LONG TIME BEHAVIOR OF SOLUTIONS TO THE 3D COMPRESSIBLE EULER EQUATIONS WITH DAMPING

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ABSTRACT. The effect of damping on the large-time behavior of solutions to the Cauchy problem for the three-dimensional compressible Euler equations is studied. It is proved that damping prevents the development of singularities in small amplitude classical solutions, using an equivalent reformulation of the Cauchy problem to obtain effective energy estimates. The full solution relaxes in the maximum norm to the constant background state at a rate of $t^{-3/2}$. While the fluid vorticity decays to zero exponentially fast in time, the full solution does not decay exponentially. Formation of singularities is also exhibited for large data.

1. INTRODUCTION

Compressible inviscid flow is governed by the Euler equations, [3, 17], the main feature of which is the development of shock waves in finite time for solutions with general initial data. This paper explores the influence of damping on the development of singularities in classical solutions of the three-dimensional compressible Euler equations. It will be shown that damping prevents the formation of singularities in small amplitude flows, but large solutions may still break down.

Shock wave formation in the undamped case is best understood in one space dimension where the method of characteristics can be successfully employed, see Courant-Friedrichs [6] and Whitham [34]. For the mathematical analysis of finite-time formation of singularities and long-time behavior of solutions of the multi-dimensional Euler equations, see Sideris [29, 30, 31], as well as Makino-Ukai-Kawashima [22], Rammaha [28], Alinhac [1, 2], Chemin [5], and the references cited therein. For the development of singularities in general systems of one-dimensional conservation laws, see [18, 15, 20, 8].

1991 *Mathematics Subject Classification.* 35L65, 76N15.

Key words and phrases. Euler equations, damping, global smooth solutions, existence, decay, singularities.

With damping, the three-dimensional compressible Euler equations for isentropic flows have the following form:

$$\begin{cases} \partial_t \rho + \nabla \cdot \rho v = 0, \\ \partial_t(\rho v) + \nabla \cdot (\rho v \otimes v) + \nabla p + a\rho v = 0, \end{cases} \tag{1.1}$$

where $\rho(x,t) \in \mathbb{R}$, $v(x,t) \in \mathbb{R}^3$ represent the density, velocity of the flow, respectively; $x \in \mathbb{R}^3$ is the space variable, $t > 0$ is the time variable; the pressure p satisfies the γ-law:

$$p = p(\rho) = A\rho^\gamma,$$

with $\gamma > 1$ the adiabatic exponent, $A > 0$ a constant; $a > 0$ is the damping constant and $1/a$ may be regarded as the relaxation time for some physical flows. In this paper we investigate the Cauchy problem of three-dimensional Euler equations (1.1) with the initial condition:

$$(\rho, v)|_{t=0} = (\rho_0, v_0). \tag{1.2}$$

We are interested in the damping effect on the regularity and large-time behavior of smooth solutions. It will be proved that the size of the smooth initial data (relative to the damping coefficient) plays the key role. If the initial data are small in an appropriate norm, then damping can prevent the development of singularities and the Cauchy problem has a unique global smooth solution which decays in the maximum norm to the background state at a rate of $t^{-3/2}$. Similar results have been previously obtained by W. Wang and T. Yang [33], but we regard our approach to be simpler. It will also be shown that the vorticity converges to zero exponentially and that the density goes to the background no faster than $t^{-3/2}$. Finally, if the initial data are large, it will be proved that the damping is not strong enough to prevent the formation of singularities in finite time, even though the initial data are smooth.

For the one-dimensional Euler equations with damping, the global existence of a smooth solution with small data was proved by Nishida [25, 26], and the behavior of the smooth solution was studied in many papers; see the excellent survey paper by Dafermos [7], the book by Hsiao [10], the papers [11, 12, 13, 14, 32, 36], and their references. For the existence of global smooth solution to general hyperbolic systems with weakly linear degeneracy, see the book [19].

In this paper we consider the multi-dimensional case which has much richer phenomena than the one-dimensional case. The method of characteristics often plays a crucial role in the analysis of one-dimensional problems, but it is of little use here. Instead, we will fall back on the method of energy estimates for symmetric hyperbolic systems. We do not work directly on the original Euler equations (1.1), however. It is much easier to first symmetrize the system by introducing the sound speed as a new dependent variable rather than the density. This reformulation of the system is valid for C^1 flows with strictly positive density. Although general weak solutions may cavitate (see [6]), for a C^1 flow, the density will remain positive if it is initially. The local existence and uniqueness of H^3 solution can be established by following the methods in Kato [16] or Majda [21]. To prove global existence of a smooth solution with small initial data, we establish global *a priori* estimates of the solution.

Using estimates for the linearized equation, we obtain the decay of the solution to the nonlinear problem in L^∞ at a rate of $t^{-3/2}$, as in the case of a diffusion equation. It is also shown that the deviation of the sound speed from its background state can not decay exponentially fast. However, we show that the vorticity decays exponentially fast to zero.

The system under consideration is an example of a hyperbolic relaxation system, the general study of which has received considerable attention, see [4, 24, 35], for example.

The break down of smooth solutions with large initial data is also demonstrated, using an adaptation of a method given in [29] for the undamped case. The argument depends on the finite propagation of compactly supported disturbances. This property holds for the damped system, by the usual local energy methods.

We organize the paper as follows. In Section 2, we reformulate the Cauchy problem for (1.1) into a symmetric hyperbolic system and discuss the positivity of the density. In Section 3, we present the local existence result and prove the finite speed of propagation. In Section 4, we establish the major energy estimates which are then used in Section 5 to prove global existence. In Section 6, we prove the algebraic decay of the smooth solution and the exponential decay of the vorticity, Finally in Section 7, we present identities and inequalitites that show that the full solution does not decay exponentially and that in the case of large data singularities may develop.

2. Reformulation of the Problem

In this section, we are going to reformulate the Cauchy problem of the compressible Euler system (1.1) with the initial condition (1.2). The main point is to obtain a symmetric system. Introduce the sound speed

$$\sigma(\rho) = \sqrt{p'(\rho)},$$

and set $\bar{\sigma} = \sigma(\bar{\rho})$ corresponding to the sound speed at a background density $\bar{\rho} > 0$. Define

$$u = \frac{2}{\gamma - 1}(\sigma(\rho) - \bar{\sigma}).$$

Then the Euler equations (1.1) are transformed into the following system for C^1 solutions:

$$\begin{cases} \partial_t u + \bar{\sigma}\nabla \cdot v = -v \cdot \nabla u - \frac{\gamma-1}{2} u \nabla \cdot v, \\ \partial_t v + \bar{\sigma}\nabla u + av = -v \cdot \nabla v - \frac{\gamma-1}{2} u \nabla u. \end{cases} \tag{2.1}$$

The initial condition (1.2) becomes

$$(u, v)|_{t=0} = (u_0(x), v_0(x)) \tag{2.2}$$

with

$$u_0 = \frac{2}{\gamma - 1}(\sigma(\rho_0) - \bar{\sigma}).$$

The proof of the following Lemma is straightforward.

