6); see Appendix A.) Since none of our estimates will depend on the distance between eigenvalues, this assumption is easily removed by a limiting argument.

Lemma 6.1. *Assume that there hold (H1)–(H3) and (A1). Then, for $|\lambda|$ sufficiently small, the eigenvalue equation $(L_\pm - \lambda)W = 0$ associated with the limiting, constant-coefficient operator $L_\pm$ has a basis of $n + r$ solutions*

$$\bar{W}_j^\pm = e^{\mu_j^\pm(\lambda)x} V_j(\lambda), \tag{6.2}$$

$\mu_j^\pm$, $V_j^\pm$, analytic in λ, consisting of r "fast" modes

$$\begin{aligned} \mu_j^\pm &= \gamma_j^\pm + \mathcal{O}(\lambda), \\ V_j^\pm &= A_\pm^{-1} S_j^\pm + \mathcal{O}(\lambda), \end{aligned} \tag{6.3}$$

$S_j^\pm = (0, s_j^{\pm t})^t$ where $\gamma_j^\pm$, $s_j^\pm$ are eigenvalues and associated right eigenvectors of $N_\pm$ defined in (A1) above, and n "slow" modes

$$\begin{aligned} \mu_{r+j}^\pm(\lambda) &:= -\lambda/a_j^\pm + \lambda^2 \beta_j^\pm / {a_j^\pm}^3 + \mathcal{O}(\lambda^3), \\ V_{r+j}^\pm(\lambda) &:= r_j^\pm + \mathcal{O}(\lambda), \end{aligned} \tag{6.4}$$

where $a_j^\pm$, $r_j^\pm$, and $\beta_j^\pm$ are as in Proposition 1.8. Likewise, the adjoint eigenvalue equation $(L_\pm - \lambda)^* Z = 0$ has a basis of solutions $\tilde{\bar{W}}_j^\pm = e^{-\mu_j^\pm(\lambda)x} \tilde{V}_j(\lambda)$, with

$$\tilde{V}_j^\pm(\lambda) = \tilde{T}_j^\pm + \mathcal{O}(\lambda) \tag{6.5}$$

and

$$\tilde{V}_{r+j}^\pm(\lambda) = l_j^\pm + \mathcal{O}(\lambda), \tag{6.6}$$

$\tilde V^\pm$ *analytic in* λ*, where* $l_j^\pm$ *are as in Proposition 1.8.*

Proof. Equivalently, we must show that $\mu_j^\pm$, $V_j^\pm$ describe eigenvalues and U-components of associated eigenvectors of the coefficient matrix of the eigenvalue ODE written in first order form. At $\lambda = 0$, there are r distinct nonzero eigenvalues $\mu_j^\pm = \gamma_j^\pm$, by assumption (A1), and an n-fold eigenvalue $\mu = 0$ corresponding to the n-dimensional kernel of Q. By spectral separation, the former vary analytically as λ is varied, with Taylor expansions (6.3) and (6.5).

The latter may be seen to vary analytically, with expansions (6.4) and (6.6), by inversion of the expansions

$$\lambda_j(i\xi) = -ia_j^{*\pm}\xi - \beta_j^{*\pm}\xi^2 + \mathcal{O}(\xi^3)$$

carried out in Appendix B for the the dispersion curves near $\xi = 0$, together with the fundamental relation $\mu = i\xi$ between roots of characteristic and dispersion equations (recall (4.7)–(4.8) of Section 3.1). Alternatively, one can carry out directly the associated matrix bifurcation problem as in [ZH,Z.3] to obtain the same result. A symmetric argument applies to the adjoint problem. $\square$

Our main result of this section is then:

Proposition 6.2. *Assume that there holds (H0)–(H4) and ($\mathcal{D}$) (equivalently, (D1)–(D2)). Then, for* $r > 0$ *sufficiently small, the resolvent kernel* G_λ *has a meromorphic extension onto* $B(0, r)$*, which may in the Lax or overcompressive case be decomposed as*

$$G_\lambda = E_\lambda + S_\lambda + R_\lambda \tag{6.7}$$

where, for $y \le 0$*:*

$$E_\lambda(x, t; y) := \lambda^{-1} \sum_{a_k^- > 0} [c_{k,-}^{j,0}] \frac{\partial}{\partial \delta_j} \begin{pmatrix} \bar u^\delta(x) \\ \bar v^\delta(x) \end{pmatrix} l_k^{-t} e^{(\lambda/a_k^\pm - \lambda^2 \beta_k^\pm / a_k^{\pm 3})y}, \tag{6.8}$$

$$S_\lambda(x, t; y) := \sum_{a_k^- > 0,\, a_j^+ > 0} [c_{k,-}^{j,+}] r_j^+ l_k^{-t} e^{(-\lambda/a_j^+ + \lambda^2 \beta_k^+ / a_k^{+3})x + (\lambda/a_k^- - \lambda^2 \beta_k^- / a_k^{-3})y} \tag{6.9}$$

for $y \le 0 \le x$*,*

$$\begin{aligned} S_\lambda(x, t; y) :=& \\ &\sum_{a_k^- > 0} r_k^- l_k^{-t} e^{(-\lambda/a_k^- + \lambda^2 \beta_k^- / a_k^{-3})(x-y)} \\ &+ \sum_{a_k^- > 0,\, a_j^- < 0} [c_{k,-}^{j,-}] r_j^- l_k^{-t} e^{(-\lambda/a_j^- + \lambda^2 \beta_j^- / a_j^{-3})x + (\lambda/a_k^- - \lambda^2 \beta_k^- / a_k^{-3})y} \end{aligned} \tag{6.10}$$

for $y \le x \le 0$, *and*

$$S_\lambda(x,t;y) := \sum_{a_k^- < 0} r_k^- {l_k^-}^t e^{(-\lambda/a_k^- + \lambda^2 \beta_k^-/{a_k^-}^3)(x-y)} + \sum_{a_k^- > 0,\, a_j^- < 0} [c_{k,-}^{j,-}] r_j^- {l_k^-}^t e^{(-\lambda/a_j^- + \lambda^2\beta_j^-/{a_j^-}^3)x + (\lambda/a_k^- - \lambda^2\beta_k^-/{a_k^-}^3)y} \tag{6.11}$$

for $x \le y \le 0$, *and* $R_\lambda = R_\lambda^E + R_\lambda^S$ *is a faster-decaying residual satisfying*

$$(\partial/\partial x)^\gamma(\partial/\partial y)^\alpha R_\lambda^E = \mathcal{O}(e^{-\theta|x-y|}) + \sum_{a_k^- > 0} e^{-\theta|x|} e^{(\lambda/a_k^\pm - \lambda^2\beta_k^\pm/{a_k^\pm}^3)y} \times \Big(\lambda^{\alpha-1}\mathcal{O}(e^{\mathcal{O}(\lambda^3)y} - 1) + \lambda^{\alpha-1}\mathcal{O}(e^{\mathcal{O}(\lambda^3)x} - 1) + \mathcal{O}(\lambda^\alpha)\Big), \tag{6.12}$$

$$\begin{aligned}(\partial/\partial x)^\gamma(\partial/\partial y)^\alpha R_\lambda^S(x,t;y) = &\sum_{a_k^- > 0,\, a_j^+ > 0} e^{(-\lambda/a_j^+ + \lambda^2\beta_k^+/{a_k^+}^3)x + (\lambda/a_k^- - \lambda^2\beta_k^-/{a_k^-}^3)y} \\ &\times \Big(\lambda^{\gamma+\alpha}\mathcal{O}(e^{\mathcal{O}(\lambda^3)y} - 1) + \lambda^{\gamma+\alpha}\mathcal{O}(e^{\mathcal{O}(\lambda^3)x} - 1) + \lambda^\alpha\mathcal{O}(e^{-\theta|x|}) + \mathcal{O}(\lambda^{1+\gamma})\Big)\end{aligned} \tag{6.13}$$

for $y \le 0 \le x$,

$$\begin{aligned}(\partial/\partial x)^\gamma(\partial/\partial y)^\alpha R_\lambda^S(x,t;y) = &\sum_{a_k^- > 0} e^{(-\lambda/a_k^- + \lambda^2\beta_k^-/{a_k^-}^3)(x-y)} \\ &\times \Big(\lambda^{\gamma+\alpha}\mathcal{O}(e^{\mathcal{O}(\lambda^3)y} - 1) + \lambda^{\gamma+\alpha}\mathcal{O}(e^{\mathcal{O}(\lambda^3)x} - 1) + \lambda^\alpha\mathcal{O}(e^{-\theta|x|}) + \mathcal{O}(\lambda^{1+\gamma})\Big) \\ &+ \sum_{a_k^- > 0,\, a_j^- < 0} e^{(-\lambda/a_j^- + \lambda^2\beta_j^-/{a_j^-}^3)x + (\lambda/a_k^- - \lambda^2\beta_k^-/{a_k^-}^3)y} \\ &\times \Big(\lambda^{\gamma+\alpha}\mathcal{O}(e^{\mathcal{O}(\lambda^3)y} - 1) + \lambda^{\gamma+\alpha}\mathcal{O}(e^{\mathcal{O}(\lambda^3)x} - 1) + \lambda^\alpha\mathcal{O}(e^{-\theta|x|}) + \mathcal{O}(\lambda^{1+\gamma})\Big)\end{aligned} \tag{6.14}$$

for $y \le x \le 0$, *and*

$$\begin{aligned}(\partial/\partial x)^\gamma(\partial/\partial y)^\alpha R_\lambda^S(x,t;y) = &\sum_{a_k^- < 0} e^{(-\lambda/a_k^- + \lambda^2\beta_k^-/{a_k^-}^3)(x-y)} \\ &\times \Big(\lambda^{\gamma+\alpha}\mathcal{O}(e^{\mathcal{O}(\lambda^3)y} - 1) + \lambda^{\gamma+\alpha}\mathcal{O}(e^{\mathcal{O}(\lambda^3)x} - 1) + \lambda^\alpha\mathcal{O}(e^{-\theta|x|}) + \mathcal{O}(\lambda^{1+\gamma})\Big) \\ &+ \sum_{a_k^- > 0,\, a_j^- < 0} e^{(-\lambda/a_j^- + \lambda^2\beta_j^-/{a_j^-}^3)x + (\lambda/a_k^- - \lambda^2\beta_k^-/{a_k^-}^3)y} \\ &\times \Big(\lambda^{\gamma+\alpha}\mathcal{O}(e^{\mathcal{O}(\lambda^3)y} - 1) + \lambda^{\gamma+\alpha}\mathcal{O}(e^{\mathcal{O}(\lambda^3)x} - 1) + \lambda^\alpha\mathcal{O}(e^{-\theta|x|}) + \mathcal{O}(\lambda^{1+\gamma})\Big)\end{aligned} \tag{6.15}$$

for $x \le y \le 0$, $0 \le \alpha, \beta \le 1$, $\alpha + \beta \le 2$, with each $\mathcal{O}(\cdot)$ term separately analytic in λ, and where $[c^{j,0}_{k,\alpha}]$, $\alpha = -, 0, +$, are constants to be determined later. Symmetric bounds hold for $y \ge 0$. Similar, but more complicated formulae hold in the undercompressive case (see Remark 6.9, below).

6.1. Normal modes (behavior in x).

We begin by relating the normal modes of the variable coefficient eigenvalue equation (4.1) to those of the limiting, constant-coefficient equations (6.1).

Lemma 6.3. *Let (H0)–(H3) hold. Then, for $\lambda \in B(0,r)$, r sufficiently small, there exist solutions $W_j^{\pm}(x;\lambda)$ of (4.1), C^1 in x and analytic in λ, satisfying*

$$\begin{aligned} W_j^{\pm}(x;\lambda) &= V_j^{\pm}(x;\lambda) e^{\mu_j^{\pm} x} \\ (\partial/\partial\lambda)^k V_j^{\pm}(x;\lambda) &= (\partial/\partial\lambda)^k V_j^{\pm}(\lambda) + \mathcal{O}(e^{-\tilde\theta |x|} |V_j^{\pm}(\lambda)|), \quad x \gtrless 0, \end{aligned} \tag{6.16}$$

for any $k \ge 0$ and $0 < \tilde\theta < \theta$, where θ is the rate of decay given in (3.2), $\mu_j^{\pm}(\lambda)$, $V_j^{\pm}(\lambda)$ are as in Lemma 6.1 above, and $\mathcal{O}(\cdot)$ depends only on k, $\tilde\theta$.

Proof. A direct consequence of Lemma 6.1 together with Lemma 3.1. □

The bases $\phi_j^{\pm}$, $\psi_j^{\pm}$ of Section 3.1 may evidently be chosen from among $W_j^{\pm}$, yielding an analytic choice of bases in λ, with the detailed description (6.16). It follows that the dual bases $\tilde\phi_j^{\pm}$, $\tilde\psi_j^{\pm}$ defined in Section 3.2 are also analytic in λ and satisfy corresponding bounds with respect to the dual solutions (6.5)–(6.6). With this observation, we have immediately, from Definition (3.8) and Proposition 4.7:

Corollary 6.4. *Let (H0)–(H3) hold. Then, the Evans function $D(\lambda)$ admits an analytic extension onto $B(0,r)$, for r sufficiently small, and the resolvent kernel $G_\lambda(x,y)$ admits a meromorphic extension, with an isolated pole of finite multiplicity at $\lambda = 0$.*

Moreover, we have close control on the (x,y) behavior of G_λ through the spectral decomposition formulae of Proposition 4.7. For *slow, dual modes*, the bounds (6.16) (in particular, the consequent bounds on first spatial derivatives) can be considerably sharpened, provided that we appropriately initialize our bases at $\lambda = 0$. This observation will be quite significant in the Lax or overcompressive case. Likewise, *fast-decaying forward modes* can be well-approximated near $\lambda = 0$ by their representatives at $\lambda = 0$, using only the basic bounds (6.16). These two categories comprise the modes determining behavior of the Green's function to lowest order.

Specifically, due to the special, conservative structure of the underlying evolution equations, the adjoint eigenvalue equation $L * \tilde W = 0$ at $\lambda = 0$ admits an n-dimensional subspace of constant solutions

$$\tilde W = (\tilde w_1^t, \tilde w_2^t)^t \equiv (\text{constant}, 0)^t; \tag{6.17}$$

this is equivalent to the observation that integral quantities in variable U are conserved under time evolution for general conservation laws. Thus, at $\lambda = 0$, we may choose, by appropriate change of coordinates if necessary, that slow-decaying dual modes $\tilde{\phi}_j^{\pm}$ and slow-growing dual modes $\tilde{\psi}_j^{\pm}$ be identically constant, Note that this does not interfere with our previous choice in Lemma 6.3, since that concerned only the choice of limiting solutions $\overline{W}_j^{\pm}$ of the asymptotic, constant coefficient equations at $x \to \pm\infty$, and not the particular representatives $W_j^{\pm}$ that approach them (which, in the case of slow modes, are specified only up to the addition of an arbitrary fast-decaying mode).

Remark 6.5. The prescription of constant dual bases just described requires that we reverse our previous approach, choosing dual bases first using the Gap Lemma, then defining forward bases using duality. Alternatively, we may rewrite the forward eigenvalue equation at $\lambda = 0$ as

$$\left(\begin{pmatrix} A_{11} & A_{12} \\ b_1 & b_2 \end{pmatrix} U'\right)' = \left(\begin{pmatrix} 0 & 0 \\ A_{12} & A_{22} \end{pmatrix} U\right)' \tag{6.18}$$

to see, after integration from $x = \pm\infty$, that fast-decaying modes satisfy the linearized traveling wave ODE (cf. Appendix A)

$$U' = \begin{pmatrix} A_{11} & A_{12} \\ b_1 & b_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ A_{12} & A_{22} \end{pmatrix} U. \tag{6.19}$$

By duality relation (4.16), requiring slow dual modes to be constant is equivalent to choosing fast-growing (forward) modes as well as fast-decaying modes from among the solutions of (6.19). (Recall: though fast-decaying modes are uniquely determined as a subspace, fast-growing modes are only determined up to the addition of faster decaying modes.)

Lemma 6.6 [MZ.1]. *Assuming (H0)–(H3), with the above choice of bases at* $\lambda = 0$, *and for* $\lambda \in B(0, r)$ *and* r *sufficiently small, slow modes* $\tilde{W}_j^{\pm}$ *satisfy*

$$\tilde{W}_j^{\pm}(y;\lambda) = e^{-\mu(\lambda)y}\bar{\tilde{V}}_j^{\pm}(0) + \lambda\tilde{\Theta}_j^{\pm}(y;\lambda), \tag{6.20}$$

where

$$\begin{aligned} |\tilde{\Theta}_j^{\pm}| &\le C|e^{-\mu(\lambda)y}| \\ |(\partial/\partial y)\tilde{\Theta}_j^{\pm}| &\le C|e^{-\mu(\lambda)y}|(|\lambda| + e^{-\theta|y|}), \end{aligned} \tag{6.21}$$

$\theta > 0$, *as* $y \to \pm\infty$, *and* $\bar{\tilde{V}}_j^{\pm} \equiv$ constant.[3] *Similarly, fast-decaying (forward) modes* $W_j^{\pm}$ *satisfy*

$$W_j^{\pm}(x;\lambda) = W_j^{\pm}(x;0) + \lambda\Theta_j^{\pm}(x;\lambda), \tag{6.22}$$

[3]In particular, this includes $|(\partial/\partial y)\tilde{W}_j^{\pm}(y;\lambda)| \le C|\lambda e^{\mu_j^{\pm}(\lambda)y}|$ in place of the general spatial-derivative bound $|(\partial/\partial y)\tilde{W}_j^{\pm}(y;\lambda)| \le C(|\mu\tilde{V}_j^{\pm}| + |(\partial/\partial y)\tilde{V}_j^{\pm}|)e^{\mu_j^{\pm}(\lambda)y} \sim C|e^{\mu_j^{\pm}(\lambda)y}|$.

where

$$|\Theta_j^{\pm}|,\ |(\partial/\partial x)\Theta_j^{\pm}| \le Ce^{-\theta|x|} \tag{6.23}$$

as $x \to \pm\infty$, *for some* $\theta > 0$.

