SOME RECENT METHODS FOR PARTIAL DIFFERENTIAL EQUATIONS OF DIVERGENCE FORM

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Abstract. Some recent methods for solving second-order nonlinear partial differential equations of divergence form and related nonlinear problems are surveyed. These methods include entropy methods via the theory of divergence-measure fields for hyperbolic conservation laws, kinetic methods via kinetic formulations for degenerate parabolic-hyperbolic equations, and free-boundary methods via free-boundary iterations for multidimensional transonic shocks for nonlinear equation of mixed elliptic-hyperbolic type. Some recent trends in this direction are also discussed.

1. Introduction

In this paper, we survey some methods, developed recently, for solving nonlinear partial differential equations of divergence form—nonlinear conservation laws— and related nonlinear problems. These equations take the following form:

$$\nabla \cdot A(y, u, Du) + B(y, u, Du) = 0, \qquad u \in \mathbb{R}^m, y \in \mathbb{R}^n, \tag{1.1}$$

where $A : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ and $B : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}^m$ are given mappings, and $Du = (\partial_{y_1} u, \dots, \partial_{y_n} u) \in \mathbb{R}^{m\times n}$ for $y = (y_1, \dots, y_n)$.

Although the form is very simple and easy to be written, many important nonlinear partial differential equations, arising from several areas of mathematics and various sciences including physics, mechanics, chemistry, and engineering sciences take this form.

Three of the most important classes of such partial differential equations are the following:

(i) *Hyperbolic Conservation Laws*:

$$\partial_t u + \nabla_x \cdot f(u) = 0, \qquad u \in \mathbb{R}^m, y = (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d, \tag{1.2}$$

where $f : \mathbb{R}^m \to \mathbb{R}^{m\times d}$ is given. Examples of (1.2) include the Euler equations for compressible fluids and various others models in fluid mechanics, geometry, and dynamic systems [5, 20, 46, 47, 93, 101, 109].

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(ii) *Degenerate Parabolic-Hyperbolic Equations*:

$$\partial_t u + \mathrm{div}_x f(u) = \mathrm{div}_x(A(u)D_x u), \qquad u \in \mathbb{R}, y = (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \tag{1.3}$$

where $f : \mathbb{R} \to \mathbb{R}^d$ and $A : \mathbb{R} \to \mathbb{R}^{d\times d}$ are given. Such equations arise in multiphase flows in porous media [28, 60], sedimentation and consolidation processes [21], and numerical analysis [77, 78, 98].

(iii) *Nonlinear Equations of Mixed Elliptic-Hyperbolic Type*:

$$\mathrm{div}\,(\rho(Du)Du) = 0, \qquad u \in \mathbb{R}, x \in \mathbb{R}^d, \tag{1.4}$$

where $\rho : \mathbb{R}^d \to \mathbb{R}$ is given. The typical example is the Euler equations for steady potential fluids, which consist of the conservation law of mass and the Bernoulli law.

One of the main difficulties to solve these nonlinear partial differential equations is the discontinuity of solutions. No matter how smooth the initial data and/or the boundary data are, the solutions (u or Du) generally develop singularity and become discontinuous functions in later time, which implies that the solutions are not in the Sobolev spaces $W^{k,p}$, $k = 1$, or 2. In general, the solutions (u or Du) are at most in BV, L^∞, L^p, and $\mathcal{M}$, and should be understood to satisfy the equations in the sense of distributions: For any $\varphi \in C_0^\infty(\Omega)$,

$$-\int_\Omega A(y,u,Du)\cdot D\varphi\, dy + \int_\Omega B(y,u,Du)\varphi\, dy = 0.$$

Across a discontinuity surface S of a weak solution, the solution and the surface must satisfy the Rankine-Hugoniot condition:

$$[A(y,u,Du)\cdot \nu]_S = 0.$$

To ensure the uniqueness of solutions, one requires an additional entropy condition to single out physical relevant solutions.

Other difficulties include concentration and cavitation which yield that solution become measures, focusing and defocusing which exclude the boundedness of solutions, and complex interactions among shock waves, rarefaction waves, contact waves, vortices, boundaries, etc. which exhibit extremely complex behavior of solutions.

Many traditional methods do not directly apply. New methods and ideas are required in order to solve these difficulties.

In this paper, we will mainly discuss some methods, developed recently, for the analysis of these major mathematical models. These methods include entropy methods via the theory of divergence-measure fields for hyperbolic conservation laws, kinetic methods via kinetic formulations for degenerate parabolic-hyperbolic equations, and free-boundary methods via free-boundary iterations for multidimensional transonic shocks for nonlinear equations of mixed elliptic-hyperbolic type. Most of the materials we present in this paper are based on the results in Chen-Frid [33, 34], Chen-Perthame [42], and Chen-Feldman [32].

This selection of topics is just an illustration of several examples of recent activities and is by no means an exhaustive treatment of all the exciting progresses that have been made in nonlinear partial differential equations of divergence form in the recent period. Even for the methods that we describe here, much more could be said about applications to other relevant problems.

In Section 2, we discuss some entropy methods and related theory of divergence-measure fields and present their applications to hyperbolic systems of conservation laws. In Section 3, we present a kinetic method via kinetic formulations for nonlinear conservation laws through anisotropic degenerate parabolic-hyperbolic equations. In Section 4, we discuss a free boundary method via free boundary iterations through a transonic shock problem. In Section 5, we discuss further methods and current trends in nonlinear partial differential equations of divergence form.

2. Entropy Methods and Divergence-Measure Fields

In this section we discuss some entropy methods and related theory of divergence-measure fields, developed recently in Chen-Frid [33], for hyperbolic systems of conservation laws and related nonlinear equations.

2.1. **Hyperbolic Conservation Laws and Divergence-Measure Fields.** One of the simplest PDEs is perhaps the following transport equations:

$$\partial_t \rho + \partial_x (v\rho) = 0, \tag{2.1}$$

where v represents the velocity and ρ represents the density. Consider the Cauchy problem:

$$\rho|_{t=0} = \rho_0(x). \tag{2.2}$$

When v is a given constant velocity, then the solution of (2.1)–(2.2), $\rho(t,x) = \rho_0(x - vt)$, just transports the initial mass $\rho_0(x)$ at the point x to the different location $x - vt$ at time $t > 0$, which keeps the same mass shape. However, when $v = v(t,x)$ is a given discontinuous function, then the solution of (2.1)–(2.2) may become a unbounded discontinuous function or a measure, although the total mass is always conserved.

Furthermore, in many physical situations, $v = v(t,x)$ is not a given function, it is governed by some other equations. For example, in isentropic fluid dynamics, the velocity $v = v(t,x)$ is usually governed by the conservation law of momentum:

$$\partial_t(\rho v) + \partial_x(\rho v^2 + p_0(\rho)) = 0, \tag{2.3}$$

which, along with (2.1), forms the system of Euler equations for isentropic fluids.

When $p_0(\rho) = 0$, system (2.1) and (2.3) becomes the pressureless Euler equations. It has been shown that even the Riemann solutions of this system contain δ-measures, as the vanishing pressure limits (e.g. [15, 41, 58, 129]).

When $p_0(\rho) = \rho^\gamma/\gamma$, there exist nonvacuum Riemann data such that the corresponding Riemann solutions consist of two rarefaction waves and one intermediate vacuum state. Under the Lagrangian coordinates (t,y) with

$$y_t(t,x) = \rho(t,x), \quad y_x(t,x) = -(\rho v)(t,x),$$

system (2.1) and (2.3) becomes

$$\partial_t \tau - \partial_y v = 0, \quad \partial_t v + \partial_y p(\tau) = 0, \quad \text{with } \; p(\tau) = p_0(\rho), \tau = 1/\rho. \tag{2.4}$$

Then the corresponding Riemann solutions of system (2.4) contain a weighted δ-measure concentrated at $y = 0$.

These simple examples of partial differential equations fit into the general framework of hyperbolic systems of conservation laws (1.2).

Definition 2.1. A function $\eta : \mathbb{R}^m \to \mathbb{R}$ is called an entropy of (1.2) if there exists $q : \mathbb{R}^m \to \mathbb{R}^d$ such that

$$\nabla q_k(u) = \nabla\eta(u)\nabla f_k(u), \qquad k = 1, 2, \ldots, d. \tag{2.5}$$

The function $q(u)$ is called the entropy flux associated with the entropy $\eta(u)$, and the pair $(\eta(u), q(u))$ is called an entropy pair. The entropy pair $(\eta(u), q(u))$ is called a convex entropy pair on the domain $K \subset \mathbb{R}^m$ if the Hessian matrix $\nabla^2\eta(u) \geq 0$, for $u \in K$. The entropy pair $(\eta(u), q(u))$ is called a strictly convex entropy pair on the domain K if $\nabla^2\eta(u) > 0$ for $u \in K$.

Friedrichs-Lax [68] observed that most of the systems of conservation laws that result from continuum mechanics are endowed with a globally defined, strictly convex entropy. Available existence theories show that solutions of (1.2) are generally in the following class of entropy solutions.

Definition 2.2. A vector function $u = u(t, x)$ is called an entropy solution if
i) $u(t, x) \in \mathcal{M}(\mathbb{R}_+ \times \mathbb{R}^d)$, or $L^p(\mathbb{R}_+ \times \mathbb{R}^d), 1 \leq p \leq \infty$;
ii) $u(t, x)$ satisfies the Lax entropy inequality:

$$\partial_t\eta(u(t, x)) + \nabla_x \cdot q(u(t, x)) \leq 0 \tag{2.6}$$

in the sense of distributions for any convex entropy pair $(\eta, q) : \mathbb{R}^m \to \mathbb{R} \times \mathbb{R}^d$ so that $\eta(u(t, x))$ and $q(u(t, x))$ are distributional functions.

