ERROR BOUNDS FOR MONOTONE APPROXIMATION SCHEMES FOR NON-CONVEX DEGENERATE ELLIPTIC EQUATIONS IN $\mathbb{R}^1$.

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Abstract. In this paper we provide estimates of the rates of convergence of monotone approximation schemes for non-convex equations in one space-dimension. The equations under consideration are the degenerate elliptic Isaacs equations with x-depending coefficients, and the results applies in particular to finite difference methods and control schemes based on the dynamic programming principle. Recently, Krylov, Barles, and Jakobsen obtained similar estimates for convex Hamilton-Jacobi-Bellman equations in arbitrary space-dimensions. Our results extend these to non-convex equations in one space-dimension and are the first results for non-convex second order equations. Furthermore for finite difference equations, we obtain better rates that Krylov and can handle more general equations than Barles and Jakobsen.

1. Introduction

In this paper we give estimates on the rate of convergence of monotone approximation schemes for Isaacs equations in one space-dimension. The Isaacs equation equation is an equation satisfied by the (upper or lower) value of a stochastic zero-sum differential game, and in one space-dimension it may take the form

$$F(x, u, u_x, u_{xx}) = 0 \quad \text{in} \quad \mathbb{R}, \tag{1.1}$$

where

$$F(x, r, p, X) = \inf_{\alpha\in\mathcal{A}} \sup_{\beta\in\mathcal{B}} \left\{ a^{\alpha,\beta}(x)X + b^{\alpha,\beta}(x)p + c^{\alpha,\beta}(x)r + f^{\alpha,\beta}(x) \right\}.$$

where a, b, c, f are bounded and continuous in all variables, and $\mathcal{A}, \mathcal{B}$ are some usually compact sets. In addition we require that $a \geq 0$, which makes the equation degenerate elliptic [10], and that $c \leq 0$. The two last conditions make the equation proper in the sense of [10]. We remark that this is a fully non-linear and non-convex equation in general.

It is known that under the conditions mentioned above, equation (1.1) need not have classical solutions. The correct notion of weak solutions has been proved to be the notion of viscosity solutions, see Fleming and Souganidis [15]. A viscosity solution u need only be continuous by its definition, and informally speaking, the equation has to be satisfied (as an inequality) by smooth test-functions ϕ only at maximum or minimum points of $u - \phi$. For more informations about viscosity solutions, we refer to the User's Guide [10] and Fleming and Soner [14].

Date: November 25, 2002.

Key words and phrases. degenerate elliptic equations, non-convex equations, Isaacs equations, numerical schemes, convergence rate, finite difference methods, differential games, dynamic programming, viscosity solutions, regularity.

Before we discuss approximation schemes for the Isaacs equation (1.1), we mention that this equation is in some sense a prototype non-convex equation. E.g. consider a simple example where $F:\mathbb{R}\to\mathbb{R}$ is locally Lipschitz continuous, then it is easy to show that $F(t)=\inf_{\alpha\in\mathcal{A}}\sup_{\beta\in\mathcal{B}}\{a^{\alpha,\beta}t+f^{\alpha,\beta}\}$ for some coefficients a,f and sets $\mathcal{A},\mathcal{B}\subset\mathbb{R}$. So

$$F(u_{xx})=g(x) \quad\Longleftrightarrow\quad \inf_{\alpha\in\mathcal{A}}\sup_{\beta\in\mathcal{B}}\{a^{\alpha,\beta}u_{xx}+f^{\alpha,\beta}-g(x)\}=0.$$

We refer to Katsoulakis [19] for a very general way of rewriting non-linear degenerate equations as Isaacs equations. We also mention that if $\mathcal{A}$ (or $\mathcal{B}$) is a singleton, then equation (1.1) reduces to the much studied convex (or concave) Hamilton-Jacobi-Bellman equation (from now on HJB equation) associated with optimal control of diffusion processes [14, 23, 24, 25].

In order to compute the solution of (1.1), numerical schemes have been derived and studied for a long time. We refer the reader to the book of Kushner [22] and the articles by Lions and Mercier [26], Crandall and Lions [12], Menaldi [27], Souganidis [28, 29], Camilli and Falcone [8], Soravia [30], and Bonnans and Zidani [5] for the derivation and properties of such schemes, including some proofs of convergence and of the rate of convergence. See also the books of Bardi and Capuzzo-Dolcetta [4], and Fleming and Soner [14] and the review article by Bardi, Falcone and Soravia [1]. The convergence can be obtained in a very general setting either by probabilistic methods (for convex HJB equations, Kushner [22]) or by viscosity solution methods (for general equations, Barles and Souganidis [3]). But until recently there were almost no results on the rate of convergence of such schemes in the degenerate elliptic case where the solution is expected to have only $C^{0,\delta}$ or $W^{1,\infty}$ regularity (see the above references). Viscosity solution methods were providing this rate of convergence only for first-order equations [28, 29, 30], i.e. for deterministic differential games, or for convex HJB equations with x-independent coefficients, see Krylov [20]. We also mention the the results for convex HJB equations obtained by Menaldi [27] requiring more regularity of the solutions.

Progress were made recently by Krylov [20, 21] and Barles and Jakobsen [2, 17] for convex HJB equations. Krylov obtained general results on the rate of convergence of finite difference schemes by combining PDE and probabilistic methods. Barles and Jakobsen then extended Krylov's results to a more general class of approximation schemes, including control schemes [8] based on the dynamical programming principle [14]. The approach of Barles and Jakobsen is a pure PDE approach and is simpler (author's opinion) than that of Krylov.

In spite of these recent advances, there are to the best of the authors knowledge, no results providing the rate of convergence when the equation is second-order and non-convex. It is the purpose of this paper to remedy this situation, at least in the one space-dimensional case.

