The Riemann problem for a class of resonant hyperbolic systems of balance laws

PAOLA GOATIN AND PHILIPPE G. LEFLOCH *

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Abstract

We solve the Riemann problem for a class of resonant hyperbolic systems of balance laws. The systems are not strictly hyperbolic and the solutions take their values in a neighborhood of a state where two characteristic speeds coincide. Our construction generalizes the ones given earlier by Isaacson and Temple for scalar equations and for conservative systems. The class of systems under consideration here includes, in particular, a model from continuum physics that describes the evolution of a fluid flow in a nozzle with discontinuous cross-section.

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1 Introduction

In this paper we study the Riemann problem for a general class of nonlinear hyperbolic systems of balance laws, which is motivated by the Euler system of compressible fluids in a nozzle with variable cross-section (see (1.9) below). Precisely, we are interested in the Riemann problem for the nonlinear hyperbolic system

$$\partial_t u + \partial_x f(u,a) = g(u,a)\,\partial_x a, \tag{1.1}$$

$$\partial_t a = 0, \tag{1.2}$$

with initial data consisting, by definition, of two constant states (a_L, u_L) and (a_R, u_R):

$$(u,a)(x,0) = \begin{cases} (u_L, a_L), & x < 0, \\ (u_R, a_R), & x > 0, \end{cases} \tag{1.3}$$

*Centre de Mathématiques Appliquées & Centre National de la Recherche Scientifique, U.M.R. 7641, Ecole Polytechnique, 91128 - Palaiseau Cedex, France; E-mail: `goatin@cmap.polytechnique.fr`, `lefloch@cmap.polytechnique.fr`

The unknowns are the two functions $u = u(x,t) \in I\!R^n$ and $a = a(x,t) \in I\!R$.

It is assumed that the flux-function $f = f(u,a)$ in (1.1) is a given smooth mapping such that, for each value $a \in I\!R$, $f(\cdot, a) : I\!R^n \mapsto I\!R^n$ is strictly hyperbolic; that is, for each $u \in I\!R^n$ the Jacobian matrix $D_u f(u,a)$ admits n real and distinct eigenvalues

$$\lambda_1(u,a) < \ldots < \lambda_n(u,a)$$

and, therefore, corresponding basis of left- and right-eigenvectors $l_i(u,a)$, $r_i(u,a)$ $(i = 1,\ldots,n)$, normalized so that

$$l_i(u,a) \cdot r_i(u,a) = 1, \quad l_i(u,a) \cdot r_j(u,a) = 0 \text{ if } i \neq j. \tag{1.4}$$

We also assume that each characteristic field of $D_u f(u,a)$ is either genuinely nonlinear or linearly degenerate, that is, for each $i = 1,\ldots,n$, the function $(u,a) \mapsto \nabla_u \lambda_i(u,a) \cdot r_i(u,a)$ never vanishes or vanishes identically, respectively.

In addition, we observe that the equation (1.2) trivially corresponds to a linearly degenerate field with eigenvalue

$$\lambda_0 := 0.$$

We are interested in studying the problem (1.1)–(1.3) when the Riemann data lie in a neighborhood of a state (u_*, a_*) at which one of the wave speeds of (1.1) also vanishes, that is, we assume that for some index k

$$\lambda_k(u_*, a_*) = \lambda_0. \tag{1.5}$$

The k-characteristic field is assumed to be genuinely nonlinear, so after normalization

$$\big(\nabla_u \lambda_k \cdot r_k\big)(u_*, a_*) > 0. \tag{1.6}$$

Throughout this paper, we restrict attention to data in the ball $\mathcal{B}(u_*, \delta_0)$ with center u_* and (small) radius $\delta_0 > 0$ and we impose on the functions f and g the following two conditions

$$\big(l_k \cdot (\partial_a f - g)\big)(u_*, a_*) \neq 0 \quad \text{(we assume that it is negative)}, \tag{1.7}$$

$$\big(l_k \cdot D_u g . r_k\big)(u_*, a_*) \neq 0 \quad \text{(we assume that it is negative)}. \tag{1.8}$$

which, as we will show, give the generic structure of elementary waves near (u_*, a_*). By continuity, we can always assume that (1.6)–(1.7) still hold for all $u \in \mathcal{B}(u_*, \delta_0)$ and $a \in \mathcal{B}(a_*, \delta_0)$.

Hyperbolic systems of balance laws, having the form of conservation laws with a source arise in many applications. Most notably, the Euler equations for a fluid flow in a nozzle with cross-sectional area $a = a(x)$ read (see for instance [5]):

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) &= -\frac{\rho u}{a}\,\partial_x a, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) &= -\frac{\rho u^2}{a}\,\partial_x a, \\ \partial_t(\rho e) + \partial_x\big((\rho e + p)\,u\big) &= -\frac{(\rho e + p)u}{a}\,\partial_x a. \end{aligned} \tag{1.9}$$

Here, $\rho > 0$ denotes the density of the fluid, p the pressure, and e is the internal energy. The equations express balance laws for the mass, momentum, and total energy of the fluid through the nozzle.

We emphasize that the model (1.1)-(1.2) has two important features. On one hand, it contains a nonconservative product $g(u,a)\,\partial_x a$ which cannot make sense within the framework of the theory of distributions. Instead, a rigorous definition of weak solutions must be based on the theory of nonconservative products due to Dal Maso, LeFloch, and Murat [6]. See also LeFloch [12] for a review of several applications, including the model (1.9) above. On the other hand, (1.1)-(1.2) is non-strictly hyperbolic; work on resonant systems goes back to Liu [14, 15] and to Isaacson and Temple [10, 11]. The construction proposed in this paper can be regarded as an extension to resonant nonconservative systems of [10] (conservative systems) and [11] (scalar, nonconservative equations). For other recent related works, especially the construction of numerical schemes for equations like (1.9), see [8, 7, 1, 2].

2 Preliminaries

By setting

$$U := (u,a), \quad F(U) := (f(u,a), 0), \quad G = (g(u,a), 0),$$

the set of equations (1.1)–(1.2) can be regarded as a nonconservative system of $n+1$ equations:

$$U_t + \big(D_U F(U) - G(U)\big)\,U_x = 0. \tag{2.1}$$

We denote by $R_i(U)$, $i = 0, \dots, n$ the corresponding right-eigenvectors of the matrix $D_U F(U) - G(U)$. Clearly, we have $R_i = (r_i, 0)$, $i = 1, \dots, n$. The vector R_0 is associated with the eigenvalue λ_0 and will be determined shortly.

First, we discuss some consequences of our assumptions (1.5)–(1.8). By the implicit function theorem, (1.6) guarantees that the equation $\lambda_k(u,a) = 0$ defines (locally) a *smooth n-dimensional manifold* of $I\!R^{n+1}$,

$$\mathcal{T} := \big\{(u,a) \,/\, \lambda_k(u,a) = 0\big\} \subset I\!R^{n+1},$$

which passes through the state $U_* := (u_*, a_*)$ and will be called the **transition manifold.** The integral curves of r_k are transversal to $\mathcal{T}$, and this property allows us to distinguish two "half-spaces"

$$\mathcal{T}^{\pm} := \big\{ U \in \mathbb{R}^{n+1} \; / \; \text{for all } V \in \mathcal{T},\, R_k(V) \cdot (U - V) \gtrless 0 \big\}.$$

We now determine the right-eigenvector $R_0(u,a)$. Thanks to (1.7) the Jacobian matrix

$$D_U F(u,a) = \begin{bmatrix} D_u f & \partial_a f - g \\ 0 & 0 \end{bmatrix}$$

has rank n on $\mathcal{T}$ in a neighborhood of U_*. We claim that the eigenvectors R_0 and R_k can be chosen outside the manifold $\mathcal{T}$ in such a way that they remain continuous across the manifold $\mathcal{T}$. This is clear for the vector field R_k. The vector $R_0 =: (r_0, b_0)$ must satisfy

$$(\partial_a f - g)\, b_0 + D_u f . r_0 = 0. \tag{2.2}$$

If we search for the component r_0 in the general form $r_0 = \sum_{i=1}^n \alpha_i r_i$ and we multiply (2.2) by each left-eigenvector l_i we obtain

$$l_i \cdot (\partial_a f - g) \; b_0 + \lambda_i \, \alpha_i = 0,$$

which determines the coefficient α_i. Hence, imposing R_0 to be a unit vector, we find

$$R_0 = c \left(\lambda_k, - \sum_{i=1}^n \frac{\lambda_k}{\lambda_i} \, l_i \cdot (\partial_a f - g)\, r_i \right), \qquad (u,a) \notin \mathcal{T},$$

where the normalization coefficient $c > 0$ is given by

$$\frac{1}{c^2} := \lambda_k^2 + \sum_{i=1}^n \left(\frac{\lambda_k}{\lambda_i} \, l_i \cdot (\partial_a f - g) \right)^2. \tag{2.3}$$

It is now easy to check that, as the state (u,a) approaches the manifold $\mathcal{T}$, we have $r_0 \to r_k$ and $b_0 \to 0$, and that R_0 can be extended smoothly to the manifold and coincides with $R_k = (r_k, 0)$ on $\mathcal{T}$, in other words,

$$R_0(u,a) - R_k(u,a) \to 0 \quad \text{ as } (u,a) \text{ approaches } \mathcal{T}. \tag{2.4}$$

In particular, provided $\delta_0 > 0$ is sufficiently small, $|R_0(U) - R_k(U)| \leq \epsilon$ for $|U - U_*| \leq \delta < \delta_0$.

