

# Small BV Solutions of Hyperbolic Non-cooperative Differential Games

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## Abstract

The paper is concerned with an  $n$ -persons differential game in one space dimension. We state conditions for which the system of Hamilton-Jacobi equations for the value functions is strictly hyperbolic. In the positive case, we show that the weak solution of a corresponding system of conservation laws determines an  $n$ -tuple of feedback strategies. These yield a Nash equilibrium solution to the non-cooperative differential game.

## 1 Introduction

This paper is concerned with the global existence of a Nash equilibrium solution for a non-cooperative  $n$ -persons differential game. The evolution of the system is governed by a differential equation of the form

$$\dot{x}(t) = \sum_{i=1}^n f_i(x, u_i), \quad (1.1)$$

say with initial data

$$x(\tau) = y. \quad (1.2)$$

Here the real valued map  $t \mapsto u_i(t)$  is the control implemented by the  $i$ -th player. Together with (1.1) we consider the cost functionals

$$J_i = J_i(\tau, y, u_1, \dots, u_n) \doteq \int_{\tau}^T h_i(x(t), u_i(t)) dt + g_i(x(T)), \quad (1.3)$$

consisting of a running cost  $h_i$  and a terminal cost  $g_i$ . The goal of the  $i$ -th player is to minimize  $J_i$ . An  $n$ -tuple of feedback strategies

$$U_i^* = U_i^*(t, x) \quad i = 1, \dots, n$$

is called a Nash equilibrium solution if the following holds. For each  $i$ , if the  $i$ -th player chooses an alternative strategy  $U_i$  while every other player sticks to his previous strategy  $U_j^*$ ,  $j \neq i$ , then the cost for the  $i$ -th player does not decrease:

$$J_i(\tau, y, U_1^*, \dots, U_{i-1}^*, U_i, U_{i+1}^*, \dots, U_n^*) \geq J_i(\tau, y, U_1^*, \dots, U_{i-1}^*, U_i^*, U_{i+1}^*, \dots, U_n^*). \quad (1.4)$$

Assume that a value function  $V = (V_1, \dots, V_n)$  exists, so that  $V_i(t, x)$  is the minimum cost for the  $i$ -th player, when everyone plays optimally but no cooperation is allowed. Under suitable regularity conditions (see [F1], p.292), the function  $V$  provides a solution to the system of Hamiltonian equations

$$\frac{\partial}{\partial t} V_i + H_i(x, \nabla V_1, \dots, \nabla V_n) = 0, \quad (1.5)$$

with terminal data

$$V_i(T, x) = g_i(x). \quad (1.6)$$

The Hamiltonian functions  $H_i$  are defined as follows. Assume that, for any given gradient vectors  $p_1, \dots, p_n$ , there exist optimal control values  $u_j^*(x, p_j)$ ,  $j = 1, \dots, n$ , such that

$$p_j \cdot f_j(x, u_j^*(x, p_j)) + h_j(x, u_j^*(x, p_j)) = \min_{\omega \in \mathbb{R}} \{p_j \cdot f_j(x, \omega) + h_j(x, \omega)\}. \quad (1.7)$$

Then

$$\begin{aligned} H_i(x, p_1, \dots, p_m) &= p_i \cdot \sum_{j=1}^m f_j(x, u_j^*(x, p_j)) + h_i(x, u_i^*(x, p_i)) \\ &= p_i \cdot \sum_{j \neq i} f_j(x, u_j^*(x, p_j)) + \min_{\omega \in \mathbb{R}} \{p_i \cdot f_i(x, \omega) + h_i(x, \omega)\}. \end{aligned} \quad (1.8)$$

In the case of a two-persons, zero-sum differential game, the value function is obtained from the scalar Bellman-Isaacs equation [F1], [I]. The analysis can thus rely on comparison principles and on the well developed theory of viscosity solutions for Hamilton-Jacobi equations, see for example [BC]. On the other hand, in the case of non-cooperative  $n$ -persons games, one has to study a highly nonlinear system of Hamilton-Jacobi equations. Little is yet known in this direction, except for particular examples as in [CR, O]. Instead, local existence results are known within the class of open-loop strategies [F1, VZ]. They apply to the case where the players cannot use any additional information on the state  $x(t)$  of the system, for  $t > 0$ .

In the one dimensional case, differentiating (1.5) w.r.t.  $x$  one obtains a system of conservation laws for the gradient functions  $p_i \doteq V_{i,x}$ , namely

$$p_{i,t} + H_i(x, p)_x = 0. \quad (1.9)$$

In recent years, considerable progress has been achieved in the understanding of weak solutions to hyperbolic systems of conservation laws in one space dimension. In particular, entropy admissible solutions with small total variation are known to be unique and

depend continuously on the initial data [B3, BLY]. Moreover, they can be obtained as the unique limits of vanishing viscosity approximations [BB].

The aim of the present paper is to investigate the relevance of these new results in the field of conservation laws toward the existence and stability of Nash equilibrium solutions, in the context of differential games. In particular, our main goals are:

- To identify the situations where the hyperbolic theory is applicable.
- In the favorable case, to derive the existence and properties of a Nash equilibrium solution.

The hyperbolicity of the system is clearly a crucial assumption, in order that the Cauchy problem for the value functions be well posed. In Section 3 we show that hyperbolicity holds provided that the derivatives of the cost functions  $g_i$  all have the same sign. In practice, this means that all players wish to steer the system in the same direction.

Granted that the system is strictly hyperbolic, the known results on systems of conservation laws can be applied. The theorem of Glimm [G], or its more general versions [BB, ILF, L]. provide then the existence of a global solution to the Hamilton-Jacobi equations, for terminal data  $g_i$  whose gradients have sufficiently small total variation.

To obtain an existence result for solutions of differential games, one has to show that, for each single player, the feedback strategy corresponding to the solution of the Hamilton-Jacobi system is indeed an optimal one. We remark that, if the value functions  $V_i$  were smooth, the optimality would be an immediate consequence of the equations. The main technical difficulty stems from the non-differentiability of these value functions.

In the literature on control theory, sufficient conditions for optimality have been obtained along two main directions. On one hand, there is the “regular synthesis” approach developed by Boltianskii [Bo], Brunovskii [Br], Sussmann and Piccoli [PS]. In this case, one typically requires that the value function be piecewise  $C^1$  and satisfy the H-J equations outside a finite or countable number of smooth manifolds  $\mathcal{M}_i$ . On the other hand, one can use the Crandall-Lions theory of viscosity solutions, and show that the value function is the unique solution of the H-J equation in the viscosity sense [BC].

None of these approaches is applicable in the present situation, because of lack of regularity. Indeed, each player now has to solve an optimal control problem for a system whose dynamics (determined by the feedbacks used by all other players) is discontinuous. Our proof of optimality will strongly rely on the special structure of BV solutions of hyperbolic systems of conservation laws. In particular, we show that the solution has bounded directional variation along a cone  $\Gamma$  bounded away from all characteristic directions. As a consequence, the value functions  $V_i$  always admit a directional derivative in the directions of the cone  $\Gamma$ . For trajectories whose speed remains inside  $\Gamma$ , the optimality can thus be tested directly from the equations. An additional argument, using Clarke’s generalized gradients [C], will rule out the optimality of trajectories whose

speed falls outside the above cone of directions.

It is interesting to observe that the entropy admissibility conditions play no role in our analysis. For example, a solution of the system of conservation laws consisting of a single, non-entropic shock still determines a Nash equilibrium solution, provided that the amplitude of the shock is small enough. There is, however, a way to distinguish entropy solutions from all others, also in the context of differential games. Indeed, entropy solutions are precisely the ones obtained as vanishing viscosity limits [BB]. They can thus be derived from a stochastic differential game of the form

$$dx = \sum_{i=1}^n f_i(x, u_i) dt + \varepsilon dw ,$$

letting the white noise parameter  $\varepsilon \rightarrow 0$ . Here  $dw$  formally denotes the differential of a Brownian motion. For a discussion of stochastic differential games we refer to [F2].

In general, a weak solution of the hyperbolic system of conservation laws uniquely determines a family of discontinuous feedback controls  $U_i^* = U_i^*(t, x)$ . Inserting these controls in (1.1) we obtain the O.D.E.

$$\dot{x} = \sum_{i=1}^n f_i(x, U_i^*(t, x)). \quad (1.10)$$

In spite of the right hand side being discontinuous, we show that the solution of the Cauchy problem is unique and depends continuously on the initial data. Indeed, every trajectory of (1.10) crosses transversally all lines of discontinuity in the functions  $f_i$ . Because of the bound on the total variation, the uniqueness result in [B1] can thus be applied.