The natural and classical way to prove a rate of convergence for approximation schemes for equation (1.1) is to look for a sequence of smooth approximate solutions v^ε of (1.1). If such a sequence $(v^\varepsilon)_\varepsilon$ exists, with precise bounds on $\|u-v^\varepsilon\|_{L^\infty(\mathbb{R})}$ and the derivatives of v^ε, then in order to obtain an estimate of $\|v_\varepsilon-u_h\|_{L^\infty(\mathbb{R})}$ one just has to plug v^ε into the scheme and use consistency along with some comparison properties of the scheme. This yields an estimate of $\|u-u_h\|_{L^\infty(\mathbb{R})}$ depending on ε and h, and the rate of convergence then follows from optimizing with respect to ε. Unfortunately, it is difficult to carry out this program, and to the best of our knowledge, nobody has been able to prove the existence of such a sequence when the data a, b, c, f depends on x. In this paper we will see that it can be done at least in one space-dimension.

A natural way to obtain a smooth approximate solution of (1.1), is to use the vanishing viscosity method. So we consider the following equation as an approximation of equation (1.1):

$$F_\varepsilon(x, u, u_x, u_{xx}) = 0 \quad \text{in} \quad \mathbb{R}, \tag{1.2}$$

where

$$F_\varepsilon(x, r, p, X) = \inf_{\alpha\in\mathcal{A}} \sup_{\beta\in\mathcal{B}} \left\{ a_\varepsilon^{\alpha,\beta}(x)X + b^{\alpha,\beta}(x)p + c^{\alpha,\beta}(x)r + f^{\alpha,\beta}(x) \right\},$$

and for some constant C,

$$a_\varepsilon \geq \varepsilon \quad \text{and} \quad \|a_\varepsilon - a\|_{L^\infty(\mathbb{R})} \leq C\varepsilon.$$

Equation (1.2) is uniformly elliptic, and it can be proved under natural assumptions on the coefficients that $\|u - u^\varepsilon\|_{L^\infty(\mathbb{R})} \leq C\sqrt{\varepsilon}$, where u, u^ε solves (1.1) and (1.2) respectively (see [18]). The main problem is that such equations need only have $\mathrm{C}^{1,\alpha}_{\mathrm{loc}}$ solutions, at least in higher space-dimensions [7, 6, 11]. And anyway, it seems difficult to obtain the necessary estimates on the higher order derivatives (they need to optimal w.r.t to their dependence on ε).

As is often the case, the situation in one space-dimension is easier than the general case. A simple observation allows us to rewrite equation (1.2) as an equivalent semi-linear equation

$$u_{xx} + \inf_{\alpha\in\mathcal{A}} \sup_{\beta\in\mathcal{B}} \left\{ \frac{b^{\alpha,\beta}(x)}{a_\varepsilon^{\alpha,\beta}(x)} u_x + \frac{c^{\alpha,\beta}(x)}{a_\varepsilon^{\alpha,\beta}(x)} u + \frac{f^{\alpha,\beta}(x)}{a_\varepsilon^{\alpha,\beta}(x)} \right\} = 0 \quad \text{in} \quad \mathbb{R}. \tag{1.3}$$

It is this rewriting of equation (1.2), that allows us to obtain the required regularity and estimates. We remark that such a rewriting of the equation is *only* possible when the equation is uniformly elliptic, it could not be done for our original Isaacs equation (1.1).

Let us now consider the approximation schemes. An approximation scheme for equation (1.1) will be written as

$$S(h, x, u_h(x), [u_h]^h_x) = 0 \quad \text{for all } x \in \mathbb{R}, \tag{1.4}$$

where h is a small parameter which measures typically the mesh size, $u_h : \mathbb{R} \to \mathbb{R}$ is the approximation of u and the solution of the scheme, $[u_h]^h_x$ is a function defined at x from u_h. Finally S is the approximation scheme. Using similar notation, we also consider approximation schemes for the vanishing viscosity equation (1.2):

$$S_\varepsilon(h, x, u^\varepsilon_h(x), [u^\varepsilon_h]^h_x) = 0 \quad \text{for all } x \in \mathbb{R}. \tag{1.5}$$

We remark that (1.5) is also a scheme for the Isaacs equation (1.1) if we choose ε to be an increasing function of h which is zero for $h = 0$. In this paper we present two types of results:

1. Assume the scheme (1.5) is consistent and monotone and that $\varepsilon = h^k$ for some $k > 0$ to be defined later. Then u^ε_h, the solution of (1.5), converges to u, the solution of the Isaacs equation (1.1) with prescribed rate of convergence.

2. Assume (1.4) and (1.5) are consistent and monotone schemes, and that the the estimate $\|u_h - u_h^\varepsilon\|_{L^\infty(\mathbb{R})} \leq C\sqrt{\varepsilon}$ holds. Then u_h, the solution of (1.4), converges to u, the solution of the Isaacs equation (1.1) with prescribed rate of convergence.

Paragraph 1 above implies that for *any* consistent, monotone scheme for equation (1.1), by adding a small viscous term to the scheme, you obtain a new scheme with prescribed convergence rate. (Of course you also need regularity assumptions on the data a, b, c, f). In particular, in this way the rate of convergence can be obtained for finite difference schemes with x-depending a's. This was a case not handled by [2, 17]. Krylov [21] has results for x-depending a's and (convex) HJB equations, but the rate of convergence obtained here is better than his. The assumptions on the scheme given in paragraph 2 above corresponds to the assumptions on the schemes given in [2, 17]. But as we will see later, the rates of convergence obtained here, are worse than what was obtained there (for convex equations). This seems to be a consequence of the regularization procedures used, see Section 3 for a brief discussion.

Finally, we mention that Deckelnick in [13] used a similar (classical) procedure to analyze a different problem: He estimates the rate of convergence of a numerical scheme for the mean curvature equation. We refer to Section 3 for some further remarks concerning this.

The outline of this paper is as follows. In Section 2 we present the notation, the main result, and the proof of the main result. Then in Section 3, we apply the main result to finite difference schemes and control schemes and obtain their rates of convergence. We also give some comments about the optimality of these rates. Finally, in Section 4, we study the vanishing viscosity equation (1.2) and obtain some estimates needed in the proof of the main result in Section 2.

