L^1 STABILITY FOR ENTROPY SOLUTIONS OF NONLINEAR DEGENERATE PARABOLIC CONVECTION-DIFFUSION EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

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ABSTRACT. We propose a Kružkov-type entropy condition for nonlinear degenerate parabolic equations with discontinuous coefficients. We establish L^1 stability, and thus uniqueness, for weak solutions satisfying the entropy condition, provided that the flux function satisfies a so called "crossing condition" and the solution satisfies a technical condition regarding the existence of traces at the jump points in the coefficients. In some important cases, we prove the existence of traces directly from the proposed entropy condition. We show that limits generated by the Engquist-Osher finite difference scheme and front tracking (for the hyperbolic equation) satisfy the entropy condition, and are therefore unique. By combining the uniqueness and L^1 stability results of this paper with previously established existence results [27, 28], we show that the initial value problem studied herein is well-posed in some important cases. Our class of equations contains conservation laws with discontinuous coefficients as well as a certain type of singular source term.

CONTENTS

1. INTRODUCTION

In this paper we are interested in entropy conditions and uniqueness for nonlinear degenerate parabolic convection-diffusion initial value problems of the type

$$\begin{cases} u_t + f(\gamma(x), u)_x = A(u)_{xx}, & (x,t) \in \Pi_T = \mathbf{R} \times (0,T), \\ u(x,0) = u_0(x), & x \in \mathbf{R}, \end{cases} \tag{1.1}$$

where $T > 0$ is fixed; $u(x,t)$ is the scalar unknown function that is sought; u_0 is a bounded and integrable initial function; and the flux function $f(\gamma, u)$ and the discontinuous coefficient vector $\gamma(x)$ are given functions to be described below.

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Among our assumptions concerning the solution u is that it is essentially bounded, providing us with a compact set $\mathcal{U}$ such that

$$u(x,t) \in \mathcal{U} \text{ for a.e. } (x,t) \in \Pi_T. \tag{1.2}$$

Regarding the diffusion function $A : \mathbf{R} \to \mathbf{R}$, we assume that it belongs to $\mathrm{Lip}(\mathcal{U})$ with Lipschitz constant $\|A'\|$ and that the following *degenerate parabolic* condition holds:

$$A(\cdot) \text{ is nondecreasing with } A(0) = 0.$$

Note that with this condition $A(\cdot)$ may be "flat", and thus one often refers to (1.1) as a mixed hyperbolic-parabolic problem. To simplify matters a bit, we assume (as we did in [28]) that the set where A degenerates (is constant) is a finite collection of disjoint intervals:

$$A'(w) = 0, \qquad \forall w \in \bigcup_{i=1}^{M'} [\alpha_i, \beta_i],$$

where $\alpha_i < \beta_i$, $i = 1, \dots, M'$, $M' \geq 1$. On these intervals, (1.1) acts as a pure hyperbolic problem. We assume that A is non-degenerate (i.e., strictly increasing) off these intervals, that is,

$$A'(w) > 0, \qquad \forall w \notin \bigcup_{i=1}^{M'} [\alpha_i, \beta_i],$$

so that (1.1) acts as a parabolic problem on $\mathcal{U} \setminus \bigcup_i [\alpha_i, \beta_i]$. Throughout this paper, we assume that there is at least one interval $[\alpha_i, \beta_i]$ on which A' is zero, so that the problem (1.1) may possess discontinuous solutions (which is the real difficulty here). We stress that with our assumptions on $A(\cdot)$ the hyperbolic conservation law is included in our setup:

$$u_t + f(\gamma(x), u)_x = 0. \tag{1.3}$$

Hyperbolic PDEs like (1.1) occur in many applications and have been widely studied in recent years, both from a mathematical and numerical point of view [1, 3, 5, 12, 13, 19, 22, 23, 24, 30, 31, 33, 34, 35, 36, 37, 38, 39, 40, 41], see also the introductory parts of [27, 28] for a review of some of this activity. Most of the results obtained for (1.3) are new, and they are presented in Section 5.

In (1.1), the connective flux function $f : \mathbf{R}^p \times \mathbf{R} \to \mathbf{R}$ has a spatial dependence through the vector-valued parameter

$$\gamma(x) = (\gamma_1(x), \dots, \gamma_p(x)).$$

The spatially varying coefficient $\gamma = (\gamma_1, \dots, \gamma_p) : \mathbf{R} \to \mathbf{R}^p$ is assumed to be piecewise C^1 with finitely many jumps (in γ and γ'), located at $\xi_1 < \xi_2 < \cdots < \xi_M$. We will assume that each γ_ν ($\nu = 1, \dots, p$) has a bounded derivative, $|\gamma_\nu'(x)| \leq \beta_\nu$, away from the points of discontinuity, with bounded (also by β_ν) one sided limits at the jumps. To indicate the magnitude of differences in the parameter vector, we use the notation

$$|\gamma - \tilde{\gamma}| := \sum_{\nu=1}^{p} |\gamma_\nu - \tilde{\gamma}_\nu| .$$

We will assume that each γ_ν belongs to $BV(\mathbf{R})$. The total variation of the vector γ is defined as

$$\mathrm{TV}(\gamma) := \sum_{\nu=1}^{p} \mathrm{TV}(\gamma_\nu) .$$

Finally, each component of $\gamma(x)$ is assumed to be bounded, i.e.,

$$\gamma(x) \in \Gamma := \prod_{\nu=1}^{p} \left[\underline{\gamma}_\nu, \overline{\gamma}_\nu\right] \subset \mathbf{R}^p, \qquad \forall x \in \mathbf{R}.$$

We should mention that our reason for working with vector-valued coefficients (this is not typical of the existing literature) is their natural occurrence in certain recent applications, see [4, 7, 5].

The convective flux $f : \Gamma \times \mathbf{R} \to \mathbf{R}$ is assumed to be Lipschitz continuous in each variable:

$$|f(\gamma, u) - f(\gamma, v)| \leq \|f_u\| \, |u - v| , \qquad \forall\, \gamma \in \Gamma, \quad \forall\, u, v \in \mathcal{U},$$

$$|f(\gamma,u)-f(\tilde{\gamma},u)| \le \|f_\gamma\|\,|\gamma-\tilde{\gamma}|\,, \qquad \forall\, u \in \mathcal{U}, \quad \forall\, \gamma,\tilde{\gamma} \in \Gamma.$$

We shall also need the following technical assumptions:

$$f(\gamma(x),0) \in L^2(\mathbf{R}),$$

and

$$\sum_{\nu=1}^{p} |f_\gamma(\gamma_\nu,u)-f_\gamma(\gamma_\nu,v)| \le \|f_{\gamma u}\|\,|u-v|\,, \qquad \forall\, u,v \in \mathcal{U}, \quad \forall\, \gamma \in \Gamma.$$

The conditions just stated for the "data" u_0, f, γ, A of the problem (1.1) will be standing assumptions throughout this paper. Additional conditions will be given wherever we need them.

Independently of the smoothness of $\gamma(x)$, if $A'(w)$ is allowed to become zero for some values of w, solutions to (1.1) are not necessarily smooth and weak solutions must be sought. A *weak solution* is here defined as follows:

Definition 1.1 (weak solution). *A function $u(x,t) \in L^1(\Pi_T)\cap L^\infty(\Pi_T)$ is a weak solution of the initial value problem* (1.1) *if it satisfies the following conditions:*

($\mathbf{D}$.1) *$A(u)$ is continuous and $A(u)_x \in L^\infty(\Pi_T)$.*

($\mathbf{D}$.2) *For all test functions $\phi \in \mathcal{D}(\Pi_T)$,*

$$\iint_{\Pi_T} \Big(u\phi_t + f(\gamma(x),u)\phi_x + A(u)\phi_{xx}\Big)\,dt\,dx = 0. \tag{1.4}$$

($\mathbf{D}$.3) *The initial condition is satisfied in the following strong L^1 sense:*

$$\operatorname*{ess\,lim}_{t\downarrow 0} \int_{\mathbf{R}} |u(x,t)-u_0(x)|\;dx \to 0.$$

In view of available existence theory [27, 28], it is natural to impose ($\mathbf{D}$.1), but it is also necessary for parts of our analysis. Thanks to ($\mathbf{D}$.1), we can replace (1.4) by

$$\iint_{\Pi_T} \Big(u\phi_t + \big(f(\gamma(x),u) - A(u)_x\big)\phi_x\Big)\,dt\,dx = 0. \tag{1.5}$$

If $A'(w)$ is zero on a whole interval $[\alpha,\beta]$, then (weak) solutions may be discontinuous and they are not uniquely determined by their initial data. Consequently, an entropy condition must be imposed to single out the physically correct solution. If γ is "smooth", a weak solution u satisfies the *entropy condition* if for all convex C^2 functions $\eta : \mathbf{R} \to \mathbf{R}$,

$$\begin{aligned} &\eta(u)_t + q(\gamma(x),u)_x + r(u)_{xx} \\ &\qquad + \gamma'(x)\cdot\big(\eta'(u)f_\gamma(\gamma(x),u) - q_\gamma(\gamma(x),u)\big) \le 0 \quad \text{in } \mathcal{D}'(\Pi_T), \end{aligned} \tag{1.6}$$

where $q : \mathbf{R}^p \times \mathbf{R} \to \mathbf{R}$ and $r : \mathbf{R} \to \mathbf{R}$ are defined by

$$q_u(\gamma(x),u) = \eta'(u)f_u(\gamma(x),u), \qquad r'(u) = \eta'(u)A'(u).$$

By a standard limiting argument, (1.6) implies that the Kružkov-type entropy condition

$$\begin{aligned} &|u-c|_t + \big[\operatorname{sign}(u-c)(f(\gamma(x),u)-f(\gamma(x),c))\big]_x + |A(u)-A(c)|_{xx} \\ &\qquad + \operatorname{sign}(u-c)f(\gamma(x),c)_x \le 0 \quad \text{in } \mathcal{D}'(\Pi_T) \end{aligned} \tag{1.7}$$

holds for all $c \in \mathbf{R}$. In what follows we will often use the notation

$$F(\gamma,u,c) := \operatorname{sign}(u-c)(f(\gamma,u)-f(\gamma,c)) \tag{1.8}$$

for the Kružkov entropy flux appearing in (1.7). Here the sign function is defined by

$$\operatorname{sign}(w) := \begin{cases} -1, & \text{if } w<0,\\ 0, & \text{if } w=0,\\ 1, & \text{if } w>0. \end{cases}$$

The entropy condition (1.7) goes back to the works by Kružkov [32], Vol'pert [43], and Vol'pert and Hudjaev [45]. Existence, uniqueness and stability results for entropy solutions of strongly degenerate parabolic equations with smooth coefficients can be found in [2, 8, 9, 10, 25, 46, 45, 44], among which we highlight the recent work by Carrillo [8].

In the class of $BV(\Pi_T)$ entropy solutions u ($u \in BV(\Pi_T)$ if and only if u_x and u_t are finite measures on Π_T), Wu and Yin [46] (see also Vol'pert and Hudjaev [45] and Volpert[44]) have derived jump conditions that generalize the well-known ones for BV entropy solutions of scalar conservation laws [43].

The notion of entropy solution described above breaks down when $\gamma(x)$ is discontinuous. We suggest here instead the following definition of an entropy solution:

Definition 1.2 (entropy solution). *A weak solution u of the initial value problem*(1.1) *is called an entropy solution, if the following Kružkov-type entropy inequality holds for all $c \in \mathbf{R}$ and all test functions $0 \le \phi \in \mathcal{D}(\Pi_T)$:*

$$\begin{aligned}&\iint_{\Pi_T} \Big(|u-c|\,\phi_t + \operatorname{sign}(u-c)(f(\gamma(x),u) - f(\gamma(x),c))\phi_x + |A(u)-A(c)|\,\phi_{xx}\Big)\,dt\,dx \\ &\quad - \iint_{\Pi_T\setminus\{\xi_m\}_{m=1}^M} \operatorname{sign}(u-c) f(\gamma(x),c)_x \phi\,dt\,dx \\ &\quad + \int_0^T \sum_{m=1}^M \left| f\left(\gamma\left(\xi_m+\right),c\right) - f\left(\gamma\left(\xi_m-\right),c\right)\right| \phi(\xi_m,t)\,dt \ge 0. \end{aligned} \tag{1.9}$$

Throughout this paper we will use the notation

$$f(\gamma(x),c)_x = \gamma'(x)\cdot f_\gamma(\gamma(x),c) = \sum_{\nu=1}^p \gamma_\nu'(x) f_{\gamma_\nu}(\gamma(x),c).$$

Again thanks to (**D**.1), we may replace $|A(u)-A(c)|\,\phi_{xx}$ in (1.9) by

$$-\left|A(u)-A(c)\right|_x \phi_x, \quad -\operatorname{sign}(A(u)-A(c))A(u)_x\phi_x, \quad \text{or} \quad -\operatorname{sign}(u-c)A(u)_x\phi_x.$$

Using this, (1.9) becomes

$$\begin{aligned}&\iint_{\Pi_T} \Big(|u-c|\,\phi_t + \operatorname{sign}(u-c)\big(f(\gamma(x),u) - f(\gamma(x),c) - A(u)_x\big)\phi_x\Big)\,dt\,dx \\ &\quad - \iint_{\Pi_T\setminus\{\xi_m\}_{m=1}^M} \operatorname{sign}(u-c) f(\gamma(x),c)_x \phi\,dt\,dx \\ &\quad + \int_0^T \sum_{m=1}^M \left| f\left(\gamma\left(\xi_m+\right),c\right) - f\left(\gamma\left(\xi_m-\right),c\right)\right| \phi(\xi_m,t)\,dt \ge 0. \end{aligned} \tag{1.10}$$

The chief goal of this paper is to prove that entropy solutions are stable in the L^1 norm, and thus unique. However, we will only be able to prove uniqueness provided the convective flux function $f(\gamma,u)$ satisfies an additional condition, which we coin the *crossing condition*, and a technical assumption on the existence of the right and left traces of the solution at the jumps in $\gamma(x)$ (see Assumption 1.2 below). We next discuss the crossing condition in some detail. For a jump in $\gamma(x)$ located at $x=\xi_m$, we will use the notation

$$\gamma_- = \lim_{x\uparrow\xi_m} \gamma(x), \qquad \gamma_+ = \lim_{x\downarrow\xi_m}\gamma(x).$$

For a fixed pair of parameters γ_- and γ_+, associated with a jump in $\gamma(x)$, the graphs of $f(\gamma_-,u)$ and $f(\gamma_+,u)$ can cross. Flux crossings do occur in important applications [12, 13, 24] and introduce some significant difficulties. There are also some important situations where no such flux crossing can occur. One example is the flux

$$f(\gamma(x),u) = \gamma_1(x)g(u),$$

where $\gamma_1(x)$ (scalar) and $g(u)$ do not change sign [27, 30, 40]. Another example is

$$f(\gamma(x),u) = g(u) + \gamma_1(x),$$

which results in a scalar conservation law with a singular source term [20, 41], see (1.13) below and Section 5. The approach of the present paper seems to apply most naturally to the situation

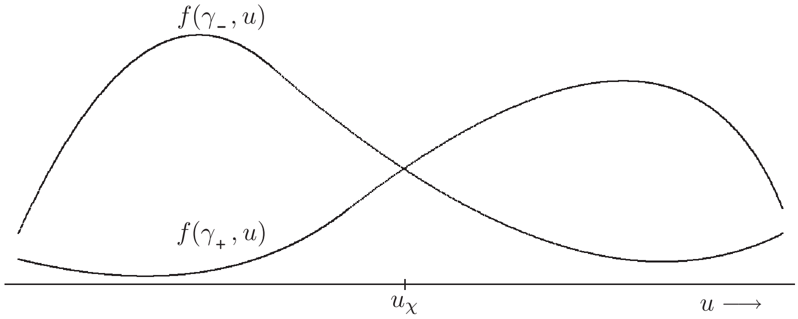

FIGURE 1. Crossing condition. The graph of $f(\gamma_-, u)$ lies above the graph of $f(\gamma_+, u)$ for u to the left of the crossing point u_χ.

where there are no flux crossings. However, we find that with a little extra effort, we are able to address flux crossings satisfying the following restriction.