Our analysis will be concerned with the spatially homogeneous case, where the functions  $f_i, h_i$  do not depend on  $x$ . In the last section we shall see what results remain valid in the non-homogeneous case, and discuss other possible extensions.

## 2 The basic framework

Consider an  $n$ -persons differential game on the real line, having the special form

$$\dot{x} = f_0 + \sum_i u_i , \quad x(\tau) = y . \quad (2.1)$$

Here the controls  $u_i$  can be any measurable, real valued functions, while  $f_0$  is a fixed real number. The cost functionals take the form

$$J_i = J_i(\tau, y, u_1, \dots, u_n) = \int_{\tau}^T h_i(u_i(t)) dt + g_i(x(T)) . \quad (2.2)$$

To simplify the problem, for the time being we thus assume that the system has the simple dynamics (2.1) and that the running costs  $h_i$  do not depend on the space variable  $x$ . In Section 6 we shall discuss how to extend the results to more general situations.

A key assumption, used throughout the paper, is that the cost functions  $h_i$  are smooth and strictly convex, with a positive second derivative:

$$\frac{\partial^2}{\partial \omega^2} h_i(\omega) > 0 \quad (2.3)$$

at every point  $\omega$ . Each player seeks a feedback strategy  $u_i = U_i^*(t, x)$  which minimizes his own cost. Call  $V_i = V_i(\tau, y)$  the value function for  $i$ -th player, and consider the spatial derivative  $p_i = V_{i,x}$ . The Hamiltonian functions  $H_i$  are defined as

$$H_i(p_1, \dots, p_n) = p_i \cdot \left( f_0 + \sum_j u_j^*(p_j) \right) + h_i(u_i^*(p_i)), \quad (2.4)$$

where the controls  $u_j^* = u_j^*(p_j)$  provide the solutions to the following minimization problems:

$$p_j u_j^* + h_j(u_j^*) = \min_{\omega} \{ p_j \cdot \omega + h_j(\omega) \} \doteq \phi_j(p_j). \quad (2.5)$$

At a point of minimum the first derivative vanishes. For every  $p_j$  we thus have

$$p_j + \frac{\partial h_j}{\partial u_j}(u_j^*(p_j)) = 0. \quad (2.6)$$

The Hamiltonian functions in (2.4) can also be written as

$$H_i(p_1, \dots, p_n) = p_i \left( f_0 + \sum_{j \neq i} u_j^*(p_j) \right) + \phi_i(p_i). \quad (2.7)$$

The corresponding Hamilton-Jacobi equation for  $V_i$  takes the form

$$V_{i,t} + H_i(V_{1,x}, \dots, V_{n,x}) = 0. \quad (2.8)$$

To determine the value functions  $V_i$ , the above system has to be solved backward in time, with data given at the terminal time  $t = T$

$$V_i(T, x) = g_i(x), \quad i = 1, \dots, n. \quad (2.9)$$

In turn, the gradients  $p_i \doteq V_{i,x}$  of the value functions satisfy the system of conservation laws

$$\frac{\partial}{\partial t} p_i + \frac{\partial}{\partial x} H_i(p_1, \dots, p_n) = 0 \quad (2.10)$$

with the terminal data

$$p_i(T, x) = g_i'(x). \quad (2.11)$$

Computing the Jacobian matrix  $A(p)$  of this system, with entries  $A_{ij} = \partial H_i / \partial p_j$ , we find

$$A_{ii} = f_0 + \sum_j u_j^*(p_j) = \dot{x}, \quad (2.12)$$

$$A_{ij} = p_i \frac{\partial u_j^*(p_j)}{\partial p_j} = p_i \frac{\partial^2 \phi_j}{\partial p_j^2}, \quad i \neq j. \quad (2.13)$$

Indeed, by (2.6) the functions  $\phi_j \doteq p_j u_j^*(p_j) + h_j(u_j^*(p_j))$  defined at (2.5) satisfy

$$\frac{\partial \phi_j}{\partial p_j} = u_j^* + p_j \frac{\partial u_j^*}{\partial p_j} + \frac{\partial h_j}{\partial u_j} \frac{\partial u_j^*}{\partial p_j} = u_j^*. \quad (2.14)$$

It will be convenient to write second derivatives as  $h_j'' \doteq \partial^2 h_j / \partial u_j^2$ ,  $\phi_j'' \doteq \partial^2 \phi_j / \partial p_j^2$ . Differentiating w.r.t.  $p_j$  the identity (2.6) we find

$$1 + h_j''(u_j^*(p_j)) \cdot \frac{\partial u_j^*}{\partial p_j}(p_j) = 0. \quad (2.15)$$

Using (2.14) and (2.15) one obtains

$$\phi_j''(p_j) = \frac{\partial u_j^*}{\partial p_j}(p_j) = -\frac{1}{h_j''(u_j^*(p_j))}. \quad (2.16)$$

Of course, this relation is well known from the theory of Legendre transforms.

### 3 Hyperbolicity conditions

In order that the Cauchy problem (2.10)-(2.11) be well posed, the system of conservation laws should be hyperbolic. It is thus important to determine in which cases the Jacobian matrix  $A(p)$  has  $n$  distinct real eigenvalues. According to (2.12)-(2.13), this matrix takes the form

$$A(p) = \begin{pmatrix} \dot{x} & p_1 \phi_2'' & p_1 \phi_3'' & \cdots & p_1 \phi_n'' \\ p_2 \phi_1'' & \dot{x} & p_2 \phi_3'' & \cdots & p_2 \phi_n'' \\ p_3 \phi_1'' & p_3 \phi_2'' & \dot{x} & \cdots & p_3 \phi_n'' \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_n \phi_1'' & p_n \phi_2'' & p_n \phi_3'' & \cdots & \dot{x} \end{pmatrix}. \quad (3.1)$$

We recall that, by (2.16) and (2.3), all second derivatives  $\phi_j''$  are strictly negative. The next lemma provides sufficient conditions on  $p_1, \dots, p_n$  for which the system of conservation laws (2.10) is hyperbolic.

**Lemma 1** *Assume that all  $p_j$  have the same sign, i.e. either  $p_j > 0$  for all  $j$ , or  $p_j < 0$  for all  $j$ . Moreover, assume that there are no distinct indices  $i \neq j \neq k$  such that*

$$p_i \phi_i'' = p_j \phi_j'' = p_k \phi_k''. \quad (3.2)$$

Then the matrix  $A(p)$  in (3.1) is strictly hyperbolic. Namely, it has  $n$  real distinct eigenvalues, all different from  $\dot{x}$ .

**Proof.** To fix the ideas, consider the case where  $p_i > 0$  for all  $i = 1, \dots, n$ . The case where  $p_i < 0$  is entirely similar.

Let  $B \doteq A - \dot{x}I$ , where  $I$  is the  $n \times n$  identity matrix, and call  $\lambda(A)$  the eigenvalues of a matrix  $A$ . Since

$$\lambda(B) = \lambda(A) - \dot{x}. \quad (3.3)$$

it suffices to show that  $B$  has  $n$  distinct real eigenvalues, all different from zero.

First we show that  $B$  has no zero eigenvalue. This is clear because

$$\begin{aligned} \det(B) &= \begin{vmatrix} 0 & p_1\phi_2'' & \cdots & p_1\phi_n'' \\ p_2\phi_1'' & 0 & \cdots & p_2\phi_n'' \\ \vdots & \vdots & \ddots & \vdots \\ p_n\phi_1'' & p_n\phi_2'' & \cdots & 0 \end{vmatrix} = \prod_{i=1}^n p_i\phi_i'' \cdot \begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{vmatrix} \\ &= (-1)^{n-1}(n-1) \prod_{i=1}^n p_i\phi_i'' \neq 0. \end{aligned}$$

As customary, the bars  $|\cdot|$  around a matrix are used here to denote its determinant.

