$W^{2,\infty}$ REGULARIZING EFFECT IN A NONLINEAR, DEGENERATE PARABOLIC EQUATION IN ONE SPACE DIMENSION.

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Abstract. In this paper we provide and analyze a non-linear *degenerate* parabolic equation in one space dimension with the follwing smoothing property: If the initial data is only uniformly continuous, at positive times, the solution has bounded second derivatives (it belongs to $W^{2,\infty}$). We call this surprising phenomenon a $W^{2,\infty}$ regularizing effect. So far, such phenomenons have only been observed in *uniformly* parabolic equations.

1. Introduction

In this paper we are going to study the following initial value problem:

$$\begin{aligned} &u_t + u_x^2 + \max(0, -u_{xx}) = 0 \quad \text{in} \quad (0,T)\times\mathbb{R} := Q_T,\\ &u(0,x) = u_0(x) \quad \text{in} \quad \mathbb{R}. \end{aligned} \tag{1.1}$$

It is an example of a problem which is non-linear and *degenerate* parabolic, but still has "smooth" solutions at positive times, even if the initial data is not "smooth". Note that the non-linearity is non-smooth. We show below that if the initial values are only uniformly continuous, then the solution $u(t,\cdot)$ belongs to the Sobolev space $W^{2,\infty}(\mathbb{R})$ for any $t \in (0,T]$. We call this a $W^{2,\infty}$ regularizing effect. Actually the solution u belongs to the parabolic Sobolev space $W^{2,\infty}([\varepsilon,T]\times\mathbb{R})$ for all $\varepsilon \in (0,T)$, and this implies that the equation is satisfied almost everywhere.

This is a surprising phenomenon, which to the best of the author's knowledge, has only been observed before in equations that are *uniformly* parabolic. In general, degenerate parabolic equations do not have smooth solutions, not even when the initial data is smooth. But regularizing effects have been studied in the subclasses of uniformly parabolic equations [5, 14, 17, 18, 19], and first order Hamilton-Jacobi equations [15, 16, 6, 3, 1]. In the first case (for convex equations), the solutions typically become continuously differentiable, twice in x and once in t. In the second case (for strictly convex Hamiltonians), the solutions typically become Lipschitz continuous and x-semiconcave.

Even though our equation is neither a first order Hamilton-Jacobi equation, nor a uniformly parabolic one, "it has regularizing effects of both types". To see what we mean, remove the second order term from the equation. It then becomes a first order HJ equation:

$$u_t + u_x^2 = 0 \quad \text{in} \quad Q_T, \tag{1.2}$$

which has Lipschitz continuous and x-semiconcave solutions for positive time. On the other hand, removing the quadratic term from (1.1) leads to the degenerate parabolic equation

$$u_t = \min(0, u_{xx}) \quad \text{in} \quad Q_T, \tag{1.3}$$

Date: October 3, 2002.

Key words and phrases. degenerate parabolic equations, Hamilton-Jacobi-Bellman equations, viscosity solutions, regularizing effects, regularity.