Proof. Applying Lemma 3.1 to the augmented variables

$$\begin{aligned}\tilde{\mathbb{W}}_j^{\pm}(y;\lambda) := \begin{pmatrix} \tilde W_j^{\pm} \\ \tilde W_j^{\pm\prime} \end{pmatrix}(y;\lambda) =: \begin{pmatrix} \tilde W_j^{\pm} \\ \tilde W_j^{\pm\prime} \end{pmatrix}(y;\lambda) \\ = e^{-\mu(\lambda)y}\begin{pmatrix} \tilde V_j^{\pm} \\ -\mu_j^{\pm}\tilde V_j^{\pm} + \tilde V_j^{\pm\prime} \end{pmatrix}(y;\lambda)\end{aligned}$$

and

$$\begin{aligned}\mathbb{W}_j^{\pm}(x;\lambda) := \begin{pmatrix} W_j^{\pm} \\ W_j^{\pm\prime} \end{pmatrix}(x;\lambda) =: e^{\mu(\lambda)x}\mathbb{V}_j^{\pm}(x;\lambda) \\ = e^{-\mu(\lambda)y}\begin{pmatrix} \tilde V_j^{\pm} \\ \mu_j^{\pm} V_j^{\pm} + V_j^{\pm\prime} \end{pmatrix}(y;\lambda)\end{aligned}$$

we obtain bounds

$$\begin{aligned}\tilde{\mathbb{W}}_j^{\pm}(x;\lambda) &= \tilde{\mathbb{V}}_j^{\pm}(x;\lambda)e^{-\mu_j^{\pm}(\lambda)x} \\ (\partial/\partial\lambda)^k\tilde{\mathbb{V}}_j^{\pm}(x;\lambda) &= (\partial/\partial\lambda)^k\tilde{\mathbb{V}}_j^{\pm}(\lambda) + \mathcal{O}(e^{-\tilde\theta|x|}|\tilde{\mathbb{V}}_j^{\pm}(\lambda)|), \quad x \gtrless 0,\end{aligned} \tag{6.24}$$

and

$$\begin{aligned}\mathbb{W}_j^{\pm}(x;\lambda) &= \mathbb{V}_j^{\pm}(x;\lambda)e^{\mu_j^{\pm}(\lambda)x} \\ (\partial/\partial\lambda)^k\mathbb{V}_j^{\pm}(x;\lambda) &= (\partial/\partial\lambda)^k\mathbb{V}_j^{\pm}(\lambda) + \mathcal{O}(e^{-\tilde\theta|x|}|\tilde{\mathbb{V}}_j^{\pm}(\lambda)|), \quad x \gtrless 0,\end{aligned} \tag{6.25}$$

$\tilde\theta > 0$, analogous to (6.16), valid for $\lambda \in B(0,r)$, where

$$\tilde{\mathbb{V}}_j^{\pm}(\lambda) = \begin{pmatrix} \tilde V(\lambda) \\ -\mu_j^{\pm}\tilde V(\lambda) \end{pmatrix}$$

and

$$\mathbb{V}_j^{\pm}(\lambda) = \begin{pmatrix} V(\lambda) \\ \mu_j^{\pm} V(\lambda). \end{pmatrix}$$

(Note: an appropriate phase space in which to apply the conjugation lemma is, e.g., (U, U', bU) for the forward equations, and similarly for the dual equations.)

By Taylor's Theorem with differential remainder, applied to $\tilde{\mathbb{V}}$, we have:

$$\tilde{\mathbb{W}}_j^{\pm}(y,\lambda) = e^{-\mu_j^{\pm}(\lambda)y}\left(\tilde{\mathbb{V}}_j^{\pm}(y;0) + \lambda(\partial/\partial\lambda)\tilde{\mathbb{V}}_j^{\pm}(y;0) + \frac{1}{2}\lambda^2(\partial/\partial\lambda)^2\tilde{\mathbb{V}}_j^{\pm}(y;\lambda_*)\right), \tag{6.26}$$

for some λ_* on the ray from 0 to λ, where, recall, $(\partial/\partial\lambda)\tilde{\mathbb{V}}_j^\pm(y;\cdot)$ and $(\partial/\partial\lambda)^2\tilde{\mathbb{V}}_j^\pm(y;\cdot)$ are uniformly bounded in $L^\infty[0,\pm\infty]$ for $\lambda\in B(0,r)$. Together with the choice $\tilde{V}_j^\pm(y,0)\equiv$ constant, this immediately gives the first (undifferentiated) bound in (6.21).

Applying now the bound (6.25) with $k=1$, we may expand the second coordinate of (6.26) as

(6.27)

$$\begin{aligned}(\partial/\partial y)\tilde{W}_j^\pm(y,\lambda) &= e^{-\mu_j^\pm(\lambda)y}\Big(-\mu_j^\pm\tilde{V}_j^\pm(y;0)+\tilde{V}_j^{\pm'}(y;0)\\ &\qquad -\lambda\big((\partial/\partial\lambda)(\mu_j^\pm\tilde{V}_j^\pm)(0)+\mathcal{O}(e^{-\theta|y|})\big)+\mathcal{O}(\lambda^2)\Big)\\ &= e^{-\mu_j^\pm(\lambda)y}\Big(-\lambda\big((\partial/\partial\lambda)\mu_j^\pm(0)\tilde{V}_j^\pm(0)+\mathcal{O}(e^{-\theta|y|})\big)+\mathcal{O}(\lambda^2)\Big),\end{aligned}$$

and subtracting off the corresponding Taylor expansion

$$\begin{aligned}&(\partial/\partial y)\big(e^{-\mu_j^\pm(\lambda)y}\tilde{V}_j^\pm(y,0)\big)\\ &\quad=\mu_j^\pm(\lambda)e^{-\mu_j^\pm(\lambda)y}\tilde{V}_j^\pm(y,0)\\ &\quad=e^{-\mu_j^\pm(\lambda)y}\Big(-\mu_j^\pm(0)\tilde{V}_j^\pm(y;0)-\lambda(\partial/\partial\lambda)\mu_j^\pm(0)\tilde{V}_j^\pm(y;0)+\mathcal{O}(\lambda^2)\Big)\\ &\quad=e^{-\mu_j^\pm(\lambda)y}\Big(-\lambda(\partial/\partial\lambda)\mu_j^\pm(0)\tilde{V}_j^\pm(y;0)+\mathcal{O}(\lambda^2)\Big),\end{aligned}\tag{6.28}$$

we obtain

$$(\partial/\partial y)\tilde{\Theta}_j^\pm(y,\lambda)=e^{-\mu_j^\pm(\lambda)y}\Big(\lambda\mathcal{O}(e^{-\theta|y|})+\mathcal{O}(\lambda^2)\Big),\tag{6.29}$$

as claimed.

Similarly, we may obtain (6.22)–(6.23) by Taylor's Theorem with differential remainder applied to $W=e^{\mu(\lambda)x}$, and the Leibnitz calculation

$$(\partial/\partial\lambda)W=(d\mu/d\lambda)xe^{\mu x}V+e^{\mu x}(\partial/\partial\lambda)V,$$

together with the observation that $|xe^{-\theta|x|}\le Ce^{-\theta|x|/2}$. □

Remark 6.7. For a more explicit derivation of (6.20)–(6.21) in the scalar case, see Lemma 2.2, (2.6)–(2.7) in [HoZ.2]. (Note: in [HoZ.2], the one-dimensional case corresponds to $\xi\equiv 0$).

This leaves only the problem of determining behavior in λ through the study of coefficients M, $d^\pm$. To this end, we make the following further observations in the Lax or overcompressive case, generalizing the corresponding observation of Lemma 4.30, [Z.3], in the strictly parabolic case:

Lemma 6.8. *For transverse ($\gamma \neq 0$) Lax and overcompressive shocks, with the above-specified choice of basis at $\lambda = 0$, fast-growing modes ψ_j^+, ψ_j^- are fast-decaying at $-\infty$, $+\infty$, respectively. Equivalently, fast-decaying modes $\tilde\psi_j^+$, $\tilde\psi_j^-$ are fast-growing at $-\infty$, $+\infty$: i.e., the only bounded solutions of the adjoint eigenvalue equation are constant solutions.*

Proof. Noting that the manifold of solutions of the r-dimensional ODE (6.19) that decay at either $x \to +\infty$ or $x \to -\infty$ is by transversality, together with Lemma 1.2, exactly $d_+ + d_- - \ell = r$, we see that *all* solutions of this ODE in fact decay at least at one infinity. This implies the first assertion, by the alternative characterization of our bases described in Remark 6.5 above; the second follows by duality, (4.16). Since the manifold of fast-decaying dual modes is uniquely determined, independent of the choice of basis, this implies that nonconstant dual modes that are bounded at one infinity must blow up at the other, hence bounded solutions must be constant as claimed. □

Remark 6.9. As noted in [LZ.2,ZH,Z.2,Z.3], the property of Lax and overcompressive shocks that the adjoint eigenvalue equation has only constant solutions has the interpretation that the only L^1 time-invariants of the evolution of the one-dimensional linearized equations about $(\bar u, \bar v)(\cdot)$ are given by *conservation of mass*, see discussion [LZ.2]. This distinguishes then from undercompressive shocks, which *do* have additional L^1 time-invariants. The presence of additional time-invariants for undercompressive shocks has significant implications for their behavior, (see discussions [LZ.2] Section 3 and [ZH], Section 10): in particular, the time-asymptotic location of a perturbed undercompressive shock (generically) evolves nonlinearly, and is not determinable by any linear functional of the initial perturbation. By contrast, the time-asymptotic location of a perturbed (stable) Lax or overcompressive shock may be determined by the mass of the initial perturbation alone.

6.2. Scattering coefficients (behavior in λ). We next turn to the estimation of scattering coefficients M_{jk}, $d_{jk}^{\pm}$. Consider coefficient M_{jk}. Expanding (4.42) using Kramer's rule, and setting $z = 0$, we obtain

$$M_{jk}^{\pm} = D^{-1} C_{jk}^{\pm}, \tag{6.30}$$

where

$$C^+ := (I, 0) \begin{pmatrix} \Phi^+ & \Phi^- \\ \Phi^{+\prime} & \Phi^{-\prime} \end{pmatrix}^{\mathrm{adj}} \begin{pmatrix} \Psi^- \\ \Psi^{-\prime} \end{pmatrix}_{|z=0} \tag{6.31}$$

and a symmetric formula holds for C^-. Here, P^{adj} denotes the *adjugate matrix* of a matrix P, i.e. the transposed matrix of minors. As the adjugate is polynomial in the entries of the original matrix, it is evident that $|C^{\pm}|$ is uniformly bounded and therefore

$$|M_{jk}^{\pm}| \leq C_1 |D^{-1}| \leq C_2 \lambda^{-\ell} \tag{6.32}$$

by $(\mathcal{D})$, where C_1, $C_2 > 0$ are uniform constants.

However, the crude bound (6.32) hides considerable cancellation, a fact that will be crucial in our analysis. Relabel the $\{\varphi^\pm\}$ so that, at $\lambda = 0$,

$$\varphi^+_{N-j+1} \equiv \varphi^-_j = (\partial/\partial\delta_j)\begin{pmatrix}\bar u^\delta\\ \bar v^\delta\end{pmatrix}, \quad j = 1, \cdots, \ell. \tag{6.33}$$

With convention (6.33), we have the sharpened bounds:

Lemma 6.10. *Let (H0)–(H4) and $(\mathcal{D})$ (equivalently, (D1)–(D2)) hold, and let $\phi^\pm_j$ be labeled as in (6.33). Then, for $|\lambda|$ sufficiently small, there hold*

$$|M^\pm_{jk}|, |d^\pm_{jk}|, |\bar d^\pm_{jk}| \le C\begin{cases}\lambda^{-1} & \textit{for } j = 1, \cdots, \ell,\\ 1 & \textit{otherwise,}\end{cases} \tag{6.34}$$

where $M^\pm$, $d^\pm$ are as defined in Proposition 4.7. Moreover,

$$\textit{Residue}_{\ \lambda=0}\, M^+_{N-j+1,k} = \textit{Residue}_{\ \lambda=0}\, d^\pm_{j,k}, \tag{6.35}$$

for $1 \le j \le \ell$, all k.

That is, blowup in M_{jk} occurs to order $\lambda^{\ell-1}|D^{-1}|$ rather than $|D^{-1}|$, and, more importantly, only in (fast-decaying) *stationary modes* $(\partial/\partial\delta_j)(\bar u^\delta, \bar v^\delta)$. Moreover, each stationary mode $(\partial/\partial\delta_j)(\bar u^\delta, \bar v^\delta)$ has consistent scattering coefficients with dual mode $\tilde\psi^-_k$, to lowest order, across all of the various representations of G_λ given in Proposition 4.7.

Proof. Formula (6.31) may be rewritten as

$$C^+_{jk} = \det\begin{pmatrix}\varphi^+_1, \cdots, \varphi^+_{j-1}, \psi^-_k, \varphi^+_j, \cdots, \varphi^+_n, & \Phi^-\\ \varphi^{+\prime}_1, \cdots, \varphi^+_{j-1}{}', \psi^-_k{}', \varphi^{+\prime}_j, \cdots, \varphi^+_n, & \Phi^{-\prime}\end{pmatrix}_{|z=0}, \tag{6.36}$$

from which we easily obtain the desired cancellation in $M^+ = C^+D^{-1}$. For example, for $j > \ell$, we have

$$\begin{aligned}C^+_{jk} &= \det\begin{pmatrix}\varphi^+_1 + \lambda\varphi^+_{1_\lambda} + \cdots, & \cdots, & \varphi^+_n + \lambda\varphi^+_{n_\lambda} + \cdots\\ \cdots, & , \cdots, & \varphi^{+\prime}_n + \lambda\varphi^+_{n_\lambda}{}' + \cdots\end{pmatrix}\\ &= \mathcal{O}(\lambda^\ell) \le C|D|,\end{aligned} \tag{6.37}$$

yielding $|M_{jk}| = |C_{jk}||D|^{-1} \le C$ as claimed, by elimination of ℓ zero-order terms, using linear dependency among fast modes at $\lambda = 0$. For $1 \le j \le \ell$, there is only an $\ell - 1$ order dependency, and we obtain instead the bound $|C_{jk}| \le C|\lambda|^{\ell-1}$, or $|M_{jk}| = |C_{jk}||D|^{-1} \le C|\lambda|^{-1}$. The bounds on $|d_{jk}|$ and $|\bar d_{jk}|$ follow similarly; likewise, (6.35) follows easily from the observation that, since columns ψ^+_j and ψ^-_{N-j+1} agree, and $\tilde\psi^-_j$ is held fixed, the expansions of the various representations by Kramer's rule yield determinants which at λ^{-1} order differ only by a transposition of columns. $\square$

In the Lax or overcompressive case, we can say a bit more:

Lemma 6.11. *Let (H0)–(H4) and ($\mathcal{D}$) (equivalently, (D1)–(D2)) hold, with $|\lambda|$ sufficiently small. Then, for Lax and overcompressive shocks, with appropriate basis at $\lambda = 0$ (i.e. slow dual modes taken identically constant), there hold*

$$|M_{jk}|, |d_{jk}|, |\bar{d}_{jk}| \leq C \tag{6.38}$$

if $\tilde{\psi}_k$ is a fast mode, and

$$|M_{jk}|, |d_{jk}|, |\bar{d}_{jk}| \leq C|\lambda| \tag{6.39}$$

if, additionally, φ_j is a slow mode.

As we shall see in the following section, this result has the consequence that only slow, constant dual modes play a role in long-time behavior of G.

Proof. Transversality, $\gamma \neq 0$, follows from (D2), so that the results of Lemma 6.8 hold. We first establish the bound (6.38). By Lemma 6.10, we need only consider $j = 1, \dots, \ell$, for which $\phi_j^- = \phi_j^+$. By Lemma 6.8, all fast-growing modes ψ_k^+, ψ_k^- lie in the fast-decaying manifolds Span $(\varphi_1^-, \cdots, \varphi_{i_-}^-)$, Span $(\varphi_1^+, \cdots, \varphi_{i_+}^+)$ at $-\infty$, $+\infty$ respectively. It follows that in the righthand side of (6.36), there is at $\lambda = 0$ a linear dependency between columns

$$\varphi_1^+, \cdots, \varphi_{j-1}^+, \psi_k^-, \varphi_{j+1}^+, \varphi_{i_+}^+ \text{ and } \varphi_j^- = \varphi_j^+ \tag{6.40}$$

i.e. an ℓ-fold dependency among columns

$$\varphi_1^+, \cdots, \psi_k^-, \cdots, \varphi_{i+}^+ \text{ and } \varphi_1^-, \cdots, \varphi_\ell^-. \tag{6.41}$$

It follows as in the proof of Lemma 6.10 that $|C_{jk}| \leq C\lambda^\ell$ for λ near zero, giving bound (6.38) for M_{jk}. If φ_j^+ is a slow mode, on the other hand, then the same argument shows that there is a linear dependency in columns

$$\varphi_1^+, \cdots, \varphi_{j-1}^+, \psi_k^-, \varphi_{j+1}^+, \varphi_{i_+}^+ \tag{6.42}$$

and an $(\ell + 1)$-fold dependency in (6.41), since the omitted slow mode φ_j^+ plays no role in either linear dependence; thus, we obtain the bound (6.39), instead. Analogous calculations yield the result for $d_{jk}^\pm$, $\bar{d}_{jk}$ as well. $\square$

6.3. Proof of Proposition 6.2. The proof of Proposition 6.2 is now just a matter of collecting the bounds of Lemmas 6.3–6.11, and substituting in the representations of Proposition 4.7. More precisely, approximating fast-decaying dual modes $\phi_1^+, \dots, \phi_\ell^+$ and $\phi_{N-\ell+1}^-, \dots, \phi_N^-$ by stationary modes $(\partial/\partial\delta_j)(\bar{u}^\delta, \bar{v}^\delta)$, and slow dual modes by $e^{-\mu(\lambda)y}\bar{\bar{V}}(\lambda)$ as described in Lemma 6.6, truncating $\mu(\lambda)$ at second order, and keeping only the lowest order terms in the Laurent expansions of scattering coefficients $M^\pm$, $d^\pm$ we obtain E_λ and S_λ, respectively, as order λ^{-1} and order one terms, except for negligible $\mathcal{O}(e^{-\theta|x-y|})$ terms which we have accounted for in error term R_λ^E.