The eigenvalues of  $B$  are the zeros of the characteristic polynomial  $\det(B - \lambda I)$ . We observe that

$$\det(B - \lambda I) = \begin{vmatrix} -\frac{\lambda}{p_1\phi_1''} & 1 & \cdots & 1 \\ 1 & -\frac{\lambda}{p_2\phi_2''} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & -\frac{\lambda}{p_n\phi_n''} \end{vmatrix} \cdot \prod_{i=1}^n p_i\phi_i''.$$

By assumption, the numbers  $c_i \doteq -p_i\phi_i''$  are all strictly positive. Up to a permutation of indices, which does not affect the determinant, we can assume that  $0 < c_1 \leq c_2 \leq \dots \leq c_n$ . The polynomial  $\det(B - \lambda I)$  has the same zeros as  $q(\lambda) \doteq \det \tilde{B}(\lambda)$ , where

$$\tilde{B}(\lambda) \doteq \begin{pmatrix} \frac{\lambda}{c_1} & 1 & \cdots & 1 \\ 1 & \frac{\lambda}{c_2} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & \frac{\lambda}{c_n} \end{pmatrix}. \quad (3.4)$$

The leading term of the polynomial  $q(\lambda)$  is easily computed:

$$q(\lambda) = \prod_{i=1}^n \frac{\lambda}{c_i} + \mathcal{O}(1) \cdot \lambda^{n-1}.$$

For some constant  $M > 0$  sufficiently large, we clearly have  $\text{sign}(q(\lambda)) = +1$  for  $\lambda > M$  and  $\text{sign}(q(\lambda)) = (-1)^n$  for all  $\lambda < -M$ . Moreover, when  $\lambda = 0$  we have  $\text{sign}(q(0)) = (-1)^{n-1}$ .

Two cases need to be considered, depending on whether all  $c_i$  are distinct, or two of them coincide.

**First case.** Assume that all the  $c_i$ 's are distinct, say  $0 < c_1 < c_2 < \dots < c_n$ . Let us compute the determinant of  $\tilde{B}(\lambda)$  at the point  $\lambda = c_i$ . In this case, the  $i$ -th row of  $\tilde{B}(\lambda)$  is identically 1, and we can subtract it from all the other rows, thus obtaining

$$\begin{aligned}
 q(\lambda) &= \begin{vmatrix} \frac{c_i}{c_1} & 1 & \dots & 1 \\ 1 & \frac{c_i}{c_2} & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ddots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & \frac{c_i}{c_n} \end{vmatrix} \leftarrow: i\text{-th row} = \begin{vmatrix} \frac{c_i}{c_1} - 1 & 0 & \dots & 0 \\ 0 & \frac{c_i}{c_2} - 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ddots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{c_i}{c_n} - 1 \end{vmatrix} \\
 &= \prod_{j \neq i} \left( \frac{c_i}{c_j} - 1 \right)
 \end{aligned}$$

Since

$$\text{sign} \left( \frac{c_i}{c_j} - 1 \right) = \begin{cases} 1, & \text{if } i > j \\ -1, & \text{if } i < j \end{cases}$$

we conclude that

$$\text{sign}(q(\lambda)) = (-1)^{n-i}, \quad \text{when } \lambda = c_i.$$

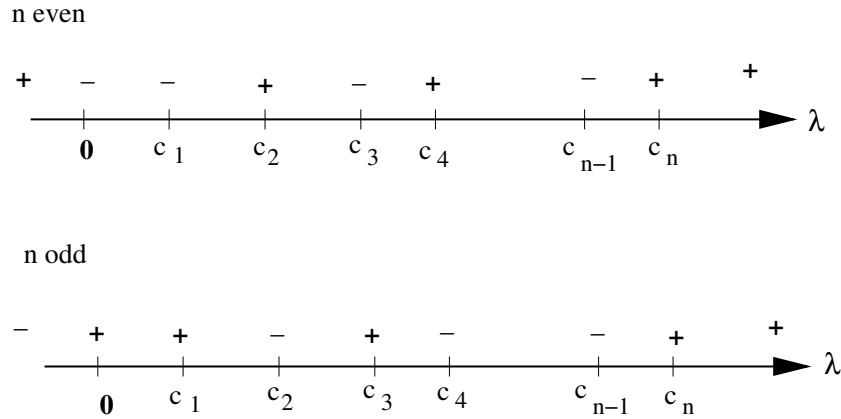


Figure 1: The sign of  $\det(\tilde{B})(\lambda)$  at various points. This shows the location of the real eigenvalues.



As shown in Figure 1, the function  $\lambda \mapsto q(\lambda)$  thus changes sign inside each one of the intervals

$$]-\infty, 0[, \quad ]c_1, c_2[, \quad \dots \quad ]c_{i-1}, c_i[, \quad \dots \quad ]c_{n-1}, c_n[.$$

By continuity, there exist  $n$  distinct real zeroes, with

$$\lambda_1 < 0 < c_1 < \lambda_2 < c_2 < \dots < c_{i-1} < \lambda_i < c_i < \dots < \lambda_n < c_n$$

Notice that we must have  $|\lambda_1| = \sum_{i=2}^n |\lambda_i|$  because the trace of  $\tilde{B}$  is zero.

**Second case.** Assume that two (but not three consecutive ones) of the numbers  $c_i$  coincide, say  $c_i = c_{i+1}$ . We claim that the polynomial  $q(\lambda)$  still has  $n$  distinct zeros, and the  $(i+1)$ -th zero is  $\lambda_{i+1} = c_i = c_{i+1}$ .

When  $\lambda = c_i = c_{i+1}$ , the matrix  $\tilde{B}(\lambda)$  has both the  $i$ -th and the  $(i+1)$ -th row identically equal to 1. Hence the determinant is zero. This shows that  $\lambda_{i+1} = c_i = c_{i+1}$  is a zero of  $q(\lambda)$ .

To prove that it is a single root, we need to check that the derivative  $q'(\lambda)$  does not vanish at  $\lambda = c_i$ . A direct computation yields

$$\begin{aligned} q'(\lambda) &= \begin{vmatrix} \frac{1}{c_1} & 1 & \dots & 1 \\ 0 & \frac{\lambda}{c_2} & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \dots & \frac{\lambda}{c_n} \end{vmatrix} + \begin{vmatrix} \frac{\lambda}{c_1} & 0 & \dots & 1 \\ 1 & \frac{1}{c_2} & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & \frac{\lambda}{c_n} \end{vmatrix} + \dots + \begin{vmatrix} \frac{\lambda}{c_1} & 1 & \dots & 0 \\ 1 & \frac{\lambda}{c_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & \frac{1}{c_n} \end{vmatrix} \\ &\doteq \det B_1(\lambda) + \det B_2(\lambda) + \dots + \det B_n(\lambda). \end{aligned}$$

When  $\lambda = c_i = c_{i+1}$  we have  $\det B_j(\lambda) = 0$  for all  $j \neq i, i+1$ , because the  $i$ -th and  $(i+1)$ -th rows of the matrix  $B_j$  are identical (all entries are 1 except the  $j$ -th entry which is 0). Moreover,  $\det B_i(\lambda) = \det B_{i+1}(\lambda)$  because  $B_i(\lambda)$  can be obtained from  $B_{i+1}(\lambda)$  by first exchanging  $i$ - and  $(i+1)$ -th rows and then exchanging  $i$ -th and  $(i+1)$ -th columns. Now we compute  $\det B_i(\lambda)$ . The entries of the  $(i+1)$ -th row are all 1 except the  $i$ -th entry which is zero. We subtract this row from all the other rows and obtain

$$\det B_i(\lambda) = \begin{vmatrix} \frac{c_i}{c_1} & \dots & 0 & 1 & \dots & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & \frac{1}{c_i} & 1 & \dots & 1 \\ 1 & \dots & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \dots & 0 & 1 & \dots & \frac{c_i}{c_n} \end{vmatrix} \begin{array}{l} \leftarrow: i\text{-th row} \\ \leftarrow: (i+1)\text{-th row} \end{array}$$

$$\begin{aligned}
&= \begin{vmatrix} \frac{c_i}{c_1} - 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \frac{1}{c_i} & 0 & \cdots & 0 \\ 1 & \cdots & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \frac{c_i}{c_n} - 1 \end{vmatrix} \begin{array}{l} \leftarrow: i\text{-th row} \\ \leftarrow: (i+1)\text{-th row} \end{array} \\
&= \frac{1}{c_i} \prod_{j \neq i, i-1} \left( \frac{c_i}{c_j} - 1 \right) \neq 0
\end{aligned}$$

Observing that  $q'(c_i) = 2 \cdot \det B_i(c_i) \neq 0$ , we conclude that  $\lambda = c_i$  is a single root.