Clearly, an entropy solution is a weak solution by choosing $\eta(u) = \pm u$ in (2.6).

One of the main issues in conservation laws is to study the behavior of solutions in this class to explore all possible information of solutions, including large-time behavior, uniqueness, stability, and traces of solutions, among others. The Schwartz lemma indicates from (2.6) that the distribution

$$\partial_t\eta(u(t, x)) + \nabla_x \cdot q(u(t, x))$$

is in fact a Radon measure, that is, the field $(\eta(u(t, x)), q(u(t, x)))$ is a divergence-measure field:

$$\mathrm{div}_{(t,x)}(\eta(u(t, x)), q(u(t, x))) \in \mathcal{M}(\mathbb{R}_+ \times \mathbb{R}^d). \tag{2.7}$$

It is clear that understanding more properties of divergence-measure fields can advance our understanding of the behavior of entropy solutions for hyperbolic conservation laws and other related nonlinear equations by selecting appropriate entropy pairs.

In general, divergence-measure fields ($\mathcal{DM}$-fields, for short) are extended vector fields, including vector fields in L^p and vector-valued Radon measures, whose divergences are Radon measures. More precisely, we have

Definition 2.3. Let $\Omega \subset \mathbb{R}^N$ be open. For $F \in L^p(\Omega; \mathbb{R}^N)$, $1 \leq p \leq \infty$, or $F \in \mathcal{M}(\Omega; \mathbb{R}^N)$, set

$$|\mathrm{div}\, F|(\Omega) := \sup\{\, \langle F, \nabla\varphi\rangle \;:\; \varphi \in C_0^1(\Omega),\ |\varphi(x)| \leq 1,\ x \in \Omega\,\}.$$

For $1 \leq p \leq \infty$, we say that F is an L^p divergence-measure field over Ω, i.e., $F \in \mathcal{DM}^p(\Omega)$, if

$$\|F\|_{\mathcal{DM}^p(\Omega)} := \|F\|_{L^p(\Omega;\mathbb{R}^N)} + |\mathrm{div}\, F|(\Omega) < \infty. \tag{2.8}$$

We say that F is an extended divergence-measure field over Ω, i.e., $F \in \mathcal{DM}^{ext}(\Omega)$, if

$$\|F\|_{\mathcal{DM}^{ext}(\Omega)} := |F|(\Omega) + |\mathrm{div}\, F|(\Omega) < \infty. \tag{2.9}$$

If $F \in \mathcal{DM}^p(\Omega)$ for any open set Ω with $\Omega \Subset D \subset \mathbb{R}^N$, we say $F \in \mathcal{DM}^p_{loc}(D)$; and, if $F \in \mathcal{DM}^{ext}(\Omega)$ for any open set Ω with $\Omega \Subset D \subset \mathbb{R}^N$, we say $F \in \mathcal{DM}^{ext}_{loc}(D)$. We denote $F \in \mathcal{DM}(\Omega)$ either $F \in \mathcal{DM}^p(\Omega)$ or $F \in \mathcal{DM}^{ext}(\Omega)$. Here, for open sets $A, B \subset \mathbb{R}^N$, the relation $A \Subset B$ means that the closure of A, $\bar{A}$, is a compact subset of B.

As we will see, these spaces under norms (2.8) and (2.9) are Banach spaces, respectively. These spaces are larger than the space of vector fields of bounded variation. The establishment of the Gauss-Green theorem, traces, and other properties of BV functions in the middle of last century (see Federer [64]) has advanced significantly our understanding of solutions of nonlinear partial differential equations and nonlinear problems in calculus of variations, differential geometry, and other areas. A natural question is whether the $\mathcal{DM}$-fields have similar properties, especially the traces and the Gauss-Green formula as for the BV functions. At a first glance, it seems unclear.

First, observe that one cannot define the traces for each component of a $\mathcal{DM}$ field over any Lipschitz boundary in general, as opposed to the case of BV fields. This fact can be easily seen through the following example.

Example 2.1: Clearly, $F(x,y) = (\sin(\frac{1}{x-y}), \sin(\frac{1}{x-y}))$ belongs to $\mathcal{DM}^\infty(\mathbb{R}^2)$. It is impossible to define any reasonable notion of traces over the line $x = y$ for the component $\sin(\frac{1}{x-y})$.

The following example indicates that the classical Gauss-Green theorem may fail.

Example 2.2: $F(x,y) = (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$ belongs to $\mathcal{DM}^1_{loc}(\mathbb{R}^2)$. As remarked in Whitney [145], for $\Omega = \{(x,y) : x^2 + y^2 < 1, y > 0\}$,

$$\int_\Omega \mathrm{div}\, F dxdy = 0 \neq \int_{\partial\Omega} F \cdot \nu \, d\mathcal{H}^1 = \pi,$$

if one understands $F \cdot \nu|_{\partial\Omega}$ in the classical sense.

Example 2.3: Let $\mu \in \mathcal{M}$. Then, for any bounded open interval $I \subset \mathbb{R}$,

$$F(x,y) = (\mu(y), 0) \in \mathcal{DM}^{ext}(I \times \mathbb{R}).$$

Some efforts for certain special cases have been made in generalizing the Gauss-Green theorem, see [4, 6, 19, 125, 151, 152]. Also see [83, 84, 88, 116, 123, 134] for related problems and references.

2.2. **Basic Properties of Divergence-Measure Fields.**

Now we list some basic properties of divergence-measure fields in the spaces $\mathcal{DM}^p(\Omega), 1 \le p \le \infty$, and $\mathcal{DM}^{ext}(\Omega)$.

Proposition 2.1. (i) *Let $\{F_j\}$ be a sequence in $\mathcal{DM}^p(\Omega)$ such that*

$$F_j \rightharpoonup F \qquad L^p_{loc}(\Omega; \mathbb{R}^N), \quad \text{for } 1 \le p < \infty, \tag{2.10}$$

$$F_j \overset{*}{\rightharpoonup} F \qquad L^\infty_{loc}(\Omega; \mathbb{R}^N), \quad \text{for } p = \infty. \tag{2.11}$$

Then

$$\|F\|_{L^p(\Omega)} \le \lim\inf_{j\to\infty} \|F_j\|_{L^p(\Omega)}, \qquad |\mathrm{div}\, F|(\Omega) \le \lim\inf_{j\to\infty} |\mathrm{div}\, F_j|(\Omega).$$

(ii) *Let* $\{F_j\}$ *be a sequence in* $\mathcal{DM}^{ext}(\Omega)$ *such that*

$$F_j \rightharpoonup F \qquad \mathcal{M}_{loc}(\Omega; \mathbb{R}^N).$$

Then

$$|F|(\Omega) \le \lim\inf_{j\to\infty} |F_j|(\Omega), \qquad |\mathrm{div}\, F|(\Omega) \le \lim\inf_{j\to\infty} |\mathrm{div}\, F_j|(\Omega).$$

This proposition immediately implies that the spaces $\mathcal{DM}^p, 1 \le p \le \infty$, and $\mathcal{DM}^{ext}(\Omega)$ are Banach spaces under norms (2.8) and (2.9), respectively.

Proposition 2.2. *Let* $\{F_j\}$ *be a sequence in* $\mathcal{DM}(\Omega)$ *satisfying*

$$\lim_{j\to\infty} |\mathrm{div}\, F_j|(\Omega) = |\mathrm{div}\, F|(\Omega)$$

and one of the following three conditions:

$$F_j \rightharpoonup F \qquad L^p_{loc}(\Omega; \mathbb{R}^N), \ \text{for } 1 \le p < \infty,$$
$$F_j \overset{*}{\rightharpoonup} F \qquad L^\infty_{loc}(\Omega; \mathbb{R}^N), \ \text{for } p = \infty,$$
$$F_j \rightharpoonup F \qquad \mathcal{M}_{loc}(\Omega; \mathbb{R}^N).$$

Then, for every open set $A \subset \Omega$,

$$|\mathrm{div}\, F|(\bar{A} \cap \Omega) \ge \lim\sup_{j\to\infty} |\mathrm{div}\, F_j|(\bar{A} \cap \Omega). \tag{2.12}$$

In particular, if $|\mathrm{div}\, F|(\partial A \cap \Omega) = 0$, *then*

$$|\mathrm{div}\, F|(A) = \lim_{j\to\infty} |\mathrm{div}\, F_j|(A). \tag{2.13}$$

We now use the so-called positive symmetric mollifiers $\omega : \mathbb{R}^N \to \mathbb{R}$ satisfying $\omega(x) \in C_0^\infty(\mathbb{R}^N)$, $\omega(x) \ge 0$, $\omega(x) = \omega(|x|)$, $\int_{\mathbb{R}^N} \omega(x)\, dx = 1$, $\mathrm{supp}\, \omega(x) \subset B_1 \equiv \{x \in \mathbb{R}^N \ : |x| < 1\}$. We denote $\omega_\varepsilon(x) = \varepsilon^{-N}\omega(\frac{x}{\varepsilon})$ and $F_\varepsilon = F * \omega_\varepsilon$, that is,

$$F^\varepsilon(x) = \varepsilon^{-N} \int_{\mathbb{R}^N} F(y)\omega\big(\frac{x-y}{\varepsilon}\big)\, dy = \int_{\mathbb{R}^N} F(x + \varepsilon y)\omega(y)\, dy. \tag{2.14}$$

Then $F^\varepsilon \in C^\infty(A; \mathbb{R}^N)$ for any $A \Subset \Omega$ when ϵ is sufficiently small. We recall that, for any $f, g \in L^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} f_\varepsilon g\, dx = \int_{\mathbb{R}^N} f g_\varepsilon\, dx. \tag{2.15}$$

The following fact for $\mathcal{DM}$ fields is analogous to a well-known property of BV functions.