Assumption 1.1 (crossing condition)**.** *For any jump in γ with associated left and right limits (γ_-, γ_+), we require that for any states u and v, the following crossing condition must hold:*

$$f(\gamma_+, u) - f(\gamma_-, u) < 0 < f(\gamma_+, v) - f(\gamma_-, v) \Longrightarrow u < v. \tag{1.11}$$

Geometrically, the crossing condition requires that either the graphs of $f(\gamma_-, \cdot)$ and $f(\gamma_+, \cdot)$ do not cross, or if they do, the graph of $f(\gamma_-, \cdot)$ lies above the graph of $f(\gamma_+, \cdot)$ to the left of any crossing point, see Figure 1 for an illustration. It is clear that there cannot be more than one crossing. A nontrivial example that satisfies the crossing condition is the ideal clarifier-thickener model, which is studied in a separate paper [7], and this provides the motivation for allowing (admittedly limited) crossings. We emphasize that we have imposed the crossing condition only to rule out situations that we cannot handle with our entropy condition (1.9). It seems that some additional condition is needed when the crossing condition is not satisfied (see also the discussion in Section 7). Our L^1 stability argument relies on jump conditions that relate limits from the right and left of the entropy solution u at jumps in the spatially varying coefficient $\gamma(x)$. More specifically, we use a Rankine-Hugoniot condition expressing conservation across each jump, and also an entropy jump inequality which is a consequence of the entropy inequality (1.9). Some type of one-sided limits (from both the right and left) are required in order for these conditions to make sense. This brings us to our next assumption.

Assumption 1.2 (existence of traces)**.** Let $u = u(x,t)$ be an entropy solution to the initial value problem (1.1). For $m = 1, \dots, M$, we assume that $u(\cdot, t)$ and $A(u)_x(\cdot, t)$ admit right and left traces at $x = \xi_m$, denoted by $u(\xi_m\pm, t)$ and $A(u)_x(\xi_m\pm, t)$, respectively. Let $W : \Pi_T \to \mathbf{R}$ be a function that belongs to $L^\infty(\Pi_T)$. By the right and left traces of $W(\cdot, t)$ at a point $x = x_0 \in \mathbf{R}$ we understand functions $t \mapsto W(x_0\pm, t) \in L^\infty(0,T)$ that satisfy for a.e. $t \in (0,T)$

$$\operatorname*{ess\,lim}_{x \downarrow x_0} |W(x,t) - W(x_0+, t)| = 0, \qquad \operatorname*{ess\,lim}_{x \uparrow x_0} |W(x,t) - W(x_0-, t)| = 0.$$

We remark that in many instances we are indeed able to prove that Assumption 1.2 holds in reasonable function classes, see Section 3.

From the point of view of degenerate parabolic equations (those not reducing to conservation laws), the well-posedness theory for the case of a discontinuous flux is still quite undeveloped, and so even with the rather strong restriction imposed by the crossing condition, this is one of the first uniqueness results that we are aware of. In [27] we studied the equation

$$u_t + (k(x)g(u))_x = A(u)_{xx} \tag{1.12}$$

with $k : \mathbf{R} \to \mathbf{R}$ possibly discontinuous and $g(u)$ genuinely nonlinear, but not necessarily convex or concave. Our flux crossing condition applies very naturally to equations of this type, specifically the case where neither k nor g has any sign changes. In [28] we allowed the convective flux to take the form $f(\gamma, u)$, with γ a scalar parameter, but now imposed a concavity condition on the mapping $u \mapsto f(\gamma, u)$. In those papers our attention was focused on establishing existence of weak solutions, first [27] using compensated compactness for a sequence of regularized equations, and then [28] using a difference scheme of the type discussed in this paper. In this paper, we will concentrate on the uniqueness issue, simply assuming that our approximations converge to a weak solution. This allows us to consider a more general class of equations than in [27, 28]. We emphasize that the existence and/or convergence issue has not yet been resolved in all cases for these wider classes.

Our emphasis in this paper is on weak solutions produced as the limits of a monotone difference scheme (although we also discuss the front tracking approach). However, for this purpose, we could also have used the regularization method applied in [27], which we now briefly illustrate on an equation of the form (1.12): Let u^ε be the unique classical solution of uniformly parabolic equation

$$u_t^\varepsilon + (k^\varepsilon(x) f(u^\varepsilon))_x = A(u^\varepsilon)_{xx} + \varepsilon u_{xx}^\varepsilon,$$

where k^ε is a smooth approximation to k for each $\varepsilon > 0$ (see Section 2 of [27] for precise statements). We proved in [27] that $u^\varepsilon \to u$ with u being a weak solution of (1.12). Although we did not prove it in [27], since each u^ε satisfies an entropy inequality of the form (1.7) it is not hard to check that the limit function u satisfies the entropy inequality (1.9) (we leave the details to the reader). Hence the results obtained herein also apply to solutions generated by the regularization method.

The present paper can be seen as continuing the investigation which we began in [27, 28], with the goal being a rather general well-posedness theory for nonlinear degenerate parabolic equations with discontinuous coefficients (containing the hyperbolic equations as special cases). Roughly speaking, the main results obtained in the present paper can be stated as follows:

Main Theorem. *With Assumptions 1.1 and 1.2, entropy solutions of the initial value problem* (1.1) *are L^1 stable, that is, for any two entropy solutions v and u with initial data v_0 and u_0, respectively, we have for a.e. $t > 0$*

$$\|v(\cdot, t) - u(\cdot, t)\|_{L^1(\mathbf{R})} \le C \|v_0 - u_0\|_{L^1(\mathbf{R})}$$

for a finite constant C. Bounded a.e. limits of the difference scheme described in Section 4 are entropy solutions, as are bounded a.e. limits of the regularization method appearing in [27]. *In the purely hyperbolic setting, limit solutions produced by the front tracking algorithm described in Section 6 are also entropy solutions. For the specific problems studied in* [27, 28, 41]*, strong traces at jumps in γ exist, i.e., Assumption 1.2 holds. Consequently, these problems are well-posed if the crossing condition (Assumption 1.1) is satisfied.*

Among the implications of our analysis is a new well-posedness result for conservation laws with source term:

$$u_t + g(u)_x = a'(x). \tag{1.13}$$

For this equation, Greenberg, Leroux, Baraille, and Noussair [20] proved existence and uniqueness for the case of convex g, with the source term a assumed piecewise smooth, but allowed to have jumps. This was accomplished by finding an explicit solution to the Riemann problem, where a jump in both a and u are allowed. The correct entropy conditions for the solution across such a jump were also found, from which it was shown that the semi-group of piecewise continuous solutions is L^1 contractive. Our results, as they apply to the problem (1.13), can be seen as a generalization of the existence and uniqueness results of [20] to the situation where g is non-convex and solutions are not necessarily piecewise smooth.

We close this section by giving a brief historical overview of the existing uniqueness results for problems with discontinuous coefficients, most of them dealing with the pure hyperbolic case, that is to say $A' \equiv 0$. For a more complete historical overview (including existence results), we refer to [27, 28]. As mentioned above, the presence of discontinuities in the coefficient γ presents substantial analytical difficulties. In the hyperbolic setting, Klingenberg and Risebro [30] observed that the Kružkov entropy inequality does not make sense in this situation, and used a so-called wave entropy inequality to prove uniqueness for an initial value problem that was the purely hyperbolic version of (1.1). Here the flux was of the form $f(\gamma(x), u) = \gamma_1(x) g(u)$ with a scalar coefficient $\gamma_1(x) > 0$ and a strictly concave nonlinearity $g(u) > 0$, so there were no flux crossings. For essentially the same problem, Towers [40] suggested an entropy condition similar to ours and established that a monotone difference scheme converges to a weak solution satisfying this entropy condition. Moreover, he proved uniqueness within the class of piecewise smooth entropy solutions. Baiti and Jenssen [1] introduced a Kružkov-type entropy condition (different from ours) and proved existence and L^1 stability for $L^1 \cap L^\infty$ entropy solutions. Their approach was however restricted to the "non-resonance" case where $u \mapsto f(\gamma, u)$ is monotone, in which case BV solutions exist. Another approach is to prove uniqueness within the class of solutions that are the limits of an equation with a smoothed coefficient γ^ε, as the smoothing parameter ε tends to zero. Klingenberg and Risebro [31] and Klausen and Risebro [29] used this approach for the case of zero diffusion, and Karlsen, Risebro, and Towers [27] used it for degenerate parabolic equations of the type (1.1). Finally, Seguin and Vovelle [38] studied a special case of the purely hyperbolic equation (1.3) with the flux function taking the form $k(x)g(u)$, so there were no flux crossings, with

$$g(u) = u(1-u), \quad k(x) = \begin{cases} k_L, & x < 0, \\ k_R, & x > 0, \end{cases}$$

for some $k_L, k_R \in \mathbf{R}$. The authors proved uniqueness of L^∞ entropy solutions, i.e., general weak solutions satisfying the purely hyperbolic variant of the entropy inequality (1.9), by the Kružkov method [32]. Note that in this case, the last integral term in (1.9) simplifies to $\int_0^T |k_R - k_L| g(c)\phi(0,t)\,dt$. Independently of Assumptions 1.1 and 1.2, the entropy condition in [40] is not sufficient for proving uniqueness of L^∞ solutions when the coefficient is varying continuously between the jump discontinuities. The entropy condition (1.9) picks out a unique solution also when the (vector-valued) coefficient $\gamma(x)$ is piecewise C^1 and possibly varying continuously between jump discontinuities.

The remaining part of this paper is organized as follows: In Section 2 we analyze in detail the behavior of an entropy solution at a jump in the parameter vector $\gamma(x)$. Starting from Theorem A.1 in the appendix, this analysis allows us to prove our main L^1 stability and uniqueness result, which is stated in Theorem 2.1. Section 3 concerns Assumption 1.2, the existence of traces. By way of demonstrating that this is a reasonable assumption, we take a specific case, and show that in this case strong traces exist for sufficiently time regular solutions of (1.1), primarily as a consequence of the entropy inequality (1.9). In Section 4 we describe a monotone difference scheme of Engquist-Osher type, and show that limits generated by this scheme satisfy the entropy inequality (1.9). This allows us to derive Theorem 4.1 stating that the initial value problem (1.1) is well-posed for an important case where the scheme is known to converge. In Section 5 we discuss (and improve some of) our results in the context of conservation laws with discontinuous coefficients as well as a certain type of singular source term. In Section 6 we discuss a front tracking method for the conservation law, while in Section 7 we make a final remark about the crossing condition. In Appendix A we prove a Kružkov-type integral inequality for the difference of two entropy solutions for test functions that vanish near the discontinuity points of the coefficients (Theorem A.1). This inequality is the starting point for proving Theorem 2.1 in Section 2, and its proof relies on recent advances [8] regarding "doubling of variables" analysis for second order PDEs (for first order PDEs this analysis is classical [32]). Finally, in Appendix B we prove a lemma regarding the initial condition, which is used in Section 5.

2. Entropy conditions at jumps in γ and L^1 stability

The ultimate goal is to prove Theorem 2.1, which establishes L^1 stability of entropy solutions. To accomplish this, we must deal with the effect of discontinuities in the solution u that are caused by the jumps at $\{\xi_m\}_{m=1}^M$ in the spatially dependent parameter vector γ. Our analysis starts with a number of technical lemmas concerning certain limits from the right and left at a jump in γ. Once we have completed this preliminary analysis, we can prove the key facts here, which are the Rankine-Hugoniot condition (Lemma 2.4) and the entropy conditions (Lemmas 2.6 and 2.7). The Rankine-Hugoniot condition is a consequence of the weak formulation (1.4), while the entropy conditions result from the Kružkov-type entropy inequality (1.9). Finally, we use these relationships to prove one more technical lemma (Lemma 2.9), and then Theorem 2.1.

Let $W = W(x)$ be any function on $\mathbf{R}$, and fix a point $x_0 \in \mathbf{R}$. We use the following notations:

$$\mathrm{L}\lim_{x \downarrow x_0} W(x) := \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{x_0}^{x_0+\varepsilon} W(x)\, dx,$$
$$\mathrm{L}\lim_{x \uparrow x_0} W(x) := \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{x_0-\varepsilon}^{x_0} W(x)\, dx.$$

We will employ the following lemma several times.