Besides the latter, we have accounted in term R_λ^E for: (i) the truncation errors involved in the aforementioned approximations (difference between stationary modes and actual fast modes, and between constant-coefficient approximant and actual slow dual modes, plus truncation errors in exponential rates); and, (ii) neglected fast-decaying forward/slow-decaying dual mode combinations (of order λ, by Lemma 6.10). A delicate point is the fact that term $\mathcal{O}(e^{-\theta|x-y|})\mathcal{O}(e^{-\theta|y|})$ arising in the dual truncation of E through the $\lambda\mathcal{O}(e^{-\theta|y|})$ portion of estimate (6.21) for $(\partial/\partial y)\tilde{\Theta}_j^{\pm}$, absorbs in the (leading) $\mathcal{O}(e^{-\theta|x-y|})$ term of (6.12). It was to obtain this key reduction that we carried out expansion (6.21) to such high order.

In R_λ^S, we have accounted for truncation errors in the slow/slow pairings approximated by term S_λ (difference between constant-coefficient approximants and actual slow forward modes, corresponding to an additional $e^{-\theta|x|}$ term in the truncation error factor, and between constant-coefficient approximants and actual slow dual modes, plus truncation errors in exponential rates). We have also accounted for the remaining, slow-decaying forward/fast-decaying dual modes, which according to Lemma 6.11 have scattering coefficient of order λ and therefore may be grouped with the difference between dual modes and their constant-coefficient approximants (of smaller order by a factor of $e^{-\theta|y|}$) in the term $\mathcal{O}(\lambda)$. □

6.4. Extended spectral theory. Finally, we cite without proof the extended spectral theory of [ZH] that we shall need in order to establish necessity of condition $(\mathcal{D})$. In the case that $(\mathcal{D})$ holds, this reduces to the simple relations (6.35). Let $C^\infty_{\exp}$ denote the space of C^∞ functions decaying exponentially in all derivatives at some sufficiently high rate.

Definition 6.12. *Let L be a (possibly degenerate type) linear ordinary differential operator for which the associated eigenvalue equation may be written as a first-order system with bounded coefficients, and let G_λ denote the Green's function of $L-\lambda I$. Further, let Ω be an open, simply connected domain intersecting the resolvent set of L, on which G_λ has a (necessarily unique) meromorphic extension. Then, for $\lambda_0 \in \Omega$, we define the* **effective eigenprojection** $\mathcal{P}_{\lambda_0} : C^\infty_{\exp}$ *by*

$$\mathcal{P}_{\lambda_0} f(x) = \int_{-\infty}^{+\infty} P_{\lambda_0}(x,y) f(y)\, dy,$$

where

$$P_{\lambda_0}(x,y) = \mathrm{Res}_{\lambda_0} G_\lambda(x,y) \tag{6.43}$$

and Res_{λ_0} *denotes residue at λ_0. We will refer to $P_{\lambda_0}(x,y)$ as the* **projection kernel**. *Likewise, we define the* **effective eigenspace** $\Sigma'_{\lambda_0}(L)$ *by*

$$\Sigma'_{\lambda_0}(L) = \text{Range }(\mathcal{P}_{\lambda_0}),$$

and the **effective point spectrum** $\sigma'_p(L)$ *of* L *in* Ω *to be the set of* $\lambda \in \Omega$ *such that* $\dim \Sigma'_{\lambda_0}(L) \neq 0$.

The above definition is the natural one from the point of view of the spectral resolution of the identity, $I = \int_\Gamma (L - \lambda I)^{-1} d\lambda$, hence also from the point of view of asymptotic behavior of solutions of PDE (see Section 6, below). Away from the essential spectrum of L, it corresponds with the usual definition [Kat,Y].

Definition 6.13. *Let* L, Ω, λ_0 *be as above, and* K *be the order of the pole of* $(L - \lambda I)^{-1}$ *at* λ_0. *For* $\lambda_0 \in \Omega$, *and* k *any integer, we define* $\mathcal{Q}_{\lambda_0,k} : C^\infty_{\exp}$ *by*

$$\mathcal{Q}_{\lambda_0,k} f(x) = \int_{-\infty}^{+\infty} Q_{\lambda_0,k}(x,y) f(y)\, dy,$$

where

$$Q_{\lambda_0,k}(x,y) = \mathrm{Res}_{\lambda_0} (\lambda - \lambda_0)^k G_\lambda(x,y).$$

For $0 \le k \le K$, *we define the* **effective eigenspace of ascent** k *by*

$$\Sigma'_{\lambda_0,k}(L) = \mathrm{Range}\ (\mathcal{Q}_{\lambda_0,K-k})$$

With the above definitions, we obtain the following, modified Fredholm Theory.

Proposition 6.14 [ZH]. *Let* L, λ_0, Ω *be as in Definition 6.12, and* K *be the order of the pole of* G_λ *at* λ_0. *Then,*

(i) The operators $\mathcal{P}_{\lambda_0}$, $\mathcal{Q}_{\lambda_0,k} : C^\infty_{\exp}$ *are* L*-invariant, with*

$$\mathcal{Q}_{\lambda_0,k+1} = (L - \lambda_0 I)\mathcal{Q}_{\lambda_0,k} = \mathcal{Q}_{\lambda_0,k}(L - \lambda_0 I) \tag{6.44}$$

for all $k \neq -1$, *and*

$$\mathcal{Q}_{\lambda_0,k} = (L - \lambda_0 I)^k \mathcal{P}_{\lambda_0} \tag{6.45}$$

for $k \ge 0$.

(ii) The effective eigenspace of ascent k *satisfies*

$$\Sigma'_{\lambda_0,k}(L) = (L - \lambda_0 I)\Sigma'_{\lambda_0,k+1}(L). \tag{6.46}$$

for all $0 \le k \le K$, *with*

$$\{0\} = \Sigma'_{\lambda_0,0}(L) \subset \Sigma'_{\lambda_0,1}(L) \subset \cdots \subset \Sigma'_{\lambda_0,K}(L) = \Sigma'_{\lambda_0}(L). \tag{6.47}$$

Moreover, each containment in (6.47) is strict.

(iii) On $\mathcal{P}^{-1}_{\lambda_0}(C^\infty_{\exp})$, $\mathcal{P}_{\lambda_0}$, $\mathcal{Q}_{\lambda_0,k}$ *all commute* ($k \ge 0$), *and* $\mathcal{P}_{\lambda_0}$ *is a projection. More generally,* $\mathcal{P}_{\lambda_0} f = f$ *for any* $f \in \Sigma_{\lambda_0}(L : C^\infty_{exp})$, *hence* $\Sigma_{\lambda_0,k}(L : C^\infty_{exp}) \subset \Sigma'_{\lambda_0,k}(L)$ *for all* $0 \le k \le K$.

(iv) *The multiplicity of the eigenvalue* λ_0*, defined as* $\dim\ \Sigma'_{\lambda_0}(L)$*, is finite and bounded by* Kn*, where* n *is the dimension of the phase space for the first-order eigenvalue ODE. Moreover, for all* $0 \le k \le K$*,*

$$\dim\ \Sigma'_{\lambda_0,k}(L) = \dim\ \Sigma'_{\lambda_0^*,k}(L^*). \tag{6.48}$$

Further, the projection kernel can be expanded as

$$P_{\lambda_0} = \sum_j \varphi_j(x)\pi_j(y), \tag{6.49}$$

where $\{\varphi_j\}$*,* $\{\pi_j\}$ *are bases for* $\Sigma'_{\lambda_0}(L)$*,* $\Sigma'_{\lambda_0^*}(L^*)$*, respectively.*

(v) *(Restricted Fredholm alternative) For* $g \in C^\infty_{\text{exp}}$ *,*

$$(L - \lambda_0 I)f = g$$

is soluble in C^∞ **if** *and soluble in* C^∞_{exp} **only if** $Q_{\lambda_0,K-1}\, g = 0$*, or equivalently*

$$g \in \Sigma'_{\lambda_0^*,1}(L^*)^\perp. \tag{6.50}$$

Proof: See [ZH]. □

Remarks: 1. For λ_0 in the resolvent set, $\mathcal{P}_{\lambda_0}$ agrees with the standard definition, hence $\Sigma'_{\lambda_0,k}(L)$ agrees with the usual L^p eigenspace of generalized eigenfunctions of ascent $\le k$, for all $p < \infty$, since C^∞_{exp} is dense in L^p, $p > 1$, and $\Sigma'_{\lambda_0,k}(L)$ is closed. In the context of stability of traveling waves (more generally, whenever the coefficients of L exponentially approach constant values at $\pm\infty$), $\Sigma'_{\lambda_0,k}(L)$ lies *between* the L^p subspace $\Sigma_{\lambda_0,k}(L)$ and the corresponding L^p_{Loc} subspace.

2. The modification from standard Fredholm Theory is primarily that here conclusions (iii) and (v) apply on restricted domains. This restriction is clearly necessary, since the various expressions occurring in their statements would otherwise not be defined. In this regard, note that $\mathcal{P}_{\lambda_0}$ is, strictly speaking, not a projection in the case that $\lambda \in \sigma_{ess}(L)$, since its domain does not then match its range. However, it maintains the projective structure (6.49), somewhat justifying our abuse of notation.

3. In [ZH], L was assumed to have C^∞ coefficients; however, this assumption was not used in the proof. On the other hand, the assumption that the eigenvalue equation may be recast as a nondegenerate first-order system is needed for conclusion (iv), but was mistakenly omitted in [ZH].

Now, suppose further (as holds in the case under consideration) that, on Ω: (i) L is a differentail operator for which both the forward and adjoint eigenvalue equations may be expressed in appropriate phase spaces as nondegenerate first-order

ODE of the same order, and for which the reduced quadratic form $\mathcal{S}$ of Lemma 4.2 is invertible; and, (ii) there exists an analytic choice of bases for stable/unstable manifolds at $+\infty/-\infty$ of the associated eigenvalue equation.

In this case, we can define an analytic Evans function $D(\lambda)$ as in (3.8), which we assume does not vanish identically. It is easily verified that the order to which D vanishes at any λ_0 is independent of the choice of analytic bases $\Phi^{\pm}$. The following result, established in [ZH], generalizes to the extended spectral framework the standard result of Gardner and Jones [GJ.1–2] in the classical setting.

Proposition 6.15 [ZH]. *Let L, λ_0 be as above. Then,*

(i) dim $\Sigma'_{\lambda_0}(L)$ *is equal to the order d to which the Evans function D_L vanishes at λ_0.*

(ii) $P_{\lambda_0}(L) = \sum_j \varphi_j(x)\pi_j(y)$, where φ_j and π_j are in $\Sigma'_{\lambda_0}(L)$ and $\Sigma'_{\lambda_0}(L^*)$, respectively, with ascents summing to $\leq K+1$, where K is the order of the pole of G_λ at λ_0.

Proof. This follows as in [ZH], with formula (6.8) of the reference replaced by the more general (4.29) applying also to operators of degenerate type. A minor omission in the argument of [ZH] was that the spanning functions f_j for $\Sigma'_{\lambda_0}(L)$ were not shown to be independent; however, this is easily remedied. Formally, these functions have the form of Jordan chains, for which the root, "genuine eigenfunctions" of L are independent by construction. Independence thus follows by a standard inductive argument, applying successively higher powers of $(L-\lambda_0)$ to the relation $\sum \alpha_j f_j = 0$ to see that $\alpha_j = 0$ for all j. □

Remark 6.16. The main novelty of the present treatment is only to point out that, thanks to (4.29) and surrounding theory, the theory developed in [ZH] for nondegenerate type operators carries over essentially unchanged to the case of degenerate type operators for which the eigenvalue equation can be written as a first-order ODE of nondegenerate type.

Section 7. Pointwise Green's function bounds.

In this section, we prove the main results of the paper, establishing pointwise bounds on the Green's function G and consequently sufficiency and necessity of condition $(\mathcal{D})$ for linearized orbital stability.

7.1. Proof of Theorem 1.8. We will establish Theorem 1.8 as a corollary of the following, more detailed result:

Proposition 7.1. *Assuming (H0)–(H4) and the spectral stability criterion $(\mathcal{D})$ (equivalently, (D1)–(D2)), the Green's function $G(x,t;y)$ associated with (1.10) may* *be decomposed as in Theorem 1.8, where, for $y \le 0$:*

(7.1)

$$\begin{aligned} R(x,t;y) &= \mathcal{O}(e^{-\eta t}e^{-|x-y|^2/Mt}) \\ &\quad + \sum_{k=1}^{n} \mathcal{O}\left((t+1)^{-1/2}e^{-\eta x^+} + e^{-\eta|x|}\right) t^{-1/2}e^{-(x-y-a_k^- t)^2/Mt} \\ &\quad + \sum_{a_k^->0,\, a_j^-<0} \chi_{\{|a_k^- t|\ge|y|\}} \mathcal{O}((t+1)^{-1/2}t^{-1/2}) e^{-(x-a_j^-(t-|y/a_k^-|))^2/Mt}e^{-\eta x^+}, \\ &\quad + \sum_{a_k^->0,\, a_j^+>0} \chi_{\{|a_k^- t|\ge|y|\}} \mathcal{O}((t+1)^{-1/2}t^{-1/2}) e^{-(x-a_j^+(t-|y/a_k^-|))^2/Mt}e^{-\eta x^-}, \end{aligned}$$

(7.2)

$$\begin{aligned} R_y(x,t;y) &= \sum_{j=1}^{J} \mathcal{O}(e^{-\eta t})\delta_{x-\bar a_j^* t}(-y) + \mathcal{O}(e^{-\eta t}e^{-|x-y|^2/Mt}) \\ &\quad + \sum_{k=1}^{n} \mathcal{O}\left((t+1)^{-1/2}e^{-\eta x^+} + e^{-\eta|x|}\right) t^{-1}e^{-(x-y-a_k^- t)^2/Mt} \\ &\quad + \sum_{a_k^->0,\, a_j^-<0} \chi_{\{|a_k^- t|\ge|y|\}} \mathcal{O}((t+1)^{-1/2}t^{-1}) e^{-(x-a_j^-(t-|y/a_k^-|))^2/Mt}e^{-\eta x^+} \\ &\quad + \sum_{a_k^->0,\, a_j^+>0} \chi_{\{|a_k^- t|\ge|y|\}} \mathcal{O}((t+1)^{-1/2}t^{-1}) e^{-(x-a_j^+(t-|y/a_k^-|))^2/Mt}e^{-\eta x^-}, \end{aligned}$$

(7.3)

$$\begin{aligned} R_x(x,t;y) &= \sum_{j=1}^{J} \mathcal{O}(e^{-\eta t})\delta_{x-\bar a_j^* t}(-y) + \mathcal{O}(e^{-\eta t}e^{-|x-y|^2/Mt}) \\ &\quad + \sum_{k=1}^{n} \mathcal{O}\left((t+1)^{-1}e^{-\eta x^+} + e^{-\eta|x|}\right) t^{-1}(t+1)^{1/2}e^{-(x-y-a_k^- t)^2/Mt} \\ &\quad + \sum_{a_k^->0,\, a_j^-<0} \chi_{\{|a_k^- t|\ge|y|\}} \mathcal{O}(t+1)^{-1/2}t^{-1}) e^{-(x-a_j^-(t-|y/a_k^-|))^2/Mt}e^{-\eta x^+} \\ &\quad + \sum_{a_k^->0,\, a_j^+>0} \chi_{\{|a_k^- t|\ge|y|\}} \mathcal{O}(t+1)^{-1/2}t^{-1}) e^{-(x-a_j^+(t-|y/a_k^-|))^2/Mt}e^{-\eta x^-}, \end{aligned}$$

and
(7.4)

$$
\begin{aligned}
R_{xy}(x,t;y) = (\partial/\partial y)\big(&\sum_{j=1}^{J}\mathcal{O}(e^{-\eta t})\delta_{x-\bar a_j^* t}(-y) + \mathcal{O}(e^{-\eta t}e^{-|x-y|^2/Mt})\big)\\
&+\sum_{k=1}^{n}\mathcal{O}\left((t+1)^{-3/2}e^{-\eta x^+}+e^{-\eta|x|}\right)t^{-3/2}(t+1)e^{-(x-y-a_k^- t)^2/Mt}\\
&+\sum_{a_k^->0,\,a_j^-<0}\chi_{\{|a_k^- t|\ge|y|\}}\mathcal{O}(t+1)^{-1/2}t^{-3/2})e^{-(x-a_j^-(t-|y/a_k^-|))^2/Mt}e^{-\eta x^+}\\
&+\sum_{a_k^->0,\,a_j^+>0}\chi_{\{|a_k^- t|\ge|y|\}}\mathcal{O}(t+1)^{-1/2}t^{-3/2})e^{-(x-a_j^+(t-|y/a_k^-|))^2/Mt}e^{-\eta x^-},
\end{aligned}
$$

for some η*,* $M>0$*, where* a_j^**,* $\bar a_j^*$ *are as in Theorem 1.8,* $x^\pm$ *denotes the positive/negative part of* x*, and indicator function* $\chi_{\{|a_k^- t|\ge|y|\}}$ *is one for* $|a_k^- t|\ge|y|$ *and zero otherwise. Symmetric bounds hold for* $y\ge 0$*.*

Remark 7.2. The generic parabolic error term $\mathcal{O}(e^{-\eta t}e^{-|x-y|^2/Mt})$ replaces a similar term $\mathcal{O}(e^{-\eta(|x-y|+t)})\chi_{\{c_-\le x-y/t\le c_+\}}$ in the hyperbolic relaxation case [MZ.1]. Note that these are essentially equivalent on their common support, both lying between $C^{-1}e^{-\eta_1 t}$ and $Ce^{-\eta_2 t}$ for some η_1, η_2, $C>0$. The difference between the two terms reflects infinite vs. finite propagation speed in hyperbolic vs. parabolic modes.

Proof. Our starting point is the representation

$$
G(x,t;y)=\frac{1}{2\pi i}\text{P.V.}\int_{\eta-i\infty}^{\eta+i\infty}e^{\lambda t}G_\lambda(x,y)d\lambda, \tag{7.5}
$$

valid by Proposition 2.4 for any η sufficiently large.