2. The Main Result.

We start by introducing the notation we need. First, by $\mathrm{C}_b(\mathbb{R})$ and $\mathrm{C}^k(\mathbb{R})$, $k = 1, 2, \ldots, \infty$, denote the spaces of bounded continuous functions and functions with k continuous derivatives respectively. For any function $f : \mathbb{R} \to \mathbb{R}$, we define the following (semi) norms

$$|f|_0 = \operatorname*{ess\,sup}_{x\in\mathbb{R}} |f(x)| \qquad \text{and} \qquad [f]_\mu = \operatorname*{ess\,sup}_{\substack{x,y\in\mathbb{R}\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\mu} \quad \text{for } \mu \in (0,1].$$

Then by $\mathrm{C}^{k,\mu}(\mathbb{R})$, with $k = 0, 1, 2, \ldots$ and $\mu \in (0, 1]$, we denote the (Sobolev) spaces of functions $f : \mathbb{R} \to \mathbb{R}$ with finite norm $|f|_0 + |\partial_x f|_0 + \cdots + |\partial_x^k f|_0 + [\partial_x^k f]_\mu$. Throughout this paper C will denote (different) constants that does not depend on any variable or parameter in the problem.

We will need the following list of assumptions concerning the Isaacs equation (1.1) and the vanishing viscosity equation (1.2):

$$a = \sigma^2 \geq 0 \text{ and } c \leq -\lambda, \text{ where } \lambda > 0 \text{ is a number,} \tag{2.1}$$

$$\begin{aligned} &\sigma, b, c, f \text{ are bounded and continuous in all arguments, and} \\ &\mathcal{A}, \mathcal{B} \text{ are compact sets.} \end{aligned} \tag{2.2}$$

For all $x, y \in \mathbb{R}$, $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$ the following inequalities hold:

$$|\sigma^{\alpha,\beta}(x) - \sigma^{\alpha,\beta}(y)| + |b^{\alpha,\beta}(x) - b^{\alpha,\beta}(y)| \leq K_1|x-y|, \tag{2.3}$$

$$|c^{\alpha,\beta}(x) - c^{\alpha,\beta}(y)| + |f^{\alpha,\beta}(x) - f^{\alpha,\beta}(y)| \leq K_2|x-y|. \tag{2.4}$$

Finally, we need to specify a^ε in equation (1.2).

$$a^\varepsilon = (\sigma + \sqrt{\varepsilon})^2 \quad \text{where } \sigma \text{ is given by (2.1).} \tag{2.5}$$

We now give a standard well-posedness result for the Isaacs equation (1.1):

Theorem 2.1 (Existence and uniqueness)**.** *Assume* (2.1)–(2.4) *and* $\lambda > K_1$. *Then the Isaacs equation* (1.1) *has a unique viscosity solution in* $\mathrm{C}^{0,1}(\mathbb{R})$.

The proof is standard, see [10, 18]. We remark that if $\lambda \in (0, K_1)$, then we have existence and uniqueness of a viscosity solution in $\mathrm{C}^{0,\mu}(\mathbb{R})$, where $\mu = \lambda/K_1 < 1$, see Lions [23] section II.4. For simplicity, in this paper we only work with Lipschitz continuous ($\mathrm{C}^{0,1}(\mathbb{R})$) solutions. But all results extend to Hölder continuous solutions, see [2].

Now consider the approximation scheme (1.4). Here we require that the following assumptions hold:

(C1) (Monotonicity) Let λ be given by (2.1). For every $h \geq 0$, $x, t \in \mathbb{R}$, $m \geq 0$ and bounded functions u, v such that $u \leq v$ then

$$S(h, x, t+m, [u+m]_x^h) \leq S(h, x, t, [v]_x^h) - \lambda m\ .$$

(C2) (Regularity) For every $h > 0$ and $\phi \in \mathrm{C}_b(\mathbb{R})$, $x \mapsto S(h, x, \phi(x), [\phi]_x^h)$ is bounded and continuous in $\mathbb{R}$ and the function $t \mapsto S(h, x, t, [\phi]_x^h)$ is uniformly continuous for bounded t, uniformly with respect to $x \in \mathbb{R}$.

(C3) (Consistency) There exist $k_1, k_2 \geq 0$ such that for every $v \in \mathrm{C}^{2,1}(\mathbb{R})$, $h \geq 0$ and $x \in \mathbb{R}$

$$|F(x, v, v_x, v_{xx}) - S(h, x, v(x), [v]_x^h)| \leq C(|\partial_x^2 v|_0 h^{k_1} + |\partial_x^3 v|_0 h^{k_2})\ .$$

(C4) (Continuous dependence) If $u_h \in \mathrm{C}_b(\mathbb{R})$ solves (1.4) and $u_h^\varepsilon \in \mathrm{C}_b(\mathbb{R})$ solves (1.5) for $\varepsilon \in (0,1)$, then

$$|u_h - u_h^\varepsilon|_0 \leq C\sqrt{\varepsilon}.$$

By (C1) $S(h, x, t, [u]_x^h)$ is non-increasing in $t \in \mathbb{R}$ and non-decreasing in $[u]_x^h$ for bounded (possibly discontinuous) functions u equipped with the usual partial ordering. Assumption (C3) implies that smooth solutions of the scheme (1.4) will converge towards the solution of equation (1.1). Assumption (C4) is the discrete version of Lemma 2.4 (b) below, which says that $|u - u^\varepsilon|_0 \leq C\sqrt{\varepsilon}$ when u and u^ε solves equations (1.1) and (1.2) respectively. Assumptions (C1) – (C3) are not sufficient for proving (C4). Loosely speaking, what is needed is a doubling of variable argument for the scheme, analogous of the argument for the equation. Such an argument can only be given for certain schemes. E.g. it can not be done for the natural monotone finite difference method for equation (1.1) when a depends on x.

In the sequel, we say that a function $u \in \mathrm{C}_b(\mathbb{R})$ is a subsolution (resp. supersolution) to the scheme (1.4) if

$$S(h, x, u(x), [u]_x^h) \geq 0 \text{ (resp. } \leq 0) \quad \text{for all } x \in \mathbb{R}\ .$$

Note the direction of the inequalities! Assumptions (C1) and (C2) imply a comparison result for bounded continuous solutions of (1.4).