To parametrize the wave curves it will be convenient to introduce at this stage a globally defined parameter $\mu_i(u) \in \mathbb{R}$ which should depend smoothly upon u and be strictly monotone along the wave curve. More precisely, we assume that the parameter μ_i is given such that

$$\nabla \mu_i(u) \cdot r_i(u) \neq 0, \qquad u \in \Omega \tag{2.5}$$

(here and in the sequel a is kept fixed and we will neglect it while it does not play a role). In view of condition (2.5), when the field is genuinely nonlinear a natural choice for μ_i is the wave speed λ_i, while there is no canonical choice for linearly degenerate fields. In particular, for $i = k$ we will set

$$\mu_k(u) = \lambda_k(u), \qquad u \in \Omega, \tag{2.6}$$

and we will reparametrize the wave curves accordingly. For some $\delta_1 < \delta_0$, $u_L \in \mathcal{B}(u_*, \delta_1)$, and $\epsilon > 0$, we will denote by $m \mapsto v_k(m; u_L)$ the Hugoniot curve (or shock curve) consisting of all right states u_R that can be connected to u_L by a k-shock of speed $\overline{\lambda}_k(m; u_L)$, and by $m \mapsto w_k(m; u_L)$ the rarefaction curve, parametrized so that we can refer to

$$m \mapsto \psi_k(m; u_L) := \begin{cases} v_k(m; u_L), & m \in (-\epsilon, \mu_k(u_L)], \\ w_k(m; u_L), & m \in [\mu_k(u_L), \epsilon) \end{cases} \tag{2.7}$$

as the k-wave curve issuing from u_L (for details, see for example [13, Chapter VI]). We will also use the notations

$$\begin{aligned} \mathcal{S}_k(u_L) &:= \Big\{ v_k(m; u_L),\ m \in (-\epsilon, \mu_k(u_L)] \Big\}, \\ \mathcal{R}_k(u_L) &:= \Big\{ w_k(m; u_L),\ m \in [\mu_k(u_L), \epsilon) \Big\}, \\ \mathcal{W}_k(u_L) &:= \mathcal{S}_k(u_L) \cup \mathcal{R}_k(u_L) = \Big\{ \psi_k(m; u_L),\ m \in (-\epsilon, \epsilon) \Big\}. \end{aligned}$$

Thanks to (2.5) and (1.6), we can choose the parameter m to coincide with μ_k, that is:

$$\mu_k(\psi_k(m; u_L)) = m. \tag{2.8}$$

In this situation, by setting $\tilde{m} := m - \mu_k(u_L)$, we have the following expansion for the shock curve

$$v_k(m; u_L) = u_L + \frac{\tilde{m}}{a(u_L)} r_k(u_L) + \frac{\tilde{m}^2}{2a(u_L)^2} \big(Dr_k.r_k + br_k \big)(u_L) + \mathcal{O}(\tilde{m}^3), \tag{2.9}$$

and the corresponding shock speed satisfies

$$\overline{\lambda}_k(m; u_L) = \lambda_k(u_L) + \frac{\tilde{m}}{2a(u_L)} \big(\nabla\lambda_k \cdot r_k \big)(u_L) + \frac{\tilde{m}^2}{6a(u_L)^2} \big(\nabla(\nabla\lambda_k \cdot r_k) \cdot r_k + c\nabla\lambda_k \cdot r_k \big)(u_L) + \mathcal{O}(\tilde{m}^3), \tag{2.10}$$

where $a = \nabla\mu_k \cdot r_k = \nabla\lambda_k \cdot r_k \neq 0$ and b, c are smooth and real-valued functions of u_L, while the expansion of the rarefaction curve $w_k(m; u_L)$ takes the same form as (2.9). Derivation of (2.10) gives the following expansions for the partial derivatives of $\overline{\lambda}_k(m; u_L)$:

$$\begin{aligned} \partial_m \overline{\lambda}_k(m; u_L) &= \frac{1}{2} + \frac{\tilde{m}}{3a^2} \big(\nabla(\nabla\lambda_k \cdot r_k) \cdot r_k + c\nabla\lambda_k \cdot r_k \big) + \mathcal{O}(\tilde{m}^2), && (2.11) \\ \nabla\overline{\lambda}_k(m; u_L) \cdot r_k &= \frac{1}{2} \nabla\lambda_k \cdot r_k - \frac{\tilde{m}}{3a} \big(\nabla(\nabla\lambda_k \cdot r_k) \cdot r_k + c\nabla\lambda_k \cdot r_k \big) + \mathcal{O}(\tilde{m}^2), && (2.12) \end{aligned}$$

(where $r_k = r_k(u_L)$, $\lambda_k = \lambda_k(u_L)$, $a = a(u_L) = \nabla\lambda_k \cdot r_k(u_L)$).

Given some state (u_-, a_-) we now investigate the properties of two important curves, that will play a central role in the construction of the solution of the Riemann problem. First we study the *standing wave curve*, denoted by $m \mapsto (\varphi, \alpha)(m; u_-, a_-)$, made of all states which can be attained by using time-independent smooth solution of (1.1)-(1.2). Second, we consider the composite *transformed standing curve*, made of states which can be reached by a standing wave followed by a shock wave with zero speed. We state the properties of these curves in the following two lemmas.

Lemma 2.1 *Given some state $(u_-, a_-) \in \mathcal{T}$ consider the standing wave solution of* (1.1)–(1.2), *denoted by $m \mapsto (\varphi, \alpha)(m; u_-, a_-)$ and determined by*

$$\begin{aligned} &(\varphi, \alpha)' = \gamma(m) R_0(\varphi, \alpha), \\ &\varphi(0) = u_-, \quad \alpha(0) = a_-, \end{aligned}$$

for some smooth scalar function $\gamma(m)$ bounded away from zero and such that $\gamma(0) = 1/a(u_-)$. Then we have

$$\alpha'(0) = 0 \quad \textit{and} \quad \alpha''(0) = -\frac{1}{a(u_-)^2} \frac{(\nabla\lambda_k \cdot r_k)(u_-, a_-)}{(l_k \cdot (\partial_a f - g))(u_-, a_-)} > 0. \tag{2.13}$$

Proof. We have

$$\begin{cases} \alpha' = \gamma(m) b_0(\varphi, \alpha), \\ \varphi' = \gamma(m) r_0(\varphi, \alpha), \end{cases} \quad \text{with} \quad \begin{cases} \alpha(0) = a_-, \\ \varphi(0) = u_-. \end{cases}$$

The standing wave satisfies

$$\partial_a f \alpha' + D_u f \, \varphi' = g \, \alpha',$$

that we rewrite as

$$(\partial_a f - g) \, \alpha' + D_u f \, \varphi' = 0. \tag{2.14}$$

We decompose the vector φ' along the right-eigenvector as

$$\varphi' = \sum_{i=1}^{n} c_i(m) \, r_i(m). \tag{2.15}$$

Since $\varphi'(0) = \big(1/a(u_-)\big) r_0 = \big(1/a(u_-)\big) r_k$, we must have $c_i(0) = 0$ for $i \neq k$ and $c_k(0) = 1/a(u_-)$. Differentiating (2.14) with respect to m we obtain

$$(\partial_{aa} f - \partial_a g)(\alpha')^2 + (\partial_a f - g) \, \alpha'' + D_u(\partial_a f - g) \, \varphi' \alpha' + \frac{d}{dm}\{D_u f \, \varphi'\} = 0. \tag{2.16}$$

We can rewrite the last term of the left-hand side using the decomposition (2.15). In fact,

$$D_u f \, \varphi' = D_u f \sum_{i=1}^{n} c_i(m) \, r_i(m) = \sum_{i=1}^{n} c_i(m) \, \lambda_i(m) \, r_i(m),$$

and thus

$$\frac{d}{dm}\{D_u f\,\varphi'\} = \sum_{i=1}^{n} c_i'(m)\,\lambda_i(m)\,r_i(m) + \sum_{i=1}^{n} c_i(m)\frac{d}{dm}\{\lambda_i(m)\,r_i(m)\}.$$

At $m = 0$ we have $\lambda_k(0) = 0$ and $\alpha'(0) = 0$, hence

$$\sum_{i=1}^{n} c_i'(0)\,\lambda_i(0)\,r_i(0) = \sum_{i\neq k} c_i'(0)\,\lambda_i(0)\,r_i(0)$$

and

$$\begin{aligned}\sum_{i=1}^{n} c_i(m)\,\frac{d}{dm}\{\lambda_i(m)\,r_i(m)\}\Big|_{m=0} &= \frac{1}{a(u_-)}\left\{\frac{1}{a(u_-)}\nabla\lambda_k\cdot r_k + \partial_a\lambda_k\alpha'(0)\right\} r_k\Big|_{(u_-,a_-)}\\ &= \frac{1}{a(u_-)^2}(\nabla\lambda_k\cdot r_k)r_k\big|_{(u_-,a_-)}.\end{aligned}$$

We now evaluate (2.16) at $m = 0$, and we obtain

$$(\partial_a f - g)\,\alpha'' + \sum_{i\neq k} c_i'\lambda_i\,r_i + \frac{1}{a(u_-)^2}(\nabla\lambda_k\cdot r_k)r_k = 0.$$

Finally, multiplying on the left by $l_k(u_-, a_-)$ yields

$$\alpha''(0) = -\frac{1}{a(u_-)^2}\;\frac{\nabla\lambda_k\cdot r_k}{l_k\cdot(\partial_a f - g)}\Big|_{(u_-,a_-)} > 0,$$

where we have used the hypothesis (1.7). □

Remark 2.2 Lemma 2.1 shows that the standing wave curve that passes through $U_- = (u_-, a_-) \in \mathcal{T}$ touches the hyperplane $a = a_-$ only at U_- and does not cross it. The sign assumptions in (1.6) and (1.7) imply that the curve lies above the hyperplane, and crosses any hyperplane $a = a_1$, $a_1 > a_-$ exactly twice in a neighborhood of U_-.

We note also that given u_-, a_-, a connecting state u_+ can always be found for $a_+ > a_-$, while for $a_+ < a_-$ this is true only as far as $a_+ \geq \alpha(0; u_-, a_-)$. This means that smooth stationary flow is always possible for expanding ducts. On the contrary, for contracting ducts the change in area must not be too large.

We now describe the transformed standing curve corresponding to a given standing wave curve. By the sign assumption in (1.6), shock curves cross $\mathcal{T}$ from $\mathcal{T}^+$ to $\mathcal{T}^-$. By the Rankine-Hugoniot condition

$$s[u] = [f(u)],$$

the 0-speed shocks ($s = 0$) cross $\mathcal{T}$ at a constant value of f.

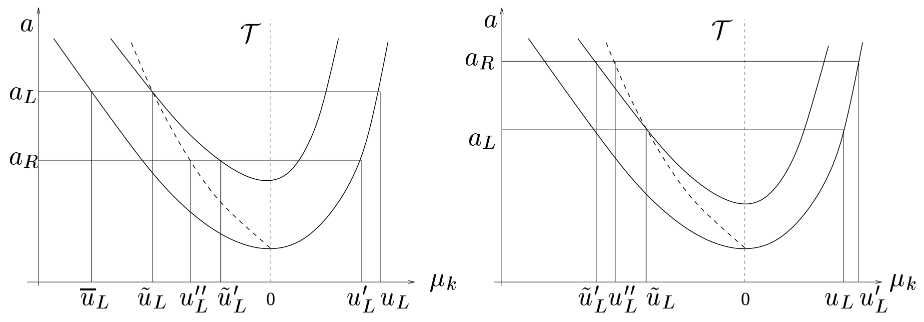


Figure 1: The mutual position of the standing wave (continuous) and the transformed standing wave (dotted).