It now remains to prove that we still have  $n$  distinct real zeros, i.e., the coincidence of the two numbers  $c_i$  and  $c_{i+1}$  does not destroy any of the other sign changes in the polynomial  $q(\lambda)$ . In particular, there is still a zero inside each interval  $]c_{i-1}, c_i[$  and  $]c_{i+1}, c_{i+2}[$ . From the previous computation we have that

$$\begin{aligned}
\text{sign}(q'(c_i)) &= (-1)^{n-i-1}, \\
\text{sign}(q(c_{i-1})) &= (-1)^{n-i+1}, \\
\text{sign}(q(c_{i+2})) &= (-1)^{n-i-2}.
\end{aligned}$$

Therefore,  $q'(c_i)$  and  $q(c_{i-1})$  have the same sign, while  $q'(c_i)$  and  $q(c_{i+2})$  have opposite signs. Looking at Figure 2 it is clear that there is one root within the interval  $]c_{i-1}, c_i[$ , another inside the interval  $]c_{i+1}, c_{i+2}[$ , while  $\lambda = c_i = c_{i+1}$  is still another root.

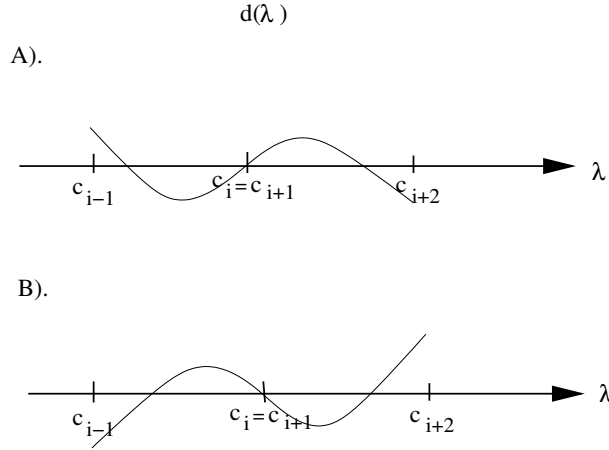


Figure 2: Checking that  $\lambda = c_i = c_{i+1}$  is a single root.

In the special case  $i = 1$  (or  $i = n - 1$ ), it can be checked in the same way that  $c_1 = c_2$  (or  $c_{n-1} = c_n$ ) is a single root, while another root lies in the interval  $]c_2, c_3[$ , (or in the interval  $]c_{n-2}, c_{n-1}[$  respectively).

At last, we need to consider the case where more than one couple of numbers  $c_i, c_{i+1}$  coincide. We claim that, in all cases, the polynomial  $q(\lambda)$  still has  $n$  distinct roots.

Indeed, in the case where the two coinciding pairs  $c_i = c_{i+1}$ ,  $c_j = c_{j+1}$  are not adjacent (i.e.,  $j \neq i+2$ ), the previous analysis applies. On the other hand, in the case where, say,  $c_{i-2} = c_{i-1} < c_i = c_{i+1}$ , the analysis of the signs of  $q'(c_{i-1})$ ,  $q'(c_i)$ ,  $q(c_{i+2})$  and  $q(c_{i-3})$  (see Figure 3) yields the desired results.

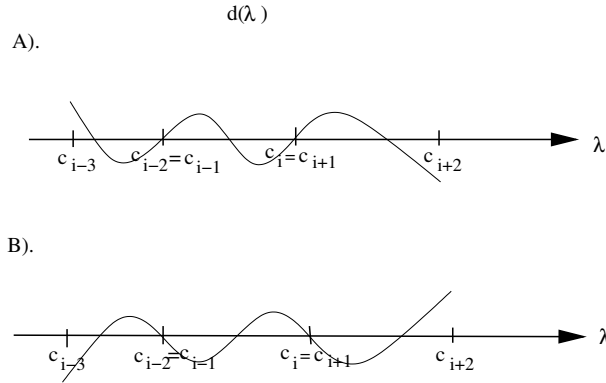


Figure 3: Two coinciding pairs next to each other.

The proof of Lemma 1 is now completed.

**Remark 1.** If three or more of the numbers  $c_j$  coincide, say  $c_{i-1} = c_i = c_{i+1}$ , then  $c_i$  becomes a multiple zero of  $\det(B - \lambda I)$ . In this case, the system of conservation laws will still be hyperbolic, but no longer strictly hyperbolic.

**Remark 2.** In the case of  $2 \times 2$  systems, the condition  $p_1 p_2 \geq 0$  is necessary for the hyperbolicity of the system. However, when  $n \geq 3$  the system (2.10) can be strictly hyperbolic also at points where  $p_1 < 0 < p_2 < p_3$ . For example: let  $n = 3$ ,  $\phi_i'' = -1$  for all  $i$ ,  $p_1 = -1$ ,  $p_2 = 5$ ,  $p_3 = 20$ . Then the characteristic polynomial  $q(\lambda) = \det \bar{B}(\lambda)$  is

$$q(\lambda) = \lambda^3 - 75\lambda + 200.$$

One can easily check that

$$q(-10) = -50, \quad q(0) = 200, \quad q(5) = -50, \quad q(10) = 450.$$

Therefore, there are three distinct real eigenvalues

$$\lambda_1 \in ]-10, 0[, \quad \lambda_2 \in ]0, 5[, \quad \lambda_3 \in ]5, 10[.$$

## 4 Review of hyperbolic systems and discontinuous O.D.E.

In this section we collect some results on hyperbolic conservation laws and discontinuous O.D.E's, which will be used in the sequel. Consider the Cauchy problem for a system of conservation laws

$$v_t + F(v)_x = 0, \quad v(0, x) = \bar{v}(x). \quad (4.1)$$

In the case where the system is strictly hyperbolic, the global existence of weak solutions with small BV initial data is well known.

**Proposition 1** *Assume that the flux function  $F : \mathbb{R}^n \mapsto \mathbb{R}^n$  is smooth and that, at some point  $v^*$ , the Jacobian matrix  $A(v^*) = DF(v^*)$  has  $n$  real distinct eigenvalues. Then there exists  $\delta > 0$  for which the following holds. If*

$$\|\bar{v}(\cdot) - v^*\|_{\mathbf{L}^\infty} < \delta, \quad \text{Tot. Var.}\{\bar{v}\} < \delta, \quad (4.2)$$

*then the Cauchy problem (4.1) admits a unique entropy weak solution  $v = v(t, x)$  defined for all  $t \geq 0$ , obtained as limit of vanishing viscosity approximations.*

In the case where each characteristic field is either linearly degenerate or genuinely nonlinear, the existence of a global weak solution was proved by Glimm [G]. The more general case was later covered in [L], [ILF] using the Glimm scheme and in [AM] using wave-front tracking. The convergence of vanishing viscosity approximations was recently proved in [BB], together with the uniqueness and Lipschitz continuous dependence of solutions on the initial data, in the  $\mathbf{L}^1$  distance. We remark that, for each time  $t$ , the function  $v(t, \cdot)$  has small total variation. Its pointwise values can be uniquely assigned by the convention

$$v(t, x) = \lim_{y \rightarrow x^+} v(t, y). \quad (4.3)$$

For applications to game theory, we shall need some additional properties of these weak solutions. By assumption, the matrix  $A(v^*)$  has distinct eigenvalues  $\lambda_1^* < \lambda_2^* < \dots < \lambda_n^*$ . By continuity, there exists  $\varepsilon > 0$  such that, for all  $v$  in the  $\varepsilon$ -neighborhood

$$\Omega_\varepsilon^* \doteq \{v; |v - v^*| \leq \varepsilon\},$$

the characteristic speeds range inside disjoint intervals

$$\lambda_j(v) \in [\lambda_j^-, \lambda_j^+]. \quad (4.4)$$

Moreover, if  $v^-, v^+ \in \Omega_\varepsilon^*$  are two states connected by a  $j$ -shock, the speed  $\lambda_j(v^-, v^+)$  of the shock remains inside the interval  $[\lambda_j^-, \lambda_j^+]$ .