Proposition 2.3. *Let* $F \in \mathcal{DM}(\Omega)$. *Let* $A \Subset \Omega$ *be open and* $|\mathrm{div}\, F|(\partial A) = 0$. *Then, for any* $\varphi \in C(\Omega; \mathbb{R})$,

$$\lim_{\varepsilon\to 0} \langle \mathrm{div}\, F^\varepsilon, \varphi\chi_A\rangle = < \mathrm{div}\, F, \varphi\chi_A > .$$

Furthermore, if $F \in \mathcal{DM}^{ext}(\Omega)$ *and* $|F|(\partial A) = 0$, *then, for any* $\varphi \in C(\Omega; \mathbb{R}^N)$,

$$\lim_{\varepsilon\to 0} < F^\varepsilon, \varphi\chi_A > = < F, \varphi\chi_A > .$$

Now we discuss some product rules for divergence-measure fields.

Proposition 2.4. *Let $F = (F_1, \ldots, F_N) \in \mathcal{DM}(\Omega)$. Let $g \in BV \cap L^\infty(\Omega)$ be such that $\partial_{x_j} g(x)$ is $|F_j|$-integrable, for each $j = 1, \ldots, N$, and the set of non-Lebesgue points of $\partial_{x_j} g(x)$ has $|F_j|$-measure zero; and $g(x)$ is $(|F| + |\mathrm{div}\, F|)$-integrable and the set of non-Lebesgue points of $g(x)$ has $(|F| + |\mathrm{div}\, F|)$-measure zero. Then $gF \in \mathcal{DM}(\Omega)$ and*

$$\mathrm{div}\,(gF) = g\,\mathrm{div}\, F + \nabla g \cdot F. \tag{2.16}$$

In particular, if $F \in \mathcal{DM}^\infty(\Omega)$, $gF \in \mathcal{DM}^\infty(\Omega)$ for any $g \in BV \cap L^\infty(\Omega)$; moreover, if g is also Lipschitz over any compact set in Ω,

$$\mathrm{div}\,(gF) = g\,\mathrm{div}\, F + F \cdot \nabla g. \tag{2.17}$$

In fact, for $F \in \mathcal{DM}^\infty(\Omega)$, one may refine the above result to yield that (2.17) holds a.e. in a more general case, not only for local Lipschitz functions. In this case, we must take the absolutely continuous part of ∇g. For $g \in BV$, let $(\nabla g)_{\mathrm{ac}}$ and $(\nabla g)_{\mathrm{sing}}$ denote the absolutely continuous part and the singular part of the Radon measure ∇g, respectively. Then

Proposition 2.5. *Given $F \in \mathcal{DM}^\infty(\Omega)$ and $g \in BV(\Omega) \cap L^\infty(\Omega)$, the identity*

$$\mathrm{div}\,(gF) = \bar{g}\mathrm{div}\, F + \overline{F \cdot \nabla g}$$

holds in the sense of Radon measures in Ω, where $\bar{g}$ is the limit of a mollified sequence for g through a positive symmetric mollifier, and $\overline{F \cdot \nabla g}$ is a Radon measure absolutely continuous with respect to $|\nabla g|$, whose absolutely continuous part with respect to the Lebesgue measure in Ω coincides with $F \cdot (\nabla g)_{ac}$ almost everywhere in Ω.

2.3. **Normal Traces and the Gauss-Green Formula.** We now discuss the generalized Gauss-Green theorem for $\mathcal{DM}$-fields over $\Omega \subset \mathbb{R}^N$ by introducing a suitable definition of normal traces over the boundary $\partial\Omega$ of a bounded open set with Lipschitz deformable boundary, established in [33].

Definition 2.4. Let $\Omega \subset \mathbb{R}^N$ be an open bounded subset. We say that $\partial\Omega$ is a deformable Lipschitz boundary, provided that

(i) $\forall\, x \in \partial\Omega$, $\exists\, r > 0$ and a Lipschitz map $\gamma : \mathbb{R}^{N-1} \to \mathbb{R}$ such that, after rotating and relabeling coordinates if necessary,

$$\Omega \cap Q(x, r) = \{y \in \mathbb{R}^N \ : \ \gamma(y_1, \ldots, y_{N-1}) < y_N\} \cap Q(x, r),$$

where $Q(x, r) = \{y \in \mathbb{R}^N \ : \ |x_i - y_i| \le r, \ i = 1, \ldots, N\}$;

(ii) $\exists\, \Psi : \partial\Omega \times [0, 1] \to \overline{\Omega}$ such that Ψ is a homeomorphism bi-Lipschitz over its image and $\Psi(\omega, 0) = \omega$ for all $\omega \in \partial\Omega$. The map Ψ is called a Lipschitz deformation of the boundary $\partial\Omega$.

Denote $\partial\Omega_s \equiv \Psi(\partial\Omega \times \{s\})$, $s \in [0, 1]$, and denote Ω_s the open subset of Ω whose boundary is $\partial\Omega_s$. We call Ψ a Lipschitz deformation of $\partial\Omega$.

Remark 2.1. The domains with deformable Lipschitz boundaries clearly include bounded domains with Lipschitz boundaries, the star-shaped domains, and the domains whose boundaries satisfy the cone property. It is also clear that, if Ω is the image through a bi-Lipschitz map of a domain $\tilde{\Omega}$ with a Lipschitz deformable boundary, then Ω itself possesses a Lipschitz deformable boundary.

For $\mathcal{DM}^p$ fields with $1 < p \le \infty$, we have

Theorem 2.1. *Let $F \in \mathcal{DM}^p(\Omega)$, $1 < p \le \infty$. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz deformable boundary. Then there exists a continuous linear functional $F\cdot\nu|_{\partial\Omega}$ over Lip$(\partial\Omega)$ such that, for any $\phi \in$ Lip$(\mathbb{R}^N)$,*

$$\langle F\cdot\nu|_{\partial\Omega}, \phi\rangle_{\partial\Omega} = \langle \operatorname{div} F,\ \phi\rangle_\Omega + \int_\Omega \nabla\phi\cdot F\,dx. \tag{2.18}$$

Moreover, let $\nu : \Psi(\partial\Omega\times[0,1]) \to \mathbb{R}^N$ be such that $\nu(x)$ is the unit outer normal to $\partial\Omega_s$ at $x \in \partial\Omega_s$, defined for a.e. $x \in \Psi(\partial\Omega\times[0,1])$. Let $h : \mathbb{R}^N \to \mathbb{R}$ be the level set function of $\partial\Omega_s$, that is,

$$h(x) := \begin{cases} 0, & \text{for } x \in \mathbb{R}^N - \overline{\Omega}, \\ 1, & \text{for } x \in \Omega - \Psi(\partial\Omega\times[0,1]), \\ s, & \text{for } x \in \partial\Omega_s, 0 \le s \le 1. \end{cases}$$

Then, for any $\psi \in$ Lip$(\partial\Omega)$,

$$\langle F\cdot\nu|_{\partial\Omega}, \psi\rangle_{\partial\Omega} = -\lim_{s\to 0}\frac{1}{s}\int_{\Psi(\partial\Omega\times(0,s))} \mathcal{E}(\psi)\,\nabla h\cdot F\,dx, \tag{2.19}$$

where $\mathcal{E}(\psi)$ is any Lipschitz extension of ψ to all $\mathbb{R}^N$.

In the case $p=\infty$, the normal trace $F\cdot\nu|_{\partial\Omega}$ is a function in $L^\infty(\partial\Omega)$ satisfying $\|F\cdot\nu\|_{L^\infty(\partial\Omega)} \le C\|F\|_{L^\infty(\Omega)}$, for some constant C independent of F. Furthermore, for any field $F \in \mathcal{DM}^\infty(\Omega)$,

$$\langle F\cdot\nu|_{\partial\Omega}, \psi\rangle_{\partial\Omega} = ess\lim_{s\to 0}\int_{\partial\Omega_s} \psi\circ\Psi_s^{-1}\,F\cdot\nu\,d\mathcal{H}^{N-1}, \qquad \text{for any } \psi \in L^1(\Omega). \tag{2.20}$$

Finally, for $F \in \mathcal{DM}^p(\Omega)$ with $1<p<\infty$, $F\cdot\nu|_{\partial\Omega}$ can be extended to a continuous linear functional over $W^{1-1/p,p}(\partial\Omega)\cap L^\infty(\partial\Omega)$.

As indicated by Example 2.2, it is more delicate for fields in $\mathcal{DM}^1$ and $\mathcal{DM}^{ext}$. Then we have to define the normal traces as functionals over the spaces Lip$(\gamma,\partial\Omega)$ with $\gamma > 1$ (see Stein [137]). For $1 < \gamma \le 2$, the elements of Lip$(\gamma,\partial\Omega)$ are $(N+1)$-components vectors, where the first component is the function itself, and the other N components are its "first-order partial derivatives". In particular, as a functional over Lip$(\gamma,\partial\Omega)$, the values of the normal trace of a field in $\mathcal{DM}^1$ or $\mathcal{DM}^{ext}$ on $\partial\Omega$ depend not only on the values of the respective functions over $\partial\Omega$, but also on the values of their first-order derivatives over $\partial\Omega$. To define the normal traces for $F \in \mathcal{DM}^1$ or $\mathcal{DM}^{ext}$, we resort to the properties of the Whitney extensions of functions in Lip$(\gamma,\partial\Omega)$ to Lip$(\gamma,\mathbb{R}^N)$. We have the following analogue of Theorem 2.1 which covers fields in $\mathcal{DM}^1$ and $\mathcal{DM}^{ext}$.