Denote by $Z(m;u_-,a_-) = (\varphi(m),\alpha(m))$ a given standing wave curve (here we will assume $Z(0;u_-,a_-) = (u_-,a_-) \in \mathcal{T}$). For a given state $(u_L,a_L) = (\varphi(m_L),\alpha(m_L)) \in \mathcal{T}^+$, $(m_L > 0)$, on this curve, define $\bar{u}_L$ and $\tilde{u}_L$ such that the states $(\bar{u}_L,a_L)$ and $(\tilde{u}_L,a_L)$ lie on the other side of $\mathcal{T}$ at the same a-level and belong, respectively, to the same standing wave curve and to the same level curve of f as the state (a_L,u_L). That is, $\bar{u}_L$ and $\tilde{u}_L$ must satisfy

$$(\bar{u}_L,a_L) = (\varphi(m_R),\alpha(m_R)) \tag{2.17}$$

(for some $m_R < 0$) and

$$f(u_L,a_L) = f(\tilde{u}_L,a_L). \tag{2.18}$$

Definition 2.3 *The* **transformed standing curve** *corresponding to a standing wave curve $Z(m;u_-,a_-) = (\varphi(m),\alpha(m))$ is the curve*

$$\{(u,a) \in \mathcal{T}^- : f(u,a) = f\big(\varphi(m),\alpha(m)\big),\quad m > 0\}.$$

Lemma 2.4 *If $l_k \cdot Dg.r_k < 0$, then for each standing wave $Z(m;u_-,a_-)$, the corresponding transformed standing curve lies closer to the transition surface $\mathcal{T}$ than Z. That is,*

$$\mu_k(\bar{u}_L) < \mu_k(\tilde{u}_L) < 0.$$

If $l_k \cdot Dg.r_k > 0$, then the corresponding transformed standing curve lies farer from the transition surface $\mathcal{T}$ than Z. That is,

$$\mu_k(\tilde{u}_L) < \mu_k(\bar{u}_L) < 0.$$

Proof. We denote the positive and negative branches of the standing wave curve as

$$(\varphi(m),\alpha(m)) = \begin{cases} (\varphi_+(m),\alpha_+(m)) & \text{if } m > 0, \\ (\varphi_-(m),\alpha_-(m)) & \text{if } m < 0, \end{cases}$$

Each of these branches can then be parametrized by a:

$$
\begin{aligned}
(\varphi_+(m), \alpha_+(m)) &\to \Phi_+(a), \qquad a \geq a_-, \\
(\varphi_-(m), \alpha_-(m)) &\to \Phi_-(a), \qquad a \geq a_-,
\end{aligned}
$$

We compute

$$
\begin{aligned}
f(\bar u_L, a_L) - f(u_L, a_L) &= f(\varphi(m_R), \alpha(m_R)) - f(\varphi(m_L), \alpha(m_L)) \\
&= \int_{m_L}^{m_R} [D_u f\, \varphi' + \partial_a f \alpha']\, dm \\
&= \int_{m_L}^{m_R} g(\varphi(m), \alpha(m))\, \alpha'(m)\, dm \\
&= \int_{m_L}^{0} g(\varphi_+(m), \alpha_+(m))\, \alpha'_+(m)\, dm + \int_0^{m_R} g(\varphi_-(m), \alpha_-(m))\, \alpha'_-(m)\, dm \\
&= \int_{a_L}^{a_-} g(\Phi_+(a), a)\, da + \int_{a_-}^{a_L} g(\Phi_-(a), a)\, da \\
&= \int_{a_-}^{a_L} [g(\Phi_-(a), a) - g(\Phi_+(a), a)]\, da.
\end{aligned}
$$

Hence

$$
l_k \cdot \big(f(\bar u_L, a_L) - f(u_L, a_L)\big) = \int_{a_-}^{a_L} \int_0^1 l_k \cdot Dg\big(s\Phi_-(a) + (1-s)\Phi_+(a), a\big). \big(\Phi_-(a) - \Phi_+(a)\big)\, ds\, da, \tag{2.19}
$$

that is different from 0 thanks to (1.8). Hence the two curves are distinct. On the other side by (2.18) we have

$$
\begin{aligned}
l_k \cdot \big(f(\bar u_L, a_L) - f(u_L, a_L)\big) &= l_k \cdot \big(f(\bar u_L, a_L) - f(\tilde u_L, a_L)\big) \\
&= \int_0^1 l_k \cdot D_u f\big(s\bar u_L + (1-s)\tilde u_L, a_L\big). (\bar u_L - \tilde u_L)\, ds \\
&\sim \lambda_k\, l_k \cdot (\bar u_L - \tilde u_L),
\end{aligned} \tag{2.20}
$$

where $\lambda_k < 0$ since $\bar u_L,\ \tilde u_L \in \mathcal{T}^-$. Identities (2.19) and (2.20) imply that

$$
l_k \cdot Dg. r_k < 0 \quad \Longrightarrow \quad \mu_k(\bar u_L) < \mu_k(\tilde u_L), \tag{2.21}
$$

$$
l_k \cdot Dg. r_k > 0 \quad \Longrightarrow \quad \mu_k(\bar u_L) > \mu_k(\tilde u_L). \tag{2.22}
$$

(see Fig.1). □

In the next section will be also useful to know the mutual position of the transformed standing curve corresponding to $Z(m; u_L, a_L) = (\varphi(m), \alpha(m))$ and the standing curve passing through $(\tilde u_L, a_L)$, that we will denote by $Z(m; \tilde u_L, a_L) = (\tilde\varphi(m), \tilde\alpha(m))$ (see again Fig. 1). For a given state a_R, define $\tilde u'_L$ and u''_L such that the states $(\tilde u'_L, a_R)$ and (u''_L, a_R) lie in

$\mathcal{T}^-$ respectively on the standing curve $Z(m;\tilde u_L,a_L)$ and on the transformed standing curve corresponding to (u_L,a_L). That is, $\tilde u_L'$ and u_L'' must satisfy

$$(\tilde u_L', a_R) = (\tilde\varphi(m'),\tilde\alpha(m')), \qquad \text{for some } m'<0,$$

and

$$f(u_L'', a_R) = f(u_L', a_R) = f(\varphi(m''),\alpha(m'')), \qquad \text{for some } m''>0.$$

Lemma 2.5 *Let us assume that $l_k\cdot Dg.\,r_k<0$. Then the following holds:*

$$a_R<a_L \quad\Longrightarrow\quad \mu_k(u_L'')<\mu_k(\tilde u_L')<0, \tag{2.23}$$

$$a_R>a_L \quad\Longrightarrow\quad \mu_k(\tilde u_L')<\mu_k(u_L'')<0. \tag{2.24}$$

PROOF. We follow closely the lines of the proof of Lemma 2.4. We compute

$$\begin{aligned}
f(\tilde u_L',a_R)-f(u_L'',a_R) &= f(\tilde\varphi(m'),\tilde\alpha(m'))-f(\varphi(m''),\alpha(m''))\\
&= \int_{\mu_k(\tilde u_L)}^{m'} [D_u f\,\varphi'+\partial_a f\alpha']\,dm + f(\tilde u_L,a_L)\\
&\qquad -\int_{\mu_k(u_L)}^{m''} [D_u f\,\varphi'+\partial_a f\alpha']\,dm - f(u_L,a_L)\\
&= \int_{\mu_k(\tilde u_L)}^{m'} g(\tilde\varphi_-(m),\tilde\alpha_-(m))\,\tilde\alpha_-'(m)\,dm\\
&\qquad -\int_{\mu_k(u_L)}^{m''} g(\varphi_+(m),\alpha_+(m))\,\alpha_+'(m)\,dm\\
&= \int_{a_L}^{a_R} g(\Phi_-(a),a)\,da - \int_{a_L}^{a_R} g(\Phi_+(a),a)\,da\\
&= \int_{a_L}^{a_R} [g(\Phi_-(a),a)-g(\Phi_+(a),a)]\,da.
\end{aligned}$$

Hence

$$l_k\cdot\big(f(\tilde u_L',a_R)-f(u_L'',a_R)\big) = \int_{a_L}^{a_R}\int_0^1 l_k\cdot Dg\big(s\Phi_-(a)+(1-s)\Phi_+(a),a\big).\,\big(\Phi_-(a)-\Phi_+(a)\big)\,ds\,da. \tag{2.25}$$

On the other hand we have

$$\begin{aligned}
l_k\cdot\big(f(\tilde u_L',a_R)-f(u_L'',a_R)\big) &= \int_0^1 l_k\cdot D_u f\big(s\tilde u_L'+(1-s)u_L'',a_R\big).\,(\tilde u_L'-u_L'')\,ds\\
&\sim \lambda_k\, l_k\cdot(\tilde u_L'-u_L''),
\end{aligned} \tag{2.26}$$

where $\lambda_k<0$ since $\tilde u_L'$, $u_L''\in\mathcal{T}^-$. Comparing identities (2.25) and (2.26) we get the conclusion. □

Remark 2.6 In the case $g \equiv 0$, the system (1.1) reduces to the fully conservative system

$$\partial_t u + \partial_x f(u,a) = 0, \tag{2.27}$$

which has been studied in [10]. Note that in this case standing wave curves and 0-speed shock curves coincide.

Remark 2.7 We consider as significant physical example the Euler equations of compressible isentropic gas flow through a nozzle

$$\begin{aligned} \partial_t \rho + \partial_x m &= -\frac{m}{a}\,\partial_x a, \\ \partial_t m + \partial_x \left(\frac{m^2}{\rho} + p(\rho)\right) &= -\frac{m^2}{a\rho}\,\partial_x a, \end{aligned} \tag{2.28}$$

where m is the momentum of the gas, $a = a(x)$ is the cross-sectional area of the duct and the pressure is given by $p(\rho) = \gamma^{-1}\rho^\gamma$, and $\gamma = 1 + 2\theta > 1$ is the adiabatic constant.