Lemma 2.2 (Comparison)**.** *Assume (C1) and (C2) hold, and let $u, v \in \mathrm{C}_b(\mathbb{R})$ be sub- and supersolutions of* (1.4) *respectively. Then $u \leq v$.*

We refer to [2] (Lemma 2.3) for the simple proof. In the following, we also assume that the scheme (1.5) (denoted S_ε) satisfies (C1) – (C3). This of course means that in (C1) – (C3), S and F have to be replaced by S_ε and F_ε where F_ε refers to the vanishing viscosity equation (1.2). The comparison principle Lemma 2.2 then also applies to sub- and supersolutions of (1.5).

Now we give the main result giving the rate of convergence of approximation schemes for the Isaacs equation (1.1). Two type of approximation schemes are considered. First the scheme (1.5), which approximates the vanishing viscosity equation (1.2). By choosing the parameter $\varepsilon = h^k$ for some $k > 0$, we obtain an approximation scheme for the original Isaacs equation (1.1). Secondly, we consider the scheme (1.4) that approximates directly the Isaacs equation (1.1).

Theorem 2.3 (Rate of convergence)**.** *Assume* (2.1)–(2.5) *hold with $\lambda > K_1$, let u denote the viscosity solution of the Isaacs equation* (1.1)*, and define*

$$\gamma = \min_{\substack{k_i \neq 0 \\ i=1,2}} \left\{ \frac{k_1}{3}, \frac{k_2}{5} \right\},$$

where the k_i's are defined in (C3).

(a) The vanishing viscosity scheme (1.5)*. Assume (C1) – (C3) hold with $S_\varepsilon, F_\varepsilon$ replacing S, F, and that u_h^ε, the solution of* (1.5)*, exists for all $\varepsilon \in (0,1)$. If $\bar{u}_h$ denotes the solution of* (1.5) *when $\varepsilon = h^{2\gamma}$, then for all $h < 1$*

$$|u - \bar{u}_h|_0 \leq Ch^\gamma.$$

(b) The "direct" scheme (1.4)*. Assume (C1) – (C4) hold for both schemes* (1.4) *and* (1.5)*, and that u_h and u_h^ε, the solutions of* (1.4) *and* (1.5)*, exists for all $\varepsilon \in (0,1)$. Then for all $h < 1$*

$$|u - u_h|_0 \leq Ch^\gamma.$$

To prove this result, we need well-posedness and certain properties of the solutions of the vanishing viscosity equation (1.2):

Lemma 2.4. *Assume* (2.1) – (2.5)*.*

(a) Equation (1.2) *has a unique viscosity solution $u^\varepsilon \in \mathrm{C}^{2,1}(\mathbb{R})$ which satisfies*

$$|u^\varepsilon|_0 + |\partial_x u^\varepsilon|_0 + \varepsilon |\partial_x^2 u^\varepsilon|_0 + \varepsilon^2 |\partial_x^3 u^\varepsilon|_0 \leq C.$$

(b) If u and u^ε are respectively the viscosity solution of (1.1) *and* (1.2)*, then*

$$|u - u^\varepsilon|_0 \leq C\sqrt{\varepsilon}.$$

This lemma will be proved in the next section, and it is the key result in this theory. The proof of Theorem 2.3 is now straightforward.

Proof of Theorem 2.3. We start by proving part (a).

1. We substitute u^ε given by Lemma 2.4 into the scheme (1.5), and use the consistency condition for (C3) (applied the scheme S_ε (1.5)) to get

$$S_\varepsilon(h, x, u^\varepsilon, [u^\varepsilon]_x^h) \leq C \left(|\partial_x^2 u^\varepsilon|_0 h^{k_1} + |\partial_x^3 u^\varepsilon|_0 h^{k_2} \right).$$

2. Applying Lemma 2.4 again, we have

$$S_\varepsilon(h,x,u^\varepsilon,[u^\varepsilon]_x^h) \le C\left(\frac{1}{\varepsilon}h^{k_1} + \frac{1}{\varepsilon^2}h^{k_2}\right) := C(h,\varepsilon),$$

and the monotonicity condition (C1) implies

$$S_\varepsilon\left(h,x,u^\varepsilon + C(h,\varepsilon)/\lambda,[u^\varepsilon + C(h,\varepsilon)/\lambda]_x^h\right) \le 0.$$

The comparison theorem for the scheme (1.5), Lemma 2.2, immediately yields

$$u_h^\varepsilon - (u^\varepsilon + C(h,\varepsilon)/\lambda) \le 0.$$

3. Using consistency (C3) once again we get $S_\varepsilon(h,x,u^\varepsilon,[u^\varepsilon]_x^h) \ge -C(h,\varepsilon)$, so monotonicity (C1) yields

$$S_\varepsilon\left(h,x,u^\varepsilon - C(h,\varepsilon)/\lambda,[u^\varepsilon - C(h,\varepsilon)/\lambda]_x^h\right) \ge 0,$$

and by comparison, Lemma 2.2, we have $(u^\varepsilon - C(h,\varepsilon)/\lambda) - u_h^\varepsilon \le 0$. We may therefore conclude by 2. and 3. that

$$|u^\varepsilon - u_h^\varepsilon|_0 \le C(h,\varepsilon)/\lambda.$$

4. Using Lemma 2.4 (b) and 3. we have

$$|u - u_h^\varepsilon|_0 \le |u-u^\varepsilon|_0 + |u^\varepsilon - u_h^\varepsilon|_0 \le C\sqrt{\varepsilon} + C(h,\varepsilon)/\lambda.$$

If we choose $\varepsilon = h^k$ and try to minimize the right hand (note $h<1$), we find that the optimal k is

$$k = \min_{\substack{k_i\neq 0\\ i=1,2}}\left\{\frac{2k_1}{3},\frac{2k_2}{5}\right\}(=2\gamma).$$

For this value of ε, $u_h^\varepsilon = \bar{u}_h$, and we have $|u-\bar{u}_h|_0 \le Ch^{k/2} = Ch^\gamma$ which completes the proof of (a).

