The stability of compressible vortex sheets in two space dimensions

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Abstract

We study the linear stability of planar compressible vortex sheets in two space dimensions. Under a supersonic condition that precludes violent instabilities, we prove an energy estimate for the linearized boundary value problem. Since the problem is characteristic, the estimate we prove exhibits a loss of control on the trace of the solution. Furthermore, the failure of the uniform Kreiss-Lopatinskii condition yields a loss of derivatives in the energy estimate.

1 Introduction

The neutral linear stability of (supersonic) vortex sheets in two space dimensions was established a long time ago, see e.g. [20]. However, the normal modes analysis performed to derive this result is far not sufficient to guarantee the existence of nonplanar vortex sheets (that is, contact discontinuities) solutions to the compressible isentropic Euler's equations. Recall that in three space dimensions, planar vortex sheets are violently unstable (see e.g. [23]) so we restrict our analysis to the two dimensional case. On the other hand, in the two dimensional case, subsonic planar vortex sheets are also violently unstable. We also refer to [10] for the study of the instability of vortex sheets when heat conduction is taken into account. In this paper, we show that supersonic planar vortex sheets are linearly stable, in the sense that the linearized system (around these particular solutions) obeys an energy estimate. This is a first step towards proving the existence of nonplanar compressible vortex sheets.

Several points need to be highlighted. First of all, the existence of compressible vortex sheets is a free boundary nonlinear hyperbolic problem. Moreover, the free boundary is characteristic with respect to both left and right states since we deal with contact discontinuities. For this reason, one can not apply Majda's analysis on shock waves (see [16, 15]) that are noncharacteristic interfaces. In some previous works devoted to weakly stable shock waves, see [8, 9], we have considered hyperbolic Initial Boundary Value Problems that did not meet the uniform Kreiss-Lopatinskii condition. In the case of vortex sheets, the analysis is closely related, with the additional difficulty that the boundary is characteristic (the present analysis thus relies more on the work of Majda and Osher [18] rather than on the work of Kreiss [12, 4]). The connection with [8, 9] is that in both cases, the analogue of the Kreiss-Lopatinskii condition is fulfilled but not in a uniform way. Furthermore, in the case of vortex sheets as in the case of shock waves, the linearized Rankine-Hugoniot conditions form an elliptic system for the unknown front. This property is a key point in our work (it is also crucial in Majda's analysis on shock waves [16, 15]).

Regarding the energy estimate for the linearized problem, the failure of the uniform Kreiss-Lopatinskii condition yields a loss of derivatives with respect to the source terms. Furthermore, because the boundary is characteristic, we expect to lose some control on the trace of the solution at the boundary. As a matter of fact, we shall see that the only loss of control is on the tangential velocity (which corresponds to the "characteristic part" of the solution). The good point is that the ellipticity of the boundary conditions for the unknown front enables us to gain one derivative for it, as in Majda's work on shock waves [16].

Let us now describe the content of the paper. In section 2, we present the nonlinear equations describing the evolution of compressible vortex sheets and introduce some notations. Then, in section 3, we shall consider the linearized equations around a planar (stationary) vortex sheet and state our main result. After several reductions, we shall detail in section 4 the normal modes analysis of this problem and construct a degenerate Kreiss' symmetrizer in order to derive the main energy estimate of this paper. Eventually, section 5 is devoted to the proof of several technical lemmas and section 6 to some remarks.

2 The nonlinear equations

We consider Euler's equations of isentropic gas dynamics in the whole plane $\mathbb{R}^2$. Denoting by $\mathbf{u}$ the velocity of the fluid and ρ the density, the equations read:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \,, \\ \partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla \, p = 0 \,, \end{cases} \tag{1}$$

where $p = p(\rho)$ is the pressure law. In all this paper, p is assumed to be a strictly increasing function of ρ, defined on $]0, +\infty[$. We also assume that p is a $\mathcal{C}^\infty$ function of ρ. The speed of sound $c(\rho)$ in the fluid is then defined by the relation

$$c(\rho) := \sqrt{p'(\rho)} \,.$$

Let $(\rho, \mathbf{u})(t, x_1, x_2)$ be a smooth function on either side of a smooth hypersurface $\Gamma := \{x_2 = \varphi(t, x_1)\}$. Then $(\rho, \mathbf{u})$ is a (weak) solution of (1) if and only if $(\rho, \mathbf{u})$ is a classical solution of (1) on both sides of Γ and the Rankine-Hugoniot conditions hold at each point of Γ:

$$\begin{aligned} \partial_t \varphi \, [\rho] - [\rho \mathbf{u} \cdot \nu] = 0 \,, \\ \partial_t \varphi \, [\rho \mathbf{u}] - [(\rho \mathbf{u} \cdot \nu) \mathbf{u}] - [p] \nu = 0 \,, \end{aligned} \tag{2}$$

where $\nu := (-\partial_{x_1} \varphi, 1)$ is a (space) normal vector to Γ. As usual, $[q] = q^+ - q^-$ denotes the jump of a quantity q across the interface Γ (see [22]).

Following Lax [13], we shall say that $(\rho, \mathbf{u})$ is a contact discontinuity if (2) is satisfied in the following way:

$$\begin{aligned} \partial_t \varphi = \mathbf{u}^+ \cdot \nu = \mathbf{u}^- \cdot \nu \,, \\ p^+ = p^- \,. \end{aligned}$$

Because p is monotone, the previous equalities read

$$\begin{aligned} \partial_t \varphi = \mathbf{u}^+ \cdot \nu = \mathbf{u}^- \cdot \nu \,, \\ \rho^+ = \rho^- \,. \end{aligned} \tag{3}$$

Since the density and the normal velocity are continuous across the interface Γ, the only jump experimented by the solution is on the tangential velocity. For this reason, a contact discontinuity is a vortex sheet and we shall make no distinction in the terminology we use. For the incompressible theory of two-dimensional vortex sheets, we refer the reader to the books [5, 17].

Note that the first two equalities above are nothing but eikonal equations: if $\Phi(t, x_1, x_2) := x_2 - \varphi(t, x_1)$ is the equation of the interface Γ, then Φ satisfies

$$\partial_t \Phi + \lambda_2(\rho^+, \mathbf{u}^+, \nabla_x \Phi) = 0 \quad \text{and} \quad \partial_t \Phi + \lambda_2(\rho^-, \mathbf{u}^-, \nabla_x \Phi) = 0\,,$$

on $\{x_2 = 0\}$, where

$$\lambda_2(\rho, \mathbf{u}, \xi) := \mathbf{u} \cdot \xi$$

is the second characteristic field of the system (1). It is linearly degenerate since the corresponding eigenvector, in the quasilinear form of (1), is given by

$$r_2(\rho, \mathbf{u}, \xi) := \begin{pmatrix} 0 \\ \xi^\perp \end{pmatrix}\,,$$

where $\xi^\perp = (-\xi_2, \xi_1)$ if $\xi = (\xi_1, \xi_2)$. Recall that the space dimension equals 2.

The interface Γ, or equivalently the function φ is part of the unknowns of the problem. We thus deal with a free boundary problem. As it is common in this kind of situation, we first straighten the unknown front in order to work in a fixed domain. More precisely, the unknowns $(\rho, \mathbf{u})$, that are smooth on either side of $\{x_2 = \varphi(t, x_1)\}$, are replaced by the functions

$$\begin{aligned} (\rho^+_\sharp, \mathbf{u}^+_\sharp)(t, x_1, x_2) &:= (\rho, \mathbf{u})(t, x_1, x_2 + \varphi(t, x_1))\,, \\ \text{and} \quad (\rho^-_\sharp, \mathbf{u}^-_\sharp)(t, x_1, x_2) &:= (\rho, \mathbf{u})(t, x_1, -x_2 + \varphi(t, x_1))\,, \end{aligned}$$

that are smooth on $\{x_2 > 0\}$. For convenience, we drop the $\sharp$ index and only keep the $+$ and $-$ exponents.

Let us denote by v and u the two components of the velocity, that is, $\mathbf{u} = (v, u)$. Then the existence of compressible vortex sheets amounts to proving the existence of smooth solutions to the following system:

$$\begin{aligned} \partial_t \rho^+ + v^+ \partial_{x_1} \rho^+ + (u^+ - \partial_t \varphi - v^+ \partial_{x_1} \varphi) \partial_{x_2} \rho^+ + \rho^+ \partial_{x_1} v^+ + \rho^+ \partial_{x_2} u^+ - \rho^+ \partial_{x_1} \varphi \, \partial_{x_2} v^+ &= 0\,, \\ \partial_t v^+ + v^+ \partial_{x_1} v^+ + (u^+ - \partial_t \varphi - v^+ \partial_{x_1} \varphi) \partial_{x_2} v^+ + \frac{p'(\rho^+)}{\rho^+} \partial_{x_1} \rho^+ - \frac{p'(\rho^+)}{\rho^+} \partial_{x_1} \varphi \, \partial_{x_2} \rho^+ &= 0\,, \\ \partial_t u^+ + v^+ \partial_{x_1} u^+ + (u^+ - \partial_t \varphi - v^+ \partial_{x_1} \varphi) \partial_{x_2} u^+ + \frac{p'(\rho^+)}{\rho^+} \partial_{x_2} \rho^+ &= 0\,, \\ \partial_t \rho^- + v^- \partial_{x_1} \rho^- - (u^- - \partial_t \varphi - v^- \partial_{x_1} \varphi) \partial_{x_2} \rho^- + \rho^- \partial_{x_1} v^- - \rho^- \partial_{x_2} u^- + \rho^- \partial_{x_1} \varphi \, \partial_{x_2} v^- &= 0\,, \\ \partial_t v^- + v^- \partial_{x_1} v^- - (u^- - \partial_t \varphi - v^- \partial_{x_1} \varphi) \partial_{x_2} v^- + \frac{p'(\rho^-)}{\rho^-} \partial_{x_1} \rho^- + \frac{p'(\rho^-)}{\rho^-} \partial_{x_1} \varphi \, \partial_{x_2} \rho^- &= 0\,, \\ \partial_t u^- + v^- \partial_{x_1} u^- - (u^- - \partial_t \varphi - v^- \partial_{x_1} \varphi) \partial_{x_2} u^- - \frac{p'(\rho^-)}{\rho^-} \partial_{x_2} \rho^- &= 0\,, \end{aligned} \tag{4}$$

in the fixed domain $\{x_2 > 0\}$, together with the boundary conditions

$$\begin{aligned} \partial_t \varphi = -v^+ \partial_{x_1} \varphi + u^+ &= -v^- \partial_{x_1} \varphi + u^-\,, \\ \rho^+ &= \rho^-\,, \end{aligned}$$

on $\{x_2 = 0\}$. For convenience, we rewrite the boundary conditions in the following way:

$$\begin{aligned}(v^+ - v^-)\partial_{x_1}\varphi - (u^+ - u^-) = 0\,,\\ \partial_t\varphi + v^+\partial_{x_1}\varphi - u^+ = 0\,,\\ \rho^+ - \rho^- = 0\,.\end{aligned} \tag{5}$$

The problem is thus the construction of (local in time) smooth solutions to (4)-(5), once initial data have been prescribed. Of course, such initial data will have to fulfill a certain number of compatibility conditions. The first step in proving such an existence result is the study of the linearized problem around a particular constant solution, and this is precisely the aim of this article.

There exist many simple solutions of (4)-(5), that correspond, in the original variables, to planar vortex sheets:

$$(\rho,\mathbf{u}) = \begin{cases}(\rho,\mathbf{u}_r) & \text{if } x_2 > \sigma t + n x_1,\\ (\rho,\mathbf{u}_l) & \text{if } x_2 < \sigma t + n x_1.\end{cases}$$

Here above, $\mathbf{u}_r, \mathbf{u}_l$ are fixed vectors in $\mathbb{R}^2$, and $\rho > 0$, σ and n are fixed real numbers. These quantities are linked by the Rankine-Hugoniot conditions:

$$\sigma = -v_r n + u_r = -v_l n + u_l\,.$$

Changing observer if necessary, we may assume without loss of generality

$$\sigma = n = u_r = u_l = 0 \quad \text{and} \quad v_r + v_l = 0 \quad (v_r \neq 0)\,.$$

In the new variables, this corresponds to the following regular solution of (4)-(5):

$$\begin{pmatrix}\rho^+\\ v^+\\ u^+\end{pmatrix} \equiv \begin{pmatrix}\rho\\ v_r\\ 0\end{pmatrix}\,, \quad \begin{pmatrix}\rho^-\\ v^-\\ u^-\end{pmatrix} \equiv \begin{pmatrix}\rho\\ v_l\\ 0\end{pmatrix}\,, \quad \varphi \equiv 0\,, \tag{6}$$

with the relation $v_r + v_l = 0$. We only consider the case $v_r \neq 0$, and without loss of generality, we assume $v_r > 0$. In the next section, we study the linearized equations around the particular solution defined by (6). Under a certain "supersonic" assumption, we shall show that the linearized equations satisfy an a priori energy estimate.