The standing waves are determined by the following system of ordinary differential equations [3]:

$$\begin{aligned} m_x &= -\frac{m}{a}\,a_x, \\ \left(\frac{m^2}{\rho} + p(\rho)\right)_x &= -\frac{m^2}{a\rho}\,a_x. \end{aligned} \tag{2.29}$$

System (2.29) can be integrated, leading to the following equations which implicitly define the standing wave curve passing through a given state (a_L, ρ_L, m_L):

$$\begin{aligned} am &= a_L m_L, \\ \frac{m^2}{\rho^2} + \frac{1}{\theta}\rho^{2\theta} &= \frac{m_L^2}{\rho_L^2} + \frac{1}{\theta}\rho_L^{2\theta}. \end{aligned} \tag{2.30}$$

Figure 1 shows the projection on the (ρ, m)-plane of the stationary curve as well as the 1-wave curve and 2-wave curve through the point $(\rho_L, m_L) = (1/5, 1/5)$, while the dotted curves $m = \pm\rho^{1+\theta}$ define the transition curves, where the eigenvalues are equal to 0.

3 The 0-k-curve

As a first step toward the construction of the solution of the Riemann problem for (1.1)-(1.1), we give in this section an accurate description of the set of all right states $u = W_{0,k}(m; u_L, a_L, a_R,)$ associated with the level a_R that can be reached from (a_L, u_L) by a solution of the Riemann problem consisting of admissible 0-waves and k-waves, only. From now on, let a_L, a_R and u_L be fixed and let us impose the following admissibility criterion on the standing waves:

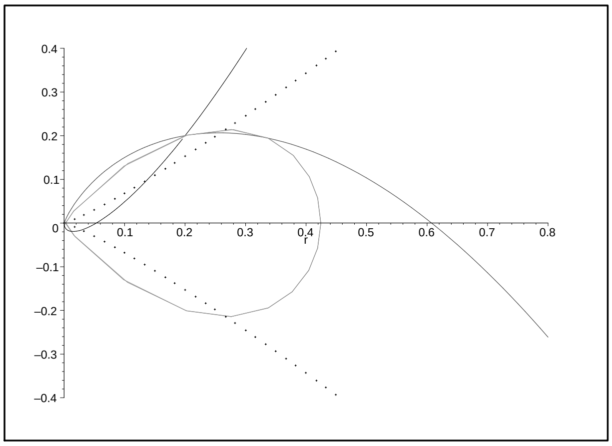


Figure 2: Significant curves for the 2×2 Euler system.

(H) A 0-wave that connects U_L to U_R on the same integral curve of R_0 by a contact discontinuity of speed zero is admissible if the integral curve of R_0 does not cross the transition surface $\mathcal{T}$ between U_L and U_R.

This admissibility condition was motivated in [10, 11] by the fact that the total variation of a in Glimm's method then does not increase. It follows from (H) that to move to the other side of the transition surface one has to use a k-wave.

To construct the 0-k-curve we will make use of the following lemma.

Lemma 3.1 *There exists $\delta_1 < \delta_0$ such that the following holds. Let $u_- \in \mathcal{B}(u_*, \delta_1)$ be given with $\mu_k(u_-) > 0$ and consider the wave curve $m \mapsto \psi_k(m; u_-)$ associated to u_-. Then there exist a (unique) point $\tilde{u}_-$ and a smooth function $\tilde{\mu}_k \le 0$ such that*

$$\tilde{u}_- = v_k(\tilde{\mu}_k(u_-); u_-) \qquad \text{and} \qquad \overline{\lambda}_k(\tilde{\mu}_k(u_-); u_-) = 0. \tag{3.1}$$

In particular, $\tilde{\mu}_k$ is a monotone decreasing function of u_- in the r_k direction:

$$\nabla \tilde{\mu}_k \cdot r_k \sim -\nabla \lambda_k \cdot r_k \qquad \text{near the transition manifold } \mathcal{T}. \tag{3.2}$$

Proof. Existence and uniqueness of the point $\tilde{u}_-$ is given by (1.6), which implies that the shock speed $\overline{\lambda}_k(m; u_-)$ is strictly increasing for small m. We are assuming here $\tilde{u}_- \in \mathcal{B}(u_*, \delta_0)$, the other case being not interesting for our purpose. The second part of the statement follows from the implicit function theorem applied to the mapping $\overline{\lambda}_k(m; u)$. Indeed, it is a smooth mapping of its arguments, and we have $\overline{\lambda}_k(\tilde{\mu}_k(u); u) = 0$ by definition. Moreover from (2.11), $\partial_m \overline{\lambda}_k(m; u)$ remains strictly positive for $m - \mu_k(u)$ small enough. From the definition (3.1) of $\tilde{\mu}_k(u)$ and (2.10) we recover

$$\big(\tilde{\mu}_k(u)-\mu_k(u)\big)+\mathcal{O}\big(\tilde{\mu}_k(u)-\mu_k(u)\big)^2=-2\mu_k(u),$$

hence $\tilde{\mu}_k(u)-\mu_k(u)$ is small if $\mu_k(u)$ is small, that is, if u is sufficiently close to the transition surface $\mathcal{T}$.

To derive (3.2) along the critical manifold we use again the implicit function theorem:

$$\nabla\tilde{\mu}_k\cdot r_k(u)=-\frac{\nabla\overline{\lambda}_k\cdot r_k(u)}{\partial_m\overline{\lambda}_k(\tilde{\mu}_k(u);u)}. \tag{3.3}$$

Using (2.11) and (2.12) to compute the derivatives in the right hand side of (3.3) and letting the state u approach the manifold $\mathcal{T}$, we get (3.2). □

The standing wave curves deserve a special treatment. As we have seen in the previous section, the standing wave through some point (u_-,a_-) is defined by the following ODE

$$Z'=\gamma(m)R_0(Z),\qquad Z(\mu_k(u_-))=U_-=(u_-,a_-) \tag{3.4}$$

(since $R_0\sim R_k$ close to $\mathcal{T}$, we can parametrize the curve with respect to the parameter m defined in (2.8)). Thanks to the regularity of R_0, (3.4) defines a curve

$$m\mapsto Z(m;U_-)=\begin{pmatrix}\varphi(m;u_-,a_-)\\ \alpha(m;u_-,a_-)\end{pmatrix},\qquad m\in(-\epsilon,\epsilon), \tag{3.5}$$

for some $\epsilon>0$, which depends smoothly upon u_-, a_- and m, and we can write the following expansions for the curve φ and its first derivative $\partial_m\varphi$:

$$\varphi(m;u_-,a_-)=u_-+\frac{\tilde{m}}{a(u_-)}r_0(u_-)+\frac{\tilde{m}^2}{2a(u_-)^2}\big(Dr_0.r_0+b_0\partial_a r_0+dr_0\big)(u_L)+\mathcal{O}(\tilde{m}^3), \tag{3.6}$$

$$\partial_m\varphi(m;u_-,a_-)=\frac{1}{a(u_-)}r_0(u_-)+\frac{\tilde{m}}{a(u_-)^2}\big(Dr_0.r_0+b_0\partial_a r_0+dr_0\big)(u_L)+\mathcal{O}(\tilde{m}^2), \tag{3.7}$$

where d is some smooth function. Moreover the R_k directional derivative with respect to the initial data is the vector V which solves the linear Cauchy problem

$$V'=\gamma(m)DR_0\big(Z(m,U_-)\big)V,\qquad V(\mu_k(u_-))=R_k$$

(see for example [9, Chapter V] for a rigorous proof). Hence, for $m-\mu_k(u_-)$ sufficiently small, we have $V(m)\sim R_k$ and, in particular,

$$D_{u_-}\varphi(m;u_-,a_-).\,r_k:=\lim_{h\to0+}\frac{\varphi(m;u_-+hr_k,a_-)-\varphi(m;u_-,a_-)}{h}\sim r_k, \tag{3.8}$$

$$\nabla_{u_-}\alpha(m;u_-,a_-).\,r_k\sim 0. \tag{3.9}$$

Similarly, we can compute the derivatives w.r. to a_-. As before, they correspond to the derivative of $Z(m;U_-)$ w.r. to the initial data, in the direction of the vector $E_1 = (1,0,\dots,0) \in \mathbb{R}^{n+1}$, that is given by the solution of

$$V' = \gamma(m) DR_0\big(Z(m,U_-)\big)V, \qquad V(\mu_k(u_-)) = E_1.$$

This means that for $m - \mu_k(u_-)$ sufficiently small, we have the following approximations

$$\partial_{a_-}\varphi(m;u_-,a_-) := \lim_{h\to 0+} \frac{\varphi(m;u_-,a_-+h) - \varphi(m;u_-,a_-)}{h} \sim 0, \tag{3.10}$$

$$\partial_{a_-}\alpha(m;u_-,a_-) \sim 1. \tag{3.11}$$

We denote by $a^T(u_-) = \alpha(0;u_-,a_-)$ the level at which the curve $Z(m;U_-)$ intersects the transition surface $\mathcal{T}$. In view of the admissibility criterion **(H)**, each fixed value $a_+ \geq a^T(u_-)$ uniquely defines the parameter value $\hat\mu(a_+;u_-,a_-)$ such that

$$\alpha(\hat\mu(a_+;u_-,a_-);u_-,a_-) = a_+. \tag{3.12}$$

Deriving (3.12) w.r to u_- and a_- gives respectively

$$D_{u_-}\hat\mu(a_+;u_-,a_-)\cdot r_k(u_-) \sim 0, \tag{3.13}$$

$$\partial_m\alpha(\hat\mu(a_+;u_-,a_-);u_-,a_-)\,\partial_{a_-}\hat\mu(a_+;u_-,a_-) \sim -1. \tag{3.14}$$

For a_L, a_R fixed, we can thus define the map

$$u \mapsto \varphi\big(\hat\mu(a_R;u,a_L);u,a_L\big), \quad \text{for } u \in \mathcal{B}(u_*,\delta_1),\ a^T(u) \leq a_R.$$

One should keep in mind that by (2.13), for $a_R < a_L$ a state u is mapped closer to the sonic line (staying on the same side of $\mathcal{T}$); the opposite is true for $a_R > a_L$.

We will distinguish two main cases, depending on whether the state U_L belongs to $\mathcal{T}^+$ or $\mathcal{T}^-$.

CASE 1. We first study the case $U_L \in \mathcal{T}^+$. The analysis will be further divided into four subcases, depending on the value of a_R.