Now we can easily prove part (b). Using paragraph 3. above, Lemma 2.4 (b) and (C4), we get

$$|u-u_h|_0 \le |u-u^\varepsilon|_0 + |u^\varepsilon - u_h^\varepsilon|_0 + |u_h^\varepsilon - u_h|_0 \le C\sqrt{\varepsilon} + C(h,\varepsilon)/\lambda + C\sqrt{\varepsilon},$$

and the conclusion follows as in (a). □

We will now consider two important examples where Theorem 2.3 applies.

3. Examples and Comments

Finite difference schemes. The first example is a monotone finite difference scheme [2, 21, 3, 28, 12]. For any function $\phi \in \mathrm{C}_b(\mathbb{R})$, we define following difference operators:

$$\Delta_h^{\pm}\phi(x) = \pm\frac{1}{h}\{\phi(x\pm h) - 2\phi(x)\},$$
$$\Delta_h^2\phi(x) = \frac{1}{h^2}\{\phi(x+h) - 2\phi(x) + \phi(x-h)\}.$$

Then we consider the following finite difference approximation for the vanishing viscosity equation (1.2):

$$\inf_{\alpha\in\mathcal{A}}\sup_{\beta\in\mathcal{B}}\Big\{a_\varepsilon^{\alpha,\beta}(x)\Delta_h^2 u_h^\varepsilon + (b^{\alpha,\beta})^+(x)\Delta_h^+ u_h^\varepsilon - (b^{\alpha,\beta})^-(x)\Delta_h^- u_h^\varepsilon + c^{\alpha,\beta}(x)u_h^\varepsilon + f^{\alpha,\beta}(x)\Big\} = 0 \quad \text{in} \quad \mathbb{R}. \tag{3.1}$$

Here $(\cdot)^+ = \max(\cdot,0)$ and $(\cdot)^- = -\min(\cdot,0)$, and a_ε is given by assumption (2.5). If ε is chosen to be h^k for some $k>0$, then this is an approximation scheme for the original Isaacs equation (1.1).

Following [2] we now want to define the symbol $S_\varepsilon(h,y,t,[\phi]_x^h)$ (for $\phi\in \mathrm{C}_b(\mathbb{R})$) corresponding to the finite difference scheme (3.1). But first we introduce the following "one step transition probabilities"

$$p_\varepsilon^{\alpha,\beta}(x,x) = 1-2a_\varepsilon^{\alpha,\beta}(x) + h|b^{\alpha,\beta}(x)|,$$
$$p_\varepsilon^{\alpha,\beta}(x,x\pm h) = a_\varepsilon^{\alpha,\beta}(x) + hb^{\alpha,\beta\pm}(x),$$

and $p_\varepsilon^{\alpha,\beta}(x,y)=0$ for all other y. Note that if

$$2a_\varepsilon^{\alpha,\beta}(x) + |b^{\alpha,\beta}(x)| \le 1 \quad \text{in} \quad \mathcal{A}\times\mathcal{B}\times\mathbb{R}, \tag{3.2}$$

and $h<1$, then $0\le p_\varepsilon^{\alpha,\beta}(x,y)\le 1$ for all α,β,x,y. Furthermore $\sum_{z\in h\mathbb{Z}} p_\varepsilon^{\alpha,\beta}(x,x+z)=1$ for all α,β,x. Assumption (3.2) may be viewed as a normalization of the coefficients in equation (1.1). We can always have this assumption satisfied by multiplying equation (1.1) by an appropriate positive constant. Now for $\phi\in C_b(\mathbb{R}^N)$, we set $[\phi]_x^h(\cdot):=\phi(x+\cdot)$ and define S_ε by

$$S_\varepsilon(h,y,t,[\phi]_x^h)$$
$$= \inf_{\alpha\in\mathcal{A}}\sup_{\beta\in\mathcal{B}}\left\{\frac{1}{h^2}\left[\sum_{z\in h\mathbb{Z}} p_\varepsilon^{\alpha,\beta}(y,y+z)[\phi]_x^h(z) - t\right] + c^{\alpha,\beta}(x)t + f^{\alpha,\beta}(y)\right\}.$$

It is not difficult to see that this S_ε correspond to the finite difference scheme (3.1), and that if (3.2) hold and $h<1$, then this scheme satisfies (C1) – (C3) with

$$|F_\varepsilon(x,v,Dv,D^2v) - S_\varepsilon(h,x,v(x),[v]_x^h)| \le Ch(|v_{xx}|_0 + |v_{xxx}|_0), \tag{3.3}$$

for any $v\in \mathrm{C}^{2,1}(\mathbb{R}^N)$. Without going into details, we claim that the scheme (3.1) has a unique $\mathrm{C}_b(\mathbb{R})$ solution if (2.1) – (2.5) and (3.2) hold and $h<1$. This follows from easy adaptations of the proofs of Proposition 4.2 in [2] or Lemma 1.7 in [20]. The main idea here is to replace $\sup(\dots)-\sup(\dots)\le\sup(\dots-\dots)$ by

$$\inf\sup(\dots)-\inf\sup(\dots)\le\sup\sup(\dots-\dots).$$

We are now in a position to use Theorem 2.3 to obtain the rate of convergence of the finite difference scheme (3.1).

Theorem 3.1. *Assume* (2.1) – (2.5) *and* (3.2) *hold with* $\lambda>K_1$, $h\le 1$, *and let* u *denote the viscosity solution of the Isaacs equation* (1.1) *and* $\bar{u}_h$ *the solution of the finite difference scheme* (3.1) *when* $\varepsilon = h^{2/5}$. *Then*

$$|u-\bar{u}_h|_0 \le Ch^{1/5}.$$

The rate of convergence obtained here, should be compared with the rates obtained for convex HJB equations (and $\varepsilon=0$): In Barles and Jakobsen [2, 17] the rate was $1/2$. But these results only apply when σ is constant in x. Without this restriction, Krylov [21] obtained the rate 1/27. For first order Isaacs equations ($\sigma \equiv 0$) it is not difficult to see that Theorem 2.3 yields the rate 1/3 for finite difference schemes. This is lower than the rate 1/2 obtained by Souganidis [28] and Crandall and Lions [12] for such equations.