3 The constant coefficients linearized system

3.1 The linearized equations

Let us denote by $\dot\rho_\pm, \dot{\mathbf{u}}_\pm, \psi$ some small perturbations of the exact solution given by (6). Up to second order, the perturbations satisfy

$$\begin{aligned}\partial_t\dot\rho_+ + v_r\,\partial_{x_1}\dot\rho_+ + \rho\,\partial_{x_1}\dot v_+ + \rho\,\partial_{x_2}\dot u_+ = 0\,,\\ \partial_t\dot v_+ + v_r\,\partial_{x_1}\dot v_+ + \frac{c^2}{\rho}\,\partial_{x_1}\dot\rho_+ = 0\,,\\ \partial_t\dot u_+ + v_r\,\partial_{x_1}\dot u_+ + \frac{c^2}{\rho}\,\partial_{x_2}\dot\rho_+ = 0\,,\\ \partial_t\dot\rho_- + v_l\,\partial_{x_1}\dot\rho_- + \rho\,\partial_{x_1}\dot v_- - \rho\,\partial_{x_2}\dot u_- = 0\,,\\ \partial_t\dot v_- + v_l\,\partial_{x_1}\dot v_- + \frac{c^2}{\rho}\,\partial_{x_1}\dot\rho_- = 0\,,\\ \partial_t\dot u_- + v_l\,\partial_{x_1}\dot u_- - \frac{c^2}{\rho}\,\partial_{x_2}\dot\rho_- = 0\,,\end{aligned} \tag{7}$$

in the domain $\{x_2 > 0\}$, together with the linearized Rankine-Hugoniot relations

$$\begin{aligned} (v_r - v_l)\,\partial_{x_1}\psi - (\dot{u}_+ - \dot{u}_-) = 0\,, \\ \partial_t\psi + v_r\,\partial_{x_1}\psi - \dot{u}_+ = 0\,, \\ \dot{\rho}_+ - \dot{\rho}_- = 0\,, \end{aligned} \tag{8}$$

on the boundary $\{x_2 = 0\}$. In (7), c denotes the sound speed corresponding to the reference density ρ.

In short, equations (7)-(8) read

$$\begin{cases} L\,U = 0\,, & \text{if } x_2 > 0, \\ B(U,\psi) = 0\,, & \text{if } x_2 = 0, \end{cases} \tag{9}$$

with $U := (\dot{\rho}_+, \dot{v}_+, \dot{u}_+, \dot{\rho}_-, \dot{v}_-, \dot{u}_-)^T$, and obvious definitions for the operators L and B. It is important to note that the interior equations do not involve the perturbation ψ, so L is an operator that only acts on U. This property also holds when one studies the linearized shock wave equations around a planar shock, see [16].

Proving an energy estimate for the linearized equations amounts to working with source terms, both in the interior domain and on the boundary. From now on, we thus consider the linear equations

$$\begin{cases} L\,U = f\,, & \text{if } x_2 > 0, \\ B(U,\psi) = g\,, & \text{if } x_2 = 0, \end{cases} \tag{10}$$

and try to estimate U and ψ in terms of f and g (in appropriate functional spaces). In order to simplify the subsequent calculations, we introduce some new unknown functions, and define the following quantities:

$$\begin{aligned} w_1 &:= \frac{1}{2}\left(-\frac{\dot{\rho}_+}{\rho} + \frac{\dot{u}_+}{c}\right)\,, \quad w_2 := \frac{1}{2}\left(\frac{\dot{\rho}_-}{\rho} + \frac{\dot{u}_-}{c}\right)\,, \\ w_3 &:= \frac{1}{2}\left(\frac{\dot{\rho}_+}{\rho} + \frac{\dot{u}_+}{c}\right)\,, \quad w_4 := \frac{1}{2}\left(-\frac{\dot{\rho}_-}{\rho} + \frac{\dot{u}_-}{c}\right)\,. \end{aligned} \tag{11}$$

We also define the following vectors:

$$\begin{aligned} W &:= (\dot{v}_+, \dot{v}_-, w_1, w_2, w_3, w_4)^T\,, \\ W^c &:= (\dot{v}_+, \dot{v}_-)^T\,, \\ W^{nc} &:= (w_1, w_2, w_3, w_4)^T\,. \end{aligned}$$

The notations W^c and W^{nc} are introduced in order to separate the "characteristic part" of the vector W and the "noncharacteristic part" of W. We shall go back to this decomposition later on.

Let us define the following 6×6 symmetric matrices:

$$\mathcal{A}_0 := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2c^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2c^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2c^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2c^2 \end{pmatrix}, \quad \mathcal{A}_1 := \begin{pmatrix} v_r & 0 & -c^2 & 0 & c^2 & 0 \\ 0 & v_l & 0 & c^2 & 0 & -c^2 \\ -c^2 & 0 & 2c^2 v_r & 0 & 0 & 0 \\ 0 & c^2 & 0 & 2c^2 v_l & 0 & 0 \\ c^2 & 0 & 0 & 0 & 2c^2 v_r & 0 \\ 0 & -c^2 & 0 & 0 & 0 & 2c^2 v_l \end{pmatrix},$$

$$\mathcal{A}_2 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2c^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2c^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2c^3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2c^3 \end{pmatrix}, \tag{12}$$

as well as the following

$$\underline{b} := \begin{pmatrix} 0 & v_r - v_l \\ 1 & v_r \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2v_r \\ 1 & v_r \\ 0 & 0 \end{pmatrix}, \quad \underline{M} := \begin{pmatrix} -c & c & -c & c \\ -c & 0 & -c & 0 \\ -1 & -1 & 1 & 1 \end{pmatrix}. \tag{13}$$

Then, using (11)-(12)-(13), the linear equations (10) equivalently read

$$\begin{cases} \mathcal{L}W = f\,, & \text{if } x_2 > 0, \\ \mathcal{B}(W^{nc}, \psi) = g\,, & \text{if } x_2 = 0, \end{cases} \tag{14}$$

with new f and g, and where we have let

$$\mathcal{L}W := \mathcal{A}_0\, \partial_t W + \mathcal{A}_1\, \partial_{x_1} W + \mathcal{A}_2\, \partial_{x_2} W\,,$$
$$\text{and} \quad \mathcal{B}(W^{nc}, \psi) := \underline{M}\, W^{nc}|_{x_2=0} + \underline{b} \begin{pmatrix} \partial_t \psi \\ \partial_{x_1} \psi \end{pmatrix}.$$

Note that the kernel of $\mathcal{A}_2$ is exactly the set of those W such that $W^{nc} = 0$ (and W^c is arbitrary). The boundary $\{x_2 = 0\}$ is thus characteristic with multiplicity 2. As already noted in earlier works, see e.g. [14, 18], we expect to lose control of the trace of W^c, that is, on the trace of the tangential velocities $(\dot{v}_+, \dot{v}_-)$. At the opposite, we expect to control the trace of W^{nc} on $\{x_2 = 0\}$, that is, we expect to control the trace of $(\dot{\rho}_+, \dot{\rho}_-, \dot{u}_+, \dot{u}_-)$.

3.2 The main result

Before stating our energy estimate for (14), we need to introduce some Sobolev weighted norms. First define the half-space

$$\Omega := \{(t, x_1, x_2) \in \mathbb{R}^3 \text{ s.t. } x_2 > 0\} = \mathbb{R}^2 \times \mathbb{R}^+ .$$

The boundary $\partial\Omega$ is identified to $\mathbb{R}^2$.

For all real number s and all $\gamma \geq 1$, define the space

$$H^s_\gamma(\mathbb{R}^2) := \{u \in \mathcal{D}'(\mathbb{R}^2) \text{ s.t. } \exp(-\gamma t) u \in H^s(\mathbb{R}^2)\}\,.$$

It is equipped with the norm

$$\|u\|_{H^s_\gamma(\mathbb{R}^2)} := \|\exp(-\gamma t) u\|_{H^s(\mathbb{R}^2)}\,.$$

Letting $\widetilde{u} := \exp(-\gamma t)u$, one has

$$\|u\|_{H^s_\gamma(\mathbb{R}^2)} \simeq \|\widetilde{u}\|_{s,\gamma} \quad \text{where} \quad \|v\|^2_{s,\gamma} := \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} (\gamma^2 + |\xi|^2)^s |\widehat{v}(\xi)|^2 \, d\xi \,,$$

where $\widehat{v}$ is the Fourier transform of any function v defined on $\mathbb{R}^2$.

For all integer k, one defines the space $H^k_\gamma(\Omega)$ in an entirely similar way.

The space $L^2(\mathbb{R}^+; H^s_\gamma(\mathbb{R}^2))$ is equipped with the norm

$$|||v|||^2_{L^2(H^s_\gamma)} := \int_0^{+\infty} \|v(\cdot, x_2)\|^2_{H^s_\gamma(\mathbb{R}^2)} \, dx_2 \,.$$

In the sequel, the variable in $\mathbb{R}^2$ is (t, x_1) while x_2 is the variable in $\mathbb{R}^+$.

Our main result is stated as follows:

Theorem 1. *Assume that the particular solution defined by* (6) *satisfies*

$$v_r - v_l > 2\sqrt{2}\, c \,. \tag{15}$$

Then there exists a positive constant C such that for all $\gamma \geq 1$ and for all $(W, \psi) \in H^2_\gamma(\Omega) \times H^2_\gamma(\mathbb{R}^2)$, the following estimate holds:

$$\gamma\, |||W|||^2_{L^2_\gamma(\Omega)} + \|W^{nc}{}_{|x_2=0}\|^2_{L^2_\gamma(\mathbb{R}^2)} + \|\psi\|^2_{H^1_\gamma(\mathbb{R}^2)} \leq C \left(\frac{1}{\gamma^3}\, |||\mathcal{L}W|||^2_{L^2(H^1_\gamma)} + \frac{1}{\gamma^2}\, \|\mathcal{B}(W^{nc}, \psi)\|^2_{H^1_\gamma(\mathbb{R}^2)} \right) . \tag{16}$$

Introducing $\widetilde{W} := \exp(-\gamma t)W$ and $\widetilde{\psi} := \exp(-\gamma t)\psi$, we easily find that (14) is equivalent to

$$\begin{cases} \mathcal{L}^\gamma \widetilde{W} := \gamma \mathcal{A}_0\, \widetilde{W} + \mathcal{L}\widetilde{W} = \exp(-\gamma t) f \,, & \text{if } x_2 > 0, \\ \mathcal{B}^\gamma(\widetilde{W}^{nc}, \widetilde{\psi}) := \underline{M}\, \widetilde{W}^{nc}_{|x_2=0} + \underline{b} \begin{pmatrix} \gamma\widetilde{\psi} + \partial_t \widetilde{\psi} \\ \partial_{x_1} \widetilde{\psi} \end{pmatrix} = \exp(-\gamma t) g \,, & \text{if } x_2 = 0. \end{cases} \tag{17}$$

Consequently, Theorem 1 admits the following equivalent formulation:

Theorem 2. *Assume that* (15) *holds. Then there exists a positive constant C such that for all $\gamma \geq 1$ and for all $(\widetilde{W}, \widetilde{\psi}) \in H^2(\Omega) \times H^2(\mathbb{R}^2)$, the following estimate holds:*

$$\gamma\, |||\widetilde{W}|||^2_0 + \|\widetilde{W}^{nc}_{|x_2=0}\|^2_0 + \|\widetilde{\psi}\|^2_{1,\gamma} \leq C \left(\frac{1}{\gamma^3}\, |||\mathcal{L}^\gamma \widetilde{W}|||^2_{1,\gamma} + \frac{1}{\gamma^2}\, \|\mathcal{B}^\gamma(\widetilde{W}^{nc}, \widetilde{\psi})\|^2_{1,\gamma} \right) . \tag{18}$$

In (18), we have used the following notations for any $u \in L^2(\mathbb{R}^+; H^s(\mathbb{R}^2))$:

$$|||v|||^2_{s,\gamma} := \int_0^{+\infty} \|v(\cdot, x_2)\|^2_{s,\gamma} \, dx_2 \,.$$

For instance, $|||\cdot|||_{0,\gamma}$ is the usual norm on $L^2(\Omega)$ and does not involve γ, so we shall denote it by $|||\cdot|||_0$. The norm $|||\cdot|||_{1,\gamma}$ is the weighted norm on $L^2(\mathbb{R}^+, H^1(\mathbb{R}^2))$.

4 Proof of the main result

4.1 Some preliminary reductions

In this paragraph, we show that it is sufficient to prove Theorem 2 in the particular case $\mathcal{L}^\gamma \widetilde{W} \equiv 0$. This first reduction simplifies many subsequent calculations. The argument we use was introduced in [18, page 636].

In order to simplify notations, we drop the tildas. Let $W \in H^2(\Omega)$ and $\psi \in H^2(\mathbb{R}^2)$. Then we define:

$$f := \mathcal{L}^\gamma W \in H^1(\Omega)\,, \quad g := \mathcal{B}^\gamma(W^{nc}, \psi) \in H^1(\mathbb{R}^2)\,.$$

Consider the following auxiliary problem:

$$\begin{cases} \mathcal{L}^\gamma W_1 = f\,, & \text{if } x_2 > 0, \\ M^{aux}\, W_1^{nc}{}_{|x_2=0} = 0\,, & \text{if } x_2 = 0, \end{cases} \tag{19}$$

where M^{aux} is defined by

$$M^{aux} := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In the auxiliary problem (19), the boundary conditions are maximally dissipative (see the formula (12) defining the matrix $\mathcal{A}_2$). Since $f \in L^2(\mathbb{R}^+; H^1(\mathbb{R}^2))$, it follows from [14] that there exists a function $W_1 \in L^2(\mathbb{R}^+; H^1(\mathbb{R}^2))$, such that the trace of W_1^{nc} on $\{x_2 = 0\}$ belongs to $H^1(\mathbb{R}^2)$, and that is a solution to (19). In particular, the function W_1 satisfies the following estimates:

$$\begin{aligned} \gamma\, |||W_1|||_0^2 \le \frac{C}{\gamma}\, |||f|||_0^2\,, \\ \|W_1^{nc}{}_{|x_2=0}\|_{1,\gamma}^2 \le \frac{C}{\gamma}\, |||f|||_{1,\gamma}^2 = \frac{C}{\gamma} \int_0^{+\infty} \|f(\cdot, x_2)\|_{1,\gamma}^2\, dx_2\,. \end{aligned} \tag{20}$$

Let us define $W_2 := W - W_1$. It satisfies

$$\begin{cases} \mathcal{L}^\gamma W_2 = 0\,, & \text{if } x_2 > 0, \\ \mathcal{B}^\gamma(W_2^{nc}, \psi) = g - \underline{M}\, W_1^{nc}{}_{|x_2=0}\,, & \text{if } x_2 = 0. \end{cases}$$

Consequently, if we manage to prove that Theorem 2 holds true as long as the interior source term is zero, we shall obtain

$$\gamma\, |||W_2|||_0^2 + \|W_2^{nc}{}_{|x_2=0}\|_0^2 + \|\psi\|_{1,\gamma}^2 \le \frac{C}{\gamma^2}\, \|g - \underline{M}\, W_1^{nc}{}_{|x_2=0}\|_{1,\gamma}^2 \le \frac{C}{\gamma^2} \left(\|g\|_{1,\gamma}^2 + \|W_1^{nc}{}_{|x_2=0}\|_{1,\gamma}^2 \right).$$

Then using (20) to estimate the H^1 norm of the trace of W_1^{nc} as well as the L^2 norm of W_1, we shall derive our main energy estimate (18). Without loss of generality, we thus assume from now on that W and ψ satisfy

$$\gamma \mathcal{A}_0\, W + \mathcal{A}_0\, \partial_t W + \mathcal{A}_1\, \partial_{x_1} W + \mathcal{A}_2\, \partial_{x_2} W = 0 \tag{21}$$

in the interior domain Ω, as well as the following boundary conditions

$$\underline{M}\, W^{nc}{}_{|x_2=0} + \underline{b} \begin{pmatrix} \gamma\psi + \partial_t \psi \\ \partial_{x_1} \psi \end{pmatrix} = g\,, \quad x_2 = 0\,. \tag{22}$$

With slight abuse of notations, we still denote by g the source term in the boundary conditions, instead of $g - \underline{M}\, W_1^{nc}$. This is of pure convenience.