Lemma 2.1. *Let $W \in L^\infty(\Pi_T)$, and fix a point $x_0 \in \mathbf{R}$. If the right and left traces $t \mapsto W(x_0\pm, t)$ exist in the sense of Assumption 1.2, then for a.e. $t \in (0,T)$ they also exist as right and left traces in the sense of Lebesgue points in L^1:*

$$\mathrm{L}\lim_{x \downarrow x_0} W(x,t) = W(x_0+, t), \qquad \mathrm{L}\lim_{x \uparrow x_0} W(x,t) = W(x_0-, t).$$

Proof. We prove the first limit as follows:

$$\begin{aligned}
&\frac{1}{\varepsilon} \int_{x_0}^{x_0+\varepsilon} |W(x,t) - W(x_0+,t)|\, dx \\
&\quad \le \frac{1}{\varepsilon} \int_{x_0}^{x_0+\varepsilon} \operatorname*{ess\,sup}_{y \in (x_0, x_0+\varepsilon)} |W(y,t) - W(x_0+,t)|\, dx \\
&\quad = \operatorname*{ess\,sup}_{y \in (x_0, x_0+\varepsilon)} |W(y,t) - W(x_0+,t)| \to 0 \text{ as } \varepsilon \downarrow 0.
\end{aligned}$$

The second limit is proved in a similar way. □

Lemma 2.2. *Let u and v be a pair of entropy solutions and suppose Assumption 1.2 holds for both u and v. Let F be the Kružkov entropy flux defined in* (1.8). *Fix one of the jumps in γ located at $x = \xi_m$. Then for a.e. $t \in (0,T)$*

$$\begin{aligned}
&\mathrm{L}\lim_{x \downarrow \xi_m} F(\gamma(x), u(x,t), v(x,t)) = F(\gamma(\xi_m+), u(\xi_m+,t), v(\xi_m+,t)), \\
&\mathrm{L}\lim_{x \uparrow \xi_m} F(\gamma(x), u(x,t), v(x,t)) = F(\gamma(\xi_m-), u(\xi_m-,t), v(\xi_m-,t)),
\end{aligned} \tag{2.1}$$

$$\begin{aligned}
&\mathrm{L}\lim_{x \downarrow \xi_m} \left|A(u) - A(v)\right|_x (x,t) \\
&= \begin{cases} \sigma(\xi_m+,t)\Big(A(u)_x(\xi_m+,t) - A(v)_x(\xi_m+,t)\Big), & \text{if } A(u(\xi_m+,t)) \neq A(v(\xi_m+,t)), \\ \left|A(u)_x(\xi_m+,t) - A(v)_x(\xi_m+,t)\right|, & \text{if } A(u(\xi_m+,t)) = A(v(\xi_m+,t)). \end{cases}
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
&\mathrm{L}\lim_{x \uparrow \xi_m} \left|A(u) - A(v)\right|_x (x,t) \\
&= \begin{cases} \sigma(\xi_m-,t)\Big(A(u)_x(\xi_m-,t) - A(v)_x(\xi_m-,t)\Big), & \text{if } A(u(\xi_m-,t)) \neq A(v(\xi_m-,t)), \\ -\left|A(u)_x(\xi_m-,t) - A(v)_x(\xi_m-,t)\right|, & \text{if } A(u(\xi_m-,t)) = A(v(\xi_m-,t)), \end{cases}
\end{aligned} \tag{2.3}$$

where

$$\sigma(\xi_m+,t) = \operatorname{sign}(u(\xi_m+,t) - v(\xi_m+,t)), \qquad \sigma(\xi_m-,t) = \operatorname{sign}(u(\xi_m-,t) - v(\xi_m-,t)).$$

Proof. Fix any $t \in (0,T)$ for which all the displayed traces in the lemma exist. Lemma 2.1 applies with $W = u, v$ and since F is Lipschitz continuous in each of its variables, it is clear that (2.1) holds. We will now compute the limits appearing in (2.2) and (2.3). We start with the case where $A(u(\xi_m+,t)) \neq A(v(\xi_m+,t))$. By continuity of $A(u)$, we must have

$$\operatorname{sign}(A(u(x,t)) - A(v(x,t))) = \operatorname{sign}(A(u(\xi_m+,t)) - A(v(\xi_m+,t))) \neq 0$$

for $x > \xi_m$ sufficiently close to ξ_m. Since A is nondecreasing, this implies that

$$\operatorname{sign}(A(u(x,t)) - A(v(x,t))) = \sigma(\xi+,t)$$

for $x > \xi_m$ sufficiently close to ξ_m. Thus for ε positive but sufficiently small,

$$\frac{1}{\varepsilon}\int_{\xi_m}^{\xi_m+\varepsilon} \left|A(u)-A(v)\right|_x(x,t)\,dx = \sigma(\xi+,t)\frac{1}{\varepsilon}\int_{\xi_m}^{\xi_m+\varepsilon}\Big(A(u)_x(x,t)-A(v)_x(x,t)\Big)\,dx.$$

Applying Lemma 2.1 with $W = A(u)_x - A(v)_x$, and letting $\varepsilon \downarrow 0$ clearly gives the first case of (2.2).

Now take the case where $A(u(\xi_m+,t)) = A(v(\xi_m+,t))$. It will simplify matters if we introduce the notation $B(x,t) := A(u(x,t)) - A(v(x,t))$. Since $B(\xi_m+,t) = 0$, we have

$$\begin{aligned}\frac{1}{\varepsilon}\int_{\xi_m}^{\xi_m+\varepsilon} \left|B(x,t)\right|_x\,dx &= \left|B(\xi_m+\varepsilon,t)\right| - \left|B(\xi_m+,t)\right| \\ &= \frac{1}{\varepsilon}\left|B(\xi_m+\varepsilon,t)-B(\xi_m+,t)\right| = \left|\frac{1}{\varepsilon}\int_{\xi_m}^{\xi_m+\varepsilon} B_x(x,t)\,dx\right|.\end{aligned}$$

Applying Lemma 2.1 with $W = |B|_x$, letting $\varepsilon \downarrow 0$ gives

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{\xi_m}^{\xi_m+\varepsilon} |B|_x(x,t)\,dx = \left|B_x(\xi_m+,t)\right|,$$

which is the second case of (2.2). A similar argument establishes the analogous limits from the left which appear in (2.3). $\square$

Lemma 2.3. *Let u be an entropy solution and suppose Assumption 1.2 holds. Let F be the Kružkov entropy flux defined in* (1.8). *Fix one of the jumps in γ located at $x = \xi_m$. For any $c \in \mathbf{R}$, we have for a.e. $t \in (0,T)$.*

$$\begin{aligned}&\operatorname{L}\lim_{x\downarrow\xi_m} F(\gamma(x),u(x,t),c) = F(\gamma(\xi_m+),u(\xi_m+,t),c),\\ &\operatorname{L}\lim_{x\uparrow\xi_m} F(\gamma(x),u(x,t),c) = F(\gamma(\xi_m-),u(\xi_m-,t),c),\end{aligned} \tag{2.4}$$

$$\operatorname{L}\lim_{x\downarrow\xi_m}\left|A(u)-A(c)\right|_x(x,t) = \begin{cases}\sigma(\xi_m+,t)A(u)_x(\xi_m+,t), & \text{if } u(\xi_m+,t)\neq c,\\ \left|A(u)_x(\xi_m+,t)\right|, & \text{if } u(\xi_m+,t)=c,\end{cases} \tag{2.5}$$

$$\operatorname{L}\lim_{x\uparrow\xi_m}\left|A(u)-A(c)\right|_x(x,t) = \begin{cases}\sigma(\xi_m-,t)A(u)_x(\xi_m-,t), & \text{if } u(\xi_m-,t)\neq c,\\ -\left|A(u)_x(\xi_m-,t)\right|, & \text{if } u(\xi_m-,t)=c,\end{cases} \tag{2.6}$$

where

$$\sigma(\xi_m-,t) = \operatorname{sign}(u(\xi_m-,t)-c), \qquad \sigma(\xi_m+,t) = \operatorname{sign}(u(\xi_m+,t)-c).$$

Proof. As in the proof of the previous lemma, (2.4) is a consequence of Lipschitz continuity of F.

If $A(u(\xi_m+,t)) \neq A(c)$, or if $u(\xi_m+,t) = c$, the proof of (2.5) is essentially the same as the proof of (2.2). To establish (2.5), the only case left to deal with is where $A(u(\xi_m+,t)) = A(c)$ but $u(\xi_m+,t) \neq c$. Assume that $u(\xi_m+,t) > c$; the case with the opposite inequality is handled similarly. Since $A(u(\xi_m+,t)) = A(c)$, we must have $u(\xi_m+,t) \in [\alpha_i,\beta_i]$, $c \in [\alpha_i,\beta_i]$ where $[\alpha_i,\beta_i]$ is one of the intervals on which $A' = 0$. If $u(\xi_m+,t) \in (\alpha_i,\beta_i)$, then

$$\frac{1}{\varepsilon}\int_{\xi_m}^{\xi_m+\varepsilon}\left|A(u)-A(c)\right|_x(x,t)\,dx = \frac{\left|A(u)-A(c)\right|(\xi_m+\varepsilon,t)}{\varepsilon} = 0,$$

for $\varepsilon > 0$ and sufficiently small, and (2.5) is proven. Thus we may assume that $u(\xi_m+,t)$ is one of the endpoints of $[\alpha_i,\beta_i]$. Since $c \in [\alpha_i,\beta_i]$ and $c < u(\xi_m+,t)$, we must have $u(\xi_m+,t) = \beta_i$, the right endpoint. For sufficiently small ε, almost all points $x \in (\xi_m,\xi_m+\varepsilon)$ will force either $u(x,t) \in [\alpha_i,\beta_i]$, or $u(x,t) \in [\beta_i,\alpha_{i+1}]$. In the former case, $A(u(x,t)) - A(\beta_i) = 0$, and in the latter case, $A(u(x,t)) - A(\beta_i) > 0$. Thus for sufficiently small, but positive ε,

$$\begin{aligned}&\frac{1}{\varepsilon}\int_{\xi_m}^{\xi_m+\varepsilon}\left|A(u)-A(c)\right|_x(x,t)\,dx\\ &= \frac{1}{\varepsilon}\int_{\xi_m}^{\xi_m+\varepsilon}\left|A(u)-A(\beta_i)\right|_x(x,t)\,dx = \frac{1}{\varepsilon}\int_{\xi_m}^{\xi_m+\varepsilon}\Big(A(u)_x - A(\beta_i)\Big)_x(x,t)\,dx.\end{aligned}$$

Using Lemma 2.1 with $W = A(u)_x$ when letting $\varepsilon \downarrow 0$, and recalling that in this case $\sigma(\xi_m+, t) = 1$, we get

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \int_{\xi_m}^{\xi_m+\varepsilon} \left|A(u) - A(c)\right|_x (x,t)\,dx = \sigma(\xi_m+, t) A(u)_x(\xi_m+, t).$$

The formula (2.6) is proven in a similar way. □

In an attempt to simplify the presentation in Lemmas 2.4, 2.5, 2.6, 2.7, and 2.8, we employ the the following notation:

$$u_\pm = u_\pm(t) = u(\xi_m\pm, t), \qquad \mathcal{A}^u_\pm = \mathcal{A}^u_\pm(t) = \mathcal{A}(\xi_m\pm, t), \qquad \mathcal{A}(x,t) = A(u)_x(x,t),$$

where it is always understood that $t \in (0,T)$ is such that the traces exist at $x = \xi_m$. We recall that the right and left traces of γ at $x = \xi_m$ are denoted by γ_+ and γ_-, respectively.

Before we continue, let us define the following compactly supported Lipschitz function ($\varepsilon > 0$)

$$\theta_\varepsilon(x) = \begin{cases} \frac{1}{\varepsilon}(\varepsilon + x), & \text{if } x \in [-\varepsilon, 0], \\ \frac{1}{\varepsilon}(\varepsilon - x), & \text{if } x \in [0, \varepsilon], \\ 0 & \text{if } |x| \geq \varepsilon, \end{cases} \tag{2.7}$$

which we will employ repeatedly.

Lemma 2.4. *Let u be an entropy solution and suppose Assumption 1.2 holds. Fix one of the jumps in γ located at $x = \xi_m$. Then the following Rankine-Hugoniot condition holds for a.e. $t \in (0,T)$:*

$$f(\gamma_+, u_+) - \mathcal{A}^u_+ = f(\gamma_-, u_-) - \mathcal{A}^u_-. \tag{2.8}$$

Proof. Let θ_ε be the function defined by (2.7). Then, since $u, f(\gamma(x), u), A(u)_x \in L^\infty(\Pi_T)$, a density argument will reveal that $\phi(x,t) = \theta_\varepsilon(x - \xi_m)\varphi(t)$ with $\varphi \in \mathcal{D}(0,T)$ can be used as an admissible test function in the weak formulation (1.5). The result is

$$\begin{aligned} &\int_0^T \int_{\mathbf{R}} u\theta_\varepsilon(x - \xi_m)\varphi'(t)\,dt\,dx \\ &\quad + \frac{1}{\varepsilon} \int_0^T \int_{\xi_m-\varepsilon}^{\xi_m} \Big(f(\gamma(x), u) - A(u)_x\Big)\varphi(t)\,dx\,dt \\ &\quad - \frac{1}{\varepsilon} \int_0^T \int_{\xi_m}^{\xi_m+\varepsilon} \Big(f(\gamma(x), u) - A(u)_x\Big)\varphi(t)\,dx\,dt = 0. \end{aligned}$$

In the limit as $\varepsilon \downarrow 0$, the contribution from the first term is zero. By Lemma 2.1 with $W = u, A(u)_x$, the bounded convergence theorem, and the Lipschitz continuity of f, the contribution from the remaining terms is

$$\int_0^T \left(f(\gamma_-, u_-) - \mathcal{A}^u_- - f(\gamma_+, u_+) + \mathcal{A}^u_+ \right) \varphi(t)\,dt = 0.$$

Since φ is an arbitrary test function on $(0,T)$, the integrand must vanish for a.e. $t \in (0,T)$. □

The following lemma provides information about the behavior of the diffusion term near a discontinuity in u due to a jump in γ. Roughly speaking, the sign of $\mathcal{A} = A(u)_x$ on either side of a discontinuity is (almost) determined by the sign of the jump in u.

Lemma 2.5. *Let u be an entropy solution and suppose Assumption 1.2 holds. Fix one of the jumps in γ located at $x = \xi_m$. The following relationships hold for a.e. $t \in (0,T)$:*

$$\operatorname{sign}(u_+ - u_-)\operatorname{sign}(\mathcal{A}^u_+) \geq 0, \quad \operatorname{sign}(u_+ - u_-)\operatorname{sign}(\mathcal{A}^u_-) \geq 0. \tag{2.9}$$

Proof. Fix a time $t \in (0,T)$ where all of the relevant right and left spatial (essential) limits exist at $x = \xi_m$. We will prove the first inequality in (2.9); the other inequality is handled in a similar way. For brevity of notation, we will suppress the dependence on t for the remainder of the proof.

If $u_+ = u_-$, the inequality is obvious, so take the case where $u_+ > u_-$. It then suffices to show that $\mathcal{A}_+^u \geq 0$. Since

$$\operatorname*{ess\,lim}_{\varepsilon \downarrow 0} u(\xi_m + \varepsilon) =: u_+ > u_-,$$

we see that

$$u(\xi_m + \varepsilon) > u_- \text{ for a.e. sufficiently small } \varepsilon > 0. \tag{2.10}$$

Applying A to both sides of (2.10), and recalling that A is nondecreasing, we have

$$A(u(\xi_m + \varepsilon)) \geq A(u_-)$$

for a.e. sufficiently small $\varepsilon > 0$. As a consequence, using the fact that $A(u_-) = A(u_+)$, we have

$$\frac{1}{\varepsilon} \int_{\xi_m}^{\xi_m+\varepsilon} A(u)_x \, dx = \frac{1}{\varepsilon} \left(A(u(\xi_m + \varepsilon)) - A(u_+) \right) \geq 0, \tag{2.11}$$

for a.e. sufficiently small $\varepsilon > 0$. Letting $\varepsilon \downarrow 0$ (along a subsequence for which (2.11) holds) yields $\mathcal{A}_+^u \geq 0$. A similar argument shows that if $u_+ < u_-$, then $\mathcal{A}_+^u \leq 0$, completing the proof. $\square$

The following lemma gives the entropy condition across a jump in γ.

