AN ENTROPY CONDITION AND GLOBAL EXISTENCE RESULT FOR HYPERBOLIC BALANCE LAWS

WEN-AN YONG

ABSTRACT. This paper presents a general result on the existence of global smooth solutions to hyperbolic systems of balance laws in several space variables. We propose an entropy condition and prove the existence of global smooth solutions for the systems with initial data close to constant equilibrium states where the entropy condition and Kawashima's condition are satisfied. In addition, we show that a system of balance laws satisfies Kawashima's condition if and only if so does its first-order approximation—a hyperbolic-parabolic system—derived from the Chapman-Enskog expansion. The result is applied to Bouchut's discrete velocity BGK models approximating hyperbolic systems of conservation laws.

1. INTRODUCTION

This work is concerned with initial value problems for nonlinear systems of balance laws in d space dimensions:

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + \sum_{j=1}^{d} F_j(u,v)_{x_j} = \begin{pmatrix} 0 \\ q(u,v) \end{pmatrix}. \tag{1.1}$$

Here u (resp. v) is an unknown $(n-r)$- (resp. r-) vector-valued function of $(x,t) \equiv (x_1, x_2, \cdots, x_d, t) \in \mathbf{R}^d \times [0, +\infty)$, $q(u,v)$ (resp. $F_j(u,v)$ $(j = 1, 2, \cdots, d)$) is a given r- (resp. n-) vector-valued smooth function of $U = (u,v) \in G$ (an open subset of $\mathbf{R}^n$ called *state space*). Below we often write $Q = Q(U)$ for the right-hand side of (1.1), i.e.

$$Q(U) = \begin{pmatrix} 0 \\ q(U) \end{pmatrix}.$$

Systems of the form (1.1) describe a large number of different physical phenomena involving various non-equilibrium processes. They also occur in numerics of conservation laws. See [1, 3, 6, 8, 11, 12, 15, 24] and references cited therein.

Our interest here is to investigate the existence of global smooth solutions to the above problem. Note that in the absence of the source term $Q(U)$, (1.1) reduces to conservation laws. In general, it is well-known that solutions develop singularities (shock waves) in finite time for nonlinear conservation laws even with smooth initial data [5]. Thus, the presence of source terms seems to be imperative for our purpose. Likewise, the $(n-r)$ conservation laws (u-equation) in (1.1) should not be decoupled from the v-equation. These and other simple considerations show that global smooth solutions can only be expected for certain properly structured balance laws.

Key words and phrases. Hyperbolic balance laws, entropy condition, global smooth solutions, Kawashima's condition, energy estimates, discrete velocity BGK models.

In this paper, we identify a class of balance laws (1.1) having unique global smooth solutions with initial data close to a constant equilibrium state. Here a constant vector $U_e \in G \subset \mathbf{R}^n$ is called *an equilibrium state* for balance laws (1.1) if $q(U_e) = 0$. One of our main results is a global existence theorem (Theorem 3.1), i.e. there exists a unique global smooth solution to (1.1) with initial data close to a constant equilibrium state U_e if the following four structural conditions are fulfilled:

(i). The Jacobian $q_v(U_e)$ is invertible.

(ii). There is a strictly convex smooth function $\eta(U)$, defined in a convex compact neighborhood G of U_e, such that $\eta_{UU}(U)F_{jU}(U)$ is symmetric for all $U \in G$ and all j.

(iii). There is a positive constant c_G, depending on the compact neighborhood G, such that for all $U \in G$,

$$[\eta_U(U) - \eta_U(U_e)]Q(U) \leq -c_G|Q(U)|^2.$$

(iv). The kernel of the Jacobian $Q_U(U_e)$, $\mathrm{Ker}\big(Q_U(U_e)\big)$, contains no eigenvectors of $\sum_j \omega_j F_{jU}(U_e)$ for each $\omega = (\omega_1, \omega_2, \cdots, \omega_d) \in \mathbf{S}^{d-1}$ (the unit sphere in $\mathbf{R}^d$).

Here and below, the subscripts $v, U, \cdots$ denote the corresponding partial derivatives with respect to these variables, $\eta_{UU}(U)$ is the Hessian matrix of $\eta(U)$, and $|Q|$ is the Euclidean norm of vector Q defined as

$$|Q| = (Q^*Q)^{1/2},$$

where the superscript * denotes the transpose. Note that Condition (i) simply excludes the case where $Q \equiv 0$ or (1.1) consists of conservation laws only.

The global existence theorem is proved with the standard continuation argument based on the local-in-time existence theory for symmetrizable hyperbolic systems (see, e.g. [13]) and a certain *a priori* estimate. The *a priori* estimate is derived with the energy method. In this derivation, we employ a *normal form* of (1.1) pertinent to the balance laws and a *basic* consequence of Conditions (i)–(iii) (Theorem 2.1). More importantly, we use the skew-symmetrizer theorem due to Shizuta and Kawashima [18] in the Fourier space.

Concerning the four conditions, we remark as follows. As a whole, (i)–(iii) is a new entropy condition for balance laws (1.1). Previously, three different entropy conditions were introduced by Chen, Levermore and Liu in [4], by Müller, Ruggeri and Boillat in [15, 2], and by the author in [22] with the aim of studying the zero relaxation limit. Indeed, that in [22] was shown therein to imply the first stability condition in [20] and therefore guarantees the desired existence of zero relaxation limit in the regime of smooth solutions. However, all the three earlier conditions seem to be too weak for the global existence. See [24] for a similar comment on Chen-Levermore-Liu's condition [4]. In addition, the entropy condition in [15, 2] is weaker than that in [4].

On the other hand, our new entropy condition is not too strong for applications. In fact, based on the facts revealed in [22], we show with Proposition 2.2 that the new entropy condition is admitted by many models from mathematical physics and numerics. These include discrete velocity models of the Boltzmann equation [11], Levermore's moment closure systems [12], Bouchut's discrete velocity BGK models approximating hyperbolic conservation laws [1, 3], and other individual models. In particular, the new entropy condition can be directly verified by taking $\eta(U)$ in (ii) and (iii) to be the entropy function $-s$ used in [24] (resp. η used in [6]) for the one-dimensional system of equations for gas dynamics in thermal non-equilibrium (resp. for the three-dimensional nonlinear Maxwell system from optics).

Besides the present global existence theory, the new entropy condition ensures several basic results asked by Müller and Ruggeri [15] concerning the balance laws (1.1) (see Chapter 1 in the book). Indeed, Theorem 2.1 shows that the new entropy condition is stronger than the second stability condition in [20, 22]. Thus, it guarantees the desirable relaxation limit [20, 21] (at least) for smooth solutions and provides a proper setting to study the existence and stability of relaxation shock profiles [23, 25, 14].

Condition (iv) was first formulated by Shizuta and Kawashima in [18] for symmetric hyperbolic-parabolic systems with source terms in order to obtain some decay estimates. It was explicitly spelt out by Zeng [24] for hyperbolic systems of balance laws and is referred to as "Kawashima's condition". Unlike these authors, we do not assume the symmetry of $\eta_{UU}(U_e)Q_U(U_e)$.

To see the nature of (iv), we recall that such a condition was proposed mainly for hyperbolic-parabolic systems. In his thesis [9] and joint works [19, 18, 10], Kawashima and his collaborators have verified that the condition is satisfied by many classical fluid flow models of hyperbolic-parabolic type like the Navier-Stokes equation. Nowadays, many people from different schools [12, 15] construct balance laws of the form (1.1) as improvements of the classical models and the Chapman-Enskog expansion has been used to connect the new and classical models.

Our another result (Theorem 2.4) says that, under the new entropy condition, a system of balance laws satisfies Kawashima's condition if and only if so does its first-order approximation (of hyperbolic-parabolic type) derived from the Chapman-Enskog expansion. For non-equilibrium statistical physics, this means that Condition (iv) should be respected when constructing the balance laws as improvements of the classical models. This result also indicates that Conditon (iv) is naturally satisfied by the models in [12, 15] which are genuine extensions of the classical models.