Recall that all matrices $\mathcal{A}_j$ are symmetric, and that $\mathcal{A}_0$ is positive definite, see (12). Taking the scalar product of (21) with W and integrating over Ω yields the following inequality:

$$\gamma \, \|W\|_0^2 \leq C \, \|W^{nc}{}_{|_{x_2=0}}\|_0^2 \, .$$

Consequently, it is sufficient to derive an estimate of the form

$$\|W^{nc}{}_{|_{x_2=0}}\|_0^2 + \|\psi\|_{1,\gamma}^2 \leq \frac{C}{\gamma^2} \, \|g\|_{1,\gamma}^2 \tag{23}$$

in order to obtain (18).

We shall derive (23) by means of a Kreiss' symmetrizer, whose construction is detailed in the next paragraphs. Once performed a Fourier transform in (t, x_1), the first step consists in "eliminating" the front in the boundary conditions (22). Then we shall detail the normal modes analysis and construct a symbolic symmetrizer.

4.2 Eliminating the front

As mentionned above, we focus on (21)-(22), and perform a Fourier transform in (t, x_1). The dual variables are denoted by (δ, η). We also define $\tau := \gamma + i\delta$. This is the Laplace dual variable (indeed, the previous manipulations amount to performing a Laplace transform with respect to t). We obtain the following system of differential equations:

$$\begin{aligned} (\tau \mathcal{A}_0 + i\eta \mathcal{A}_1) \, \widehat{W} + \mathcal{A}_2 \, \frac{d\widehat{W}}{dx_2} = 0 \, , \quad x_2 > 0 \, , \\ b(\tau, \eta) \widehat{\psi} + \underline{M} \, \widehat{W^{nc}}(0) = \widehat{g} \, , \end{aligned} \tag{24}$$

where $b(\tau, \eta)$ is simply defined by

$$b(\tau, \eta) := \underline{b} \begin{pmatrix} \tau \\ i\eta \end{pmatrix} = \begin{pmatrix} 2iv_r \eta \\ \tau + iv_r \eta \\ 0 \end{pmatrix} . \tag{25}$$

Recall that $\underline{b}$ and $\underline{M}$ are defined by (13). Observe that $b(\tau, \eta)$ is homogeneous (of degree 1) with respect to (τ, η). In order to take such homogeneity properties into account, we define the hemisphere

$$\Sigma := \{ (\tau, \eta) \in \mathbb{C} \times \mathbb{R} \text{ s.t. } |\tau|^2 + v_r^2 \eta^2 = 1 \text{ and } \operatorname{Re} \tau \geq 0 \} \, .$$

Recall that τ is the Laplace dual variable of the time variable t, while η is the Fourier dual variable of x_1, so τ/η is a velocity. Our definition of Σ takes this property into account.

We denote by Ξ the set

$$\Xi := \{ (\gamma, \delta, \eta) \in [0, +\infty[\times \mathbb{R}^2 \text{ s.t. } (\gamma, \delta, \eta) \neq (0,0,0) \} =]0, +\infty[\cdot \Sigma \, .$$

It is the set of "frequencies" we shall consider in the sequel. We always identify $(\gamma, \delta) \in \mathbb{R}^2$ and $\tau = \gamma + i\delta \in \mathbb{C}$.

One crucial remark is that the symbol $b(\tau, \eta)$ is elliptic, that is, it does not vanish on the closed hemisphere Σ. More precisely, we have the following Lemma:

Lemma 1. *There exists a $\mathcal{C}^\infty$ mapping Q defined on Ξ such that Q has values in $GL_3(\mathbb{C})$, is homogeneous of degree* 0*, and satisfies*

$$\forall\,(\tau,\eta)\in\Xi\,,\quad Q(\tau,\eta)\,b(\tau,\eta)=\begin{pmatrix}0\\0\\ \vartheta(\tau,\eta)\end{pmatrix}\,,$$

where ϑ is $\mathcal{C}^\infty$, homogeneous of degree 1*, and has the additional property:*

$$\min_{(\tau,\eta)\in\Sigma}|\vartheta(\tau,\eta)|>0\,.$$

Proof. We shall define the mapping Q on Σ and then extend Q by homogeneity. If we define

$$\forall\,(\tau,\eta)\in\Sigma\,,\quad Q(\tau,\eta):=\begin{pmatrix}0&0&1\\ \tau+iv_r\eta&-2iv_r\eta&0\\ -2iv_r\eta&\overline{\tau}-iv_r\eta&0\end{pmatrix}\,,$$

then we check that Q has all the required properties. The corresponding ϑ is given by

$$\forall\,(\tau,\eta)\in\Sigma\,,\quad \vartheta(\tau,\eta):=|\tau+iv_r\eta|^2+4v_r^2\eta^2\,.$$

□

Let us multiply the boundary conditions in (24) by the matrix $Q(\tau,\eta)$. We obtain:

$$\begin{pmatrix}0\\0\\ \vartheta(\tau,\eta)\end{pmatrix}\widehat{\psi}+\begin{pmatrix}\beta(\tau,\eta)\\ \ell(\tau,\eta)\end{pmatrix}\widehat{W^{nc}}(0)=Q(\tau,\eta)\,\widehat{g}\,,\tag{26}$$

where β has two rows while ℓ has one row and

$$\begin{pmatrix}\beta(\tau,\eta)\\ \ell(\tau,\eta)\end{pmatrix}:=Q(\tau,\eta)\,\underline{M}\,.$$

The exact expression of ℓ is useless, but we shall use the expression of β:

$$\forall\,(\tau,\eta)\in\Sigma\,,\quad \beta(\tau,\eta)=\begin{pmatrix}-1&-1&1&1\\ -c(\tau+iv_l\eta)&c(\tau+iv_r\eta)&-c(\tau+iv_l\eta)&c(\tau+iv_r\eta)\end{pmatrix}\,.\tag{27}$$

Both β and ℓ are homogeneous of degree 0 and $\mathcal{C}^\infty$ on Ξ.

The last equation in (26) is

$$\vartheta(\tau,\eta)\,\widehat{\psi}+\ell(\tau,\eta)\,\widehat{W^{nc}}(0)=Q_3(\tau,\eta)\,\widehat{g}\,,$$

where Q_3 is the last row of Q. Using the ellipticity property of ϑ (see Lemma 1) together with a uniform bound for ℓ and Q_3, we obtain

$$(|\tau|^2+v_r^2\eta^2)|\widehat{\psi}|^2\le C\left(|\widehat{W^{nc}}(0)|^2+|\widehat{g}|^2\right)\,.$$

Let us now integrate this last inequality with respect to $(\delta,\eta)\in\mathbb{R}^2$. (Recall that δ is the imaginary part of τ). Using Plancherel's Theorem, we obtain

$$\|\psi\|_{1,\gamma}^2\le C\left(\|W^{nc}{}_{|x_2=0}\|_0^2+\|g\|_0^2\right)\le C\left(\|W^{nc}{}_{|x_2=0}\|_0^2+\frac{1}{\gamma^2}\|g\|_{1,\gamma}^2\right)\,.\tag{28}$$

In order to derive (23), it is therefore sufficient to derive an estimate of the trace of W^{nc}. Consequently, we focus on the reduced problem

$$(\tau \mathcal{A}_0 + i\eta \mathcal{A}_1)\,\widehat{W} + \mathcal{A}_2\,\frac{d\widehat{W}}{dx_2} = 0\,, \quad x_2 > 0\,,$$
$$\beta(\tau,\eta)\widehat{W^{nc}}(0) = \widehat{h}\,,$$

and try to derive an estimate for $\widehat{W^{nc}}(0)$. One has to remember that the source term $\widehat{h} \in \mathbb{C}^2$ is easily estimated by $\widehat{g}$, see (26).

In the next paragraph, we recall that under the assumption made in Theorems 1 and 2, the above boundary problem satisfies the Kreiss-Lopatinskii condition but violates the uniform Kreiss-Lopatinskii condition.

4.3 The normal modes analysis

Writing $W = (v_+, v_-, w_1, w_2, w_3, w_4)$, the two first equations in (24) are:

$$(\tau + iv_r\eta)\,\widehat{v_+} - ic^2\eta\,\widehat{w_1} + ic^2\eta\,\widehat{w_3} = 0\,,$$
$$(\tau + iv_l\eta)\,\widehat{v_-} - ic^2\eta\,\widehat{w_4} + ic^2\eta\,\widehat{w_2} = 0\,.$$

They do not involve derivation with respect to the normal variable x_2. For Re $\tau > 0$, we obtain an expression for $\widehat{v_+}$ and $\widehat{v_-}$ that we plug in the last four equations. This operation yields a system of ordinary differential equations of the following form:

$$\begin{cases} \dfrac{d\widehat{W^{nc}}}{dx_2} = \mathcal{A}(\tau,\eta)\,\widehat{W^{nc}}, & \text{if } x_2 > 0, \\ \beta(\tau,\eta)\widehat{W^{nc}}(0) = \widehat{h}\,, & \text{if } x_2 = 0. \end{cases} \tag{29}$$

The matrix $\mathcal{A}(\tau,\eta)$ in (29) is given by

$$\mathcal{A}(\tau,\eta) := \begin{pmatrix} \mu_r & 0 & -m_r & 0 \\ 0 & \mu_l & 0 & -m_l \\ m_r & 0 & -\mu_r & 0 \\ 0 & m_l & 0 & -\mu_l \end{pmatrix}, \quad \mu_{r,l} := \frac{\dfrac{1}{c}\,(\tau + iv_{r,l}\eta)^2 + \dfrac{c}{2}\eta^2}{\tau + iv_{r,l}\eta}\,, \quad m_{r,l} := \frac{\dfrac{c}{2}\eta^2}{\tau + iv_{r,l}\eta}\,. \tag{30}$$

A well-known result, that is due to Hersh [11] in the noncharacteristic case (see [18] for the extension to the characteristic case), asserts that the matrix $\mathcal{A}(\tau,\eta)$ has no purely imaginary eigenvalue as long as Re $\tau > 0$. As a consequence, the stable subspace of $\mathcal{A}(\tau,\eta)$ has constant dimension when Re $\tau > 0$. This dimension equals the number of characteristics going out of the discontinuity. In our case, those theoretical results can be checked directly by computing the eigenvalues and the stable subspace of $\mathcal{A}(\tau,\eta)$. The following Lemma gives an expression of the stable subspace:

Lemma 2. *Let $\tau \in \mathbb{C}$ and $\eta \in \mathbb{R}$ with Re $\tau > 0$ and $(\tau,\eta) \in \Sigma$. The eigenvalues of $\mathcal{A}(\tau,\eta)$ are the roots ω of the dispersion relations*

$$\omega^2 = \mu_r^2 - m_r^2 = \frac{1}{c^2}(\tau + iv_r\eta)^2 + \eta^2\,, \tag{31a}$$

$$\omega^2 = \mu_l^2 - m_l^2 = \frac{1}{c^2}(\tau + iv_l\eta)^2 + \eta^2\,. \tag{31b}$$

In particular, (31a) *[resp.* (31b)*] admits a unique root ω_r [resp. ω_l] of negative real part. The other root of* (31a) *[resp.* (31b)*] is $-\omega_r$ [resp. $-\omega_l$] and has positive real part. The stable subspace $\mathcal{E}^-(\tau,\eta)$ of $\mathcal{A}(\tau,\eta)$ has dimension 2 and is spanned by the two following vectors:*

$$
\begin{aligned}
E_r(\tau,\eta) &:= \left(\frac{c}{2}\eta^2, 0, \frac{1}{c}(\tau+iv_r\eta)^2+\frac{c}{2}\eta^2-(\tau+iv_r\eta)\omega_r, 0\right)^T ,\\
E_l(\tau,\eta) &:= \left(0, \frac{c}{2}\eta^2, 0, \frac{1}{c}(\tau+iv_l\eta)^2+\frac{c}{2}\eta^2-(\tau+iv_l\eta)\omega_l\right)^T .
\end{aligned}
\tag{32}
$$

Both ω_r and ω_l admit a continuous extension to any point (τ,η) such that Re $\tau=0$ and $(\tau,\eta)\in\Sigma$. This allows to extend both vectors E_r and E_l in (32) *to the whole hemisphere Σ. Those two vectors are linearly independent for any value of $(\tau,\eta)\in\Sigma$.*

The symbol $\mathcal{A}(\tau,\eta)$ is diagonalizable as long as both ω_r and ω_l do not vanish, that is, when $\tau\neq(\pm v_r\pm c)i\eta$. Away from such points, $\mathcal{A}$ admits a $\mathcal{C}^\infty$ basis of eigenvectors.