Now consider an open cone of the form

$$\Gamma \doteq \{(t, x); t > 0, a < x/t < b\}. \quad (4.5)$$

Following [B1] we define the *directional variation* of the function  $(t, x) \mapsto v(t, x)$  along the cone  $\Gamma$  as

$$\sup \left\{ \sum_{i=1}^N |v(t_i, x_i) - v(t_{i-1}, x_{i-1})| \right\}, \quad (4.6)$$

where the supremum is taken over all finite sequences  $(t_0, x_0), (t_1, x_1), \dots, (t_N, x_N)$  such that

$$(t_i - t_{i-1}, x_i - x_{i-1}) \in \Gamma \quad \text{for every } i = 1, \dots, N. \quad (4.7)$$

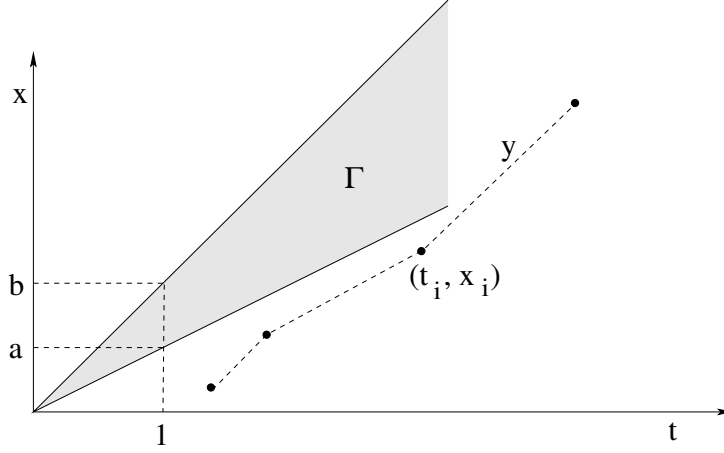


Figure 4: Directional variation along the cone  $\Gamma$ .

(See Fig. 4). We now show that the weak solution  $v = v(t, x)$  has bounded directional variation along a suitable cone  $\Gamma$ .

**Lemma 2** *Let  $v = v(t, x)$  be an entropy weak solution of (4.1) taking values inside the domain  $\Omega_\varepsilon^*$ . Assume that  $\lambda_{k-1}^+ < a < b < \lambda_k^-$  for some  $k$ . Then  $v$  has bounded directional variation along the cone  $\Gamma$  in (4.5).*

**Proof.** Fix any finite sequence of points  $(t_i, x_i)$ ,  $i = 0, \dots, N$ , satisfying (4.7). It is not restrictive to assume that  $t_0 = 0$ . Call  $T \doteq t_N$  and define  $y : [0, T] \mapsto \mathbb{R}$  the polygonal line with nodes at the points  $(t_i, x_i)$  (Fig. 4). Clearly  $y$  is a.e. differentiable, with  $\dot{y}(t) \in ]a, b[$ . From the theory of conservation laws, it is well known that the entropy weak solution  $v$  can be obtained as the limit of a sequence of front tracking approximate solutions  $v_\nu$ . For each  $\nu \geq 1$ , one can derive a uniform bound on the total variation of the map  $t \mapsto v_\nu(t, y(t))$ . Indeed, call  $V^y(t)$  the total strength of all wave-fronts in  $v_\nu(t, \cdot)$  approaching  $y(t)$  at time  $t$ , i.e.,

$$V^y(t) \doteq \sum_{\alpha \in \mathcal{A}(y)} |\sigma_\alpha|. \quad (4.8)$$

Here  $\sigma_\alpha$  denotes the strength of the wave-front in  $v_\nu(t, \cdot)$  located at  $x_\alpha$ . Observing that  $\lambda_k^+ < \dot{y} < \lambda_{k+1}^-$ , the above summation will include the following fronts:

- The fronts of a family  $k_\alpha \leq k$  located at a point  $x_\alpha > y$ ,
- The fronts of a family  $k_\alpha > k$  located at a point  $x_\alpha < y$ .

We now call

$$Q(t) = \sum_{\alpha, \beta \in \mathcal{A}} |\sigma_\alpha| |\sigma_\beta| \quad (4.9)$$

the *interaction potential* of  $v_\nu(t, \cdot)$ , i.e. the sum of products of all couples of approaching waves in  $v_\nu(t, \cdot)$ . Assuming that the total variation of the solution remains small, for some constant  $C_0$  the positive functional

$$\Upsilon(t) \doteq V^y(t) + C_0 Q(t)$$

is non-increasing in time. Moreover, at each time  $\tau$  where a wave front of strength  $\sigma_\alpha$  crosses  $y(\cdot)$ , we have

$$\Upsilon(\tau+) - \Upsilon(\tau-) = -|\sigma_\alpha|.$$

Therefore, the total strength of all wave-fronts in  $v_\nu$  which cross the polygonal line  $y(\cdot)$  is bounded by

$$V^y(0) + C_0 Q(0) = \mathcal{O}(1) \cdot \text{Tot.Var.}\{\bar{v}\}.$$

This proves that the total variation of the maps  $t \mapsto v_\nu(t, y(t))$  is uniformly bounded for all  $\nu \geq 1$ . To get the desired estimate for the solution  $v$ , we now let  $\nu \rightarrow \infty$ . If we have the pointwise convergence  $v_\nu(t_i, x_i) \rightarrow v(t_i, x_i)$  for every  $i = 0, \dots, N$ , we can immediately conclude

$$\sum_{i=1}^N |v(t_i, x_i) - v(t_{i-1}, x_{i-1})| \leq \limsup_{\nu \rightarrow \infty} |v_\nu(t_i, x_i) - v_\nu(t_{i-1}, x_{i-1})| = \mathcal{O}(1) \cdot \text{Tot.Var.}\{\bar{v}\},$$

proving our claim. However, if  $v$  is discontinuous at some point  $(t_i, x_i)$ , the pointwise convergence may not hold. To achieve the result also in the general case we observe that, for each time  $\tau$ , we have the convergence  $v_\nu(\tau, x) \rightarrow v(\tau, x)$  for a.e.  $x \in \mathbb{R}$ . Using the right continuity of the functions  $v(t_i, \cdot)$ , we can find points  $x'_i$  sufficiently close to  $x_i$  such that

$$|v(t_i, x'_i) - v(t_i, x_i)| < 1/N, \quad (t_i - t_{i-1}, x'_i - x'_{i-1}) \in \Gamma,$$

and such that  $v_\nu(t_i, x'_i) \rightarrow v(t_i, x'_i)$  for every  $i$ . This yields the estimate

$$\begin{aligned} \sum_{i=1}^N |v(t_i, x_i) - v(t_{i-1}, x_{i-1})| &\leq \sum_{i=1}^N \{ |v(t_i, x'_i) - v(t_{i-1}, x'_{i-1})| + |v(t_i, x'_i) - v(t_i, x_i)| \\ &\quad + |v(t_i, x'_{i-1}) - v(t_{i-1}, x_{i-1})| \} \\ &\leq 2 + \limsup_{\nu \rightarrow \infty} \sum_{i=1}^N |v_\nu(t_i, x_i) - v_\nu(t_{i-1}, x_{i-1})| \\ &= 2 + \mathcal{O}(1) \cdot \text{Tot.Var.}\{\bar{v}\}, \end{aligned}$$

proving the lemma.

Together with  $\Gamma$  we now consider a strictly smaller cone, say

$$\Gamma' \doteq \{(t, x); t > 0, a' < x/t < b'\}. \quad (4.10)$$

with  $a < a' < b' < b$ . A standard theorem in real analysis states that a BV function of one real variable admits left and right limits at every point. We now prove an analogous result for functions with bounded directional variation.