Furthermore, our existence result is applicable to many other models with a kinetic origin. In Section 5, we discuss Bouchut's discrete velocity BGK models [1, 3] and give a natural constraint on the discrete velocities such that the BGK models satsify our new entropy conditon and Condition (iv). In addition, as shown in [18, 11] and the references cited therein, many discrete velocity models of the Boltzmann equation satisfy Kawashima's condition.

However, unlike the entropy condition (i)–(iii), (iv) is not satisfied by some physical models. An example is the system of equations for gas dynamics in thermal non-equilibrium (see [24, 22]). However, a global existence theorem was established in Zeng [24] for this model. This indicates the unnecessity of Condition (iv) for the global existence. It would be interesting to weaken that condition while having a global existence theorem. This and the large-time behavior of solutions are under our current considerations.

The paper is organized as follows. In Section 2 we present some important algebraic consequences of Conditions (i)-(iv). Section 3 contains the exact statement of our global existence theorem and several calculus inequalities in Sobolev spaces. The *a priori* estimate is derived in Section 4. The last section deals with Bouchut's discrete velocity BGK models as an application of the global existence theory developed in this paper.

After the completion of this work mentioned in [22] (see page 281 and reference [59] therein), the author got to know that the global existence result and Kawashima's condition have attracted some attention [7, 16] recently. Precisely, Hanouzet and Natalini in [7] obtained a similar existence result for one-dimensional problems. By constructing an interesting Liapunov functional involving conditions of the form (i)–(iv), Ruggeri and Serre in [16] showed the time-asymptotic L^2-stability of constant equilibrium states by assuming the global existence of certain weak entropy solutions. Unlike these papers, ours is for multi-dimensional problems and contains Theorem 2.4 on the inherence of Kawashima's condition. Moreover, this paper present a concise and compact entropy condition (i)–(iii), which singles out a possibly meaningful class of partial differential equations.

2. On Structural Conditions

This section is devoted to some algebraic consequences of the structural conditions (i)–(iv) introduced in the previous section, which will be used later.

We start with Condition (ii), which is the classical Lax entropy condition [5] for hyperbolic systems of conservation laws. By the Poincaré lemma, $\eta_{UU}(U)F_{jU}(U)$ is symmetric if and only if there is a smooth function $\psi_j(U)$ such that

$$\eta_U F_{jU} = \psi_{jU} \tag{2.1}$$

provided that G is convex.

Considering the invertibility of $q_v(U_e)$ given by Condition (i), we introduce

$$V = \begin{pmatrix} u \\ q(U) \end{pmatrix} \equiv V(U) \tag{2.2}$$

and set $V_e = V(U_e)$. Since $q_v(U_e)$ is invertible, $V = V(U)$ has an inverse $U = U(V)$ for U close to U_e. In particular, v can be viewed as a function $v(u,q)$ of u and q. Set $A_j(V) = U_V^{-1}(V)F_{jU}(U(V))U_V(V)$. Then, for smooth solutions, (1.1) can be rewritten as

$$V_t + \sum_{j=1}^{d} A_j(V)V_{x_j} = V_U(U)Q. \tag{2.3}$$

This is the *normal form* of our balance laws. Note that the transformation (2.2) pertains to the balance laws (1.1) and does not destroy the conservation form of the u-equation therein. Moreover, we have

$$V_U(U)Q = \operatorname{diag}\big(0, q_v(U)\big)V. \tag{2.4}$$

The equivalent version (2.3) of (1.1) enables us to use the Chapman-Enskog expansion (or the Maxwell iteration [15]) easily to get a first-order approximation thereof. To do this, we partition an $n \times n$-matrix A as

$$A = \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix}$$

corresponding to the decomposition of V in (2.2). With this partition, we rewrite the q-equation in (2.3) as

$$q = \epsilon q_v^{-1}(U(V))\Big[q_t + \sum_j A_j^{21}(V)u_{x_j} + \sum_j A_j^{22}(V)q_{x_j}\Big]$$

with positive parameter

$$\epsilon = \begin{cases} = 1, & \text{for the present problem} \\ \ll 1, & \text{for the zero relaxation limit problem} \end{cases},$$

then *iterate* it once to obtain

$$q = \epsilon q_v^{-1}(U(u,0)) \sum_j A_j^{21}(u,0) u_{x_j} + O(\epsilon^2).$$

Substituting the truncation into the u-equation in (2.3), we arrive at

$$u_t + \sum_j A_j^{11}(u,0) u_{x_j} + \sum_{jk} A_j^{12}(u,0) \left[q_v^{-1}(U(u,0)) A_k^{21}(u,0) u_{x_k} \right]_{x_j} = 0. \tag{2.5}$$

Under Condition (iii), $[\eta_U(U) - \eta_U(U_e)]Q(U)$ achieves its maximum value at $U = U_e$. Thus, the Hessian matrix is non-positive at $U = U_e$, that is,

$$0 \geq \big([\eta_U - \eta_U(U_e)]Q\big)_{UU}(U_e) = \eta_{UU}(U_e) Q_U(U_e) + Q_U(U_e)^* \eta_{UU}(U_e). \tag{2.6}$$

Here we state the following *basic* theorem, which has been proved for the most part in [20] (see also [21]):

Theorem 2.1. *Assume (i), (ii) and (2.6). Then $U_V^* \eta_{UU} U_V$ evaluated at $U = U_e$ is a block-diagonal matrix (with the aforesaid partition) and there is a positive semidefinite $r \times r$-matrix $\Lambda(U_e)$ such that*

$$\eta_{UU}(U_e) Q_U(U_e) + Q_U^*(U_e) \eta_{UU}(U_e) = -V_U^*(U_e) \begin{pmatrix} 0 & 0 \\ 0 & \Lambda(U_e) \end{pmatrix} V_U(U_e). \tag{2.7}$$

Furthermore, if either $r = 1$ or $\eta_{UU}(U_e) Q_U(U_e)$ is symmetric or (iii) holds, then $\Lambda(U_e)$ is positive definite.

Proof. This is Theorem 2.2 in [21], apart from the conclusion that Condition (iii) implies the positive definiteness of $\Lambda(U_e)$. In order to show this new conclusion, we observe that

$$\begin{aligned} &2[\eta_U(U) - \eta_U(U_e)]Q(U) = 2[\eta_U(U(V)) - \eta_U(U(V_e))]Q \\ =&(V - V_e)^* U_V^*(V_e) \left[\eta_{UU}(U_e) Q_U(U_e) + Q_U^*(U_e) \eta_{UU}(U_e) \right] U_V(V_e)(V - V_e) + o(|V - V_e|^2) \\ =& -q^* \Lambda(U_e) q + o(|V - V_e|^2). \end{aligned}$$

Here the second equality is due to (2.7). Thus, it follows from Condition (iii) that

$$q^* \Lambda(U_e) q + o(|V - V_e|^2) \geq 2c_G |q|^2$$

for all $V = (u, q)$ close to $(u_e, 0)$. Hence $\Lambda(U_e) \geq 2c_G I_r$ and the proof is complete. $\square$

This theorem says that Conditions (i)–(iii) imply the second stability condition in [20, 21, 22] and therefore ensure the desired existence of zero relaxation limit. On the other hand, it was shown in [22] that many models from mathematical physics and numerics have the form (1.1) and admit (i), (ii), (2.7) with $\eta_{UU}(U_e) Q_U(U_e)$ being symmetric (hence $\Lambda(U_e) > 0$ due to the above theorem). Moreover, it was shown (but not spelt out explicitly) that for those models it holds $\eta_U(U_e) Q(U) \equiv 0$ for any U_e in equilibrium and for any U. The following proposition shows that those models fulfill Condition (iii), too.