Case I. $|x-y|/t$ **large.** We first treat the simpler case that $|x-y|/t\ge S$, S sufficiently large. Fixing x, y, t, set $\lambda=\eta+i\xi$. Applying Proposition 5.2, we obtain, for $\eta>0$ sufficiently large, the decomposition
(7.6)

$$
\begin{aligned}
G(x,t;y)=&\frac{1}{2\pi i}\text{P.V.}\int_{\eta-i\infty}^{\eta+i\infty}e^{\lambda t}H_\lambda(x,y)d\lambda+\frac{1}{2\pi i}\text{P.V.}\int_{\eta-i\infty}^{\eta+i\infty}e^{\lambda t}\Theta_\lambda^H(x,y)d\lambda\\
&+\frac{1}{2\pi i}\text{P.V.}\int_{\eta-i\infty}^{\eta+i\infty}e^{\lambda t}P_\lambda(x,y)d\lambda+\frac{1}{2\pi i}\text{P.V.}\int_{\eta-i\infty}^{\eta+i\infty}e^{\lambda t}\Theta_\lambda^P(x,y)d\lambda\\
=:&\, I+II+III+IV,
\end{aligned}
$$

where
(7.7)

$$
H_\lambda(x,y)=\begin{cases}-\sum_{j=K+1}^{J}a_j^*(y)^{-1}e^{-\int_y^x\lambda/a_j^*(z)\,dz}R_j^*(x)\tilde\zeta_j^*(x,y)L_j^{*t}(y) & x>y,\\ \sum_{j=1}^{K}a_j^*(y)^{-1}e^{-\int_y^x\lambda/a_j^*(z)\,dz}R_j^*(x)\tilde\zeta_j^*(x,y)L_j^{*t}(y) & x<y,\end{cases}
$$

$$\Theta_\lambda^H(x,y) := \lambda^{-1}B(x,y;\lambda) + \lambda^{-1}(x-y)C(x,y;\lambda), \tag{7.8}$$

and

$$\Theta_\lambda^P(x,y) := \lambda^{-2}D(x,y;\lambda), \tag{7.9}$$

with ζ_j, B, C, D as defined in Proposition 5.2. For definiteness taking $x > y$, we estimate each term in turn.

Term I. Term I of (7.6) contributes to integral (7.5) the explicitly evaluable quantity

(7.10)

$$\begin{aligned}
\frac{1}{2\pi i}\text{P.V.}\int_{\eta-i\infty}^{\eta+i\infty} e^{\lambda t}H_\lambda(x,y)d\lambda &= \sum_{j=K+1}^{J}\Big(a_j(y)^{-1}\text{P.V.}\int_{-\infty}^{+\infty} e^{i\xi\left(t-\int_y^x (1/a_j(z))\,dz\right)}\,d\xi\Big) \\
&\quad\times e^{\eta\left(t-\int_y^x (1/a_j(z))\,dz\right)} r_j(x)\tilde\zeta_j(x,y)l_j^t(y) \\
&= \sum_{j=K+1}^{J}\Big(a_j(y)^{-1}\delta\big(t-\int_y^x (1/a_j(z))\,dz\big)\Big) r_j(x)\tilde\zeta_j(x,y)l_j^t(y) \\
&= \sum_{j=K+1}^{J}\delta_{x-\bar a_j t}(-y)r_j(x)\zeta_j(y,t)l_j^t(y) \\
&= H(x,t;y),
\end{aligned}$$

where $\bar a_j$, ζ_j, and H are as defined in Theorems 1.8 and 7.1. Note: in the final equality of (7.10) we have used the general fact that

$$f_y(x,y,t)\delta\big(f(x,y,t)\big) = \delta_{h(x,t)}(y), \tag{7.11}$$

provided $f_y \neq 0$, where $h(x,t)$ is defined by $f(x,h(x,t),t) \equiv 0$. This term clearly vanishes for x outside $[z_1(y,t), z_J(y,t)]$, hence makes no contribution for S sufficiently large.

Term II. Similar calculations show that the "hyperbolic error term" II in (7.6) also vanishes. For example, the term $e^{\lambda t}\lambda^{-1}B(x,y;\lambda)$ contributes

(7.12)

$$\frac{1}{2\pi i}\sum_{j=K+1}^{K}\Big(\text{P.V.}\int_{-\infty}^{+\infty}(\eta+i\xi)^{-1}e^{i\xi\left(t-\int_y^x (1/a_j(z))\,dz\right)}\,d\xi\Big)\, e^{\eta\left(t-\int_y^x (1/a_j(z))\,dz\right)} b_j^+(x,y).$$

The factor

$$\text{P.V.}\int_{-\infty}^{+\infty}(\eta+i\xi)^{-1}e^{i\xi\left(t-\int_y^x (1/a_j(z))\,dz\right)}\,d\xi, \tag{7.13}$$

though not absolutely convergent, is integrable and uniformly bounded as a principal value integral, for all real η bounded from zero, by explicit computation. On the other hand,

$$e^{\eta\left(t-\int_y^x (1/a_j(z))\,dz\right)} \le Ce^{-\eta\, d(x,[z_1(y,t),z_J(y,t)])/\min_{j,x}|a_j(x)|} \to 0 \tag{7.14}$$

as $\eta \to +\infty$, for each $K+1 \le j \le J$, since $a_j > 0$ on this range. Thus, taking $\eta \to +\infty$, we find that the product (7.12) goes to zero, giving the result. Likewise, the contributions of terms $e^{\lambda t}C(x,y;\lambda)$ and $e^{\lambda t}D(x,y;\lambda)$ split into the product of a convergent, uniformly bounded integral in ξ, a (constant) factor depending only on (x,y), and a factor $\alpha(x,y,t,\eta)$ going to zero as $\eta \to 0$ at rate (7.14). Thus, each of these terms vanishes also as $\eta \to +\infty$, as claimed.

Term III. The parabolic term III may be treated exactly as in the strictly parabolic case [ZH]. Namely, using analyticity of P_λ within the sector Ω_P defined in (5.7), we may for any fixed η deform the contour in the principal value integral to

$$\int_{\Gamma_1\cup\Gamma_2} e^{\lambda t}P_\lambda(x,y)\,d\lambda, \tag{7.15}$$

where $\Gamma_1 := \partial B(0,R)\cap \bar\Omega_P$ and $\Gamma_2 := \partial\Omega_P \setminus B(0,R)$ plus an error term of order not greater than

$$|\xi|^{-1/2}\int_\eta^{-\infty} e^{wt}dw \to 0$$

as $|\xi| \to \infty$, where

$$R := \bar\alpha^2, \quad \bar\alpha := \frac{\theta|x-y|}{2t}, \tag{7.16}$$

θ is as in (5.16).

By estimate (5.16) we have for all $\lambda \in \Gamma_1\cup\Gamma_2$ that

$$|P_\lambda(x,y)| \le C|\lambda^{-1/2}|e^{-\theta|\lambda^{1/2}||x-y|} \tag{7.17}$$

Further, we have

$$\begin{aligned} Re\lambda &\le R(1-\eta\omega^2), \quad \lambda\in\Gamma_1,\\ Re\lambda &\le Re\lambda_0 - \eta(|\mathrm{Im}\ \lambda| - |\mathrm{Im}\ \lambda_0|), \quad \lambda\in\Gamma_2 \end{aligned} \tag{7.18}$$

for R sufficiently large, where ω is the argument of λ and λ_0 and λ_0^* are the two points of intersection of Γ_1 and Γ_2, for some $\eta > 0$ independent of $\bar\alpha$.

Combining (7.17), (7.18), and (7.16), we obtain

$$\begin{aligned}
|\int_{\Gamma_1} e^{\lambda t} P_\lambda d\lambda| &\le \int_{\Gamma_1} C|\lambda^{-1/2}|\, e^{Re\lambda t - \theta|\lambda^{1/2}||x-y|}\, d\lambda \\
&\le Ce^{-\bar\alpha^2 t} \int_{-L}^{+L} R^{-1/2} e^{-R\eta\omega^2 t}\, Rd\omega \\
&\le Ct^{-1/2} e^{-\bar\alpha^2 t}.
\end{aligned}$$

Likewise,

$$\begin{aligned}
|\int_{\Gamma_2} e^{\lambda t} P_\lambda d\lambda| &\le \int_{\Gamma_2} C|\lambda^{-1/2}|\, Ce^{Re\lambda t - \theta|\lambda^{1/2}||x-y|}\, d\lambda \\
&\le Ce^{Re(\lambda_0)t - \theta|\lambda_0^{1/2}||x-y|} \int_{\Gamma_2} |\lambda^{-1/2}| e^{(Re\lambda) - Re\lambda_0)t}\ |d\lambda| \\
&\le Ce^{-\bar\alpha^2 t} \int_{\Gamma_2} |Im\,\lambda|^{-1/2} e^{-\eta|Im\,\lambda - Im\,\lambda_0|t}\ |d\,Im\lambda| \\
&\le Ct^{-1/2} e^{-\bar\alpha^2 t}.
\end{aligned} \tag{7.19}$$

Combining these last two estimates, and recalling (7.16), we have

$$III \le Ct^{-1/2} e^{-\bar\alpha^2 t/2} e^{-\theta^2 (x-y)^2/8t} \le Ct^{-1/2} e^{-\eta t} e^{-(x-y)^2/Mt}, \tag{7.20}$$

for $\eta > 0$, $M > 0$ independent of $\bar\alpha$. Observing that

$$\frac{|x-at|}{2t} \le \frac{|x-y|}{t} \le \frac{2|x-at|}{t}$$

for any bounded a, for $|x-y|/t$ sufficiently large, we find that III can be absorbed in the residual term $\mathcal{O}(e^{-\eta t} e^{-|x-y|^2/Mt})$ for $t \ge \epsilon$, any $\epsilon > 0$, and by any summand $\mathcal{O}(t^{-1/2}(t+1)^{-1/2} e^{-(x-y-a_k^{\pm}t)^2/Mt}) e^{-\eta x\pm} e^{-\eta y\pm}$ for t small.

Likewise, differentiating within the absolutely convergent integral (7.15), we find that $|III_x|$ and $|III_y|$ can be absorbed in the corresponding summands of (7.3) and (7.2), while III_{xy} can be absorbed in terms $(\partial/\partial y)\mathcal{O}(e^{-\eta t} e^{-|x-y|^2/Mt})$ and

$$\mathcal{O}(t^{-1/2}(t+1)^{-3/2} e^{-(x-y-a_k^{\pm}t)^2/Mt}) e^{-\eta x\pm} e^{-\eta y\pm},$$

respectively.

Note. A delicate point about the derivative bounds to first move the contour to (7.15) and then differentiate. Differentiating within the original principal value integral yields an integral that is not obviously convergent, and for which it is not clear that the contour may be so deformed.

Term IV. Similarly as in the treatment of term III, the principal value integral for the "parabolic error term" IV may be shifted to $\eta = R = \bar\alpha^2$, $\bar\alpha$ as in (7.16), at the cost of an error term that vanishes as $|\xi| \to \infty$. But, this yields an estimate

$$|IV| \le Ce^{-\bar\alpha^2 t} \int_{-\infty}^{+\infty} |\eta_0 + i\xi|^{-2} d\xi \le e^{-\bar\alpha^2 t}$$

that by (7.20) may be absorbed in $\mathcal{O}(e^{-\eta t}e^{-|x-y|^2/Mt})$ for all t. Derivatives IV_x and IV_y may be similarly absorbed, while IV_{xy} may be absorbed in

$$(\partial/\partial y)\mathcal{O}(e^{-\eta t}e^{-|x-y|^2/Mt}).$$

Case II. $|x-y|/t$ bounded. We now turn to the critical case that $|x-y|/t \le S$ for some fixed S. In this regime, note that any contribution of order $e^{\theta t}$, $\theta > 0$, may be absorbed in the residual (error) term R (resp. R_x, R_y); we shall use this observation repeatedly.

Decomposition of the contour. We begin by converting contour integral (7.5) into a more convenient form decomposing high, intermediate, and low frequency contributions.

Observation 7.3. *Assuming (H0)–(H4), there holds the representation*

$$\begin{aligned} G(x,t;y) &= I_a + I_b + I_c + II_a + II_b \\ &:= \frac{1}{2\pi i}\text{P.V.}\int_{\eta-i\infty}^{\eta+i\infty} e^{\lambda t} H_\lambda(x,y)d\lambda \\ &+ \frac{1}{2\pi i}\text{P.V.}\left(\int_{-\eta_1-i\infty}^{-\eta_1-iR} + \int_{-\eta_1+iR}^{-\eta_1+i\infty}\right) e^{\lambda t}(G_\lambda - H_\lambda - P_\lambda)(x,y)d\lambda \\ &+ \frac{1}{2\pi i}\oint_{\Gamma_2} e^{\lambda t}P_\lambda(x,y)d\lambda \\ &+ \frac{1}{2\pi i}\oint_{\Gamma} e^{\lambda t}G_\lambda(x,y)d\lambda - \frac{1}{2\pi i}\int_{-\eta_1-iR}^{-\eta_1+iR} e^{\lambda t}H_\lambda(x,y)d\lambda, \end{aligned} \tag{7.21}$$

$$\Gamma := [-\eta_1 - iR, \eta - iR] \cup [\eta - iR, \eta + iR] \cup [\eta + iR, -\eta_1 + iR], \tag{7.22}$$

$$\Gamma_2 := \partial\Omega_P \setminus \Omega \tag{7.23}$$

Ω_P as defined in (5.7), for any $\eta > 0$ such that (7.5) holds, R sufficiently large, and $-\eta_1 < 0$ as in (5.8). (Note that we have not here assumed ($\mathcal{D}$)).

Proof. We first observe that, by Proposition 5.2, L has no spectrum on the portion of Ω lying outside of the rectangle

$$\mathcal{R} := \{\lambda : -\eta_1 \le \text{Re }\lambda \le \eta,\ -R \le \text{Im }\lambda \le R\} \tag{7.24}$$

for $\eta > 0$, $R > 0$ sufficiently large, hence G_λ is analytic on this region. Since, also, H_λ is analytic on the whole complex plane, contours involving either one of these contributions may be arbitrarily deformed within $\Omega \backslash \mathcal{R}$ without affecting the result, by Cauchy's Theorem. Likewise, P_λ is analytic on $\Omega_P \backslash \mathcal{R}$, and so contours involving this contribution may be arbitrarily deformed within this region.

Recalling, further, that

$$|G_\lambda - H_\lambda - P_\lambda| = \mathcal{O}(\lambda^{-1})$$

by Proposition 5.2, we find that

$$\frac{1}{2\pi i}\text{P.V.}\int_{\eta-i\infty}^{\eta+i\infty} e^{\lambda t}(G_\lambda - H_\lambda - P_\lambda)(x,y)d\lambda$$

may be deformed to

$$\frac{1}{2\pi i}\text{P.V.}\oint_{\partial(\Omega\backslash\mathcal{R})} e^{\lambda t}(G_\lambda - H_\lambda - P_\lambda)(x,y)d\lambda.$$

And, similarly as in the treatment of term III of the large $|x-y|/t$ case, we find using bounds (5.16) that

$$\frac{1}{2\pi i}\text{P.V.}\int_{\eta-i\infty}^{\eta+i\infty} e^{\lambda t}P_\lambda(x,y)d\lambda$$

may be deformed to the absolutely convergent integral

$$\frac{1}{2\pi i}\oint_{\partial(\Omega_P\backslash\mathcal{R})} e^{\lambda t}P_\lambda(x,y)d\lambda.$$

Noting, finally, that

$$+\frac{1}{2\pi i}\oint_{\partial\mathcal{R}} e^{\lambda t}H_\lambda(x,y)d\lambda = 0,$$

by Cauchy's Theorem, we obtain, finally,

(7.25)

$$\begin{aligned}
&\frac{1}{2\pi i}\text{P.V.}\int_{\eta-i\infty}^{\eta+i\infty} e^{\lambda t}G_\lambda(x,y)d\lambda = \\
&\quad \frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty} e^{\lambda t}H_\lambda(x,y)d\lambda + \frac{1}{2\pi i}\text{P.V.}\int_{\eta-i\infty}^{\eta+i\infty} e^{\lambda t}(G_\lambda - H_\lambda - P_\lambda)(x,y)d\lambda \\
&\qquad + \frac{1}{2\pi i}\oint_{\partial(\Omega_P\backslash\mathcal{R})} e^{\lambda t}P_\lambda(x,y)d\lambda \\
&\quad = \frac{1}{2\pi i}\text{P.V.}\int_{\eta-i\infty}^{\eta+i\infty} e^{\lambda t}H_\lambda(x,y)d\lambda + \frac{1}{2\pi i}\text{P.V.}\oint_{\partial(\Lambda\backslash\mathcal{R})} e^{\lambda t}(G_\lambda - H_\lambda - P_\lambda)(x,y)d\lambda \\
&\qquad + \frac{1}{2\pi i}\oint_{\partial\mathcal{R}} e^{\lambda t}H_\lambda(x,y)d\lambda + \frac{1}{2\pi i}\oint_{\partial(\Omega_P\backslash\mathcal{R})} e^{\lambda t}P_\lambda(x,y)d\lambda,
\end{aligned}$$

from which the result follows by combining the second third and fourth contour integrals along their common edges Γ. $\square$

Observation 7.4. *Assuming ($\mathcal{D}$) together with (H0)–(H4), we may replace (7.21) by*

$$G(x,t;y) = I_a + I_b + I_c + II_{\tilde a} + II_b + III,$$

where I_a, I_b, I_c, and II_b are as in (7.21), and

$$II_{\tilde a} := \frac{1}{2\pi}\left(\int_{-\eta_1 - iR}^{-\eta_1 - ir/2} + \int_{-\eta_1+ir/2}^{-\eta_1+iR}\right) e^{\lambda t} G\lambda(x,y)dxi,$$

$$III := \frac{1}{2\pi i}\oint_{\tilde\Gamma} e^{\lambda t} G_\lambda(x,y)d\lambda, \tag{7.26}$$

and

$$\tilde\Gamma := [-\eta_1 - ir/2, \eta - ir/2] \cup [\eta - ir/2, \eta + ir/2] \cup [\eta + ir/2, -\eta_1 + ir/2], \tag{7.27}$$

for any η, $r > 0$, and η_1 sufficiently small with respect to r.