The proof follows from straightforward computations, and we shall omit it.

We point out that the stable subspace $\mathcal{E}^-(\tau,\eta)$ is defined for all $(\tau,\eta)\in\Sigma$, while the matrix $\mathcal{A}(\tau,\eta)$ has some poles on the boundary of Σ, see (30). The poles are exactly those points $(\tau,\eta)\in\Sigma$ verifying $\tau=-iv_{r,l}\eta=\mp iv_r\eta$ (recall that we have $v_r=-v_l\neq0$).

Following Majda and Osher [18], we define the Lopatinskii determinant associated with the boundary conditions β in the following way:

$$
\Delta(\tau,\eta) := \det\left[\beta(\tau,\eta)\,\begin{pmatrix} E_r(\tau,\eta) & E_l(\tau,\eta)\end{pmatrix}\right] ,
\tag{33}
$$

with β defined by (27) and (E_r,E_l) defined by (32). We emphasize that the Lopatinskii determinant Δ is defined on the whole hemisphere Σ and is continuous with respect to (τ,η). The first step in proving an energy estimate for (29) consists in determining whether Δ vanishes on Σ. The answer is given in the following result:

Proposition 1. *Assume that* (15) *holds. Then there exists a positive number V_1 such that for any $(\tau,\eta)\in\Sigma$, one has $\Delta(\tau,\eta)=0$ if and only if*

$$\tau=0 \quad \textit{or} \quad \tau=\pm iV_1\eta\,.$$

Each of these roots is simple. For instance, there exists a neighborhood $\mathcal{V}$ of $(0,1/v_r)$ in Σ and a $\mathcal{C}^\infty$ function h defined on $\mathcal{V}$ such that

$$\forall\,(\tau,\eta)\in\mathcal{V}\,,\quad \Delta(\tau,\eta)=\tau\,h(\tau,\eta) \quad \textit{and} \quad h(0,1/v_r)\neq0\,.$$

A similar result holds near $(0,-1/v_r)$ or near the points $(\pm iV_1\eta,\eta)\in\Sigma$.

The number V_1 is given by

$$V_1^2 := c^2+v_r^2-c\sqrt{c^2+4v_r^2}\,.$$

In particular, one has:

$$0<V_1<v_r-c\,.$$

When $\tau=0$ or $\tau=\pm iV_1\eta$, both eigenmodes ω_r and ω_l are purely imaginary.

We postpone the proof of Proposition 1 to section 5. We simply note here that the three *critical* speeds $-V_1,0,V_1$ are exactly the speeds of the kink modes exhibited in the work by Artola and Majda [1]. As a matter of fact, Artola and Majda used a “geometric optics approach”, while we have followed here a “normal modes analysis approach”. However, the calculations are similar in both cases (in [1], the number η equals 1).

4.4 Constructing a symmetrizer: the interior points

We now turn to the construction of our degenerate Kreiss' symmetrizer. The construction is microlocal and is achieved near any point $(\tau_0, \eta_0) \in \Sigma$. The analysis is rather long since one has to distinguish between five different cases. In the end, we shall consider a partition of unity to patch things together and derive our energy estimate.

In all the rest of the article, the letter κ denotes a generic positive constant (typically, though not necessarily, a rather small one).

We first consider the case $(\tau_0, \eta_0) \in \Sigma$ with Re $\tau_0 > 0$. Then the matrix $\mathcal{A}(\tau, \eta)$ is diagonalizable for all (τ, η) close to (τ_0, η_0). A smooth (that is, $\mathcal{C}^\infty$) basis of eigenvectors is given by the following family:

$$E_r(\tau,\eta), \quad E_l(\tau,\eta), \quad \left(\frac{c}{2}\eta^2, 0, \frac{1}{c}(\tau+iv_r\eta)^2+\frac{c}{2}\eta^2+(\tau+iv_r\eta)\omega_r, 0\right)^T,$$
$$\text{and} \quad \left(0, \frac{c}{2}\eta^2, 0, \frac{1}{c}(\tau+iv_l\eta)^2+\frac{c}{2}\eta^2+(\tau+iv_l\eta)\omega_l\right)^T.$$

Both vectors E_r and E_l are defined by (32). Therefore, there exists a $\mathcal{C}^\infty$ mapping $T(\tau, \eta)$, defined on a neighborhood $\mathcal{V}$ of (τ_0, η_0) in Σ, with values in $GL_4(\mathbb{C})$, and such that

$$\forall\,(\tau,\eta)\in\mathcal{V}, \quad T(\tau,\eta)\mathcal{A}(\tau,\eta)T(\tau,\eta)^{-1} = \begin{pmatrix} \omega_r & 0 & 0 & 0 \\ 0 & \omega_l & 0 & 0 \\ 0 & 0 & -\omega_r & 0 \\ 0 & 0 & 0 & -\omega_l \end{pmatrix}.$$

The first two columns of the matrix $T(\tau, \eta)^{-1}$ are the vectors E_r and E_l. In the neighborhood $\mathcal{V}$, we define the symmetrizer r in the usual way:

$$\forall\,(\tau,\eta)\in\mathcal{V}, \quad r(\tau,\eta) := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & K & 0 \\ 0 & 0 & 0 & K \end{pmatrix},$$

where $K \geq 1$ is a positive real number, to be fixed large enough. In what follows, we use the standard notation

$$\text{Re } M := \frac{M+M^*}{2}$$

for all square matrix M with complex entries (M^* is the classical adjoint matrix).

The matrix r defined just above is hermitian, and we have

$$\forall\,(\tau,\eta)\in\mathcal{V}, \quad \text{Re } \left(r(\tau,\eta)T(\tau,\eta)\mathcal{A}(\tau,\eta)T(\tau,\eta)^{-1}\right) \geq \kappa\, I \geq \kappa\gamma\, I\,, \tag{34}$$

for some positive constant κ. This is because ω_r and ω_l have negative real part when $(\tau, \eta) \in \mathcal{V}$, and $\gamma \leq 1$ when $(\tau, \eta) \in \Sigma$.

We now simply need to fix $K \geq 1$ in order to recover an estimate for the trace of W^{nc}. We show that for K sufficiently large, the following inequality holds:

$$\forall\,(\tau,\eta)\in\mathcal{V}, \quad r(\tau,\eta) + C\,\left(\widetilde{\beta}(\tau,\eta)\right)^* \widetilde{\beta}(\tau,\eta) \geq I\,, \tag{35}$$

where C is a positive constant and

$$\widetilde{\beta}(\tau,\eta) := \beta(\tau,\eta)\,T(\tau,\eta)^{-1}.$$

Let $Z = (Z^-, Z^+)^T \in \mathbb{C}^4$, with $Z^-, Z^+ \in \mathbb{C}^2$. We write

$$\widetilde{\beta}(\tau,\eta)\, Z = \beta(\tau,\eta)\, T(\tau,\eta)^{-1} \begin{pmatrix} Z^- \\ 0 \end{pmatrix} + \widetilde{\beta}(\tau,\eta) \begin{pmatrix} 0 \\ Z^+ \end{pmatrix} ,$$

and recall that the first two columns of $T(\tau,\eta)^{-1}$ are E_r and E_l. Because the Lopatinskii determinant does not vanish at (τ_0,η_0), see (33) and Proposition 1, we obtain an estimate of the form

$$|Z^-|^2 \leq C_0 \left(|Z^+|^2 + |\widetilde{\beta}(\tau,\eta) Z|^2 \right) ,$$

for a suitable constant C_0 that is independent of $(\tau,\eta) \in \mathcal{V}$. With C_0 satisfying this inequality, the definition of r yields

$$\langle r(\tau,\eta) Z, Z\rangle_{\mathbb{C}^4} + 2C_0\, |\widetilde{\beta}(\tau,\eta) Z|^2 = -|Z^-|^2 + K|Z^+|^2 + 2C_0\, |\widetilde{\beta}(\tau,\eta) Z|^2 \geq |Z^-|^2 + (K - 2C_0)|Z^+|^2 .$$

This gives (35) for K large enough (e.g., $K = 2C_0 + 1$).

4.5 Constructing a symmetrizer: the boundary points (case 1)

We now turn to the construction of the symmetrizer near those points $(\tau,\eta) \in \Sigma$ such that Re $\tau = 0$. We first prove a general result on the behavior of the eigenmodes $\omega_{r,l}$ in the neighborhood of such points:

Lemma 3. *Let $(\tau_0,\eta_0) \in \Sigma$ so that Re $\tau_0 = 0$ and $\tau_0 \neq (-v_r \pm c) i \eta_0$. In particular, ω_r depends analytically on (τ,η) near (τ_0,η_0). Then the two following cases may occur:*

1. *The eigenmode ω_r has negative real part at (τ_0,η_0), and, in a suitable neighborhood $\mathcal{V}$ of (τ_0,η_0), one has*

$$\text{Re } \omega_r \leq -\kappa < 0 .$$

2. *The eigenmode ω_r is purely imaginary at (τ_0,η_0). In this case, the derivative $\partial_\gamma \omega_r$ calculated at (τ_0,η_0) is a nonzero real number. In a suitable neighborhood $\mathcal{V}$ of (τ_0,η_0) in Σ, we have*

$$\text{Re } \omega_r \leq -\kappa \gamma .$$

A completely similar result holds for ω_l near all points $(\tau_0,\eta_0) \in \Sigma$ satisfying Re $\tau_0 = 0$ and $\tau_0 \neq (-v_l \pm c) i \eta_0$.

Proof. Because $\tau_0 \neq (-v_r \pm c) i \eta_0$, the eigenmode ω_r is not zero at (τ_0,η_0) and it depends analytically on (τ,η) by the implicit functions Theorem.

The first case in Lemma 3 simply follows from the continuity of ω_r with respect to (τ,η). In the second case, we use (31a) to derive

$$\omega_r\, \frac{\partial \omega_r}{\partial \gamma} = \frac{1}{c^2}\, (\tau + i v_r \eta) .$$

When $\tau = \tau_0$ and $\eta = \eta_0$, one has $\omega_r \in i\mathbb{R} \setminus \{0\}$ and $(\tau_0 + i v_r \eta_0) \in i\mathbb{R}$. This proves that the derivative $\partial_\gamma \omega_r$ is real. We now remark that $\tau_0 \neq -i v_r \eta_0$ for, in such a case, ω_r has negative real part. Consequently, the derivative $\partial_\gamma \omega_r$ is not zero. The estimate on Re ω_r is obtained by performing a Taylor expansion of ω_r at (τ_0,η_0). $\square$

According to Proposition 1, there are exactly four types of points on the boundary of Σ:

1. Those points (τ_0, η_0) where $\mathcal{A}(\tau_0, \eta_0)$ is diagonalizable and the Lopatinskii condition is satisfied at (τ_0, η_0).

2. Those points (τ_0, η_0) where $\mathcal{A}(\tau_0, \eta_0)$ is diagonalizable and the Lopatinskii condition breaks down at (τ_0, η_0).

3. Those points (τ_0, η_0) where $\mathcal{A}(\tau_0, \eta_0)$ is not diagonalizable, that is, $\tau_0 = (\pm v_r \pm c) i \eta_0$. In this case, Proposition 1 asserts that the Lopatinskii condition is satisfied at (τ_0, η_0).

4. Those points (τ_0, η_0) that are the poles of $\mathcal{A}$, that is, $\tau_0 = \pm i v_r \eta_0$. At those points, the Lopatinskii condition is satisfied.

As a matter of fact, an immediate consequence of Proposition 1 is that whenever the Lopatinskii condition fails at (τ_0, η_0), then (τ_0, η_0) is not a pole and the symbol $\mathcal{A}(\tau, \eta)$ is (smoothly) diagonalizable in a neighborhood of (τ_0, η_0). The three first categories of boundary points can thus be treated as in [4, 8, 12, 21], provided the technical assumptions of [8] near instability points hold. We are going to show that such technical assumptions hold true. The last category of boundary points (the poles of the symbol $\mathcal{A}$) requires special attention.

We now deal with the construction of our symmetrizer in case 1: $(\tau_0, \eta_0) \in \Sigma$ is such that $\mathcal{A}(\tau_0, \eta_0)$ is diagonalizable and the Lopatinskii condition is satisfied at (τ_0, η_0), that is, $\Delta(\tau_0, \eta_0) \neq 0$.