Proposition 2.2. *Assume $\eta_U(U_e) Q(U) \equiv 0$ for any U_e in equilibrium and for any U. Then (iii) is implied by (i), (ii) and (2.7) with $\Lambda(U_e)$ being positive definite.*

Proof. From (2.7) we deduce that

$$\eta_{vv}(U_e)q_v^{-1}(U_e) + q_v^{-*}(U_e)\eta_{vv}(U_e) = -\Lambda(U_e) < 0. \tag{2.8}$$

On the other hand, recall that v can be expressed in terms of $V = (u, q)$, that is, $v = v(u, q)$ for (u, q) close to $(u_e, 0)$ and thereby $q_v^{-1} = v_q$. It follows from the assumption of this proposition and (2.8) that

$$\begin{aligned}
&[\eta_U(U) - \eta_U(U_e)]Q(U) = \eta_v(u, v(u, q))q = [\eta_v(u, v(u, q)) - \eta_v(u, v(u, 0))]q \\
&= q^*\Big(\int_0^1 \eta_{vv}(u, v(u, \theta q))v_q(u, \theta q)d\theta\Big)q \\
&= \frac{1}{2}q^*\Big(\int_0^1 \big[\eta_{vv}(u, v(u, \theta q))v_q(u, \theta q) + v_q^*(u, \theta q)\eta_{vv}(u, v(u, \theta q))\big]d\theta\Big)q \\
&\le -q^*\Lambda(U_e)q/4 \le -c|q|^2
\end{aligned}$$

with some $c > 0$ and for (u, q) close to $(u_e, 0)$. This gives Condition (iii). □

Remark 2.1. As a new entropy condition, (i)–(iii) is stronger than the third stability condition proposed in [22], which pertains to the zero relaxation limit and consists of the following three parts:

$$q_v(U_e) \textit{ has no non-zero imaginary eigenvalues}, \text{ (ii) and } [\eta_U(U) - \eta_U(U_e)]Q(U) \le 0.$$

As remarked in [22], the third stability condition can be viewed as a revised version of the entropy conditions both in [4] due to Chen, Levermore and Liu for balance laws (1.1) and in [15, 2] due to Boillat, Müller and Ruggeri for extended thermodynamics. However, the first part of the third stability condition is not included in these papers [4, 15, 2]. We mention that the first part is implied by (i) and (iii), but not by (i) and the third part.

Finally we turn to Condition (iv). Recall the derivation of (2.3). We see that Q_U and F_{jU} are simultaneously similar to $\mathrm{diag}(0, q_v)$ and A_j, respectively. Thus, (iv) is equivalent to

(iv)'. *Ker*$\big(diag(0, q_v(U_e))\big)$ *contains no eigenvectors of* $A(\omega) := \sum_j \omega_j A_j(V_e)$ *for each* ω.

On the other hand, it holds $\mathrm{Ker}\big(\mathrm{diag}(0, q_v(U_e))\big) = \mathbf{R}^{n-r} \times \{0\}$ thanks to (i). Thus, (iv) is equivalent to

(iv)". $A^{11}(\omega)$ *has no eigenvectors in Ker*$\big(A^{21}(\omega)\big)$ *for each* $\omega \in \mathbf{S}^{d-1}$.

With these preparations, we formulate the following skew-symmetrizer theorem due essentially to Shizuta and Kawashima [18].

Theorem 2.3. *(Shizuta-Kawashima) Assume (i) and (ii). Condition (iv) is equivalent to the existence of a constant* $c_S > 0$ *and a skew-symmetric real matrix* $K = K(\omega) \in C^\infty(\mathbf{S}^{d-1})$ *satisfying* $K(-\omega) = -K(\omega)$ *and*

$$K(\omega)A(\omega) - A^*(\omega)K(\omega) \ge 2c_S I_n - 2\,diag(0, I_r)$$

for each $\omega \in \mathbf{S}^{d-1}$. *Here* I_n *denotes the unit matrix of order* n.

Proof. Set $A_0 = U_V^*(V_e)\eta_{UU}(U_e)U_V(V_e)$ and note that $A_0A_j(V_e)$ is symmetric due to Condition (ii). We consider the degenerate linear parabolic system

$$A_0V_t + \sum_j A_0A_j(V_e)V_{x_j} = \mathrm{diag}(0, I_r)\sum_j V_{x_jx_j}.$$

With Condition (i), (iv) is equivalent to (iv)” as shown above. Moreover, (iv)” is Condition (3) of Theorem 1.1 in [18] for the above parabolic system. Thus, by the same theorem, (iv)” and hence (iv) is equivalent to the existence of an $n\times n$-matrix $\tilde{K} = \tilde{K}(\omega) \in C^\infty(\mathbf{S}^{d-1})$ with $\tilde{K}(-\omega) = -\tilde{K}(\omega)$, $\tilde{K}(\omega)A_0$ being skew-symmetric and the sum

$$\tilde{K}(\omega)A_0A(\omega) + [\tilde{K}(\omega)A_0A(\omega)]^* + 2\mathrm{diag}(0, I_r)$$

being positive definite. Thanks to the smoothness of $\tilde{K}(\omega)$ on the compact sphere $\mathbf{S}^{d-1}$, there is a positive constant c_S such that the sum is larger than $2c_SI_n$. Hence, with $K(\omega) = \tilde{K}(\omega)A_0$ the proof is complete. □

Remark 2.2. Condition (iv) is called “Kawashima’s condition” and was first formulated by Shizuta and Kawashima in [18] for symmetric hyperbolic-parabolic systems including (2.3) in order to obtain some decay estimates. Recently, this condition was explicitly spelt out in [24] for one-dimensional systems of the form (2.3) to study the global existence and asymptotic decay of solutions. Note that the aforementioned works both require the symmetry of $\eta_{UU}(U_e)Q_U(U_e)$ (see Theorem 2.1).

We conclude this section with a theorem from [22] on an equivalent statement of Condition (iv), which is related to the first-order approximation (2.5) of (1.1) derived by using the Chapman-Enskog expansion. From [18] we know that, for the hyperbolic-parabolic system (2.5) linearized at $u = u_e$, Kawashima’s condition is:

(*). *$A^{11}(\omega)$ has no eigenvectors in $Ker\big(A^{12}(\omega)q_v^{-1}(U_e)A^{21}(\omega)\big)$ for each $\omega \in \mathbf{S}^{d-1}$.*

The theorem reads as

Theorem 2.4. *Under the entropy condition (i)–(iii), a system of balance laws (1.1) satisfies Kawashima’s condition if and only if so does its first-order approximation (2.5).*

Proof. Since Condition (iv) is equivalent to (iv)” and

$$Ker\big(A^{21}(\omega)\big) \subset Ker\big(A^{12}(\omega)q_v^{-1}(U_e)A^{21}(\omega)\big),$$

it remains to show that $Ker\big(A^{21}(\omega)\big) \supset Ker\big(A^{12}(\omega)q_v^{-1}(U_e)A^{21}(\omega)\big)$. To do so, we recall from Theorem 2.1 that $A_0 := U_V^*(V_e)\eta_{UU}(U_e)U_V(V_e)$ is a positive definite block-diagonal matrix $\mathrm{diag}(A_0^{11}, A_0^{22})$ and note that $A_0A_j(V_e)$ is symmetric due to Condition (ii). Then we have $A_0^{11}A^{12}(\omega) = A^{21*}(\omega)A_0^{22}$ and $A_0^{22}q_v(U_e) + q_v^*(U_e)A_0^{22} = -\Lambda(U_e)$ due to (2.7). For

$y \in Ker\big(A^{12}(\omega)q_v^{-1}(U_e)A^{21}(\omega)\big)$, we have $A^{12}(\omega)q_v^{-1}(U_e)A^{21}(\omega)y = 0$ and

$$\begin{aligned}
0 =& y^* A_0^{11} A^{12}(\omega) q_v^{-1}(U_e) A^{21}(\omega) y = y^* A^{21*}(\omega) A_0^{22} q_v^{-1}(U_e) A^{21}(\omega) y \\
=& \frac{1}{2} y^* A^{21*}(\omega) q_v^{-*}(U_e) \big[q_v^*(U_e) A_0^{22} + A_0^{22} q_v(U_e)\big] q_v^{-1}(U_e) A^{21}(\omega) y \\
=& -\frac{1}{2}\big(q_v^{-1}(U_e) A^{21}(\omega) y\big)^* \Lambda(U_e) \big(q_v^{-1}(U_e) A^{21}(\omega) y\big) \\
\le& -c_G |q_v^{-1}(U_e) A^{21}(\omega) y|^2.
\end{aligned}$$