Proof. By assumption ($\mathcal{D}$), L has no spectrum on the region between contour Γ and the union of contour $\tilde\Gamma$ and the contour of term $II_{\tilde a}$, hence G_λ is analytic on that region, and

$$II_a = II_{\tilde a} + III$$

by Cauchy's Theorem, giving the result. □

Using the final decomposition (7.26), we shall estimate in turn the high frequency contributions I_a, I_b, and I_c, the intermediate frequency contributions $II_{\tilde a}$ and II_b, and the low frequency contributions III.

High frequency contribution. We first carry out the straightforward estimation of the high-frequency terms I_a, I_b, and I_c. The principal term I_a has already been computed in (7.10) to be $H(x,t;y)$. Likewise, calculations similar to those of (7.12)–(7.14) show that the error term

$$\begin{aligned} I_b &= \frac{1}{2\pi i}\text{P.V.}\left(\int_{-\eta_1 - i\infty}^{-\eta_1 - R}\int_{-\eta_1+iR}^{-\eta_1+i\infty}\right) e^{\lambda t}\Theta_\lambda(x,y)d\lambda \\ &= \frac{1}{2\pi i}\text{P.V.}\left(\int_{-\eta_1 - i\infty}^{-\eta_1 - R}\int_{-\eta_1+iR}^{-\eta_1+i\infty}\right) e^{\lambda t}\big(\lambda^{-1}B(x,y;\lambda) \\ &\qquad + \lambda^{-1}(x-y)C(x,y;\lambda) + \lambda^{-2}D(x,y;\lambda)\big)d\lambda \end{aligned}$$

is time-exponentially small.

For example, for $x > y$ the term $e^{\lambda t}\lambda^{-1}B(x,y;\lambda)$ contributes

$$\sum_{j=K+1}^{K} \frac{1}{2\pi i}\text{P.V.}\left(\int_{-\infty}^{-R}\int_{R}^{+\infty}\right)(-\eta_1+i\xi)^{-1}e^{i\xi\left(t-\int_y^x (1/a_j(z))\,dz\right)}\,d\xi \\ \times e^{-\eta_1\left(t-\int_y^x(1/a_j(z))\,dz\right)}b_j^+(x,y), \tag{7.28}$$

where

$$\frac{1}{2\pi i}\text{P.V.}\left(\int_{-\infty}^{-R}\int_{R}^{+\infty}\right)(\eta+i\xi)^{-1}e^{i\xi\left(t-\int_y^x(1/a_j(z))\,dz\right)}\,d\xi<\infty, \tag{7.29}$$

and

$$e^{\eta_1\int_y^x(1/a_j(z))\,dz}b(x,y)\le C_1e^{\eta_1\int_y^x(1/a_j(z))\,dz-\theta|x-y|}\le C_2, \tag{7.30}$$

for η_1 sufficiently small. This may be absorbed in the first term of R, (7.1). Likewise, the contributions of terms $e^{\lambda t}C(x,y;\lambda)$ and $e^{\lambda t}D(x,y;\lambda)$ split into the product of a convergent, uniformly bounded integral in ξ, a bounded factor analogous to (7.30), and the factor $e^{-\eta_1 t}$, giving the result.

The term I_c may be estimated exactly as was term III in the large $|x-y|/t$ case, to obtain contribution $\mathcal{O}(t^{-1/2}e^{-\eta_1 t})$ absorbable again in the residual term $\mathcal{O}(e^{-\eta t}e^{-|x-y|^2/Mt})$ for $t\ge\epsilon$, any $\epsilon>0$, and by any summand $\mathcal{O}(t^{-1/2}(t+1)^{-1/2}e^{-(x-y-a_k^{\pm}t)^2/Mt})e^{-\eta x\pm}e^{-\eta y\pm}$ for t small.

Derivative bounds. Derivatives $(\partial/\partial x)I_b$ (resp. $(\partial/\partial y)I_b$, $(\partial/\partial y)(\partial/\partial x)I_b$) may be treated in identical fashion using (5.17) (resp. (5.18), to show that they are absorbable in the estimates given for R_x (resp. R_y, R_{xy}). We point out that the integral arising from term B_x^0 (resp. B_y^0) of (5.17) (resp. (5.18)) corresponds to the first, delta-function term in (7.3) (resp. (7.2)), while the integral arising from term $(x-y)C_x^0$ (resp. $(x-y)C_y^0$) vanishes, as the product of $(x-y)$ and a delta function $\delta_z(y)$ with $z\ne x$. Derivatives of term I_c may be treated exactly as were derivatives of term III in the large $|x-y|/t$ case.

Intermediate frequency contribution. Error term II_b is time-exponentially small for η_1 sufficiently small, by the same calculation as in (7.28)–(7.30), hence negligible. Likewise, term $II_{\tilde a}$ by the basic estimate (4.26) is seen to be time-exponentially small of order $e^{-\eta_1 t}$ for any $\eta_1>0$ sufficiently small that the associated contour lies in the resolvent set of L.

Low frequency contribution. It remains to estimate the low-frequency term III, which is of essentially the same form as the low-frequency contribution analyzed in [ZH,Z.2,Z.3] in the strictly parabolic case, in that the contour is the same and the resolvent kernel G_λ satisfies identical bounds in this regime. Thus, we may conclude from these previous analyses that III gives contribution $E+S+R$, as claimed, exactly as in the strictly parabolic case. For completeness, we indicate the main features of the argument here.

Case $t \le 1$. First observe that estimates in the short-time regime $t \le 1$ are trivial, since then $|e^{\lambda t}G_\lambda(x,y)|$ is uniformly bounded on the compact set $\tilde\Gamma$, and we have $|G(x,t;y)| \le C \le e^{-\theta t}$ for $\theta > 0$ sufficiently small. But, likewise, E and S are uniformly bounded in this regime, hence time-exponentially decaying. As observed previously, all such terms are negligible, being absorbable in the error term R. Thus, we may add $E+S$ and subtract G to obtain the result

Case $t \ge 1$. Next, consider the critical (long-time) regime $t \ge 1$. For definiteness, take $y \le x \le 0$; the other two cases are similar. Decomposing

(7.31)
$$G(x,t;y) = \frac{1}{2\pi i}\oint_{\tilde\Gamma} e^{\lambda t}E_\lambda(x,y)d\lambda + \frac{1}{2\pi i}\oint_{\tilde\Gamma} e^{\lambda t}S_\lambda(x,y)d\lambda + \frac{1}{2\pi i}\oint_{\tilde\Gamma} e^{\lambda t}R_\lambda(x,y)d\lambda,$$

with E_λ, S_λ, and R_λ as defined in Proposition 6.2, we consider in turn each of the three terms on the righthand side.

E_λ *term.* Let us first consider the dominant term

$$\frac{1}{2\pi i}\oint_{\tilde\Gamma} e^{\lambda t}E_\lambda(x,y)d\lambda, \tag{7.32}$$

which, by (6.8), is given by

$$\sum_{a_k^- > 0} [c_{k,-}^{j,0}]\frac{\partial}{\partial\delta_j}\begin{pmatrix}\bar u^\delta(x)\\ \bar v^\delta(x)\end{pmatrix} l_k^{-t}\alpha_k(x,t;y), \tag{7.33}$$

where

$$\alpha_k(x,t;y) := \frac{1}{2\pi i}\oint_{\tilde\Gamma} \lambda^{-1}e^{\lambda t}e^{(\lambda/a_k^- - \lambda^2\beta_k^-/a_k^{-3})y}\,d\lambda. \tag{7.34}$$

Using Cauchy's Theorem, we may move the contour $\tilde\Gamma$ to obtain

$$\begin{aligned}\alpha_k(x,t;y) = {} & \frac{1}{2\pi}\text{P.V.}\int_{-r/2}^{+r/2}(i\xi)^{-1}e^{i\xi t}e^{(i\xi/a_k^- + \xi^2\beta_k^-/a_k^{-3})y}\,d\xi \\ & + \frac{1}{2\pi i}\left(\int_{-\eta_1 - ir/2}^{-ir/2} + \int_{ir/2}^{-\eta_1 + ir/2}\right)\lambda^{-1}e^{\lambda t}e^{(\lambda/a_k^- - \lambda^2\beta_k^-/a_k^{-3})y}\,d\lambda \\ & + \frac{1}{2}\text{Residue }_{\lambda=0}\, e^{\lambda t}e^{(\lambda/a_k^- - \lambda^2\beta_k^-/a_k^{-3})y},\end{aligned} \tag{7.35}$$

or, rearranging and evaluating the final, residue term:

$$\begin{aligned}\alpha_k(x,t;y) &= \left(\frac{1}{2\pi}\text{P.V.}\int_{-\infty}^{+\infty}(i\xi)^{-1}e^{i\xi(t+y/a_k^-)}e^{\xi^2(\beta_k^-/a_k^{-3})y}\,d\xi+\frac{1}{2}\right)\\ &\quad-\frac{1}{2\pi}\left(\int_{-\infty}^{-r/2}+\int_{r/2}^{+\infty}\right)(i\xi)^{-1}e^{i\xi(t+y/a_k^-)}e^{\xi^2(\beta_k^-/a_k^{-3})y}\,d\xi\\ &\quad+\frac{1}{2\pi i}\left(\int_{-\eta_1-ir/2}^{-ir/2}+\int_{ir/2}^{-\eta_1+ir/2}\right)\lambda^{-1}e^{\lambda t}e^{(\lambda/a_k^- -\lambda^2\beta_k^-/a_k^{-3})y}\,d\lambda.\end{aligned}\tag{7.36}$$

The first term in (7.36) may be explicitly evaluated to give

$$\text{errfn}\left(\frac{y+a_k^-t}{\sqrt{4\beta_k^-|y/a_k^-|}}\right),\tag{7.37}$$

where

$$\text{errfn }(z):=\frac{1}{2\pi}\int_{-\infty}^{z}e^{-y^2}\,dy,\tag{7.38}$$

whereas the second and third terms are clearly time-exponentially small for $t\le C|y|$ and η_1 sufficiently small relative to r (see detailed discussion of similar calculations below, under R_λ *term*). In the trivial case $t\ge C|y|$, $C>0$ sufficiently large, we can simply move the contour to $[-\eta_1-ir/2,-\eta_1+ir/2]$ to obtain (complete) residue 1 plus a time-exponentially small error corresponding to the shifted contour integral, which result again agrees with (7.37) up to a time-exponentially small error.

Expression (7.37) may be rewritten as

$$\text{errfn}\left(\frac{y+a_k^-t}{\sqrt{4\beta_k^-|y/a_k^-|}}\right)\tag{7.39}$$

plus error

$$\begin{aligned}&\text{errfn}\left(\frac{y+a_k^-t}{\sqrt{4\beta_k^-|y/a_k^-|}}\right)-\text{errfn}\left(\frac{y+a_k^-t}{\sqrt{4\beta_k^-t}}\right)\\ &\sim \text{errfn}'\left(\frac{y+a_k^-t}{\sqrt{4\beta_k^-t}}\right)\left(-\frac{4\beta_k^-}{2}(y+a_k^-t)^2(4\beta_k^-t)^{-3/2}\right)\\ &=\mathcal{O}(t^{-1}e^{(y+a_k^-t)^2/Mt}),\end{aligned}\tag{7.40}$$

for $M > 0$ sufficiently large, and similarly for x- and y-derivatives. Multiplying by

$$[c_{k,-}^{j,0}]\frac{\partial}{\partial\delta_j}\begin{pmatrix}\bar u^\delta(x)\\ \bar v^\delta(x)\end{pmatrix} l_k^{-t} = \mathcal{O}(e^{-\theta|x|}),$$

we find that term (7.39) gives contribution

$$[c_{k,-}^{j,0}]\frac{\partial}{\partial\delta_j}\begin{pmatrix}\bar u^\delta(x)\\ \bar v^\delta(x)\end{pmatrix} l_k^{-t}\text{errfn}\left(\frac{y+a_k^- t}{\sqrt{4\beta_k^-|y/a_k^-|}}\right), \tag{7.41}$$

whereas term (7.40) gives a contribution absorbable in R (resp. R_x, R_y); see Remark 7.5 below for detailed discussion of a similar calculation.

Finally, observing that

$$[c_{k,-}^{j,0}]\frac{\partial}{\partial\delta_j}\begin{pmatrix}\bar u^\delta(x)\\ \bar v^\delta(x)\end{pmatrix} l_k^{-t}\text{errfn}\left(\frac{y-a_k^- t}{\sqrt{4\beta_k^- t}}\right) \tag{7.42}$$

is time-exponentially small for $t \ge 1$, since $a_k^- > 0$, $y < 0$, and $(\partial/\partial\delta_j)(\bar u^\delta, \bar v^\delta) \le Ce^{-\theta|x|}$, $\theta > 0$, we may subtract and add this term to (7.41) to obtain a total of $E(x,t;y)$ plus terms absorbable in R (resp. R_x, R_y).

S_λ term. Next, consider the second-order term

$$\frac{1}{2\pi i}\oint_{\tilde\Gamma} e^{\lambda t} S_\lambda(x,y)d\lambda, \tag{7.43}$$

which, by (6.10), is given by

$$\sum_{a_k^->0} r_k^- l_k^{-t}\alpha_k(x,t;y) + \sum_{a_k^->0,\, a_j^-<0}[c_{k,-}^{j,-}]r_j^- l_k^{-t}\alpha_{jk}(x,t;y) \tag{7.44}$$

where

$$\alpha_k(x,t;y) := \frac{1}{2\pi i}\oint_{\tilde\Gamma} e^{\lambda t} e^{(-\lambda/a_k^- + \lambda^2\beta_k^-/a_k^{-3})(x-y)}\,d\lambda. \tag{7.45}$$

and

$$\alpha_{jk}(x,t;y) := \frac{1}{2\pi i}\oint_{\tilde\Gamma} e^{\lambda t} e^{(-\lambda/a_j^- + \lambda^2\beta_j^-/a_j^{-3})x + (\lambda/a_k^- - \lambda^2\beta_k^-/a_k^{-3})y}\,d\lambda. \tag{7.46}$$

Similarly as in the treatment of the E_λ term, just above, by deforming the contour $\tilde\Gamma$ to

$$\Gamma' := [-\eta_1 - ir/2, -ir/2] \cup [-ir/2, +ir/2] \cup [+ir/2, -\eta_1 + ir/2], \tag{7.47}$$

these may be transformed modulo time-exponentially decaying terms to the elementary Fourier integrals

$$\begin{aligned}(7.48)\qquad &\frac{1}{2\pi}\mathrm{P.V.}\int_{-\infty}^{+\infty} e^{i\xi(t-(x-y)/a_k^-)}e^{\xi^2(-\beta_k^-/a_k^{-3})(x-y)}\,d\xi\\ &\qquad= (4\pi\beta_k^- t)^{-1/2}e^{-(x-y-a_k^- t)^2/4\beta_k^- t}\end{aligned}$$

and

$$\begin{aligned}(7.49)\qquad &\frac{1}{2\pi}\mathrm{P.V.}\int_{-\infty}^{+\infty} e^{i\xi\left((t-|y/a_k^-|)-x/a_j^-\right)}e^{-\xi^2\left((\beta_k^-/a_k^{-3})|y|+(\beta_j^-/a_j^{-3})|x|\right)}\,d\xi\\ &\qquad= (4\pi\bar\beta_{jk}^- t)^{-1/2}e^{-(x-z_{jk}^-)^2/4\bar\beta_{jk}^- t},\end{aligned}$$

respectively, where $\bar\beta_{jk}-^{+-}$ and $z_{jk}^{\pm}$ are as defined in (1.32) and (1.31). These correspond to the first and third terms in expansion (4.13), the latter of which has an additional factor $e^{-x}/(e^x+e^{-x})$. Noting that the second and fourth terms of (4.13) are time-exponentially small for $t\ge 1$, $y\le x\le 0$, and that

$$\begin{aligned}&|(4\pi\bar\beta_{jk}^- t)^{-1/2}e^{-(x-z_{jk}^-)^2/4\bar\beta_{jk}^- t}(1-e^{-x}/(e^x+e^{-x}))|\\ &\qquad\le |(4\pi\bar\beta_{jk}^- t)^{-1/2}e^{-(x-z_{jk}^-)^2/4\bar\beta_{jk}^- t}e^{-\theta|x|}\end{aligned}$$

for some $\theta>0$, so is absorbable in error term R, we find that the total contribution of this term, modulo terms absorbable in R, is S.

R_λ *term.* Finally, we briefly discuss the estimation of error term

$$(7.50)\qquad \frac{1}{2\pi i}\oint_\Gamma e^{\lambda t}R_\lambda(x,y)d\lambda,$$

which decomposes into the sum of integrals involving the various terms of R_λ^E and R_λ^S given in (6.12) and (6.14). Since each of these are separately analytic, they may be split up and estimated sharply via the *Riemann saddlepoint method* (method of steepest descent), as described at great length in [ZH,HoZ.2]. That is, for each summand $\alpha_\lambda(x,y)\sim e^{\beta(\lambda)x+\gamma(\lambda)y}$ in R_λ we deform the contour $\tilde\Gamma$ to a new, contour in Λ that is a mini-max contour for the modulus

$$m_\alpha(x,y,\lambda):=|e^{\lambda t}\alpha_\lambda(x,y)|=e^{\mathrm{Re}\ \lambda t+\mathrm{Re}\ \beta x+\mathrm{Re}\ \gamma y},$$

passing through an appropriate saddlepoint/critical point of $m_\alpha(x,y,\cdot)$: necessarily lying on the real axis, by the underlying complex symmetry resulting from reality of operator L.

