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A lagrangian Discontinuous Galerkin type method on unstructured meshes to solve hydrodynamics problems

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SUMMARY

This paper concerns a new lagrangian Discontinuous Galerkin type method to solve 2D fluid flows on unstructured meshes. By using a basis of Bernstein polynomials of degree m in each triangle, we define a diffusion process which ensures positivity and stability of the scheme. The discontinuities of the physical variables at the interfaces between cells are solved with an acoustic Riemann solver. A remeshing process is performed with a particle method: this remeshing is locally conservative and its accuracy can be adapted to the accuracy of the numerical method. Copyright © 2000 John Wiley & Sons, Ltd.

KEY WORDS: Lagrangian method; Discontinuous Galerkin method; remeshing process; hydrodynamics problem; isentropic compression

1. INTRODUCTION

In this paper a new lagrangian Discontinuous Galerkin type method in 2D is developped. This method is devoted to multidimensional hydrodynamics problems on unstructured meshes in lagrangian coordinates. The first section gives an overview of the lagrangian system of conservation laws. The hierarchical numerical method is presented in the second section of this paper. The third section presents the locally conservative remeshing process made with a particle method. The fourth section presents several numerical test cases in 1D and 2D.

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2. 2D LAGRANGIAN SYSTEM OF CONSERVATION LAWS

Let $(\xi,\eta)^t$ be the 2D lagrangian coordinates and $\left(X\left((\xi,\eta),t\right), Y\left((\xi,\eta),t\right)\right)^t$ the eulerian ones. They are linked to each other by the transformation ($(u,v)^t$ is the fluid velocity)

$$X((\xi,\eta),t=0)=\xi, \qquad Y((\xi,\eta),t=0)=\eta, \tag{1}$$

$$\frac{\partial}{\partial t}X=u, \qquad \frac{\partial}{\partial t}Y=v. \tag{2}$$

The jacobian matrix describes the time evolution of these two systems of coordinates. The jacobian is the determinant of this matrix and can be associated with a local compression/expansion rate. This jacobian may not change sign during time evolution in order to define a bijection between lagrangian and eulerian coordinates

$$\mathcal{J}=\begin{pmatrix} \partial_\xi X & \partial_\xi Y \\ \partial_\eta X & \partial_\eta Y \end{pmatrix} \qquad \left\{ \begin{array}{l} J((\xi,\eta),t)=\partial_\xi X\partial_\eta Y-\partial_\xi Y\partial_\eta X \\ J((\xi,\eta),0)=1. \end{array}\right. \tag{3}$$

The 2D lagrangian system of conservation laws can be written as (see [6])

$$\begin{aligned}
\frac{\partial}{\partial t}J-\overrightarrow{\nabla_\xi}.\left(u\left(\overrightarrow{\nabla_\xi Y}\right)_\perp - v\left(\overrightarrow{\nabla_\xi X}\right)_\perp\right) &= 0 && (4)\\
\frac{\partial}{\partial t}(\rho J) &= 0 && (5)\\
\frac{\partial}{\partial t}(\rho u J)+\overrightarrow{\nabla_\xi}.\left(p\left(\overrightarrow{\nabla_\xi Y}\right)_\perp\right) &= 0 && (6)\\
\frac{\partial}{\partial t}(\rho v J)+\overrightarrow{\nabla_\xi}.\left(-p\left(\overrightarrow{\nabla_\xi X}\right)_\perp\right) &= 0 && (7)\\
\frac{\partial}{\partial t}(\rho e J)+\overrightarrow{\nabla_\xi}.\left(pu\left(\overrightarrow{\nabla_\xi Y}\right)_\perp - pv\left(\overrightarrow{\nabla_\xi X}\right)_\perp\right) &= 0 && (8)
\end{aligned}$$

$$p=(\gamma-1)\rho(e-\frac{1}{2}(u^2+v^2)). \tag{9}$$

We use the notation $\left(\overrightarrow{\nabla_\xi A}\right)_\perp=\begin{pmatrix} -\partial_\eta A \\ \partial_\xi A \end{pmatrix}$ for all scalar A. (9) can be replaced with any other equation of state. A more concise notation of the previous system leads to a *physical system of conservation laws*

$$\frac{\partial}{\partial t}(UJ)+\overrightarrow{\nabla_\xi}.\overrightarrow{\varphi}_\perp=0 \tag{10}$$

$$U=(\rho,\rho u,\rho v,\rho e,1)^t \tag{11}$$

$$\overrightarrow{F}(U)=\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}=\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix},\begin{pmatrix} p \\ 0 \end{pmatrix},\begin{pmatrix} 0 \\ p \end{pmatrix},p\begin{pmatrix} u \\ v \end{pmatrix},-\begin{pmatrix} u \\ v \end{pmatrix}\right) \tag{12}$$

$$\overrightarrow{\varphi}_\perp=F_1\left(\overrightarrow{\nabla_\xi Y}\right)_\perp - F_2\left(\overrightarrow{\nabla_\xi X}\right)_\perp \tag{13}$$

and to a *geometrical system of conservation laws*

$$\frac{\partial}{\partial t}\overrightarrow{\nabla_\xi X}-\overrightarrow{\nabla_\xi u}=0, \qquad \frac{\partial}{\partial t}\overrightarrow{\nabla_\xi Y}-\overrightarrow{\nabla_\xi v}=0. \tag{14}$$

In [7] and [1] hyperbolicity studies of the above systems can be found. Both systems are hyperbolic but their union (called complete system) is only weakly hyperbolic: even if all the eigenvalues are real, there is no complete basis of eigenvectors. However if the numerical method does not have to determine a complete basis of eigenvectors then this system can be considered to be hyperbolic.