Case 1a: $a_R \geq a_L$**.** While moving along a 0-k-curve, one has at most three possibilities:

(A) follow the standing wave curve up to level a_R, and then move along the k-wave with nonnegative speed;

(B) use a k-wave with nonpositive speed at constant level a_L followed by a standing wave;

(C) a more complex pattern, move first along the standing wave curve up to an intermediate level a_M, jump on the other side of $\mathcal{T}$ by means of a 0-speed k-shock and then use another standing wave to reach a_R.

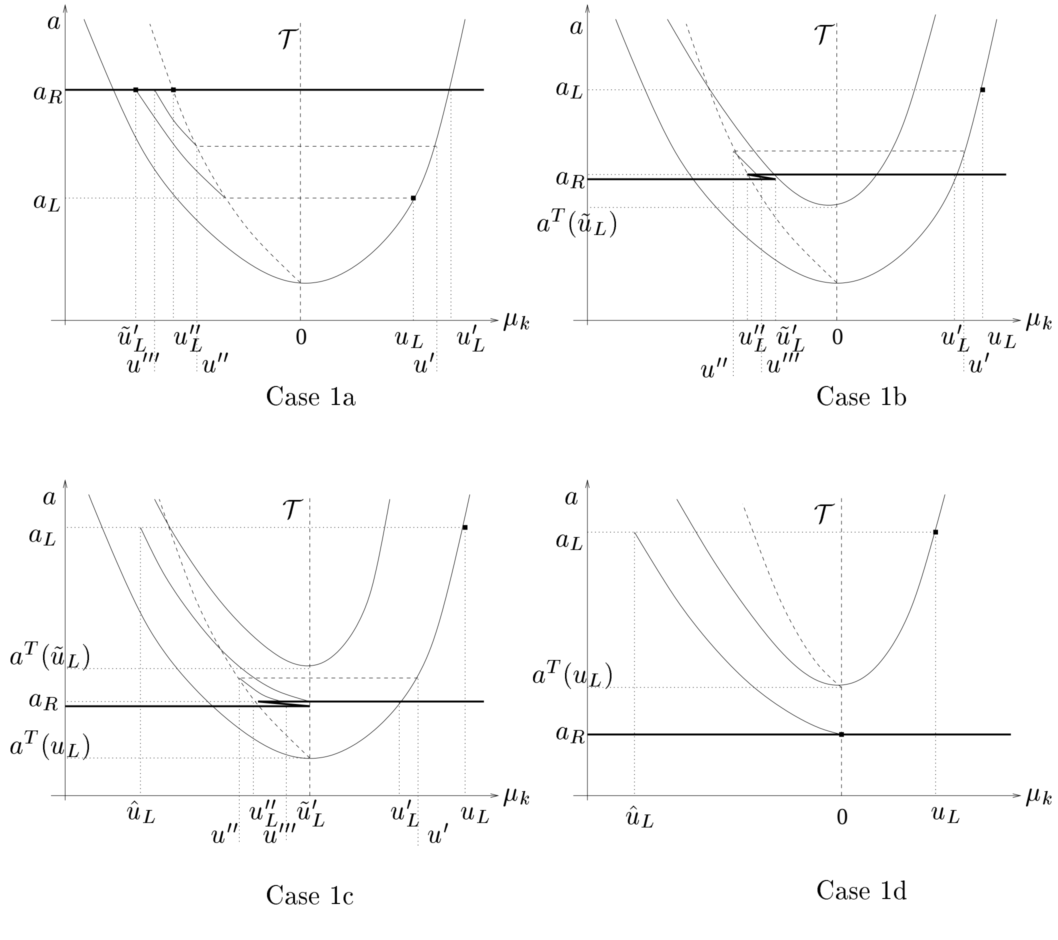


Figure 3: CASE 1.

These three cases define three different branches of the curve

$$\widetilde{\mathcal{W}}_k(u_L) := \Big\{ W_{0,k}(m; u_L, a_L, a_R,),\ m \in (-\epsilon, \epsilon) \Big\}.$$

Case (A) defines a first branch of the curve,

$$\widetilde{\mathcal{W}}_k^+(u_L) := \Big\{ \psi_k\big(m; \varphi(\hat{\mu}(a_R; u_L, a_L); u_L, a_L)\big),\ m \in \big[\tilde{\mu}_k\big(\varphi(\hat{\mu}(a_R; u_L, a_L); u_L, a_L)\big), \epsilon\big) \Big\},$$

while another branch is found following case (B)

$$\widetilde{\mathcal{W}}_k^-(u_L) := \Big\{ \varphi\big(\hat{\mu}(a_R; v_k(m; u_L), a_L); v_k(m; u_L), a_L\big),\ m \in \big(-\epsilon, \tilde{\mu}_k(u_L)\big] \Big\}.$$

Finally, case (C) can be described by

$$\begin{aligned} \widetilde{\mathcal{W}}_k^M(u_L) := \Big\{ & \varphi\Big(\hat{\mu}\big(a_R; u'', \alpha(m; u_L, a_L)\big); u'', \alpha(m; u_L, a_L)\Big), \\ & u'' = v_k\Big(\tilde{\mu}_k\big(\varphi(m; u_L, a_L)\big); \varphi(m; u_L, a_L)\Big),\ \mu_k(u_L) \le m \le \hat{\mu}(a_R; u_L, a_L) \Big\}. \end{aligned}$$

In the following we will also set

$$\begin{aligned} u' &= \varphi(m; u_L, a_L), \quad \min\{a_L, a_R\} \le \alpha(m; u_L, a_L) \le \max\{a_L, a_R\}, \\ u''' &= \varphi\Big(\hat{\mu}\big(a_R; u'', \alpha(m; u_L, a_L)\big); u'', \alpha(m; u_L, a_L)\Big), \\ u'_L &= \varphi\big(\hat{\mu}(a_R; u_L, a_L); u_L, a_L\big), \\ u''_L &= v_k\big(\tilde{\mu}_k(u'_L); u'_L\big), \\ \tilde{u}'_L &= \varphi\big(\hat{\mu}(a_R; \tilde{u}_L, a_L); \tilde{u}_L, a_L\big) \end{aligned}$$

(see Fig. 3).

The setting of the following lemma is general as to cover the next cases.

Lemma 3.2 *There exists $\delta_1 < \delta_0$ such that for $u_L \in \mathcal{B}(u_*, \delta_1)$ with $\mu_k(u_L) > 0$, and $|a_L - a_R| \le \delta_1$, the parameter m is strictly monotone along each branch of the curve $\widetilde{\mathcal{W}}_k(u_L)$. More precisely*

(i) $m \mapsto \lambda_k\big(\psi_k(m; u'_L)\big)$ *is strictly increasing for* $m \in \big[\tilde{\mu}_k(u'_L), \epsilon\big)$*;*

(ii) $m \mapsto \lambda_k\Big(\varphi\big(\hat{\mu}(a_R; v_k(m; u_L), a_L); v_k(m; u_L), a_L\big)\Big)$ *is strictly increasing for* $m \in \big(-\epsilon, \tilde{\mu}_k(u_L)\big]$*;*

(iii) if moreover $a_R \ge a^T(\tilde{u}_L)$*, then* $m \mapsto \lambda_k(u''')$ *is strictly increasing for* $m \in \big[\mu_k(u_L), \hat{\mu}(a_R; u_L, a_L)\big]$*.*

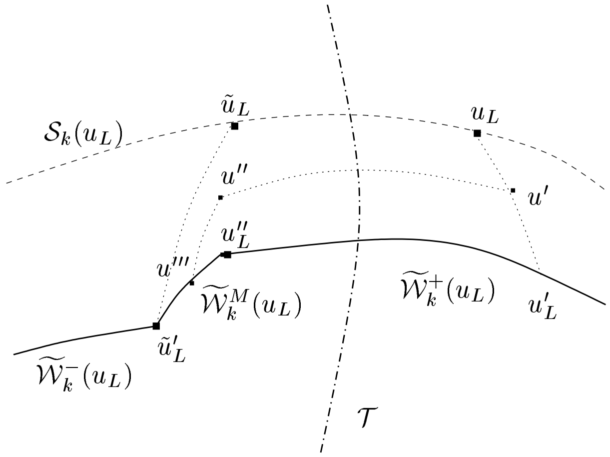


Figure 4: The curve $\widetilde{\mathcal{W}}_k(u_L)$ for $a_R > a_L$ (case 1a).

Proof. By definition (2.8), m is strictly increasing along $\widetilde{\mathcal{W}}_k^+(u_L)$, and (i) is proved. To prove (ii), we use (3.7), (3.13), (2.9) and (3.8) to compute

$$\begin{aligned}
&\frac{d}{dm}\varphi\big(\hat\mu(a_R; v_k(m;u_L), a_L); v_k(m;u_L), a_L\big)\\
&\quad = \big(\partial_m\varphi\, D_{u_-}\hat\mu + D_{u_-}\varphi\big).\,\partial_m v_k(m;u_L)\\
&\quad \sim \left(\frac{1}{a(u_L)}\, D_{u_-}\hat\mu + D_{u_-}\varphi\right).\left(\frac{1}{a(u_L)} r_k(u_L) + \mathcal{O}\big(m-\mu_k(u_L)\big)\right)\\
&\quad \sim \frac{1}{a(u_L)}\, D_{u_-}\varphi.\, r_k(u_L)\\
&\quad \sim \frac{1}{a(u_L)}\, r_k(u_L),
\end{aligned}$$

for u_L sufficiently close to the transition surface, and $|a_L - a_R|$ sufficiently small.