Since terms of each type appearing in R have been sharply estimated in [ZH], we shall omit the details, only describing two sample calculations to illustrate the method:

Example 1: $e^{-\theta|x-y|}$. Contour integrals of form

$$\frac{1}{2\pi i}\oint_{\tilde\Gamma} e^{\lambda t} e^{-\theta|x-y|}d\lambda, \tag{7.51}$$

arising through the pairing of fast forward and fast dual modes, may be deformed to

$$\frac{1}{2\pi i}\int_{-\eta_1-ir/2}^{-\eta_1+ir/2} e^{\lambda t} e^{-\theta|x-y|}d\lambda \tag{7.52}$$

and estimated as $\mathcal{O}(e^{-\eta_1 t-\theta|x-y|})$, a negligible, time-exponentially decaying contribution. $\square$

Example 2: $\lambda^r e^{(-\lambda/a_k^- +\lambda^2\beta_k^-/{a_k^-}^3)(x-y)}$. Contour integrals of form

$$\frac{1}{2\pi i}\oint_{\tilde\Gamma} e^{\lambda t}\lambda^r e^{(-\lambda/a_k^- +\lambda^2\beta_k^-/{a_k^-}^3)(x-y)}d\lambda, \tag{7.53}$$

$a_k^->0$, arising through the pairing of slow forward and slow dual modes, may be deformed to contour

$$\Gamma' := [-\eta_1-ir/2,\eta_*-ir/2]\cup[\eta_*-ir/2,\eta_*+ir/2]\cup[\eta_*+ir/2,-\eta_1+ir/2], \tag{7.54}$$

where saddlepoint η_* is defined as

$$\eta_*(x,y,t) := \begin{cases} \frac{\bar\alpha}{p} & \text{if } |\frac{\bar\alpha}{p}|\le\varepsilon \\ \pm\varepsilon & \text{if } \frac{\bar\alpha}{p}\gtrless\varepsilon, \end{cases} \tag{7.55}$$

with

$$\bar\alpha := \frac{x-y-a_k^- t}{2t}, \quad p := \frac{\beta_k^-(x-y)}{(a_k^-)^2 t} > 0, \tag{7.56}$$

and η_1, $\varepsilon>0$ are chosen sufficiently small with respect to r, to yield, modulo time-exponentially decaying terms, the estimate

$$e^{-(x-y-a_k^- t)^2/4\beta^- t}\int_{-\infty}^{+\infty}\mathcal{O}(|\eta_*|^r+|\xi|^r)e^{-\theta\xi^2 t}\,d\xi = \mathcal{O}(t^{-(r+1)/2}e^{-(x-y-a_k^- t)^2/Mt}) \tag{7.57}$$

if $|\frac{\bar\alpha}{p}|\le\varepsilon$, and

$$e^{-\varepsilon t/M}\int_{-\infty}^{+\infty}\mathcal{O}(|\eta_*|^r+|\xi|^r)e^{-\theta\xi^2 t}\,d\xi = \mathcal{O}(t^{-(r+1)/2}e^{-\eta t}). \tag{7.58}$$

if $|\frac{\bar\alpha}{p}| \geq \varepsilon$, $\theta > 0$. In either case, the main contribution lies along the central portion $[\eta_* - ir/2, \eta_* + ir/2]$ of contour Γ'.

To see this, note that $|x-y|$ and t are comparable when $|\bar\alpha/p|$ is bounded, whence the evident spatial decay $e^{-\theta\xi^2|x-y|}$ of the integrand along contour $\lambda = \eta_* + i\xi$ may be converted to the $e^{-\xi^2 t}$ decay displayed in (7.57)–(7.58); likewise, temporal growth in $e^{\lambda t}$ on the horizontal portions of the contour, of order $\leq e^{(|\eta_1|+|\varepsilon|)t}$, is dominated by the factor $e^{-\theta\xi^2} = e^{-\theta r^2/4}$, provided $|\eta_1| + |\varepsilon|$ is sufficiently small with respect to r^2. For $|\bar\alpha/p|$ large, on the other hand, η_* is uniformly negative, and also $|x-y|$ is negligible with respect to t, whence the estimate (7.58) holds trivially.

We point out that η_* is easily determined, as the minimal point on the real axis of the quadratic function

$$f_{x,y,t}(\lambda) := \lambda t(-\lambda/a_k^- + \lambda^2\beta_k^-/a_k^{-3})(x-y), \tag{7.59}$$

the argument of the integrand of (7.53). □

Other terms may be treated similarly: All "constant-coefficient" terms $\phi_k\tilde\phi_k^*$ or $\psi_k\tilde\psi_k^*$ are of either the form treated in Example 1 (fast modes) or in example 2 (slow modes). Scattering pairs involving slow forward and slow dual modes from different families (i.e., terms with coefficients M_{jk}, $d_{jk}^{\pm}$) may be treated similarly as in Example 2. Scattering pairs involving two fast modes are of the form already treated in Example 1, since both modes of a scattering pair are decaying. Scattering pairs involving one fast and one slow mode may be factored as the product of a term of the form treated in example 2 and a term that is uniformly exponentially decaying in either x or y; factoring out the exponentially decay, we may treat such terms as in Example 2.

Scattering coefficients. The relation (1.33) may now be deduced, a posteriori, from conservation of mass in the linearized flow (1.10). For, by inspection, all terms save E and S in the decomposition (1.25) of G decay in L^1, whence these terms must, time-asymptotically, carry exactly the mass in U of the initial perturbation $\delta_y(x)I_n$, i.e.,

$$\lim_{t\to\infty}\int_{-\infty}^{+\infty}\big(E(x,t;y) + S(x,t;y)\big)\,dx = I_n. \tag{7.60}$$

Taking $y \leq 0$ for definiteness, and right-multiplying both sides of (7.60) by r_k^-, we thus obtain the result from definitions (4.9)–(4.13).

Moreover, rewriting (1.33) as

$$
(r_1^-, \dots, r_{n-i_-}^-, r_{i_+}^+, \dots, r_n^+, m_1, \dots, m_\ell) \begin{pmatrix} [c_{k,-}^{1,-}] \\ \vdots \\ [c_{k,-}^{n-i_-,-}] \\ [c_{k,-}^{i_+,+}] \\ \vdots \\ [c_{k,-}^{n,+}] \\ [c_{k,-}^{1,0}] \\ \vdots \\ [c_{k,-}^{\ell,0}] \end{pmatrix} = r_k^-, \tag{7.61}
$$

where

$$
m_j := \int_{-\infty}^{+\infty} (\partial/\partial\delta_j)\bar{U}^\delta(x)\, dx, \tag{7.62}
$$

we find that, under assumption $(\mathcal{D})$, it uniquely determines the scattering coefficients $[c_{k,-}^{j,\pm}]$. For, the determinant

$$
\det\, (r_1^-, \dots, r_{n-i_-}^-, r_{i_+}^+, \dots, r_n^+, m_1, \dots, m_\ell) \tag{7.63}
$$

of the matrix on the righthand side of (7.61) is exactly the inviscid stability coefficient Δ defined in [ZS,Z.3], hence is nonvanishing by the equivalence of $(\mathcal{D}2)$ and (D2) (recall discussion of Section 1.2, just below Theorem 1.4). Note that (7.63) reduces to the Liu–Majda determinant (1.16) in the Lax case, $\ell = 1$, with $\bar{U}^\delta(x)$ parametrized as $\bar{U}^\delta(x) := \bar{U}(x - \delta_1 x)$, for which m_1 reduces to $[U]$.

Relation (1.34) follows from the observation that

$$
\begin{aligned}
P_\lambda(x,y) &:= \text{Residue}\,_{\lambda=0} G_\lambda(x,y) \\
&= \sum_{j=1}^{\ell} (\partial/\partial\delta_j)\bar{U}^\delta(x)\, \pi_j,
\end{aligned}
$$

with π_j defined by the first, or second expressions appearing in (1.34), according as y is less than or equal to/greater than or equal to zero. For, the extended spectral theory of Section 5.4 then implies that π_j are the left effective eigenfunctions associated with right eigenfunctions $(\partial/\partial\delta_j)\bar{U}^\delta$. Alternatively, (1.34) may be deduced by linear-algebraic manipulation directly from (7.61) and its counterpart for $y \geq 0$ (the same identity with $k, -$ everywhere replaced by $k, +$).

Remark 7.5. As pointed out in [LZ.2], the left eigenfunctions (constant vectors) $\pi_j = (\pi_{j,u}, 0)$ may themselves be determined by the relation

$$\pi^t_{j,u}(r_1^-, \dots, r^-_{n-i_-}, r^+_{i_+}, \dots, r^+_n, m_1, \dots, m_\ell) = (0, e_j^t), \tag{7.64}$$

from which (1.34) is easily deduced, where e_j here denotes the jth standard basis element in $\mathbb{R}^\ell$. This may be deduced a priori through the extended spectral theory of Section 5.4; it would interesting also to derive (7.61) a priori, directly through the relations (4.40)–(4.41).

□

Remark 7.6 (The undercompressive case). In the undercompressive case, the result of Lemma 6.8 is false, and consequently the estimates of Lemma 6.11 do not hold. This fact has the implication that shock dynamics are not governed solely by conservation of mass, as in the Lax or overcompressive case, but by more complicated dynamics of front interaction; for related discussion, see [LZ.2,Z.2]. At the level of Proposition 6.2, it means that the simple representations of E_λ and S_λ in terms of slow dual modes alone (corresponding to characteristics that are incoming to the shock) are no longer valid in the undercompressive case, and there appear new terms involving rapidly decaying dual modes $\sim e^{-\theta|y|}$ related to inner layer dynamics. Though precise estimates can nonetheless be carried out, we have not found a similarly compact representation of the resulting bounds as that of the Lax/overcompressive case, and so we shall not state them here. We mention only that this rapid variation in the y-coordinate precludes the L^p stability arguments used here and in [Z.2], requiring instead detailed pointwise bounds as in [HZ.2,Z.5]. See [LZ.1–2,ZH,Z.2,Z.5] for further discussion of this interesting case.

7.2. Inner layer dynamics. Similarly as in [ZH], we now investigate dynamics of the inner shock layer, in the case that $(\mathcal{D})$ does not necessarily hold, in the process establishing the necessity of $(\mathcal{D})$ for linearized orbital stability.

Proposition 7.7. *Let (H0)–(H4) hold. Then, there exists $\eta > 0$ such that, for x, y restricted to any bounded set, and t sufficiently large,*

$$G(x, y; t) = \sum_{\lambda \in \sigma'_p(L) \cap \{Re(\lambda) \geq 0\}} e^{\lambda t} \sum_{k \geq 0} t^k (L - \lambda I)^k P_\lambda(x, y) + \mathcal{O}(e^{-\eta t}), \tag{7.65}$$

where $P_\lambda(x, y)$ is the effective projection kernel described in Definition 6.12, and $\sigma'_-(L)$ the effective point spectrum.

Proof. Similarly as in the proof of Proposition 7.1, decompose G into terms I_a, I_b, I_c, II_a, and II_b of (7.21). Then, the same argument (Case II. $|x-y|/t$ bounded) yields that $I_a = H(x, t; y)$, while I_b and II_b are time-exponentially small. Observing that $H = 0$ for x, y bounded, and t sufficiently large, we have reduced the problem to the study of

$$II_a := \frac{1}{2\pi i} \oint_\Gamma e^{\lambda t} G_\lambda(x, y) d\lambda, \tag{7.66}$$

where, recall,

$$\Gamma := [-\eta_1 - iR, \eta - iR] \cup [\eta - iR, \eta + iR] \cup [\eta + iR, -\eta_1 + iR]. \tag{7.67}$$

Note that the high-frequency bounds of Proposition 5.2 imply that L has no point spectrum in Ω outside of the rectangle $\mathcal{R}$ enclosed by $\Gamma \cup [-\eta_1 - iR, -\eta + iR]$. Choosing η_1 sufficiently small, therefore, we may ensure that no effective point spectrum lies within the strip $-\eta_1 \le \text{Re } \lambda < 0$, by compactness of $\mathcal{R}$ together with the fact that effective eigenvalues are isolated from one another, as zeroes of the analytic Evans function.

Recall that G_λ is meromorphic on Ω. Thus, we may express (7.66), using Cauchy's Theorem, as

$$\frac{1}{2\pi i}\int_{-\eta_1 - iR}^{-\eta_1 + iR} e^{\lambda t} G_\lambda(x,y)d\lambda + \text{Residue}_{\ \lambda\in\mathcal{R}}\, e^{\lambda t} G_\lambda(x,t;y). \tag{7.68}$$

By Definition 6.12 and Proposition 6.14,

$$\begin{aligned}
&\text{Res}_{\lambda\in\{\text{Re }\lambda\ge 0\}} e^{\lambda t} G_\lambda(x,y)\\
&\qquad = \sum_{\lambda_0\in\sigma_p'(L)\cap\{\text{Re }\lambda\ge 0\}} e^{\lambda_0 t}\text{Res}_{\lambda_0} e^{(\lambda-\lambda_0)t} G_\lambda(x,y)\\
&\qquad = \sum_{\lambda_0\in\sigma_p'(L)\cap\{\text{Re }\lambda\ge 0\}} e^{\lambda_0 t}\sum_{k\ge 0}(t^k/k!)\text{Res}_{\lambda_0}(\lambda-\lambda_0)^k G_\lambda(x,y)\\
&\qquad = \sum_{\lambda_0\in\sigma_p'(L)\cap\{\text{Re }\lambda\ge 0\}} e^{\lambda_0 t}\sum_{k\ge 0}(t^k/k!) Q_{\lambda_0,k}(x,y)\\
&\qquad = \sum_{\lambda_0\in\sigma_p'(L)\cap\{\text{Re }\lambda\ge 0\}} e^{\lambda_0 t}\sum_{k\ge 0}(t^k/k!)(L-\lambda_0 I)^k P_{\lambda_0}(x,y)).
\end{aligned} \tag{7.69}$$

On the other hand, for x, y bounded and t sufficiently large, t dominates $|x|$ and $|y|$ and we obtain from Propositions 4.4 and 6.2 that

$$|G_\lambda(x,t;y)| \le C$$

for all $\lambda \in \Omega$, for η_1 sufficiently small, hence

$$\frac{1}{2\pi i}\int_{-\eta_1 - iR}^{-\eta_1 + iR} e^{\lambda t} G_\lambda(x,y)d\lambda \le 2CRe^{-\eta_1 t}. \tag{7.70}$$

Combining (7.68), (7.69), and (7.70), we obtain the result. □

Corollary 7.8. *Let (H0)–(H4) hold. Then, $(\mathcal{D})$ is necessary for linearized orbital stability with respect to compactly supported initial data, as measured in any L^p norm, $1 \le p \le \infty$.*

Proof: From (7.65), we find that $\bar U(\cdot)$ is linearly orbitally stable only if $P_\lambda = 0$ for all $\lambda \in \{\mathrm{Re}\ \lambda \ge 0\} \setminus \{0\}$ and Range $\mathcal{P}_0 = \mathrm{Span}\ \{(\partial/\partial\delta_j)\bar U^\delta\}$. By Propositions 6.14 and 6.15, this is equivalent to $(\mathcal{D})$. $\square$

Section 8. Linearized Stability.

Linearized orbital stability now follows immediately from the pointwise bounds we have established. Similarly as in [Z.2], define the *linear instantaneous projection*:

$$\varphi(x,t) := \int_{-\infty}^{+\infty} E(x,t;y)U_0(y)\,dy =: \delta(t)(\partial/\partial\delta)\bar U^\delta(x), \tag{8.1}$$

where U_0 is the initial data for the linearized evolution equation (1.10). The coordinates $\delta \in \mathbb{R}^{1\times\ell}$ may be expressed, alternatively, as

$$\delta(t) = \int_{-\infty}^{+\infty} e(y,t)v_0(y)\,dy,$$

where

$$E(x,t;y) =: (\partial/\partial\delta)\bar U^\delta(x)e(y,t), \tag{8.2}$$

i.e.,

$$e(y,t) := \sum_{a_k^-}\left(\mathrm{errfn}\ \left(\frac{y+a_k^-t}{\sqrt{4\beta_k^-t}}\right) - \mathrm{errfn}\ \left(\frac{y-a_k^-t}{\sqrt{4\beta_k^-t}}\right)\right) l_k^- \tag{8.3}$$

for $y \le 0$, and symmetrically for $y \ge 0$.

Then, the solution U of (1.10) satisfies

$$U(x,t) - \phi(x,t) = \int_{-\infty}^{+\infty} (H+\tilde G)(x,t;0)U_0(y)\,dy, \tag{8.4}$$

where

$$\tilde G := S + R \tag{8.5}$$

is the regular part and H the singular part of the time-decaying portion of the Green's function G.

Lemma 8.1. *Under assumptions (H0)–(H4) and ($\mathcal{D}$), there hold:*

$$|\int_{-\infty}^{+\infty} \tilde{G}(\cdot,t;y)f(y)dy|_{L^p} \le C(1+t)^{-\frac{1}{2}(1-1/r)}|f|_{L^q}, \tag{8.6}$$

$$|\int_{-\infty}^{+\infty} \tilde{G}(\cdot,t;y)(0,I_r)^t f(y)dy|_{L^p} \le C(1+t)^{-\frac{1}{2}(1-1/r)-1/2}|f|_{L^q}, \tag{8.7}$$

$$|\int_{-\infty}^{+\infty} \tilde{G}_y(\cdot,t;y)f(y)dy|_{L^p} \le C(1+t)^{-\frac{1}{2}(1-1/r)-1/2}|f|_{L^q} + Ce^{-\eta t}|f|_{L^p}, \tag{8.8}$$

and

$$|\int_{-\infty}^{+\infty} H(\cdot,t;y)f(y)dy|_{L^p} \le Ce^{-\eta t}|f|_{L^p}, \tag{8.9}$$

for all $t \ge 0$, some C, $\eta > 0$, for any $1 \le r \le p$ and $f \in L^q$ (resp. L^p), where $1/r + 1/q = 1 + 1/p$.