Lemma 2.1. *For any $T > 0$, if $(\rho, v) \in C^1(\mathbb{R}^3 \times [0, T])$ is a solution of (1.1) with $\rho > 0$, then $(u, v) \in C^1(\mathbb{R}^3 \times [0, T])$ is a solution of (2.1) with $\frac{\gamma-1}{2}u + \bar{\sigma} > 0$.*

Conversely, if $(u, v) \in C^1(\mathbb{R}^3 \times [0, T])$ is a solution of (2.1) with $\frac{\gamma-1}{2}u + \bar{\sigma} > 0$ and $\rho = \sigma^{-1}(\frac{\gamma-1}{2}u + \bar{\sigma})$, then $(\rho, v) \in C^1(\mathbb{R}^3 \times [0, T])$ is a solution of (1.1) with $\rho > 0$.

The positivity of the density in the above lemma follows from the corresponding positivity of the initial density.

Lemma 2.2. *If $(\rho, v) \in C^1(\mathbb{R}^3 \times [0, T])$ is a uniformly bounded solution of (1.1) with $\rho(x, 0) > 0$, then $\rho(x, t) > 0$ on $\mathbb{R}^3 \times [0, T]$.*

If $(u, v) \in C^1(\mathbb{R}^3 \times [0, T])$ is a uniformly bounded solution of (2.1) with $\frac{\gamma-1}{2}u(x, 0) + \bar{\sigma} > 0$, then $\frac{\gamma-1}{2}u(x, t) + \bar{\sigma} > 0$ on $\mathbb{R}^3 \times [0, T]$.

Proof. Since v is uniformly bounded, for any given $y \in \mathbb{R}^3$ and $s \in [0, T]$ the particle trajectory $x = x(t; y, s)$ starting at y at time s is defined for $0 \leq t \leq T$ by

$$\frac{dx}{dt} = v(x, t), \quad x|_{t=s} = y.$$

Along the particle trajectory $x = x(t; y, s)$, the directional derivative of the density is

$$\begin{aligned} \frac{d}{dt}\rho(x(t; y, s), t) &= \partial_t \rho(x(t; y, s), t) + v(x(t; y, s), t) \cdot \nabla \rho(x(t; y, s), t) \\ &= -\rho(x(t; y, s), t)\nabla \cdot v(x(t; y, s), t) \end{aligned}$$

using the first equation in (1.1). Solving this ordinary differential equation gives

$$\rho(x(t;y,s),t) = \rho(x_0,0)\exp\left(-\int_0^t \nabla\cdot v(x(\tau;y,s),\tau)d\tau\right) > 0$$

for any $t \in [0,T]$, where $x_0 = x(0;y,s)$. In particular

$$\rho(y,s) = \rho(x(s;y,s),s) > 0.$$

Since (y,s) is arbitrary, the first part of the lemma is proved. The second part is equivalent to the first part, via Lemma 2.1. $\square$

We remark that the positivity of the density for C^1 solutions in the above lemma is generally not true if the solution has singularities or shock waves, that is, a vacuum state may develop for weak solutions; see Courant-Friedrichs [6].

3. Local Existence and Finite Propagation Speed

Set $U(x,t) = (u(x,t), v(x,t))$ and

$$U_0(x) = U(x,0) = (u(x,0), v(x,0)) = (u_0(x), v_0(x)).$$

For the Cauchy problem (2.1) and (2.2), we first have the local existence result which can be obtained using the arguments in [16, 21].

Lemma 3.1. *If $U_0 = U(x,0) = (u_0(x), v_0(x)) \in H^3$, then there exists a unique local solution $U(x,t)$ of the Cauchy problem* (2.1)*and* (2.2) *in $C([0,T), H^3) \cap C^1([0,T), H^2)$ for some finite $T > 0$.*

We also have the following property about the finite speed of propagation of the solution.

Lemma 3.2. *Suppose that $U_0 \in H^3$ and $U \in C([0,T), H^3) \cap C^1([0,T), H^2)$ is a solution to the Cauchy problem* (2.1) *and* (2.2) *for any given $T > 0$. If* $\operatorname{supp} U_0 \subset \{|x| \le R\}$, *for some $R > 0$, then* $\operatorname{supp} U(\cdot,t) \subset \{|x| \le R + \bar\sigma t\}$, *for $0 \le t < T$.*

Proof. Looking at (2.1) we multiply the first equation by u, the second by v. Then we add them together and upon rearranging the nonlinear terms we get

$$\begin{aligned} &\frac{1}{2}\partial_t u^2 + \frac{1}{2}\partial_t |v|^2 + \bar\sigma \nabla\cdot(uv) \\ &= -a|v|^2 - \frac{1}{2}\Big(v\cdot\nabla(u^2+|v|^2) + (\gamma-1)u\nabla\cdot(uv)\Big). \end{aligned} \tag{3.1}$$

For a given $(x,t) \in \mathbb{R}^3 \times (0,T]$, take any $\tau \in [0,t)$, and define the truncated cone

$$C_\tau = \{(y,s) : \ |y-x| \le \bar\sigma(t-s),\ 0 \le s \le \tau\}.$$

We integrate (3.1) over C_τ. For the terms from the left side of the equation (3.1), the divergence theorem gives

$$\frac{1}{2}\int_{|y-x|\leq\bar\sigma(t-\tau)}(u^2+|v|^2)(y,\tau)dy-\frac{1}{2}\int_{|y-x|\leq\bar\sigma t}(u^2+|v|^2)(y,0)dy$$
$$+\frac{1}{\sqrt{\bar\sigma^2+1}}\int_0^\tau\int_{|y-x|=\bar\sigma(t-s)}\left(\frac{\bar\sigma}{2}(u^2+|v|^2)+\frac{y-x}{|y-x|}\cdot\bar\sigma uv\right)dS_y ds.$$

The integral along the sides, i.e., the third integral, is nonnegative because

$$\left|\frac{y-x}{|y-x|}\cdot\bar\sigma uv\right|\leq\bar\sigma|uv|\leq\frac{\bar\sigma}{2}(u^2+|v|^2).$$

Using integration by parts and the inequality $|uv|\leq(u^2+|v|^2)/2$, the terms on the right side of (3.1) yield

$$\iint_{C_\tau}\left(-a|v|^2-\frac{1}{2}\left(v\cdot\nabla(u^2+|v|^2)+(\gamma-1)u\nabla\cdot(uv)\right)\right)dyds$$
$$\leq-\iint_{C_\tau}\left(uv\cdot\nabla u+v\cdot(v\cdot\nabla v)+\frac{\gamma-1}{2}(u^2\nabla\cdot v+u\nabla u\cdot v)\right)dyds$$
$$\leq C\iint_{C_\tau}|\nabla U|(u^2+|v|^2)dyds$$

for some constant $C>0$. Therefore, combining the above estimates, we see that

$$\frac{1}{2}\int_{|y-x|\leq\bar\sigma(t-\tau)}(u^2+|v|^2)(y,\tau)dy-\frac{1}{2}\int_{|y-x|\leq\bar\sigma t}(u^2+|v|^2)(y,0)dy$$
$$\leq C\int_0^\tau\int_{|y-x|\leq\bar\sigma(t-s)}|\nabla U|(u^2+|v|^2)(y,s)dyds.$$