In order to establish (iii), we need (2.9) and (3.7), which give

$$\begin{aligned}
&\frac{d}{dm} v_k\Big(\tilde\mu_k\big(\varphi(m;u_L,a_L)\big); \varphi(m;u_L,a_L)\Big) =\\
&= \partial_m v_k(\tilde\mu(u');u')\,\nabla\tilde\mu_k(u')\cdot\partial_m\varphi + D_u v_k(\tilde\mu(u');u').\,\partial_m\varphi\\
&= \left(\frac{1}{a(u')}\, r_k(u') + \mathcal{O}\big(\tilde\mu_k(u') - \mu_k(u')\big)\right)\nabla\tilde\mu_k(u')\cdot\left(\frac{1}{a(u_L)}\, r_0(u_L) + \mathcal{O}(m-\mu_k(u_L))\right)\\
&\qquad + D_u v_k(\tilde\mu(u');u').\left(\frac{1}{a(u_L)}\, r_0(u_L) + \mathcal{O}(m-\mu_k(u_L))\right)\\
&\sim \frac{1}{a(u')}\, r_k(u')\ \frac{1}{a(u_L)}\nabla\tilde\mu_k(u')\cdot r_0(u_L) + \frac{1}{a(u_L)}\, D_u v_k(\tilde\mu(u');u').\, r_0(u_L)
\end{aligned} \tag{3.15}$$

near $\mathcal{T}$. Moreover we compute

$$u' - u_L = \varphi(m;u_L,a_L) - \varphi(\mu_k(u_L);u_L,a_L)$$

$$
\begin{aligned}
&= \partial_m\varphi(\mu_k(u_L);u_L,a_L)\,\big(m-\mu_k(u_L)\big)+\mathcal{O}\big(m-\mu_k(u_L)\big)^2\\
&= \frac{1}{a(u_L)}r_0(u_L)\,\big(m-\mu_k(u_L)\big)+\mathcal{O}\big(m-\mu_k(u_L)\big)^2
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
\tilde\mu(u') &= \tilde\mu(u_L)+\nabla\tilde\mu_k(u_L)\cdot(u'-u_L)+\mathcal{O}\big(|u'-u_L|^2\big)\\
&\sim \tilde\mu(u_L)+\nabla\tilde\mu_k(u_L)\cdot\left(\frac{1}{a(u_L)}r_0(u_L)\,\big(m-\mu_k(u_L)\big)\right)\\
&\sim \tilde\mu(u_L)+\frac{1}{a(u_L)}\,\nabla\tilde\mu_k(u_L)\cdot r_0(u_L)\,\big(m-\mu_k(u_L)\big)\\
&\sim \tilde\mu(u_L)-\frac{1}{a(u_L)}\,\nabla\lambda_k(u_L)\cdot r_k(u_L)\,\big(m-\mu_k(u_L)\big)\\
&\sim \tilde\mu(u_L)-\big(m-\mu_k(u_L)\big),
\end{aligned}
$$

where we have used (3.2), which also gives

$$
\begin{aligned}
\nabla\tilde\mu_k(u')\cdot r_0(u_L) &\sim \nabla\tilde\mu_k(u_L)\cdot r_0(u_L)-\nabla\big(m-\mu_k(u_L)\big)\cdot r_0(u_L)\\
&\sim \nabla\tilde\mu_k(u_L)\cdot r_0(u_L)+\nabla\lambda_k(u_L)\cdot r_0(u_L)\sim 0.
\end{aligned}
\tag{3.16}
$$

Using (3.16) in (3.15) we get

$$
\frac{d}{dm}v_k\Big(\tilde\mu_k\big(\varphi(m;u_L,a_L)\big);\varphi(m;u_L,a_L)\Big)\sim\frac{1}{a(u_L)}\,r_0(u_L). \tag{3.17}
$$

near $\mathcal{T}$. Thus, together with (3.13) and (3.14), we get the following estimate

$$
\begin{aligned}
&\frac{d}{dm}\varphi\Big(\hat\mu\big(a_R;v_k(\tilde\mu_k(u');u'),\alpha(m;u_L,a_L)\big);v_k(\tilde\mu_k(u');u'),\alpha(m;u_L,a_L)\Big)=\\
&= \partial_m\varphi\left((\nabla_{u_-}\hat\mu)\,\frac{d}{dm}v_k+(\partial_{a_-}\hat\mu)\,\partial_m\alpha\right)+D_{u_-}\varphi.\,\frac{d}{dm}v_k+\partial_{a_-}\varphi\,\partial_m\alpha\\
&\sim \partial_m\varphi\left(\frac{1}{a(u_L)}\,\nabla_{u_-}\hat\mu\cdot r_k(u_L)-C\frac{b_0\big(u',\alpha(m;u_L,a_L)\big)}{b_0(u''',a_R)}\right)+\frac{1}{a(u_L)}\,D_{u_-}\varphi.\,r_0(u_L)\\
&\sim -C_1\partial_m\varphi\,\frac{\lambda_k\big(u',\alpha(m;u_L,a_L)\big)}{\lambda_k(u''',a_R)}+\frac{1}{a(u_L)}\,r_0(u_L)\\
&\sim C_2\frac{1}{a(u'')}\,r_0(u'')+\frac{1}{a(u_L)}\,r_0(u_L)
\end{aligned}
$$

for some smooth functions $C, C_1, C_2\sim 1$. Here we have used the fact that $\lambda_k\big(u',\alpha(m;u_L,a_L)\big)$ and $\lambda_k(u''',a_R)$ have opposite signs, since $\big(u',\alpha(m;u_L,a_L)\big)$ and (u''',a_R) lie on opposite sides of $\mathcal{T}$. □

It is clear that when $a_L=a_R$, the curve $\widetilde{\mathcal{W}}_k(u_L)$ coincides with $\mathcal{W}_k(u_L)$. Lemma 2.5 allows us to determine the mutual positions of the singular points $\tilde u'_L$ and u''_L (see Figure 4). In this case the curve $\widetilde{\mathcal{W}}_k(u_L)$ is monotone w.r. to m.

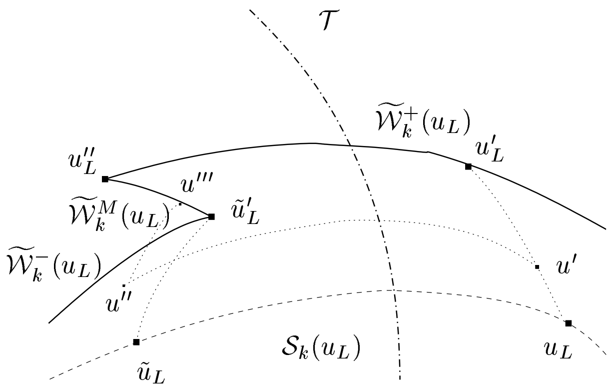


Figure 5: The curve $\widetilde{\mathcal{W}}_k(u_L)$ for $a^T(\tilde{u}_L) \le a_R < a_L$ (case 1b).

Case 1b: $a^T(\tilde{u}_L) \le a_R < a_L$**.** We still have the three branches defined by (A), (B) and (C). By Lemma 2.5 we have $\mu_k(u''_L) < \mu_k(\tilde{u}'_L)$, hence the curve $\widetilde{\mathcal{W}}_k(u_L)$ is no more monotone w.r.t. to m, but presents a bifurcation. More precisely, $\widetilde{\mathcal{W}}_k^M(u_L)$ is now described by

$$\widetilde{\mathcal{W}}_k^M(u_L) := \Big\{ \varphi\Big(\hat{\mu}\big(a_R; u'', \alpha(m; u_L, a_L)\big); u'', \alpha(m; u_L, a_L)\Big), \\ u'' = v_k\Big(\tilde{\mu}_k\big(\varphi(m; u_L, a_L)\big); \varphi(m; u_L, a_L)\Big),\ \hat{\mu}(a_R; u_L, a_L) \le m \le \mu_k(u_L)\Big\}.$$

By Lemma 3.2, the map $m \to \lambda_k(u''')$ is now strictly decreasing when we move from $\mu_k(u_L)$ to $\hat{\mu}(a_R; u_L, a_L)$ (Fig. 5).

Case 1c: $a^T(u_L) \le a_R < a^T(\tilde{u}_L)$**.** Let us define the point $(\hat{u}_L, a_L) \in \mathcal{T}^-$ such that

$$a^T(\hat{u}_L) = a_R. \tag{3.18}$$

By construction, we have

$$\mu_k(\bar{u}_L) \le \mu_k(\hat{u}_L) < \mu_k(\tilde{u}_L) = \tilde{\mu}_k(u_L)$$

(see Lemma 2.5 and Fig. 3). Moreover we define the value $\overline{a} \in [a_L, a_R]$ such that

$$f\Big(\varphi_+\big(\hat{\mu}(\overline{a}; u_L, a_L); u_L, a_L\big), \overline{a}\Big) = f\Big(\varphi_-\big(\hat{\mu}(\overline{a}; \hat{u}_L, a_L); \hat{u}_L, a_L\big), \overline{a}\Big).$$

That is, $\overline{a}$ is the level at which the standing wave issuing from $(\hat{u}_L, a_L)$ intersects the transformed standing wave corresponding to $Z(\cdot; u_L, a_L)$.

Case (A) still holds, while we have cases (B) and (C) changed into

(B') use a k-wave with speed $\lambda_k \le \overline{\lambda}_k(\mu_k(\hat{u}_L); u_L)$ at constant level a_L followed by a standing wave;

(C') move first along the standing wave curve up to an intermediate level $a_M < \overline{a}$, jump on the other side of $\mathcal{T}$ by means of a 0-speed k-shock and then use another standing wave to reach a_R.

Hence branches $\widetilde{\mathcal{W}}_k^-(u_L)$ and $\widetilde{\mathcal{W}}_k^M(u_L)$ become

$$
\begin{aligned}
\widetilde{\mathcal{W}}_k^-(u_L) &:= \Big\{\varphi\big(\hat\mu(a_R; v_k(m;u_L), a_L); v_k(m;u_L), a_L\big),\ m \in \big(-\epsilon, \mu_k(\hat u_L)\big]\Big\},\\
\widetilde{\mathcal{W}}_k^M(u_L) &:= \Big\{\varphi\Big(\hat\mu\big(a_R; u'', \alpha(m;u_L,a_L)\big); u'', \alpha(m;u_L,a_L)\Big),\\
&\qquad u'' = v_k\Big(\tilde\mu_k\big(\varphi(m;u_L,a_L)\big); \varphi(m;u_L,a_L)\Big),\ \hat\mu(a_R;u_L,a_L) \le m \le \hat\mu(\overline{a};u_L,a_L)\Big\}.
\end{aligned}
$$

Lemma 3.2 still applies, showing that even in this case the curve $\widetilde{\mathcal{W}}_k(u_L)$ presents a bifurcation, since the map $m \to \lambda_k(u''')$ is now strictly decreasing when we move from $\hat\mu(\overline{a};u_L,a_L)$ to $\hat\mu(a_R;u_L,a_L)$ (Fig. 5).