Theorem 2.2. *Let $F \in \mathcal{DM}^1(\Omega)$ or $\mathcal{DM}^{ext}(\Omega)$. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz deformable boundary. Then there exists a continuous linear functional $F\cdot\nu|_{\partial\Omega}$ over Lip$(\gamma,\partial\Omega)$ for any $\gamma > 1$ such that, for any $\phi \in$ Lip$(\gamma,\mathbb{R}^N)$,*

$$\langle F\cdot\nu|_{\partial\Omega}, \phi\rangle_{\partial\Omega} = \langle \operatorname{div} F, \phi\rangle_\Omega + \langle F, \nabla\phi\rangle_\Omega. \tag{2.21}$$

Moreover, let $h : \mathbb{R}^N \to \mathbb{R}$ be the level set function as in Theorem 2.1; and, in the case that $F \in \mathcal{DM}^{ext}(\Omega)$, we also assume that $\partial_{x_i}h$ is $|F_i|$-measurable and its set of non-Lebesgue points has $|F_i|$-measure zero, $i=1,\dots,N$. Then, for any $\psi \in$ Lip$(\gamma,\partial\Omega)$, $\gamma > 1$,

$$\langle F\cdot\nu|_{\partial\Omega}, \psi\rangle_{\partial\Omega} = -\lim_{s\to 0}\frac{1}{s}\langle F, \mathcal{E}(\psi)\,\nabla h\rangle_{\Psi(\partial\Omega\times(0,s))}, \tag{2.22}$$

where $\mathcal{E}(\psi) \in Lip\,(\gamma, \mathbb{R}^N)$ *is the Whitney extension of* ψ *on* $\partial\Omega$ *to* $\mathbb{R}^N$.

Remark 2.2. In general, for $F \in \mathcal{DM}^1(D)$ or $\mathcal{DM}^{ext}(D)$, the normal traces $F \cdot \nu|_{\partial\Omega}$ may be no longer functions. This can be seen in Example 2.2 for $F \in \mathcal{DM}^1_{loc}(\mathbb{R}^2)$ with $\Omega = \{(x,y) \,:\, x^2+y^2<1, y>0\}$, for which $F\cdot\nu|_{\partial\Omega}$ is a measure.

As a corollary of the generalized Gauss-Green formula in $\mathcal{DM}^\infty$, we have

Proposition 2.6. *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded open set with Lipschitz boundary and* $F_1 \in \mathcal{DM}^\infty(\Omega)$, $F_2 \in \mathcal{DM}^\infty(\mathbb{R}^N - \bar{\Omega})$. *Then*

$$F(y) = \begin{cases} F_1(y), & y \in \Omega, \\ F_2(y), & y \in \mathbb{R}^N - \bar{\Omega} \end{cases} \tag{2.23}$$

belongs to $\mathcal{DM}^\infty(\mathbb{R}^N)$, *and*

$$\|F\|_{\mathcal{DM}^\infty(\mathbb{R}^N)} \le \|F_1\|_{\mathcal{DM}^\infty(\Omega)} + \|F_2\|_{\mathcal{DM}^\infty(\mathbb{R}^N-\bar{\Omega})} + \|F_1\cdot\nu - F_2\cdot\nu\|_{L^\infty(\partial\Omega)}\mathcal{H}^{N-1}(\partial\Omega).$$

Some entropy methods based on the theory of divergence-measure fields presented above have been developed and applied to solving various nonlinear problems for conservation laws and related nonlinear equations. These problems especially include (1) the stability of Riemann solutions, which may contain rarefaction waves, contact discontinuities, and/or vacuum states, in the class of entropy solutions of the Euler equations for gas dynamics in [34, 35, 44]; (2) the decay of periodic entropy solutions for hyperbolic conservation laws in [34]; (3) Initial and boundary layer problems for hyperbolic conservation laws in [43, 141]; (4) initial-boundary value problems for hyperbolic conservation laws in [34]; and (5) nonlinear degenerate parabolic-hyperbolic equations in [22, 111].

One of the entropy methods is to identify Lyapunov-type functionals and employ the Gauss-Green formula to establish the uniqueness and stability of entropy solutions; see [34, 35, 44]. In this regard, some related Lyapunov-type functionals have been identified for small BV solutions obtained by the Glimm scheme, the wave-front tracking scheme, and the vanishing viscosity method; see Biachini-Bressan [12], Bressan [17], Liu-Yang [107], LeFloch [96], and the references cited therein for the details.

It would be interesting to apply the theory of divergence-measure fields to develop more efficient entropy methods for solving more various problems in partial differential equations and related areas whose solutions are only measures or L^p functions.

3. Kinetic Methods and Kinetic Formulations

In this section, we present a kinetic method, developed recently in Chen-Perthame [42], for nonlinear conservation laws through anisotropic degenerate parabolic-hyperbolic equations (1.3), that is,

$$\partial_t u + \mathrm{div}_x f(u) = \mathrm{div}_x(A(u)D_x u), \qquad u \in \mathbb{R},$$

where $f : \mathbb{R} \longrightarrow \mathbb{R}^d$ and $A : \mathbb{R} \to \mathbb{R}^{d\times d}$ with $A(u)_{d\times d} \ge 0$, symmetric, and hence

$$A(u) = (a_{ij}(u)) = (\sum_{k=1}^{K} \sigma_{ki}(u)\sigma_{kj}(u)).$$

We assume

$$f \in \mathrm{Lip}_{loc}(\mathbb{R}; \mathbb{R}^d), \quad \sigma_{kj} \in L^{\infty}_{loc}(\mathbb{R}; \mathbb{R}). \tag{3.1}$$

Equation (1.3) and its variants model degenerate diffusion-convection motions of ideal fluids and arise in a wide variety of important applications including two phase flows in porous media and sedimentation-consolidation processes. There are a large literature for the design and analysis of various numerical methods for such equations. The well posedness for such equations has been in great demands not only for the mathematical theory of degenerate parabolic equations, but also for various applications.

When $A(u) \equiv 0$, equation (1.3) is a scalar hyperbolic conservation law. The well posedness in BV, L^∞, and L^1 was established in [92, 93, 118, 142] and [55, 104].

When $f(u) \equiv 0$, equation (1.3) is a degenerate parabolic equation. Many efforts have been made, for example, see [18, 24, 51, 52] and the references cited therein.

When $A(u) = \alpha'(u)I, \alpha'(u) \geq 0$, the isotropic case, DiBenedetto [51] showed the regularity of solutions, that is, $\alpha(u) \in C$. Only recently, the well-posedness for the isotropic degenerate parabolic-hyperbolic equation was established for L^∞ solutions in bounded domain in [27] (also see [89]) and for unbounded solutions for the Cauchy problem in [31]. Other early related references can been found in [73].

Similar to the argument in Section 2.1, we can derive the following entropy inequality for $u \in L^\infty$: For any $\eta(u), \eta''(u) \geq 0$,

$$\partial_t \eta(u) + \mathrm{div}_x(q(u) - A(u)D_x\eta(u)) \leq -\eta''(u)(D_x u)^\top A(u) D_x u \leq 0,$$

which implies

$$-\mathrm{div}_{(t,x)}(\eta(u), q(u) - A(u)D_x\eta(u)) = \mu \in \mathcal{M}_+.$$

Then the theory of divergence-measure fields and related entropy methods can naturally be applied to studying solution behaviors of degenerate parabolic-hyperbolic equations by using the nonnegativity of μ.

The methods in Section 2 are based on the macroscopic setting for the macroscopic variables $\eta(u)$ and $q(u) - A(u)D_x\eta(u)$; the framework is very general and can be applied to many important situations. However, in some cases, more detailed information about the Radon measure μ and the entropy pairs is very useful, if available, for studying further behavior of solutions. In the last decade, kinetic methods, along with kinetic formulations, have been developed for the hyperbolic case, by exploring more information about the Radon measure and entropy pairs. One of the basic ideas of kinetic methods is to explore some additional information about the measure μ, that is, microscopic or mescroscopic information, by introducing an additional variable, usually the kinetic velocity variable. For the hyperbolic case, see Lions-Perthame-Tadmor [104] and Perthame [121]. The main motivation is that the Euler equations for compressible fluids can be derived from the Boltzmann equation.

Now we illustrate a kinetic method in Chen-Perthame [42] through establishing the well-posedness in L^1 for (1.3) with

$$u|_{t=0} = u_0 \in L^1(\mathbb{R}^d), \qquad x \in \mathbb{R}^d. \tag{3.2}$$

One of the main difficulties for the well-posedness problem is the coupling between convection and degenerate diffusions, that is, singularity against regularity.

One requires a uniform approach that works for both the hyperbolic and parabolic phases.

Introduce the kinetic function χ on $\mathbb{R}^2$:

$$\chi(\xi;u)=\begin{cases} +1 & \text{for } \ 0<\xi<u, \\ -1 & \text{for } \ u<\xi<0, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 3.1. A kinetic solution is a function $u(t,x)\in L^\infty\left([0,\infty);L^1(\mathbb{R}^d)\right)$ such that
(i) The kinetic equation

$$\partial_t\,\chi(\xi;u)+f'(\xi)\cdot D_x\chi(\xi;u)-\mathrm{div}_x(A(\xi)D_x\chi(\xi;u))=\partial_\xi(m+n)(t,x,\xi), \tag{3.3}$$

holds in $\mathbb{D}'(\mathbb{R}^+\times\mathbb{R}^{d+1})$ with initial data

$$\chi(\xi;u)|_{t=0}=\chi(\xi;u_0),$$

for some measures $m(t,x,\xi)\geq 0$ and $n(t,x,\xi)\geq 0$:

$$\int_{\mathbb{R}} n(t,x,\xi)\,\psi(\xi)\,d\xi=\sum_{k=1}^{K}\Big(\mathrm{div}_x\beta_k^\psi(u(t,x))\Big)^2,$$

for any $\psi\in C_0^\infty(\mathbb{R})$ with $\psi\geq 0$, where $\beta_k^\psi(u)=\int_0^u\sqrt{\psi(w)}\sigma_k(w)dw\in\mathbb{R}^d$ for $\sigma_k(u)=(\sigma_{k1}(u),\ldots,\sigma_{kd}(u))$;
(ii) There exists $\mu(\xi)\in L^\infty$ with $\mu(\xi)\to 0$ as $|\xi|\to\infty$ such that

$$\int_0^\infty\int_{\mathbb{R}^d}(m+n)(t,x,\xi)\,dx\,dt\leq\mu(\xi);$$

(iii) For any nonnegative $\psi\in C_0^\infty(\mathbb{R})$,

$$\mathrm{div}_x\beta_k^\psi(u)\in L^2([0,\infty)\times\mathbb{R}^d),\quad k=1,\ldots,K;$$

(iv) For any nonnegative $\psi_1,\,\psi_2\in C_0^\infty(\mathbb{R})$,

$$\sqrt{\psi_1(u(t,x))}\ \mathrm{div}_x\beta_k^{\psi_2}(u(t,x))=\mathrm{div}_x\beta_k^{\psi_1\psi_2}(u(t,x)),\qquad a.e.$$

Remark 3.1. This notion of *kinetic solutions* applies to more general situations, especially when a solution u is only in L^1. The advantage of this notion is that the kinetic equation is well defined even though the macroscopic fluxes $q(u)$ are not locally integrable so that L^1 is a natural space on which kinetic solutions are posed.