Control schemes. The second example is a control scheme based on the dynamical programming principle [8, 27, 2, 30, 1]. It is defined as follows

$$u_h(x) = \inf_{\alpha\in\mathcal{A}}\sup_{\beta\in\mathcal{B}}\Big\{\big(1+hc^{\alpha,\beta}(x)\big)\,\Pi_h^{\alpha,\beta}u_h(x) + hf^{\alpha,\beta}(x)\Big\} \quad \text{in} \quad \mathbb{R}, \tag{3.4}$$

where for any function $\phi \in \mathrm{C}_b(\mathbb{R})$, the operator $\Pi_h^{\alpha,\beta}$ is defined as

$$\Pi_h^{\alpha,\beta}\phi(x) = \frac{1}{2}\Big\{\phi\Big(x + hb^{\alpha,\beta}(x) + \sqrt{h}\sigma^{\alpha,\beta}(x)\Big) + \phi\Big(x + hb^{\alpha,\beta}(x) - \sqrt{h}\sigma^{\alpha,\beta}(x)\Big)\Big\}.$$

Following [2] we now define the symbol $S(h,y,t,[\phi]_x^h)$ corresponding to the scheme (3.4) for $\phi \in \mathrm{C}_b(\mathbb{R})$. First, for any $x,z\in\mathbb{R}$, we set $[\phi]_x^h(z) = \phi(x+z)$ and then

$$\begin{aligned} &S(h,y,t,[\phi]_x^h)\\ &\quad= \inf_{\alpha\in\mathcal{A}}\sup_{\beta\in\mathcal{B}}\left\{\frac{1+hc^{\alpha,\beta}(y)}{h}\Big(A(h,\alpha,\beta,y,[\phi]_x^h) - t\Big) + c^{\alpha,\beta}(y)t + f^{\alpha,\beta}(y)\right\}, \end{aligned} \tag{3.5}$$

where A is given by

$$\begin{aligned} &A(h,\alpha,\beta,y,[\phi]_x^h)\\ &= \frac{1}{2}\Big([\phi]_x^h(hb^{\alpha,\beta}(y) + \sqrt{h}\sigma^{\alpha,\beta}(y)) + [\phi]_x^h(hb^{\alpha,\beta}(y) - \sqrt{h}\sigma^{\alpha,\beta}(y))\Big). \end{aligned}$$

It is not difficult to see that this scheme S is the same as the control scheme (3.4) and that it satisfies (C1) – (C3) with

$$|F(x,v,Dv,D^2v) - S(h,x,v(x),[v]_x^h)| \le C(|v_{xx}|_0 h + |v_{xxx}|_0 h^{1/2}),$$

for any $v \in \mathrm{C}^{2,1}(\mathbb{R}^N)$ and h satisfying $h\sup_{\alpha,\beta}|c^{\alpha,\beta}|_0 \le 1$. Without going into details, we claim that the scheme (3.4) has a unique $\mathrm{C}^{0,1}(\mathbb{R})$ solution if (2.1) – (2.4) hold with

$$\lambda \ge K_3 \qquad \text{and} \qquad h\sup_{\alpha,\beta}|c^{\alpha,\beta}|_0 \le 1.$$

This follows from easy adaptations of the proofs in Camilli and Falcone [8]. Note that the constant K_3 here may have to be slightly bigger than K_1 in assumption (2.3), see [2].

We also need the corresponding scheme (1.5) (denoted S_ε) for the vanishing viscosity equation (1.2). It is simply the control scheme (3.4) with σ replaced by $\sigma + \sqrt{\varepsilon}$. The scheme S_ε will have the same properties as S, so (C1) – (C3) are satisfied with $S_\varepsilon, F_\varepsilon$ replacing S, F, and existence and regularity results carries over from the scheme S. What remains to be to checked is assumption (C4). But this assumption is direct consequence of the continuous dependence result for control schemes Theorem A.1 in [2]:

Lemma 3.2. *Assume* (2.1) – (2.5) *hold and that* $u_h, u_h^\varepsilon \in \mathrm{C}^{0,1}(\mathbb{R})$ *solves respectively the control schemes* S *and* S_ε. *Then*

$$|u_h - u_h^\varepsilon|_0 \le C\sqrt{\varepsilon}.$$

We have showed that the control scheme (3.4) satisfies all the assumptions of Theorem 2.3 (b), and hence we have the following result on the rate of convergence:

Theorem 3.3. *Assume* (2.1) – (2.5) *hold with* $\lambda > K_3$ *(*K_3 *defined above), and let* u *denote the viscosity solution of the Isaacs equation* (1.1) *and* u_h *the solution of the control scheme* (3.4). *Then*

$$|u - u_h|_0 \leq C h^{1/10} \qquad \text{for} \qquad h \leq \min\left\{1, \frac{1}{\sup_{\alpha,\beta} |c^{\alpha,\beta}|_0}\right\}.$$

The rate of convergence obtained here, should be compared with the rates obtained for convex HJB equations: In Barles and Jakobsen [2, 17] the rate was 1/4, and in Menaldi [27], with further regularity of the solution, the rate was 1/2. For first order Isaacs equations ($\sigma \equiv 0$) it is not difficult to see that Theorem 2.3 yields the rate 1/3 for control schemes. This is lower than the rate 1/2 obtained by Soravia [30] for such equations and schemes.