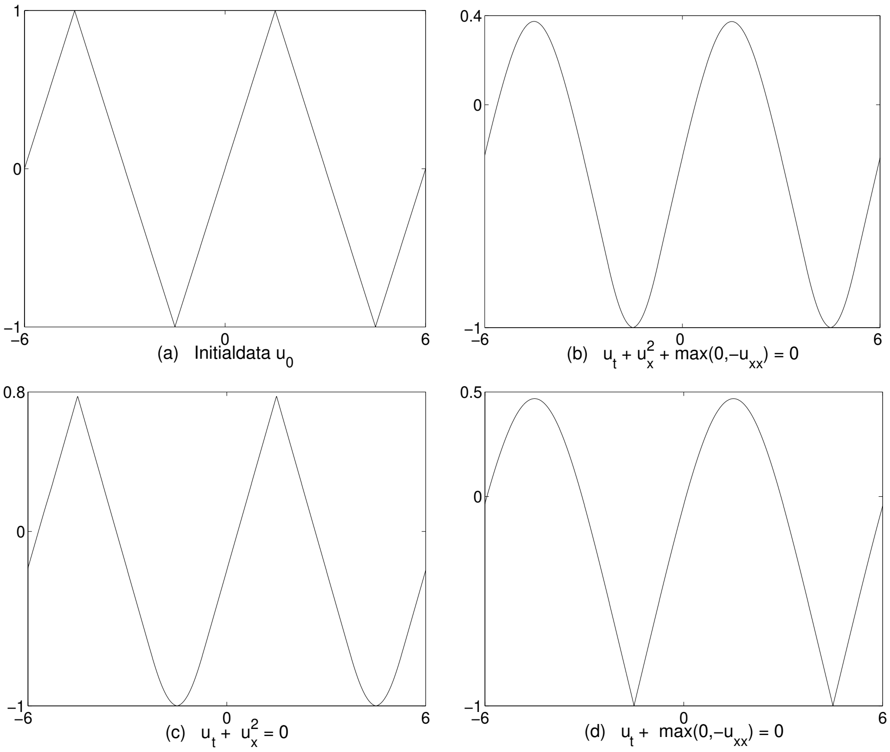

FIGURE 1. Solution of $u_t + f(u_x, u_{xx}) = 0$, $u|_{t=0} = u_0$ at $t = 0.5$: (b) $f = u_x^2 + \max(0, -u_{xx})$, (c) $f = u_x^2$, (d) $f = \max(0, -u_{xx})$.

which has x-semiconvex solutions for positive time. It turns out that adding both terms to the equation preserves both regularizing effects! That is, the solution of the full equation becomes both x-semiconvex and x-semiconcave, which means it belongs to $W^{2,\infty}(\mathbb{R})$ for any fixed positive time.

These regularizing effects are illustrated in Figure 1. Here a Lipschitz continuous initial data (a) is evolved according to equations (1.1) – (1.3), see (b) – (d). It is easy to see that the solutions ploted in (b), (c), and (d) are $W^{2,\infty}$, semiconcave, and semiconvex respectively.

The rest of the paper is organized as follows: In Section 2 the main result is given along with a corollary and some comments. The proof of the main result is given in Section 3, and finally there is an appendix containing the proofs of results which are either standard or technical.

Acknowledgement. I am grateful to Peter Lindqvist, Harald Hanche-Olsen, and Kenneth Karlsen for useful discussions, comments, and advice.

2. THE MAIN RESULT.

We start by remarking that equation (1.1) is a convex Hamilton-Jacobi-Bellman equation, as can be seen more clearly when it is written in the following form:

$$u_t + \max_{\substack{\alpha \in \{0,1\} \\ \beta \in \mathbb{R}}} \{-\alpha u_{xx} + 2\beta u_x - \beta^2\} = 0 \quad \text{in} \quad Q_T. \tag{2.1}$$

It is well-known that such equations need not have smooth solutions, not even when the initial values are smooth. The correct notion of weak solutions for these equations, is the notion of viscosity solutions. We refer to the User's Guide [4] for a detailed presentation, and to the book by Fleming & Soner [7] for more information about Hamilton-Jacobi-Bellman equations and connections to the viscosity solutions theory.

Let us introduce some notation. For functions $f : \overline{Q} \to \mathbb{R}$ and numbers $\delta \in (0, 1]$ we define the following (semi) norms:

$$
\begin{aligned}
|f|_0 &= \operatorname*{ess\,sup}_{(t,x)\in\overline{Q}_T} |f(t,x)|,\\
[f]_\delta &= \operatorname*{ess\,sup}_{\substack{(t,x),(s,y)\in\overline{Q}_T\\ x\neq y, t\neq s}} \frac{|f(t,x)-f(s,y)|}{|x-y|^\delta + |t-s|^{\delta/2}},\\
|f|_\delta &= |f|_0 + [f]_\delta.
\end{aligned}
$$

Let BUC denote spaces of bounded uniformly continuous functions, and $\mathcal{C}^{0,\delta}$ spaces of functions with finite $|\cdot|_\delta$ norm. The last space contains bounded functions which are Hölder-δ continuous in x and Hölder-$\delta/2$ continuous in t. Furthermore, let $W^{2,\infty}(K)$ denote the space of functions $f : K \to \mathbb{R}$ for $K \subset \overline{Q}_T$, with finite norm $|f|_0 + |f_x|_0 + |f_t|_0 + |f_{xx}|_0$. Finally, by C we denote constants independent of t, x and any smoothness parameters like ε, ϵ (see below).

Now we proceed to give an existence, uniqueness, and regularity theorem for solutions of (1.1).