3. THE HIERARCHICAL NUMERICAL SCHEMES - A LAGRANGIAN DISCONTINUOUS GALERKIN APPROACH

Let $\Omega = \bigcup_{c=1}^{N_T} T_c$ with $T_c \equiv T$ being the triangle under consideration, be a non structured mesh. The nodes of T are locally labeled $1, 2, 3$, the side h in front of the node number i ($i = 1, 2, 3$) is side number i and the normal unit output vector to h is called $\overrightarrow{\nu}_h$, the boundary of T is ∂T, and $\overset{o}{T} = T/\partial T$. T_h is the neighborhood triangle having h as common side.
In every triangle T one can define a Bernstein basis of polynomials of degree m: $\{\sigma_k\}_{k=1}^{B_m}$. If $\lambda_1, \lambda_2, \lambda_3 = 1 - \lambda_1 - \lambda_2$ are the barycentric coordinates in T then the Bernstein basis of polynomials are the monomials of the development

$$(\lambda_1 + \lambda_2 + \lambda_3)^m \equiv 1. \tag{15}$$

For example if $m = 1$ the basis consists of 3 Bernstein polynomials

$$\sigma_1 = \lambda_1,\ \sigma_2 = \lambda_2,\ \sigma_3 = \lambda_3, \tag{16}$$

For $m = 2$ the basis is made up of 6 Bernstein polynomials

$$\sigma_1 = \lambda_1^2,\ \sigma_2 = 2\lambda_1\lambda_2,\ \sigma_3 = \lambda_2^2,\ \sigma_4 = 2\lambda_2\lambda_3,\ \sigma_5 = \lambda_3^2,\ \sigma_6 = 2\lambda_1\lambda_3. \tag{17}$$

The number of Bernstein's polynomials of degree m is equal to B_m (see [1] for details).
Every component of equation (10) is multiplied with every σ_k to give the moment equations.

$$\int_\Omega \left(\frac{\partial}{\partial t}(UJ) + \left\{ \overrightarrow{\nabla}_\xi . \overrightarrow{\varphi}_\perp \right\} \right) \sigma_k d\xi d\eta = 0. \tag{18}$$

These $\{\bullet\}$ means that $\bullet$ has to be considered in the sense of a distribution. The Discontinuous Galerkin principles imply that the functions are polynomials in each cell, such that on a triangle T the divergence of $\overrightarrow{\varphi}_\perp$ is splitted into two parts:

$$\left\{ \overrightarrow{\nabla}_\xi . \overrightarrow{\varphi}_\perp \right\} \equiv \begin{cases} \overrightarrow{\nabla}_\xi . \overrightarrow{\varphi}_\perp & \text{if} \quad \overrightarrow{\xi} \in \overset{o}{T} \\ [\overrightarrow{\varphi}_\perp|_{T_h} - \overrightarrow{\varphi}_\perp|_T] . \overrightarrow{\nu}_h & \text{if} \quad \overrightarrow{\xi} \in h. \end{cases} \tag{19}$$

Equation (18) is equivalent to ($T_c \equiv T$)

$$\sum_{c=1}^{M} \left\{ \int_T \frac{\partial}{\partial t}(UJ)\sigma_k d\xi d\eta + \int_{\overset{o}{T}} \overrightarrow{\nabla}_\xi . \overrightarrow{\varphi}_\perp \sigma_k d\xi d\eta + \sum_{h \in \partial T} \int_h [\overrightarrow{\varphi}_\perp|_{T_h} . \overrightarrow{\nu}_h - \overrightarrow{\varphi}_\perp|_T . \overrightarrow{\nu}_h] \sigma_k d\mu \right\} = 0. \tag{20}$$

The discontinuity on h is splitted into two contributions by using the intermediate value $\overline{\int_h \overset{*}{\overrightarrow{\varphi}_\perp} \sigma_k d\mu}$

$$\int_h \left[\overrightarrow{\varphi}_\perp|_{T_h} - \overrightarrow{\varphi}_\perp|_T\right] \sigma_k d\mu \quad = \quad \underbrace{\left[\int_h \overrightarrow{\varphi}_\perp|_{T_h} \sigma_k d\mu - \overline{\int_h \overset{*}{\overrightarrow{\varphi}_\perp} \sigma_k d\mu}\right]}_{\text{contribution for } T_h} + \underbrace{\left[\overline{\int_h \overset{*}{\overrightarrow{\varphi}_\perp} \sigma_k d\mu} - \int_h \overrightarrow{\varphi}_\perp|_T \sigma_k d\mu\right]}_{\text{contribution for } T}$$

Whatever the intermediate value conservation will be respected. The same process is repeated on each side h such that the moment equations are finally given by

$$\int_T \frac{\partial}{\partial t}(UJ)\sigma_k d\xi d\eta + \int_{\overset{\circ}{T}} \overrightarrow{\nabla}_\xi . \overrightarrow{\varphi}_\perp \sigma_k d\xi d\eta + \sum_{h\in\partial T} \left[\overline{\int_h \overset{*}{\overrightarrow{\varphi}_\perp . \overrightarrow{\nu}_h} \sigma_k d\mu} - \int_h \overrightarrow{\varphi}_\perp|_T . \overrightarrow{\nu}_h \sigma_k d\mu\right] = 0 \tag{21}$$

A Green formula for the second term yields

$$\int_T \frac{\partial}{\partial t}(UJ)\sigma_k d\xi d\eta - \int_{\overset{\circ}{T}} \overrightarrow{\varphi}_\perp . \overrightarrow{\nabla_\xi \sigma_k} d\xi d\eta + \sum_{h\in\partial T} \overline{\int_h \overset{*}{\overrightarrow{\varphi}_\perp . \overrightarrow{\nu}_h} \sigma_k d\mu} = 0. \tag{22}$$