Hence, $q_v^{-1}(U_e)A^{21}(\omega)y = 0$ or $y \in Ker\big(A^{21}(\omega)\big)$ and the proof is complete. □

As a simple application of Theorem 2.4, we consider the well-known Jin-Xin model

$$\begin{aligned}
u_t + v_x &= 0, \\
v_t + a^2 u_x &= f(u) - v,
\end{aligned}$$

which was introduced in [8] for approximating conservation laws $u_t + f(u)_x = 0$. Here u, v and $f(u)$ are all r-vectors and a is a constant. This model can be viewed as an example of the discrete velocity BGK models constructed in [1, 3] and thus satisfies Conditions (i)-(iii) under the well-known subcharacteristic condition (see Section 5). Here we use Theorem 2.4 to show that it also satisfies (iv). To do so, we write down the first-order approximation (2.5) for the present model:

$$u_t + f(u)_x = \big(a^2 u_x - f_u(u) f(u)_x\big)_x.$$

This is a strictly parabolic system if the subcharacteristic condition $a^2 I_r > f_u^2(u)$ holds and therefore it admits Kawashima's condition (*). By Theorem 2.4, the one-dimensional Jin-Xin model satisfies Condition (iv).

3. A Global Existence Theorem

In this section we present our global existence theorem and recall several calculus inequalities in Sobolev spaces, which will be used in the next section as tools in deriving a key *a priori* estimate.

First of all, let us agree on some notations. L^2 is the space of square integrable (vector- or matrix-valued) functions on $\mathbf{R}^d$ and its norm is denoted by $\|\cdot\|$. In case of A being a function of another variable t as well as $x \in \mathbf{R}^d$, we write $\|A(t)\|$ or $\|A(\cdot, t)\|$ to remind the reader that the norm is computed with respect to x while t is viewed as a parameter. Similar notations will be adopted for the function spaces introduced below. For a non-negative integer s, the Sobolev space $H^s = H^s(\mathbf{R}^d)$ is defined as the space of functions whose distributional derivatives of order $\le s$ are all in L^2. We use $\|\cdot\|_s$ to denote the norm of H^s. Furthermore, $C\big(0, T; H^s\big)$ denotes the space of continuous functions on $[0, T]$ (or $[0, T)$ if $T = \infty$) with values in H^s, and ∇ the gradient operator with respect to x.

Our main result is

Theorem 3.1. *Let $s \ge s_0 + 1 \equiv [d/2] + 2$ be an integer and U_e a constant equilibrium state where Conditions (i)–(iv) hold. Then there are two constants c_1, c_2 such that if $U_0 = U_0(x) \in H^s(\mathbf{R}^d)$ satisfies*

$$\|U_0 - U_e\|_s \le c_1,$$

then the system of balance laws (1.1) with U_0 as its initial value has a unique global solution $U = U(x,t) \in C\big(0,\infty; H^s(\mathbf{R}^d)\big)$ satisfying

$$\|U(\cdot,T) - U_e\|_s^2 + \int_0^T \|Q(U)(\cdot,t)\|_s^2 dt + \int_0^T \|\nabla U(\cdot,t)\|_{s-1}^2 dt \le c_2\|U_0 - U_e\|_s^2 \tag{3.1}$$

for any $T > 0$.

To prove this theorem, we recall the well-known embedding inequality to obtain

$$|U_0(x) - U_e| \le C\|U_0 - U_e\|_{s_0} \le C\|U_0 - U_e\|_s \le Cc_1, \tag{3.2}$$

where C depends on d only. Therefore, $U_0 = U_0(x)$ takes values in a compact neighborhood of U_e if c_1 is sufficiently small. On the other hand, it follows from Condition (ii) that (1.1) is a symmetrizable hyperbolic system. Thus, from the local-in-time existence theory of H^s-solutions for such systems (see [5, 13]) we know that there exists $T > 0$, depending only on $\|U_0 - U_e\|_s$ and the compact neighborhood, such that (1.1) with U_0 as its initial value has a unique solution

$$U \in C(0,T;H^s).$$

Now we suppose the (*a priori*) estimate in (3.1) to be verified for all $T < \infty$, where $U \in C(0,T;H^s)$ is a solution. It follows from (3.1) that

$$\|U(\cdot,T) - U_e\|_s \le c_1\sqrt{c_2}$$

which is independent of T. Thus, by using the just mentioned local-in-time existence theory, the above solution can be easily continued up to $T = \infty$.

Thus, the proof of Theorem 3.1 is reduced to verification of the *a priori* estimate in (3.1). For this purpose, we need several calculus inequalities in Sobolev spaces (see, e.g., [13]). Their proofs can be found in [20] and references cited therein.

Lemma 3.2. *(calculus inequalities) Let s, s_1 and s_2 be three nonnegative integers.*

1. *If $s_3 = \min\{s_1, s_2, s_1 + s_2 - s_0\} \ge 0$, then $H^{s_1}H^{s_2} \subset H^{s_3}$. Here the inclusion symbol $\subset$ also means the continuity of the embedding.*
2. *Suppose $s \ge s_0 + 1$ and $\nabla A, U \in H^{s-1}$. Then for all multi-indices α with $|\alpha| \le s$, the commutator $[\partial^\alpha, A]U := \partial^\alpha(AU) - A\partial^\alpha U \in L^2$ and*
$$\|[\partial^\alpha, A]U\| \le C_s\|\nabla A\|_{s-1}\|U\|_{|\alpha|-1}.$$
3. *Suppose $s \ge s_0$, $V \in H^s$ with values in G, and $A \in C^s(G)$ with $A(0) = 0$. Then $A(V(\cdot)) \in H^s$ and*
$$\|A(V(\cdot))\|_s \le C_s|A|_s\|V\|_s(1 + \|V\|_s^{s-1}).$$

Here and below, C_s denotes a generic constant depending only on s and d, and $|A|_s$ stands for $\sup_{\{U\in G, 1\le|\alpha|\le s\}} |\partial_U^\alpha A(U)|$.

The *a priori* estimate (3.1) will be derived in the following section. We conclude this section by showing that the estimate implies

$$\sup_{x\in\mathbf{R}^d} \big|\partial^\alpha\big(U(x,t) - U_e\big)\big| \to 0 \qquad \text{as} \quad t \to \infty \tag{3.3}$$

for all multi-indices α with $|\alpha| \le s - s_0 - 1$. In fact, as a function of t, $\|\nabla U(\cdot,t)\|_{s-2}$ is uniformly bounded and square integrable over $t \ge 0$. Moreover, it is uniformly Lipschitz continuous, which can be easily deduced as follows. For $s \ge s_0 + 1$ it follows from Lemma 3.2 that

$$
\begin{aligned}
&\|\nabla Q(U)\|_{s-1} = \|Q_U(U)\nabla U\|_{s-1} \\
\le&\|Q_U(U_e)\nabla U\|_{s-1} + \|(Q_U(U) - Q_U(U_e))\nabla U\|_{s-1} \\
\le&C\|\nabla U\|_{s-1}\big(1 + \|Q_U(U) - Q_U(U_e)\|_{s-1}\big) \\
\le&C\|\nabla U\|_{s-1}\big(1 + |Q_U(U) - Q_U(U_e)|_{s-1}\|U - U_e\|_{s-1}(1 + \|U - U_e\|\|_{s-1}^{s-2})\big).
\end{aligned}
\tag{3.4}
$$

Here and below C denotes a generic constant. With (3.4), we deduced from the equation in (1.1) and the estimate in (3.1) that

$$
\begin{aligned}
&|f(t_2) - f(t_1)| \le \|\nabla U(t_2) - \nabla U(t_1)\|_{s-2} \\
\le&\left\| \int_{t_1}^{t_2} \nabla Q(U(\tau))d\tau - \int_{t_1}^{t_2} \sum_j \nabla F_j(U(\tau))_{x_j} d\tau \right\|_{s-2} \\
\le&|t_2 - t_1| \max_\tau \left(\|\nabla Q(U(\tau))\|_{s-2} + \sum_j \|\nabla F_j(U(\tau))\|_{s-1} \right) \\
\le&C|t_2 - t_1|.
\end{aligned}
$$

Thus, it follows that $\|\nabla U(\cdot, t)\|_{s-2} \to 0$ as $t \to \infty$ and the conclusion in (3.3) follows from the celebrated Gagliardo-Nirenberg inequality

$$
\sup_{x \in \mathbf{R}^d} |U(x,t) - U_e| \le C\|U(\cdot,t)\|^{1-d/(2s_0)} \|\nabla U(\cdot,t)\|_{s_0-1}^{d/(2s_0)}.
$$

4. An *a priori* Estimate

In this section, we derive the *a priori* estimate in (3.1). Thanks to the equivalence between U and V with $V = V(U)$ defined in (2.2), we will work with variable V for convenience.