Comments. Note that the consistency relation (C3) do not allow for higher than third order derivatives. If we could use also fourth order derivatives, the consistency relations of the above mentioned schemes would take the following (optimal) forms: For any $v \in \mathrm{C}^{3,1}(\mathbb{R})$

$$|S_\varepsilon(h, x, v, [v]^h_x) - F_\varepsilon(x, v, v_x, v_{xx})| \leq C(h|v_{xx}|_0 + h^2|v_{xxxx}|_0)$$

for the finite difference scheme (3.1), and

$$|S(h, x, v, [v]^h_x) - F(x, v, v_x, v_{xx})| \leq Ch(|v_{xx}|_0 + |v_{xxx}|_0 + |v_{xxxx}|_0)$$

for the control scheme (3.4). See also [17]. Such consistency relations would improve the rates of convergence obtained above. The reason they can not be used, is that we have not been able to bound the fourth order derivative of the solution of the vanishing viscosity equation (1.2) or any smoothed version of this equation. It is to this smooth solution the consistency relation is applied.

An other remark is that the rates we obtain using the vanishing viscosity method to obtain smooth approximate solutions, are worse than the rates obtained using other methods *where these methods can be compared*. For first order equations the method of this paper can be compared with the methods of Souganidis [28], Soravia [30], and Barles and Jakobsen [2, 17] (convex equations) and others. And for first order equations, the optimal consistency relations only use second order derivatives, so they can be used together with the methods of this paper (see the discussion above). But still the other methods yield better rates of convergence! They give 1/2 compared with 1/3 obtained here. This loss of rate seems to be a consequence of the regularization procedures used. To see this, consider [2, 17] (and [20, 21]) where the equations are convex, so that mollification may be used in a clever way. By this procedure the estimate corresponding to our estimate $\|u^\varepsilon - u\|_{L^\infty(\mathbb{R})} \leq C\sqrt{\varepsilon}$ (see Lemma 2.4 (b)), becomes $\|u^\varepsilon - u\|_{L^\infty(\mathbb{R})} \leq C\varepsilon$, while all the other estimates remains of the same order in ε as for the vanishing viscosity method used here. A consequence of this, is that the vanishing viscosity method leads to lower rates of convergence than you would get if you could use a mollification procedure.

Here we also mention the article by Deckelnick [13] on the rate of convergence of a finite difference scheme for the mean curvature equation. In that article a similar (classical) idea is adopted as in this paper, i.e. to discretize not the original equation directly but instead an approximate equation with smooth solutions. The point is that the rate obtained in that paper is also low, essentially $\Delta x^{1/5-\varepsilon}$ for any $\varepsilon \in (0, 1/5)$. That is, the rate of convergence there is more or less of the same order as for finite difference schemes in this paper!

Finally, we mention that it seems difficult to extend the techniques of this paper to Isaacs equations in $\mathbb{R}^N$, $N > 1$. First of all it is only a subclass of such equations that are equivalent to semi-linear equations. Secondly, when $N > 1$

$$\Delta u = f(x) \quad \text{in} \quad \mathbb{R}^N$$

need not have solutions $u \in \mathrm{C}^{2,1}(\mathbb{R}^N)$ when $f \in \mathrm{C}^{0,1}(\mathbb{R}^N)$. However, $u \in \mathrm{C}^{2,\varepsilon}(\mathbb{R}^N)$ for any $\varepsilon \in (0, 1)$ (see [16]), but this is not good enough. Of course we can mollify the equation to obtain arbitrary smooth solutions, but then we need some extra continuous dependence properties of the schemes to complete the arguments. Such properties holds for control schemes but not for finite difference schemes.

4. The vanishing viscosity equation.

In this section we study the properties of the solutions of the vanishing viscosity equation (1.2). Lemma 2.4 is a consequence of the results of this section. Let us start by a well-posedness result which is a corollary to Theorem 2.1.

Proposition 4.1. *Assume* (2.1)–(2.5) *and* $\lambda > K_1$. *Then the vanishing viscosity equation* (1.2) *has a unique viscosity solution* $u^\varepsilon \in \mathrm{C}^{0,1}(\mathbb{R})$ *satisfying*

$$|u^\varepsilon|_0 + |u^\varepsilon_x|_0 \leq C.$$

Next, we give a result showing that solutions of the vanishing viscosity equation (1.2) converges to solutions of the Isaacs equation (1.1) as $\varepsilon \to 0$.

Proposition 4.2. *If* $u, u^\varepsilon \in \mathrm{C}^{0,1}(\mathbb{R})$ *are respectively the viscosity solutions of* (1.1) *and* (1.2)*, then*

$$|u - u^\varepsilon|_0 \leq C\sqrt{\varepsilon}.$$

This result is a special case of Theorem 3.4 in [18]. Since $u, u^\varepsilon \in \mathrm{C}^{0,1}(\mathbb{R})$ by Theorem 2.1 and Proposition 4.1, this proposition implies Lemma 2.4 (b). We proceed to show that the solutions of (1.2) are classical $\mathrm{C}^2(\mathbb{R})$ solutions, and then we derive bounds on the second and third order derivative of these solutions. What makes this possible, is the fact that equation (1.2) is equivalent to the semi-linear equation (1.3). To show this, we use the following simple lemma.

Lemma 4.3. *Let* $\{a^{\alpha,\beta}\}_{\alpha,\beta}$ *and* $\{b^{\alpha,\beta}\}_{\alpha,\beta}$ *be subsets of* $\mathbb{R}$. *If* $a^{\alpha,\beta} \geq k > 0$ *for all* α, β *and* $\inf_\alpha \sup_\beta a^{\alpha,\beta} b^{\alpha,\beta} = 0$*, then* $\inf_\alpha \sup_\beta b^{\alpha,\beta} = 0$.

Proof. Assume $\inf_\alpha \sup_\beta b^{\alpha,\beta} > 0$. Then $0 = \inf_\alpha \sup_\beta a^{\alpha,\beta} b^{\alpha,\beta} \geq k \inf_\alpha \sup_\beta b^{\alpha,\beta}$, which is a contradiction. On the other hand if $\inf_\alpha \sup_\beta b^{\alpha,\beta} < 0$, then $0 = \inf_\alpha \sup_\beta a^{\alpha,\beta} b^{\alpha,\beta} \leq k \inf_\alpha \sup_\beta b^{\alpha,\beta}$, also a contradiction. □

We state the result:

Proposition 4.4. *Assume* a_ε, b, c, f *are continuous in all arguments and* $a_\varepsilon \geq \varepsilon > 0$. *Then a function* u *is a viscosity solution of equation* (1.2)*, iff it is a viscosity solution of equation* (1.3).