Lemma 2.6. *Let u be an entropy solution and suppose Assumption 1.2 holds. Let F be the Kružkov entropy flux defined in* (1.8) *Fix one of the jumps in γ located at $x = \xi_m$. For every $c \in \mathbf{R}$, the following entropy inequality holds for a.e. $t \in (0, T)$:*

$$\begin{aligned} &\Big(F(\gamma_+, u_+, c) - \operatorname{sign}(u_+ - c)\mathcal{A}_+^u \Big) \\ &\quad - \Big(F(\gamma_-, u_-, c) - \operatorname{sign}(u_- - c)\mathcal{A}_-^u \Big) \leq \left| f(\gamma_+, c) - f(\gamma_-, c) \right|. \end{aligned} \tag{2.12}$$

Proof. We proceed as in the proof of Lemma 2.4, but this time we choose $\varphi \geq 0$. Inserting the test function $\phi(x,t) = \theta_\varepsilon(x - \xi_m)\varphi(t)$ in the entropy inequality (1.10) yields

$$\begin{aligned} &\int_0^T \int_{\mathbf{R}} |u - c| \, \theta_\varepsilon(x - \xi_m)\varphi'(t) \, dx \, dt \\ &\quad + \frac{1}{\varepsilon} \int_0^T \int_{\xi_m - \varepsilon}^{\xi_m} \Big(F(\gamma(x), u, c) - |A(u) - A(c)|_x \Big) \varphi(t) \, dx \, dt \\ &\quad - \frac{1}{\varepsilon} \int_0^T \int_{\xi_m}^{\xi_m+\varepsilon} \Big(F(\gamma(x), u, c) - |A(u) - A(c)|_x \Big) \varphi(t) \, dx \, dt \\ &\quad - \iint_{\Pi_T \setminus \{\xi_m\}_{m=1}^M} \operatorname{sign}(u - c) f(\gamma(x), c)_x \theta_\varepsilon(x - \xi_m)\varphi(t) \, dt \, dx \\ &\quad + \int_0^T \left| f\left(\gamma\left(\xi_m+\right), c\right) - f\left(\gamma\left(\xi_m-\right), c\right) \right| \varphi(t) \, dt \geq 0. \end{aligned} \tag{2.13}$$

When $\varepsilon \downarrow 0$, the the first and fourth lines of (2.13) vanish. By Lemma 2.3 and the bounded convergence theorem, the second and third lines in (2.13) converge to

$$\int_0^T \Big(F(\gamma_-, u_-, c) - \operatorname*{L\,lim}_{x \uparrow \xi_m} |A(u) - A(c)|_x - F(\gamma_+, u_+, c) + \operatorname*{L\,lim}_{x \downarrow \xi_m} |A(u) - A(c)|_x \Big) \varphi(t) \, dt. \tag{2.14}$$

Recalling that $\phi \geq 0$ was an arbitrary test function on $(0, T)$, we have that for a.e. $t \in (0, T)$,

$$\begin{aligned} &F(\gamma_+, u_+, c) - \operatorname*{L\,lim}_{x \downarrow \xi_m} |A(u) - A(c)|_x - F(\gamma_-, u_-, c) + \operatorname*{L\,lim}_{x \uparrow \xi_m} |A(u) - A(c)|_x \\ &\quad \leq \left| f\left(\gamma\left(\xi_m+\right), c\right) - f\left(\gamma\left(\xi_m-\right), c\right) \right|. \end{aligned} \tag{2.15}$$

Now fix a time $t \in (0, T)$ where (2.15) holds, and for the moment, assume that $u_+ \neq c$, $u_- \neq c$. When this is the case, we have by Lemma 2.3

$$\operatorname*{L\,lim}_{x \downarrow \xi_m} |A(u) - A(c)|_x = \operatorname{sign}(u_+ - c)\mathcal{A}_+^u, \quad \operatorname*{L\,lim}_{x \uparrow \xi_m} |A(u) - A(c)|_x = \operatorname{sign}(u_- - c)\mathcal{A}_-^u.$$

Thus (2.12) holds with the additional assumption that $u_+ \neq c$, $u_- \neq c$.

	$f(\gamma_-,c) \le f(\gamma_+,c)$	$f(\gamma_-,c) \ge f(\gamma_+,c)$
$u_- \le c \le u_+$	$f(\gamma_+,u_+) - \mathcal{A}^u_+ \le f(\gamma_+,c)$	$f(\gamma_-,u_-) - \mathcal{A}^u_- \le f(\gamma_-,c)$
$u_+ \le c \le u_-$	$f(\gamma_-,u_-) - \mathcal{A}^u_- \ge f(\gamma_-,c)$	$f(\gamma_+,u_+) - \mathcal{A}^u_+ \ge f(\gamma_+,c)$

TABLE 1. Entropy jump conditions.

To remove this restriction, take the case where $c = u_-$, the other case being similar. We then claim that (2.12) is simply a consequence of the Rankine-Hugoniot condition (2.8) and (2.9). With $c = u_-$, we start off from the left side of (2.12), which now reads

$$\operatorname{sign}(u_+ - u_-)\left(f(\gamma_+,u_+) - f(\gamma_+,u_-) - \mathcal{A}^u_+\right),$$

and calculate as follows:

$$\begin{aligned}
&\operatorname{sign}(u_+ - u_-)\left(f(\gamma_+,u_+) - f(\gamma_+,u_-) - \mathcal{A}^u_+\right)\\
&\overset{(2.8)}{=} \operatorname{sign}(u_+ - u_-)\left(f(\gamma_-,u_-) - f(\gamma_+,u_-) - \mathcal{A}^u_-\right)\\
&\le |f(\gamma_-,u_-) - f(\gamma_+,u_-)| - \operatorname{sign}(u_+ - u_-)\mathcal{A}^u_-\\
&\overset{(2.9)}{\le} |f(\gamma_-,u_-) - f(\gamma_+,u_-)|.
\end{aligned}$$

This concludes the proof of the lemma. □

The jump condition (2.12), along with the Rankine-Hugoniot condition (2.8), implies a set of geometric entropy conditions, which we collect in Table 1 and state in the next lemma.

Lemma 2.7. *With the same hypotheses as in Lemma 2.6, the appropriate inequality in Table 1 holds for all c lying between u_- and u_+.*

Proof. The proof is a study in cases. First take the case in the upper left entry of Table 1, i.e., $u_- \le c \le u_+$, $f(\gamma_-,c) \le f(\gamma_+,c)$. Assume for now that $c \in (u_-,u_+)$. In this case, inequality (2.12) becomes

$$\begin{aligned}
&f(\gamma_+,u_+) - f(\gamma_+,c) + f(\gamma_-,u_-) - f(\gamma_-,c) - \mathcal{A}^u_+ - \mathcal{A}^u_-\\
&\qquad \le f(\gamma_+,c) - f(\gamma_-,c).
\end{aligned} \tag{2.16}$$

Canceling $f(\gamma_-,c)$ from both sides, then applying the Rankine-Hugoniot condition (2.8) and dividing by two gives the upper left entry in the table.

Now assume that c coincides with one of the endpoints, say $c = u_+$, the case where $c = u_-$ being similar. Then inequality (2.12) becomes

$$f(\gamma_-,u_-) - f(\gamma_-,c) - \mathcal{A}^u_- \le f(\gamma_+,c) - f(\gamma_-,c).$$

Canceling $f(\gamma_-,c)$ from both sides and using the Rankine-Hugoniot condition (2.8) gives

$$f(\gamma_+,u_+) - \mathcal{A}^u_+ \le f(\gamma_+,c).$$

The other three entries in Table 1 are derived in an analogous way. □

Remark 2.1. The entropy inequalities in Table 1 hold independently of whether the flux satisfies the crossing condition.

Lemma 2.8. *Let u be an entropy solution, and suppose Assumption 1.2 holds for u. Fix one of the jumps in γ located at $x = \xi_m$. If $u_- \ne u_+$, then*

$$A'(w) = 0, \qquad \text{for } w \text{ between } u_- \text{ and } u_+, \tag{2.17}$$

and thus $A(\cdot)$ is constant on the interval connecting u_- to u_+:

$$A(w) = A(u_-) = A(u_+), \qquad \text{for } w \text{ between } u_- \text{ and } u_+. \tag{2.18}$$

Proof. From continuity (**D**.1),

$$\int_{u_-}^{u_+} A'(w)\,dw = A(u_+) - A(u_-) = 0.$$

Since $u_- \neq u_+$, this proves (2.17). Assertion (2.18) is immediate from (2.17). $\square$

Before proceeding to the main theorem, we collect in the following lemma some technical facts that will be required in its proof.

Lemma 2.9. *Let u and v be a pair of entropy solutions and suppose Assumption 1.2 holds for both u and v. Fix a jump in γ located at $x = \xi_m$ and a time $t \in (0, T)$ where all of the relevant right and left traces exist. Introduce the notation*

$$\begin{aligned} u_\pm &= u_\pm(t) = u(\xi_m\pm, t), \quad \mathcal{A}^u = A(u)_x, \qquad \mathcal{A}^u_\pm = \mathcal{A}^u_\pm(t) = \mathcal{A}^u(\xi_m\pm, t),\\ v_\pm &= v_\pm(t) = v(\xi_m\pm, t), \qquad \mathcal{A}^v = A(v)_x \quad \mathcal{A}^v_\pm = \mathcal{A}^v_\pm(t) = \mathcal{A}^v(\xi_m\pm, t), \end{aligned}$$

and (as usual) $\gamma_\pm = \gamma(\xi_m\pm)$. Assume that

$$u_- > v_-, \quad u_+ < v_+.$$

If $u_+ \le u_-$, then

$$\begin{aligned} &v_- \in [u_+, u_-),\ f(\gamma_+, v_-) \ge f(\gamma_-, v_-) \Longrightarrow f(\gamma_-, v_-) - \mathcal{A}^v_- \le f(\gamma_-, u_-) - \mathcal{A}^u_-,\\ &v_+ \in (u_+, u_-],\ f(\gamma_-, v_+) \ge f(\gamma_+, v_+) \Longrightarrow f(\gamma_+, v_+) - \mathcal{A}^v_+ \le f(\gamma_+, u_+) - \mathcal{A}^u_+. \end{aligned} \tag{2.19}$$

If $v_- \le v_+$, then

$$\begin{aligned} &u_- \in (v_-, v_+],\ f(\gamma_+, u_-) \le f(\gamma_-, u_-) \Longrightarrow f(\gamma_-, v_-) - \mathcal{A}^v_- \le f(\gamma_-, u_-) - \mathcal{A}^u_-,\\ &u_+ \in [v_-, v_+),\ f(\gamma_+, u_+) \ge f(\gamma_-, u_+) \Longrightarrow f(\gamma_+, v_+) - \mathcal{A}^v_+ \le f(\gamma_+, u_+) - \mathcal{A}^u_+. \end{aligned} \tag{2.20}$$

Proof. Take the first assertion in (2.19). We claim that if v_- lies in the interior of $[u_+, u_-]$, then $\mathcal{A}^v_- = 0$. Since $v_- \in (u_+, u_-)$, there is an $\varepsilon > 0$ such that

$$u_+ < v(x, t) < u_-, \quad \text{for a.e. } x \in (\xi_m - \varepsilon, \xi_m).$$

Since A is nondecreasing, we also have

$$A(u_+) \le A(v(x, t)) \le A(u_-), \quad \text{for a.e. } x \in (\xi_m - \varepsilon, \xi_m).$$

By Lemma 2.8, $A(u_+) = A(u_-)$, and so

$$A(u_+) = A(v(x, t)) = A(u_-), \quad \text{for a.e. } x \in (\xi_m - \varepsilon, \xi_m).$$

Since $A(u)$ is continuous (see (**D**.1) of Definition 1.1), this last inequality must hold for all x in $(\xi_m - \varepsilon, \xi_m)$, implying that $\mathcal{A}^v_- = 0$.

The remaining possibility is that $v_- = u_+$. In this case, $v_- < v_+$, and hence $\mathcal{A}^v_- \ge 0$ by Lemma 2.5. In either of these cases $\mathcal{A}^v_- \ge 0$, and so by Table 1 (with $c = v_-$),

$$f(\gamma_-, v_-) - \mathcal{A}^v_- \le f(\gamma_-, v_-) \le f(\gamma_-, u_-) - \mathcal{A}^u_-.$$

The other assertion in (2.19), as well as the two assertions in (2.20) are proven in a similar way. $\square$

We are finally in a position to state and prove our main main L^1 stability theorem:

Theorem 2.1 (L^1 stability and uniqueness)**.** *Let v and u be two entropy solutions to the initial value problem (1.1) with initial data $v_0, u_0 \in L^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$, respectively. If f satisfies the crossing condition (Assumption 1.1), and we assume the existence of traces (Assumption 1.2) then, for a.e. $t \in (0, T)$,*

$$\|v(\cdot, t) - u(\cdot, t)\|_{L^1(\mathbf{R})} \le C\|v_0 - u_0\|_{L^1(\mathbf{R})}, \tag{2.21}$$

for some finite constant $C > 0$. If $\gamma'(x) = 0$ for a.e. $x \in \mathbf{R}$, then $C = 1$.

Proof. For any two entropy solutions $v = v(x,t)$ and $u = u(x,t)$ the following L^1 contraction property can be derived:

$$\begin{aligned} &-\iint_{\Pi_T} \Big(|v-u|\,\varphi_t + F(\gamma(x),v,u)\varphi_x + |A(v)-A(u)|\,\varphi_{xx} \Big)\,dt\,dx \\ &\qquad \le \text{Const.} \iint_{\Pi_T} |v-u|\,\phi\,dt\,dx, \end{aligned} \tag{2.22}$$

for any $0 \le \varphi \in \mathcal{D}\left(\Pi_T \setminus \{\xi_m\}_{m=1}^M\right)$. The proof of (2.22) is based on the "doubling of variables" argument [32, 8] and can be found in Appendix A.

We start by removing the assumption in (2.22) that ϕ vanishes near each $x = \xi_m$. For $h > 0$, introduce the Lipschitz function

$$\mu_h(x) = \begin{cases} \frac{1}{h}(x+2h), & x \in [-2h,-h], \\ 1, & x \in [-h,h], \\ \frac{1}{h}(2h-x), & x \in [h,2h], \\ 0, & |x| \ge 2h. \end{cases}$$

We now define $\Psi_h(x) = 1 - \sum_{m=1}^M \mu_h(x-\xi_m)$, and note that $\Psi_h \to 1$ in $L^1(\mathbf{R})$. Moreover, Ψ_h vanishes in a neighborhood of $x = \xi_m$, $m = 1,\dots,M$. For any $0 \le \phi \in \mathcal{D}(\Pi_T)$, we can check that $\varphi = \phi\Psi_h$ is an admissible test function in (2.22). Inserting such φ in (2.22) and then doing a spatial integration by parts, we get

$$\begin{aligned} &-\iint_{\Pi_T} \Big(|v-u|\,\phi_t\Psi_h + F(\gamma(x),v,u)\phi_x\Psi_h - \operatorname{sign}(v-u)\big(A(v)_x - A(u)_x\big)\phi_x\Psi_h \Big)\,dt\,dx \\ &\quad \underbrace{-\iint_{\Pi_T} \Big(F(\gamma(x),v,u) - |A(v)-A(u)|_x \Big)\phi(x,t)\,\Psi_h'(x)\,dt\,dx}_{=:J(h)} \\ &\quad \le \text{Const} \iint_{\Pi_T} |v-u|\,\phi\Psi_h\,dt\,dx. \end{aligned} \tag{2.23}$$

Sending $h \downarrow 0$ in (2.23) and then doing a spatial integration by parts, we end up with

$$\begin{aligned} &-\iint_{\Pi_T} \Big(|v-u|\,\phi_t + F(\gamma(x),v,u)\phi_x - |A(v)-A(u)|\,\phi_{xx} \Big)\,dt\,dx \\ &\qquad \le \text{Const} \iint_{\Pi_T} |v-u|\,\phi\,dt\,dx + \lim_{h\downarrow 0} J(h). \end{aligned} \tag{2.24}$$

Using linear changes of variables and the bounded convergence theorem, it easy to check that

$$\begin{aligned} \lim_{h\downarrow 0} J(h) &= \sum_{m=1}^M \lim_{h\downarrow 0} \frac{1}{h} \int_0^T \int_{\xi_m+h}^{\xi_m+2h} \Big(F(\gamma(x),v,u) - |A(v)-A(u)|_x \Big)\,dx\,dt \\ &\quad - \sum_{m=1}^M \lim_{h\downarrow 0} \frac{1}{h} \int_0^T \int_{\xi_m-2h}^{\xi_m-h} \Big(F(\gamma(x),v,u) - |A(v)-A(u)|_x \Big)\,dx\,dt \\ &= \sum_{m=1}^M \int_0^T \Big[F(\gamma(x),v,u) - |A(v)-A(u)|_x \Big]_{x=\xi_m-}^{x=\xi_m+} \phi(\xi_m,t)\,dt, \end{aligned} \tag{2.25}$$

where the notation indicates limits from the right and left at $x = \xi_m$.