We derive the L^2-estimate of the solution by using an equation for the entropy $\eta = \eta(U)$. Let U_e be an equilibrium state. We introduce

$$
E[U, U_e] = \eta(U) - \eta(U_e) - \eta_U(U_e)(U - U_e).
$$

Since η is strictly convex, $E[U, U_e]$ is equivalent to the quadratic function $|U - U_e|^2$ and hence to $|V - V(U_e)|^2$ for V close to $V_e = V(U_e)$. Namely, there is a positive constant C such that, for V close to V_e,

$$
C^{-1}|V - V_e|^2 \le E[U, U_e] \le C|V - V_e|^2. \tag{4.1}
$$

With ψ_j defined in (2.1), we compute from (1.1) the equation satisfied by $E = E[U, U_e]$:

$$
E_t + \sum_{j=1}^d \big[\psi_j(U) - \eta_U(U_e)F_j(U)\big]_{x_j} = [\eta_U(U) - \eta_U(U_e)]Q(U).
$$

Integration of this equation over $\mathbf{R}^d \times [0, T]$ yields

$$
\int_{\mathbf{R}^d} E[U(x,T), U_e]dx - \int_0^T \int_{\mathbf{R}^d} [\eta_U(U) - \eta_U(U_e)]Q(U)dxdt = \int_{\mathbf{R}^d} E[U_0, U_e]dx.
$$

Using (4.1) and Condition (iii) we arrive at

$$\|V(\cdot,T)-V_e\|^2+\int_0^T\|Q\|^2dt\le C\|V_0-V_e\|^2. \tag{4.2}$$

Next we show the L^2-estimate for the derivatives of the solution by using the normal form (2.3). Differentiating (2.3) with respect to ∂_x^α for an multi-index α with $1\le|\alpha|\le s$ yields

$$V_{\alpha t}+\sum_{j=1}^d A_j(V)V_{\alpha x_j}=V_U(U)Q_\alpha+[\partial_x^\alpha,V_U(U)]Q+\sum_{j=1}^d[A_j(V),\partial_x^\alpha]V_{x_j}.$$

Set $A_0(V)=U_V^*(V)\eta_{UU}(U(V))U_V(V)$ and note that $A_0(V)A_j(V)$ is symmetric due to Condition (ii). Multiplying the last equation by $2V_\alpha^*A_0(V)$ we obtain

$$\begin{aligned}&\big[V_\alpha^*A_0(V)V_\alpha\big]_t+\sum_{j=1}^d\big[V_\alpha^*A_0(V)A_j(V)V_\alpha\big]_{x_j}\\ =&2\mathrm{Re}V_\alpha^*A_0(V)V_U(U)Q_\alpha+2\mathrm{Re}V_\alpha^*A_0(V)[\partial_x^\alpha,V_U(U)]Q\\ &+2\mathrm{Re}\sum_j V_\alpha^*A_0(V)[A_j(V),\partial_x^\alpha]V_{x_j}+V_\alpha^*\Big[\partial_tA_0(V)+\sum_j\partial_{x_j}\big(A_0(V)A_j(V)\big)\Big]V_\alpha\end{aligned}$$

or

$$\begin{aligned}&\big[V_\alpha^*A_0(V)V_\alpha\big]_t+\sum_{j=1}^d\big[V_\alpha^*A_0(V)A_j(V)V_\alpha\big]_{x_j}\\ =&2\mathrm{Re}V_\alpha^*A_0(V_e)V_U(U_e)Q_\alpha+2\mathrm{Re}V_\alpha^*[A_0(V)V_U(U)-A_0(V_e)V_U(U_e)]Q_\alpha\\ &+2\mathrm{Re}V_\alpha^*A_0(V_e)[\partial_x^\alpha,V_U(U)]Q+2\mathrm{Re}V_\alpha^*[A_0(V)-A_0(V_e)][\partial_x^\alpha,V_U(U)]Q\\ &+2\mathrm{Re}\sum_j V_\alpha^*A_0(V)[A_j(V),\partial_x^\alpha]V_{x_j}+V_\alpha^*\Big[\partial_tA_0(V)+\sum_j\partial_{x_j}\big(A_0(V)A_j(V)\big)\Big]V_\alpha.\end{aligned} \tag{4.3}$$

To bound the terms in (4.3), we recall (2.4) and Theorem 2.1, which says that $A_0(V_e)$ is block-diagonal and

$$A_0(V_e)\mathrm{diag}(0,q_v(U_e))+\mathrm{diag}(0,q_v^*(U_e))A_0(V_e)\le-\mathrm{diag}(0,2cI_r)$$

for some positive constant $c>0$. Thus it is easy to see that

$$\begin{aligned}2\mathrm{Re}V_\alpha^*A_0(V_e)V_U(U_e)Q_\alpha=&2\mathrm{Re}Q_\alpha^*A_0(V_e)V_U(U_e)Q_\alpha\le-2c|Q_\alpha|^2,\\ 2\mathrm{Re}V_\alpha^*A_0(V_e)[\partial_x^\alpha,V_U(U)]Q=&2\mathrm{Re}Q_\alpha^*A_0(V_e)[\partial_x^\alpha,V_U(U)]Q\\ \le&c|Q_\alpha|^2+C|[\partial_x^\alpha,V_U(U)]Q|^2.\end{aligned}$$

In addition, for V close to V_e there is a constant C such that

$$\begin{aligned}C^{-1}I\le A_0(V)\le CI,&\\ 2\mathrm{Re}V_\alpha^*[A_0(V)V_U(U)-A_0(V_e)V_U(U_e)]Q_\alpha\le&C|V-V_e||V_\alpha|^2,\\ 2\mathrm{Re}V_\alpha^*[A_0(V)-A_0(V_e)][\partial_x^\alpha,V_U(U)]Q\le&C|V-V_e||V_\alpha|^2+C|[\partial_x^\alpha,V_U(U)]Q|^2,\\ \partial_tA_0(V)+\sum_j\partial_{x_j}\big(A_0(V)A_j(V)\big)\le&C\big(|V_t|+\sum_j|V_{x_j}|\big).\end{aligned}$$

Using these inequalities, we integrate (4.3) over $\mathbf{R}^d \times [0, T]$ to obtain

$$
\begin{aligned}
\|V_\alpha(T)\|^2 + \int_0^T \|Q_\alpha\|^2 dt \le{}& C\|V_{0\alpha}\|^2 + C\int_0^T \|[\partial_x^\alpha, V_U(U)]Q\|^2 dt \\
&+ C\sum_j \int_0^T \|V_\alpha\|\|[A_j(V), \partial_x^\alpha]V_{x_j}\| dt \\
&+ C\int_0^T \int_{\mathbf{R}^d} \Big\{|V - V_e| + |V_t| + \sum_j |V_{x_j}|\Big\}|V_\alpha|^2 dx dt.
\end{aligned}
\tag{4.4}
$$