Proposition 2.1. *If $u_0 \in \mathrm{BUC}(\mathbb{R})$, then there exists a unique viscosity solution $u \in \mathrm{BUC}(\overline{Q}_T)$ of* (1.1). *Furthermore if $u_0 \in \mathcal{C}^{0,\delta}(\mathbb{R})$ for some $\delta \in (0, 1]$, then $u \in \mathcal{C}^{0,\delta}(\overline{Q}_T)$.*

This is a standard result holding for a large class of equations, and the regularity here is the best you can hope for in the general case (without regularizing effects). In fact the result is *optimal* w.r.t. global regularity or regularity up to the boundary. The (global) result is optimal even for the linear heat equation. See e.g. [11] for general regularity results of this type for nonlinear equations. For completeness, we provide a proof of this result in the Appendix. We just remark that the first thing to do, is to prove a so-called strong comparison principle for (1.1) [4, 2]. Existence of a bounded solution then follows by Perron's method [9, 4]. Uniqueness and regularity are then (for this problem) rather simple consequences of the comparison principle.

We now give the main result, saying that problem (1.1) has a $W^{2,\infty}$ regularizing effect. In other words, for $t > 0$ the solution becomes $W^{2,\infty}$. So for $t > 0$ it is more regular than the initial data and hence more regular than suggested by Proposition 2.1.

Theorem 2.2 ($W^{2,\infty}$ regularizing effect)**.** *Let u be the viscosity solution of* (1.1)*, and assume $u_0 \in \mathrm{BUC}(\mathbb{R})$. Then $u \in W^{2,\infty}([\varepsilon, T] \times \mathbb{R})$ for all $\varepsilon \in (0, T)$.*

Note that functions in $W^{2,\infty}([\varepsilon, T] \times \mathbb{R})$ are differentiable a.e. twice w.r.t. x and once w.r.t. t. Since viscosity solutions are point-wise solutions at such points of differentiability, the following corollary is immediate:

Corollary 2.3. *If $u_0 \in \mathrm{BUC}(\mathbb{R})$, then the viscosity solution u of equation* (1.1) *satisfies the equation a.e., i.e. it is a so-called strong solution of* (1.1).

We remark that the proof of the main result given below, only works in one space-dimension. To see this, consider the $\mathbb{R}^N$ version of (1.1):

$$w_t + |Dw|^2 + \max(0, -\Delta w) = 0 \quad \text{in} \quad (0, T) \times \mathbb{R}^N.$$

It turns out that using the techniques of this paper, you can prove that for $t > 0$, w is Lipschitz continuous in t and x and semiconcave in x. However, our proof of semiconvexity leads to the following inequality

$$\Delta w(t,x) \geq w_t(t,x) \geq -C(t) \quad \text{a.e.},$$

where $C(t)$ is a positive function satisfying $\lim_{t\to 0^+} C(t) = \infty$. It is only in $\mathbb{R}^1$ that this inequality implies semiconvexity. In $\mathbb{R}^N$ we get two-sided control only of the Laplacian.

The result and proofs can be generalized to more general equations. An easy generalization is to consider equations of the form

$$w_t + H(w_x) = f(w_{xx}) \quad \text{in} \quad Q_T,$$

where H is strictly convex and f is increasing, Lipschitz continuous, concave, and satisfies $f(X) \leq CX$ for $X \in \mathbb{R}$. If $f = 0$, it is a well-known that the equation has Lipschitz continuous and x-semiconcave solutions. Other generalizations are also possible, see e.g. Chapter 13 in Lions [15] for how to treat more general first order terms.

Finally, numerical simulations seem to indicate that the following equation also exhibits a $W^{2,\infty}$ regularizing effect:

$$w_t + \max(w_x^2, -w_{xx}) = 0 \quad \text{in} \quad Q_T.$$

It would be interesting to have a proof in this (more difficult) case too. So far we have not been able to do it.

3. Proof of the main result.

We divide the proof into three parts. First we use the regularizing properties of the u_x^2 term, to prove that for positive time, the solution is Lipschitz continuous and semiconcave in x (Proposition 3.1). Then we show that the solution is Lipschitz in t for positive time (Proposition 3.4). This regularizing effect seems to be due to the u_t term in the equation, and it occurs in a large class of parabolic equations [12]. Finally, we show that the solution is semiconvex in x (Proposition 3.7), using above mentioned two results and the regularizing properties of the $\max(0, -u_{xx})$ term. Taken together, these results imply Theorem 2.2.