The goal now is to prove that

$$\lim_{h\downarrow 0} J(h) \le 0. \tag{2.26}$$

It is clear that for a.e. $t \in (0,T)$, the contribution to $\lim_{h\downarrow 0} J(h)$ at the jump $x = \xi_m$ is

$$S := \Big[F(\gamma(x),v,u) - |A(v)-A(u)|_x \Big]_{x=\xi_m-}^{x=\xi_m+}. \tag{2.27}$$

Let us fix a jump in γ at $x = \xi_m$, for some $m = 1, \dots, M$, and a time $t \in (0,T)$ where all of the relevant essential right and left limits exist. If we can show that the quantity S associated with (ξ_m, t) is non-positive, we will have the desired result (2.26). We will use the notation

$$u_\pm = u_\pm(t) = u(\xi_m \pm, t), \qquad \mathcal{A}^u = A(u)_x, \qquad \mathcal{A}^u_\pm = \mathcal{A}^u_\pm(t) = \mathcal{A}^u(\xi_m \pm, t),$$
$$v_\pm = v_\pm(t) = v(\xi_m \pm, t), \qquad \mathcal{A}^v = A(v)_x \qquad \mathcal{A}^v_\pm = \mathcal{A}^v_\pm(t) = \mathcal{A}^v(\xi_m \pm, t).$$

and $\gamma_\pm = \gamma(\xi_m \pm)$. There are a number of cases to deal with.

Case 1 ($v_- = u_-$, $v_+ = u_+$). Then

$$F(\gamma_+, v_+, u_+) = 0, \quad F(\gamma_-, v_-, u_-) = 0,$$

and by Lemma 2.2, S reduces to

$$S = -\left|\mathcal{A}^u_+ - \mathcal{A}^v_+\right| - \left|\mathcal{A}^u_- - \mathcal{A}^v_-\right| \le 0,$$

completing the discussion of this case.

Case 2 ($v_- = u_-, u_+ \neq v_+$). Assume that $v_+ > u_+$. In this case

$$S = f(\gamma_+, v_+) - f(\gamma_+, u_+) - \mathcal{A}^v_+ + \mathcal{A}^u_+ - \left|\mathcal{A}^v_- - \mathcal{A}^u_-\right|, \tag{2.28}$$

where we have used the fact that $f(\gamma_-, v_-) = f(\gamma_-, u_-)$. Via the Rankine-Hugoniot condition, and another application of $f(\gamma_-, u_-) = f(\gamma_-, v_-)$, we get

$$f(\gamma_+, v_+) - f(\gamma_+, u_+) - \mathcal{A}^v_+ + \mathcal{A}^u_+ = -\mathcal{A}^v_- + \mathcal{A}^u_-.$$

Substituting this into (2.28) gives

$$S = -\mathcal{A}^v_- + \mathcal{A}^u_- - \left|\mathcal{A}^v_- - \mathcal{A}^u_-\right| \le 0.$$

The situation where $v_+ < u_+$ is handled similarly.

Case 3 ($v_+ = u_+$, $u_- \neq v_-$). The proof of this case is similar to that of the previous case, and is omitted.

Case 4 ($u_- < v_-$, $u_+ < v_+$). In this case (2.27) becomes

$$\begin{aligned} S = &\Big(f(\gamma_+, v_+) - f(\gamma_+, u_+) - \mathcal{A}^v_+ + \mathcal{A}^u_+\Big) \\ &- \Big(f(\gamma_-, v_-) - f(\gamma_+, u_+) - \mathcal{A}^v_- + \mathcal{A}^u_-\Big), \end{aligned}$$

which equals zero, by the Rankine-Hugoniot condition (2.8).

Case 5 ($u_- > v_-$, $u_+ > v_+$). As in the preceding case, the quantity S vanishes, by a similar calculation.

Case 6 ($u_- > v_-$, $u_+ < v_+$). In this case, (2.27) becomes

$$\begin{aligned} S &= f(\gamma_-, v_-) + f(\gamma_+, v_+) - f(\gamma_-, u_-) - f(\gamma_+, u_+) \\ &\quad - \mathcal{A}^v_- - \mathcal{A}^v_+ + \mathcal{A}^u_- + \mathcal{A}^u_+ \\ &= 2(f(\gamma_+, v_+) - \mathcal{A}^v_+) - 2(f(\gamma_+, u_+) - \mathcal{A}^u_+) && (2.29) \\ &= 2(f(\gamma_-, v_-) - \mathcal{A}^v_-) - 2(f(\gamma_-, u_-) - \mathcal{A}^u_-), && (2.30) \end{aligned}$$

where (2.29) and (2.30) follow from the Rankine-Hugoniot condition (2.8).

There are two sub-cases to consider, depending on whether or not $f(\gamma_+, v_-) \ge f(\gamma_-, v_-)$. We will show that either

$$f(\gamma_-, v_-) - \mathcal{A}^v_- \le f(\gamma_-, u_-) - \mathcal{A}^u_- \tag{2.31}$$

or

$$f(\gamma_+, v_+) - \mathcal{A}^v_+ \le f(\gamma_+, u_+) - \mathcal{A}^u_+ \tag{2.32}$$

holds, which implies, using (2.29) or (2.30), that $S \le 0$.

Subcase A. Assume that $f(\gamma_+, v_-) \ge f(\gamma_-, v_-)$. With the assumption that $u_- > v_-, u_+ < v_+$, it is easy to check that one of

$$u_+ \le v_- < u_- \quad \text{or} \quad v_+ > u_+ \ge v_- \tag{2.33}$$

must hold. If $u_+ \le v_- < u_-$, then by Lemma 2.9, inequality (2.31) holds.

Now take the case where $v_+ > u_+ \geq v_-$. If there is no flux crossing between v_- and u_+, then $f(\gamma_+, u_+) \geq f(\gamma_-, u_+)$. If there is a crossing between v_- and u_+, then the crossing condition (Assumption 1.1) forces v_- to be on the right side. Since $u_+ \geq v_-$, u_+ must also be to the right of the crossing point. Again, $f(\gamma_+, u_+) \geq f(\gamma_-, u_+)$, and so in either case, Lemma 2.9 yields (2.32).

Subcase B. Assume that $f(\gamma_+, v_-) \leq f(\gamma_-, v_-)$. It follows from the assumption $u_- > v_-$, $u_+ < v_+$ that one of the following must hold:

$$u_+ < v_+ \leq u_- \quad \text{or} \quad v_- < u_- \leq v_+.$$

Take the case where $u_+ < v_+ \leq u_-$. If there is no flux crossing between v_+ and v_-, then $f(\gamma_+, v_+) \leq f(\gamma_-, v_+)$, and so Lemma 2.9 gives inequality (2.32). If there is a crossing, then v_- lies to the left and v_+ lies to the right of the crossing point, thanks to the crossing condition, Assumption 1.1. Lemma 2.7 implies that the horizontal line connecting $(v_-, f(\gamma_-, v_-) - \mathcal{A}_-^v)$ to $(v_+, f(\gamma_+, v_+) - \mathcal{A}_+^v)$ lies below the crossing. Since $u_+ \leq v_- \leq v_+ \leq u_-$, the horizontal line connecting $(u_+, f(\gamma_+, u_+) - \mathcal{A}_+^u)$ to $(u_-, f(\gamma_-, u_-) - \mathcal{A}_-^u)$ lies above the crossing, by Lemma 2.7 again. We conclude that inequality (2.31) holds in this case.

Consider the case where $v_- < u_- \leq v_+$. If there is no crossing between u_- and v_-, it follows that $f(\gamma_+, u_-) \leq f(\gamma_-, u_-)$, and so by Lemma 2.9, inequality (2.31) is satisfied.

Now suppose that there is a flux crossing between u_- and v_-. Due to the crossing condition, u_- is on the right side of the crossing point, and v_- is on the left side. If u_+ is on the right side of the crossing point, then $f(\gamma_+, u_+) \geq f(\gamma_-, u_+)$ and $v_- \leq u_+ < v_+$. Then by Lemma 2.9 inequality (2.32) holds. If u_+ is on the left side of the crossing point, then $u_+ \leq u_-$, and so the horizontal line connecting $(u_+, f(\gamma_+, u_+) - \mathcal{A}_+^u)$ to $(u_-, f(\gamma_-, u_-) - \mathcal{A}_-^u)$ lies above the crossing. At the same time, the horizontal line connecting $(v_-, f(\gamma_-, v_-) - \mathcal{A}_-^v)$ to $(v_+, f(\gamma_+, v_+) - \mathcal{A}_+^v)$ lies below the crossing. Thus inequality (2.31) is satisfied, and the proof of Case 7 is complete.

Case 7 ($u_- < v_-$, $u_+ > v_+$). The proof for this case is identical to that of the previous case; we merely switch the roles of u and v, noting that the quantity S is symmetric in u and v. Of course, wherever Lemma 2.9 is invoked, we use the version that results by everywhere interchanging u and v.

We hence conclude that $S \leq 0$, which implies that (2.26) holds since m and t were arbitrary.

Let us now conclude the proof in the standard way. Our starting point is

$$\begin{aligned} -\iint_{\Pi_T} \Big(|v-u|\, \phi_t + F(\gamma(x), v, u)\phi_x + |A(v) - A(u)|\, \phi_{xx} \Big)\, dt\, dx \\ \leq \text{Const.} \iint_{\Pi_T} |v-u|\, \phi\, dt\, dx, \end{aligned} \tag{2.34}$$

for some constant depending on f, and $\gamma'(x)$ away from discontinuities. The inequality (2.34) holds for any $0 \leq \phi \in \mathcal{D}(\Pi_T)$. For $r > 1$, let $\alpha_r : \mathbf{R} \to \mathbf{R}$ be a C^∞ function which takes values in $[0,1]$ and satisfies

$$\alpha_r(x) = \begin{cases} 1, & \text{if } |x| \leq r, \\ 0, & \text{if } |x| \geq r+1. \end{cases} \tag{2.35}$$

Fix any s_0 and s such that $0 < s_0 < s < T$. For any $\tau > 0$ and $\kappa > 0$ with $0 < s_0 + \tau < s + \kappa < T$, let $\beta_{\tau,\kappa} : [0,T] \to \mathbf{R}$ be a Lipschitz function that is linear on $[s_0, s_0+\tau] \cup [s, s+\kappa]$ and satisfies

$$\beta_{\tau,\kappa}(t) = \begin{cases} 0, & \text{if } t \in [0, s_0] \text{ or } t \in [s+\kappa, T], \\ 1, & \text{if } t \in [s_0+\tau, s]. \end{cases}$$

Then it is not hard to check via a standard regularization argument that $\phi(x,t) = \alpha_r(x)\beta_{\tau,\kappa}(t)$ is an admissible test function in (2.34). Inserting this ϕ into (2.34) gives

$$\begin{aligned}
&\frac{1}{\kappa}\int_s^{s+\kappa}\int_{\mathbf{R}} |v(x,t)-u(x,t)|\,\alpha_r(x)\,dx\,dt - \frac{1}{\tau}\int_{s_0}^{s_0+\tau}\int_{\mathbf{R}} |v(x,t)-u(x,t)|\,\alpha_r(x)\,dx\,dt \\
&\qquad \le \text{Const.} \int_{s_0}^{s+\kappa}\int_{\mathbf{R}} |v(x,t)-u(x,t)|\,\alpha_r(x)\,dx\,dt \\
&\qquad\quad + \|\alpha_r'\|_{L^\infty(\mathbf{R})}\int_{s_0}^{s+\kappa}\int_{r\le|x|\le r+1} F(\gamma(x),v(x,t),u(x,t))\,dx\,dt \\
&\qquad\quad + \|\alpha_r''\|_{L^\infty(\mathbf{R})}\int_{s_0}^{s+\kappa}\int_{r\le|x|\le r+1} |A(v(x,t))-A(u(x,t))|\,dx\,dt.
\end{aligned} \tag{2.36}$$

Sending $s_0 \downarrow 0$ and then using the triangle inequality, we get

$$\begin{aligned}
\frac{1}{\kappa}\int_s^{s+\kappa}\int_{-r}^{r} |v(x,t)-u(x,t)|\,dx\,dt &\le \int_{-r}^{r} |v_0(x)-u_0(x)|\,dx \\
&\quad + \frac{1}{\tau}\int_0^\tau\int_{-r}^{r} |v(x,t)-v_0(x)|\,dx\,dt + \frac{1}{\tau}\int_0^\tau\int_{-r}^{r} |u(x,t)-u_0(x)|\,dx\,dt \\
&\quad + \text{Const.}\int_0^{t+\tau}\int_{\mathbf{R}} |v(x,t)-u(x,t)|\,\alpha_r(x)\,dx\,dt + o\left(\frac{1}{r}\right),
\end{aligned} \tag{2.37}$$

where $o(1/r)$ is a collective symbol for terms that tend to zero as $r \uparrow \infty$ (uniformly in τ, κ). Observe that by (**D**.3) the second and third terms on the right-hand side of the inequality sign in (2.37) tend to zero as $\tau \downarrow 0$ (uniformly in r). By sending first $\tau \downarrow 0$ and second $r \uparrow \infty$, we thus produce $\frac{1}{\kappa}\int_s^{s+\kappa}\mathcal{E}(t)\,dt \le \mathcal{E}(0) + \text{Const.}\int_0^{s+\kappa}\mathcal{E}(t)\,dt$, where $\mathcal{E}(t) := \int_{\mathbf{R}}|v(x,t)-u(x,t)|\,dx$. Let s be an arbitrary Lebesgue point of the L^1 function $\mathcal{E} : (0,T) \to \mathbf{R}$. Sending $\kappa \downarrow 0$ yields $\mathcal{E}(s) \le \mathcal{E}(0) + \text{Const.}\int_0^s \mathcal{E}(t)\,dt$. Since the set of Lebesgue points of $\mathcal{E}(\cdot)$ has full measure, an application of Gronwall's inequality gives the desired result $\mathcal{E}(s) \le \text{Const.}\,\mathcal{E}(0)$ for a.e. $s \in (0,T)$, thereby finishing the proof of (2.21). □

3. Existence of traces at jumps in γ for a particular case

In Section 1 we mentioned the issue of existence of traces at jumps in γ. Via Assumption 1.2, our approach was mainly to simply assume their existence in Section 2. However, in some important situations it is possible to establish directly from the entropy inequality (1.9) that sufficiently regular entropy solutions have traces at jumps in γ. We provide such an example here. We single out this example because such entropy solutions are known to exist [28]. This is a consequence of the fact that solutions of these problems can be generated as limits of the monotone difference scheme described in Section 4, where we show that such limits are entropy solutions. As we mentioned in Section 1, the vanishing viscosity approach can also be used to produce entropy solutions. In Section 5, we discuss another method, namely front tracking, which is applicable in the purely hyperbolic setting.