Remark 3.2. Any bounded kinetic solution $u\in L^\infty$ satisfies the entropy inequality. This can be seen as follows: For any $\eta\in C^2$ with $\eta''(u)\geq 0$, multiplying the kinetic equation (3.3) by $\eta'(\xi)$ and then integrating with respect to $\xi\in\mathbb{R}$ yield

$$\partial_t\eta(u)+\mathrm{div}_x(q(u)-A(u)D_x\eta(u))=-\int_{\mathbb{R}}\eta''(\xi)(m+n)(t,x,\xi)d\xi\leq 0$$

in the sense of distributions.

Remark 3.3. If $u_0\in W^{2,1}\cap H^1\cap L^\infty(\mathbb{R}^d)$, then it can be shown that there exists a global kinetic solution $u=u(t,x)$ of the Cauchy problem (1.3) and (3.2), via the vanishing viscosity method.

Remark 3.4. For the isentropic case, $A(u)=\alpha'(u)I$, condition (iv) in Definition 3.1 can be removed as a direct corollary of a standard chain rule (see [42]).

For kinetic solutions, we have first the following stability theorem.

Theorem 3.1. *Assume that* (3.1) *holds. Then*
(i) *For any kinetic solution* $u \in L^\infty([0,\infty); L^1(\mathbb{R}^d))$ *with initial data* $u_0 \in L^1$,

$$\|u(t) - u_0\|_{L^1(\mathbb{R}^d)} \to 0, \qquad \textit{when } t \to 0.$$

(ii) *If* $u, v \in L^\infty([0,\infty); L^1(\mathbb{R}^d))$ *are kinetic solutions to* (1.3) *and* (3.2) *with initial data* $u_0, v_0 \in L^1$, *respectively, then*

$$\|u(t) - v(t)\|_{L^1(\mathbb{R}^d)} \le \|u_0 - v_0\|_{L^1(\mathbb{R}^d)}. \tag{3.4}$$

(iii) *Furthermore, if* $u \in L^\infty([0,\infty) \times \mathbb{R}^d)$, *this kinetic solution is the entropy solution.*

We now give a formal proof to present the kinetic method for establishing the stability of kinetic solutions.

Formal Proof. Consider two solutions $u(t,x)$ and $v(t,x)$. Denote by $m(t,x,\xi)$ the kinetic defect measure and by

$$n(t,x,\xi) := \delta(\xi - u(t,x)) \sum_{k=1}^{K} \left(\sum_{i=1}^{d} \partial_{x_i} \sigma_{ik}(u(t,x)) \right)^2, \tag{3.5}$$

the parabolic defect measure, which are associated with $u(t,x)$; and by $p(t,x,\xi)$ the kinetic defect measure and by

$$q(t,x,\xi) := \delta(\xi - v(t,x)) \sum_{k=1}^{K} \left(\sum_{i=1}^{d} \partial_{x_i} \sigma_{ik}(v(t,x)) \right)^2, \tag{3.6}$$

the parabolic defect measure, which are associated with $v(t,x)$.

Then we use the following microscopic contraction functional introduced in [121]:

$$Q(t,x,\xi) = |\chi(\xi; u(t,x))| + |\chi(\xi; v(t,x))| - 2\,\chi(\xi; u(t,x))\ \chi(\xi; v(t,x)) \ge 0, \tag{3.7}$$

which is useful for deriving a contraction principle since

$$\int_{\mathbb{R}} Q(t,x,\xi)\, d\xi = |u(t,x) - v(t,x)|.$$

Notice that

$$\begin{aligned} \partial_t |\chi(\xi; u(t,x))| + a(\xi) \cdot D_x |\chi(\xi; u(t,x))| - \sum_{i,j=1}^{d} \partial^2_{x_i x_j} \left(a_{ij}(\xi) |\chi(\xi; u(t,x))| \right) \\ = \operatorname{sgn}(\xi)\, \partial_\xi (m+n)(t,x,\xi)\,, \end{aligned}$$

which yields

$$\frac{d}{dt} \int_{\mathbb{R}^{d+1}} |\chi(\xi; u(t,x))|\, dx\, d\xi = -2 \int_{\mathbb{R}^d} (m+n)(t,x,0) dx.$$

A similar identity holds for $v(t,x)$.

Furthermore, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^{d+1}} \chi(\xi;u(t,x))\ \chi(\xi;v(t,x))\,dx\,d\xi \\
&\qquad +2\int_{\mathbb{R}^{d+1}}\sum_{i,j=1}^{d} a_{ij}(\xi)\ \partial_{x_i}\chi(\xi;u(t,x))\partial_{x_j}\chi(\xi;v(t,x))\,dx\,d\xi \\
&= \int_{\mathbb{R}^{d+1}} ((m+n)(t,x,\xi)(\delta(\xi-v(t,x))-\delta(\xi)) \\
&\qquad +(p+q)(t,x,\xi)(\delta(\xi-u(t,x))-\delta(\xi)))\,dx\ d\xi.
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^{d+1}} Q(t,x,\xi)\,dx\,d\xi \\
&= 4\int_{\mathbb{R}^{d+1}}\sum_{i,j=1}^{d} a_{ij}(\xi)\ \partial_{x_i}\chi(\xi;u(t,x))\ \partial_{x_j}\chi(\xi;v(t,x))\,dx\,d\xi \\
&\quad -2\int_{\mathbb{R}^{d+1}} ((m+n)(t,x,\xi)\delta(\xi-v(t,x))+(p+q)(t,x,\xi)\delta(\xi-u(t,x)))\,dx\ d\xi \\
&\le 4\int_{\mathbb{R}^{d+1}}\sum_{i,j=1}^{d} a_{ij}(\xi)\ \partial_{x_i}u(t,x)\ \partial_{x_j}v(t,x)\,\delta(\xi-u(t,x))\ \delta(\xi-v(t,x))\,dx\,d\xi \\
&\quad -2\int_{\mathbb{R}^{d+1}} (n(t,x,\xi)\delta(\xi-v(t,x))+q(t,x,\xi)\delta(\xi-u(t,x)))\,dx\ d\xi,
\end{aligned}$$

since $m(t,x,\xi)$ and $p(t,x,\xi)$ are nonnegative.

Using (3.5) and (3.6) yields

$$\begin{aligned}
&\int_{\mathbb{R}^{d+1}} (n(t,x,\xi)\delta(\xi-v(t,x))+q(t,x,\xi)\delta(\xi-u(t,x)))\,dx\ d\xi \\
&= \sum_{k=1}^{K}\int_{\mathbb{R}^{d+1}} \delta(\xi-u(t,x))\ \delta(\xi-v(t,x)) \\
&\qquad\qquad \times\left(\left(\sum_{i=1}^{d}\partial_{x_i}\sigma_{ik}(u(t,x))\right)^2+\left(\sum_{i=1}^{d}\partial_{x_i}\sigma_{ik}(v(t,x))\right)^2\right)\ dx\ d\xi \\
&\ge 2\sum_{k=1}^{K}\int_{\mathbb{R}^{d+1}} \delta(\xi-u(t,x))\ \delta(\xi-v(t,x)) \\
&\qquad\qquad \times\left(\sum_{i=1}^{d}\partial_{x_i}\sigma_{ik}(u(t,x))\right)\ \left(\sum_{j=1}^{d}\partial_{x_j}\sigma_{jk}(v(t,x))\right)\ dx\ d\xi \\
&= 2\sum_{k=1}^{K}\sum_{i,j=1}^{d}\int_{\mathbb{R}^{d+1}} \delta(\xi-u(t,x))\ \delta(\xi-v(t,x))\ \sigma_{ik}(u(t,x))\ \sigma_{jk}(v(t,x)) \\
&\qquad\qquad \times\ \partial_{x_i}u(t,x)\,\partial_{x_j}v(t,x)\ dx\ d\xi \\
&= 2\sum_{i,j=1}^{d}\int_{\mathbb{R}^{d+1}} a_{ij}(\xi)\ \partial_{x_i}u(t,x)\,\partial_{x_j}u(t,x)\ \delta(\xi-u(t,x))\ \delta(\xi-v(t,x))\ dx\ d\xi.
\end{aligned}$$

Therefore, we end up with

$$\frac{d}{dt}\int_{\mathbb{R}^{d+1}} Q(t,x,\xi)\,dx\,d\xi \le 0,$$

which implies that $\|u(t)-v(t)\|_{L^1(\mathbb{R}^d)}$ is non-increasing. This concludes the contraction property (3.4).