Letting

$$e(\tau)=\frac{1}{2}\int_{|y-x|\leq\bar\sigma(t-\tau)}(u^2+|v|^2)(y,\tau)dy,$$

we have

$$e(\tau)\leq e(0)+C\max_{C_\tau}|\nabla U|\int_0^\tau e(s)ds.$$

By Gronwall's inequality we see that

$$e(\tau)\leq e(0)\exp\left(C\max_{C_\tau}|\nabla U|t\right).$$

Therefore, if $U(y,0)=0$ for $|y-x|\leq\bar\sigma t$, then $U(y,\tau)=0$ for $|y-x|\leq\bar\sigma(t-\tau)$ and any $\tau\in[0,t)$. This implies that if $U(x,0)=0$ for $|x|>R$ then $U(x,t)=0$ for $|x|>R+\bar\sigma t$. The proof of the lemma is complete. $\square$

Lemma 3.2 on finite propagation speed will be used later to show the non-exponential decay of the density and the formation of singularities.

4. Energy Estimates

In this section, we establish the energy estimates which will be used for proving the global existence of solution in the next section.

We will use the notation $\|\cdot\|$ for the norm in $L^2(\mathbb{R}^3)$ and $|\cdot|_\infty$ for the norm in $L^\infty(\mathbb{R}^3)$. In order to distinguish time and space derivatives, let ∂ denote the vector of all first spatial and time derivatives and let ∇ denote only the spatial derivatives, thus $\partial = (\nabla, \partial_t)$. We will use C to denote a generic positive constant which may depend only on γ. The energy of a function is

$$E[u](t) = \|u(\cdot,t)\|^2_{H^3(\mathbb{R}^3)} + \|\partial_t u(\cdot,t)\|^2_{H^2(\mathbb{R}^3)}. \tag{4.1a}$$

It is also convenient to introduce the quantity

$$X[u](t) = \sum_{|\alpha|\le 2} \|\partial\nabla^\alpha u(\cdot,t)\|^2. \tag{4.1b}$$

Notice that

$$E[u](t) = X[u](t) + \|u(\cdot,t)\|^2. \tag{4.1c}$$

We have the following estimates.

Lemma 4.1. *If $U = (u,v) \in C([0,T), H^3) \cap C^1([0,T), H^2)$ is a solution of (2.1) for any given $T > 0$, then*

$$\frac{1}{2}\frac{d}{dt}\|U(\cdot,t)\|^2 + a\|v(\cdot,t)\|^2 \le C|U(\cdot,t)|_\infty \, \|v(\cdot,t)\| \, \|\nabla U(\cdot,t)\|, \tag{4.2a}$$

$$\frac{1}{2}\frac{d}{dt}X[U](t) + aX[v](t) \le C|\partial U(\cdot,t)|_\infty \, X[U](t), \tag{4.2b}$$

and as a consequence

$$\frac{1}{2}\frac{d}{dt}E[U](t) + aE[v](t) \le CE[U](t)^{1/2}\Big([X[u](t) + E[v](t)\Big). \tag{4.2c}$$

Proof. Multiplying the first equation of (2.1) by u, the second by v and adding them together, we obtain (3.1). Integrate the equation (3.1) and use integration by parts for the last terms on the left and right side to get, with the aid of the Cauchy-Schwarz inequality,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|U(\cdot,t)\|^2 + a\|v(\cdot,t)\|^2 \\
&= -\frac{1}{2}\int v\cdot\nabla(u^2+|v|^2)dx + \frac{\gamma-1}{2}\int uv\cdot\nabla u dx \\
&\le C\|v\| \left(\|\nabla(u^2+|v|^2)\| + \|u\nabla u\|\right) \le C\|v\| \, |U|_\infty \|\nabla U\|,
\end{aligned}$$

where and from now on, we denote $\int_{\mathbb{R}^3}$ by $\int$ for simplicity of notation. Estimate (4.2a) follows.

We prove estimate (4.2b) in a similar manner. First we take $\partial\nabla^\alpha$ derivatives of equations (2.1) and multiply the equations by $\partial\nabla^\alpha U$. After integrating, summing on $|\alpha| \leq 2$, and adding the two expressions, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}X[U](t) + aX[v](t) \\
&= -\sum_{|\alpha|\leq 2}\int \Big(\partial\nabla^\alpha(v\cdot\nabla u)\partial\nabla^\alpha u + \partial\nabla^\alpha(v\cdot\nabla v)\cdot\partial\nabla^\alpha v\Big)dx \\
&\quad - \frac{\gamma-1}{2}\sum_{|\alpha|\leq 2}\int \Big(\partial\nabla^\alpha(u\nabla\cdot v)\partial\nabla^\alpha u + \partial\nabla^\alpha(u\nabla u)\cdot\partial\nabla^\alpha v\Big)dx.
\end{aligned} \tag{4.3}$$

The worst case is when $|\alpha| = 2$. Here there are three main possibilities with the product $\partial\nabla^\alpha(U\partial U)$ from (4.3). First, exactly one derivative or all three derivatives can fall on the first term. This case can be handled directly with the Cauchy-Schwarz inequality, and hence

$$\int \partial U\nabla^3 U\partial\nabla^2 U dx \leq C|\partial U|_\infty X[U],$$

$$\int \nabla U\partial\nabla^2 U\partial\nabla^2 U dx \leq C|\partial U|_\infty X[U].$$

Secondly, exactly one derivative can fall on the second term of the product $\partial\nabla^\alpha(U\partial U)$ from (4.3). Here we need to first use Hölder's inequality to get

$$\int \partial\nabla U\nabla^2 U\partial\nabla^2 U dx \leq \int |\partial\nabla U|^2|\partial\nabla^2 U|dx \leq \|\partial\nabla U\|_{L^4}^2\|\partial\nabla^2 U\|. \tag{4.4}$$

Using integration by parts, one has

$$\begin{aligned}
\|\partial\nabla U\|_{L^4}^4 &= \int |\nabla\partial U|^2\nabla\partial U\cdot\nabla\partial U dx \\
&= -\int \partial U\nabla\left(|\nabla\partial U|^2\nabla\partial U\right)dx \\
&\leq C|\partial U|_\infty\int |\nabla\partial U|^2|\nabla^2\partial U|dx \\
&\leq C|\partial U|_\infty\|\nabla\partial U\|_{L^4}^2\|\nabla^2\partial U\|,
\end{aligned}$$

and then we obtain the Gagliado-Nirenberg type inequality:

$$\|\partial\nabla U\|_{L^4}^2 \leq C|\partial U|_\infty\|\partial\nabla^2 U\|. \tag{4.5}$$

Thus (4.4) and (4.5) imply that

$$\begin{aligned}\int \partial\nabla U \nabla^2 U \partial \nabla^2 U dx &\le C|\partial U|_\infty \|\partial\nabla^2 U\|^2 \\ &\le C|\partial U|_\infty X[U].\end{aligned}$$