Note that $s \ge s_0 + 1$. It follows from Lemma 3.2 that

$$
\begin{aligned}
\|[\partial_x^\alpha, V_U(U)]Q\| &\le C\|\nabla V_U\|_{s-1}\|Q\|_{|\alpha|-1}, \\
\|[A_j(V), \partial_x^\alpha]V_{x_j}\| &\le C\|\nabla A_j\|_{s-1}\|V_{x_j}\|_{|\alpha|-1}.
\end{aligned}
$$

Moreover, we deduce from equation (2.3) and $Q(U_e) = 0$ that

$$
|V - V_e| + |V_t| + \sum_j |V_{x_j}| \le |V - V_e| + C|Q| + C\sum_j |V_{x_j}| \le C\|V - V_e\|_s.
$$

Substituting these three inequalities into (4.4) yields

$$
\begin{aligned}
&\|V_\alpha(T)\|^2 + \int_0^T \|Q_\alpha\|^2 dt \\
\le{}& C\|V_{0\alpha}\|^2 + C\int_0^T \|\nabla V_U\|_{s-1}^2 \|Q\|_{|\alpha|-1}^2 dt \\
&+ C\sum_j \int_0^T \|\nabla A_j\|_{s-1}\|\nabla V\|_{|\alpha|-1}^2 dt + C\int_0^T \|V - V_e\|_s \|V_\alpha\|^2 dt \\
\le{}& C\|V_{0\alpha}\|^2 + CN_s(T)\int_0^T \big(\|\nabla V\|_{|\alpha|-1}^2 + \|Q\|_{|\alpha|-1}^2\big) dt.
\end{aligned}
\tag{4.5}
$$

Here

$$
\begin{aligned}
N_s(t) = \sup_{0\le\tau\le t}\Big(&\|V(\cdot, \tau) - V_e\|_s + \|\nabla V_U\|_{s-1}^2 \\
&+ \sum_j \|\nabla A_j\|_{s-1} + \sum_j \|A_j(V) - A_j(V_e)\|_{s-1}\Big).
\end{aligned}
\tag{4.6}
$$

Now we combine (4.2) with the sum of (4.5) for all α with $1 \le |\alpha| \le s$ to obtain

$$
\begin{aligned}
&\|V(T) - V_e\|_s^2 + \int_0^T \|Q\|_s^2 dt \\
\le{}& C\|V_0 - V_e\|_s^2 + CN_s(T)\int_0^T \big(\|\nabla V\|_{s-1}^2 + \|Q\|_{s-1}^2\big) dt.
\end{aligned}
\tag{4.7}
$$

To control the term $\int_0^T \|\nabla V\|_{s-1}^2 dt$ in (4.7), we rewrite equation (2.3) as

$$
W_t + \sum_j A_j(V_e)W_{x_j} = V_U(U)Q + \sum_j (A_j(V_e) - A_j(V))W_{x_j} \equiv V_U(U)Q + h
$$

with $W = V - V_e$. Application of the Fourier transform to the last equation yields

$$\hat{W}_t + i\sum_j \xi_j A_j(V_e)\hat{W} = \widehat{V_U(U)Q} + \hat{h}. \tag{4.8}$$

Here $\hat{W} = \hat{W}(\xi, t)$ denotes the Fourier transform of $W(x,t)$ with respect to x. Let $K = K(\frac{\xi}{|\xi|})$ be the skew-symmetric real matrix from Theorem 2.3, which says

$$K\sum_j \xi_j A_j(V_e) - \sum_j \xi_j A_j^*(V_e)K \geq 2|\xi|(c_S I - \mathrm{diag}(0, I_r)). \tag{4.9}$$

We multiply (4.8) with $-i\hat{W}^*K$ to obtain

$$-i\hat{W}^*K\hat{W}_t + \hat{W}^*K\sum_j \xi_j A_j(V_e)\hat{W} = -i\hat{W}^*K[\widehat{V_U(U)Q} + \hat{h}].$$

Since

$$2\mathrm{Im}\hat{W}^*K[\widehat{V_U(U)Q} + \hat{h}] \leq c_S|\xi||\hat{W}|^2 + C|\xi|^{-1}(|\widehat{V_U(U)Q}|^2 + |\hat{h}|^2),$$

it follows from (4.9) and the last equation that

$$-i\big(\hat{W}^*K\hat{W}\big)_t + c_S|\xi||\hat{W}|^2 \leq 2|\xi||\hat{q}|^2 + C|\xi|^{-1}(|\widehat{V_U(U)Q}|^2 + |\hat{h}|^2).$$

Moreover, for $k \geq 1$ we have

$$-i|\xi|^{2k-1}\big(\hat{W}^*K\hat{W}\big)_t + c_S|\xi|^{2k}|\hat{W}|^2 \leq 2|\xi|^{2k}|\hat{q}|^2 + C|\xi|^{2k-2}(|\widehat{V_U(U)Q}|^2 + |\hat{h}|^2).$$

Integrating this over $(\xi, t) \in \mathbf{R}^d \times [0, T]$ yields

$$\begin{aligned} c_S\int_0^T\int |\xi|^{2k}|\hat{W}|^2 d\xi dt \leq & 2\int_0^T\int |\xi|^{2k}|\hat{q}|^2 d\xi dt + C\int |\xi|^{2k-1}|\hat{W}(\xi, T)|^2 d\xi \\ & + C\int |\xi|^{2k-1}|\hat{W}_0|^2 d\xi + C\int_0^T\int |\xi|^{2k-2}(|\hat{h}|^2 + |\widehat{V_U(U)Q}|^2)d\xi dt. \end{aligned}$$

Since $2|\xi| \leq 1 + |\xi|^2$ and $k \geq 1$, we get

$$\begin{aligned} c_S\int_0^T \sum_{|\alpha|=k} \|W_\alpha(\cdot, t)\|^2 dt \leq & 2\int_0^T \sum_{|\alpha|=k} \|q_\alpha(\cdot, t)\|^2 dt + C\|W(\cdot, T)\|_k^2 + C\|W_0\|_k^2 \\ & + C\int_0^T \sum_{|\alpha|=k-1} (\|h_\alpha\|^2 + \|\big(V_U(U)Q\big)_\alpha\|^2)dt. \end{aligned}$$

Moreover, we sum up this inequality for all α with $1 \leq |\alpha| \leq s$ to obtain

$$\begin{aligned} c_S\int_0^T \|\nabla V\|_{s-1}^2 dt \leq & 2\int_0^T \|\nabla_x q\|_{s-1}^2 dt + C\|V(\cdot, T) - V_e\|_s^2 \\ & + C\|V_0 - V_e\|_s^2 + C\int_0^T (\|h\|_{s-1}^2 + \|V_U(U)Q\|_{s-1}^2)dt. \end{aligned} \tag{4.10}$$

On the other hand, we recall that $h = \sum_j (A_j(V_e) - A_j(V))V_{x_j}$. It follows from Lemma 3.2 that

$$\begin{aligned}
\|h\|_{s-1} \le & C\sum_j \|A_j(V_e) - A_j(V)\|_{s-1}\|V_{x_j}\|_{s-1} \le CN_s(T)\|\nabla V\|_{s-1}, \\
\|V_U(U)Q\|_{s-1} \le & \|V_U(U_e)Q\|_{s-1} + \|[V_U(U) - V_U(U_e)]Q\|_{s-1} \\
\le & C(1 + \|V_U(U) - V_U(U_e)\|_{s-1})\|Q\|_{s-1} \\
\le & C(1 + \|V - V_e\|_{s-1} + \|V - V_e\|_{s-1}^{s-1})\|Q\|_{s-1}.
\end{aligned}$$

Similarly, $N_s(T)$ defined in (4.6) can be estimated as

$$N_s(T) \le C\|V - V_e\|_s(1 + \|V - V_e\|_s^{2s-1}).$$

Thus, in case of $N_s(T) \le C$ we arrive at

$$\begin{aligned}
c_S\int_0^T \|\nabla V\|_{s-1}^2 dt \le & C\int_0^T \|q\|_s^2 dt + C\|V(\cdot,T) - V_e\|_s^2 \\
& + C\|V_0 - V_e\|_s^2 + CN_s(T)\int_0^T \|\nabla V\|_{s-1}^2 dt.
\end{aligned} \tag{4.11}$$

Combining (4.7) with (4.11) gives

$$\begin{aligned}
& \|V(T) - V_e\|_s^2 + \int_0^T \|q\|_s^2 dt + \int_0^T \|\nabla V\|_{s-1}^2 dt \\
\le & C\|V_0 - V_e\|_s^2 + CN_s(T)\int_0^T \big(\|\nabla V\|_{s-1}^2 + \|q\|_{s-1}^2\big) dt.
\end{aligned}$$

Thus, in case of $N_s(T) \le 1/(2C)$, we get

$$\|V(T) - V_e\|_s^2 + \int_0^T \|q\|_s^2 dt + \int_0^T \|\nabla V\|_{s-1}^2 dt \le C\|V_0 - V_e\|_s^2.$$

This is the *a priori* estimate in (3.1) and the proof is complete.