If we define $M_{UJ,k} := \int_T (UJ)\sigma_k d\xi d\eta$ to be the moment of UJ, $M_{\nabla\varphi,k} := \int_{\overset{\circ}{T}} \overrightarrow{\nabla}_\xi . \overrightarrow{\varphi}_\perp \sigma_k d\xi d\eta$ to be the interior cell term, $\sum_{h\in\partial T}\left[\overset{*}{M}_{\varphi,k,h} - M_{\varphi,k,h}\right] := \sum_{h\in\partial T}\left[\overline{\int_h \overset{*}{\overrightarrow{\varphi}_\perp . \overrightarrow{\nu}_h} \sigma_k d\mu} - \int_h \overrightarrow{\varphi}_\perp|_T . \overrightarrow{\nu}_h \sigma_k d\mu\right]$ to be the border terms then the system (21) becomes

$$\frac{\partial}{\partial t} M_{UJ,k} + M_{\nabla\varphi,k} + \sum_{h\in\partial T}\left[\overset{*}{M}_{\varphi,k,h} - M_{\varphi,k,h}\right] = 0. \tag{23}$$

Let $[t_n, t_{n+1}] \subset [0,T]$ with $T>0$ and $\Delta t = t_{n+1} - t_n$, then the time discretisation is given as

$$\frac{\partial}{\partial t}(M_{UJ,k}) \simeq \frac{M_{UJ,k}^{n+1} - M_{UJ,k}}{\Delta t}, \tag{24}$$

superscript $^{n+1}$ is the variable at time $t = t_{n+1}$ and the absence of a superscript implies that the variable has to be taken at time $t = t_n$. Equation (23) becomes

$$M_{UJ,k}^{n+1} = M_{UJ,k} - \Delta t M_{\nabla\varphi,k} - \Delta t \sum_{h\in\partial T}\left[\overset{*}{M}_{\varphi,k,h} - M_{\varphi,k,h}\right]. \tag{25}$$

Interior cell terms approximation. For all $k = 1, \cdots, B_m$

$$M_{\nabla\varphi,k} = \int_{\overset{\circ}{T}} \overrightarrow{\nabla}_\xi . \overrightarrow{\varphi}_\perp \sigma_k d\xi d\eta, \quad \text{with} \quad \overrightarrow{\varphi}_\perp = F_1\left(\overrightarrow{\nabla_\xi Y}\right)_\perp - F_2\left(\overrightarrow{\nabla_\xi X}\right)_\perp. \tag{26}$$

Suppose that $\overrightarrow{\varphi}_\perp$ is a vector where components are polynomials of degre m, then

$$\overrightarrow{\varphi}_\perp(\overrightarrow{\xi}) = \sum_{i=1}^{B_m} \overrightarrow{\varphi}_{\perp,i} \sigma_i(\overrightarrow{\xi}) \implies M_{\nabla\varphi,k} = \sum_{i=1}^{B_m} \overrightarrow{\varphi}_{\perp,i} . \int_{\overset{\circ}{T}} \overrightarrow{\nabla_\xi \sigma_i} \sigma_k d\xi d\eta, \tag{27}$$

with $\overrightarrow{\varphi}_{\perp,i}$ the i-component in the Bernstein basis. To determine (27) we have to compute $\overrightarrow{\varphi}_{\perp,i}$ for all i. As moments of UJ are known at $t = t_n$, we can define

$$\widehat{U}(\overrightarrow{\xi}) = \sum_{i=1}^{B_m} \widehat{U}_i \sigma_i(\overrightarrow{\xi}) \quad \text{with} \quad \widehat{U}_i = \left(\frac{M_{UJ,i}}{M_{J,i}}\right), \tag{28}$$

then an approximation of $\overrightarrow{F}(U)$ is given by

$$\overrightarrow{F}(U)(\overrightarrow{\xi}) \simeq \overrightarrow{F}(\widehat{U})(\overrightarrow{\xi}) = \sum_{i=1}^{B_m} \widehat{\overrightarrow{F}}_i \sigma_i(\overrightarrow{\xi}) \quad \text{with} \quad \widehat{\overrightarrow{F}}_i = \overrightarrow{F}(\widehat{U}_i). \tag{29}$$

If we suppose that $\overrightarrow{\nabla_\xi X}$ and $\overrightarrow{\nabla_\xi Y}$ are polynomials then

$$\overrightarrow{\nabla_\xi X} \simeq \overrightarrow{\widehat{\nabla_\xi X}}(\overrightarrow{\xi}) = \sum_{i=1}^{B_m} \overrightarrow{\widehat{\nabla_\xi X}}_i \sigma_i(\overrightarrow{\xi}), \tag{30}$$

$$\overrightarrow{\nabla_\xi Y} \simeq \overrightarrow{\widehat{\nabla_\xi Y}}(\overrightarrow{\xi}) = \sum_{i=1}^{B_m} \overrightarrow{\widehat{\nabla_\xi Y}}_i \sigma_i(\overrightarrow{\xi}). \tag{31}$$

A $\mathbb{P}_m$ approximation of $\overrightarrow{\varphi}_\perp$ is then computed as a (m)-diffusive Bernstein polynomial

$$\overrightarrow{\varphi}_\perp \simeq \sum_{i=1}^{B_m} \widehat{\overrightarrow{\varphi}}_{\perp,i} \sigma_i(\xi) \quad \text{with} \quad \widehat{\overrightarrow{\varphi}}_{\perp,i} = \left(\widehat{F}_{1,i}\left(\overrightarrow{\widehat{\nabla_\xi Y}}\right)_{\perp,i} - \widehat{F}_{2,i}\left(\overrightarrow{\widehat{\nabla_\xi X}}\right)_{\perp,i}\right). \tag{32}$$

Remark 1. *The definition of a (m)-diffusive Bernstein polynomial of a function $A \in L^1(T)$ is*

$$\forall (x,y)^t \in T \qquad \widehat{A}(x,y) = \sum_{i=1}^{6} \left(\frac{\int_T A(\xi,\eta)\sigma_i(\xi,\eta) d\xi d\eta}{\int_T \sigma_i(\xi,\eta) d\xi d\eta}\right) \sigma_i(x,y), \tag{33}$$

In [1] some properties of such polynomials are shown †. By using (m)-diffusive polynomials numerical diffusion is addded to the scheme and stability is increased. Moreover (m)-diffusive polynomials ensure that the jacobian, the density and the pressure remain positive.