We consider now the problem studied in reference [28], which is a special case of the class of equations discussed herein. In that paper, the vector of coefficients γ reduced to a scalar, and the convective flux $u \mapsto f(\gamma,u)$ had a single maximum $u^*(\gamma)$. With a slight abuse of notation, in this section we will identify the one-dimensional vector γ with its single component γ^1.

We now describe the assumptions concerning the data of the problem that we required in [28] in addition to those that we have already given in Section 1. For the convective flux function f, we assumed that

$$f(\gamma,0) = f_0 \in \mathbf{R} \text{ for all } \gamma \text{ and } f(\gamma,1) = f_1 \in \mathbf{R} \text{ for all } \gamma. \tag{3.1}$$

The purpose of this assumption is to guarantee that a solution initially in the interval $[0,1]$ remains in $[0,1]$ for all subsequent times.

We also require (in addition to the assumptions given in Section 1) the technical assumption that f_u was Lipschitz continuous as a function of γ, with Lipschitz constant $L_{u\gamma}$. For example, if $f(\gamma,u)=\gamma g(u)$, where $g\in C^1([0,1])$, this Lipschitz assumption will hold with $L_{u\gamma}=\|g'\|_\infty$.

We assumed in [28] that for each $\gamma\in[\underline{\gamma},\overline{\gamma}]$, there was a unique maximum $u^*(\gamma)\in[0,1]$ such that $f(\gamma,\cdot)$ was strictly increasing for $u<u^*(\gamma)$ and strictly decreasing for $u>u^*(\gamma)$, and that for each fixed $\gamma\in[\underline{\gamma},\overline{\gamma}]$, $|f_u(\gamma,u)|>0$ for almost all $u\in[0,1]$. In this section, we will make the simplifying assumption that u^* does not depend on γ (we used a somewhat weaker condition on u^* in [28]), an important example being the case where the flux has the form $f(\gamma,u)=\gamma g(u)$, where $g(u)$ has a single maximum.

Finally, we assumed that the initial function u_0 satisfied

$$
\begin{cases} u_0\in L^1(\mathbf{R})\bigcap BV(\mathbf{R}); \;\; u_0(x)\in[0,1]\ \forall x\in\mathbf{R};\\ A(u_0) \text{ is absolutely continuous in } \mathbf{R};\\ A(u_0)_x\in BV(\mathbf{R}). \end{cases} \tag{3.2}
$$

Our assumption that $A(u_0)$ is absolutely continuous requires that any jump in u_0 must be contained within one of the intervals $[\alpha_i,\beta_i]$ where A is constant.

We established in [28] that with the assumptions of that paper, weak solutions in the sense of Definition 1.1 exist. Although we did not explicitly state it there, such solutions also satisfy $u\in BV_t(\Pi_T)$ (this space is defined below); this is an easy consequence of the results in that paper.

Turning now to the task of establishing the existence of traces for this problem, we denote by $\mathcal{M}(\Pi_T)$ the Radon measures on Π_T. We recall that a measure μ on Π_T is a Radon measure if μ is Borel regular and $\mu(K)<\infty$ for each compact set $K\subset\Pi_T$. The space $BV(\Pi_T)$ of functions of bounded variation is defined as the set of locally integrable functions $W:\Pi_T\to\mathbf{R}$ for which $\partial_x W,\partial_t W\in\mathcal{M}(\Pi_T)$. Below we use the space $BV_t(\Pi_T)$ of locally integrable functions $W:\Pi_T\to\mathbf{R}$ for which only $\partial_t W\in\mathcal{M}(\Pi_T)$. Of course, we have $BV(\Pi_T)\subset BV_t(\Pi_T)$. We can also define the space $BV_x(\Pi_T)$ by replacing the condition $\partial_t W\in\mathcal{M}(\Pi_T)$ by $\partial_x W\in\mathcal{M}(\Pi_T)$. The following lemma will be used later:

Lemma 3.1. *Suppose $u\in BV_t(\Pi_T)$ is an entropy solution in the sense of Definition 1.2. Then, for any $c\in\mathbf{R}$,*

$$\Big(f(\gamma(x),u)-f(\gamma(x),c)-(A(u)-A(c))_x\Big)_x\in\mathcal{M}(\Pi_T).$$

and

$$\Big(\operatorname{sign}(u-c)(f(\gamma(x),u)-f(\gamma(x),c))-\left|A(u)-A(c)\right|_x\Big)_x\in\mathcal{M}(\Pi_T).$$

Proof. The first part of the lemma is a straightforward consequence of (1.5) and that the total variation of each component of $\gamma(x)$ is bounded. For $\phi\in\mathcal{D}(\Pi_T)$, define $\widetilde{\mathcal{L}}(\phi)$ as the left side of the inequality (1.10). From (1.10), the functional $\widetilde{\mathcal{L}}:\mathcal{D}(\Pi_T)\to\mathbf{R}$ is linear and nonnegative. The Riesz representation theorem then provides us with a Radon measure $\widetilde{\mu}$ on Π_T such that $\widetilde{\mathcal{L}}(\phi)=\int_{\Pi_T}\phi\,d\widetilde{\mu}$ for all $\phi\in\mathcal{D}(\Pi_T)$. Denote by $\mathcal{L}:\mathcal{D}(\Pi_T)\to\mathbf{R}$ the functional defined by

$$\mathcal{L}(\phi)=\iint_{\Pi_T}\left(|u-c|\,\phi_t+\Big(\operatorname{sign}(u-c)(f(\gamma(x),u)-f(\gamma(x),c))-\left|A(u)-A(c)\right|_x\Big)\phi_x\right)dt\,dx.$$

Denote by $\widehat{\mathcal{L}}:C_c(\Pi_T)\to\mathbf{R}$ the functional $\widehat{\mathcal{L}}:=\widetilde{\mathcal{L}}-\mathcal{L}$. Since the total variation of each component of $\gamma(x)$ is finite, there is a finite constant $C>0$ such that $\left|\widehat{\mathcal{L}}(\phi)\right|\le C$ for all $\phi\in C_c(\Pi_T)$ with $|\phi|\le 1$. The Riesz representation theorem then provides us with another Radon measure $\widetilde{\mu}$ on Π_T such that $\widehat{\mathcal{L}}(\phi)=\int_{\Pi_T}\phi\,d\widehat{\mu}$ for all $\phi\in C_c(\Pi_T)$. Setting $\mu:=\widetilde{\mu}-\widehat{\mu}$, we have

$$|u-c|_t+\Big(\operatorname{sign}(u-c)(f(\gamma(x),u)-f(\gamma(x),c))-\left|A(u)-A(c)\right|_x\Big)_x=-\mu\in\mathcal{M}(\Pi_T).$$

It is clear from this and the assumption $u_t\in\mathcal{M}(\Pi_T)$ that the second part of lemma holds true. $\square$

Before continuing we record the following technical lemma, whose proof is standard and is therefore omitted.

Lemma 3.2. *If $W = W(x,t) \in L^\infty(\Pi_T)$ and $W_x \in \mathcal{M}(\Pi_T)$, then for any $x_0 \in \mathbf{R}$ the following right and left limits exist for a.e. $t \in (0,T)$:*

$$W(x_0+,t) = \operatorname*{ess\,lim}_{x \downarrow x_0} W(x,t), \qquad W(x_0-,t) = \operatorname*{ess\,lim}_{x \uparrow x_0} W(x,t).$$

Moreover,

$$t \mapsto W(x_0 \pm, t) \in L^\infty(0,T).$$

We call $W(x_0+,t)$ and $W(x_0-,t)$ the right and left traces of $W(\cdot,t)$ at $x = x_0$, respectively.

We recall that a singular mapping Ψ that was used to establish convergence for the difference scheme discussed in [28]. Letting $\mathcal{S}(\cdot)$ denote the characteristic function for $\bigcup_{i=1}^{M'} [\alpha_i, \beta_i]$, we define Ψ by

$$\Psi(\gamma,u) = \int_0^u \mathcal{S}(w)\,|f_u(\gamma,w)|\;dw + A(u) =: \mathcal{F}(\gamma,u) + A(u). \tag{3.3}$$

In [28] we showed that the mapping $\Psi(\gamma,u)$ is strictly increasing as a function of u. Furthermore, both Ψ and $\mathcal{F}$ belong to $\mathrm{Lip}\left([\underline{\gamma},\overline{\gamma}] \times \mathbf{R}\right)$.

Lemma 3.3. *For the particular case of the initial value problem* (1.1) *discussed in* [28], *and with the additional assumption that $u^\star$ is independent of γ, entropy solutions u in the sense of Definition 1.2 satisfying $u \in BV_t(\Pi_T)$ admit right and left traces at jumps in γ.*

Proof. For $i = 1, \ldots, M'$, let $c \in \mathbf{R}$ and define

$$\alpha_i^L = u^\star \wedge \alpha_i, \quad \alpha_i^R = u^\star \vee \alpha_i, \quad \beta_i^L = u^\star \wedge \beta_i, \quad \beta_i^R = u^\star \vee \beta_i, \tag{3.4}$$

where $a \wedge b = \max(a,b)$ and $a \vee b = \min(a,b)$. Note that

$$\begin{aligned}
\left[\alpha_i^L, \beta_i^L\right] &= [\alpha_i,\beta_i] \cap [0,u^\star] \subseteq [0,u^\star]\,,\\
\left[\alpha_i^R, \beta_i^R\right] &= [\alpha_i,\beta_i] \cap [u^\star,1] \subseteq [u^\star,1]\,,\\
\left[\alpha_i^L, \beta_i^L\right] \cup \left[\alpha_i^R, \beta_i^R\right] &= [\alpha_i,\beta_i]\,.
\end{aligned}$$

We decompose the hyperbolic portion $\mathcal{F}$ of Ψ using the intervals $[\alpha_i,\beta_i]$:

$$\mathcal{F}(\gamma,u) = \sum_{i=1}^{M'} \mathcal{F}_i(\gamma,u), \qquad \mathcal{F}_i(\gamma,u) = \int_{u^\star}^u \chi_{(\alpha_i,\beta_i]}(w)\,|f_u(\gamma,w)|\;dw.$$

Now fixing any $1 \le i \le M'$, our immediate goal is to show that $\mathcal{F}_i(\gamma,u)_x \in \mathcal{M}(\Pi_T)$. Consider the two expressions

$$P(x,t,c) := \operatorname{sign}(u-c)(f(\gamma(x),u) - f(\gamma(x),c)) - \left|A(u) - A(c)\right|_x, \tag{3.5}$$

and

$$Q(x,t,c) := (f(\gamma(x),u) - f(\gamma(x),c)) - (A(u) - A(c))_x. \tag{3.6}$$

By Lemma 3.1,

$$P(x,t,c)_x \quad \text{and} \quad Q(x,t,c)_x \in \mathcal{M}(\Pi_T).$$

Adding (3.5) and (3.6), then dividing by two gives

$$\Upsilon^R(x,t,c) := \operatorname{sign}_+(u-c)(f(\gamma(x),u) - f(\gamma(x),c)) - \Big(\operatorname{sign}_+(u-c)(A(u)-A(c))\Big)_x, \tag{3.7}$$

where $\operatorname{sign}_+(w) = \max(\operatorname{sign}(w),0)$. Similarly, subtracting (3.6) from (3.5), then dividing by two gives

$$\Upsilon^L(x,t,c) := \operatorname{sign}_-(u-c)(f(\gamma(x),u) - f(\gamma(x),c)) - \Big(\operatorname{sign}_-(u-c)(A(u)-A(c))\Big)_x, \tag{3.8}$$

where $\operatorname{sign}_-(w) = \min(\operatorname{sign}(w),0)$. It is clear that for each $c \in \mathbf{R}$,

$$\Upsilon^R(x,t,c)_x, \Upsilon^L(x,t,c)_x \in \mathcal{M}(\Pi_T).$$

Next, we use the following elementary relationships

$$\operatorname{sign}_+(u-c)(f(\gamma,u)-f(\gamma,c)) = -\int_{u^*}^{u} \chi_{(c,1]}(w)\,|f_u(\gamma(x),w)|\;dw, \qquad \text{for } c\in[u^\star,1],$$
$$\operatorname{sign}_+(u-\alpha_i^R)-\operatorname{sign}_+(u-\beta_i^R) = \chi_{(\alpha_i^R,\beta_i^R]}(u),$$
$$A(u)=A(\alpha_i^R)=A(\beta_i^R), \qquad \text{for } u\in\left[\alpha_i^R,\beta_i^R\right],$$

to derive

$$\begin{aligned}\Upsilon^R(x,t,\beta_i^R)-\Upsilon^R(x,t,\alpha_i^R) &= \int_{u^*}^{u}\chi_{[\alpha_i^R,\beta_i^R]}(w)\,|f_u(\gamma(x),w)|\;dw\\ &\qquad+\Big(\chi_{[\alpha_i^R,\beta_i^R]}(u)(A(u)-A(\alpha_i^R))\Big)_x\\ &= \int_{u^*}^{u}\chi_{[\alpha_i^R,\beta_i^R]}(w)\,|f_u(\gamma(x),w)|\;dw.\end{aligned}$$

Here we have also used the fact that $\left[\alpha_i^R,\beta_i^R\right]\subseteq[\alpha_i,\beta_i]$ and recalled that A is constant on $[\alpha_i,\beta_i]$. Similar observations yield

$$\begin{aligned}\Upsilon^L(x,t,\beta_i^L)-\Upsilon^L(x,t,\alpha_i^L) &= \int_{u^*}^{u}\chi_{[\alpha_i^L,\beta_i^L]}(w)\,|f_u(\gamma(x),w)|\;dw\\ &\qquad+\Big(\chi_{[\alpha_i^L,\beta_i^L]}(u)(A(u)-A(\alpha_i^L))\Big)_x\\ &= \int_{u^*}^{u}\chi_{[\alpha_i^L,\beta_i^L]}(w)\,|f_u(\gamma(x),w)|\;dw.\end{aligned}$$

Finally, we form the combination

$$\begin{aligned}&\Upsilon^R(x,t,\beta_i^R)-\Upsilon^R(x,t,\alpha_i^R)+\Upsilon^L(x,t,\beta_i^L)-\Upsilon^L(x,t,\alpha_i^L)\\ &\quad=\int_{u^*}^{u}\Big(\chi_{[\alpha_i^R,\beta_i^R]}(w)+\chi_{[\alpha_i^L,\beta_i^L]}(w)\Big)\,|f_u(\gamma(x),w)|\;dw\\ &\quad=\int_{u^*}^{u}\chi_{[\alpha_i,\beta_i]}(w)\,|f_u(\gamma(x),w)|\;dw=\mathcal{F}_i(\gamma(x),u).\end{aligned}\tag{3.9}$$

Since $\Upsilon^{L,R}(x,t,\alpha_i^{L,R})_x$, $\Upsilon^{L,R}(x,t,\beta_i^{L,R})_x\in\mathcal{M}(\Pi_T)$, we see from (3.9) that

$$\mathcal{F}_i(\gamma(x),u)_x\in\mathcal{M}(\Pi_T), \qquad i=1,\dots,M'.$$

From our assumption that $A(u)_x\in L^\infty(\Pi_T)$, we also have $A(u)_x\in\mathcal{M}(\Pi_T)$. With the observation that $\Psi(\gamma,u)$ is a combination of $\mathcal{F}_i$ and A, we conclude that

$$\Psi(\gamma(x),u)_x\in\mathcal{M}(\Pi_T).$$

Using the facts that Ψ is Lipschitz continuous, $\gamma(x)\in\Gamma$ for all x, and $u\in L^\infty(\Pi_T)$, we also have $\Psi(\gamma(x),u)\in L^\infty(\Pi_T)$. An application of Lemma 3.2 now guarantees that $\Psi(\gamma,u)$ has right and left traces at each jump in γ. Finally, the existence of right and left traces of u now follows from the fact that Ψ is continuous and strictly increasing as a function of u. □