We start by stating

Proposition 3.1. *If $u \in \mathrm{BUC}(\overline{Q}_T)$ is the viscosity solution of* (1.1), *then*

$$[u(t,\cdot)]_1 \leq \left(\frac{|u|_0}{t}\right)^{1/2} \qquad \textit{and} \qquad u_{xx}(t,x) \leq \frac{2}{t} \quad \textit{a.e.}.$$

In order to prove this result, we need to consider a regularized version of (1.1). We regularize by mollifying the max function and the initial data, and adding a viscous term to the equation. So let ρ be a positive, smooth function with mass 1, and define the mollifier ρ_ε as $\rho_\varepsilon(x) := \frac{1}{\varepsilon}\rho(\frac{x}{\varepsilon})$ for $x \in \mathbb{R}$ and $\varepsilon > 0$. Furthermore, define

$$f^\varepsilon(x) = \int_{\mathbb{R}} \min(0,y)\rho_\varepsilon(x-y)dy \qquad \text{and} \qquad u_0^\varepsilon(x) = \int_{\mathbb{R}} u_0(y)\rho_\varepsilon(x-y)dy.$$

Then the regularized problem in consideration can be written

$$\begin{aligned} &u_t^\varepsilon + (u_x^\varepsilon)^2 = f^\varepsilon(u_{xx}) + \varepsilon u_{xx}^\varepsilon \quad \text{in} \quad Q_T, \\ &u^\varepsilon(0,x) = u_0^\varepsilon(x) \quad \text{in} \quad \mathbb{R}. \end{aligned} \tag{3.1}$$

Note that $f^\varepsilon, u_0^\varepsilon$ are smooth and have bounded derivatives of all orders, and that f^ε is nondecreasing and concave. This new equation is a smooth and uniformly parabolic Hamilton-Jacobi-Bellman equation. The properties of its solutions are given in the next lemma whose proof will be given in the appendix.

Lemma 3.2. *There exists a unique classical solution u^ε to the initial value problem* (3.1) *(i.e. u^ε is continuous differentiable twice w.r.t. x and once w.r.t. t). Moreover, u^ε_{xxxx} and u^ε_{txx} are continuous functions.*

Proof of Proposition 3.1. Let u^ε be the solution of the regularized problem (3.1) provided by Lemma 3.2. We claim that

$$u^\varepsilon_{xx} \le \frac{1}{2t} \quad \text{in} \quad Q_T. \tag{3.2}$$

To see this, let $w = tu^\varepsilon_{xx}$, differentiate twice w.r.t. x in (3.1) (ok by Lemma 3.2), and multiply by t to obtain

$$\begin{aligned} L(w) := w_t + 2u^\varepsilon_x w_x - (\varepsilon + (f^\varepsilon)')w_{xx} &= \frac{1}{t}w - \frac{2}{t}w^2 + \frac{1}{t}(f^\varepsilon)'' w_x^2 \\ &\le \frac{1}{t}\left(w - 2w^2\right) \end{aligned} \tag{3.3}$$

The inequality follows since $(f^\varepsilon)'' \le 0$ by concavity of f^ε. Now suppose $\sup_{\overline{Q}_T} w > 0$. By Kruzkov [13] we may assume that u_0 has compact support. This implies that $\sup_{\overline{Q}_T} w$ is achieved at some point (t_0, x_0) (still by [13]). At this maximum point $L(w)\mid_{(t_0,x_0)} \ge 0$ (the maximum principle), so by taking also into account (3.3) we have

$$0 \le \left(w - 2w^2\right)\mid_{(t_0,x_0)}.$$

We may therefore conclude that $\sup_{\overline{Q}_T} w = w(t_0, x_0) \le 1/2$, which is (3.2).

It can be proved that u^ε converges uniformly to u as $\varepsilon \to 0$ (see e.g. [11] for case of $u_0 \in \mathcal{C}^{0,\delta}(\mathbb{R})$). So it immediately follows that u satisfies the bound (3.2) in the sense of distributions and hence a.e..

Since $u \in \mathrm{BUC}(\bar{Q}_T)$ satisfies $u_{xx} \le C_0$ in the sense of distributions (with $C_0 = (2t)^{-1}$), a standard interpolation inequality (see e.g. [15, p. 240]) yields

$$[u(t,\cdot)]_1 \le (2|u|_0 C_0)^{1/2} \quad \text{for all} \quad t \in (0, T].$$

This completes the proof. □

Remark 3.3. This proof is taken from Kruzkov [13] where first order Hamilton-Jacobi equations were considered, see also Lions [15] Chapter 13.