Proof. Bounds (8.6)–(8.8) follow by the Hausdorff-Young inequality together with bounds (4.13) and (7.1)–(7.2), precisely as in [Z.2]. Bound (8.9) follows by direct computation and the fact that particle paths $z_j(y,t)$ satisfy uniform bounds

$$1/C \le |(\partial/\partial y)z_j| < C,$$

for all y, t, by the fact that characteristic speeds $a_j(x)$ converge exponentially as $x \to \pm\infty$ to constant states. $\square$

Corollary 8.2. *Let (H0)–(H4) hold. Then, ($\mathcal{D}$) is sufficient for $L^1 \cap L^p \to L^p$ linearized orbital stability, for any $p > 1$. More precisely, for initial data $U_0 \in L^1 \cap L^p$, the solution $U(x,t)$ of the linearized evolution equations (1.10) satisfies the linear decay bounds:*

$$|U(\cdot,t) - \varphi(\cdot,t)|_{L^p} \le C(1+t)^{-\frac{1}{2}(1-1/p)}(|U_0|_{L^1} + |U_0|_{L^p}). \tag{8.10}$$

Proof. Immediate, from (8.4) and bounds (8.6) and (8.9), with $q = p$. $\square$

Proof of Theorem 1.4. Theorem 1.4 now follows immediately from Proposition 7.8 together with Corollary 8.2. $\square$

Appendices: Auxiliary Results/Calculations.

Appendix A. Structure of Viscous Profiles.

In this appendix, we prove the results cited in the introduction concerning structure of viscous profiles.

Proof of Lemma 1.1. Integrating (1.6) from $-\infty$ to x and rearranging, we may write (1.5)–(1.6) in the alternative form

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} f_u & f_v \\ b_1 & b_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ g - g_- \end{pmatrix}. \tag{9.1}$$

Linearizing (9.1) about $U_\pm$, we obtain

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} f_u & f_v \\ b_1 & b_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ g_u & g_v \end{pmatrix}_{|(U_\pm)} \begin{pmatrix} u \\ v \end{pmatrix}, \tag{9.2}$$

or, setting

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} := \begin{pmatrix} f_u & f_v \\ b_1 & b_2 \end{pmatrix}_{|(U_\pm)} \begin{pmatrix} u \\ v \end{pmatrix},$$

the pair of equations

$$z_1' = 0$$

and

$$z_2' = \begin{pmatrix} g_u & g_v \end{pmatrix} \begin{pmatrix} f_u & f_v \\ b_1 & b_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I_r \end{pmatrix}_{|(U_\pm)} z_2, \tag{9.3}$$

the latter of which evidently describes the linearized ODE on manifold (1.5).

Observing that

$$\det \begin{pmatrix} g_u & g_v \end{pmatrix} \begin{pmatrix} f_u & f_v \\ b_1 & b_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I_r \end{pmatrix}_{|(U_\pm)} = \det \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix} \begin{pmatrix} f_u & f_v \\ b_1 & b_2 \end{pmatrix}^{-1}_{|(U_\pm)} \neq 0$$

by (H2) and (H1)(i), we find that the coefficient matrix of (9.3) has no zero eigenvalues. On the other hand, it can have no nonzero purely imaginarly eigenvalues $i\xi$, since otherwise

$$\begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix} \begin{pmatrix} f_u & f_v \\ b_1 & b_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ v \end{pmatrix} = i\xi \begin{pmatrix} 0 \\ v \end{pmatrix},$$

and thus

$$\Big[-i\xi \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix} - \xi^2 \begin{pmatrix} 0 & 0 \\ b_1 & b_2 \end{pmatrix} \Big] \Big[\begin{pmatrix} f_u & f_v \\ b_1 & b_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ v \end{pmatrix} \Big] = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for $\xi \neq 0 \in \mathbb{R}$, in violation of (H3). Thus, we find that $U_\pm$ are hyperbolic rest points, from which (3.2) follows. $\square$

Proof of Lemma 1.2. We follow the notation of Sections 1.2 and 3.1. Equating by consistent splitting/continuous extension to $\lambda = 0$ the dimensions of the stable subspace S^+ of the limiting coefficient matrix $\mathbb{A}_+(\lambda)$ appearing in eigenvalue equation (3.1) at $\lambda = 0$ and at $\lambda \to +\infty$, we obtain

$$\dim \mathcal{U}(A_+) + d_+ = \dim \mathcal{U}(A_{*+}) + r,$$

or

$$(n - i_+) + d_+ = \dim \mathcal{U}(A_{*+}) + r,$$

and similarly at $x \to -\infty$, where the lefthand side follows from Lemma 6.1 and the righthand side from the proof of Lemma 4.1. Rearranging, we obtain (1.8). If there exists a connecting profile, then we have by (H1)(iii) that dim $\mathcal{S}(A_*)$ and dim $\mathcal{U}(A_*)$ are constant along the profile and sum to $n-r$. Thus, dim $\mathcal{S}(A_{*-})$ + dim $\mathcal{U}(A_{*+}) = n - r$, and we obtain $n - i = r - d$ by summing relations (1.8). $\square$

Appendix B. Expansion of the Fourier Symbol.

In this appendix, we record the Taylor expansions about $\xi = 0$ and $\xi = \infty$ of the Fourier symbol $P(i\xi) = -i\xi A(x) - \xi^2 B(x)$ of the frozen, constant-coefficient operator at a given x.

Low frequency expansion. By straightforward matrix perturbation theory [Kat], the second-order expansion at $i\xi = 0$ of $P(i\xi)$ with respect to $i\xi$ is just

$$P = -i\xi A - \xi^2 R \,\mathrm{diag}\,\{\beta_1, \dots, \beta_n\} L,$$

$$\mathrm{diag}\,\{\beta_1, \dots, \beta_n\} := LBR,$$

where L, R are composed of left, right eigenvectors of A, as described in the introduction.

High frequency expansion. We next consider the expansion of P at infinity with respect to $(i\xi)^{-1}$. Recall that

$$B := \begin{pmatrix} 0 & 0 \\ b_1 & b_2 \end{pmatrix}, \quad A := \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad A_* = A_{11} - A_{12} b_1 b_2^{-1}. \tag{9.4}$$

Claim. To zeroth order, "hyperbolic" dispersion relations

$$\lambda(i\xi) \in \sigma(-\xi^2 B - i\xi A), \quad \lambda \sim \xi, \tag{9.5}$$

are given by

$$\lambda_j(\xi) = -i\xi a_j^* - \eta_j^* + \dots, \quad j = 1, \cdots n - r, \tag{9.6}$$

with corresponding right and left eigenvectors

$$R_j^*(i\xi) = \begin{pmatrix} r_j^* \\ -b_2^{-1}b_1 r_j^* \end{pmatrix} + \dots \quad \text{and} \quad L_j^*(i\xi) = \begin{pmatrix} l_j^* \\ 0 \end{pmatrix} + \dots, \tag{9.7}$$

where

$$\eta_j^* := l_j^* D^* r_j^*, \qquad D_* := A_{12} b_2^{-1} \Big[A_{21} - A_{22} b_2 b_1 + b_2^{-1} b_1 A_* \Big], \tag{9.8}$$

and a_j^*, r_j^*, l_j^* are eigenvalues and associated right and left eigenvectors of A_*. The remaining r "parabolic" relations $\lambda \sim \xi^2$ are given to second order by

$$\lambda_{n-r+j}(\xi) = -\xi^2 \gamma_j + \dots, \quad j = 1, \cdots r, \tag{9.9}$$

with corresponding right and left eigenvectors

$$R_{n-r+j}(i\xi) = \begin{pmatrix} 0 \\ s_j \end{pmatrix} + \dots \quad \text{and} \quad L_{n-r+j}(i\xi) = \begin{pmatrix} \gamma_j^{-1} b_1^{tr} t_j \\ t_j \end{pmatrix}, \tag{9.10}$$

where γ_j, s_j, t_j are the eigenvalues and right and left eigenvectors of b_2.

Proof. Expressing $P(i\xi) = -\xi^2(B+\epsilon A)$, $\epsilon := (i\xi)^{-1}$ and expanding around $\epsilon = 0$ by standard matrix perturbation theory, we obtain the results after rearrangement; for details, see the dual calculation carried out in Appendix A2, [MZ.1] in the relaxation case. Of the stated relations, only (9.8) requires calculation, and this is included only for the interest of the reader; in fact, we do not require the explicit form of η_j^* anywhere in our analysis. □

Remark. Note that we obtain as a consequence of (H3) the "strict dissipativity" condition Re $\eta_j^* > 0$, $j = 1, \dots, n$. In the symmetrizable case, this is equivalent to the simple conditions that $B \geq 0$ and no vector in $ker(B)$ is an eigenvector of A; see [Kaw].

Appendix C. Asymptotic ODE estimates.

For completeness, we prove in this appendix the asymptotic ODE estimates cited in Section 3. We first recall the gap lemma of [GZ,KS,ZH].

Proposition 9.1 (the gap lemma). *Consider (3.1), under assumption (h0), with $\bar{\theta} < \theta$. If $V^-(\lambda)$ is an eigenvector of $\mathbb{A}_-$ with eigenvalue $\mu(\lambda)$, both analytic in λ, then there exists a solution of (3.1) of form*

$$W(\lambda, x) = V(x, \lambda) e^{\mu(\lambda) x},$$

where V is C^1 in x and locally analytic in λ and for each $j = 0, 1, \dots$ satisfies

$$(\partial/\partial\lambda)^j V(x, \lambda) = (\partial/\partial\lambda)^j V^-(\lambda) + \mathcal{O}(e^{-\bar{\theta}|x|} |V^-(\lambda)|), \quad x < 0, \tag{9.11}$$

for all $\bar\theta < \theta$. Moreover, if $\mathrm{Re}\ \mu(\lambda) > \mathrm{Re}\ \widetilde{\mu}(\lambda) - \theta$ *for all (other) eigenvalues $\widetilde{\mu}$ of $\mathbb{A}_-$, then also W is uniquely determined by (9.11), and (9.11) holds for $\bar\theta = \theta$.*

Proof. Setting $W(x) = e^{\mu x}V(x)$, we can rewrite $W' = \mathbb{A}W$ as

$$V' = (\mathbb{A}_- - \mu I)V + \theta V, \qquad \theta := (\mathbb{A} - \mathbb{A}_-) = \mathcal{O}(e^{-\theta|x|}), \tag{9.12}$$

and seek a solution $V(x,\lambda) \to V^-(x)$ as $x \to \infty$. Set $\bar\theta = \bar\theta < \theta_1 < \theta_2 < \bar\theta < \theta$. Fixing a base point λ_0, we can define on some neighborhood of λ_0 to the complementary $\mathbb{A}_-$-invariant projections $P(\lambda)$ and $Q(\lambda)$ where P projects onto the direct sum of all eigenspaces of $\mathbb{A}_-$ with eigenvalues $\widetilde{\mu}$ satisfying $\mathrm{Re}\ (\widetilde{\mu}) < \mathrm{Re}\ (\mu) + \theta_2$, and Q projects onto the direct sum of the remaining eigenspaces, with eigenvalues satisfying $\mathrm{Re}\ (\widetilde{\mu}) \ge \mathrm{Re}\ (\mu) + \theta_2 > \mathrm{Re}\ (\mu) + \theta_1$. By basic matrix perturbation theory (eg. [Kat]) it follows that P and Q are analytic in a neighborhood of λ_0, with

$$\left|e^{(\mathbb{A}_- - \mu I)x}P\right| = \mathcal{O}(e^{\bar\theta_2 x}), \quad x > 0, \qquad \left|e^{(\mathbb{A}_- - \mu I)x}Q\right| = \mathcal{O}(e^{\bar\theta_1 x}), \quad x < 0. \tag{9.13}$$

Thus, for $M > 0$ sufficiently large, the map $\mathcal{T}$ defined by

$$\mathcal{T}V(x) = V^- + \int_{-\infty}^{x} e^{(\mathbb{A}_- - \mu I)(x-y)}P\theta(y)V(y)dy - \int_{x}^{-M} e^{(\mathbb{A}_- - \mu I)(x-y)}Q\theta(y)V(y)dy \tag{9.14}$$

is a contraction on $L^\infty(-\infty, -M]$; for, applying (9.13), we have

$$\begin{aligned} |\mathcal{T}V_1 - \mathcal{T}V_2|_{(x)} &\le \mathcal{O}(1)|V_1 - V_2|_\infty \left(\int_{-\infty}^{x} e^{\bar\theta_2(x-y)}e^{\theta y}dy + \int_{x}^{-M} e^{\bar\theta_1(x-y)}e^{\theta y}dy \right) \\ &= \mathcal{O}(1)|V_1 - V_2|_\infty \left(e^{\bar\theta_2 x}e^{(\theta - \bar\theta_2)y}|_{-\infty}^{x} + e^{\bar\theta_1 x}e^{(\theta - \bar\theta_1)y}|_{x}^{-M} \right) \\ &= \mathcal{O}(1)|V_1 - V_2|_\infty e^{-\bar\theta M} < \frac{1}{2}|V_1 - V_2|_\infty. \end{aligned} \tag{9.15}$$

By iteration, we thus obtain a solution $V \in L^\infty(-\infty, -M]$ of $V = \mathcal{T}V$ with $V = \mathcal{O}(|V^-|)$; since $\mathcal{T}$ clearly preserves analyticity $V(\lambda, x)$ is analytic in λ as the uniform limit of analytic iterates (starting with $V_0 = 0$). Differentiation shows that V is a bounded solution of $V = \mathcal{T}V$ if and only if it is a bounded solution of (9.12). Further, taking $V_1 = V$, $V_2 = 0$ in (9.15), we obtain from the second to last equality that

$$|V - V^-| = |\mathcal{T}(V) - \mathcal{T}(0)| = \mathcal{O}(1)e^{\bar\theta x}|V| = \mathcal{O}(e^{\bar\theta x})|V^-|, \tag{9.16}$$

giving (9.11) for $j = 0$. Derivative bounds, $j > 0$, follow by standard interior estimates, or, alternatively, by differentiating (9.14) with respect to λ and repeating the same argument. Analyticity, and the bounds (9.11), extend to $x < 0$ by standard analytic dependence for the initial value problem at $x = -M$. Finally, if Re $(\mu(\lambda)) >$ Re $(\tilde{\mu}(\lambda)) - \frac{\theta}{2}$ for all other eigenvalues, then $P = I$, $Q = 0$, and $V = \mathcal{T}V$ must hold for any V satisfying (9.11), by Duhamel's principle. Further, the only term appearing in (9.15) is the first integral, giving (9.16) with $\bar{\theta} = \theta$. $\square$

Proof of Lemma 3.1. Substituting $W = P_+ Z$ into (3.1), equating to (3.3), and rearranging, we obtain the defining equation

$$P_+' = \mathbb{A}_+ P_+ - P_+ \mathbb{A}, \qquad P_+ \to I \quad \text{as} \quad x \to +\infty. \tag{9.17}$$

Viewed as a vector equation, this has the form $P_+' = \mathcal{A}P_+$, where $\mathcal{A}$ approaches exponentially as $x \to +\infty$ to its limit $\mathcal{A}_+$, defined by $\mathcal{A}_+ P := \mathbb{A}_+ P - P\mathbb{A}_+$. The limiting operator $\mathcal{A}_+$ evidently has analytic eigenvalue, eigenvector pair $\mu \equiv 0$, $P_+ \equiv I$, whence the result follows by Lemma 9.1 for $j = k = 0$. The x-derivative bounds $0 < k \le K + 1$ then follow from the ODE and its first K derivatives, and the λ-derivative bounds from standard interior estimates for analytic functions. A symmetric argument gives the result for P_-. $\square$

Proof of Proposition 3.9. Setting $\Phi_2 := \psi_1 \psi_2^{-1}$, $\psi_1 \in \mathbb{C}^{(N-k)\times k}$, $\psi_2 \in \mathbb{C}^{k \times k}$, where $(\psi_1^t, \psi_2^t)^t \in \mathbb{C}^{N \times k}$ satisfies (3.12), we find after a brief calculation that Φ_2 satisfies

$$\Phi_2' = (M_1 \Phi_2 - \Phi_2 M_2) + \delta Q(\Phi_2), \tag{9.18}$$

where Q is the quadratic matrix polynomial $Q(\Phi) := \Theta_{12} + \Theta_{11}\Phi - \Phi\Theta_{22} + \Phi\Theta_{21}\Phi$. Viewed as a vector equation, this has the form

$$\Phi_2' = \mathcal{M}\Phi_2 + \delta Q(\Phi_2), \tag{9.19}$$

with linear operator $\mathcal{M}\Phi := M_1\Phi - \Phi M_2$. Note that a basis of solutions of the decoupled equation $\Phi' = \mathcal{M}\Phi$ may be obtained as the tensor product $\Phi = \phi\tilde{\phi}^*$ of bases of solutions of $\phi' = M_1\phi$ and $\tilde{\phi}' = -M_2^*\tilde{\phi}$, whence we obtain from (7.33) the bound

$$|\mathcal{F}^{y\to x}| \le |\tilde{\mathcal{F}}_1^{y\to x}||\tilde{\mathcal{F}}_2^{y\to x}| = |\tilde{\mathcal{F}}_1^{y\to x}||(\mathcal{F}_2^{y\to x})^{-1}| \le Ce^{-\eta|x-y|} \tag{9.20}$$

for $x > y$, where $\mathcal{F}$ denotes the flow of the full decoupled matrix-valued equation, $\mathcal{F}_j$ the flow of $\phi' = M_j\phi$ and $\tilde{\mathcal{F}}_j$ the adjoint flow associated with equation $\tilde{\phi}' = -M_j^*\tilde{\phi}$. (Recall the standard duality relation $\tilde{\mathcal{F}}_j = \mathcal{F}_j^{-1}$).

That is, $\mathcal{F}$ is uniformly exponentially decaying in the forward direction. Thus, assuming only that Φ_2 is bounded at $-\infty$, we obtain by Duhamel's principle the integral fixed-point equation

$$\Phi_2(x) = \mathcal{T}\Phi_2(x) := \delta \int_{-\infty}^{x} \mathcal{F}^{y\to x} Q(\Phi_2)(y)\, dy. \tag{9.21}$$

Using (9.20), we find that $\mathcal{T}$ is a contraction of order $\mathcal{O}(\delta/\eta)$, hence (9.21) determines a unique solution for δ/η sufficiently small, which, moreover, is order δ/η as claimed. The bound on $\partial_x \Phi_j^\epsilon$ then follows by differentiation of (9.21), using the fact that $\partial_x \mathcal{F}^{y\to x} \le C\mathcal{F}^{y\to x}$, since the coefficients of the decoupled (linear) flow are bounded. A symmetric argument establishes existence of Φ_1. □

Remarks 9.2. 1. Note that the above stable/unstable manifold construction requires only boundedness of coefficients at infinity, and not any special (e.g., asymptotically constant or periodic) structure. This reflects the local nature of large frequency/short time estimates in the applications.