Because $\mathcal{A}(\tau_0, \eta_0)$ is diagonalizable, we have $\tau_0 \neq (\pm v_r \pm c) i \eta_0$. Hence there exists a neighborhood $\mathcal{V}$ of (τ_0, η_0) in Σ and a smooth basis of eigenvectors of $\mathcal{A}$ defined on $\mathcal{V}$. The smooth basis is the same as in the case of interior points (see the preceeding paragraph). We thus have

$$\forall\, (\tau, \eta) \in \mathcal{V}\,, \quad T(\tau, \eta) \mathcal{A}(\tau, \eta) T(\tau, \eta)^{-1} = \begin{pmatrix} \omega_r & 0 & 0 & 0 \\ 0 & \omega_l & 0 & 0 \\ 0 & 0 & -\omega_r & 0 \\ 0 & 0 & 0 & -\omega_l \end{pmatrix} \,,$$

where, once again, the two first columns of $T(\tau, \eta)^{-1}$ are exactly E_r and E_l. We choose r as in the case of interior points:

$$\forall\, (\tau, \eta) \in \mathcal{V}\,, \quad r(\tau, \eta) := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & K & 0 \\ 0 & 0 & 0 & K \end{pmatrix} \,,$$

with $K \geq 1$ to be fixed large enough. Clearly, r is a hermitian matrix. Using Lemma 3, we can already conclude that

$$\forall\, (\tau, \eta) \in \mathcal{V}\,, \quad \mathrm{Re}\, \left(r(\tau, \eta) T(\tau, \eta) \mathcal{A}(\tau, \eta) T(\tau, \eta)^{-1} \right) \geq \kappa \gamma\, I\,. \tag{36}$$

Because the Lopatinskii condition is satisfied at (τ_0, η_0), it is possible to choose K large enough so that the following estimate holds:

$$\forall\, (\tau, \eta) \in \mathcal{V}\,, \quad r(\tau, \eta) + C\, \left(\widetilde{\beta}(\tau, \eta) \right)^* \widetilde{\beta}(\tau, \eta) \geq I\,. \tag{37}$$

In (37), we have let, as usual,

$$\widetilde{\beta}(\tau, \eta) := \beta(\tau, \eta)\, T(\tau, \eta)^{-1}\,.$$

The analysis is the same as what we have done for interior points. The estimates (36)-(37) are similar (but not exactly identical) to (34)-(35).

4.6 Constructing a symmetrizer: the boundary points (case 2)

In this paragraph, we consider a point $(\tau_0, \eta_0) \in \Sigma$ such that the Lopatinskii determinant Δ vanishes at (τ_0, η_0). From Proposition 1, we know that the symbol $\mathcal{A}$ is smoothly diagonalizable on a neighborhood $\mathcal{V}$ of (τ_0, η_0) in Σ. In other words, there exists a smooth mapping $T(\tau, \eta)$ with values in $GL_4(\mathbb{C})$, and such that

$$\forall\, (\tau, \eta) \in \mathcal{V}\,, \quad T(\tau, \eta)\mathcal{A}(\tau, \eta)T(\tau, \eta)^{-1} = \begin{pmatrix} \omega_r & 0 & 0 & 0 \\ 0 & \omega_l & 0 & 0 \\ 0 & 0 & -\omega_r & 0 \\ 0 & 0 & 0 & -\omega_l \end{pmatrix} .$$

The first two columns of $T(\tau, \eta)^{-1}$ are E_r and E_l. In this case, we define our symmetrizer r in the following (degenerate) way:

$$\forall\, (\tau, \eta) \in \mathcal{V}\,, \quad r(\tau, \eta) := \begin{pmatrix} -\gamma^2 & 0 & 0 & 0 \\ 0 & -\gamma^2 & 0 & 0 \\ 0 & 0 & K & 0 \\ 0 & 0 & 0 & K \end{pmatrix} ,$$

with $K \geq 1$ to be fixed large enough. The matrix $r(\tau, \eta)$ is hermitian and we have

$$\forall\, (\tau, \eta) \in \mathcal{V}\,, \quad \mathrm{Re}\, \left(r(\tau, \eta)T(\tau, \eta)\mathcal{A}(\tau, \eta)T(\tau, \eta)^{-1}\right) \geq \kappa\gamma \begin{pmatrix} \gamma^2 & 0 & 0 & 0 \\ 0 & \gamma^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{38}$$

It only remains to fix K appropriately in order to recover an estimate on the boundary. The choice of K relies on the following Lemma:

Lemma 4. *Let $(\tau_0, \eta_0) \in \Sigma$ be a point where the Lopatinskii determinant vanishes. Then there exists a neighborhood $\mathcal{V}$ of (τ_0, η_0) in Σ and a constant $\kappa_0 > 0$ such that the following estimate holds for all $(\tau, \eta) \in \mathcal{V}$:*

$$\forall\, Z^- \in \mathbb{C}^2\,, \quad |\beta(\tau, \eta) \begin{pmatrix} E_r(\tau, \eta) & E_l(\tau, \eta) \end{pmatrix} Z^-|^2 \geq \kappa_0\, \gamma^2\, |Z^-|^2 . \tag{39}$$

Before proving Lemma 4, let us first show that it enables us to obtain an estimate between r and the boundary matrix β. More precisely, we are going to show the following estimate:

$$\forall\, (\tau, \eta) \in \mathcal{V}\,, \quad r(\tau, \eta) + C\, \left(\widetilde{\beta}(\tau, \eta)\right)^* \widetilde{\beta}(\tau, \eta) \geq \gamma^2\, I\,, \tag{40}$$

for an appropriate positive constant C. The definition of $\widetilde{\beta}(\tau, \eta)$ is the same as in the preceeding cases.

Let $Z = (Z^-, Z^+)^T \in \mathbb{C}^4$, where Z^- and Z^+ belong to $\mathbb{C}^2$. Once again, we write

$$\widetilde{\beta}(\tau, \eta)\, Z = \beta(\tau, \eta)\, T(\tau, \eta)^{-1} \begin{pmatrix} Z^- \\ 0 \end{pmatrix} + \widetilde{\beta}(\tau, \eta) \begin{pmatrix} 0 \\ Z^+ \end{pmatrix} ,$$

and we recall that the first two columns of $T(\tau, \eta)^{-1}$ are E_r and E_l. Using Lemma 4, we obtain

$$\kappa_0\, \gamma^2\, |Z^-|^2 \leq C_0 \left(|Z^+|^2 + |\widetilde{\beta}(\tau, \eta)\, Z|^2 \right) .$$

We thus derive

$$\begin{aligned}\langle r(\tau,\eta)Z,Z\rangle_{\mathbb{C}^4} + \frac{2C_0}{\kappa_0}\,|\widetilde{\beta}(\tau,\eta)Z|^2 &= -\gamma^2\,|Z^-|^2 + K|Z^+|^2 + \frac{2C_0}{\kappa_0}\,|\widetilde{\beta}(\tau,\eta)Z|^2 \\ &\geq \gamma^2\,|Z^-|^2 + \left(K - \frac{2C_0}{\kappa_0}\right)|Z^+|^2 .\end{aligned}$$

Choosing $K = 2C_0/\kappa_0 + 1$ yields

$$\langle r(\tau,\eta)Z,Z\rangle_{\mathbb{C}^4} + \frac{2C_0}{\kappa_0}\,|\widetilde{\beta}(\tau,\eta)Z|^2 \geq \gamma^2\,|Z^-|^2 + |Z^+|^2 \geq \gamma^2\,|Z|^2 .$$

Here we have much weakened our estimate for the last two components Z^+. However, an inequality like (40) is simpler to use since it does not distinguish between the different coordinates of the vector Z. One should remember that the real loss of control is on the modes ω_r and ω_l, and not on the modes $-\omega_r$ and $-\omega_l$.

The proof of Lemma 4 is postponed to the section 5. It relies on the fact that the roots of the Lopatinskii determinant Δ are simple (see Proposition 1).

4.7 Constructing a symmetrizer: the boundary points (case 3)

In this paragraph, we consider a point $(\tau_0,\eta_0) \in \Sigma$ such that $\tau_0 = -iv_r\eta_0 \pm ic\eta_0$. (The case $\tau_0 = -iv_l\eta_0 \pm ic\eta_0$ is entirely similar and we shall not detail it). Because $v_l = -v_r$ and $v_r > c\sqrt{2}$, we have $\tau_0 \neq -iv_l\eta_0 \pm ic\eta_0$, and therefore, the eigenmode ω_l depends smoothly on (τ,η) in a neighborhood $\mathcal{V}$ of (τ_0,η_0). Oppositely, ω_r is only continuous with respect to (τ,η) near (τ_0,η_0), but ω_r^2 is $\mathcal{C}^\infty$ near (τ_0,η_0). This is because (31a) has a double root when $(\tau,\eta) = (\tau_0,\eta_0)$.

When (τ,η) is close to (τ_0,η_0), the following family is a $\mathcal{C}^\infty$ basis of $\mathbb{C}^4$:

$$(m_r,0,-m_r,0)^T , \quad (-i,0,0,0)^T ,$$
$$E_l(\tau,\eta) \quad \text{and} \quad \left(0, \frac{c}{2}\eta^2, 0, \frac{1}{c}(\tau + iv_l\eta)^2 + \frac{c}{2}\eta^2 + (\tau + iv_l\eta)\omega_l\right)^T .$$

Recall that m_r is defined by (30). Let $T(\tau,\eta)^{-1}$ be the (regular) matrix whose columns are those four vectors. We compute

$$T(\tau,\eta)\mathcal{A}(\tau,\eta)T(\tau,\eta)^{-1} = \begin{pmatrix} a_r & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \omega_l & 0 \\ \mathbf{0} & 0 & -\omega_l \end{pmatrix} ,$$

where a_r is the 2×2 matrix defined as follows:

$$a_r(\tau,\eta) := \begin{pmatrix} \dfrac{-c\,\omega_r^2}{\tau + iv_r\eta} & i \\[2ex] \dfrac{2im_r\,c\,\omega_r^2}{\tau + iv_r\eta} & \dfrac{c\,\omega_r^2}{\tau + iv_r\eta} \end{pmatrix} .$$

In particular, we have

$$a_r(\tau_0,\eta_0) = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} =: iN .$$

Moreover, we make the following observations:

For all (τ,η) close to (τ_0,η_0) so that $\tau \in i\mathbb{R}$, a_r has purely imaginary coefficients.

The lower left coefficient ϑ_r of a_r satisfies

$$\frac{\partial \vartheta_r}{\partial \gamma}\,(\tau_0,\eta_0) \in \mathbb{R} \setminus \{0\}\,.$$

Here we have exhibited a basis in which Ralston's result [21] applies. Readers who are familiar with the theory will have recognized the "block structure condition", that is a consequence here of the strict hyperbolicity of (1), see [19].

We are looking for a symmetrizer r under the form

$$r(\tau,\eta) = \begin{pmatrix} s(\tau,\eta) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -1 & 0 \\ \mathbf{0} & 0 & K \end{pmatrix}\,,$$

where $K \geq 1$ is a real number, to be fixed large enough, and s is some 2×2 hermitian matrix, depending smoothly on (τ,η). More precisely, we are looking for the matrix s under the following form

$$s(\tau,\eta) = \underbrace{\begin{pmatrix} 0 & e_1 \\ e_1 & e_2 \end{pmatrix}}_{E} + \underbrace{\begin{pmatrix} f(\tau,\eta) & 0 \\ 0 & 0 \end{pmatrix}}_{F(\tau,\eta)} - i\gamma \underbrace{\begin{pmatrix} 0 & -g \\ g & 0 \end{pmatrix}}_{G}\,,$$

where e_1, e_2 and g are real numbers, and f is a real valued $\mathcal{C}^\infty$ mapping that vanishes at (τ_0,η_0), see [4, 12, 21]. We choose e_1 in the following way:

$$e_1 := \left(\frac{\partial \vartheta_r}{\partial \gamma}\,(\tau_0,\eta_0) \right)^{-1} \in \mathbb{R} \setminus \{0\}\,,$$

where ϑ_r is defined just above. This choice may look surprising but it will be justified later on.

Now we observe that our choice of s yields ($\gamma_0 = 0$):

$$r(\tau_0,\eta_0) = \begin{pmatrix} 0 & e_1 & 0 & 0 \\ e_1 & e_2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & K \end{pmatrix}\,.$$

Moreover, the first and third columns of $T(\tau_0,\eta_0)^{-1}$ are nothing but $E_r(\tau_0,\eta_0)$ and $E_l(\tau_0,\eta_0)$. This is due to the equality $\mu_r = -m_r$ when $(\tau,\eta) = (\tau_0,\eta_0)$. Because Lopatinskii's condition is satified at (τ_0,η_0), we can choose e_2 and K large enough such that the following estimate holds:

$$r(\tau_0,\eta_0) + C\,\left(\widetilde{\beta}(\tau_0,\eta_0)\right)^*\,\widetilde{\beta}(\tau_0,\eta_0) \geq I\,.$$

As was done before, we have let

$$\widetilde{\beta}(\tau,\eta) := \beta(\tau,\eta)\,T(\tau,\eta)^{-1}\,.$$

Up to shrinking $\mathcal{V}$, we have thus derived the following estimate

$$\forall\,(\tau,\eta) \in \mathcal{V}\,, \quad r(\tau,\eta) + C\,\left(\widetilde{\beta}(\tau,\eta)\right)^*\,\widetilde{\beta}(\tau,\eta) \geq \frac{1}{2}\,I\,, \tag{41}$$

for a suitable constant C.