Finally, in the case that all of the derivatives fall on the second term in the product $\partial\nabla^\alpha(U\partial U)$ from (4.3), integration by parts gives

$$\begin{aligned}&\int \left(v\cdot\partial\nabla^3 u\partial\nabla^2 u + v\cdot\partial\nabla^3 v\partial\nabla^2 v\right) dx \\ &= \frac{1}{2}\int v\cdot\nabla\left(|\partial\nabla^2 u|^2 + |\partial\nabla^2 v|^2\right) dx \\ &= -\frac{1}{2}\int \left(|\partial\nabla^2 u|^2 + |\partial\nabla^2 v|^2\right)\nabla\cdot v dx \le C|\partial U|_\infty X[U],\end{aligned}$$

and

$$\begin{aligned}&\int \left(u\partial\nabla^2\nabla\cdot v\partial\nabla^2 u + u\partial\nabla^2\nabla u\cdot\partial\nabla^2 v\right) dx \\ &= \int u\nabla(\partial\nabla^2 v\partial\nabla^2 u)\, dx = -\int \nabla u(\partial\nabla^2 v\partial\nabla^2 u)\, dx \\ &\le C|\partial U|_\infty X[U].\end{aligned}$$

If $|\alpha| < 2$ the situation is similar to one of these three cases and thus the details are omitted here. Estimate (4.2b) is proved.

Finally to show (4.2c) we simply add (4.2a) and (4.2b) together and use (4.1c) to get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}E[U](t) + aE[v](t) \\ &\le C|U(\cdot,t)|_\infty\|v(\cdot,t)\|\,\|\nabla U(\cdot,t)\| + C|\partial U((t))|_\infty X[U](t).\end{aligned} \tag{4.6}$$

Now using the Sobolev inequality we see that $|U|_\infty \le C\|U\|_{H^2} \le CE[U]^{1/2}$ and hence

$$\begin{aligned}|U|_\infty\|v\|\,\|\nabla U\| &\le CE[U]^{1/2}\ E[v]^{1/2}\ X[U]^{1/2} \\ &\le CE[U]^{1/2}\ \left(X[U]+E[v]\right) \le CE[U]^{1/2}\ \left(X[u]+E[v]\right).\end{aligned}$$

For the remaining term in (4.6) use the Sobolev inequality again to see that $|\partial U|_\infty \le C\|\partial U\|_{H^2} \le CE[U]^{1/2}$. Hence

$$\begin{aligned}|\partial U|_\infty X[U] &\le CE[U]^{1/2}\left(X[u]+X[v]\right) \\ &\le CE[U]^{1/2}\left(X[u]+E[v]\right).\end{aligned}$$

Then estimate (4.2c) follows. This completes the proof of Lemma 4.1. $\square$

Inequality (4.2c) is the major energy estimate we need for proving the global existence in the next section. The following further estimate on the relation between X and E is also necessary:

Lemma 4.2. *If* $U = (u, v) \in C([0,T), H^3) \cap C^1([0,T), H^2)$ *is a solution of* (2.1) *for any given* $T > 0$*, then*

$$X[u] \le CE[u]X[u] + \big(C + CE[u]\big)E[v]. \tag{4.7}$$

Proof. Using (2.1) we can write

$$\begin{aligned}\partial_t u &= -\left(\bar{\sigma}\nabla\cdot v + v\cdot\nabla u + \frac{\gamma-1}{2}u\nabla\cdot v\right),\\ \nabla u &= -\frac{1}{\bar{\sigma}}\left(\partial_t v + av + v\cdot\nabla v + \frac{\gamma-1}{2}u\nabla u\right).\end{aligned}$$

If we take α space derivatives of these equations in L^2, sum on $|\alpha| \le 2$ and add them together, we see that

$$\begin{aligned}&X[u]\\ &\le C\sum_{|\alpha|\le 2}\Big(\|\nabla^\alpha\nabla\cdot v\|^2 + \|\nabla^\alpha(v\cdot\nabla u)\|^2 + \|\nabla^\alpha(u\nabla\cdot v)\|^2\\ &\qquad + \|\nabla^\alpha\partial_t v\|^2 + \|\nabla^\alpha v\|^2 + \|\nabla^\alpha(v\cdot\nabla v)\|^2 + \|\nabla^\alpha(u\nabla u)\|^2\Big)\\ &\le CE[v] + C\sum_{|\alpha|\le 2}\Big(\|\nabla^\alpha(v\cdot\nabla u)\|^2 + \|\nabla^\alpha(u\nabla\cdot v)\|^2 + \|\nabla^\alpha(u\nabla u)\|^2\Big).\end{aligned} \tag{4.8a}$$

Using the Sobolev inequality

$$\begin{aligned}&|u|_\infty \le C\|u\|_{H^2} \le CE[u]^{1/2}, \quad |v|_\infty \le C\|v\|_{H^2} \le CE[v]^{1/2}\\ &|\nabla u|_\infty \le C\|\nabla u\|_{H^2} \le CE[u]^{1/2}, \quad |\nabla v|_\infty \le C\|\nabla v\|_{H^2} \le CE[v]^{1/2},\end{aligned}$$

we have the following estimates

$$\begin{aligned}\sum_{|\alpha|\le 2}\|\nabla^\alpha(u\nabla u)\|^2 &\le \sum_{|\alpha|\le 2}|(u,\nabla u)|_\infty^2\|\nabla^\alpha\nabla u\|^2\\ &\le CE[u]\,X[u],\end{aligned} \tag{4.8b}$$

$$\begin{aligned}&\sum_{|\alpha|\le 2}\|\nabla^\alpha(v\cdot\nabla u)\|^2\\ &\le \sum_{|\alpha|\le 2}\Big(|\nabla u|_\infty^2\|\nabla^\alpha v\|^2 + |\nabla v|_\infty^2\|\nabla^\alpha u\| + |v|_\infty^2\|\nabla^\alpha\nabla u\|^2\Big)\\ &\le CE[u]E[v],\end{aligned} \tag{4.8c}$$

and similarly

$$\sum_{|\alpha|\le 2} \|\nabla^\alpha (u\nabla\cdot v)\|^2 \le CE[u]E[v]. \tag{4.8d}$$

Then estimate (4.7) follows from (4.8a)-(4.8d). The proof of Lemma 4.2 is complete. ☐

5. Global Existence

In this section, we prove that, if the initial energy is sufficiently small, the Cauchy problem (2.1) and (2.2) has a unique global smooth solution. For the common constant $C > 0$ in (4.2c) and (4.7), define

$$\delta_0 = \min\left\{\frac{a^2}{16C^2(C+1)^2}, \frac{1}{2C}\right\}, \quad \text{and} \quad \mu = \frac{a}{2(2C+1)}.$$

First we have the following estimate on the energy of the solution:

Theorem 5.1. *For any given $T > 0$, suppose that $U = (u, v)$ is the solution of the Cauchy problem* (2.1) *and* (2.2) *defined for $(x,t) \in \mathbb{R}^3 \times [0,T)$, with $U \in C([0,T), H^3) \cap C^1([0,T), H^2)$. If $E[U](t) \le \delta_0$, then the following estimate holds for all $t \in [0,T)$:*

$$E[U](t) + \mu \int_0^t \Big([X[u](s) + E[v](s)\Big) ds \le E[U](0). \tag{5.1}$$