2. Equation (9.18) could alternatively be derived from the point of view of Lemma 3.1, as defining a change of coordinates

$$P := \begin{pmatrix} I & \Phi_2 \\ \Phi_1 & I \end{pmatrix}$$

conjugating (3.12) to diagonal form. The result of Proposition 3.9 shows that there is a unique such transformation that is bounded on the whole line.

3. If desired, one can obtain a full, and explicit Neumann series expansion in powers of δ/η from the fixed-point equation (9.21), using our explicit description of the flow $\mathcal{F}$ as the tensor product $\Phi = \phi\tilde{\phi}^*$ of bases of solutions of $\phi' = M_1\phi$ and $\tilde{\phi}' = -M_2^*\tilde{\phi}$. For our purposes, it is only the existence and not the precise form of the expansion that is important.

Proof of Corollary 3.11. Similarly as in the proof of Proposition 3.9 just above, we may obtain a fixed-point representation

$$\mathcal{F}_j^{y\to x} = \mathcal{T}\mathcal{F}_j^{y\to x} := \bar{\mathcal{F}}_1^{y\to x} + \delta\int_y^x \bar{\mathcal{F}}^{y\to z}\big(\Theta_{11} + \Theta_{12}\Phi_2\big)(z,\epsilon)\mathcal{F}_j^{z\to x}\,dz,$$

where $\mathcal{T}$ is a contraction of order $\mathcal{O}(\delta/\eta)$, yielding the result. □

Lemma 9.3. *Let $d_1 \in \mathbb{C}^{m_1\times m_1}$ and $d_2 \in \mathbb{C}^{m_2\times m_2}$ have norm bounded by C_1 and respective spectra separated by $1/C_2 > 0$. Then, the matrix commutator equation*

$$d_1 X - X d_2 = F, \tag{9.22}$$

$X \in \mathbb{C}^{m_1\times m_2}$, is soluble for all $F \in \mathbb{C}^{m_1\times m_2}$, with $|X| \le C(C_1, C_2)|F|$.

Proof. Consider (9.22) as a matrix equation $\mathcal{D}\mathcal{X} = \mathcal{F}$ where $\mathcal{X}$ and $\mathcal{F}$ are vectorial representations of the $m_1 \times m_2$ dimensional quantities X and F, and $\mathcal{D}$ is the matrix representation of the linear operator (commutator) corresponding to the lefthand side. It is readily seen that $\sigma(\mathcal{D})$ is just the difference $\sigma(d_1) - \sigma(d_2)$ between the spectra of d_1 and d_2, with associated eigenvectors of form $r_1 l_2^*$, where r_1 are right eigenvectors associated with d_1 and l_2 are left eigenvectors associated with d_2. (Here as elsewhere, $*$ denotes adjoint, or conjugate transpose of a matrix or vector). By assumption, therefore, the spectrum of $\mathcal{D}$ has modulus uniformly bounded below by $1/C_2$, whence the result follows. ∎

Proof of Proposition 3.12. Substituting $W = T\tilde{W}$ into (3.26) and rearranging, we obtain

$$\begin{aligned}\tilde{W}' &= (T^{-1}AT - T^{-1}T')\tilde{W}\\ &= (T^{-1}AT - \varepsilon T^{-1}T_y)\tilde{W},\end{aligned}$$

yielding the defining relation

$$(T^{-1}AT - \varepsilon T^{-1}T_y) = D \tag{9.23}$$

for T.

By (h2), there exists a uniformly well-conditioned family of matrices $T_0(x)$ such that $T_0^{-1}A_0T_0 = D_0$, D_0 as in (3.25); moreover, these may be chosen with the same regularity in y as A_0 (i.e., the full regularity of A). Expanding

$$T^{-1}(y,\varepsilon) = \left(I - \big(\sum_{j=2}^{p} \varepsilon^j T_0^{-1}T_j\big) + \big(\sum_{j=2}^{p} \varepsilon^j T_0^{-1}T_j\big)^2 - \dots \right) T_0^{-1} \tag{9.24}$$

by Neumann series, and matching terms of like order ε^j, we obtain a hierarchy of systems of linear equations of form:

$$D_0T_0^{-1}T_j - T_0^{-1}T_jD_0 - F_j = D_j, \tag{9.25}$$

where F_j depends only on A_k for $0 \le k \le j$ and T_k, $(d/dy)T_k$ for $0 \le k \le j-1$.

On off-diagonal blocks (k,l), (9.25) reduces to

$$d_l[T_0^{-1}T_j]^{(k,l)} - [T_0^{-1}T_j]^{(k,l)}d_k = F_j^{(k,l)}, \tag{9.26}$$

uniquely determining $[T_0^{-1}T_j]^{(k,l)}$, by Lemma 9.3. On diagonal blocks (k,k), we are free to set $[T_0^{-1}T_j]^{(k,k)} = 0$, whereupon (9.25) reduces to

$$[D_j]^{(k,k)} = -[F_j]^{(k,k)}, \tag{9.27}$$

determining the remaining unknown $[D_j]^{(k,k)}$.

Thus, we may solve for D_j, T_j at each successive stage in a well-conditioned way. Moreover, it is clear in the general case $A_0 \not\equiv$ constant that the regularity of D_j, T_j is as claimed, since we lose one order of regularity at each stage, through the dependence of F_j on derivatives of lower order T_k, $k < j$. In the case that $A_0, \dots, A_s$ are constant in y, we may choose also $T_0, \dots, T_s$ to be constant in y, again recovering the claimed regularity. ∎

References

[AGJ] J. Alexander, R. Gardner and C.K.R.T. Jones, *A topological invariant arising in the analysis of traveling waves*, J. Reine Angew. Math. 410 (1990) 167–212.

[BE] A.A. Barmin and S.A. Egorushkin, *Stability of shock waves,* Adv. Mech. 15 (1992) No. 1–2, 3–37.

[BSZ] S. Benzoni–Gavage, D. Serre, and K. Zumbrun, *Alternate Evans functions and viscous shock waves,* to appear, SIAM J. Math. Anal.

[CS] C. Conley and J. Smoller, *On the structure of magnetohydrodynamic shock waves,* Comm. Pure Appl. Math. 28 (1974), 367–375.

[CH] R. Courant and D. Hilbert, *Methods of mathematical physics. Vol. I,* Interscience Publishers, Inc., New York, N.Y., 1953. xv+561 pp.

[Er] J. J. Erpenbeck, *Stability of step shocks,* Phys. Fluids 5 (1962) no. 10, 1181–1187.

[E.1] J.W. Evans, *Nerve axon equations: I. Linear approximations,* Ind. Univ. Math. J. 21 (1972) 877–885.

[E.2] J.W. Evans, *Nerve axon equations: II. Stability at rest,* Ind. Univ. Math. J. 22 (1972) 75–90.

[E.3] J.W. Evans, *Nerve axon equations: III. Stability of the nerve impulse,* Ind. Univ. Math. J. 22 (1972) 577–593.

[E.4] J.W. Evans, *Nerve axon equations: IV. The stable and the unstable impulse,* Ind. Univ. Math. J. 24 (1975) 1169–1190.

[Ev] L.C. Evans, *Partial differential equations,* Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 1998. xviii+662 pp. ISBN: 0-8218-0772-2.

[Fre.1] H. Freistühler, *Small amplitude intermediate magnetohydrodynamic shock waves,* Phys. Scripta T74 (1998) 26–29.

[Fre.2] H. Freistühler, *Profiles for small Laxian shock waves in Kawashima type systems of conservation laws,* Preprint (2000).

[FreS] H. Freistühler and P. Szmolyan, *Existence and bifurcation of viscous profiles for all intermediate magnetohydrodynamic shock waves,* SIAM J. Math. Anal. 26 (1995), no. 1, 112–128.

[FreZ] H. Freistühler and K. Zumbrun, *Examples of unstable viscous shock waves,* unpublished note, Institut für Mathematik, RWTH Aachen, February 1998.

[Fr] A. Friedman, *Partial differential equations of parabolic type*, Prentice-Hall, Englewood Cliffs, NY (1964), Reprint Ed. (1983).

[Fri.1] C. Fries, *Stability of viscous shock waves associated with non-convex modes,* Arch. Ration. Mech. Anal. 152 (2000), no. 2, 141–186.

[Fri.2] C. Fries, *Nonlinear asymptotic stability of general small-amplitude viscous Laxian shock waves,* J. Differential Equations 146 (1998), no. 1, 185–202.

[GJ.1] R. Gardner and C.K.R.T. Jones, *A stability index for steady state solutions of boundary value problems for parabolic systems*, J. Diff. Eqs. 91 (1991), no. 2, 181–203.

[GJ.2] R. Gardner and C.K.R.T. Jones, *Traveling waves of a perturbed diffusion equation arising in a phase field model*, Ind. Univ. Math. J. 38 (1989), no. 4, 1197–1222.

[GZ] R. Gardner and K. Zumbrun, *The Gap Lemma and geometric criteria for instability of viscous shock profiles*, Comm. Pure Appl. Math. 51 (1998), no. 7, 797–855.

[Gi] D. Gilbarg, *The existence and limit behavior of the one-dimensional shock layer,* Amer. J. Math. 73, (1951). 256–274.

[Go.1] J. Goodman, *Nonlinear asymptotic stability of viscous shock profiles for conservation laws,* Arch. Rational Mech. Anal. 95 (1986), no. 4, 325–344.

[Go.2] J. Goodman, *Remarks on the stability of viscous shock waves*, in: Viscous profiles and numerical methods for shock waves (Raleigh, NC, 1990), 66–72, SIAM, Philadelphia, PA, (1991).

[He] D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Mathematics, Springer–Verlag, Berlin (1981), iv + 348 pp.

[Ho] D. Hoff, *Global well-posedness of the Cauchy problem for the Navier-Stokes equations of nonisentropic flow with discontinuous initial data,* J. of Differential Equations 95 (1992), 33-73.

[HoZ] D. Hoff and K. Zumbrun, *Multi-dimensional diffusion waves for the Navier-Stokes equations of compressible flow,* Indiana Univ. Math. J. 44 (1995), no. 2, 603–676.

[HZ.1] P. Howard and K. Zumbrun, *A tracking mechanism for one-dimensional viscous shock waves,* preprint (1999).

[HZ.2] P. Howard and K. Zumbrun, *Pointwise estimates for dispersive-diffusive shock waves,* to appear, Arch. Rational Mech. Anal.

[HuZ] J. Humpherys and K. Zumbrun, *Spectral stability of small amplitude shock profiles for dissipative symmetric hyperbolic–parabolic systems,* to appear ZAMP (2001).

[Ka] Ya. Kanel, *On a model system of equations of of one-dimensional gas motion,* Diff. Eqns. 4 (1968) 374–380.

[K.1] T. Kapitula, *Stability of weak shocks in* λ–ω *systems*, Indiana Univ. Math. J. 40 (1991), no. 4, 1193–12.

[K.2] T. Kapitula, *On the stability of travelling waves in weighted* L^∞ *spaces*, J. Diff. Eqs. 112 (1994), no. 1, 179–215.

[Kaw] S. Kawashima, *Systems of a hyperbolic–parabolic composite type, with applications to the equations of magnetohydrodynamics*, thesis, Kyoto University (1983).

[KM] S. Kawashima and A. Matsumura, *Asymptotic stability of traveling wave solutions of systems for one-dimensional gas motion,* Comm. Math. Phys. 101 (1985), no. 1, 97–127.

[KMN] S. Kawashima, A. Matsumura, and K. Nishihara, *Asymptotic behavior of solutions for the equations of a viscous heat-conductive gas,* Proc. Japan Acad. Ser. A Math. Sci. 62 (1986), no. 7, 249–252.

[LU] O.A. Ladyzenskaja and N.N. Ural'tseva, *On linear and quasij-linear equations and systems of elliptic and parabolic types,* 1963 Outlines Joint Sympos. Partial Differential Equations (Novosibirsk, 1963) 146–150 Acad. Sci. USSR Siberian Branch, Moscow.

[LSU] O.A. Ladyzenskaja, V.A. Solonnikov, and N.N. Ural'tseva, *Linear and quasi-linear equations of parabolic type,* Translations of Math. Monographs 23, AMS, Providence, RI (1968).

[La] P.D. Lax, *Hyperbolic systems of conservation laws and the mathematical theory of shock waves*, Conference Board of the Mathematical Sciences Regional Conference Series in Applied Mathematics, No. 11. Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1973. v+48 pp.

[Le] E.E. Levi, *Sulle equazioni lineari totalmente ellittiche alle derivate parziali,* Rend. Circ. Mat. Palermo 24 (1907) 275–317.

[L.1] T.-P. Liu, *Pointwise convergence to shock waves for viscous conservation laws*, Comm. Pure Appl. Math. 50 (1997), no. 11, 1113–1182.

[L.2] T.-P. Liu, *Shock waves for compressible Navier–Stokes equations are stable,* Comm. Pure Appl. Math. 39 (1986), no. 5, 565–594.

[L.3] T.-P. Liu, *The Riemann problem for general systems of conservation laws,* J. Diff. Eqs. 18 (1975) 218–234.

[LZ.1] T.P. Liu and K. Zumbrun, *Nonlinear stability of an undercompressive shock for complex Burgers equation,* Comm. Math. Phys. 168 (1995), no. 1, 163–186.

[LZ.2] T.P. Liu and K. Zumbrun, *On nonlinear stability of general undercompressive viscous shock waves,* Comm. Math. Phys. 174 (1995), no. 2, 319–345.

[LZe] T.-P. Liu and Y. Zeng, *Large time behavior of solutions for general quasilinear hyperbolic–parabolic systems of conservation laws*, AMS memoirs 599 (1997).

[Ly] G. Lyng, *One dimensional stability of detonation waves,* doctoral thesis, Indiana University (2002).

[LyZ] G. Lyng and K. Zumbrun, *Stability of detonation waves,* in preparation.

[M.1], A. Majda, *The stability of multi-dimensional shock fronts – a new problem for linear hyperbolic equations,* Mem. Amer. Math. Soc. 275 (1983).

[M.2], A. Majda, *The existence of multi-dimensional shock fronts,* Mem. Amer. Math. Soc. 281 (1983).

[M.3] A. Majda, *Compressible fluid flow and systems of conservation laws in several space variables,* Springer-Verlag, New York (1984), viii+ 159 pp.

[MP] A. Majda and R. Pego, *Stable viscosity matrices for systems of conservation laws*, J. Diff. Eqs. 56 (1985) 229–262.

[MN] A. Matsumura and K. Nishihara, *On the stability of travelling wave solutions of a one-dimensional model system for compressible viscous gas,* Japan J. Appl. Math. 2 (1985), no. 1, 17–25.

[MZ.1] C. Mascia and K. Zumbrun, *Pointwise Green's function bounds and stability of relaxation shocks,* preprint (2001).

[MZ.2] C. Mascia and K. Zumbrun, *Stability of viscous shock profiles for symmetrizable hyperbolic–parabolic systems,* preprint (2001), available: math.indiana.edu/home/kzumbrun.

[P.1] R.L. Pego, *Stable viscosities and shock profiles for systems of conservation laws,* Trans. Amer. Math. Soc. 282 (1984) 749–763.

[P.2] R.L. Pego, *Phase transitions in one-dimensional nonlinear viscoelasticity: admissibility and stability,* Arch. Rational Mech. Anal. 97 (1987) 353–394.

[Pa] A. Pazy, *Semigroups of linear operators and applications to partial differential equations,* Applied Mathematical Sciences, 44, Springer-Verlag, New York-Berlin, (1983) viii+279 pp. ISBN: 0-387-90845-5.

[Sat] D. Sattinger, *On the stability of waves of nonlinear parabolic systems*, Adv. Math. 22 (1976) 312–355.

[SZ] D. Serre and K. Zumbrun, *Boundary layer stability in real-vanishing viscosity limit,* to appear, Comm. Math. Phys. (2001).

[Sm] J. Smoller, *Shock waves and reaction–diffusion equations,* Second edition, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 258. Springer-Verlag, New York, 1994. xxiv+632 pp. ISBN: 0-387-94259-9.

[St] J.C. Strikwerda, *Finite difference schemes and partial differential equations,* (Chapman and Hall, New York (1989) xii+ 386 pp.

[Ze.1] Y. Zeng, L^1 *asymptotic behavior of compressible, isentropic, viscous* 1-*d flow,* Comm. Pure Appl. Math. 47 (1994) 1053–1092.

[ZH] K. Zumbrun and P. Howard, *Pointwise semigroup methods and stability of viscous shock waves,* Indiana Mathematics Journal V47 (1998) no. 4, 741–871; *Errata,* to appear, Indiana Mathematics Journal (2002) available math.indiana.edu/home/kzumbrun.

[Z.1] K. Zumbrun, *Stability of viscous shock waves*, Lecture Notes, Indiana University (1998).

[Z.2] K. Zumbrun, *Refined wave–tracking and nonlinear stability of viscous lax shocks,* Methods Appl. Anal. 7 (2000) 747–768.

[Z.3] K. Zumbrun, *Multidimensional stability of planar viscous shock waves,* Advances in the theory of shock waves, 307–516, Progr. Nonlinear Differential Equations Appl., 47, Birkhuser Boston, Boston, MA, 2001.

[Z.4] K. Zumbrun, *Stability index for relaxation and real viscosity systems,* unpublished note (revised appendix for ref. [Z.3]), (2002) available math.indiana.edu/home/kzumbrun.

[ZS] K. Zumbrun and D. Serre, *Viscous and inviscid stability of multidimensional planar shock fronts,* Indiana Univ. Math. J. 48 (1999), no. 3, 937–992.

Dipartimento di Matematica "G. Castelnuovo", Università di Roma "La Sapienza", P.le Aldo Moro, 2 - 00185 Roma (ITALY)
E-mail address: `mascia@ mat.uniroma1.it`

Department of Mathematics, Indiana University, Bloomington, IN 47405-4301
E-mail address: `kzumbrun@indiana.edu`