Lemma 3.4. *Let $u\in BV_t(\Pi_T)$ be an entropy solution in the sense of Definition 1.2 for the particular case of the initial value problem* (1.1) *discussed in* [28]*, and with the additional assumption that u^* is independent of γ. Then the functions $|A(u)-A(c)|_x$ and $A(u)_x$ admit right and left traces at $x=\xi_m$ for $m=1,\dots,M$.*

Proof. For each fixed $c\in\mathbf{R}$, the quantity

$$\Phi(x,t):=F(\gamma(x),u,c)-|A(u)-A(c)|_x$$

satisfies $\Phi_x\in\mathcal{M}(\Pi_T)$, thanks to Lemma 3.1. Since $u,A(u)_x\in L^\infty(\Pi_T)$, $\Phi\in L^\infty(\Pi_T)$. Lemma 3.2 now guarantees the existence of right and left traces for Φ. We also have traces for $F(\gamma(x),u,c)$, which follows from the existence of traces for u, along with the Lipschitz continuity of F. Now since

$$|A(u)-A(c)|_x=F(\gamma,u,c)-\Phi(x,t),$$

the existence of traces for the quantity on the left side of this equation follows from that of each of the quantities on the right side. Finally, we select $c = -\|u\|_{L^\infty(\Pi_T)}$. Recalling that A is nondecreasing, we get

$$|A(u) - A(c)|_x = (A(u) - A(c))_x = A(u)_x.$$

It is now clear that $A(u)_x$ also has strong traces at a jump in γ, since $|A(u) - A(c)|_x$ does. □

4. A monotone difference scheme

At this point, it is natural to ask whether solutions to which we can apply our L^1 stability theory exist. The purpose of this section is to present one method of constructing such entropy solutions. We will focus on the difference scheme approach, but we stress that the vanishing viscosity/smoothing approach [27] also produces entropy solutions. For conservation laws, yet another method is available, namely the front tracking approach, see Section 5.

In [28], we used a simple difference scheme for generating approximate solutions to the initial value problem (1.1) for the case of a scalar parameter $\gamma(x)$, and flux with a single maximum (see [28] for a complete list of assumptions concerning the data of the problem). Our main result was convergence to a weak solution in the sense of Definition 1.1. A multidimensional version of this scheme was used in [26] for the situation where γ was allowed to be "rough" but continuous. In this section, we use a discrete entropy inequality (see (4.9) below) derived in that paper to show that convergent limits of our scheme satisfy the analogous Kružkov-type entropy inequality (1.9), and are thus entropy solutions.

We next recall the definition of the difference scheme as it was presented in [28], generalizing slightly to the case where γ is a vector. The clarifier-thickener model discussed in [4, 7] provides an example where the vector version of the scheme has been applied. We begin by discretizing the spatial domain $\mathbf{R}$ into cells

$$I_j = [x_j - \Delta x/2, x_j + \Delta x/2) = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}),$$

where $x_j = j\Delta x$ for $j \in \mathbf{Z}$. Similarly, the time interval $[0, T]$ is is discretized via $t_n = n\Delta t$ for $n = 0, \ldots, N$, where the integer N is chosen such that $N\Delta t = T$, resulting in the time strips

$$I^n = [t_n, t_{n+1}).$$

Here $\Delta x > 0$ and $\Delta t > 0$ denote the spatial and temporal discretization parameters respectively.

Let $\chi_j(x)$ and $\chi^n(t)$ be the characteristic functions for the intervals I_j and I^n, respectively. Define $\chi_j^n(x,t) = \chi_j(x)\chi^n(t)$ to be the characteristic function for the rectangle

$$R_j^n = I_j \times I^n.$$

We will use U_j^n to denote the finite difference approximation of $u(j\Delta x, n\Delta t)$. The iteration (4.4) is started by setting

$$U_j^0 = \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_0(x)\, dx, \tag{4.1}$$

and the discretization of γ is *staggered* with respect to that of u:

$$\gamma_{j+\frac{1}{2}} = \gamma\left(\hat{x}_{j+\frac{1}{2}}+\right), \tag{4.2}$$

where $\hat{x}_{j+\frac{1}{2}}$ is any point lying in the interval $I_{j+\frac{1}{2}} = (x_j, x_{j+1})$. This is meaningful since each component of the coefficient vector γ has bounded total variation. The difference solution $\{U_j^n\}$ is extended to all of Π_T by defining

$$u^\Delta(x,t) = \sum_{n=0}^{N} \sum_{j\in\mathbf{Z}} \chi_j^n(x,t) U_j^n, \qquad (x,t) \in \Pi_T, \tag{4.3}$$

where $\Delta = (\Delta x, \Delta t)$. Similarly, the discrete coefficient vector $\{\gamma_{j+\frac{1}{2}}\}$ is extended to all of $\mathbf{R}$ by defining

$$\gamma^\Delta(x) = \sum_{j\in\mathbf{Z}} \chi_{j+\frac{1}{2}}(x)\gamma_{j+\frac{1}{2}}, \qquad x \in \mathbf{R},$$

where $\chi_{j+\frac{1}{2}}$ is the characteristic function for the interval $I_{j+\frac{1}{2}}$.

To simplify the presentation, we use Δ_+ and Δ_- to designate the difference operators in the x direction, e.g.,

$$\Delta_+ f(\gamma_j, U_j^n) = f\left(\gamma_{j+1}, U_{j+1}^n\right) - f\left(\gamma_j, U_j^n\right) = \Delta_- f\left(\gamma_{j+1}, U_{j+1}^n\right).$$

We will also use the difference operators Δ_+^u and Δ_-^u that leave γ fixed, e.g.,

$$\Delta_+^u f(\gamma_j, U_j^n) = f\left(\gamma_j, U_{j+1}^n\right) - f\left(\gamma_j, U_j^n\right) = \Delta_-^u f\left(\gamma_j, U_{j+1}^n\right).$$

The difference scheme uses a monotone flux for the convection part and centered differencing for the parabolic part. With $\lambda = \Delta t/\Delta x$ and $\mu = \Delta t/\Delta x^2$, the algorithm takes the following (conservation) form

$$U_j^{n+1} = U_j^n - \lambda\,\Delta_- h\left(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n\right) + \mu\,\Delta_+\Delta_- A(U_j^n). \tag{4.4}$$

Specifically, the scheme is based on the Engquist-Osher (EO henceforth) numerical flux [14]

$$h(\gamma, v, u) = \frac{1}{2}(f(\gamma, u) + f(\gamma, v)) - \frac{1}{2}\int_u^v |f_u(\gamma, w)|\,dw. \tag{4.5}$$

This generalization of the EO scheme, as it applies to a number of situations, has been discussed in [4, 26, 28, 40, 41]. The EO flux $h(\gamma, v, u)$ is monotone with respect to v and u, and Lipschitz continuous with respect to all variables. By monotone, we mean that for fixed γ, h is nondecreasing in u and non-increasing in v. We also note that the numerical flux function is consistent, i.e.,

$$h(\gamma, u, u) = f(\gamma, u). \tag{4.6}$$

Because of the EO flux our algorithm is a so-called upwind scheme, meaning that the differencing of the convective flux is biased in the direction of incoming waves, making it possible to resolve shocks without excessive smearing.

A finite difference scheme such as the scheme (4.4) is monotone [11, 21] if

$$U_j^n \le V_j^n \quad \forall j \quad \Longrightarrow \quad U_j^{n+1} \le V_j^{n+1} \quad \forall j. \tag{4.7}$$

We assume that there is an interval $[\underline{u}, \overline{u}]$ such that if $U_j^0 \in [\underline{u}, \overline{u}]$ for all j and the scheme (4.4) is applied with the parameters λ and μ chosen so that the following CFL condition is satisfied

$$\lambda \|f_u\| + \mu\,|A'| \le \frac{1}{2}, \tag{4.8}$$

then the computed solutions U_j^n remain in the interval $[\underline{u}, \overline{u}]$, the CFL condition (4.8) holds for each succeeding time step, and the scheme (4.4) is monotone. For example, if

$$f(\gamma, \underline{u}) = f(\gamma, \overline{u}) = 0 \text{ for all } \gamma,$$

it is not hard to show that this condition holds, see [28]. In other situations, obtaining an L^∞ bound may be more involved and require analytical techniques that are specific to the particular situation. Here we simply assume that such a bound is available.

In the purely hyperbolic case, the CFL condition becomes $\lambda\|f_u\| \le 1/2$. The factor of 2 (implying a halving of the allowable time step from the normal situation) is needed because with a flux of the form $f(\gamma(x), u)$, the more relaxed CFL condition $\lambda\|f_u\| \le 1$ does not seem to be sufficient. When the flux has the form $\gamma(x)f(u)$ where $\gamma(x)$ is a scalar, $\lambda\|f_u\| \le 1$ is sufficient.

We stress that the discretization of γ is staggered with respect to that of the conserved variable u. This results in a significant reduction in complexity compared with the alternative of aligning the two discretizations. In the latter case, a more complicated 2×2 Riemann problem has to be solved (exactly or approximately) [33, 34, 19, 30, 31]. Staggering the discretizations also simplifies the analysis, making it possible to apply, with some allowances for the parabolic terms, some of the analytical techniques developed for monotone difference schemes for purely hyperbolic problems.

The scheme discussed herein has conservation form, i.e., it is shock capturing in the purely hyperbolic regime where $A' = 0$. This is an important property. For the case of constant γ, the reference [15] provides numerical evidence that differencing (1.1) directly (i.e., not in conservation form) results in wrong solutions, specifically shocks may move with the wrong speed.

In a number of important cases [7, 28, 40, 41], it has been possible to prove convergence of the scheme (4.4) to a weak solution of the initial value problem (1.1). In each case, the convergence proof depended on the close functional relationship between the viscosity of the EO flux and a certain singular mapping that plays an important role in establishing compactness of the approximations.

The following lemma provides us with a discrete entropy inequality.

Lemma 4.1. *For any $c \in \mathbf{R}$, the following cell entropy inequality is satisfied by approximate solutions U_j^n generated by the scheme* (4.4)*:*

$$\begin{aligned} \left|U_j^{n+1} - c\right| \le \left|U_j^n - c\right| - \Delta_- \left(\lambda H^n_{j+\frac12} - \mu\, \Delta_+ \left|A\left(U_j^n\right) - A(c)\right|\right) \\ - \lambda \operatorname{sign}(U_j^{n+1} - c)\Delta_+ f\left(\gamma_{j-\frac12}, c\right), \end{aligned} \tag{4.9}$$

where the numerical entropy flux $H_{j+\frac12}$ is defined by

$$H^n_{j+\frac12} = h\left(\gamma_{j+\frac12}, U^n_{j+1} \vee c, U_j^n \vee c\right) - h\left(\gamma_{j+\frac12}, U^n_{j+1} \wedge c, U_j^n \wedge c\right), \tag{4.10}$$

and h is defined in (4.5)*.*

Proof. Let $\mathcal{S}(U^n; j)$ denote the right-hand side (4.4), and put $W_j^n = U_j^n \vee c$, $V_j^n = U_j^n \wedge c$, and $W_j^{n+1} = \mathcal{S}(W^n; j)$, $V_j^{n+1} = \mathcal{S}(V_j^n)$, $C_j^{n+1} = \mathcal{S}(c; j)$. Observe that

$$\begin{aligned} \left|U_j^{n+1} - C_j^{n+1}\right| &= \left|U_j^{n+1} - c + \lambda\, \Delta_+ f\left(\gamma_{j-\frac12}, c\right)\right| \\ &\ge \operatorname{sign}(U_j^{n+1} - c)\left(U_j^{n+1} - c + \lambda\, \Delta_+ f\left(\gamma_{j-\frac12}, c\right)\right) \\ &= \left|U_j^{n+1} - c\right| + \lambda \operatorname{sign}(U_j^{n+1} - c)\Delta_+ f\left(\gamma_{j-\frac12}, c\right). \end{aligned}$$

By monotonicity of the scheme, $W_j^{n+1} \ge U_j^{n+1} \ge V_j^{n+1}$ and $V_j^{n+1} \le C_j^{n+1} \le W_j^{n+1}$, which implies $\left|U_j^{n+1} - C_j^{n+1}\right| \le W_j^{n+1} - C_j^{n+1}$. Moreover,

$$W_j^{n+1} - V_j^{n+1} = \left|U_j^n - c\right| - \Delta_- \left(\lambda H^n_{j+\frac12} - \mu\, \Delta_+ \left|A\left(U_j^n\right) - A(c)\right|\right).$$

By combining the above inequalities, we get (4.9). □

Lemma 4.2. *Let u be a limit of approximations u^Δ generated by the scheme* (4.4)*, converging boundedly a.e. to a weak solution of the type specified in Definition 1.1. Then u satisfies the following entropy inequality for all $c \in \mathbf{R}$ and all test functions $0 \le \phi \in \mathcal{D}(\Pi_T)$:*

$$\begin{aligned} \iint_{\Pi_T} \Big(|u - c|\, \phi_t + \operatorname{sign}(u - c)(f(\gamma(x), u) - f(\gamma(x), c))\phi_x - \left|A(u) - A(c)\right|_x \phi_x\Big)\, dt\, dx \\ + \iint_{\Pi_T \setminus \{\xi_m\}_{m=1}^M} \left|f(\gamma(x), c)_x\right| \phi\, dt\, dx \\ + \int_0^T \sum_{m=1}^M \left|f\left(\gamma\left(\xi_m+\right), c\right) - f\left(\gamma\left(\xi_m-\right), c\right)\right| \phi(\xi_m, t)\, dt \ge 0. \end{aligned} \tag{4.11}$$

Proof. Let $0 \le \phi \in \mathcal{D}(\Pi_T)$. Fix $X > 0$ such that ϕ vanishes for $|x| \ge X$. Let $\phi_j^n = \phi\left(U_j^n\right)$ and $A_j^n = A\left(U_j^n\right)$. Multiply the cell entropy inequality (4.9) by $\phi_j^n \Delta x$. Then summing by parts gives

$$\begin{aligned} &- \Delta x \Delta t \sum_{j,n} \left|U_j^{n+1} - c\right| \left(\phi_j^{n+1} - \phi_j^n\right)/\Delta t \\ &\quad - \Delta x \Delta t \sum_{j,n} H^n_{j-\frac12}\left(\Delta_- \phi_j^n/\Delta x\right) - \Delta x \Delta t \sum_{j,n} \left|A_j^n - c\right| \left(\Delta_- \Delta_+ \phi_j^n/\Delta x^2\right) \\ &\quad \le \Delta x \Delta t \sum_{j,n} \left|\Delta_+ f\left(\gamma_{j-\frac12}, c\right)/\Delta x\right| \phi_j^n, \end{aligned}$$

where $J\Delta x = X$ and $N\Delta t = T$, and $j \in \{-J,\dots,J\}$, $n \in \{0,\dots,N\}$. The sum in the first line converges to

$$\iint_{\Pi_T} |u-c|\,\phi_t\,dt\,dx$$

by the bounded convergence theorem. Similarly, the sum involving A_j^n converges to its integral counterpart:

$$\Delta x\Delta t\sum_{j,n}\left|A_j^n - A(c)\right|\left(\Delta_-\Delta_+\phi_j^n/\Delta x^2\right) \to \iint_{\Pi_T} |A(u)-A(c)|\,\phi_{xx}\,dt\,dx.$$