Remark 3.5. Employing Theorem 3.1, we can establish the existence of kinetic solutions in L^1 for initial data $u_0 \in L^1(\mathbb{R}^d)$. This can be achieved by approximating u_0 by $u_0^\epsilon \in W^{2,1}\cap H^1 \cap L^\infty(\mathbb{R}^d)$ and by using the L^1 contraction property for the corresponding kinetic solutions u^ϵ that yields the convergence of u^ϵ in L^1 as $\epsilon \to 0$.

Remark 3.6. Our uniqueness result implies that, if $u_0 \in L^\infty \cap L^1(\mathbb{R}^d)$, then the kinetic solution is the unique entropy solution and $|u(t,x)| \le \|u_0\|_{L^\infty}$. Therefore, the two notions are equivalent for solutions in L^∞, although the notion of kinetic solutions is more general.

Using this kinetic method, a general L^1–framework for continuous dependence and error estimates for quasilinear, anisotropic degenerate parabolic equations has also been developed, and applications of our general L^1–framework have been made in establishing an explicit estimate for continuous dependence on the nonlinearities and an optimal error estimate for the vanishing anisotropic viscosity method, without the requirement of bounded variation of the approximate solutions, for anisotropic degenerate parabolic equations. For more details, see Chen-Karlsen [38].

4. Free Boundary Methods and Free Boundary Iterations

Besides well-posedness problems for partial differential equations of divergence form, many PDE problems arising from various areas are more specific and require the behavior of specific solutions. Such problems include free-boundary problems for partial differential equations of divergence form for two phase flows. In many cases, given one phase $u^-(x)$, we seek a surface S determined by some physical laws and a solution determined by the partial differential equations for the other phase. Such problems are called one-phase free boundary problems. In particular, across a discontinuity surface S, a weak solution $u(y)$ of (1.1) should satisfy the Rankine-Hugoniot conditions:

$$A(y,u,Du)\cdot\nu|_S = G(y,\nu) := A(y,u^-,Du^-)\cdot\nu|_S \quad \text{Lipschitz w.r.t. } y,\nu, \tag{4.1}$$

$$u|_S = u^-|_S, \tag{4.2}$$

which are the free boundary conditions, where ν is the unit normal to S in the direction of the unknown phase.

In this section we present a free boundary method via a free boundary iteration developed in Chen-Feldman [32] through a multidimensional transonic shock problem.

Consider inviscid steady potential fluid flows, which are governed by the Euler equations consisting of the conservation law of mass and the Bernoulli law for the velocity. Then the Euler equations for the velocity potential $\varphi : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ can be formulated into the form (1.4) with $u = \varphi$, that is,

$$\operatorname{div}\left(\rho(|D\varphi|^2)D\varphi\right) = 0, \tag{4.3}$$

where the density function $\rho(q^2)$ has the form:

$$\rho(q^2) = \left(1 - \theta q^2\right)^{\frac{1}{2\theta}}, \tag{4.4}$$

with $\theta = \frac{\gamma-1}{2}$ and the adiabatic exponent $\gamma > 1$.

The second order nonlinear equation (4.3) is elliptic at $D\varphi$ with $|D\varphi| = q$ if

$$\rho(q^2) + 2q^2\rho'(q^2) > 0, \tag{4.5}$$

and is hyperbolic if

$$\rho(q^2) + 2q^2\rho'(q^2) < 0. \tag{4.6}$$

The elliptic regions of equation (4.3) correspond to the subsonic flow, and the hyperbolic regions to the supersonic flow. Shocks are jump discontinuities in the velocity $D\varphi$. We study weak solutions of (4.3) with transonic (hyperbolic-elliptic) shocks. Classical linear models for the equations of mixed type include the Tricomi equation and the Keldysh equation (see [11, 147, 148])

Shiffman [130], Bers [8], and Finn-Gilbarg [67] studied subsonic (elliptic) solutions of (4.3) outside an obstacle; also see Dong [57]. Alt-Caffarelli-Friedman [3] studied free boundary problems for subsonic flows.

Morawetz in [113] showed that the flows of (4.3) past an obstacle may contain transonic shocks in general. One exception is that, when the obstacle forms a wedge with small angle, the uniform outflow with supersonic speed may produce a non-transonic shock (hyperbolic-hyperbolic shock); see [45, 99, 100, 127, 149]. Transonic shocks also arise in many other situations of physical importance. For example, when a plane shock hits a wedge, a reflected flow may form a transonic shock; see Glimm-Majda [76] and Morawetz [114]. Steady transonic shocks are also very useful for solving some unsteady important problems; see Chen-Glimm [36].

Canic-Keyfitz-Lieberman [25] studied perturbations of steady transonic shocks between two uniform flows for the two-dimensional transonic small-disturbance (TSD) equation, which governs the behavior of the first non-trivial term in the geometric optics expansion to (4.3) near a certain physical point. The TSD model can be written as a second order nonlinear equation of mixed type with coefficients depending only upon the unknown function itself, but independent of its gradient as in (4.3). This feature is essential for the approach in [25] (also see [26]).

Majda [108] studied the existence and stability of multidimensional shock fronts, locally in time, for the Euler equations for compressible fluids (also see Métivier [112]).

We now discuss several recent results on the existence and stability of multidimensional transonic shocks for (4.3) in unbounded domains and present a nonlinear method in [32] for establishing these results.

4.1. **Transonic Shocks.** We first introduce multidimensional transonic shocks for (4.3). A function $\varphi \in W^{1,\infty}(\Omega)$ is a weak solution of (4.3) in an unbounded domain Ω if

(i) $|D\varphi(x)| \le 1/\sqrt{\theta}$ a.e.

(ii) For any $\zeta \in C_0^\infty(\Omega)$,

$$\int_\Omega \rho(|D\varphi|^2) D\varphi \cdot D\zeta \, dx = 0. \tag{4.7}$$

We are interested in weak solutions with shocks. Let Ω^+ and Ω^- be open subsets of Ω such that

$$\Omega^+ \cap \Omega^- = \emptyset, \qquad \overline{\Omega^+} \cup \overline{\Omega^-} = \overline{\Omega},$$

and $S = \partial\Omega^+ \cap \Omega$. Let $\varphi \in W^{1,\infty}(\Omega)$ be a weak solution of (4.3) and be in $C^2(\Omega^\pm) \cap C^1(\overline{\Omega^\pm})$ so that $D\varphi$ experiences a jump across S that is an $(n-1)$-dimensional smooth surface. Then the condition that $\varphi(x)$ is a weak solution of (4.3) implies the continuity of $\varphi(x)$ across S:

$$\varphi^+ = \varphi^- \qquad \text{on } S \tag{4.8}$$

and the following Rankine-Hugoniot condition on S:

$$\Big[\rho(|D\varphi|^2) D\varphi \cdot \nu\Big]_S = 0, \tag{4.9}$$

where ν is the unit normal to S in the direction of Ω^+, and the bracket denotes the difference between the values of the function along S on the $\Omega^\pm$ sides, respectively. We can also write (4.9) as

$$\rho(|D\varphi^+|^2) D\varphi^+ \cdot \nu = \rho(|D\varphi^-|^2) D\varphi^- \cdot \nu \qquad \text{on} \quad S. \tag{4.10}$$

Let $K > 0$. Then the function

$$\Phi_K(p) := \left(K - \theta p^2\right)^{\frac{1}{2\theta}} p, \tag{4.11}$$

defined for $p \in \left[0, \sqrt{K/\theta}\right]$, satisfies

$$\lim_{p \to 0+} \Phi_K(p) = \lim_{p \to \sqrt{K/\theta}-} \Phi_K(p) = 0, \tag{4.12}$$

$$\Phi_K(p) > 0 \quad \text{for } p \in \left(0, \sqrt{K/\theta}\right), \tag{4.13}$$

$$0 < \Phi_K'(p) < K^{\frac{1}{2\theta}} \ \text{ on } (0, p_{sonic}^K) \ \text{ and } \ \Phi_K'(p) < 0 \quad \text{on} \quad \left(p_{sonic}^K, \sqrt{K/\theta}\right), \tag{4.14}$$

$$\Phi_K''(p) < 0 \quad \text{on} \quad (0, p_{sonic}^K], \tag{4.15}$$

where

$$p_{sonic}^K = \sqrt{K/(\theta+1)}. \tag{4.16}$$

Suppose that $\varphi(x)$ is a solution satisfying

$$|D\varphi(x)| < p_{sonic}^1 = \frac{1}{\sqrt{\theta+1}} \quad \text{in } \Omega^+, \qquad |D\varphi(x)| > p_{sonic}^1 \quad \text{in } \Omega^-, \tag{4.17}$$

besides (4.8) and (4.9). Then $\varphi(x)$ is a *transonic shock solution* with *transonic shock* S dividing Ω into the *subsonic region* Ω^+ and the *supersonic region* Ω^- and satisfying the physical entropy condition (see Courant-Friedrichs [47]; also see Lax [93, 94]):

$$\rho(|D\varphi_-|^2) < \rho(|D\varphi_+|^2) \ \text{ and } \ D\varphi^\pm \cdot \nu > 0 \qquad \text{along } S, \tag{4.18}$$

that is, the density ρ increases across a shock in the flow direction. Note that the equation (4.3) is elliptic in the subsonic region and hyperbolic in the supersonic region.

4.2. **Multidimensional Transonic Shocks near Flat Shocks.** Let (x', x_n) be the coordinates in $\mathbb{R}^n$, where $x' = (x_1, \dots, x_{n-1}) \in \mathbf{R}^{n-1}$ and $x_n \in \mathbb{R}$.