Interior cell terms are computed as

$$M_{\nabla\varphi,k} \simeq \sum_{j=1}^{B_m} \widehat{\overrightarrow{\varphi}}_{\perp,j} . \int_{\overset{\circ}{T}} \overrightarrow{\nabla_\xi \sigma_j} \sigma_k d\xi d\eta. \tag{34}$$

Approximation of the border terms. The border terms are splitted into two parts

$$\sum_{h \in \partial T} \left[\overset{*}{M}_{\varphi,k,h} - M_{\varphi,k,h}\right] = \sum_{h \in \partial T} \left[\overline{\int_h \overrightarrow{\varphi}_\perp . \overrightarrow{\nu}_h \sigma_k d\mu}^{\,*} - \int_h \overrightarrow{\varphi}_\perp|_T . \overrightarrow{\nu}_h \sigma_k d\mu\right]. \tag{35}$$

† (i) $\int_T \widehat{A} d\xi d\eta = \int_T A d\xi d\eta$, (ii) $A > 0 \Rightarrow \widehat{A} > 0$, (iii) $||\widehat{A} - A||_p \overset{m\to\infty}{\longrightarrow} 0$ for all $1 \leq p < \infty$

An approximation of $\overrightarrow{\varphi}_\perp$ has already been given therefore

$$\int_h \overrightarrow{\varphi}_\perp|_T.\overrightarrow{\nu}_h\sigma_k d\mu \simeq \sum_{j=1}^{B_m} \overrightarrow{\widehat{\varphi}}_{\perp,j}.\overrightarrow{\nu}_h \int_h \sigma_i\sigma_k d\mu. \tag{36}$$

The star term is the solution of the moment Riemann problem on the side h in the direction $\overrightarrow{n}_h$. $\overrightarrow{n}_h$ is the outgoing normal vector to ∂K the image of ∂T in the transformation from lagrangian coordinates to eulerian ones. The physical moments on both sides of h are given by

$$\int_{\partial K} U(S)\sigma_k(S)dS \quad = \quad \int_{\partial T} U\sigma_k\sqrt{\left(\left(\overrightarrow{\nabla_\xi Y}\right)_\perp.\overrightarrow{\nu}_{\partial T}\right)^2 + \left(-\left(\overrightarrow{\nabla_\xi X}\right)_\perp.\overrightarrow{\nu}_{\partial T}\right)^2}d\mu. \tag{37}$$

The approximation of (37) uses (m)-diffusive polynomials

$$\widehat{UG}(\mu) := \sum_{i=1}^{B_m}\widehat{UG}_i\sigma_i(\mu) \quad \widehat{UG}_i = \widehat{U}_i\sqrt{\left(\left(\overrightarrow{\widehat{\nabla_\xi Y}}\right)_{\perp,i}.\overrightarrow{\nu}_{\partial T}\right)^2 + \left(-\left(\overrightarrow{\widehat{\nabla_\xi X}}\right)_{\perp,i}.\overrightarrow{\nu}_{\partial T}\right)^2}, \tag{38}$$

$\left(\overrightarrow{\widehat{\nabla_\xi X}}\right)_{\perp,i}$ and $\left(\overrightarrow{\widehat{\nabla_\xi Y}}\right)_{\perp,i}$ are given due to (30) and (31) (see (51) and (52)).

Moment Riemann problem. The physical states on both sides of h are given by

$$\widehat{\widehat{U^T_{h,k}}} := \frac{\sum_{i=1}^{B_m}\widehat{UG}_i\int_{h\in\partial T}\sigma_i\sigma_k d\mu}{\sum_{i=1}^{B_m}\widehat{G}_i\int_{h\in\partial T}\sigma_i\sigma_k d\mu} \qquad UG, G \text{ defined in } T, \tag{39}$$

$$\widehat{\widehat{U^{T_h}_{h,k}}} := \frac{\sum_{i=1}^{B_m}\widehat{UG}_i\int_{h\in\partial T_h}\sigma_i\sigma_k d\mu}{\sum_{i=1}^{B_m}\widehat{UG}_i\int_{h\in\partial T_h}\sigma_i\sigma_k d\mu} \qquad UG, G \text{ defined in } T_h. \tag{40}$$

Using (39) and (40) two densities $(\rho^T_{h,k}, \rho^{T_h}_{h,k})$, two pressures $(p^T_{h,k}, p^{T_h}_{h,k})$, two sound velocities $(c^T_{h,k}, c^{T_h}_{h,k})$ and two velocities $(\nu^T_{h,k}, \nu^{T_h}_{h,k})$ can be computed for each Bernstein basis σ_k on the side h. The Riemann problem on these two states is solved with the following acoustic riemann solver

$$RP(\widehat{\widehat{U^T_{h,k}}}, \widehat{\widehat{U^{T_h}_{h,k}}}, \overrightarrow{n}_{\partial K}) \begin{array}{l} \nearrow \quad \overset{*}{\nu}_{h,k} = \frac{p^{T_h}_{h,k} - p^T_{h,k} + (\rho c\nu)^T_{h,k} + (\rho c\nu)^{T_h}_{h,k}}{(\rho c)^T_{h,k} + (\rho c)^{T_h}_{h,k}} \\ \searrow \quad \overset{*}{p}_{h,k} = p^T_{h,k} - (\rho c)^T_{h,k}(\overset{*}{\nu}_{h,k} - \nu^T_{h,k}). \end{array} \tag{41}$$