Now, we show how to choose the real valued function f and the real number g. We first write

$$a_r(\tau,\eta) = a_r(\tau_0,\eta_0) + (a_r(i\delta,\eta) - a_r(\tau_0,\eta_0)) + (a_r(\tau,\eta) - a_r(i\delta,\eta))\,,$$

and then use Taylor's formula. We obtain

$$a_r(\tau,\eta) = iN + (a_r(i\delta,\eta) - a_r(\tau_0,\eta_0)) + \gamma\,\frac{\partial a_r}{\partial\gamma}(i\delta,\eta) + \gamma^2\,M(\tau,\eta)\,,$$

for a suitable continuous function M. Because a_r has purely imaginary coefficients when τ is purely imaginary, we have

$$a_r(i\delta,\eta) - a_r(\tau_0,\eta_0) = \begin{pmatrix} ib_1(i\delta,\eta) & 0 \\ ib_2(i\delta,\eta) & ib_3(i\delta,\eta)\end{pmatrix} =: iB_r(i\delta,\eta)\,,$$

for some $\mathcal{C}^\infty$, real valued mappings b_1, b_2, b_3. Those three mappings obviously vanish at (τ_0,η_0). We choose

$$f(\tau,\eta) := e_1\,(b_1(i\delta,\eta) - b_3(i\delta,\eta)) + e_2\,b_2(i\delta,\eta)\,,$$

so that f is a $\mathcal{C}^\infty$ real valued mapping that vanishes at (τ_0,η_0). Moreover, this choice of f implies that the matrix

$$(E + F(\tau,\eta))(N + B_r(i\delta,\eta))$$

is real and symmetric for all (τ,η). Consequently, we obtain

$$\mathrm{Re}\ (s(\tau,\eta)a_r(\tau,\eta)) = \gamma\ \mathrm{Re}\ \left(GN + E\,\frac{\partial a_r}{\partial\gamma}(i\delta,\eta)\right) + \gamma\,L(\tau,\eta)\,,$$

where L is continuous (it is even $\mathcal{C}^\infty$) and $L(\tau_0,\eta_0) = 0$. Our choice for e_1 yields

$$\mathrm{Re}\ \left(GN + E\,\frac{\partial a_r}{\partial\gamma}(\tau_0,\eta_0)\right) = \begin{pmatrix} 0 & 0 \\ 0 & g\end{pmatrix} + \begin{pmatrix} 1 & * \\ * & *\end{pmatrix}\,,$$

where the $*$ are coefficients that only depend on e_1, e_2 (that have already been fixed) and (τ_0,η_0). Choosing g large enough, and shrinking $\mathcal{V}$ if necessary, we end up with

$$\mathrm{Re}\ (s(\tau,\eta)a_r(\tau,\eta)) \geq \frac{1}{4}\,\gamma\,I\,,$$

and we thus obtain

$$\forall\,(\tau,\eta)\in\mathcal{V}\,,\quad \mathrm{Re}\ (r(\tau,\eta)T(\tau,\eta)\mathcal{A}(\tau,\eta)T(\tau,\eta)^{-1}) \geq \kappa\,\gamma\,I\,. \tag{42}$$

4.8 Constructing a symmetrizer: the boundary points (case 4)

We now consider the last case, which is $(\tau_0,\eta_0)\in\Sigma$ with $\tau_0 = -iv_r\eta_0$. (We shall not detail the case $\tau_0 = -iv_l\eta_0$ that is entirely similar). The symbol $\mathcal{A}$ is not defined at (τ_0,η_0), while the stable subspace $\mathcal{E}^-$ of $\mathcal{A}$ admits a continuous extension at this point. The family (E_r, E_l) is a $\mathcal{C}^\infty$ basis of $\mathcal{E}^-$ near (τ_0,η_0), see Lemma 2, and Lopatinskii's condition is satisfied near (τ_0,η_0), see Proposition 1.

The eigenmode ω_r has negative real part when (τ,η) is close to (τ_0,η_0), see (31a). Oppositely, the eigenmode ω_l is purely imaginary when (τ,η) is close to (τ_0,η_0) and Re $\tau = 0$, see (31b). Furthermore, both ω_r and ω_l depend analytically on (τ,η) in a neighborhood $\mathcal{V}$ of (τ_0,η_0).

The matrix $\mathcal{A}$ is not smoothly diagonalizable on $\mathcal{V}$. This is because the eigenvector associated with the eigenvalue $-\omega_r$ tends to be parallel to the eigenvector associated with the eigenvalue ω_r. Consequently, Majda and Osher's construction of a symmetrizer in this case involves a singularity in the symmetrizer, see [18]. We prefer to avoid this singularity in the symmetrizer and construct a smooth (that is, $\mathcal{C}^\infty$) symmetrizer in the whole neighborhood $\mathcal{V}$. This is possible if we go back to the original system:

$$(\tau\mathcal{A}_0 + i\eta\mathcal{A}_1)\,\widehat{W} + \mathcal{A}_2\,\frac{d\widehat{W}}{dx_2} = 0\,,\quad x_2 > 0\,, \tag{43a}$$

$$\beta(\tau,\eta)\widehat{W^{nc}}(0) = \widehat{h}\,, \tag{43b}$$

The following analysis is inspired from [3]. For (τ,η) in a neighborhood $\mathcal{V}$ of (τ_0,η_0), the following matrix is regular (that is, invertible):

$$\mathcal{T}(\tau,\eta)^{-1} := \begin{pmatrix} 1 & 0 & -ic\eta(\tau+iv_r\eta-c\omega_r) & 0 & 0 & 0 \\ 0 & 1 & 0 & ic\eta(\tau+iv_l\eta-c\omega_l) & 0 & ic\eta(\tau+iv_l\eta+c\omega_l) \\ 0 & 0 & \dfrac{c}{2}\eta^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{c}{2}\eta^2 & 0 & \dfrac{c}{2}\eta^2 \\ 0 & 0 & (\tau+iv_r\eta)(\mu_r-\omega_r) & 0 & 1 & 0 \\ 0 & 0 & 0 & (\tau+iv_l\eta)(\mu_l-\omega_l) & 0 & (\tau+iv_l\eta)(\mu_l+\omega_l) \end{pmatrix}. \tag{44}$$

Indeed, the determinant of $\mathcal{T}(\tau,\eta)^{-1}$ is given by

$$\det \mathcal{T}(\tau,\eta)^{-1} = \frac{c^2}{2}\,\eta^4\,(\tau+iv_l\eta)\,\omega_l\,,$$

and it is easy to check that this quantity does not vanish near (τ_0,η_0). This justifies our notation $\mathcal{T}^{-1}$. Using the definition (30) for μ_r and μ_l, we get

$$\xi_r := (\tau+iv_r\eta)(\mu_r-\omega_r) = \frac{1}{c}\,(\tau+iv_r\eta)^2 + \frac{c}{2}\,\eta^2 - (\tau+iv_r\eta)\omega_r\,, \tag{45a}$$

$$\xi_l^\pm := (\tau+iv_l\eta)(\mu_l\pm\omega_l) = \frac{1}{c}\,(\tau+iv_l\eta)^2 + \frac{c}{2}\,\eta^2 \pm (\tau+iv_l\eta)\omega_l\,. \tag{45b}$$

Hence, the matrix $\mathcal{T}^{-1}$ depends smoothly on (τ,η) in the whole neighborhood $\mathcal{V}$. It has no pole.

The main idea now is that (43a) has a simple expression if we decompose the vector $\widehat{W}$ in the basis defined by $\mathcal{T}^{-1}$. In other words, the matrices

$$(\tau\mathcal{A}_0 + i\eta\mathcal{A}_1)\mathcal{T}(\tau,\eta)^{-1} \quad\text{and}\quad \mathcal{A}_2\mathcal{T}(\tau,\eta)^{-1}$$

have a rather similar structure. Performing some manipulations on the rows of these two matrices, we shall transform (43a) into an "almost diagonal" system of differential equations.

After some simplifications, we obtain the following expressions:

$$(\tau\mathcal{A}_0+i\eta\mathcal{A}_1)\mathcal{T}(\tau,\eta)^{-1}=\begin{pmatrix}\tau+iv_r\eta & 0 & 0 & 0 & ic^2\eta & 0\\ 0 & \tau+iv_l\eta & 0 & 0 & 0 & 0\\ -ic^2\eta & 0 & c^4\eta^2\omega_r & 0 & 0 & 0\\ 0 & ic^2\eta & 0 & c^4\eta^2\omega_l & 0 & -c^4\eta^2\omega_l\\ ic^2\eta & 0 & -2c^3\omega_r\xi_r & 0 & 2c^2(\tau+iv_r\eta) & 0\\ 0 & -ic^2\eta & 0 & -2c^3\omega_l\xi_l^- & 0 & 2c^3\omega_l\xi_l^+\end{pmatrix},$$

$$\mathcal{A}_2\mathcal{T}(\tau,\eta)^{-1}=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&-c^4\eta^2&0&0&0\\0&0&0&-c^4\eta^2&0&-c^4\eta^2\\0&0&2c^3\xi_r&0&2c^3&0\\0&0&0&2c^3\xi_l^-&0&2c^3\xi_l^+\end{pmatrix},$$

where ξ_r and $\xi_l^\pm$ are defined by (45a)-(45b). These expressions may look somehow complicated, but it is not so hard to simplify them multiplying on the left by a suitable symmetrizer. Indeed, for all (τ,η) in a neighborhood $\mathcal{V}$ of (τ_0,η_0), let us define the following matrix:

$$R_1(\tau,\eta):=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&-2ic^4\eta(\tau+iv_l\eta+c\omega_l)&0&2c^3\xi_l^+&0&c^4\eta^2\\2ic^4\eta(\tau+iv_r\eta-c\omega_r)&0&2c^3\xi_r&0&c^4\eta^2&0\\0&-2ic^4\eta(\tau+iv_l\eta-c\omega_l)&0&2c^3\xi_l^-&0&c^4\eta^2\end{pmatrix}. \tag{46}$$

First we obtain

$$R_1(\tau,\eta)\mathcal{A}_2\mathcal{T}(\tau,\eta)^{-1}=\text{ diag }(0,0,-c^4\eta^2,-4c^7\eta^2(\tau+iv_l\eta)\omega_l,2c^7\eta^2,4c^7\eta^2(\tau+iv_l\eta)\omega_l)\,,$$

where diag $(a_1,\dots,a_p)$ stands for the diagonal matrix whose diagonal elements are $a_1,\dots,a_p$. We also obtain the following expression:

$$R_1(\tau,\eta)(\tau\mathcal{A}_0+i\eta\mathcal{A}_1)\mathcal{T}(\tau,\eta)^{-1}=$$
$$\begin{pmatrix}\tau+iv_r\eta&0&0&0&ic^2\eta&0\\0&\tau+iv_l\eta&0&0&0&0\\-ic^2\eta&0&c^4\eta^2\omega_r&0&0&0\\0&0&0&4c^7\eta^2\omega_l^2(\tau+iv_l\eta)&0&0\\0&0&0&0&2c^7\eta^2\omega_r&0\\0&0&0&0&0&4c^7\eta^2\omega_l^2(\tau+iv_l\eta)\end{pmatrix}.$$

Recall that ω_l, η and $(\tau+iv_l\eta)$ do not vanish in the neighborhood $\mathcal{V}$ of (τ_0,η_0) (up to shrinking $\mathcal{V}$). The following matrix is therefore a $\mathcal{C}^\infty$ mapping of (τ,η) on $\mathcal{V}$:

$$R_2(\tau,\eta):=\text{ diag }\left(1,1,1,\frac{1}{4c^7\eta^2\omega_l(\tau+iv_l\eta)},\frac{1}{2c^7\eta^2},\frac{1}{4c^7\eta^2\omega_l(\tau+iv_l\eta)}\right). \tag{47}$$

We define

$$S(\tau,\eta):=R_2(\tau,\eta)\,R_1(\tau,\eta)\,,$$

where R_1 and R_2 are defined by (46)-(47). It is now easy to derive the following equalities for all (τ,η) in $\mathcal{V}$:

$$S(\tau,\eta)\mathcal{A}_2\mathcal{T}(\tau,\eta)^{-1} = \text{ diag }(0,0,-c^4\eta^2,-1,1,1)\,, \tag{48a}$$

$$S(\tau,\eta)(\tau\mathcal{A}_0+i\eta\mathcal{A}_1)\mathcal{T}(\tau,\eta)^{-1} = \begin{pmatrix} \tau+iv_r\eta & 0 & 0 & 0 & ic^2\eta & 0 \\ 0 & \tau+iv_l\eta & 0 & 0 & 0 & 0 \\ -ic^2\eta & 0 & c^4\eta^2\omega_r & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_l & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega_r & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega_l \end{pmatrix}. \tag{48b}$$

In particular, it is important to observe that the matrix $S\mathcal{A}_2\mathcal{T}^{-1}$ is hermitian, and both matrices S and $\mathcal{T}^{-1}$ are $\mathcal{C}^\infty$ on the whole neighborhood $\mathcal{V}$. Up to shrinking $\mathcal{V}$, we may assume that ω_r has negative real part for all $(\tau,\eta)\in\mathcal{V}$. This is possible because $\omega_r=-|\eta|$ at (τ_0,η_0).

Though a little complicated, the preceeding calculations are based on the simple idea that the differential equations (43a) have an easy expression if we replace the standard coordinates by the coordinates on the stable subspace. The difficulty comes from the fact that we deal with the differential system satisfied by $\widehat{W}$ and not with the system satisfied by $\widehat{W^{nc}}$. Because the boundary is characteristic, (43a) is not an *ordinary* differential equation, and we thus need to diagonalize simultaneously $(\tau\mathcal{A}_0+i\eta\mathcal{A}_1)$ and $\mathcal{A}_2$. Though the matrix $S(\tau\mathcal{A}_0+i\eta\mathcal{A}_1)\mathcal{T}^{-1}$ is not diagonal, we shall see that the reduced expressions (48a)-(48b) are sufficient to derive energy estimates in such a neighborhood $\mathcal{V}$ of the pole (τ_0,η_0).

4.9 Derivation of the energy estimate

We now turn to the derivation of the estimate (23). Recall that we are considering a function $W\in H^1(\Omega)$ such that

$$(\tau\mathcal{A}_0+i\eta\mathcal{A}_1)\,\widehat{W}+\mathcal{A}_2\,\frac{d\widehat{W}}{dx_2}=0\,,\quad x_2>0\,,$$
$$\beta(\tau,\eta)\widehat{W^{nc}}(0)=\widehat{h}\,,$$

where $\widehat{h}$ is obtained from the source term $\widehat{g}$ in (24) after eliminating the unknown front:

$$\forall\,(\tau,\eta)\in\Sigma\,,\quad \widehat{h}=\begin{pmatrix} 0 & 0 & 1 \\ \tau+iv_r\eta & -2iv_r\eta & 0\end{pmatrix}\widehat{g}\,.$$

Thanks to (28), it is sufficient to get an estimate of the trace of W^{nc} on $\{x_2=0\}$ in order to derive (23).