5. Discrete Velocity BGK Models

In this section we apply our global existence theorem (Theorem 3.1) to discrete velocity BGK models from [1, 3] approximating hyperbolic systems of conservation laws:

$$u_t + \sum_{j=1}^d g_j(u)_{x_j} = 0, \tag{5.1}$$

where $u \in \mathcal{U} \subset \mathbf{R}^p$ (a convex open set) is unkown and g_j maps $\mathcal{U}$ to $\mathbf{R}^p$. As in [3], we assume that this system of conservation laws is endowed with a strictly convex entropy function $\tilde\eta = \tilde\eta(u)$, i.e. the $p \times p$-matrix $\tilde\eta_{uu}(u)g_{ju}(u)$ is symmetric for each j and each $u \in \mathcal{U}$.

The BGK models constructed in [1, 3] are of the form

$$f_{kt} + a(k)\cdot\nabla_x f_k = M_k(u) - f_k \tag{5.2}$$

for $k \in \Xi := \{1, 2, \cdots, N\}$. Here $f_k = f_k(x, t) \in \mathbf{R}^p$ is unknown, $a(k) \in \mathbf{R}^d$ is a constant vector with components $a_j(k)$ $(j = 1, 2, \cdots, d)$, $a(k) \cdot \nabla_x = \sum_{j=1}^d a_j(k)\partial_{x_j}$, $u = \sum_{k=1}^N f_k$, and $M_k : \mathcal{U} \to \mathbf{R}^p$ satisfies the consistency relations

$$\sum_{k=1}^{N} M_k(u) = u, \qquad \sum_{k=1}^{N} a_j(k) M_k(u) = g_j(u), \qquad \forall u \in \mathcal{U}. \tag{5.3}$$

In order for $M_k(u)$ to be well defined, the state space is taken to be

$$G = \Big\{U = (f_1, f_2, \cdots, f_N) \in \mathbf{R}^{Np} : \ \sum_{k=1}^{N} f_k \in \mathcal{U}\Big\}.$$

Note that G is convex and open if so is $\mathcal{U}$.

Clearly, a BGK model of the form (5.2) is determined by choosing the $a(k)$'s and $M_k(u)$'s so that the consistency relations in (5.3) hold. We mention that, in [3], Ξ is not necessarily a finite set. However, since the concerned balance laws (1.1) consist of a finite numbers of equations, we consider only the case where Ξ is finite.

Using the first consistency relation in (5.3) and $v = (f_2, f_3, \cdots, f_N)$, we can rewrite BGK models (5.2) as the form (1.1) with $n = Np$ and $r = n - p$. In doing so, we see immediately that $q_v = -I_r$ and therefore Condition (i) is satisfied.

Our next lemma visualizes Condition (iv) for the present BGK models. Note that the visualization implies $N \geq d + 1$ and is closely related to the solvability of the $M_k(u)$'s from the consistency relations in (5.3).

Lemma 5.1. *Assume $M_{ku}(u_e)$ is invertible for each k. Then the BGK model (5.2)-(5.3) satisfies Condition (iv) if and only if the $d \times (N-1)$-matrix*

$$\big[a(2) - a(1), a(3) - a(1), \cdots, a(N) - a(1)\big]$$

is of rank d.

Proof. Assume the $d \times (N-1)$-matrix is of rank d. For each $\omega \in \mathbf{S}^{d-1}$, consider the characteristic matrix

$$diag(a(1) \cdot \omega I_p, a(2) \cdot \omega I_p, \cdots, a(N) \cdot \omega I_p).$$

It is not difficult to see that the $a(k) \cdot \omega$'s take at least two distinct values. Thus, for each right eigenvector $(y_1, y_2, \cdots, y_N)^*$ with y_k a p-dimensional row vector of the characteristic matrix there is a k_0 such that $y_{k_0} = 0$. On the other hand, assume

$$Q_U(U_e)(y_1, y_2, \cdots, y_N)^* = 0.$$

It is easy to see that $M_{ku}(u_e)\sum_j y_j = y_k$ for each k. Since $M_{k_0u}(u_e)$ is invertible, we have $\sum_j y_j = M_{k_0u}^{-1}(u_e) y_{k_0} = 0$ and thus $y_k = M_{ku}(u_e)\sum_j y_j = 0$ for each k. Hence the characteristic matrix has no eigenvectors in $\mathrm{Ker}\big(Q_U(U_e)\big)$.

With the above argument, the opposite becomes obvious and hence the proof is complete. $\square$

To verify Conditions (ii) and (iii), we follow Bouchut [3] to choose $M_k(u)$ so that $\tilde{\eta}_{uu}(u) M_{ku}(u)$ is symmetric positive definite. A simple way to obtain such M_k is to take linear combinations of u and the $g_j(u)$'s:

$$M_k(u) = \alpha_{k0} u + \sum_{j=1}^{d} \alpha_{kj} g_j(u).$$

The coefficients $\alpha_{kj}(j = 0, 1, \cdots, d)$ are determined so that the consistency relations in (5.3) hold and the matrix $\alpha_{k0} I_p + \sum_{j=1}^{d} \alpha_{kj} g_{ju}(u)$ has only positive eigenvalues. The reader is recommended to derive the one-dimensional Jin-Xin model in Section 2 and the subcharacteristic condition in this way.

With the above choice of the $M_k(u)$'s, Condition (ii) was verified in [3, 17] by constructing a strictly convex entropy $\eta = \eta(U)$ for the BGK models. The following theorem shows that Condition (iii) holds with the same convex function.

Theorem 5.2. *If each $\tilde{\eta}_{uu}(u) M_{ku}(u)$ is symmetric positive definite in a convex neighborhood $\mathcal{U}$ of u_e, then the BGK models in (5.2)-(5.3) satisfy Conditions (ii) and (iii).*

Note that, in the present setting, we do not require $M_k(u)$ to have an inverse, since only a neighborhood of u_e is involved.