The increments are then defined as

$$U = \rho u \quad : \quad \overset{*}{M}_{\varphi,k,h} = \overset{*}{p}_{h,k}\int_{h\in\partial T}\overrightarrow{\widehat{\nabla_\xi Y}}_\perp.\overrightarrow{\nu}_h\sigma_k d\mu, \tag{42}$$

$$U = \rho v \quad : \quad \overset{*}{M}_{\varphi,k,h} = \overset{*}{p}_{h,k}\int_{h\in\partial T} -\overrightarrow{\widehat{\nabla_\xi X}}_\perp.\overrightarrow{\nu}_h\sigma_k d\mu, \tag{43}$$

$$U = \rho e \quad : \quad \overset{*}{M}_{\varphi,k,h} = \overset{*}{p}_{h,k}\overset{*}{\nu}_{h,k}\int_{h\in\partial T}\widehat{G}\sigma_k d\mu, \tag{44}$$

$$U = 1 \quad : \quad \overset{*}{M}_{\varphi,k,h} = \overset{*}{\nu}_{h,k}\int_{h\in\partial T}\widehat{G}\sigma_k d\mu. \tag{45}$$

The explicit scheme can now be written as

$$M^{n+1}_{UJ,k} = M_{UJ,k} - \Delta t \sum_{j=1}^{B_m} \overrightarrow{\widehat{\varphi}}_{\perp,j} . \int_{\overset{\circ}{T}} \overrightarrow{\nabla_\xi \sigma_j} \sigma_k d\xi d\eta - \Delta t \sum_{h\in\partial T} \left[\overset{*}{M}_{\varphi,k,h} - \sum_{j=1}^{B_m} \overrightarrow{\widehat{\varphi}}_{\perp,j} . \overrightarrow{\nu}_h \int_{h\in\partial T} \sigma_j \sigma_k d\mu \right] . \tag{46}$$

The explicit discretisation of the geometrical system yields

$$M^{n+1}_{\overrightarrow{\nabla_\xi X},j} = M_{\overrightarrow{\nabla_\xi X},j} + \Delta t \int_T \overrightarrow{\widehat{\nabla_\xi u}} \sigma_j d\xi d\eta, \tag{47}$$

$$M^{n+1}_{\overrightarrow{\nabla_\xi Y},j} = M_{\overrightarrow{\nabla_\xi Y},j} + \Delta t \int_T \overrightarrow{\widehat{\nabla_\xi v}} \sigma_j d\xi d\eta, \tag{48}$$

such that

$$M^{n+1}_{\overrightarrow{\nabla_\xi X},j} = M_{\overrightarrow{\nabla_\xi X},j} + \Delta t \sum_{k=1}^{B_m} \widehat{u}_k \int_T \overrightarrow{\nabla_\xi \sigma_k} \sigma_j d\xi d\eta, \tag{49}$$

$$M^{n+1}_{\overrightarrow{\nabla_\xi Y},j} = M_{\overrightarrow{\nabla_\xi Y},j} + \Delta t \sum_{k=1}^{B_m} \widehat{v}_k \int_T \overrightarrow{\nabla_\xi \sigma_k} \sigma_j d\xi d\eta. \tag{50}$$

As $\overrightarrow{\widehat{\nabla_\xi X}}_j = \frac{\int_T \overrightarrow{\nabla_\xi X} \sigma_j d\xi d\eta}{\int_T \sigma_j d\xi d\eta}$ and $\overrightarrow{\widehat{\nabla_\xi Y}}_j = \frac{\int_T \overrightarrow{\nabla_\xi Y} \sigma_j d\xi d\eta}{\int_T \sigma_j d\xi d\eta}$ then the gradients of X and Y are given by

$$\overrightarrow{\widehat{\nabla_\xi X}}^{n+1}_j = \overrightarrow{\widehat{\nabla_\xi X}}_j + \Delta t \sum_{k=1}^{B_m} \widehat{u}_k \frac{\int_T \overrightarrow{\nabla_\xi \sigma_k} \sigma_j d\xi d\eta}{\int_T \sigma_j d\xi d\eta}, \tag{51}$$

$$\overrightarrow{\widehat{\nabla_\xi Y}}^{n+1}_j = \overrightarrow{\widehat{\nabla_\xi Y}}_j + \Delta t \sum_{k=1}^{B_m} \widehat{v}_k \frac{\int_T \overrightarrow{\nabla_\xi \sigma_k} \sigma_j d\xi d\eta}{\int_T \sigma_j d\xi d\eta}. \tag{52}$$

Step size. Δt is limited by two conditions: a lagrangian CFL type condition and a Jacobian condition. Because the jacobian has to remain positive we enforce the Δt value to be such that $M^{n+1}_{J,k}$ will remain positive for all k (as will be $\widehat{J}$). A Runge-Kutta (RK) method has been used to increase the value of Δt.

Slope type limiting. The use of (m)-diffusive polynomials adds some numerical diffusion to the method in such a way that the positivity (of jacobian, density, pressure) is ensured and the scheme is stable without any slope type limiting for $m \leq 2$, aside from strong shock waves. In the general case a slope type limiter is added. Its philosophy is: *"if the compression is maximum in T (with respect to its three neighbours), then all variables UJ are replaced by their (m)-diffusive polynomial"*.

Boundary conditions are simulated by prescribing to pressure $\overset{*}{p}_{h,k}$ (or the velocity $\overset{*}{\nu}_{h,k}$) on the side in contact with the external world.