The previous analysis shows that for all $(\tau_0,\eta_0)\in\Sigma$, there exists a neighborhood $\mathcal{V}$ of (τ_0,η_0) in Σ and mappings defined on this neighborhood that satisfy certain properties. Because Σ is a $\mathcal{C}^\infty$ compact manifold, there exists a finite covering $(\mathcal{V}_1,\dots,\mathcal{V}_I)$ of Σ by such neighborhoods, and a smooth partition of unity $(\chi_1,\dots,\chi_I)$ associated with this covering. Namely, the χ_i's are nonnegative, $\mathcal{C}^\infty$, and satisfy

$$\text{Supp }\chi_i\subset\mathcal{V}_i \quad\text{and}\quad \sum_{i=1}^I\chi_i^2\equiv 1\,.$$

There are three different cases.

• In the first case, $\mathcal{V}_i$ is a neighborhood of an interior point or a neighborhood of a boundary point corresponding to cases 1 and 3 above (boundary points that are not poles and for which the Lopatinskii condition is satisfied). On such a neighborhood, there exist two $\mathcal{C}^\infty$ mappings r_i and T_i such that

r_i is hermitian,

T_i has values in $GL_4(\mathbb{C})$,

the following estimate holds for all $(\tau,\eta) \in \mathcal{V}_i$:

$$\begin{aligned} \text{Re } \left(r_i(\tau,\eta)T_i(\tau,\eta)\mathcal{A}(\tau,\eta)T_i(\tau,\eta)^{-1}\right) \geq \kappa_i \gamma\, I\,, \\ r_i(\tau,\eta) + C_i \, \left(\beta(\tau,\eta)T_i(\tau,\eta)^{-1}\right)^* \beta(\tau,\eta)T_i(\tau,\eta)^{-1} \geq I\,. \end{aligned} \tag{49}$$

The inequalities (49) are direct consequences of (34)-(35)-(36)-(37)-(41)-(42).

We define

$$U_i(\tau,\eta,x_2) := \chi_i(\tau,\eta)\, T_i(\tau,\eta)\, \widehat{W^{nc}}(\delta,\eta,x_2)\,.$$

Both mappings r_i and T_i are not defined on Σ but only on the neighborhood $\mathcal{V}_i$. However, only the values of these mappings on the support of χ_i will be involved in the subsequent calculations, so we choose for convenience to extend these mappings to the whole hemishpere Σ. Then we extend χ_i, r_i, T_i to the whole set of frequencies Ξ as homogeneous mappings of degree 0 with respect to (τ,η).

Because $\mathcal{V}_i$ does not contain any pole of the symbol $\mathcal{A}$, defined by (30), the preceeding calculations show that U_i satisfies

$$\begin{aligned} \frac{dU_i}{dx_2} = T_i(\tau,\eta)\mathcal{A}(\tau,\eta)T_i(\tau,\eta)^{-1}U_i\,, \quad x_2 > 0\,, \\ \beta(\tau,\eta)T_i(\tau,\eta)^{-1}U_i(0) = \chi_i\, \widehat{h}\,. \end{aligned}$$

We take the scalar product of this later ordinary differential equation with r_iU_i and integrate with respect to x_2 on $\mathbb{R}^+$. Then we take the real part of the obtained equality and use (49). These operations yield the classical Kreiss' estimate:

$$\kappa_i\gamma \int_0^{+\infty} |U_i(\tau,\eta,x_2)|^2\, dx_2 + \frac{1}{2}\, |U_i(\tau,\eta,0)|^2 \leq C_i\, \chi_i(\tau,\eta)^2\, |\widehat{h}(\tau,\eta)|^2\,.$$

Now we use the definition of U_i and a uniform bound for $\|T_i(\tau,\eta)^{-1}\|$ on the support of χ_i to derive

$$\gamma\, \chi_i(\tau,\eta)^2 \int_0^{+\infty} |\widehat{W^{nc}}(\delta,\eta,x_2)|^2\, dx_2 + \chi_i(\tau,\eta)^2\, |\widehat{W^{nc}}(\delta,\eta,0)|^2 \leq C_i\, \chi_i(\tau,\eta)^2\, |\widehat{h}|^2\,. \tag{50}$$

• In the second case, $\mathcal{V}_i$ is a neighborhood of a zero of the Lopatinskii determinant. On such a neighborhood, there exist two $\mathcal{C}^\infty$ mappings r_i and T_i such that

r_i is hermitian,

T_i has values in $GL_4(\mathbb{C})$,

the following estimate holds for all $(\tau,\eta) \in \mathcal{V}_i$:

$$\begin{aligned} \text{Re } \left(r_i(\tau,\eta)T_i(\tau,\eta)\mathcal{A}(\tau,\eta)T_i(\tau,\eta)^{-1}\right) \geq \kappa_i \gamma^3\, I\,, \\ r_i(\tau,\eta) + C_i \, \left(\beta(\tau,\eta)T_i(\tau,\eta)^{-1}\right)^* \beta(\tau,\eta)T_i(\tau,\eta)^{-1} \geq \gamma^2\, I\,. \end{aligned} \tag{51}$$

These inequalities correspond to (38)-(40).

As was done before, we first extend r_i and T_i as $\mathcal{C}^\infty$ mappings on the whole hemisphere Σ. Then we extend T_i and χ_i as homogeneous mappings of degree 0 with respect to (τ,η), and we extend r_i as a homogeneous mapping of degree 2 with respect to (τ,η). Thus (51) reads

$$\begin{aligned}\text{Re}\,\left(r_i(\tau,\eta)T_i(\tau,\eta)\mathcal{A}(\tau,\eta)T_i(\tau,\eta)^{-1}\right) \geq \kappa_i\gamma^3\,I\,,\\ r_i(\tau,\eta)+C_i(|\tau|^2+v_r^2\eta^2)\,\left(\beta(\tau,\eta)T_i(\tau,\eta)^{-1}\right)^*\beta(\tau,\eta)T_i(\tau,\eta)^{-1}\geq\gamma^2\,I\,,\end{aligned} \tag{52}$$

for all $(\tau,\eta)\in\mathbb{R}^+\cdot\mathcal{V}_i$. Once again, we define

$$U_i(\tau,\eta,x_2):=\chi_i(\tau,\eta)\,T_i(\tau,\eta)\,\widehat{W^{nc}}(\delta,\eta,x_2)\,.$$

Because $\mathcal{V}_i$ does not contain any pole of $\mathcal{A}$, we still have

$$\begin{aligned}\frac{dU_i}{dx_2}=T_i(\tau,\eta)\mathcal{A}(\tau,\eta)T_i(\tau,\eta)^{-1}U_i\,,\quad x_2>0\,,\\ \beta(\tau,\eta)T_i(\tau,\eta)^{-1}U_i(0)=\chi_i\,\widehat{h}\,.\end{aligned}$$

We perform the same calculations as above (in the first case), but now we use (52) instead of (49). We obtain

$$\gamma\,\chi_i(\tau,\eta)^2\int_0^{+\infty}|\widehat{W^{nc}}(\delta,\eta,x_2)|^2\,dx_2+\chi_i(\tau,\eta)^2\,|\widehat{W^{nc}}(\delta,\eta,0)|^2\leq\frac{C_i}{\gamma^2}\,\chi_i(\tau,\eta)^2\,|\widehat{h}|^2\,(|\tau|^2+v_r^2\eta^2)\,. \tag{53}$$

• In the third and last case, $\mathcal{V}_i$ is a neighborhood of a pole of the symbol $\mathcal{A}$. For instance, $\mathcal{V}_i$ is a neighborhood of a point $(-iv_r\eta_0,\eta_0)\in\Sigma$. In this case, we have shown that there exists $\mathcal{C}^\infty$ mappings $\mathcal{T}_i$ and S_i defined on $\mathcal{V}_i$ such that

$\mathcal{T}_i$ has values in $GL_6(\mathbb{C})$,

both relations (48a)-(48b) hold on $\mathcal{V}_i$.

Recall that ω_r has negative real part on $\mathcal{V}_i$.

Here, we extend $\chi_i,\mathcal{T}_i$ and S_i as homogeneous mappings of degree 0 with respect to (τ,η). Moreover, we shall make as if $\mathcal{T}_i$ and S_i were defined on the whole hemisphere Σ (this is of pure convenience since only the values on the support of χ_i are involved in what follows). We define

$$U_i(\tau,\eta,x_2):=\chi_i(\tau,\eta)\,\mathcal{T}_i(\tau,\eta)\,\widehat{W}(\delta,\eta,x_2)\in\mathbb{C}^6\,.$$

The components of the vector U_i are denoted as follows

$$U_i=(U_{i,1},U_{i,2},\dots,U_{i,6})^T\,,$$

and we also define

$$U_i^{nc}:=(U_{i,3},U_{i,4},U_{i,5},U_{i,6})^T\in\mathbb{C}^4\,.$$

We decompose the matrix $\mathcal{T}_i(\tau,\eta)^{-1}$, see (44), in the following way:

$$\mathcal{T}_i(\tau,\eta)^{-1}=\begin{pmatrix}I_2 & * \\ 0 & T_i(\tau,\eta)^{-1}\end{pmatrix}\,,\quad T_i(\tau,\eta)^{-1}\in GL_4(\mathbb{C})\,.$$

From these definitions, we get

$$(\tau\mathcal{A}_0 + i\eta\mathcal{A}_1)\,\mathcal{T}_i(\tau,\eta)^{-1}U_i + \mathcal{A}_2\mathcal{T}_i(\tau,\eta)^{-1}\frac{dU_i}{dx_2} = 0\,,\quad x_2 > 0\,,$$
$$\beta(\tau,\eta)T_i(\tau,\eta)^{-1}U_i^{nc}(0) = \chi_i(\tau,\eta)\,\widehat{h}\,.$$

Multiplying the equation in $\{x_2 > 0\}$ by $S_i(\tau,\eta)$ and using (48a)-(48b), we obtain the following system of equations:

$$(\tau + iv_r\eta)\,U_{i,1} + ic^2\eta\,U_{i,5} = 0\,, \tag{54a}$$
$$(\tau + iv_l\eta)\,U_{i,2} = 0\,, \tag{54b}$$
$$-ic^2\eta\,U_{i,1} + \frac{c^4\eta^2}{|\tau|^2 + v_r^2\eta^2}\,\omega_r\,U_{i,3} - \frac{c^4\eta^2}{|\tau|^2 + v_r^2\eta^2}\,\frac{dU_{i,3}}{dx_2} = 0\,, \tag{54c}$$
$$\omega_l\,U_{i,4} - \frac{dU_{i,4}}{dx_2} = 0\,, \tag{54d}$$
$$\omega_r\,U_{i,5} + \frac{dU_{i,5}}{dx_2} = 0\,, \tag{54e}$$
$$\omega_l\,U_{i,6} + \frac{dU_{i,6}}{dx_2} = 0\,. \tag{54f}$$

Recall that when (τ,η) belongs to the conical set $\mathbb{R}^+\cdot\mathcal{V}_i$, one has

$$\mathrm{Re}\;\omega_r \le -\kappa(|\tau|^2 + v_r^2|\eta|^2)^{1/2} \quad \text{and} \quad \mathrm{Re}\;\omega_l \le -\kappa\gamma\,,$$

for a suitable constant $\kappa > 0$. Because $U_{i,5}(x_d)$ and $U_{i,6}(x_d)$ belong to $L^2(\mathbb{R}^+)$, for $\gamma > 0$, (54e) and (54f) imply

$$U_{i,5} \equiv 0 \quad \text{and} \quad U_{i,6} \equiv 0\,.$$

Using (54a) and (54b), we also obtain

$$U_{i,1} \equiv 0 \quad \text{and} \quad U_{i,2} \equiv 0\,.$$

Eventually, (54c) and (54d) reduce to

$$\omega_r\,U_{i,3} - \frac{dU_{i,3}}{dx_2} = 0 \quad \text{and} \quad \omega_l\,U_{i,4} - \frac{dU_{i,4}}{dx_2} = 0\,.$$

Because the two first columns of $T_i(\tau,\eta)^{-1}$ are nothing but the vectors E_r and E_l defined in Lemma 2, see (44), the boundary conditions for $U_{i,3}$ and $U_{i,4}$ read:

$$\beta(\tau,\eta)\,\begin{pmatrix} E_r(\tau,\eta) & E_l(\tau,\eta)\end{pmatrix}\begin{pmatrix} U_{i,3}(0) \\ U_{i,4}(0)\end{pmatrix} = \chi_i(\tau,\eta)\,\widehat{h}\,. \tag{55}$$

Using the properties of ω_r and ω_l on the conical set $\mathbb{R}^+\cdot\mathcal{V}_i$, we derive

$$(|\tau|^2 + v_r^2|\eta|^2)^{1/2}\int_0^{+\infty}|U_{i,3}(\tau,\eta,x_2)|^2\,dx_2 \le C\,|U_{i,3}(\tau,\eta,0)|^2\,,$$
$$\gamma\int_0^{+\infty}|U_{i,4}(\tau,\eta,x_2)|^2\,dx_2 \le C\,|U_{i,4}(\tau,\eta,0)|^2\,.$$

Because the uniform Lopatinskii condition is satisfied on $\mathcal{V}_i$, (55) gives the following estimate:

$$\left|\begin{pmatrix} U_{i,3}(\tau,\eta,0) \\ U_{i,4}(\tau,\eta,0)\end{pmatrix}\right|^2 \le C\,\chi_i(\tau,\eta)^2\,|\widehat{h}|^2\,.$$

Eventually, we obtain

$$\gamma \int_0^{+\infty} |U_i^{nc}(\tau,\eta,x_2)|^2 \, dx_2 + |U_i^{nc}(\tau,\eta,0)|^2 \le C \, \chi_i(\tau,\eta)^2 \, |\widehat{h}|^2 \, .$$

We now use the definition of the vector U_i^{nc} to derive

$$\gamma \, \chi_i^2 \int_0^{+\infty} |\widehat{W^{nc}}(\delta,\eta,x_2)|^2 \, dx_2 + \chi_i^2 \, |\widehat{W^{nc}}(\delta,\eta,0)|^2 \le C \, \chi_i^2 \, |\widehat{h}|^2 \, . \tag{56}$$

- We now add up (50)-(53)-(56), and use that the χ_i's form a partition of unity. We obtain

$$\gamma \int_0^{+\infty} |\widehat{W^{nc}}(\delta,\eta,x_2)|^2 \, dx_2 + |\widehat{W^{nc}}(\delta,\eta,0)|^2 \le \frac{C}{\gamma^2} \, |\widehat{h}|^2 \, (|\tau|^2 + v_r^2 \eta^2) \, .$$

We have already recalled that $\widehat{h}$ is simply obtained from the source term $\widehat{g}$ in (24) by multiplying by a uniformly bounded matrix. Thus integrating the previous inequality with respect to $(\delta,\eta) \in \mathbb{R}^2$ and using Plancherel's Theorem yields the desired estimate:

$$\gamma \, \|W^{nc}\|_0^2 + \|W^{nc}{}_{|x_2=0}\|_0^2 \le \frac{C}{\gamma^2} \, \|g\|_{1,\gamma}^2 \, .$$

Combining with (28), we have finished to prove (23).