Let us proceed to

Proposition 3.4. *If $u \in \mathrm{BUC}(\overline{Q}_T)$ is the viscosity solution of* (1.1)*, then $|u_t(t,\cdot)|_0 \le C(t) < \infty$ for any $t \in (0, T]$.*

Remark 3.5. It follows from the proof below, that $C(t) \to \infty$ as $t \to 0^+$, and that $C(t)$ can be chosen to be a decreasing function.

Proof. Here we need to work with a globally x-Lipschitz solution. This can be achieved by considering the restriction of u to the interval $[\epsilon, T]$ for any fixed $\epsilon > 0$. If v denotes this function, then v satisfies

$$\begin{aligned} &v_t + (v_x)^2 + \max(0, -v_{xx}) = 0 \quad \text{in} \quad (\epsilon, T) \times \mathbb{R}, \\ &v(\epsilon, x) = u(\epsilon, x) \quad \text{in} \quad \mathbb{R}. \end{aligned} \tag{3.4}$$

Obviously v is globally x-Lipschitz and $|v_x|_0 \le C/\epsilon^{1/2}$, since $|u_x(t,\cdot)|_0 \le C/t^{1/2}$ by Proposition 3.1. Keeping the form (2.1) of equation (1.1) in mind, we may therefore rewrite equation (3.4) in the following way:

$$v_t + \max_{\substack{\alpha \in \{0,1\} \\ |\beta| \le |v_x|_0}} \{-\alpha v_{xx} + 2\beta v_x - \beta^2\} = 0 \quad \text{in} \quad Q_T. \tag{3.5}$$

Here the controls α, β take values in compact sets, so the coefficients in the above equation are bounded. This again implies that the function

$$F(p,X) = \max_{\substack{\alpha\in\{0,1\}\\ |\beta|\leq |u_x|_0}} \{-\alpha X + 2\beta p - \beta^2\}$$

is Lipschitz continuous in both arguments. Since $v_t + F(v_x, v_{xx}) = 0$ in the viscosity sense, and $v(\epsilon,\cdot)$ is x-Lipschitz continuous, the conditions of Theorem 4.2 (or Theorem 6.1) in [12] are satisfied. We may therefore conclude that $|(t-\epsilon)v_t|_0 \leq C_\epsilon$ for some constant C_ϵ depending on ϵ, and $t > \epsilon$. Since $\epsilon > 0$ was arbitrary, and $u = v$ for $t \in [\epsilon, T]$, the proof is complete. □

Remark 3.6. Theorem 6.1 in Katsoulakis [12] essentially says that the solution u of the equation

$$\begin{aligned} &u_t + f(t,x,u,Du,D^2u) = 0 \quad \text{in} \quad (0,T)\times\mathbb{R}^N, \\ &u(0,x) = u_0(x) \quad \text{in} \quad \mathbb{R}^N, \end{aligned}$$

satisfies $|tu_t| \leq C$ in the sense of distributions if the following conditions holds:

(i) u_0 is bounded and Lipschitz continuous.

(ii) f is continuous in all variables.

(iii) $f(t,x,r,p,X) \leq f(t,x,r,p,Y)$ for $Y \leq X$.

(iv) $|f(t,x,r,p,X) - f(t,y,r,p,X)| \leq C|x-y|(1+|r|+|p|)$.

(v) $|f(t,x,r,p,X) - f(t,x,s,q,Y)| \leq C(|r-s|+|p-q|+|X-Y|)$.

The proof of this result relies on rewriting the equation as an Isaacs equation and studying the associated stochastic differential game. It turns out that the upper (or lower) value function of this game satisfies the Isaacs equation in the viscosity sense [12, 8]. Moreover this value function has by its definition a stochastic (Feynman-Kac like) representation formula. The result is then proved working directly with this stochastic formula.

The proof of Theorem 2.2 is completed by giving

Proposition 3.7. *If* $u \in \mathrm{BUC}(\overline{Q}_T)$ *is the viscosity solution of* (1.1)*, then*

$$u_{xx}(t,x) \geq -C(t) \quad \textit{a.e.},$$

where $C(t)$ *is defined in Proposition 3.4.*

Proof. By Proposition 3.1, u is semiconcave for $t > 0$, and hence it is twice differentiable a.e. w.r.t. x. Furthermore, by Proposition 3.4 u is differentiable a.e. w.r.t. t. At such points of differentiability, the viscosity solution u is also a pointwise solution. This means that the equation holds a.e. Using the a.e. equation in combination with the bound on the time-derivative from Proposition 3.4, yields

$$u_{xx} \geq \min(0, u_{xx}) = u_t + u_x^2 \geq -C(t) \quad \text{a.e.},$$

and the proof is completed. □

Appendix A. Proofs of secondary results.