4. REMAPPING/REMESHING PROCESS - A PARTICLE METHOD

Lagrangian methods are known to be accurate because the mesh is refined near the compression area and unrefined near the expansion area. However one of the drawbacks of lagrangian methods is the fact that for highly disturbed flows positivity of the jacobian can not be guaranted. The previous numerical method ensures jacobian positivity by reducing Δt. But if Δt tends to 0 the solution is to remap/remesh the domain.
The remapping/remeshing process is a critical phase in a lagrangian hydrodynamics code because it can destroy all the efforts made by the scheme to be conservative and accurate. The point is to build a remeshing process (i) locally conservative, (ii) with the same accuracy as the scheme.
If $f \in \mathbb{P}_m(\Omega)$, the accuracy of the remeshing process is measured as its capability to give a polynomial approximation of f called $\widetilde{f} \in \mathbb{P}_m(\Omega)$ such that $\widetilde{f} = f$ after several remeshing phases.

4.1. Projection on particles

Definition 1. *A particle is a Dirac function $\delta_{\overrightarrow{\xi}_p}(\overrightarrow{\xi})$ located in $\overrightarrow{\xi}_p$ carrying 6 weights*

$$(\omega U)_p = (\omega\rho, \omega\rho u, \omega\rho v, \omega\rho e, \omega)_p, \qquad U_p = \frac{(\omega U)_p}{\omega_p}.$$

A particle p belongs to T if $\overrightarrow{\xi}_p \in T$. So we can define discrete moments with respect to the Bernstein basis

Definition 2. *Discrete moments of degree (m) are the (B_m)-vectors (vector of size B_m)*

$$(M_{\omega,.}) \equiv \begin{pmatrix} M_{\omega U,1} \\ M_{\omega U,2} \\ \vdots \\ M_{\omega U,B_m} \end{pmatrix} = \begin{pmatrix} \sum_{p\in T} \omega_p U_p \sigma_1^{(n)}(\xi_p) \\ \sum_{p\in T} \omega_p U_p \sigma_2^{(n)}(\xi_p) \\ \vdots \\ \sum_{p\in T} \omega_p U_p \sigma_{B_m}^{(n)}(\xi_p) \end{pmatrix}$$

Lemma 1. *Let $N \geq B_m$ be distinct points in T called $\overrightarrow{\xi}_p$, $p = 1, \ldots, N$. Let $(M_{UJ,.})$ be the m moments given by the numerical scheme. If the weights $(\omega U)_p$ (associated to the position $\overrightarrow{\xi}_p$) are located on a polynomial of degree m then a unique population of N particles exists such that for all $k = 1, \ldots, B_m$*

$$M_{\omega U,k} = M_{UJ,k} \tag{53}$$

Proof: see [1].
The point (i) is now fullfilled because equation (53) implies that all the information carryied by the moments has been projected on particle weights, no hypothesis has been made on the position of the particles. Then any polynomial of degre less than m is exactly integrated with this method if the mesh does not change (for example to plot the results).

4.2. Projection from particles

In the case we want to remesh/remap the domain we have to define a more accurate numerical integration. Indeed if the N particles are located at the Gauss points $\overrightarrow{\xi}_g$ with Gauss weights

w_g then the numerical integration

$$M_{UJ,k} := \int_T (UJ)(\overrightarrow{\xi})\sigma_k(\overrightarrow{\xi})d\xi d\eta = \sum_{g=1}^{N} w_g (UJ)(\overrightarrow{\xi}_g)\sigma_k(\overrightarrow{\xi}_g) \tag{54}$$

is exact if $N \geq B_m$ in T. The same idea as Gauss quadrature is used but we will not enforce the particle positions.
Let the current triangle in the new mesh be τ. Bernstein polynomials have changed in the new cells but the positions of the particles and their weights $(UJ)(\overrightarrow{\xi}_p)$ have not changed. Suppose there are $\mathcal{N}$ particles in τ ($\mathcal{N}$ can be different from N). Then we have the lemma

Lemma 2. *Let $\overrightarrow{\xi}_g$, $g = 1, \dots, \mathcal{N}$ be $\mathcal{N}$ distinct points in τ, then the quadrature problem*

$$\mathcal{M}_{UJ,k} := \int_\tau (UJ)(\overrightarrow{\xi})\sigma_k(\overrightarrow{\xi})d\xi d\eta \equiv \sum_{g=1}^{\mathcal{N}} \mathcal{W}_g (UJ)(\overrightarrow{\xi}_g)\sigma_k(\overrightarrow{\xi}_g) \tag{55}$$

for all $k = 1, \dots, B_m$ and for all polynomial $(UJ) \in \mathbb{P}_m(\Omega)$ has an unique solution if

$$(i)\;\; \mathcal{N} \geq B_{m+1}\;; \quad (ii)\;\; \forall g = 1, 2, \dots \mathcal{N} \quad \mathcal{W}_g = \widetilde{\mathcal{W}}(\overrightarrow{\xi}_g),\; \textit{with } \widetilde{\mathcal{W}} \in \mathbb{P}_{2m}(\tau).$$

Proof: see [1].
If enough particles are in each triangle τ then the numerical integration of any polynomial of degree m is exact. Therefore the numerical diffusion due to the remeshing process is very low and the accuracy of the remeshing can be adapted to the accuracy of the scheme.
The moments in triangle τ give an initialisation on the new mesh for the numerical method

$$\forall 1 \leq k \leq B_m \quad \mathcal{M}_{UJ,k} := \sum_{q=1}^{\mathcal{N}} \mathcal{W}_q (UJ)_q \sigma_k(\xi_q, \eta_q). \tag{56}$$

4.3. Eulerian particle position

The lagrangian particle position $(\xi_p, \eta_p)^t \in T$ is by definition fixed in time. The eulerian position $(X_p, Y_p)^t$ changes with the velocity defined in T (recall $\widehat{UJ} \in \mathbb{P}_m(T)$)

$$\widehat{u}_p^{n+1} = \frac{\widehat{\rho u J}^{n+1}(\xi_p, \eta_p)}{\widehat{\rho J}^{n+1}(\xi_p, \eta_p)}, \tag{57}$$

$$\widehat{v}_p^{n+1} = \frac{\widehat{\rho v J}^{n+1}(\xi_p, \eta_p)}{\widehat{\rho J}^{n+1}(\xi_p, \eta_p)}, \tag{58}$$

then ($(\Delta t)_k$ is the k-th timestep value)

$$X_p^{n+1} = \xi_p + \sum_{k=1}^{n+1} (\Delta t)_k \; \widehat{u}_p^k, \tag{59}$$

$$Y_p^{n+1} = \eta_p + \sum_{k=1}^{n+1} (\Delta t)_k \; \widehat{v}_p^k. \tag{60}$$