Let $q_0^- \in \left(p^1_{sonic}, 1/\sqrt{\theta}\right)$ and $\varphi_0^-(x) := q_0^- x_n$. Then $\varphi_0^-(x)$ is a supersonic solution in Ω. According to (4.12)–(4.14), there exists a unique $q_0^+ \in (0, p^1_{sonic})$ such that

$$(1 - \theta(q_0^+)^2)^{\frac{1}{2\theta}}\, q_0^+ = (1 - \theta(q_0^-)^2)^{\frac{1}{2\theta}}\, q_0^-. \tag{4.19}$$

Define $\varphi_0^+(x) := q_0^+ x_n$ in Ω. Then the function

$$\varphi_0(x) = \min(\varphi_0^+(x), \varphi_0^-(x)) \tag{4.20}$$

is a plane transonic shock solution in Ω, Ω_0^+ and $\Omega_0^- := \Omega \setminus \overline{\Omega_0^+}$ are its subsonic and supersonic regions, respectively, and $S_0 = \{x_n = 0\}$ is a transonic shock.

We first focus on an infinite cylinder $(0, a)^{n-1} \times (-\infty, \infty)$. Since it is not necessary to require that the supersonic perturbation $\varphi^-(x)$ be defined in the whole infinite cylinder, we introduce a finite subcylinder $\Omega_1 := (0, a)^{n-1} \times (-1, 1)$ and focus on the cylinder domain $\Omega := (0, a)^{n-1} \times (-1, \infty)$ without loss of generality. Then our multidimensional transonic shock problem can be formulated into the following form:

> **Multidimensional Transonic Shock Problem (MTS).** Given a supersonic weak solution $\varphi^-(x)$ of (4.3) in Ω_1, which is a $C^{1,\alpha}$ perturbation of $\varphi_0^-(x)$, for some $\alpha \in (0, 1)$:
>
> $$\|\varphi^- - \varphi_0^-\|_{1,\alpha,\Omega_1} \le \sigma, \tag{4.21}$$
>
> with $\sigma > 0$ small, and
>
> $$\varphi_\nu^- = 0 \qquad \text{on} \quad \partial(0, a)^{n-1} \times [-1, 1], \tag{4.22}$$
>
> find a transonic shock solution $\varphi(x)$ in Ω such that $\Omega^- := \Omega \setminus \overline{\Omega^+} = \{x \in \Omega \ : \ |D\varphi(x)| > p^1_{sonic}\} \subset \Omega_1$, $\varphi = \varphi^-$ in Ω^-, and
>
> $$\varphi = \varphi^-, \ \varphi_{x_n} = \varphi^-_{x_n} \qquad \text{on } (0, a)^{n-1} \times \{-1\}, \tag{4.23}$$
>
> $$\varphi_\nu = 0 \qquad \text{on } \partial(0, a)^{n-1} \times [-1, \infty), \tag{4.24}$$
>
> $$\varphi(\cdot, x_n) - \omega x_n \to 0 \qquad \text{uniformly on } (0, a)^{n-1} \text{ as } x_n \to \infty, \tag{4.25}$$
>
> for some constant $\omega \in (0, p^1_{sonic})$ which is not prescribed.

In order to deal with the multidimensional transonic shock problem in the unbounded domain Ω, we define the following weighted Hölder semi-norms and norms in a domain $\mathcal{D} \subset \mathbb{R}^n$: Let $x \to \delta_x$ be a given non-negative function defined on $\mathcal{D}$. Let $\delta_{x,y} := \min(\delta_x, \delta_y)$ for $x, y \in \mathcal{D}$. For $k \in \mathbb{R}$, $\alpha \in (0, 1)$, and $m \in \mathbf{N}$ (the set of nonnegative integers), we define

$$\begin{aligned}
&[u]^{(k)}_{m;0;\mathcal{D}} = \sum_{|\beta|=m} \sup_{x \in \mathcal{D}} \left(\delta_x^{m+k} |D^\beta u(x)|\right), \\
&[u]^{(k)}_{m;\alpha;\mathcal{D}} = \sum_{|\beta|=m} \sup_{x,y \in \mathcal{D}, x \ne y} \left(\delta_{x,y}^{m+\alpha+k} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x - y|^\alpha}\right), \\
&\|u\|^{(k)}_{m;0;\mathcal{D}} = \sum_{j=0}^{m} [u]^{(k)}_{j;0;\mathcal{D}}, \qquad \|u\|^{(k)}_{m;\alpha;\mathcal{D}} = \|u\|^{(k)}_{m;0;\mathcal{D}} + [u]^{(k)}_{m;\alpha;\mathcal{D}},
\end{aligned} \tag{4.26}$$

where $D^\beta = \partial_{x_1}^{\beta_1} \dots \partial_{x_n}^{\beta_n}$, $\beta = (\beta_1, \dots, \beta_n)$ is a multi-index with $\beta_j \geq 0, \beta_j \in \mathbf{N}$, $|\beta| = \beta_1 + \dots + \beta_n$, and the weight function

$$\delta_x = 1 + |x_n|.$$

We denote by $\|u\|_{m,\alpha,\mathcal{D}}$ the standard (non-weighted) Hölder norms in a domain $\mathcal{D}$, i.e., the norms defined as above with $\delta_x = \delta_{x,y} = 1$.

Then we have

Theorem 4.1. *Let $q_0^+ \in (0, p_{sonic}^1)$ and $q_0^- \in \left(p_{sonic}^1, 1/\sqrt{\theta}\right)$ satisfy (4.19), and let $\varphi_0(x)$ be the plane transonic shock solution (4.20). Then there exist $\sigma_0 > 0$ and C_0 depending only on n, a, α, γ, and q_0^+ such that, for every $\sigma \in (0, \sigma_0)$ and any supersonic solution $\varphi^-(x)$ of (4.3) satisfying the conditions stated in Problem* (MTS)*, the following hold:*

(i) *There exists $q^+ \in (0, p_{sonic}^1)$ that is the unique solution of the equation*

$$\rho((q^+)^2) q^+ = Q^+$$

with

$$Q^+ = \frac{1}{a^{n-1}} \int_{(0,a)^{n-1}} \rho(|D\varphi^-(x', -1)|^2) \varphi_{x_n}^-(x', -1) dx', \tag{4.27}$$

and satisfies

$$|q^+ - q_0^+| \leq C_0 \sigma, \tag{4.28}$$

such that, for any solution $\varphi(x)$ of Problem (MTS) *with the subsonic region $\Omega^+(\varphi) := \{x \in \Omega \ : \ |D\varphi(x)| < p_{sonic}^1\}$ of the form:*

$$\Omega^+(\varphi) = \{x_n > f(x')\} \cap \Omega \quad \text{with} \quad f \in C^{1,\alpha}([0,a]^{n-1}),\ Df|_{\partial(0,a)^{n-1}} = 0, \tag{4.29}$$

the constant ω in (4.25) must be q^+:

$$\omega = q^+. \tag{4.30}$$

(ii) *There exists a solution $\varphi(x)$ of Problem* (MTS) *satisfying*

$$\|\varphi - q_0^+ x_n\|_{1;\alpha;\Omega^+(\varphi)}^{(-1)} \leq C_0 \sigma, \tag{4.31}$$

and

$$\varphi(\cdot, x_n) - q^+ x_n \to 0 \qquad \text{uniformly on } (0,a)^{n-1} \text{ when } x_n \to \infty. \tag{4.32}$$

This solution also satisfies (4.29) with

$$\|f\|_{1,\alpha,[0,a]^{n-1}} \leq C_0 \sigma, \tag{4.33}$$

and, for every $k = 1, 2, \dots$,

$$\|\varphi - q^+ x_n\|_{1;\alpha;\Omega^+(\varphi)}^{(k)} \leq C_k \sigma, \tag{4.34}$$

where C_k depends only on k, n, a, γ, and q_0^+. That is, when $x_n \to \infty$, $\varphi(x)$ uniformly converges to the linear function $q^+ x_n$ with respect to $x' \in (0,a)^{n-1}$ at a rate faster than any algebraic order.

(iii) *If $\varphi^- \in C^{2,\alpha}(\overline{\Omega_1})$, in addition to the previous assumptions, then $\varphi \in C^{2,\alpha}(\overline{\Omega^+(\varphi)})$ and $f \in C^{2,\alpha}([0,a]^{n-1})$, and*

$$\|D^2 f\|_{0,\alpha,[0,a]^{n-1}} \leq C(n, a, \alpha, \gamma, q_0^+, \sigma, \|D^2\varphi^-\|_{0,\alpha,\Omega_1}) < \infty, \tag{4.35}$$

$$\|D^2 \varphi\|_{0;\alpha;\Omega^+(\varphi)}^{(k)} \leq C(n, a, \alpha, \gamma, q_0^+, \sigma, \|D^2\varphi^-\|_{0,\alpha,\Omega_1}, k) < \infty, \tag{4.36}$$

for $k = 1, 2, \ldots$. Moreover, if $\|\varphi^- - q_0^- x_n\|_{2;\alpha;\Omega_1} \le C\sigma$, then a solution satisfying (4.31) *and* (4.32) *is unique.*

Remark 4.1. Theorem 4.1 indicates that, given any supersonic solution $\varphi^-(x)$ in the upstream region, the necessary and sufficient condition for the existence of a transonic shock solution in an infinite cylinder Ω is that the uniform velocity state $\omega\nu_0, \nu_0 = (0, \ldots, 0, 1)$, at infinity in the downstream direction must be $q^+\nu_0$ determined uniquely by $\varphi^-(x)$. This means that, for this problem, one can apriori prescribe the uniformity condition of the flow, but can not prescribe a velocity state, at infinity in the downstream direction in general; otherwise the problem is overdetermined.

Furthermore, as a consequence of the uniqueness, non-degeneracy, and regularity of solutions of Problem (MTS), we have the following stability theorem.