5 Proof of intermediate results

5.1 The proof of Proposition 1

Using (27) and (32), we obtain

$$\beta \, \begin{pmatrix} E_r & E_l \end{pmatrix} = \begin{pmatrix} (\tau + iv_r\eta)(c^{-1}(\tau + iv_r\eta) - \omega_r) & (\tau + iv_l\eta)(c^{-1}(\tau + iv_l\eta) - \omega_l) \\ -c\omega_r(\tau + iv_l\eta)(c\omega_r - (\tau + iv_r\eta)) & c\omega_l(\tau + iv_r\eta)(c\omega_l - (\tau + iv_l\eta)) \end{pmatrix}$$

for all $(\tau,\eta) \in \Sigma$. This gives the following expression for the Lopatinskii determinant:

$$\Delta(\tau,\eta) = -c^2(\tau + iv_r\eta - c\omega_r)(\tau + iv_l\eta - c\omega_l)(\omega_r\omega_l - \eta^2)(\omega_r + \omega_l) \, .$$

Recall that ω_r and ω_l have negative real part when τ has positive real part, and satisfy the dispersion relations (31a)-(31b). Because $v_r = -v_l$, we have the identity

$$\omega_r(\tau,\eta) = \omega_l(\tau,-\eta) \, .$$

With the above expression for Δ, it is easy to check that $\Delta(\tau,\eta) = \Delta(\tau,-\eta)$. We shall thus only consider nonnegative values of η in all this section: $\eta \ge 0$.

One first checks that both expressions

$$(\tau + iv_r\eta - c\omega_r) \quad \text{and} \quad (\tau + iv_l\eta - c\omega_l)$$

do not vanish for any $(\tau,\eta) \in \Sigma$, because of (31a)-(31b).

Clearly, the sum $\omega_r + \omega_l$ can not vanish when τ has positive real part, since both numbers have negative real part. When τ is purely imaginary, one extends $\omega_{r,l}$ by continuity. If $\tau = i\delta \in i\mathbb{R}$ satisfies $(\delta + v_r\eta)^2 \le c^2\eta^2$, one has

$$\omega_r = -\sqrt{\eta^2 - \frac{1}{c^2}(\delta + v_r\eta)^2} \in \mathbb{R} \, .$$

If $(\delta + v_r \eta)^2 > c^2\eta^2$, we use Cauchy-Riemann relations to derive

$$\omega_r = -i\,\mathrm{sgn}(\delta + v_r\eta)\sqrt{\frac{1}{c^2}(\delta + v_r\eta)^2 - \eta^2} \in i\mathbb{R}\,.$$

The calculations are almost the same as those done in [2]. For ω_l, one just changes v_r into $v_l = -v_r$ and derives similar formulas. Then using the dispersion relations (31a)-(31b), we easily check that $\omega_r + \omega_l$ vanishes if and only if $\tau = 0$ (and therefore $\eta \neq 0$). For $\eta > 0$, this gives the following values for the eigenmodes:

$$\omega_r = -i\eta\sqrt{\frac{v_r^2}{c^2} - 1} = -\omega_l \in i\mathbb{R}\,.$$

Recall that $v_r > c\sqrt{2}$.

It now remains to determine whether the expression $(\omega_r\omega_l - \eta^2)$ may vanish. If $\eta = 0$, one has $\omega_r = \omega_l = -\tau/c$, and, therefore, $\omega_r\omega_l \neq 0$. We thus assume $\eta \neq 0$ (that is, $\eta > 0$) and introduce the reduced expressions

$$V := \frac{\tau}{i\eta}\,, \quad \Omega_{r,l} := \frac{\omega_{r,l}}{i\eta}\,.$$

Assume that $\Omega_r\Omega_l = -1$. Using (31) and $(\Omega_r\Omega_l)^2 = 1$, we obtain the following polynomial equation for V:

$$V^4 - 2(c^2 + v_r^2)V^2 + v_r^2(v_r^2 - 2c^2) = 0\,.$$

This is a polynomial equation of degree 2 for the unknown V^2, whose roots are real and distinct. If (15) holds, both roots are positive. Let us denote these roots by V_1^2 and V_2^2, where $0 < V_1 < V_2$. One has

$$V_1^2 = c^2 + v_r^2 - c\sqrt{c^2 + 4v_r^2}\,,$$
$$V_2^2 = c^2 + v_r^2 + c\sqrt{c^2 + 4v_r^2}\,.$$

We first show that the root V_2 does not yield any instability. If $V = V_2$, one has $V \pm v_r \geq c$. Because $\eta > 0$, we obtain

$$\Omega_r = -\sqrt{\frac{1}{c^2}(V_2 + v_r)^2 - 1} \quad \text{and} \quad \Omega_l = -\sqrt{\frac{1}{c^2}(V_2 - v_r)^2 - 1}\,,$$

so $\Omega_r\Omega_l \neq -1$. The Lopatinskii determinant Δ does not vanish when $V = V_2$. A similar argument shows that Δ does not vanish either for $V = -V_2$.

Now we show that $V = V_1$ is a root of the Lopatinskii determinant. One first checks that $V_1 + v_r > c$, and $V_1 - v_r < -c$. Hence, for $\eta > 0$ and $\tau = iV_1\eta$, we find

$$\Omega_r = -\sqrt{\frac{1}{c^2}(V_1 + v_r)^2 - 1} \quad \text{and} \quad \Omega_l = \sqrt{\frac{1}{c^2}(V_1 - v_r)^2 - 1}\,.$$

These relations yield $\Omega_r\Omega_l < 0$ and $(\Omega_r\Omega_l)^2 = 1$, that is, we have $\Omega_r\Omega_l = -1$. For $V = V_1$, the Lopatinskii determinant vanishes. The same argument holds for $V = -V_1$.

This completes the first part of the proof of Proposition 1. What remains to show is that the roots of the Lopatinskii determinant are simple. We first show that near $\tau = 0$ and $\eta = 1/v_r$ (this is the only point of Σ such that $\tau = 0$ and $\eta > 0$), we have

$$\omega_r + \omega_l = \tau\, h(\tau, \eta)\,,$$

for an appropriate $\mathcal{C}^\infty$ function h. Since $\eta \neq 0$ near $\tau = 0$, we have

$$\omega_r + \omega_l = i\eta(\Omega_r + \Omega_l)\,,$$

where the notations are those introduced earlier. Near $V = 0$, both functions Ω_r and Ω_l are analytic with respect to V and satisfy

$$\Omega_r^2 = \frac{1}{c^2}(V + v_r)^2 - 1\,, \quad \Omega_l^2 = \frac{1}{c^2}(V - v_r)^2 - 1\,.$$

We thus obtain

$$\left(\frac{d\Omega_r}{dV} + \frac{d\Omega_l}{dV}\right)|_{V=0} = \frac{2v_r}{c\,\Omega_r(0)} \neq 0\,.$$

Using a classical factorization property of holomorphic functions, we obtain

$$\Omega_r + \Omega_l = V\,H(V)\,,$$

where H is holomorphic near 0 and $H(0) \neq 0$. This yields

$$\omega_r + \omega_l = \tau\,H\left(\frac{\tau}{i\eta}\right)\,.$$

The factorization result for Δ is proved near the root $\tau = 0$.

As regards the situation near those roots of the form $(\pm iV_1\eta, \eta) \in \Sigma$, it is entirely similar and we shall not detail the proof. (The proof is similar because Ω_r and Ω_l are still holomorphic with respect to V in a neighborhood of V_1 and $-V_1$). The result is that the Lopatinskii determinant Δ admits a factorization that reads

$$\Delta(\tau,\eta) = (\tau - iV_1\eta)h(\tau,\eta) \quad \text{or} \quad \Delta(\tau,\eta) = (\tau + iV_1\eta)h(\tau,\eta)\,,$$

where h is $\mathcal{C}^\infty$ and does not vanish near the roots of Δ. This completes the proof.

5.2 The proof of Lemma 4

In the proof of Proposition 1, we have seen that the matrix $\beta\,(E_r\ \ E_l)$ has the following expression:

$$\beta\,(E_r \quad E_l) = \begin{pmatrix} (\tau + iv_r\eta)(c^{-1}(\tau + iv_r\eta) - \omega_r) & (\tau + iv_l\eta)(c^{-1}(\tau + iv_l\eta) - \omega_l) \\ -c\omega_r(\tau + iv_l\eta)(c\omega_r - (\tau + iv_r\eta)) & c\omega_l(\tau + iv_r\eta)(c\omega_l - (\tau + iv_l\eta)) \end{pmatrix}$$

for all $(\tau,\eta) \in \Sigma$. We have also seen that the quantity $(\tau + iv_r\eta - c\omega_r)$ does not vanish for $(\tau,\eta) \in \Sigma$. Let us now consider a neighborhood $\mathcal{V}$ of a point $(\tau_0,\eta_0) \in \Sigma$ such that $\tau_0 = 0$. Up to shrinking $\mathcal{V}$, the quantity $(\tau + iv_r\eta)$ does not vanish in $\mathcal{V}$. As a consequence, the upper left corner coefficient of $\beta\,(E_r\ E_l)$ does not vanish in $\mathcal{V}$. We write

$$\beta\,(E_r \quad E_l) = \begin{pmatrix} \zeta_1 & \zeta_2 \\ \zeta_3 & \zeta_4 \end{pmatrix}\,.$$

Then, relation (33) can be rewritten as

$$\Delta = \zeta_1\zeta_4 - \zeta_2\zeta_3\,,$$

and ζ_1 does not vanish in $\mathcal{V}$. The identity

$$\begin{pmatrix} 1/\zeta_1 & 0 \\ -\zeta_3/\zeta_1 & 1 \end{pmatrix} \beta\,(E_r \quad E_l) \begin{pmatrix} 1 & -\zeta_2 \\ 0 & \zeta_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \Delta \end{pmatrix}$$

is a straightforward verification. In particular, this identity yields the estimate

$$|\beta\ \big(E_r \quad E_l\big)\ Z^-|^2 \geq \kappa\ \min(1, |\Delta|^2)\, |Z^-|^2\,,$$

for all $Z^- \in \mathbb{C}^2$. Using Proposition 1, the Lopatinskii determinant Δ can be factorized near (τ_0, η_0):

$$\Delta(\tau, \eta) = \tau\, h(\tau, \eta)\,, \quad h(\tau_0, \eta_0) \neq 0\,.$$

Since γ is the real part of τ, we obtain

$$|\Delta(\tau, \eta)| \geq \kappa\, \gamma\,,$$

for a suitable constant $\kappa > 0$ (still up to shrinking the neighborhood $\mathcal{V}$). This last inequality yields

$$|\beta\ \big(E_r \quad E_l\big)\ Z^-|^2 \geq \kappa\, \gamma^2\, |Z^-|^2\,,$$

for all $Z^- \in \mathbb{C}^2$ and all $(\tau, \eta) \in \mathcal{V}$.

Lemma 4 is thus proved when (τ_0, η_0) satisfies $\tau_0 = 0$. The other points where Δ vanishes are those points (τ_0, η_0) such that $\tau_0 = \pm i V_1 \eta_0$. Near those points, the upper left corner coefficient of $\beta\, (E_r\ E_l)$ still does not vanish. We can again conclude that in an appropriate neighborhood $\mathcal{V}$ of (τ_0, η_0) (with, for instance, $\tau_0 = i V_1 \eta_0$), one has

$$|\beta\ \big(E_r \quad E_l\big)\ Z^-|^2 \geq \kappa\ \min(1, |\Delta|^2)\, |Z^-|^2\,.$$

Now we use the factorization

$$\Delta(\tau, \eta) = (\tau - i V_1 \eta)\, h(\tau, \eta)\,, \quad h(\tau_0, \eta_0) \neq 0\,,$$

to conclude. This completes the proof of Lemma 4.

6 Concluding remarks

In this paper, we have proved a linear stability result for a wide class of planar compressible vortex sheets. To prove the local in time existence of nonplanar vortex sheets, the next step will be to extend our stability result to a variable coefficients linearized problem. The construction developed here is a key tool to derive such a result, since it provides with a symbolic symmetrizer for the "frozen coefficients problem". Consequently, the variable coefficients linearized problem might be attacked using a similar procedure to what was done in [9]. This is postponed to a future work.

The one-dimensional stability of contact discontinuities has received a general treatment in [6] and [7]. Unfortunately, the isentropic Euler's equations do not admit contact discontinuities in one space dimension. However, it would be interesting to determine whether the present analysis extends to some contact discontinuities for the general Euler's equations, and see the connections with the one-dimensional analysis.

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