**Lemma 3** Let  $v = v(t, x)$  be a function with bounded directional variation along the cone  $\Gamma$  in (4.5), and consider the smaller cone  $\Gamma' \subset \Gamma$  in (4.10), with  $a < a' < b' < b$ . Then, at every point  $P = (t, x)$  there exist the directional limits

$$v^+(P) \doteq \lim_{Q \rightarrow P, Q-P \in \Gamma'} v(Q), \quad v^-(P) \doteq \lim_{Q \rightarrow P, P-Q \in \Gamma'} v(Q). \quad (4.11)$$

**Proof.** If the first limit does not exist, we can find two sequences  $Q'_\nu \rightarrow P$ ,  $Q''_\nu \rightarrow P$  with  $Q'_\nu - P \in \Gamma'$ ,  $Q''_\nu - P \in \Gamma'$  for every  $\nu \geq 1$ , along which the function  $v$  converges to distinct limits:

$$v(Q'_\nu) \rightarrow v', \quad v(Q''_\nu) \rightarrow v'',$$

with  $v' \neq v''$ . Since  $\Gamma'$  is strictly smaller than  $\Gamma$ , by induction we can select two subsequences

$$Q'_{\nu(1)}, Q'_{\nu(2)}, \dots \quad Q''_{\nu(1)}, Q''_{\nu(2)}, \dots$$

such that

$$Q'_{\nu(j)} - Q''_{\nu(j)} \in \Gamma, \quad Q''_{\nu(j)} - Q'_{\nu(j+1)} \in \Gamma$$

for every  $j$ . In this case

$$\lim_{N \rightarrow \infty} \sum_{i=1}^N |v(Q'_{\nu(j)}) - v(Q''_{\nu(j)})| = \infty,$$

in contrast with the assumption of bounded directional variation. This proves the existence of the first limit in (4.11). The second one is entirely similar.

Next, we recall some results on differential equations with discontinuous right hand sides. Let  $f = f(t, x)$  be a bounded function. By a *Caratheodory solution* of the O.D.E.

$$\dot{x}(t) = f(t, x(t)) \quad (4.12)$$

we mean an absolutely continuous function  $t \mapsto x(t)$  which satisfies the equation (4.12) at a.e. time  $t$ .

In the case where  $f$  is discontinuous, it is well known that the Cauchy problem may have no Caratheodory solutions. One can then relax the concept of solution, introducing multivalued regularizations of  $f$ . For example, consider the multifunction

$$F(t, x) \doteq \bigcap_{\varepsilon > 0} \overline{\text{co}} \{f(s, y); |s - t| \leq \varepsilon, |y - x| \leq \varepsilon\} \quad (4.13)$$

where  $\overline{\text{co}}$  denotes the convex closure of a set. Following [H], by a *Krasovskii solution* of (4.12) we mean an absolutely continuous function  $t \mapsto x(t)$  which satisfies the differential inclusion

$$\dot{x}(t) \in F(t, x(t)) \quad (4.14)$$

at a.e. time  $t$ . Another concept of solution, proposed by Filippov, relates to the multifunction

$$F^*(t, x) \doteq \bigcap_{\varepsilon > 0} \bigcap_{\text{meas}(\mathcal{N})=0} \overline{\text{co}} \{f(s, y); |s - t| \leq \varepsilon, |y - x| \leq \varepsilon, (s, y) \notin \mathcal{N}\}, \quad (4.15)$$

obtained as in (4.13), neglecting the behavior of  $f$  of sets of measure zero. An absolutely continuous function  $t \mapsto x(t)$  which satisfies a.e. the differential inclusion

$$\dot{x}(t) \in F^*(t, x(t)) \quad (4.16)$$

is called a *Filippov solution* of (4.12). Notice that  $F^* \subseteq F$ . Moreover, the multifunction  $F^*(t, x)$  is not affected if the function  $f$  is modified on sets of measure zero.

Under the only assumption that  $f$  is bounded, it is well known that the multifunctions  $F, F^*$  are both upper semicontinuous, with compact convex values [AC]. Hence the Cauchy problem

$$\dot{x}(t) = f(t, x(t)), \quad x(s) = y \quad (4.17)$$

admits at least one solution, according to the definitions of Filippov and of Krasovskii. In the case where the function  $f$  has directionally bounded variation, a much stronger result can be proved.

**Lemma 4** *Assume that the function  $f$  has bounded directional variation along the cone  $\Gamma$  in (4.5). Moreover, assume that*

$$a < a' \leq f(t, x) \leq b' < b$$

*for all  $t, x$ . Then the Cauchy problem (4.17) has a unique Caratheodory solution, depending Lipschitz continuously on the initial data  $(s, y)$ . Such a solution is also the unique Krasovskii and Filippov solution of the Cauchy problem.*

**Proof.** The existence, uniqueness and continuous dependence of the Caratheodory solution was proved in [B1]. For directionally continuous vector fields, the equivalence between Caratheodory, Filippov and Krasovskii solutions was shown in [B2], p.26.

We conclude this section by proving a simple result from non-smooth analysis.

**Lemma 5** *Consider a Lipschitz continuous function  $V = V(t, x)$  and call  $(\phi, \psi) \doteq (V_t, V_x)$  its partial derivatives, defined at a.e. point  $(t, x)$ . Let  $\Gamma$  be the cone at (4.5). Assume that, at a given point  $(\bar{t}, \bar{x})$  there exists the directional limit*

$$(\bar{\phi}, \bar{\psi}) = \lim_{\substack{(t,x) \rightarrow (\bar{t}, \bar{x}) \\ (t-\bar{t}, x-\bar{x}) \in \Gamma}} (\phi(t, x), \psi(t, x)) . \quad (4.18)$$

*Moreover, consider a continuous function  $t \mapsto x(t)$  which is differentiable at  $t = \bar{t}$  and assume*

$$x(\bar{t}) = \bar{x}, \quad \dot{x}(\bar{t}) \in ]a, b[ .$$

*Then the composite function admits the one-sided derivative*

$$\lim_{h \rightarrow 0^+} \frac{V(\bar{t} + h, x(\bar{t} + h)) - V(\bar{t}, \bar{x})}{h} = \bar{\phi} + \bar{\psi} \cdot \dot{x}(\bar{t}) . \quad (4.19)$$



**Proof.** From the theory of generalized gradients [C] it follows that, for  $h > 0$  small,

$$V(\bar{t} + h, x(\bar{t} + h)) - V(\bar{t}, \bar{x}) \in h \cdot \bar{c} \bar{o} \{ \phi(t, x); \bar{t} < t < t + h, (t - \bar{t}, x - \bar{x}) \in \Gamma \} \\ + h \cdot \bar{c} \bar{o} \{ \psi(t, x); \bar{t} < t < t + h, (t - \bar{t}, x - \bar{x}) \in \Gamma \} \cdot [x(t) - \bar{x}].$$

Letting  $h \rightarrow 0+$  and using (4.18) one obtains (4.19).

## 5 Optimal feedback strategies

The analysis in Section 3 has identified conditions which ensure that the system of conservation laws (2.10) is strictly hyperbolic in a neighborhood of a point  $p = (p_1, p_2, \dots, p_n)$ . In this case, assuming that the terminal condition (2.11) has small total variation, one can apply Glimm's theorem and obtain the global existence of a weak solution. We shall now prove that the components of this solution determine a family of feedback strategies  $u_i = U_i^*(t, x)$ , which provide a Nash equilibrium solution to the non-cooperative differential game.

**Theorem 1** *Consider the differential game (2.1)-(2.2), where the cost functions  $h_i$  are smooth and satisfy the convexity assumption (2.3). In connection with the functions  $\phi_j$  at (2.5), let  $p^* \doteq (p_1^*, \dots, p_n^*)$  be a point where the assumptions of Lemma 1 are satisfied. Then there exists  $\delta > 0$  such that the following holds. If*

$$\|g'_i - p_i^*\|_{L^\infty} < \delta, \quad \text{Tot. Var. } \{g'_i(\cdot)\} < \delta \quad (5.1)$$

*then for any  $T > 0$  the terminal value problem (2.10)-(2.11) has a weak solution  $p : [0, T] \times \mathbb{R} \mapsto \mathbb{R}^n$ . The (possibly discontinuous) feedback controls  $U_j^*(t, x) \doteq u_j^*(p(t, x))$  defined at (2.5) provide a Nash equilibrium solution to the differential game. The trajectories  $t \mapsto x(t)$  are Lipschitz continuous functions of the initial data  $(\tau, y)$ .*