Theorem 4.2. *There exist a nonnegative function $\Psi \in C([0,\infty))$ satisfying $\Psi(0) = 0$ and $\sigma_0 > 0$ depending only upon n, a, α, γ, and q_0^+ such that, if $\sigma < \sigma_0$, $\varphi^-(x)$ and $\hat\varphi^-(x)$ satisfies $\|\varphi^- - \hat\varphi_0^-\|_{2,\alpha,\Omega_1} \le \sigma$ and*

$$\|\varphi^- - \hat\varphi^-\|_{1,\alpha,\Omega_1} \le \kappa, \tag{4.37}$$

with $\kappa < \sigma$, then the unique solutions $\varphi(x)$ and $\hat\varphi(x)$ of Problem (MTS) *for $\varphi^-(x)$ and $\hat\varphi^-(x)$, respectively, satisfy*

$$\|f_\varphi - f_{\hat\varphi}\|_{1,\alpha,(0,a)^{n-1}} \le \Psi(\kappa), \tag{4.38}$$

where $f_\varphi(x')$ and $f_{\hat\varphi}(x')$ are the free boundary functions in (4.29) *with $\varphi(x)$ and $\hat\varphi(x)$, respectively.*

Similarly, we can establish the existence and stability of multidimensional transonic shocks near flat shocks in the whole space and near spherical transonic shocks in $\mathbb{R}^n, n \ge 3$.

4.3. Free Boundary Iterations for the Transonic Shock Problem (MTS).

The transonic shock problems can be formulated into a one-phase free boundary problem for a nonlinear elliptic equation: Given $\varphi^- \in C^{1,\alpha}(\overline{\Omega})$, find a function $\varphi(x)$ that is continuous in Ω and satisfies

$$\varphi \le \varphi^- \qquad \text{in } \bar\Omega; \tag{4.39}$$

the nonlinear equation (4.3) and the ellipticity condition (4.5) in the non-coincidence set

$$\Omega^+ = \{\varphi < \varphi^-\};$$

the free boundary condition (4.10) on the boundary $S = \partial\Omega^+ \cap \Omega$; and the prescribed conditions (4.23)–(4.25) on the fixed boundary $\partial\Omega$ and at infinity.

The free boundary is the location of the shock, and the free boundary condition (4.10) is the Rankine-Hugoniot condition (4.9). Note that condition (4.39) is motivated by the similar properties (4.20) of non-perturbed shocks; and, locally on the shock, (4.39) is equivalent to the entropy condition (4.18). Condition (4.39) transforms the transonic shock problem, in which the subsonic region Ω^+ is determined by the gradient condition $|D\varphi(x)| < p^1_{sonic}$, into a free boundary problem, in which Ω^+ is the non-coincidence set.

In order to solve this free boundary problem, we first modify equation (4.3) to make it uniformly elliptic and, correspondingly, modify the free boundary condition

(4.10). Then we solve this modified free boundary problem. Since $\varphi^-(x)$ is a small $C^{1,\alpha}$ perturbation of $\varphi_0^-(x)$, we show that the solution $\varphi(x)$ of the free boundary problem is a small $C^{1,\alpha}$ perturbation of the given subsonic shock solution $\varphi_0^+(x)$ in Ω^+. In particular, the gradient estimate implies that $\varphi(x)$ in fact satisfies the original free boundary problem, hence the transonic shock problem.

The modified free boundary problem does not directly fit into the variational framework of Alt-Caffarelli [1] and Alt-Caffarelli-Friedman [2] and the regularization framework of Berestycki-Caffarelli-Nirenberg [9] via the penalty method. Also, the nonlinearity of (4.40) makes it difficult to apply the Harnack inequality approach of Caffarelli [23]. In particular, a boundary comparison principle for positive solutions of nonlinear elliptic equations in Lipschitz domains is unavailable yet for the equations that are not homogeneous with respect to D^2u, Du, and u, which however is our case.

The free boundary method we develop is an iteration scheme based on the nondegeneracy of the free boundary condition: the jump of the normal derivative of a solution across the free boundary has a strictly positive lower bound. Our iteration process is as follows: We start with Ω_0^+ and $S_0 = \partial\Omega_0^+ \setminus \partial\Omega$. Suppose the domain Ω_k^+ is given so that $S_k := \partial\Omega_k^+ \setminus \partial\Omega$ is $C^{1,\alpha}$. We solve the oblique derivative problem in Ω_k^+ obtained by rewriting the (modified) equation (4.3) and the free boundary condition (4.10) in terms of the function $u := \varphi - \varphi_0^+$. Then the problem has the following form:

$$\begin{aligned} &\operatorname{div} A(x, Du) = F(x) && \text{in } \ \Omega_k^+ := \{u > 0\} \cap \Omega, \\ &A(x, Du) \cdot \nu = G(x, \nu) && \text{on } \ S := \partial\Omega_k^+ \setminus \partial\Omega, \end{aligned} \tag{4.40}$$

in addition to the fixed boundary conditions on $\partial\Omega_k^+ \cap \partial\Omega$ and the conditions at infinity. The equation is quasilinear, uniformly elliptic, $A(x, 0) \equiv 0$, while $G(x, \nu)$ has a certain structure. Let $u_k \in C^{1,\alpha}(\overline{\Omega_k^+})$ be the solution of (4.40). We estimate that $\|u_k\|_{C^{1,\alpha}(\Omega_k^+)}$ is small if the perturbation is small, where we use appropriate weighted Hölder norms in the unbounded domains. Then we extend the function $\varphi_k := \varphi_0^+ + u_k$ from Ω_k^+ to Ω so that the $C^{1,\alpha}$ norm of $\varphi_k - \varphi_0^+$ in Ω is controlled by $\|u_k\|_{C^{1,\alpha}}$. We define $\Omega_{k+1}^+ := \{x \in \Omega \, : \, \varphi^k(x) < \varphi^-(x)\}$ for the next step. Note that, since $\|\varphi_k - \varphi_0^+\|_{C^{1,\alpha}(\Omega)}$ and $\|\varphi^- - \varphi_0^-\|_{C^{1,\alpha}(\Omega)}$ are small, we have $|D\varphi^-| - |D\varphi_k| \ge \delta > 0$ in Ω, and this nondegeneracy implies that $S_{k+1} := \partial\Omega_{k+1}^+ \setminus \partial\Omega$ is $C^{1,\alpha}$ and its norm is estimated in terms of the data of the problem.

The fixed point Ω^+ of this process determines a solution of the free boundary problem since the corresponding solution $\varphi(x)$ satisfies $\Omega^+ = \{\varphi < \varphi^-\}$ and the Rankine-Hugoniot condition (4.10) holds on $S := \partial\Omega^+ \cap \Omega$.

On the other hand, the elliptic estimates alone are not sufficient to get the existence of a fixed point, because the right-hand side of the boundary condition in problem (4.40) depends on the unit normal ν of the free boundary. We use the following feature of the flat or spherical shocks:

$$\rho(|D\varphi_0^+|^2) D\varphi_0^+ = \rho(|D\varphi_0^-|^2) D\varphi_0^- \quad \text{in } \ \Omega \tag{4.41}$$

to obtain better estimates for the iteration and to prove the existence of a fixed point. Note that this is a vector identity, and the Rankine-Hugoniot condition (4.10) is the normal part of (4.41) on the non-perturbed free boundary S_0.

We remark in passing that the method we described above is mainly for handling the nonlinear free boundary problems in some bounded domains with appropriate fixed boundary conditions. For a unbounded domain, say the infinite cylinder $(0,a)^{n-1} \times (-\infty, \infty)$, we first use the free boundary method to construct the approximate solutions of the free boundary problem in appropriate bounded approximate domains, then make some uniform estimates of the solutions independent of the size of the approximate domains, and finally prove that the approximate solutions strongly converge to a solution of the original free boundary problem in the unbounded domain.

The uniqueness and stability of solutions for the transonic shock problem is obtained by using the regularity and nondegeneracy of solutions.

For more details, see Chen-Feldman [32].

5. Further Methods and Trends

In Sections 2–4, we have discussed some of selected recent methods to deal with nonlinear partial differential equations of divergence form. These methods are only samples and do not reflect fully the scope of nonlinear partial differential equations of divergence form. For example, some recent methods for nonlinear dispersive equations and Navier-Stokes equations were not covered; see [95, 91, 13], [85, 101, 140, 65, 66], and references cited therein. It will also be emphasized that some current trends go far beyond the materials discussed here. It is important to develop further the methods; some of them have briefly been mentioned above.

We would like to conclude with a few more examples of such developments, which are not covered above.

Compensated compactness methods have still great potential to be further developed to solve various important nonlinear partial differential equations, for example, some recent results on the existence and compactness of entropy solutions of the Euler equations with general pressure laws in [39] and for isothermal fluids in [87], and a new understanding of the relation between the large-time behavior of entropy solutions and the compactness of solution operators for hyperbolic conservation laws in [34] (also see [30, 37]). The theory of divergence-measure fields was motivated by the ideas of compensated compactness. The averaging compactness methods have similar connections [79, 80, 81, 122]. Other related methods include H-measures of Tartar [139] and Gérard [69] and the related Wigner measures by Lions-Paul [102] and Gérard [70]; also see Gérard-Markowich-Mauser-Poupaud [71].

Test function methods have become more important to handle with solutions of partial differential equations of divergence form that are not continuous. The doubling variables techniques were first developed by Kruzhkov [92] to establish the well posedness in L^∞ for scalar hyperbolic conservation laws; see [16, 31, 38, 27, 89, 63] for recent developments and applications. The perturbed test function method, which entails various modifications of the test functions by lower order correctors; applications include homogenization for elliptic PDEs of divergence form and approximations of quasilinear parabolic PDEs by systems of Hamilton-Jacobi equations; see Evans [61] for the details. The Holmgren method and the Haar's method have been applied to solve the uniqueness and stability of weak entropy solutions for hyperbolic conservation laws; see [50, 86, 97, 105, 126, 128, 118, 119] and the references cited therein.

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