Note well that in this section we only give proofs, please refer to earlier sections for notation and statements of the results.

Proof of Proposition 2.1. This result is mainly a consequence of the two lemmas below. But first we need some notation: For functions $\phi : \overline{Q}_T \to \mathbb{R}$, $G[\phi] := \phi_t + \phi_x^2 + \max(0, -\phi_{xx})$ in the viscosity sense, and BUSC denotes the space of bounded upper semicontinuous functions.

Lemma A.1 ("Strong" comparison principle)**.** *Let* $u, -v \in \mathrm{BUSC}(\overline{Q}_T)$ *satisfy* $G[u] \leq 0$, $G[v] \geq 0$, $u(0,x) \leq v(0,x)$, *and* $u(0,x), v(0,x) \in \mathrm{BUC}(\mathbb{R})$. *Then* $u \leq v$ *in* $[0,T) \times \mathbb{R}$.

Proof. Define

$$\begin{aligned}
&\phi(t,x,y) = \alpha|x-y|^2 + \varepsilon(|x|^2 + |y|^2) + \frac{\varepsilon}{T-t},\\
&\psi(t,x,y) = u(t,x) - v(t,y) - \phi(t,x,y),\\
&\sigma = \sup_{t,x,y} \psi(t,x,y) - \sup_{x,y} \psi(0,x,y)^+,
\end{aligned}$$

where the suprema are taken over the set $[0,T) \times \mathbb{R} \times \mathbb{R}$. We will derive an upper bound on σ, so we may assume that $\sigma > 0$. By classical arguments there is a maximum point $(t_0, x_0, y_0) \in [0,T) \times \mathbb{R} \times \mathbb{R}$ of $\psi(t,x,y) - \sigma t$. Furthermore, $t_0 > 0$ since otherwise $\sigma \leq 0$. Using the maximum principle for semicontinuous functions (Theorem 8.3 in [4]), and subtracting the inequalities that follows from the definitions of viscosity solutions, we obtain

$$\phi_t(t_0,x_0,y_0) + \sigma \leq (\phi_y(t_0,x_0,y_0))^2 - (\phi_x(t_0,x_0,y_0))^2 + \max(0,-Y) - \max(0,-X),$$

where $X, Y \in \mathbb{R}$ satisfies $X \leq Y + k\varepsilon$ for some number $k > 0$. Simplify this expression to get

$$\frac{\varepsilon}{(T-t_0)^2} + \sigma \leq 2\varepsilon(x_0+y_0)\alpha(x_0-y_0) + \varepsilon^2(y_0^2 - x_0^2) + k\varepsilon \leq \varepsilon^{1/2} C \left(1 + \alpha^{1/2}\right).$$

The last inequality follows since the inequality $\psi(t_0,x_0,y_0) \geq \psi(t_0,0,0)$, and boundedness of u and v, yields the bound $\alpha|x_0-y_0|^2 + \varepsilon(|x_0|^2 + |y_0|^2) \leq C$. Since $u_0 \in \mathrm{BUC}(\mathbb{R})$ it is easy to get the estimate $\sup_{x,y} \psi(0,x,y)^+ \leq \omega\left(\alpha^{-1} + \varepsilon\right)$ for some modulus ω. By the definition of σ, for any $(t,x) \in [0,T) \times \mathbb{R}$ we now have

$$u(t,x) - v(t,x) \leq \sigma + \sup_{x,y} \psi(0,x,y)^+ + 2\varepsilon|x|^2 + \frac{\varepsilon}{T-t}.$$

By the above estimates we can conclude the proof by first sending ε to 0, and then sending α to ∞. □

We give a standard and immediate corollary:

Corollary A.2. *Assume* $u_0 \in \mathrm{BUC}(\mathbb{R})$, *and let* u *be a viscosity solution of* (1.1). *Then* u *is unique. Furthermore, if* $h \in \mathbb{R}$ *and* $t \in [0,T]$, *then*

$$\sup_{x\in\mathbb{R}} |u(t,x) - u(t,x+h)| \leq \sup_{x\in\mathbb{R}} |u(0,x) - u(0,x+h)|.$$

This immediately tells us that the x-regularity of the solution, is the same as the regularity of the initial function u_0. So if u_0 belongs to $\mathrm{BUC}(\mathbb{R})$ or $\mathcal{C}^{0,\delta}(\mathbb{R})$ for $\delta \in (0,1]$, then $u(t,\cdot)$ (if it exists) belongs to $\mathrm{BUC}(\mathbb{R})$ or $\mathcal{C}^{0,\delta}(\mathbb{R})$ for $t \in (0,T]$. Regularity in t can be obtained from the comparison principle if one knows the t-regularity at $t = 0$. This again can be obtained using an approximation and comparison argument. See [10] for the details in the Hölder continuous case.