Case 1d: $a_R < a^T(u_L)$**.** We take $(\hat u_L, a_L) \in \mathcal{T}^-$ as defined in (3.18), except that now we have

$$
\mu_k(\hat u_L) < \mu_k(\bar u_L).
$$

This time we can distinguish only two paths:

(B") use a k-wave with speed $\lambda_k \le \overline{\lambda}_k(\mu_k(\hat u_L); u_L)$ at constant level a_L followed by a standing wave;

(C") use a shock of speed $\lambda_k = \overline{\lambda}_k(\mu_k(\hat u_L); u_L)$ at level a_L, then move along the standing wave curve up to $(\varphi(\hat\mu(a_R;\hat u_L, a_L); \hat u_L, a_L), a_R) \in \mathcal{T}$, and finally follow the rarefaction curve $w_k\big(m; \varphi(\hat\mu(a_R;\hat u_L,a_L);\hat u_L,a_L)\big)$ with positive speed at level a_R.

These define respectively the following two branches for $\widetilde{\mathcal{W}}_k(u_L)$

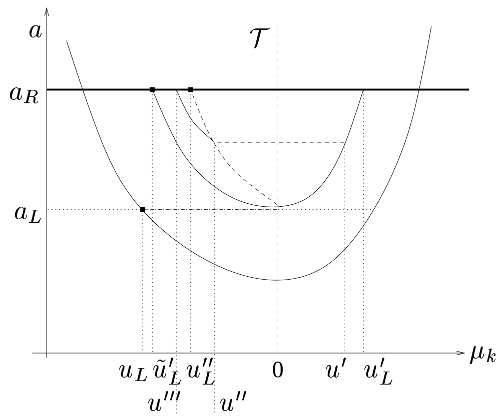
$$
\begin{aligned}
\widetilde{\mathcal{W}}_k^-(u_L) &:= \Big\{\varphi\big(\hat\mu(a_R; v_k(m;u_L), a_L); v_k(m;u_L), a_L\big),\ m \in \big(-\epsilon, \mu_k(\hat u_L)\big]\Big\},\\
\widetilde{\mathcal{W}}_k^+(u_L) &:= \Big\{w_k\big(m; \varphi(\hat\mu(a_R;\hat u_L,a_L);\hat u_L,a_L)\big),\ m \in [0,\epsilon)\Big\},
\end{aligned}
$$

which are monotone increasing w.r. to m as in Lemma 3.2.

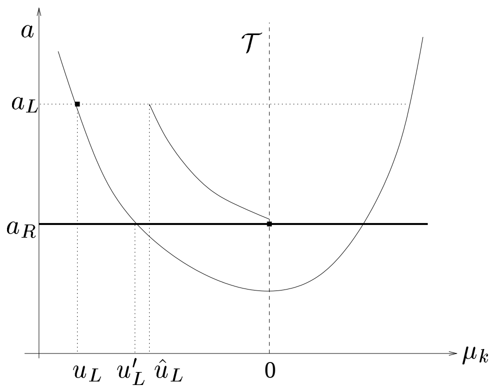
CASE 2. We now study the case $U_L \in \mathcal{T}^-$. The main difference in the construction of the 0-k-waves is that in this case we cannot "jump" on the other side of $\mathcal{T}$ by means of a k-shock. The analysis will be divided into three subcases, depending on the value of a_R.

Case 2a: $a_R \ge a_L$**.** We can use the following three paths:

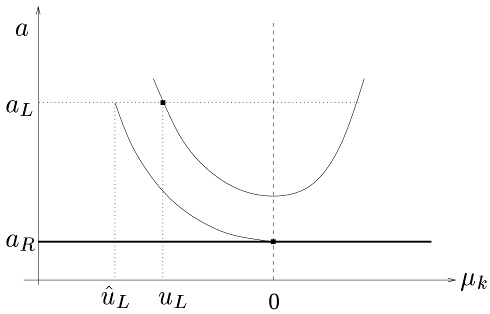
(A) use a k-rarefaction with nonpositive speed at constant level a_L up to $(w_k(0;u_L), a_L) \in \mathcal{T}$, followed by the positive branch of the standing wave up to level a_R, and finally move along a k-wave with nonnegative speed at constant level a_R;



Case 2a



Case 2b



Case 2c

Figure 6: CASE 2.

(B) use a k-wave with nonpositive speed at constant level a_L followed by a standing wave;

(C) move first along the k-rarefaction with nonpositive speed up to the point $(w_k(0;u_L),a_L) \in \mathcal{T}$, then enter $\mathcal{T}^+$ following the positive branch of the standing wave curve up to an intermediate level a_M , jump on the other side of $\mathcal{T}$ by means of a 0-speed k-shock and then use another standing wave to reach a_R.

These three cases define respectively

$$
\begin{aligned}
\widetilde{\mathcal{W}}_k^+(u_L) &:= \Big\{\psi_k\big(m;\varphi_+(\hat\mu(a_R;w_k(0;u_L),a_L);w_k(0;u_L),a_L)\big),\\
&\qquad\qquad m \in \big[\tilde\mu_k\big(\varphi_+(\hat\mu(a_R;w_k(0;u_L),a_L);w_k(0;u_L),a_L)\big),\epsilon\big)\Big\},\\
\widetilde{\mathcal{W}}_k^-(u_L) &:= \Big\{\varphi\big(\hat\mu(a_R;\psi_k(m;u_L),a_L);\psi_k(m;u_L),a_L\big),\ m \in \big(-\epsilon,0\big]\Big\},\\
\widetilde{\mathcal{W}}_k^M(u_L) &:= \Big\{\varphi\Big(\hat\mu\big(a_R;u'',\alpha_+(m;w_k(0;u_L),a_L)\big);u'',\alpha_+(m;w_k(0;u_L),a_L)\Big),\\
&\qquad u'' = v_k\Big(\tilde\mu_k\big(\varphi_+(m;w_k(0;u_L),a_L)\big);\varphi_+(m;w_k(0;u_L),a_L)\Big),\\
&\qquad\qquad m \in \big[0,\hat\mu_+(a_R;w_k(0;u_L),a_L)\big]\Big\},
\end{aligned}
$$

where $\hat\mu_+$ means that we are moving along (φ_+,α_+). As in CASE 1, we will also set

$$
\begin{aligned}
u' &= \varphi_+(m;w_k(0;u_L),a_L), \quad a_L \le \alpha_+(m;w_k(0;u_L),a_L) \le a_R,\\
u''' &= \varphi\Big(\hat\mu\big(a_R;u'',\alpha_+(m;w_k(0;u_L),a_L)\big);u'',\alpha_+(m;w_k(0;u_L),a_L)\Big),\\
u'_L &= \varphi_+\big(\hat\mu(a_R;w_k(0;u_L),a_L);w_k(0;u_L),a_L\big),\\
u''_L &= v_k\big(\tilde\mu_k(u'_L);u'_L\big),\\
\tilde u'_L &= \varphi_-\big(\hat\mu(a_R;w_k(0;u_L),a_L);w_k(0;u_L),a_L\big)
\end{aligned}
$$

(see Fig. 6). The proof of the following lemma is very similar to the proof of Lemma 3.2.

Lemma 3.3 *There exists $\delta_1 < \delta_0$ such that for $u_L \in \mathcal{B}(u_*,\delta_1)$ with $\mu_k(u_L) < 0$, and $|a_L - a_R| \le \delta_1$, the parameter m is strictly monotone along each branch of the curve $\widetilde{\mathcal{W}}_k(u_L)$. More precisely*

(i) $m \mapsto \lambda_k\big(\psi_k(m;u'_L)\big)$ is strictly increasing for $m \in \big[\tilde\mu_k(u'_L),\epsilon\big)$;

(ii) $m \mapsto \lambda_k\Big(\varphi\big(\hat\mu(a_R;\psi_k(m;u_L),a_L);\psi_k(m;u_L),a_L\big)\Big)$ is strictly increasing for $m \in (-\epsilon,0]$;

(iii) $m \mapsto \lambda_k(u''')$ is strictly increasing for $m \in \big[0,\hat\mu(a_R;w_k(0;u_L),a_L)\big]$.

Moreover, by Lemma 2.4 and Lemma 2.5, the singular points $\tilde u'_L$ and u''_L are placed so that $\mu_k(\tilde u'_L) \le \mu_k(u''_L) < 0$. This shows that the curve $\widetilde{\mathcal{W}}_k(u_L)$ is monotone w.r. to m.

Case 2b: $a^T(u_L) \le a_R < a_L$. In this case we have only two branches, defined by cases

(A') use a k-rarefaction with nonpositive speed at constant level a_L up to $\hat{u}_L$ (defined as in (3.18)), then the standing wave up to level a_R, and finally move along a k-rarefaction with nonnegative speed at constant level a_R;

(B') use a k-wave with nonpositive speed at constant level a_L followed by a standing wave;

We obtain respectively the following two branches

$$\begin{aligned}\widetilde{\mathcal{W}}_k^+(u_L) &:= \Big\{ w_k\big(m; \varphi(\hat{\mu}(a_R; \hat{u}_L, a_L); \hat{u}_L, a_L)\big),\ m \in [0, \epsilon) \Big\}, \\ \widetilde{\mathcal{W}}_k^-(u_L) &:= \Big\{ \varphi\big(\hat{\mu}(a_R; \psi_k(m; u_L), a_L); \psi_k(m; u_L), a_L\big),\ m \in \big(-\epsilon, \mu_k(\hat{u}_L)\big] \Big\}.\end{aligned}$$

Again, by Lemma 3.3, the 0-k-curve $\widetilde{\mathcal{W}}_k(u_L)$ is monotone.

Case 2c: $a_R < a^T(u_L)$**.** It is very similar to the previous case, apart from the position of $\hat{u}_L$, which makes (A') changed into

(A") use a k-shock with nonpositive speed at constant level a_L up to $\hat{u}_L$, then the standing wave up to level a_R, and finally move along a k-rarefaction with nonnegative speed at constant level a_R.

$\widetilde{\mathcal{W}}_k(u_L)$ is still monotone.

4 The Riemann problem

We are now ready to solve the Riemann problem (1.1)-(1.3). First of all, since the parametrization of the 0-k-curve exhibit jumps at the points connecting together the various branches (see the points $\tilde{u}'_L$, u''_L or the intersection point with $\mathcal{T}$ in Section 3), it is convenient to reparametrize the curve by choosing a global parameter s so that

$$\mu_k\big(W_{0,k}(s; u_L, a_L, a_R)\big) = s$$

(we set $s = m$ for the curves belonging to families $i \neq k$).