**Proof.** The proof will be given in several steps.

**Step 1.** By the assumptions, the system of conservation laws

$$\frac{\partial}{\partial t} v_i - \frac{\partial}{\partial x} H_i(v_1, \dots, v_n) = 0 \quad (5.2)$$

is strictly hyperbolic in a neighborhood of the point  $p^*$ . Given the initial data  $v_i(0, x) = g'_i(x)$  with sufficiently small total variation, by Proposition 1 the Cauchy problem admits a weak solution  $v = v(t, x)$ , defined for all  $t \geq 0$ . Reversing time, we thus obtain a weak solution  $p(t, x) \doteq v(T - t, x)$  of the terminal value problem (2.10)-(2.11). For each time  $t$ , the map  $x \mapsto p(t, x)$  has small total variation. Its pointwise values can be uniquely assigned by the convention

$$p(t, x) = \lim_{y \rightarrow x^+} p(t, y). \quad (5.3)$$

**Step 2.** By strict hyperbolicity and continuity, there exists  $\varepsilon > 0$  such that, for all  $p$  in the  $\varepsilon$ -neighborhood

$$\Omega_\varepsilon^* \doteq \{p; |p - p^*| \leq \varepsilon\} \subset \mathbb{R}^n \quad (5.4)$$

the following holds. The characteristic speeds for (2.10) range inside disjoint intervals

$$\lambda_j(p) \in [\lambda_j^-, \lambda_j^+]. \quad (5.5)$$

Moreover, the speed  $\dot{x}$  at (2.1) remains bounded away from all characteristic speeds. Namely, there exists an index  $k \in \{1, \dots, n-1\}$  and numbers  $\epsilon > 0$  and  $a < b$  such that

$$\lambda_k^+ < a - \epsilon < b + \epsilon < \lambda_{k+1}^-, \quad (5.6)$$

and

$$f_0 + \sum_j u_j^*(p_j) \in [a + \epsilon, b - \epsilon] \quad (5.7)$$

whenever  $p \in \Omega_\varepsilon^*$ .

Together with the cone  $\Gamma$  at (4.5) we now define

$$\begin{aligned} \Gamma_\epsilon^+ &\doteq \{(t, x); t > 0, a - \epsilon \leq x/t \leq b + \epsilon\}, \\ \Gamma_\epsilon^- &\doteq \{(t, x); t > 0, a + \epsilon \leq x/t \leq b - \epsilon\}. \end{aligned}$$

Clearly,  $\Gamma_\epsilon^- \subset \Gamma \subset \Gamma_\epsilon^+$ . By Lemma 2, each  $p_i = p_i(t, x)$  has bounded variation in the direction of the cone  $\Gamma$ . By the assumptions, the maps  $p_j \mapsto u_j^*(p_j)$  in (2.5) are locally Lipschitz continuous. Hence, for  $i = 1, \dots, n$ , all the composed maps  $(t, x) \mapsto u_i^*(p_i(t, x))$  also have bounded directional variation along the cone  $\Gamma$ . By (5.7) we can thus apply Lemma 4, showing that the Cauchy problem for the O.D.E.

$$\dot{x}(t) = f_0 + \sum_j u_j^*(p_j(t, x)) \quad (5.8)$$

has a unique Caratheodory (equivalently: Filippov or Krasovskii) solution, depending Lipschitz continuously on initial data  $(\tau, y)$  in (1.2).

**Step 3.** We now construct the value functions  $V_i$ , corresponding to the feedback strategies  $U_j^*(t, x) \doteq u_j^*(p_j(t, x))$ . For  $j = 1, \dots, n$ , define the cost functions

$$h_j^*(t, x) \doteq h_j(u_j^*(p_j(t, x))).$$

Given a point  $(\tau, \xi)$ , let  $t \mapsto x(t; \tau, \xi)$  be the trajectory of (5.8) passing through  $(\tau, \xi)$ . For each  $i = 1, \dots, n$  we define

$$V_i(\tau, \xi) \doteq \int_\tau^T h_i^*(t, x(t; \tau, \xi)) dt + g_i(x(T; \tau, \xi)). \quad (5.9)$$

By the same arguments as in [B1] one can show that the functions  $V_i = V_i(t, x)$  are Lipschitz continuous, hence a.e. differentiable. At every point where the differential exists, by construction one has

$$\frac{\partial V_i}{\partial t} + \dot{x} \cdot \frac{\partial V_i}{\partial x} = -h_i^*(t, x). \quad (5.10)$$

To derive further properties of the gradient of  $V_i$ , fix a time  $\tau$  and any two points  $\xi' < \xi''$ . We claim that

$$V_i(\tau, \xi'') - V_i(\tau, \xi') = \int_{\xi'}^{\xi''} p_i(\tau, x) dx. \quad (5.11)$$

Indeed, let  $x'(\cdot)$  and  $x''(\cdot)$  be the two trajectories of (5.8) which start from the initial points  $x'(\tau) = \xi'$  and  $x''(\tau) = \xi''$  respectively. Consider the region  $\Delta \subset \mathbb{R}^2$  defined as

$$\Delta \doteq \{(t, x); \tau \leq t \leq T, x'(t) \leq x \leq x''(t)\}.$$

Applying the divergence theorem to the vector field  $\mathbf{v} \doteq (p_i, H_i)$  on the domain  $\Delta$  and using the conservation equation (2.10), we obtain

$$\begin{aligned} \int_{x'(T)}^{x''(T)} p_i(T, x) dx &= \int_{\xi'}^{\xi''} p_i(\tau, x) dx + \int_{\tau}^T \{ [p_i \cdot \dot{x}' + h_i^*(t, x'(t))] - p_i \cdot \dot{x}' \} dt \\ &\quad - \int_{\tau}^T \{ [p_i \cdot \dot{x}'' + h_i^*(t, x''(t))] - p_i \cdot \dot{x}'' \} dt. \end{aligned}$$

Observing that  $p_i(T, x) = g_{i,x}(x) = V_{i,x}(T, x)$ , we conclude

$$g_i(x''(T)) - g_i(x'(T)) = \int_{\xi'}^{\xi''} p_i(\tau, x) dx + \int_{\tau}^T [h_i^*(t, x'(t)) - h_i^*(t, x''(t))] dt.$$

The two above equalities yield (5.11). Since  $\xi', \xi''$  are arbitrary, this in turn implies

$$V_{i,x} = p_i \quad (5.12)$$

at a.e. point  $(t, x)$ . Together with (5.10), this yields

$$V_{i,t} = -p_i \cdot \left( f_0 + \sum_j u_j^*(p_j) \right) - h_i^*. \quad (5.13)$$

Therefore, the value functions  $(V_1, \dots, V_n)$  satisfy a.e. the system of Hamilton-Jacobi equations

$$V_{i,t} + V_{i,x} \cdot \left( f_0 + \sum_j u_j^*(V_{j,x}) \right) + h_i(u_i^*(V_{i,x})) = 0. \quad (5.14)$$

We recall that  $u_j^* = u_j^*(p_j)$  are the optimal control values defined at (2.5).

**Step 4.** We now conclude the proof, showing that the feedback strategies  $U_j^*(t, x) \doteq u_j^*(p_j(t, x))$  represent a Nash equilibrium solution. Fix an index  $i \in \{1, \dots, n\}$  and consider the optimal control problem for the  $i$ -th player:

$$\min_{z(\cdot)} \left\{ \int_{\tau}^T h_i(z(t)) dt + g_i(T, x(T)) \right\}, \quad (5.15)$$

$$\dot{x}(t) = f_0 + \sum_{j \neq i} U_j^*(t, x) + z(t) \quad x(\tau) = y. \quad (5.16)$$

We claim that the minimum cost is precisely  $V_i(\tau, y)$ . Consider any absolutely continuous trajectory  $x : [\tau, T] \mapsto \mathbb{R}$  with  $x(\tau) = y$ . It suffices to show that, for a.e.  $t \in [\tau, T]$ ,

$$\frac{d}{dt}V_i(t, x(t)) \geq -h_i(z(t)), \quad (5.17)$$

where the control function  $z(\cdot)$  implemented by the  $i$ -th player is

$$z(t) \doteq \dot{x}(t) - f_0 - \sum_{j \neq i} U_j^*(t, x).$$

Indeed, if (5.17) holds then

$$\int_{\tau}^T h_i(z(t)) dt + g_i(T, x(T)) \geq \int_{\tau}^T \left\{ -\frac{d}{dt}V_i(t, x(t)) \right\} dt + V_i(T, x(T)) = V_i(\tau, y),$$

as claimed.