Proof. Since $E[U](t) \le \delta_0$, by Lemma 4.2, we have

$$X[u] \le \frac{1}{2}X[u] + \left(C + \frac{1}{2}\right) E[v],$$

then

$$X[u] \le (2C+1)E[v]. \tag{5.2}$$

Using the estimate (5.2), the assumption $E[U](t) \le \delta_0$, and the definition of δ_0, Lemma 4.1 yields

$$\frac{1}{2}\frac{d}{dt}E[U] + aE[v] \le CE[U]^{1/2}\Big(X[u] + E[v]\Big) \le \frac{a}{2}E[v],$$

thus

$$\frac{d}{dt}E[U](t) + aE[v](t) \le 0.$$

By (5.2),

$$\frac{a}{2(2C+1)}X[u] \le \frac{a}{2}E[v],$$

and then

$$\frac{d}{dt}E[U](t) + \mu\left(X[u] + E[v]\right) \le 0.$$

Therefore, (5.1) follows from integrating this inequality over $[0,t]$.

☐

Now we have the result on global existence of solutions.

Theorem 5.2. *Given the initial condition* $U_0(x) = (u_0(x), v_0(x)) \in H^3$ *with* $E[U](0) < \delta_0$*, there exists a unique global solution* $U(x,t) = (u,v)(x,t)$ *of the Cauchy problem* (2.1) *and* (2.2) *in* $C([0,\infty), H^3) \cap C^1([0,\infty), H^2)$ *satisfying the energy estimate* (5.1).

Proof. The problem is locally well-posed in H^3. Since $E[U](0) < \delta_0$, Theorem 5.1 implies that $E[U](t) < \delta_0$ as long as the solution exists. This bound ensures that the solution can be continued globally. □

Remark 5.1. If we consider the Euler system (1.1) in a bounded domain Ω with smooth boundary, we can study the initial-boundary value problem of (1.1) with the initial condition (1.2) and the following boundary condition:

$$v \cdot \nu|_{\partial\Omega} = 0,$$

where ν is the unit outward normal of the boundary $\partial\Omega$. The trouble of this initial-boundary value problem is that the spatial derivative of v is not known on the boundary. This can be overcome by first looking at the estimates of the time derivatives of the solution since $\partial_t^k v|_{\partial\Omega} = 0$ for any positive integer k. Using the ideas of this paper combined with the arguments in other papers on the boundary problems, e.g., [23], one may work on the equivalent system (2.1) to obtain a similar global existence result. The details are omitted.

6. Decay Estimates

In this section, we make estimates to obtain the decay rates of the solution constructed in Theorem 5.2. We also show the exponential decay of the vorticity.

To obtain the decay estimates, without loss of generality, we take $\bar{\sigma} = 1$ in (2.1) and consider the linear system

$$\begin{cases} \partial_t u + \nabla \cdot v = 0, \\ \partial_t v + \nabla u + av = 0. \end{cases} \tag{6.1}$$

The Fourier transform of (6.1) yields $\partial_t \hat{U}(\xi,t) = A(\xi)\hat{U}(\xi,t)$, with $\hat{U}(\xi,t) = (\hat{u}(\xi,t), \hat{v}(\xi,t))^\top$ and

$$A(\xi) = \begin{bmatrix} 0 & -i\xi \\ -i\xi^\top & -a\mathbf{I}_3 \end{bmatrix},$$

where $\top$ denotes the transpose of a row vector and $\mathbf{I}_3$ is the 3×3 identity matrix. The eigenvalues of $A(\xi)$ are: $-a, -a, -\lambda_1(\xi), -\lambda_2(\xi)$

with

$$\lambda_1(\xi) = \frac{1}{2}(a+\Delta), \quad \lambda_2(\xi) = \frac{1}{2}(a-\Delta), \quad \Delta = \sqrt{a^2 - 4|\xi|^2}.$$

The eigenspace corresponding to the eigenvalue $-a$ is the subspace of vectors $(0,\eta)$ with $\eta \in \mathbb{R}^3$ and $\xi \cdot \eta = 0$. The vector $h = (i\lambda_2, \xi)^\top$ is an eigenvector for the eigenvalue $-\lambda_1$. Define the orthonormal set:

$$u_1 = (0,\eta_1)^\top,\ u_2 = (0,\eta_2)^\top,\ u_3 = h/|h|,$$

where η_1, η_2, ξ are mutually orthogonal row vectors in $\mathbb{R}^3$. Then choose $u_4 \in \mathbb{R}^4$ so as to form an orthonormal basis $\{u_j\}_{j=1}^4$ in $\mathbb{R}^4$. Let $R(\xi)$ be the unitary matrix whose columns are $u_1, \cdots, u_4$. Then $A(\xi)R(\xi) = R(\xi)B(\xi)$, where

$$B(\xi) = \begin{bmatrix} -a & 0 & 0 & 0 \\ 0 & -a & 0 & 0 \\ 0 & 0 & -\lambda_1 & z \\ 0 & 0 & 0 & -\lambda_2 \end{bmatrix}, \text{ and } z = \begin{cases} -a, & \text{if } a^2 - 4|\xi|^2 < 0, \\ -2\lambda_1, & \text{if } a^2 - 4|\xi|^2 \geq 0. \end{cases}$$

We find that

$$\hat{T}(t) = \begin{bmatrix} e^{-at} & 0 & 0 & 0 \\ 0 & e^{-at} & 0 & 0 \\ 0 & 0 & e^{-\lambda_1 t} & \frac{e^{-\lambda_1 t} - e^{-\lambda_2 t}}{\lambda_2 - \lambda_1} z \\ 0 & 0 & 0 & e^{-\lambda_2 t} \end{bmatrix},$$

satisfies $\hat{T}'(t) = B(\xi)\hat{T}(t)$ and $\hat{T}(0) = I$, so $\hat{T}(t) = \exp(tB(\xi))$. It follows that

$$\hat{S}(t) \equiv \exp(tA(\xi)) = R(\xi)\exp(tB(\xi))R^*(\xi) = R(\xi)\hat{T}(t)R^*(\xi).$$

In conclusion, the solution of the linear system (6.1) with initial data U_0 is $S(t)U_0$, where $S(t) = \mathcal{F}^{-1}\hat{S}(t)\mathcal{F}$ and $\mathcal{F}$ is the Fourier transform.

Lemma 6.1. *Given $U_0 \in L^1(\mathbb{R}^3) \cap H^2(\mathbb{R}^3)$, we have the estimates:*

$$|S(t)U_0|_\infty \leq C\left((1+t)^{-\frac{3}{2}}\|U_0\|_{L^1} + e^{-\beta t}\|\nabla^2 U_0\|\right), \tag{6.2a}$$

$$\|\nabla^k S(t)U_0\| \leq C\left((1+t)^{-\frac{3}{4}-\frac{k}{2}}\|U_0\|_{L^1} + e^{-\beta t}\|\nabla^k U_0\|\right),\ k = 0, 1, \tag{6.2b}$$

where the constants C and β depend only on a.