Lemma A.3. *If* $u_0 \in \mathrm{BUC}(\mathbb{R}^N)$ *then* (1.1) *has a solution* $u \in \mathrm{BUC}(\overline{Q}_T)$ *satisfying* $|u|_0 \leq |u_0|_0$.

Outline of proof. Here, we need to consider discontinuous viscosity solutions and initial values in the viscosity sense, meaning that a subsolution v and supersolution $\bar{v}$ has to satisfy in the viscosity sense

$$\begin{aligned}
&G[v^*] \leq 0 \quad \text{and} \quad G[\bar{v}_*] \geq 0 \quad \text{in} \quad Q_T,\\
&\min\left(G[v^*], v^* - u_0^*\right) \leq 0 \quad \text{in} \quad \{0\} \times \mathbb{R},\\
&\max\left(G[\bar{v}_*], \bar{v}_* - u_{0*}\right) \geq 0 \quad \text{in} \quad \{0\} \times \mathbb{R}.
\end{aligned}$$

Here $v^*(v_*)$ denotes the upper (lower) semicontinuous envelopes of the possibly discontinuous function v.

To continue, we note that the constants $|u_0|_0$ and $-|u_0|_0$ are smooth sub- and supersolutions of the initial value problem in the viscosity sense. Then by Perron's method, the following function is a discontinuous viscosity solution of this problem,

$$u(t,x) = \sup\{w(t,x) \ : \ -|u_0|_0 \leq w \leq |u_0|_0 \ \text{ and } \ G[w^*] \leq 0 \text{ in } Q_T\}.$$

From Lemma 4.7 in [2] it follows that $u(0,x) \leq u_0^*(x)$ and $u(0,x) \geq u_{0*}(x)$. Since $u_0 = u_0^* = u_{0*}$ (u_0 is continuous), this means that $u(0,x) = u_0(x) \in \mathrm{BUC}(\mathbb{R})$. By the "strong" comparison principle (Lemma A.1) it follows that $u^* \leq u_*$. Since the opposite inequality is always true, $u = u^* = u_*$, implying that u is continuous. It is immediate from Corollary A.2 that u is uniformly continuous in x, and uniform continuity in time follows as explained below Corollary A.2. □

This completes the proof of Proposition 2.1. □

Proof of Lemma 3.2. This result follows from the two lemmas below. But first we need some notation. Let $C^{1,\alpha}(Q_T)$ and $C^{2,\alpha}(Q_T)$ denote the space of functions $f : \overline{Q}_T \to \mathbb{R}$ with finite norms $|f|_0+|f_x|_\alpha$ and $|f|_0+|f_x|_0+|f_{xx}|_\alpha+|f_t|_\alpha$ respectively. Furthermore, let $C^{1,\alpha}_{\mathrm{loc}}(Q_T)$ and $C^{2,\alpha}_{\mathrm{loc}}(Q_T)$ be their local versions (the norms being finite on compact sets).

Lemma A.4. *The initial value problem* (3.1) *has a unique solution* $u^\varepsilon \in C^{2,\beta}_{\mathrm{loc}}(Q_T)$, *for some* $\beta \in (0,1)$.

Proof. 1. By viscosity methods there exists a unique viscosity solution $u^\varepsilon \in \mathcal{C}^{0,1}(\overline{Q}_T)$. The proof of this is almost exactly the same as the proof of Proposition 2.1. The regularity is a consequence of the Lipschitz regularity of the initial data u_0^ε and the comparison principle for this equation.

2. It immediately follows that u^ε also is the (unique) viscosity solution of the following truncated equation

$$u_t^\varepsilon + H_R(u_x^\varepsilon) = f^\varepsilon(u_{xx}) + \varepsilon u_{xx}^\varepsilon \quad \text{in} \quad Q_T,$$

where $H_R(t)$ is t^2 for $|t| < R$ and R otherwise for some fixed $R > |u_x^\varepsilon|_0$. Alternatively, and in view of the global Lipschitz continuity of f^ε, we may write

$$u_t^\varepsilon + \max_{\substack{\alpha\in[0,1]\\ \beta\in[-R,R]}} \{-F_1^\varepsilon(\alpha)u_{xx} - F_2^\varepsilon(\alpha) + 2\beta u_x^\varepsilon - \beta^2\} = \varepsilon u_{xx}^\varepsilon \quad \text{in} \quad Q_T,$$

for some appropriately defined functions F_1^ε and F_2^ε, bounded on $[0,1]$. Now we have achieved a Hamilton-Jacobi-Bellman equation with bounded coefficients.