123	Left	Right
ρ	1	1
u	-2	2
p	0.4	0.4

Blast	L.	Mid.	R.
ρ	1	1	1
u	0	0	0
p	1000	0.01	100

Sod	L.	R.
ρ	1	0.125
u	0	0
p	1	0.1

Table I. Initial data of the 123 problem, the blastwave problem, the 2D Sod tube problem.

5. NUMERICAL TESTS

The present method can be written for any m, with a different time discretization (explicit, RK, implicit), with or without slope type limiting on unstructured meshes and has been tested in 1D and 2D on several classical test cases, see [1] and [8] for details of the test cases.
All the tests have been performed with $m = 2$ and a 2-step Runge-Kutta time discretization. No remeshing was necessary even if a strong compression has been observed (see [1] for the validation of the remeshing process).

123 problem in 1D. The internal energy for 100 and 250 cells is shown in figure 1. No limiter has been used.

Blastwave in 1D. In figure 2 density and velocity are presented for 400 cells. The limiter is active.

Sod shock tube in 1D. The density and the pressure are presented in figure 3. The limiter is active.

Isentropic compression of a ring. The *Stiffened Gaz* equation of state $p_{SG} = p - \gamma\pi_\infty$ (p is the pressure from the perfect EOS, $\gamma = 3.5$, $\pi_\infty = 350.10^9$) is used. The initial density is $\rho = 7.82$, the initial pressure is $p_{SG} = 0$, the velocity is centripetal and $||\vec{V}|| = 2.10^5$. The radii are $R^+ = 10$ and $R^- = 9.5$. As vacuum is around ($p_{ext} = 0$), the ring is moving towards its center and in the same time is becoming thicker. Both phenomena should compensate each other and the average density on the ring should remain close to 7.82. Afterwards the compression will be much more higher and the average density will increase until the ring closes on itself. An unstructured mesh (51 points on circumferences of 11 concentrical circles, 2040 cells, 1071 mesh points) is used. The density on particles are presented for two times ($T_1 = 1.43\ 10^{-6}$ and $T_2 = 38.8\ 10^{-6}$) in figure 4. In figure 5 the average density on the ring is shown for the present method and the 1D cylindrical Godunov scheme with 100 cells. Numerical oscillations for the two schemes are due to the fact that acoustic waves are generated due to the boundary conditions at the very begining of the computation. Even if the mesh is unstructured the cylindricity of the phenomenom is respected.

6. CONCLUSION

In this paper a new lagrangian Discontinuous Galerkin type method on unstructured meshes has been presented. A non classical lagrangian formulation of the Euler equations is solved. Using Bernstein polynomials we are able to define (m)-diffusive polynomials which add some numerical diffusion to the scheme during the computation of the increment terms. We show that this numerical diffusion is sufficient to ensure the conservation of the positivity. An acoustic Riemann solver is used to treat the discontinuity at the interfaces and a Runge-Kutta (or implicit) method is used for time discretisation. Such a method defines a hierarchical class of numerical schemes depending on the degree of the Bernstein basis. The remeshing phase is treated with a particle method: this process is locally conservative and its accuracy can be adapted to the accuracy of the scheme. In 1D and 2D some test cases show the efficiency of this method to treat shock waves, contact discontinuities, expansion waves or cylindrical phenomena without the need for remeshing.

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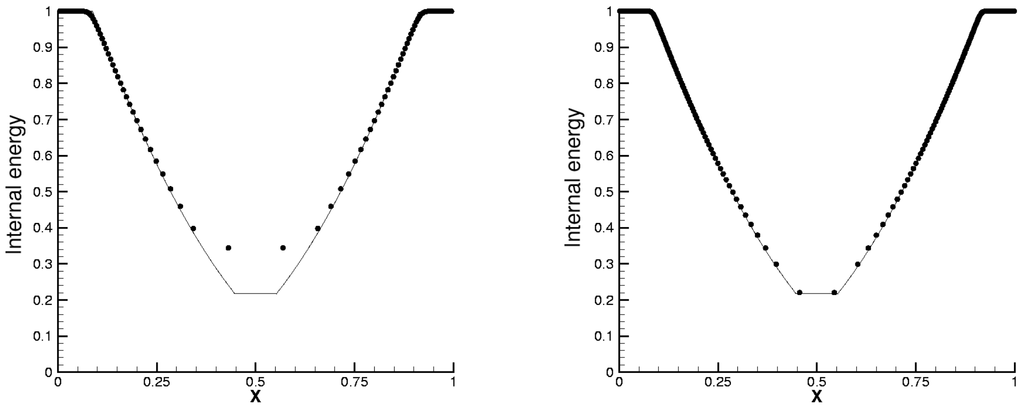

Figure 1. 123 problem - Internal energy given by the 1D method for $m = 2$ with 100 cells (left) and 250 cells (right) *vs* exact solution.