Proof. If $a^2 - 4|\xi|^2 \geq 0$, then

$$\frac{e^{-\lambda_1 t} - e^{-\lambda_2 t}}{\lambda_2 - \lambda_1} = e^{-a/2t}\,\frac{\sinh(t\Delta/2)}{\Delta/2}.$$

For $0 \le \Delta < a/2$, this is bounded by $Cte^{-at/4} \le Ce^{-at/8}$. For $a/2 \le \Delta \le a$, we have the bound $Ce^{-t|\xi|^2/a}$. On the other hand, if $a^2 - 4|\xi|^2 < 0$, then

$$\frac{e^{-\lambda_1 t} - e^{-\lambda_2 t}}{\lambda_2 - \lambda_1} = e^{-a/2t} \frac{\sin(t|\Delta|/2)}{|\Delta|/2},$$

and this is bounded by $Cte^{-at/2} \le Ce^{-at/4}$. Since z is uniformly bounded, the off-diagonal element of $\hat{T}(t)$ is bounded by,

$$\begin{cases} Ce^{-t|\xi|^2/a}, & \text{if} \quad |\xi| < \alpha \equiv \sqrt{3}a/4, \\ Ce^{-a/4t}, & \text{if} \quad |\xi| > \alpha. \end{cases} \tag{6.3}$$

A similar bound holds for the diagonal elements. Thus the operator $\hat{S}(t)$ is bounded by the expressions in (6.3). Now we have, for some $\beta > 0$,

$$\begin{aligned}
|S(t)U_0|_\infty &\le \|\hat{S}(t)\hat{U}_0\|_{L^1} \\
&\le C\int_{|\xi|<\alpha} e^{-t|\xi|^2/a}|\hat{U}_0|d\xi + C\int_{|\xi|>\alpha} e^{-\beta t}|\hat{U}_0|d\xi \\
&\le C|\hat{U}_0|_\infty \int\limits_{|\xi|<\alpha} e^{-t|\xi|^2/a}d\xi \\
&\qquad + Ce^{-\beta t}\left(\int_{|\xi|>\alpha} |\xi|^{-4}d\xi\right)^{1/2}\left(\int_{|\xi|>\alpha} |\xi|^4|\hat{U}_0|^2d\xi\right)^{1/2} \\
&\le C(1+t)^{-3/2}\|U_0\|_{L^1} + Ce^{-\beta t}\|\nabla^2 U_0\|.
\end{aligned}$$

and

$$\begin{aligned}
&\left\|\nabla^k S(t)U_0\right\|^2 = \left\| \, |\xi|^k \hat{S}(t)\hat{U}_0 \right\|^2 \\
&\le C\int_{|\xi|<\alpha} |\xi|^{2k}e^{-2t|\xi|^2}|\hat{U}_0|^2d\xi + C\int_{|\xi|>\alpha} e^{-2\beta t}|\xi|^{2k}|\hat{U}_0|^2d\xi \\
&\le C|\hat{U}_0|_\infty^2 \int_{|\xi|<\alpha} |\xi|^{2k}e^{-2t|\xi|^2}d\xi + Ce^{-2\beta t}\int_{|\xi|>\alpha} |\xi|^{2k}|\hat{U}_0|^2d\xi \\
&\le C(1+t)^{-\frac{3}{2}-k}\|U_0\|_{L^1}^2 + Ce^{-2\beta t}\|\nabla^k U_0\|^2.
\end{aligned}$$

The proof of Lemma 6.1 is complete. □

Suppose that $U_0 \in L^1(\mathbb{R}^3) \cap H^3(\mathbb{R}^3)$ and $E[U](0) < \delta$ with δ sufficiently small. The nonlinear problem (2.1)-(2.2) has a unique global solution $U(x,t)$ as constructed in Theorem 5.2. Define the quantities

$$L_\infty(t) = \sup_{0\le s\le t}(1+s)^{\frac{3}{2}}|U(\cdot,s)|_\infty, \quad L_0(t) = \sup_{0\le s\le t}(1+s)^{\frac{3}{4}}\|U(\cdot,s)\|,$$

$$L_1(t) = \sup_{0\le s\le t}(1+s)^{\frac{5}{4}}\|\nabla U(\cdot,s)\|, \quad \mathcal{E}(t) = \sup_{0\le s\le t}\|U(\cdot,s)\|_{H^3}.$$

Lemma 6.2. *Suppose that* $\|U_0\|_{L^1} + \|U_0\|_{H^3} < \delta$. *Then*

$$L_\infty(t) + L_0(t) + L_1(t) \le C\left(\delta + L_0(t)L_1(t) + L_\infty(t)\mathcal{E}(t)\right). \tag{6.4}$$

Proof. Using the Duhamel principle, we write

$$U(x,t) = S(t)U_0(x) + \int_0^t S(t-s)G(U,\nabla U)(x,s)ds, \tag{6.5}$$

where $G(U,\nabla U) = (-v\cdot\nabla u - \frac{\gamma-1}{2}u\nabla\cdot v,\, -v\cdot\nabla v - \frac{\gamma-1}{2}u\nabla u)^\top$ stands for the nonlinear terms in (2.1). For $0\le s\le t$, we have, using the Sobolev inequality,

$$\|G(U,\nabla U)(\cdot,s))\|_{L^1} \le \|U(\cdot,s)\|\|\nabla U(\cdot,s)\| \le L_0(t)L_1(t)(1+s)^{-2}, \tag{6.6a}$$

$$\|G(U,\nabla U)(\cdot,s))\| \le L_\infty(t)\mathcal{E}(t)(1+s)^{-\frac{3}{2}}, \tag{6.6b}$$

$$\begin{aligned}\|\nabla G(U,\nabla U)(\cdot,s))\| &\le |U(\cdot,s)|_\infty\|\nabla^2U(\cdot,s)\| + \|\nabla U(\cdot,s)\|_{L^4}^2\\ &\le C|U(\cdot,s)|_\infty\|\nabla^2U(\cdot,s)\| \le CL_\infty(t)\mathcal{E}(t)(1+s)^{-\frac{3}{2}},\end{aligned} \tag{6.6c}$$

$$\begin{aligned}\|\nabla^2 G(U,\nabla U)(\cdot,s))\| &\le |U(\cdot,s)|_\infty\|\nabla^3U(\cdot,s)\| + \|\nabla U\nabla^2U(\cdot,s)\|\\ &\le C|U(\cdot,s)|_\infty\|\nabla^3U(\cdot,s)\| \le CL_\infty(t)\mathcal{E}(t)(1+s)^{-\frac{3}{2}}.\end{aligned} \tag{6.6d}$$

Now from (6.5), (6.2a), and (6.6a), (6.6d), we have

$$\begin{aligned}|U(\cdot,t)|_\infty &\le C(1+t)^{-\frac{3}{2}}\delta + C\int_0^t(1+t-s)^{-\frac{3}{2}}\|G(U(\cdot,s))\|_{L^1}ds\\ &\quad + C\int_0^t e^{-(t-s)}\|\nabla^2G(U(\cdot,s))\|ds\\ &\le C(1+t)^{-\frac{3}{2}}\delta + CL_0(t)L_1(t)\int_0^t(1+t-s)^{-\frac{3}{2}}(1+s)^{-2}ds\\ &\quad + CL_\infty(t)\mathcal{E}(t)\int_0^t e^{-(t-s)}(1+s)^{-\frac{3}{2}}ds.\end{aligned}$$

Subdividing these integrals at $s=t/2$, simple estimates bound them by $(1+t)^{-\frac{3}{2}}$, and so

$$|U(\cdot,t)|_\infty \le C(1+t)^{-\frac{3}{2}}\left(\delta + L_0(t)L_1(t) + L_\infty(t)\mathcal{E}(t)\right).$$

It follows that $L_\infty(t)$ is bounded by the right-hand side of (6.4).