3. For the equation in 2, consider the Cauchy-Dirichlet problem on cylinders for the form $[0,T)\times B(x,r)$ where $x \in \mathbb{R}$ and $r > 0$. At the parabolic boundary, we require that the solution equals the function u^ε given by 1. We are now in a situation where the $C^{1,\alpha}$-theory of [5] applies. All the requirements of Theorem 9.3 of [5] are satisfied for the problem on the cylinder, and this implies the existence of a unique viscosity solution $\bar{u} \in C^{1,\bar{\alpha}}_{\mathrm{loc}}$ for all $\bar{\alpha} \in (0,1)$. Uniqueness implies that $\bar{u} = u^\varepsilon$ on the cylinder, and since x, r was arbitrary, $u^\varepsilon \in C^{1,\bar{\alpha}}_{\mathrm{loc}}(Q_T)$.

4. By 3, u^ε actually solves the following equation

$$u_t = f^\varepsilon(u_{xx}) + \varepsilon u_{xx} + g(t,x) \quad \text{in} \quad Q_T,$$

where $g(t,x) = -(u_x^\varepsilon(t,x))^2$ is a $\mathcal{C}^{0,\bar{\alpha}}_{\mathrm{loc}}(Q_T)$ function. An argument similar to the one in 3, using this time Evans-Krylov $C^{2,\alpha}$-theory, see Theorem 14.10 in [14], yields $u^\varepsilon \in C^{2,\beta}_{\mathrm{loc}}(Q_T)$ for some $\beta \in (0,1)$. □

Lemma A.5. *If $u^\varepsilon \in C^{2,\beta}_{\mathrm{loc}}(Q_T)$ for some $\beta \in (0,1)$, then u^ε_{xxxx} and u^ε_{txx} belong to $\mathcal{C}^{0,\alpha}_{\mathrm{loc}}(Q_T)$.*

Proof. Here we will use difference quotients to obtain higher regularity of the solution. Let $h \in \mathbb{R}$ be non-zero, and define

$$w(t,x) = \frac{u^\varepsilon(t,x+h) - u^\varepsilon(t,x)}{h}.$$

Simple calculations show that for $t > 0$, w satisfies

$$\begin{aligned} &w_t + \left(\int_0^1 2\big(su^\varepsilon_x(t,x+h) + (1-s)u^\varepsilon_x(t,x)\big)ds\right) w_x \\ &\quad = \left(\int_0^1 f^{\varepsilon\prime}\big(su^\varepsilon_{xx}(t,x+h) + (1-s)u^\varepsilon_{xx}(t,x)\big)ds\right) w_{xx} + \varepsilon w_{xx}. \end{aligned} \tag{A.1}$$

Since $f^{\varepsilon\prime} \geq 0$, this is a linear uniformly elliptic equation in w. Furthermore, on each compact subset of Q_T, the coefficients are as Hölder continuous as are $u^\varepsilon_x, u^\varepsilon_{xx}$ with bounds independent of h. Linear Schauder theory then implies that $w \in C^{2,\beta}_{\mathrm{loc}}(Q_T)$ with bounds independent of h. This in turn implies that $u^\varepsilon_{xxx} \in \mathcal{C}^{0,\beta}_{\mathrm{loc}}(Q_T)$. A similar but simpler argument, using the *linear* equation (A.1), shows that

$$\bar{w}(t,x) = \frac{w(t,x+k) - w(t,x)}{k}$$

belongs to $C^{2,\beta}_{\mathrm{loc}}(Q_T)$ with bounds independent of k (and h), and we can conclude that $u^\varepsilon_{xxxx} \in \mathcal{C}^{0,\beta}_{\mathrm{loc}}(Q_T)$. The regularity of u^ε_{txx} is obtained by differentiating the equation and using the x-regularity of u^ε. $\square$

Remark A.6. u^ε is actually infinitely differentiable, as can be seen by continuing the above argument.

This completes the proof of Lemma 3.2. $\square$

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