Proof. Since each $\tilde{\eta}_{uu}(u) M_{ku}(u)$ is symmetric and $\mathcal{U}$ is convex, the Poincaré lemma ensures the existence of a smooth scalar function $H_k(u)$ defined for $u \in \mathcal{U}$ such that

$$H_{ku}(u) = \tilde{\eta}_u(u) M_{ku}(u).$$

On the other hand, the positiveness of $\tilde{\eta}_{uu}(u) M_{ku}(u)$ implies the invertibility of $M_{ku}(u)$. Thus, it follows from the implicit function theorem that $M_k = M_k(u)$ has an inverse, say $V_k = V_k(M_k) = u$, in a convex neighborhood $\mathcal{U}$ of u_e. Define

$$\eta(U) = \sum_{k=1}^{N} H_k(V_k(f_k)). \tag{5.4}$$

Since $M_k(V_k(f_k)) = f_k$ and thereby $M_{ku}(V_k(f_k)) V_{kf_k}(f_k) = I_p$, we see that

$$\eta_{f_k}(U) = H_{ku}(V_k(f_k)) V_{kf_k}(f_k) = \tilde{\eta}_u(V_k(f_k)) M_{ku}(V_k(f_k)) V_{kf_k}(f_k) = \tilde{\eta}_u(V_k(f_k))$$

and

$$\eta_{UU} = \mathrm{diag}(\tilde{\eta}_{uu}(V_1(f_1)) V_{1f_1}(f_1), \tilde{\eta}_{uu}(V_2(f_2)) V_{2f_2}(f_2), \cdots, \tilde{\eta}_{uu}(V_N(f_N)) V_{Nf_N}(f_N))$$

is a symmetrizer for the diagonal hyperbolic systems in (5.2). Note that, in equilibrium, $f_k = M_k(u), V_k(M_k(u)) = u$, $\eta_{f_k}(U) = \tilde{\eta}_u(u)$, and

$$\eta_{UU} = \mathrm{diag}(\tilde{\eta}_{uu}(u) M_{1u}^{-1}(u), \tilde{\eta}_{uu}(u) M_{2u}^{-1}(u), \cdots, \tilde{\eta}_{uu}(u)(M_{Nu}^{-1}(u))$$

is positive definite. Thus, (ii) is verified with $\eta(U)$ defined in (5.4) and G a small neighborhood of U_e.

Furthermore, for any $U \in G$ and U_e with $Q(U_e) = 0$, it follows from the first consistency relation in (5.3) that

$$\eta_U(U_e) Q_U(U) = \sum_{k=1}^{N} \eta_{f_k}(U_e)(M_k(u) - f_k) = \sum_{k=1}^{N} \eta_u(u_e)(M_k(u) - f_k) \equiv 0.$$

Moreover, from (5.3) and the strict convexity of $\tilde\eta$ we deduce that

$$\begin{aligned}[\eta_U(U) - \eta_U(U_e)] \cdot Q(U) &= \sum_{k=1}^{N} \eta_{f_k}(U)(M_k(u) - f_k) \\ &= \sum_{k=1}^{N} [\tilde\eta_u(V_k(f_k)) - \tilde\eta_u(u)](M_k(u) - f_k) \\ &= \sum_{k=1}^{N} [\tilde\eta_u(V_k(f_k)) - \tilde\eta_u(V_k(M_k(u)))](M_k(u) - f_k) \\ &\leq -c \sum_{k=1}^{N} |M_k(u) - f_k|^2\end{aligned}$$

in a neighborhood of U_e and for some $c > 0$. This completes the proof. □

In conclusion, the global existence theorem (Theorem 3.1) applies to the BGK models in (5.2)-(5.3), provided that

(a). the approximated conservation laws in (5.1) are endowed with a strictly convex entropy function $\tilde\eta = \tilde\eta(u)$;
(b). the $M_k(u)$'s are chosen so that each $\tilde\eta_{uu}(u)M_{ku}(u)$ is symmetric positive definite;
(c). the $a(k)$'s are chosen so that the following $d \times (N-1)$-matrix

$$\big[a(2) - a(1), a(3) - a(1), \cdots, a(N) - a(1)\big]$$

is of rank d.

ACKNOWLEDGEMENTS

The author owes thanks to Alexander Dressel for useful discussions and for carefully reading draft vesions of this paper. This work was supported by the Deutsche Forschungsgemeinschaft through the Schwerpunktprogramm ANumE and SFB 359 at the University of Heidelberg, and by the European TMR-Network “Hyperbolic and Kinetic Equations”.

References

[1] D. Aregba-Driollet & R. Natalini, *Discrete kinetic schemes for multidimensional systems of conservation laws*, SIAM J. Numer. Anal. **37** (2000), 1973-2004.
[2] G. Boillat & T. Ruggeri, *Hyperbolic principal subsystems: entropy convexity and subcharacteristic conditions*, Arch. Rat. Mech. Anal. **137** (1997), 305-320.
[3] F. Bouchut, *Construction of BGK models with a family of kinetic entropies for a given system of conservation laws*, J. Statist. Phys. **95** (1999), 113-170.
[4] G.-Q. Chen & C. D. Levermore & T.-P. Liu, *Hyperbolic conservation laws with stiff relaxation terms and entropy*, Comm. Pure Appl. Math. **47** (1994), 787-830.
[5] C. M. Dafermos, *Hyperbolic conservation laws in continuum physics*, Springer, Berlin, 2000.
[6] B. Hanouzet & P. Huynh, *Approximation par relaxation d'un système de Maxwell non linéaire*, C. R. Acad. Sci., Paris, Sér. I, Math. **330** (2000), 193-198.

[7] B. Hanouzet & R. Natalini, *Global existence of smooth solutions for partially dissipative hyperbolic systems with a convex entropy*, to appear in Arch. Rat. Mech. Anal.

[8] S. Jin & Z. Xin, *The relaxation schemes for systems of conservation laws in arbitrary space dimensions*, Comm. Pure Appl. Math. **48** (1995), 235-276.

[9] S. Kawashima, *Systems of a hyperbolic-parabolic composite type, with applications to the equations of magnetohydrodynamics*, Ph.D. Thesis, Kyoto University, 1984.

[10] S. Kawashima, *Large-time behaviour of solutions to hyperbolic-parabolic systems of conservation laws and applications*, Proc. R. Soc. Edinb., Sect. **A 106** (1987), 169-194.

[11] S. Kawashima, *Asymptotic stability of Maxwellians of the discrete Boltzmann equation*, Transp. Theory Stat. Phys. **16** (1987), 781-793.

[12] C. D. Levermore, *Moment closure hierarchies for kinetic theories*, J. Statist. Phys. **83** (1996), 1021-1065.

[13] A. Majda, *Compressible fluid flow and systems of conservation laws in several space variables*, Springer, New York, 1984.

[14] C. Mascia & K. Zumbrun, *Pointwise Green's function bounds and stability of relaxation shocks*, Preprint (2001).

[15] I. Müller & T. Ruggeri, *Rational extended thermodynamics*, Springer, New York, 1998.

[16] T. Ruggeri & D. Serre, *Stability of constant equilibrium state for dissipative balance laws system with a convex entropy*, to appear in Q. Appl. Math.

[17] D. Serre, *Relaxation semi-linéaire et cinétique des systèmes de lois de conservation*, Ann. Inst. H. Poincaré Anal. Non Linéaire **17** (2000), 169-192.

[18] Y. Shizuta & S. Kawashima, *Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation*, Hokkaido Math. J. **14** (1985), 249-275.

[19] T. Umeda & S. Kawashima & Y. Shizuta, *On the decay of solutions to the linearized equations of electro-magneto- fluid dynamics*, Japan J. Appl. Math. **1** (1984), 435-457.

[20] W.-A. Yong, *Singular perturbations of first-order hyperbolic systems*, Ph.D. Thesis, Universität Heidelberg, 1992.

[21] W.-A. Yong, *Singular perturbations of first-order hyperbolic systems with stiff source terms*, J. Differ. Equations **155** (1999), 89-132.

[22] W.-A. Yong, *Basic aspects of hyperbolic relaxation systems*, in "Advances in the Theory of Shock Waves", ed. H. Freistühler et al, Progress in Nonlinear Differential Equations and Their Applications 47, Birkhäuser, Boston, 2001, 259-305.

[23] W.-A. Yong & K. Zumbrun, *Existence of relaxation shock profiles for hyperbolic conservation laws*, SIAM J. Appl. Math. **60** (2000), 1565-1575.

[24] Y. Zeng, *Gas dynamics in thermal nonequilibrium and general hyperbolic systems with relaxation*, Arch. Rat. Mech. Anal. **150** (1999), 225-279.

[25] K. Zumbrun, *Multidimensional stability of planar viscous shock waves*, in "Advances in the Theory of Shock Waves", ed. H. Freistühler et al, Progress in Nonlinear Differential Equations and Their Applications 47, Birkhäuser, Boston, 2001, 307-516.

IWR, Universität Heidelberg, Im Neuenheimer Feld 294, 69120 Heidelberg, Germany
E-mail address: yong.wen-an@iwr.uni-heidelberg.de