Going back to (6.5), the corresponding bound for $L_0(t)$ is proved using (6.2b), (6.6a), (6.6b), as follows:

$$\begin{aligned}\|U(\cdot,t)\| &\le C(1+t)^{-\frac{3}{4}}\delta + C\int_0^t (1+t-s)^{-\frac{3}{4}}\|G(U,\nabla U)(\cdot,s))\|_{L^1}ds\\ &\quad + C\int_0^t e^{-(t-s)}\|G(U,\nabla U)(\cdot,s))\|ds\\ &\le C(1+t)^{-\frac{3}{4}}\delta + CL_0(t)L_1(t)\int_0^t (1+t-s)^{-\frac{3}{4}}(1+s)^{-2}ds\\ &\quad + CL_\infty(t)\mathcal{E}(t)\int_0^t e^{-(t-s)}(1+s)^{-\frac{3}{2}}ds\\ &\le C(1+t)^{-\frac{3}{4}}\left(\delta + L_0(t)L_1(t) + L_\infty(t)\mathcal{E}(t)\right).\end{aligned}$$

A similar argument using (6.6a), (6.6c) yields the bound for $L_1(t)$. ☐

Theorem 6.1. *Suppose $\|U_0\|_{L^1}+\|U_0\|_{H^3} < \delta$, with δ sufficiently small. Then the quantities $L_\infty(t)$, $L_0(t)$, $L_1(t)$ remain bounded for all time, and so the following decay estimates hold:*

$$|U(t)|_\infty \le C(1+t)^{-\frac{3}{2}},\ \|U(t)\| \le C(1+t)^{-\frac{3}{4}},\ \|\nabla U(t)\| \le C(1+t)^{-\frac{5}{4}}. \tag{6.7}$$

Proof. Set $Q(t) = L_\infty(t) + L_0(t) + L_1(t)$. It follows from Lemma 6.2 and the smallness of $\mathcal{E}(t)$ that $Q(t) \le C\delta + CQ(t)^2$. Now if $\delta < 1/4C^2$, the function $f(x) = C\delta - x + Cx^2$ has positive roots $0 < r_1 < r_2$. By Lemma 6.1, $Q(0) < C\delta < r_1$. But since $f(Q(t)) \ge 0$ and $Q(t)$ is continuous, we must have $Q(t) \le r_1$, for all $t > 0$. ☐

We now show the exponential decay of the vorticity.

Theorem 6.2. *The vorticity $\omega = \nabla\times v$ decays exponentially in L^2.*

Proof. In three space dimensions, the curl of the velocity equation in (2.1) gives

$$\partial_t\omega + a\omega + v\cdot\nabla\omega - \omega\cdot\nabla v = 0.$$

Thus

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int|\omega|^2dx + a\int|\omega|^2dx &\le C\int\left(|\omega|^2|\nabla\cdot v| + |\omega\cdot\nabla v\cdot\omega|\right)dx\\ &\le C|\nabla v|_\infty\int|\omega|^2dx. \quad (6.8)\end{aligned}$$

Since $|\nabla v|_\infty \le CE[v]^{1/2}$ is small, we have

$$\frac{1}{2}\frac{d}{dt}\int |\omega|^2 dx + \frac{a}{2}\int |\omega|^2 dx \le 0.$$

This implies the exponential decay of $\|\omega(\cdot,t)\|$. $\square$

7. Lower Bounds and Formation of Singularities

In this section, we derive some simple identities and differential inequalities which are used to show that smooth solutions of the Cauchy problem for the damped Euler equation do not decay exponentially in time and may blow up in finite time if the data is sufficiently large.

Fix $R > 0$ and define

$$\begin{aligned}
&F(t) = \int_{\mathbb{R}^3} x\cdot \rho v dx, \quad M(t) = \int_{\mathbb{R}^3} (\rho - \bar\rho) dx,\\
&B(t) = \{|x| < R + \bar\sigma t\}, \quad |B(t)| = \text{volume of } B(t),\\
&A(t) = (R+\bar\sigma t)^2 (M(0) + \bar\rho |B(t)|),\\
&D(t) = 3\bar\sigma^2 M(0) A(t) - (aA(t)/2)^2, \ \text{ and}\\
&K(t) = \begin{cases} -\frac{1}{3}\frac{D(t)}{A(t)^2} t, & \text{if } D(t) > 0\\ -\frac{1}{3}\frac{D(t)}{A(t)^2} t(1 + 2\frac{(-D(t))^{1/2} t}{A(t)^2}), & \text{if } D(t) \le 0. \end{cases}
\end{aligned}$$

Then we have the following theorem.

Theorem 7.1. *Suppose that* $U_0 = (u_0, v_0) \in H^3$, supp $U_0 \subset \{|x| \le R\}$ *for some* $R > 0$, *and* $\frac{\gamma-1}{2}u_0 + \bar\sigma > 0$. *Let* $\rho_0 = \sigma^{-1}(\frac{\gamma-1}{2}u_0 + \bar\sigma)$ *and* $\bar\rho = \sigma^{-1}(\bar\sigma)$. *Assume that* $M(0) = \int (\rho_0 - \bar\rho)\, dx > 0$.

If $E[U](0)$ *is sufficiently small so that, by Theorem 5.2 and Lemmas 2.1, 2.2, the initial value problems* (1.1), (1.2) *and* (2.1), (2.2) *have global classical solutions, then for sufficiently large* t

$$\|u(t)\| \ge C_0 (R + \bar\sigma t)^{-3/2} \tag{7.1a}$$

$$\|(\rho - \bar\rho)(t)\| \ge C_0 (R + \bar\sigma t)^{-3/2} \tag{7.1b}$$

$$\|v(t)\| \ge C_0 (R + \bar\sigma t)^{-5/2}, \tag{7.1c}$$

for some constant $C_0 > 0$.

Suppose that (ρ, v) *is a solution of the Cauchy problem* (1.1), (1.2), *with* $(\rho - \bar\rho, v) \in C([0,\tau), H^3) \cap C^1([0,\tau), H^2)$, *for some* $\tau > 0$. *For any fixed* $T > 0$, *if either*

$$F(0) > \exp(aT)A(T)/T, \tag{7.2a}$$

or

$$F(0) > A(T)(T^{-1} + a/2 + K(T)) \tag{7.2b}$$

then $\tau < T$.