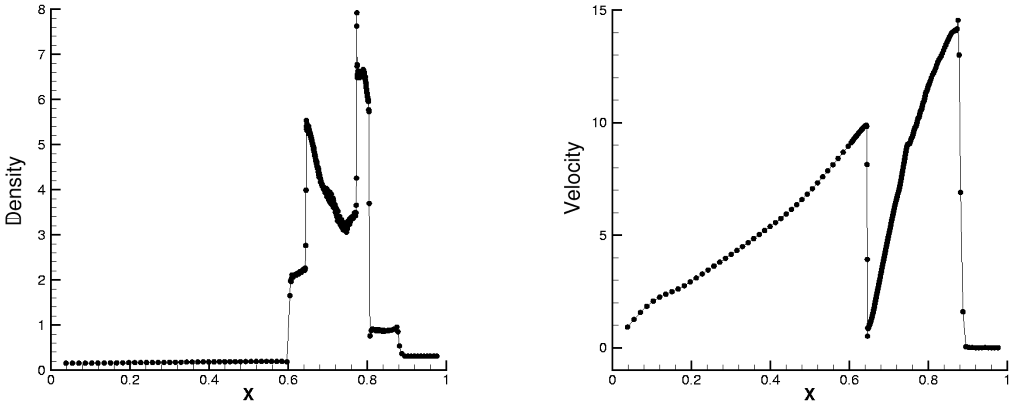

Figure 2. Blastwave - Density and velocity given by the 1D method for $m = 2$ with 400 cells. The line joins the average values between two adjacent cells.

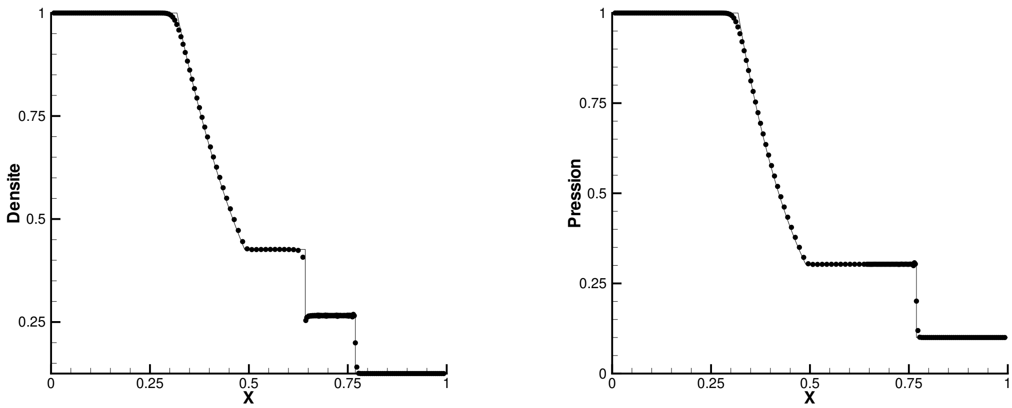

Figure 3. Sod tube - Density and pressure given by the 1D method for $m = 2$ with 200 cells

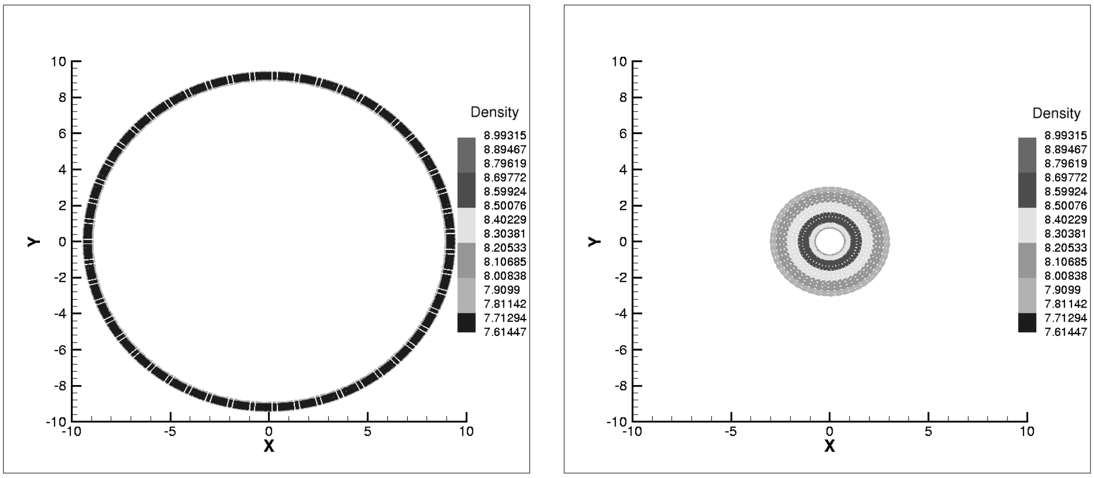

Figure 4. Isentropic compression of a ring - Density on particles given by the 2D method for $m = 2$ at $T_1 = 1.43\ 10^{-6}$ and $T_2 = 38.8\ 10^{-6}$

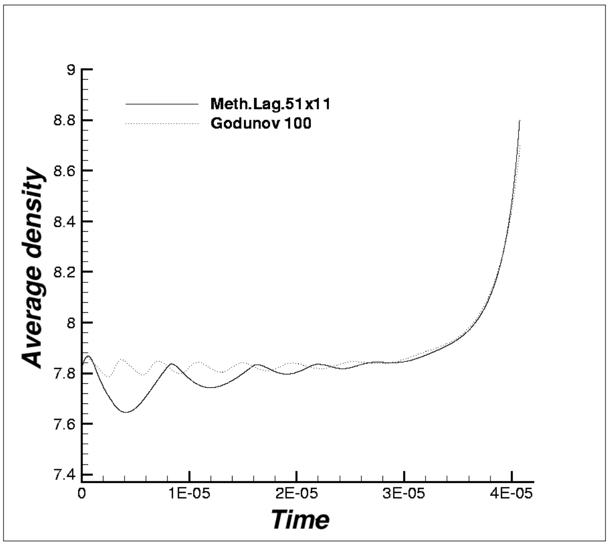

Figure 5. Isentropic compression of a ring - Time evolution of the average density on the ring (present method and 1D cylindrical Godunov scheme with 100 cells)