We now give a proof of (5.17), assuming that the total variation of the functions  $g_j'(\cdot)$  is sufficiently small. In connection with the vector  $p^* = (p_1^*, \dots, p_n^*)$  considered in (5.1), define the constant controls

$$\omega_j^* \doteq u_j^*(p_j^*).$$

Moreover, recalling (5.13), set

$$q_i^* \doteq -p_i^* \cdot \left( f_0 + \sum_j \omega_j^* \right) - h_i(\omega_i^*).$$

Choose  $\varepsilon_1 > 0$  small enough so that, if  $|u_j - \omega_j^*| \leq \varepsilon_1$  for all  $j$ , then

$$\dot{x} = f_0 + \sum_j u_j \in [a + \varepsilon, b - \varepsilon]. \quad (5.18)$$

Observe that our definitions imply

$$q_i^* + p_i^* \cdot \left( f_0 + \sum_{j \neq i} \omega_j^* \right) + \min_{\omega} \{ p_i^* \omega + h_i(\omega) \} = 0.$$

By the strict convexity of the cost function  $h_i$  at (2.5), there exists  $\delta' > 0$  such that

$$q_i^* + p_i^* \cdot \left( f_0 + \sum_{j \neq i} \omega_j^* \right) + p_i^* \omega + h_i(\omega) > \delta'$$

whenever  $|\omega - \omega_i^*| \geq \varepsilon_1$ . By continuity, there exists  $\varepsilon_2 > 0$  such that, if

$$|q_j - q_j^*| \leq \varepsilon_2, \quad |p_j - p_j^*| \leq \varepsilon_2 \quad j = 1, \dots, n, \quad (5.19)$$

$$|\omega - \omega_i^*| \geq \varepsilon_1, \quad |u_j - \omega_j^*| \leq \varepsilon_2 \quad j \neq i, \quad (5.20)$$

then

$$q_i + p_i \cdot \left( f_0 + \sum_{j \neq i} u_j \right) + p_i \omega + h_i(\omega) > 0. \quad (5.21)$$

Choosing  $\delta > 0$  in (5.1) sufficiently small, we can assume that the partial derivatives  $p_j = V_{j,x}$ ,  $q_j = V_{j,t}$  satisfy a.e. all the bounds in (5.19). Moreover, for  $j \neq i$ , the functions  $u_j = u_j^*(p_j)$  satisfy the bounds in (5.20). Observe that a.e. time  $\bar{t} \in [0, T]$  is in the Lebesgue set of all three measurable functions

$$z(t), \quad \frac{d}{dt}x(t), \quad \frac{d}{dt}V_i(t, x(t)).$$

(see [Fo], p.92). Choose any such Lebesgue point  $\bar{t}$  and call  $\bar{x} = x(\bar{t})$ . To prove (5.17) we consider two alternatives.

**Case 1:**  $|z(\bar{t}) - \omega_i^*| \leq \varepsilon_1$ .

In this case (5.18) holds. Define the “one-sided” partial derivatives of  $V_i$  at  $(\bar{t}, \bar{x})$

$$(\bar{\phi}_i, \bar{\psi}_i) = \lim_{\substack{(t,x) \rightarrow (\bar{t}, \bar{x}) \\ (t-\bar{t}, x-\bar{x}) \in \Gamma}} (V_{i,t}(t, x), V_{i,x}(t, x)). \quad (5.22)$$

Notice that these directional limits exist, because of (5.12)-(5.13) and the directional continuity of all functions  $p_j$ . Since (5.14) holds almost everywhere, we have

$$\bar{\phi}_i + \bar{\psi}_i \left( f_0 + \sum_{j \neq i} u_j^*(\bar{\psi}_j) \right) + \min_{\omega} \{ \bar{\psi}_i \omega + h_i(\omega) \} = 0. \quad (5.23)$$

By the assumptions, the function  $t \mapsto V_i(t, x(t))$  is differentiable at  $t = \bar{t}$ . Its derivative can be computed by taking the one-sided limit in (4.19). Using Lemma 5 together with (5.23) we obtain

$$\begin{aligned} \left. \frac{d}{dt}V_i(t, x(t)) \right|_{t=\bar{t}} &= \bar{\phi}_i + \bar{\psi}_i \dot{x}(t) \\ &= -h_i(z(t)) + \{ \bar{\psi}_i z(t) + h_i(z(t)) \} - \min_{\omega} \{ \bar{\psi}_i \omega + h_i(\omega) \} \\ &\geq -h_i(z(t)). \end{aligned}$$

Hence (5.17) holds.

**Case 2:**  $|z(\bar{t}) - \omega_i^*| > \varepsilon_1$ . In this case, by a non-smooth version of the chain rule [C], there exist numbers

$$\phi_i \in \overline{\text{co}} \{ V_{i,t}(t, x); t \in [0, T], x \in \mathbb{R} \},$$

$$\psi_i \in \overline{\text{co}} \{ V_{i,x}(t, x); t \in [0, T], x \in \mathbb{R} \},$$

such that, at  $t = \bar{t}$ ,

$$\left. \frac{d}{dt}V_i(t, x(t)) \right|_{t=\bar{t}} = \phi_i + \psi_i \dot{x}(\bar{t}). \quad (5.24)$$

The previous assumptions now imply

$$|\phi_i - q_j^*| \leq \varepsilon_2, \quad |\psi_i - p_i^*| \leq \varepsilon_2.$$

Hence, by (5.21),

$$\left. \frac{d}{dt} V_i(t, x(t)) \right|_{t=\bar{t}} = \phi_i + \psi_i \left( f_0 + \sum_{j \neq i} u_j^*(p_j) \right) + \psi_i z(\bar{t}) > -h_i(z(\bar{t})), \quad (5.25)$$

showing that (5.17) holds also in this case. This completes the proof of Theorem 1. We remark that, if the other players adopt the feedback strategies  $u_j = U_j^*(t, x)$ , the choice  $u_i = U_i^*(t, x)$  is the unique optimal strategy for the  $i$ -th player.

## 6 Concluding Remarks

In this final section we point out some possible extensions of our previous results. Consider a differential game with the more general form

$$\dot{x} = f_0 + \sum_{i=1}^n f_i(\tilde{u}_i) \quad (6.1)$$

and cost functionals

$$J_i \doteq \int_{\tau}^T \tilde{h}_i(\tilde{u}_i(t)) dt + g_i(x(T)). \quad (6.2)$$

Assume that each  $f_i$  is a homeomorphism from a (possibly unbounded) open interval  $]a_i, b_i[$  into  $\mathbb{R}$ , with a smooth inverse  $f_i^{-1} : \mathbb{R} \mapsto ]a_i, b_i[$ . Then the reparametrization of the control functions  $u_i \doteq f_i(\tilde{u}_i)$  puts the system (6.1)-(6.2) in the standard form (2.1)-(2.2), with  $h_i(\omega) \doteq \tilde{h}_i(f_i^{-1}(\omega))$ . Of course, the key assumption (2.3) must now be carefully checked.

If the functions  $f_i$  in (6.1) and the running costs  $h_i$  in (6.2) also depend on  $x$ , then the corresponding system of conservation laws (1.9) will also depend on the space variable  $x$ . Assuming that the system is strictly hyperbolic, one can then use the results in [DH], and obtain the local existence of weak solutions, on a time interval  $[0, T]$  suitably small. A similar analysis as in the previous sections would now provide the existence of a Nash equilibrium solution in feedback form, but only locally in time.

Another possible extension of our results is to the case where the data  $g_i'(\cdot)$  have large total variation. Using the local existence theorem of Schochet [Sc], one can still construct a weak solution to the system of conservation laws (2.10), at least on a short time interval  $[0, T]$ . For large BV solutions, however, checking that the feedbacks  $U_i^*(t, x) \doteq u^*(p_i(t, x))$  at (2.5) yield a Nash equilibrium solution to the differential game will require a more accurate analysis. Furthermore, it is not clear whether, for large BV initial data, the solution to the system of conservation laws (5.2) can blow up in finite time. For

general hyperbolic systems this can indeed happen [J]. The particular form of the flux functions  $H_i$ , however, may prevent such blow up. To understand the matter, a more detailed analysis is again required.

The basic assumption in Theorem 1 was the hyperbolicity of the Hamiltonian system, in a neighborhood of the reference point  $p^*$ . When this condition is violated, searching for a Nash equilibrium in feedback form leads to an elliptic Cauchy problem. It is well known that this is ill posed [Lx]. Indeed, by elementary Fourier analysis one checks that even the constant solutions are linearly unstable. It thus appears that, in the elliptic regime, the model provided by non-cooperative games must be revised. A concept of “partially cooperative” solution should be considered, in order to recover the well posedness of the problem. This will be the content of a forthcoming paper [BS].

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