Proof. We will use repeatedly the finite propagation speed given in Lemma 3.2: supp $(\rho - \bar{\rho}, v) \subset B(t)$, as long as the solution is defined.

¿From the first equation of (1.1), we see that

$$M(t) = M(0), \tag{7.3a}$$

and likewise using (1.1) and integration by parts we derive

$$F'(t) + aF(t) = \int \Big(\rho|v|^2 + 3(p(\rho) - p(\bar{\rho}))\Big)dx.$$

By the convexity of $p = A\rho^\gamma$ for $\gamma > 1$, we get

$$\int (p(\rho) - p(\bar{\rho}))dx \geq \int_{\mathbb{R}^3} p'(\bar{\rho})(\rho - \bar{\rho})dx = \bar{\sigma}^2 M(0). \tag{7.3b}$$

Thus, from (7.3b), we find that

$$F'(t) + aF(t) \geq \int \rho|v|^2 dx + 3\bar{\sigma}^2 M(0). \tag{7.3c}$$

With these preliminaries, we now prove the first part of the theorem. Thus, we assume that the solution is globally defined. To prove (7.1b), we use (7.3a) and the Cauchy-Schwarz inequality:

$$\begin{aligned} 0 < M(0) &= \left| \int_{B(t)} (\rho - \bar{\rho})dx \right| \\ &\leq \|\rho - \bar{\rho}\| \left(\frac{4\pi}{3}(R + \bar{\sigma}t)^3 \right)^{1/2}. \end{aligned}$$

Next, take $\rho = \varphi(u) \equiv \sigma^{-1}(\frac{\gamma-1}{2}u + \bar{\sigma})$ and then $\bar{\rho} = \varphi(0)$. Since $E[U](t)$ is uniformly bounded, the same is true for $|u(t)|_\infty$. Therefore, we have that pointwise

$$|(\rho - \bar{\rho})| = |\varphi(u) - \varphi(0)| \leq |u| \int_0^1 |\varphi'(su)|\, ds \leq C|u|,$$

and so

$$\|u\| \geq \frac{1}{C}\|\rho - \bar{\rho}\| \geq C_2(R + \bar{\sigma}t)^{-3/2}.$$

This proves (7.1a).

Discarding the first term on the right in (7.3c), simple integration yields the lower bound

$$F(t) \geq [F(0) - 3\bar{\sigma}^2 M(0)/a] \exp(-at) + 3\bar{\sigma}^2 M(0)/a. \tag{7.4a}$$

Using the Cauchy-Schwarz inequality, finite propagation speed, and the fact that $|\rho| \le \bar\rho + C|u|$ is uniformly bounded, we have that

$$F(t) = \int_{B(t)} x\cdot \rho v dx \le C(R+\bar\sigma t)^{5/2}\left(\int |v|^2 dx\right)^{1/2}. \tag{7.4b}$$

Together, (7.4a), (7.4b) imply (7.1c).

We now turn to the proof of the second half of the theorem. So now the solution is assumed to exist on the time interval $[0,\tau)$. Use the finite propagation speed and the Cauchy-Schwarz inequality to obtain

$$F(t)^2 = \left(\int_{B(t)} x\cdot\rho v dx\right)^2 \le \left(\int_{B(t)} |x|^2\rho\, dx\right)\left(\int_{B(t)} \rho|v|^2 dx\right). \tag{7.5a}$$

Using (7.3a) we see that

$$\begin{aligned}\int_{B(t)} |x|^2\rho\, dx &\le (R+\bar\sigma t)^2\int_{B(t)}\rho dx\\ &= (R+\bar\sigma t)^2\left(M(t)+\int_{B(t)}\bar\rho dx\right)\\ &= (R+\bar\sigma t)^2\left(M(0)+\bar\rho|B(t)|\right) = A(t).\end{aligned} \tag{7.5b}$$

Combining (7.3c), (7.5a), (7.5b), we obtain the following differential inequality:

$$F'(t) \ge \frac{F(t)^2}{A(t)} - aF(t) + 3\bar\sigma^2 M(0),$$

which is valid for $t\in[0,\tau)$. Since $A(t)$ is increasing, the inequality

$$F'(t) \ge \frac{F(t)^2}{A(\tau)} - aF(t) + 3\bar\sigma^2 M(0), \tag{7.6}$$

holds for $t\in[0,\tau)$.

If the condition (7.2a) holds, then using $M(0)\ge 0$ and setting $G(t) = e^{at}F(t)$, one has

$$G'(t) \ge \frac{G(t)^2}{e^{a\tau}A(\tau)}$$

for any $t\in[0,\tau)$. Then

$$G(t) \ge \frac{F(0)e^{a\tau}A(\tau)}{e^{a\tau}A(\tau) - F(0)t}.$$

Thus $\tau < T$.

Suppose that the weaker condition (7.2b) holds. For simplicity we write $A = A(\tau)$ and $D = D(\tau)$. Completing the square in (7.6) and letting $H(t) = F(t) - aA/2$, we obtain a simpler differential inequality for $H(t)$:

$$H'(t) \geq \frac{1}{A}(H(t)^2 + D)$$

for $t \in [0, \tau)$. In order that $H(t)$ is increasing we require that $H(0)^2 + D > 0$ which is implied by (7.2b). Now let $\ell^2 = |D|$. Then

$$H'(t) \geq \begin{cases} \frac{1}{A}(H(t)^2 + \ell^2), & \text{if } D > 0 \\ \frac{1}{A}(H(t)^2 - \ell^2), & \text{if } D \leq 0. \end{cases}$$

Integrating from 0 to τ we get

$$H(0) \leq \begin{cases} \ell \cot(\ell\tau/A), & \text{if } D > 0 \\ \ell \coth(\ell\tau/A), & \text{if } D \leq 0. \end{cases}$$

Next we use the facts that $\cot x < \frac{1}{x} - \frac{1}{3}x$ for $x \in (0, \pi)$, and $\coth x < \frac{1}{x} + \frac{1}{3}x(1 + 2x)$. (Note, if $D > 0$ then $\ell T/A < \pi$.) Hence we get that

$$H(0) \leq \begin{cases} \frac{A(\tau)}{\tau} - \frac{1}{3}\frac{\ell^2\tau}{A(\tau)}, & \text{if } D > 0 \\ \frac{A(\tau)}{\tau} + \frac{1}{3}\frac{\ell^2\tau}{A(\tau)}(1 + 2\frac{\ell\tau}{A(\tau)}), & \text{if } D \leq 0. \end{cases}$$

Remembering that $H(t) = F(t) - aA/2$ we see that $\tau < T$, otherwise assumption (7.2b) is contradicted.

This completes the proof of Theorem 7.1. □

We remark that conditions (7.2a) and (7.2b) can easily be satisfying by first fixing the radius of the support R, then choosing the initial density ρ_0, and finally choosing v_0 so as to make $F(0)$ sufficiently large.

Acknowledgments

The research of T. C. Sideris and B. Thomases was supported in part by the National Science Foundation. Dehua Wang's research was supported in part by the National Science Foundation and the Office of Naval Research.

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