It is clear from the construction performed in Section 3 that the 0-k-curve is merely Lipschitz continuous at the points $\tilde{u}'_L$ and u''_L (Cases 1a, 1b, 1c and 2a) or at $s = 0$ (Cases 1d, 2b and 2c), even when there is no bifurcation phenomena. So, it is necessary to rely on the implicit function theorem for Lipschitz continuous mapping [4] to obtain existence (and uniqueness) of the solution. See also [10, 13].

In addition to the lack of regularity of the wave curves, we have to handle the bifurcation phenomena. So, we propose here to extend smoothly each branch of the wave curve in the

r_k direction. The corresponding curves are denoted by

$$\begin{aligned}
\widetilde{\mathcal{W}}_k^+(u) &= \left\{ \widetilde{\psi}_k^+(s;u),\ s \in (-\epsilon, \epsilon) \right\}, \\
\widetilde{\mathcal{W}}_k^-(u) &= \left\{ \widetilde{\psi}_k^-(s;u),\ s \in (-\epsilon, \epsilon) \right\}, \\
\widetilde{\mathcal{W}}_k^M(u) &= \left\{ \widetilde{\psi}_k^M(s;u),\ s \in (-\epsilon, \epsilon) \right\}.
\end{aligned}$$

We set $\tilde{s}_i = s_i - \mu_i(u)$ for $i \neq k$, et $\tilde{s}_k = s_k - \mu_i(u')$, where

$$u' = \begin{cases} \varphi\big(\hat{\mu}(a_R; u, a_L); u, a_L\big), & \text{if } a_R \geq a^T(u), \\ W_{0,k}(0; u, a_L, a_R), & \text{otherwise.} \end{cases}$$

Hence each mapping

$$\begin{aligned}
\tilde{s} = (\tilde{s}_1, \ldots, \tilde{s}_n) \in (-\epsilon, \epsilon)^n \to \Psi_k^+(\tilde{s}) &= \psi_n(\tilde{s}_n) \circ \ldots \circ \widetilde{\psi}_k^+(\tilde{s}_k) \circ \ldots \circ \psi_1(\tilde{s}_1)(u_L), \\
\tilde{s} = (\tilde{s}_1, \ldots, \tilde{s}_n) \in (-\epsilon, \epsilon)^n \to \Psi_k^-(\tilde{s}) &= \psi_n(\tilde{s}_n) \circ \ldots \circ \widetilde{\psi}_k^-(\tilde{s}_k) \circ \ldots \circ \psi_1(\tilde{s}_1)(u_L), \\
\tilde{s} = (\tilde{s}_1, \ldots, \tilde{s}_n) \in (-\epsilon, \epsilon)^n \to \Psi_k^M(\tilde{s}) &= \psi_n(\tilde{s}_n) \circ \ldots \circ \widetilde{\psi}_k^M(\tilde{s}_k) \circ \ldots \circ \psi_1(\tilde{s}_1)(u_L)
\end{aligned}$$

is a $\mathcal{C}^2$ diffeomorphism from a neighborhood of $0 \in I\!\!R^n$ onto a neighborhood of u'_L. This follows from the implicit function theorem since the partial derivatives at $\tilde{s} = 0$ are

$$\begin{aligned}
\partial_i \Psi^{\pm,M}(0) &= \alpha_i r_i(u_L, a_L), \quad i < k, \\
\partial_k \Psi^{\pm,M}(0) &\sim \alpha_k r_k(u'_L, a_L), \\
\partial_i \Psi^{\pm,M}(0) &\sim \alpha_i r_i(u'_L, a_R), \quad i > k,
\end{aligned}$$

$\alpha_i \neq 0$. By the strict hyperbolicity and the continuity of $D_u f(u,a)$ the differentials $D\Psi^{\pm,M}(0)$ are invertible $n \times n$ matrices. Hence there exists $\delta > 0$ such that, for all $u_L \in \mathcal{B}(u_*, \delta_0)$, if $|u_R - u_L| \leq \delta$ and $|a_R - a_L| \leq \delta$ then

$$\begin{aligned}
u_R &= \psi_n(s_n^+) \circ \ldots \circ \widetilde{\psi}_k^+(s_k^+) \circ \ldots \circ \psi_1(s_1^+)(u_L), \\
u_R &= \psi_n(s_n^-) \circ \ldots \circ \widetilde{\psi}_k^-(s_k^-) \circ \ldots \circ \psi_1(s_1^-)(u_L), \\
u_R &= \psi_n(s_n^M) \circ \ldots \circ \widetilde{\psi}_k^M(s_k^M) \circ \ldots \circ \psi_1(s_1^M)(u_L),
\end{aligned}$$

for some $s_1^{\pm,M}, \ldots, s_n^{\pm,M}$. Equivalently, if the points $\omega_0^{\pm,M}, \ldots, \omega_n^{\pm,M}$ are inductively defined by

$$\omega_0 = u_L, \qquad \omega_i^{\pm,M} = \psi_i(s_i^{\pm,M})(\omega_{i-1}),$$

in particular

$$\omega_k^{\pm,M} = \widetilde{\psi}_k^{\pm,M}(s_k^{\pm,M})(\omega_{k-1}),$$

then $\omega_n^{\pm,M} = u_R$. At this stage we keep into account only the solutions that are physically admissible, that is, we keep the values $s_1, \ldots, s_n$ for which the point ω_k belongs to one of the original branches of $\widetilde{\mathcal{W}}_k(\omega_{k-1})$. Due to the transversality of each characteristic field (we recall that each branch of $\widetilde{\mathcal{W}}_k(\omega_{k-1})$ is essentially parallel to r_k, and then transverse to any other curve $\mathcal{W}_i(\omega_i)$, $i \neq k$), we can distinguish the following cases:

- if $\widetilde{\mathcal{W}}_k(\omega_{k-1})$ has no bifurcations (Cases 1a, 1d and CASE 2), the solution is unique, due to the monotonicity of the parameter s in the r_k direction;
- in Cases 1b and 1c we may have up to three solutions (with one point ω_k on each branch of $\widetilde{\mathcal{W}}_k(\omega_{k-1})$), which reduce to two for $s_k = \mu_k(\tilde{u}'_L)$ or $s_k = \mu_k(u''_L)$.

Now assume that

$$u_R = \psi_n(s_n) \circ \ldots \circ \widetilde{\psi}_k^{\pm,M}(s_k) \circ \ldots \circ \psi_1(s_1)(u_L).$$

When $i \neq k$, each Riemann problem with initial data

$$u(x,0) = \begin{cases} \omega_{i-1}, & x < 0, \\ \omega_i, & x > 0, \end{cases} \qquad a(x,0) = \begin{cases} a_L, & i < k, \\ a_R, & i > k, \end{cases} \tag{4.1}$$

has an entropy-admissible, self-similar (that is, $u = u(x/t)$) solution made of two constant states separated by a contact discontinuity, a shock or a rarefaction fan.

For $i = k$, the Riemann problem with initial data

$$(u,a)(x,0) = \begin{cases} (\omega_{k-1}, a_L), & x < 0, \\ (\omega_k, a_R), & x > 0, \end{cases} \tag{4.2}$$

has a self-similar solution made of two or more constant states separated by shocks, rarefactions or standing waves, that can have the same speed (equal to 0), and then be superposed. More precisely

- if $\omega_k \in \widetilde{\mathcal{W}}_k^+(\omega_{k-1})$:

CASE 1 $\omega_{k-1} \to \omega'_{k-1}$ by a standing contact discontinuity,
$\omega'_{k-1} \to \omega_k$ by a shock or a rarefaction with nonnegative speed (one may have $\omega'_{k-1} = \omega_k$);

CASE 2 $\omega_{k-1} \to \omega'_{k-1}$ by a shock or a rarefaction with nonpositive speed,
$\omega'_{k-1} \to \omega''_{k-1}$ by a standing contact discontinuity,
$\omega''_{k-1} \to \omega_k$ by a shock or a rarefaction with nonnegative speed;

- if $\omega_k \in \widetilde{\mathcal{W}}_k^-(\omega_{k-1})$: $\omega_{k-1} \to \omega'_{k-1}$ by a shock or a rarefaction with nonpositive speed, $\omega'_{k-1} \to \omega_k$ by a standing contact discontinuity;
- if $\omega_k \in \widetilde{\mathcal{W}}_k^M(\omega_{k-1})$:

CASE 1 $\omega_{k-1} \to \omega'_{k-1}$ by a standing contact discontinuity,
$\omega'_{k-1} \to \omega''_{k-1}$ by a zero speed shock,
$\omega''_{k-1} \to \omega_k$ by a standing contact discontinuity;

CASE 2 $\omega_{k-1} \to \omega'_{k-1}$ by a rarefaction with nonpositive speed,
$\omega'_{k-1} \to \omega''_{k-1}$ by a standing contact discontinuity,
$\omega''_{k-1} \to \omega'''_{k-1}$ by a zero speed shock,
$\omega'''_{k-1} \to \omega_k$ by a standing contact discontinuity;

The solution to the original problem (1.1)-(1.3) can now be constructed by piecing together the solutions of the n Riemann problems (4.1)-(4.2) on different sectors of the (x,t) plane. Indeed for $\tilde{s}_1, \ldots, \tilde{s}_n$ sufficiently small, and a_R sufficiently close to a_L, the speed of each wave remains close to the corresponding eigenvalue $\lambda_i(u_L, a_L)$ of the matrix $D_u f(u_L, a_L)$. By the strict hyperbolicity and continuity properties, we can thus assume that the wave speeds remain distinct.

In conclusion, we have proved the following:

Theorem 4.1 (The Riemann solver.) *Suppose that, in $\mathcal{B}(u_*, \delta_0)$, the system* (1.1) *is strictly hyperbolic and admits only genuinely nonlinear or linearly degenerate fields. Under the assumptions* (1.5), (1.6), (1.7), (1.8), *there exists $\delta_1 < \delta_0$ such that the following holds. Given any $u_L, u_R \in \mathcal{B}(u_*, \delta_1)$, the Riemann problem* (1.1)-(1.3) *admits at most three self-similar solutions made up of $n+1$ constant states*

$$\omega_0 = u_L, \omega_1, \ldots, \omega_n = u_R$$

separated by elementary waves (shocks, rarefactions or contact discontinuities). Moreover, the states ω_{k-1} and ω_k are connected by at most three intermediate states $\{\omega_k^i\}_{1 \le i \le j,\ j \le 3}$ separated by a standing wave and possibly shocks or rarefactions.

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