For classical solutions, the result is an immediate consequence of Lemma 4.3, and it is easy to check that the result also holds for viscosity solutions. Using this result, we now prove that the viscosity solutions of (1.2) belong to $C^2(\mathbb{R})$.

Proposition 4.5. *Assume* (2.1)–(2.5) *and* $\lambda > K_1$. *Then the viscosity solution of vanishing viscosity equation* (1.2) *belong to* $\mathrm{C}^2(\mathbb{R})$.

Proof. Let u^ε be the solution provided by Proposition 4.1. On every ball $B \subset \mathbb{R}^N$ u^ε solves

$$u_t + \inf_{\alpha\in\mathcal{A}} \sup_{\beta\in\mathcal{B}} \left\{ a_\varepsilon^{\alpha,\beta}(x)u_{xx} + b^{\alpha,\beta}(x)u_x + c^{\alpha,\beta}(x)u + f^{\alpha,\beta}(x) \right\} = 0$$
$$\text{in} \quad B \times (0,T),$$
$$u = u^\varepsilon \mid_{\partial P} \quad \text{in} \quad \partial P,$$

where $\partial P = B\times\{0\}\cup\partial B\times(0,T)$ is the parabolic boundary. This problem satisfies all the assumptions of Theorem 9.3 in [11], so we may conclude existence and uniqueness of a $\mathrm{C}^{1,\bar\alpha}_{\mathrm{loc}}(B\times(0,T))$ viscosity solution for some $\bar\alpha \in (0,1)$. Uniqueness in $B\times(0,T)$ and arbitrariness of B then implies that u^ε belongs to $\mathrm{C}^{1,\bar\alpha}_{\mathrm{loc}}(\mathbb{R})$. By Proposition 4.4 we then have

$$u^\varepsilon_{xx} = \tilde f(x) \quad \text{in} \quad \mathbb{R},$$

where $\tilde f$ is the first order term in the semi-linear equation (1.3). By the regularity of u^ε and a_ε, b, c, f, it follows that $\tilde f$ belongs to $\mathrm{C}^{0,\bar\alpha}_{\mathrm{loc}}(\mathbb{R})$. The classical theory of the Poisson equation [16] now shows that u^ε belongs to $\mathrm{C}^{2,\bar\alpha}_{\mathrm{loc}}(\mathbb{R}) \subset \mathrm{C}^2(\mathbb{R})$. □

The result implies that viscosity solutions of (1.2) are classical solutions. We end this section by proving that the solution of equation (1.2) belongs to $\mathrm{C}^{2,1}(\mathbb{R})$ and provide bounds on the second and third derivatives of this solution depending on ε.

Proposition 4.6. *Assume* (2.1) – (2.5), *and* $\lambda > K_1$. *Then the solution* u^ε *of* (1.2) *belongs to* $C^{2,1}(\mathbb{R})$ *and satisfies the following bound*

$$\varepsilon\|u^\varepsilon_{xx}\|_\infty + \varepsilon^2\|u^\varepsilon_{xxx}\|_\infty \le C.$$

Proof. Use Proposition 4.4 to rewrite equation (1.2) as the semi-linear equation (1.3). According to Propositions 4.1 and 4.5, there is a classical solution u^ε of (1.3) (and (1.2)), and according to Proposition 4.1 (again), $\|u^\varepsilon\|_\infty + \|u^\varepsilon_x\|_\infty \le C$ independent of ε. Then it is immediate from the boundedness of the coefficients (2.1) and (2.2), that equation (1.3) implies

$$|u^\varepsilon_{xx}|_0 \le \frac{C}{\varepsilon}.$$

We proceed to estimating u^ε_{xxx}. Let $w(x) = \frac{1}{h}(u^\varepsilon_{xx}(x+h) - u^\varepsilon_{xx}(x))$ and use the semi-linear equation (1.3) to get

$$\begin{aligned} |w(x)| \le \sup_{\alpha\in\mathcal{A},\beta\in\mathcal{B}} \Bigg\{ & \frac{1}{\varepsilon^2}\|a^{\alpha,\beta}_{\varepsilon,x}\|_\infty \Big(\|b^{\alpha,\beta}\|_\infty\|u^\varepsilon_x\|_\infty + \|c^{\alpha,\beta}\|_\infty\|u^\varepsilon\|_\infty + \|f^{\alpha,\beta}\|_\infty \Big) \\ & + \frac{1}{\varepsilon}\Big(\|b^{\alpha,\beta}_x\|_\infty\|u^\varepsilon_x\|_\infty + \|b^{\alpha,\beta}\|_\infty\|u^\varepsilon_{xx}\|_\infty \\ & \qquad + \|c^{\alpha,\beta}_x\|_\infty\|u^\varepsilon\|_\infty + \|c^{\alpha,\beta}\|_\infty\|u^\varepsilon_x\|_\infty + \|f^{\alpha,\beta}_x\|_\infty \Big) \Bigg\} \end{aligned}$$

By the previous bounds on u and conditions (2.1)–(2.5), we see that $|w(x)| \leq C/\varepsilon^2$ independent of h. We may therefore conclude that

$$\|u^\varepsilon_{xxx}\|_\infty \leq \frac{C}{\varepsilon^2}.$$

□

At this point, we note that we cannot get higher regularity of the solution of equation (1.2), because the equation itself is only Lipschitz continuous. The $C^{2,1}(\mathbb{R})$ regularity obtained here should be contrasted with the fact that in general, you can only expect $\mathrm{C}^{1,\alpha}_{\mathrm{loc}}$ solutions of non-convex uniformly elliptic equations [6, 11]. Finally, we mention that Lemma 2.4 (a) is a consequence of Propositions 4.1 and 4.6.

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