Next, we address the sum involving $H_{j-\frac12}^n$. We will use the easily checked fact that the numerical entropy flux H inherits Lipschitz continuity from the numerical flux h. Using the identity, $H\left(\gamma_j, U_j^n, U_j^n\right) = F\left(\gamma_j, U_j^n, c\right)$, and letting $\|H_\gamma\|$ and $\|H_u\|$ denote Lipschitz constants for H, it is clear that

$$\begin{aligned}\left|H_{j-\frac12}^n - F\left(\gamma_j, U_j^n, c\right)\right| &\le \left|H\left(\gamma_{j-\frac12}, U_j^n, U_{j-1}^n\right) - H\left(\gamma_j, U_j^n, U_{j-1}^n\right)\right| \\ &\quad + \left|H\left(\gamma_j, U_j^n, U_{j-1}^n\right) - H\left(\gamma_j, U_j^n, U_j^n\right)\right| \\ &\le \|H_\gamma\|\sum_{\nu=1}^p\left|\gamma_{j-\frac12}^\nu - \gamma_j^\nu\right| + \|H_u\|\left|U_j^n - U_{j-1}^n\right|. \end{aligned} \tag{4.12}$$

Thus,

$$\Delta x\Delta t\sum_{j,n} H_{j-\frac12}^n\left(\Delta_-\phi_j^n/\Delta x\right) = \Delta x\Delta t\sum_{j,n} F\left(\gamma_j, U_j^n, c\right)\left(\Delta_-\phi_j^n/\Delta x\right) + E(\Delta),$$

where

$$|E(\Delta)| \le \|\phi_x\|_{L^\infty(\Pi_T)}\Delta x\Delta t\sum_{j,n}\left(\|H_\gamma\|\sum_{\nu=1}^p\left|\gamma_{j-\frac12}^\nu - \gamma_j^\nu\right| + \|H_u\|\left|U_j^n - U_{j-1}^n\right|\right).$$

Since

$$\Delta x\Delta t\sum_{j,n} F\left(\gamma_j, U_j^n, c\right)\left(\Delta_-\phi_j^n/\Delta x\right) \to \iint_{\Pi_T} F(\gamma(x), u, c)\phi_x\,dt\,dx,$$

it remains to show that

$$\Delta x\Delta t\sum_{j,n}\left|\gamma_{j-\frac12}^\nu - \gamma_j^\nu\right| \to 0, \quad \nu = 1,\dots,p, \qquad \Delta x\Delta t\sum_{j,n}\left|U_j^n - U_{j-1}^n\right| \to 0. \tag{4.13}$$

That the first limit holds is immediate from the fact that $TV(\gamma^\nu) < \infty$, $\nu = 1,\dots,p$. For the second limit, observe that

$$\begin{aligned}\Delta x\Delta t\sum_{j,n}\left|U_j^n - U_{j-1}^n\right| &= \int_0^T\int_{-X}^X \left|u^\Delta(x,t) - u^\Delta(x-\Delta x,t)\right|\,dx\,dt \\ &\le \int_0^T\int_{-X}^X \left|u^\Delta(x,t) - u(x,t)\right|\,dx\,dt \\ &\quad + \int_0^T\int_{-X}^X |u(x,t) - u(x-\Delta x,t)|\,dx\,dt \\ &\quad + \int_0^T\int_{-X}^X \left|u(x-\Delta x,t) - u^\Delta(x-\Delta x,t)\right|\,dx\,dt. \end{aligned}$$

The first and third of the integrals on the right side of this last inequality tend to zero by the bounded convergence theorem. That the second integral vanishes as $\Delta \to 0$ is a standard fact from real analysis, since u is integrable.

For the remaining term,

$$
\begin{aligned}
&\Delta x \Delta t \sum_{j,n} \phi_j^n \left| \frac{1}{\Delta x} \Delta_+ f\left(\gamma_{j-\frac{1}{2}}, c\right) \right| \\
&\qquad \to \iint_{\Pi_T \setminus \{\xi_m\}} |f(\gamma(x), c)_x| \, \phi(x,t) \, dt \, dx \\
&\qquad\qquad + \int_0^T \sum_{m=1}^{M} |f(\gamma(\xi_m+), c) - f(\gamma(\xi_m-), c)| \, \phi(\xi_m, t) \, dt.
\end{aligned} \tag{4.14}
$$

Let us detail the argument when there is just one jump in γ, located at $x = \xi_1$. It will become clear that our argument generalizes to multiple jumps. Due to our method of discretizing the coefficient vector γ, for each $\Delta > 0$ there is an index j_0 (whose dependence on Δ we suppress) such that

$$\hat{x}_{j_0 - \frac{1}{2}} < \xi_1 \leq \hat{x}_{j_0 + \frac{1}{2}},$$

from which it follows that

$$
\begin{aligned}
&\left| \gamma_{j_0+\frac{1}{2}} - \gamma(\xi_1+) \right| \leq \|\gamma'\|_{L^\infty(\xi_1,\infty)} \Delta x, \quad \gamma \text{ is } C^1 \text{ on } (\xi_1, \infty) \\
&\left| \gamma_{j_0-\frac{1}{2}} - \gamma(\xi_1-) \right| \leq \|\gamma'\|_{L^\infty(-\infty,\xi_1)} \Delta x, \quad \gamma \text{ is } C^1 \text{ on } (-\infty, \xi_1).
\end{aligned} \tag{4.15}
$$

This allows us to write

$$
\begin{aligned}
&\Delta x \Delta t \sum_{j,n} \phi_j^n \left| \frac{1}{\Delta x} \Delta_+ f\left(\gamma_{j-\frac{1}{2}}, c\right) \right| = \Delta t \sum_n \phi_{j_0}^n \left| f(\gamma_{j_0+\frac{1}{2}}, c) - f(\gamma_{j_0-\frac{1}{2}}, c) \right| \\
&\quad + \Delta x \Delta t \sum_n \sum_{j<j_0} \phi_j^n \left| \frac{1}{\Delta x} \Delta_+ f\left(\gamma_{j-\frac{1}{2}}, c\right) \right| + \Delta x \Delta t \sum_n \sum_{j>j_0} \phi_j^n \left| \frac{1}{\Delta x} \Delta_+ f\left(\gamma_{j-\frac{1}{2}}, c\right) \right|.
\end{aligned} \tag{4.16}
$$

By (4.15),

$$\Delta t \sum_{n=0}^{N} \phi_{j_0}^n \left| f(\gamma_{j_0+\frac{1}{2}}, c) - f(\gamma_{j_0-\frac{1}{2}}, c) \right| = \Delta t \sum_{n=0}^{N} \phi_{j_0}^n \left| f(\gamma(\xi_1+), c) - f(\gamma(\xi_1-), c) \right| + E(\Delta x),$$

where

$$|E(\Delta x)| \leq 2T \, \|f_\gamma\| \, \|\gamma'\|_{L^\infty(\mathbf{R} \setminus \{\xi_1\})} \, \|\phi\|_{L^\infty(\Pi_T)} \Delta x.$$

From this it is clear that as $\Delta \to 0$, the first sum on the right side of (4.16) converges to

$$\int_0^T |f(\gamma(\xi_1+), c) - f(\gamma(\xi_1-), c)| \, \phi(\xi_1, t) \, dt.$$

Since γ is C^1 on $\mathbf{R} \setminus \{\xi_1\}$, an application of the mean value theorem, along with the bounded convergence theorem, shows that the second line of (4.16) converges to

$$\iint_{\Pi_T \setminus \{\xi_1\}} |f(\gamma(x), c)_x| \, \phi(x,t) \, dt \, dx,$$

and the proof of the claim is complete. $\square$

In order to prove that our scheme generates entropy solutions as its limits (Lemma 4.4), we will need the following technical lemma.

Lemma 4.3. *Let $\Omega \subset \mathbf{R}^d$ be a bounded open set, $g \in L^1(\Omega)$, and suppose that $g_\nu(x) \to g(x)$ for a.e. $x \in \Omega$. Then there exists a set Θ, which is at most countable, such that for any $c \in \mathbf{R} \setminus \Theta$*

$$\operatorname{sign}(g_\nu(x) - c) \to \operatorname{sign}(g(x) - c) \quad \text{a.e. in } \Omega.$$

Let $c \in \Theta$, and define

$$\mathcal{E}_c = \left\{ x \in \Omega \,\middle|\, g(x) = c \right\}.$$

It is possible to construct sequences $\{\underline{c}_\nu\}_{\nu=1}^\infty$, $\{\overline{c}_\nu\}_{\nu=1}^\infty$ *such that* $\underline{c}_\nu, \overline{c}_\nu \in \mathbf{R}\setminus\Theta$ *and*

$$
\begin{cases} \underline{c}_\nu < c, \quad \underline{c}_\nu \uparrow c \text{ as } \nu \to \infty, \\ \operatorname{sign}\left(g(x) - \underline{c}_\nu\right) \to \operatorname{sign}(g(x)-c) \text{ as } \nu \to \infty \text{ for a.e. } x \in \Omega \setminus \mathcal{E}_c\ , \end{cases}
\qquad
\begin{cases} \overline{c}_\nu > c, \quad \overline{c}_\nu \downarrow c \text{ as } \nu \to \infty, \\ \operatorname{sign}\left(g(x) - \overline{c}_\nu\right) \to \operatorname{sign}(g(x)-c) \text{ as } \nu \to \infty \text{ for a.e. } x \in \Omega \setminus \mathcal{E}_c. \end{cases}
\tag{4.17}
$$

Proof. Fix $c \in \mathbf{R}$ and a point $x \in \Omega$ where $g_\nu(x) \to g(x)$, and $g(x) \neq c$. For ν sufficiently large, $\operatorname{sign}\left(g_\nu(x) - c\right) = \operatorname{sign}(g(x)-c)$, i.e., $\operatorname{sign}\left(g_\nu(x)-c\right)$ is constant and so converges to $\operatorname{sign}(g(x)-c)$. We have shown that for each $c \in \mathbf{R}$, $\operatorname{sign}\left(g_\nu - c\right) \to \operatorname{sign}(g-c)$ a.e. in $\Omega \setminus \mathcal{E}_c$. The proof will be done if we can show that all but at most countably many of the sets $\mathcal{E}_c$ have measure $\operatorname{meas}(\mathcal{E}_c) = 0$. To this end, define

$$\mathcal{C}_n = \left\{ c \in \mathbf{R} \,\middle|\, \operatorname{meas}(\mathcal{E}_c) \geq \frac{1}{n} \right\}.$$

Since Ω is bounded, $\mathcal{C}_n$ can only contain a finite number of points. Then

$$\left\{ c \in \mathbf{R} \,\middle|\, \operatorname{meas}(\mathcal{E}_c) > 0 \right\} = \bigcup_{n>0} \mathcal{C}_n$$

is an at most countable set.

For the first assertion in (4.17), fix $c \in \Theta$. It is clear that since Θ is at most countable, we can construct a sequence $\{\underline{c}_\nu\}_{\nu=1}^\infty$ such that $\underline{c}_\nu < c$ and $\underline{c}_\nu \to c$ as $\nu \to \infty$. Let $x \in \Omega \setminus \mathcal{E}_c$. Since $g(x) \neq c$, $\operatorname{sign}\left(g(x) - \underline{c}_\nu\right) = \operatorname{sign}(g(x)-c)$ for ν sufficiently large, and thus $\operatorname{sign}\left(g(x) - \underline{c}_\nu\right)$ converges to $\operatorname{sign}(g(x)-c)$. The second assertion in (4.17) is proven in a similar way. □

Lemma 4.4. *Let u be a limit of approximations u^Δ generated by the scheme* (4.4)*, converging boundedly a.e. to a weak solution of the type specified in Definition 1.1. Then u satisfies the entropy inequality* (1.9).

Proof. Let $0 \leq \phi \in \mathcal{D}\left(\Pi_T\right)$. For each sufficiently small $\varepsilon > 0$ we can find $\rho^\varepsilon, \sigma^\varepsilon \in \mathcal{D}\left(\Pi_T\right)$ such that any nonnegative test function ϕ is written as a sum of two test functions, one having support away from the compact set $\{\xi_m\}_{m=1}^M$, and the other with support in the vicinity of $\{\xi_m\}_{m=1}^M$. Concretely

$$\phi(x,t) = \rho^\varepsilon(x,t) + \sigma^\varepsilon(x,t), \qquad 0 \leq \rho^\varepsilon(x,t) \leq \phi(x,t), \quad 0 \leq \sigma^\varepsilon(x,t) \leq \phi(x,t)$$

where ρ^ε has support located around the jumps in γ:

$$\begin{gathered} \operatorname{supp}\,(\rho^\varepsilon) \subseteq \left((\xi_1 - \varepsilon, \xi_1 + \varepsilon) \cup \cdots \cup (\xi_M - \varepsilon, \xi_M + \varepsilon) \right) \times [0,T], \\ \rho^\varepsilon(\xi_m, t) = \varphi(\xi_m, t), \quad m = 1, \dots, M, \quad \varepsilon > 0, \end{gathered}$$

and σ^ε vanishes around the jumps in γ, i.e.,

$$\operatorname{supp}\,(\sigma^\varepsilon) \subseteq \left(\mathbf{R} \setminus \{\xi_m\}_{m=1}^M \right) \times [0,T].$$

We can accomplish this decomposition in such a way that

$$\sigma^\varepsilon \to \phi \text{ in } L^1(\Pi_T), \quad \rho^\varepsilon \to 0 \text{ in } L^1(\Pi_T). \tag{4.18}$$

With ρ^ε as the test function in the entropy inequality (4.11), we get

$$
\begin{aligned}
&\iint_{\Pi_T} \Big(|u-c|\, \rho^\varepsilon_t + \operatorname{sign}(u-c)(f(\gamma(x),u) - f(\gamma(x),c))\rho^\varepsilon_x + |A(u) - A(c)|\, \rho^\varepsilon_{xx} \Big)\, dt\, dx \\
&\qquad + \iint_{\Pi_T \setminus \{\xi_m\}_{m=1}^M} |f(\gamma(x),c)_x|\, \rho^\varepsilon\, dt\, dx \\
&\qquad + \int_0^T \sum_{m=1}^M \left| f\left(\gamma\left(\xi_m+\right), c\right) - f\left(\gamma\left(\xi_m-\right), c\right) \right| \rho^\varepsilon(\xi_m, t)\, dt \geq 0.
\end{aligned}
\tag{4.19}
$$

Next we multiply the cell entropy inequality (4.9) by σ^ε, and then proceed as in the proof of Lemma 4.2. This time, instead of simply bounding the integral involving $f(\gamma(x),c)_x$ by taking the absolute value, we apply the first part of Lemma 4.3. This gives us the following entropy inequality, which holds for all $c \in \mathbf{R} \setminus \Theta$, where Θ is at most countable (recall that σ^ε has support away from $\{\xi_m\}_{m=1}^M$):

$$\begin{aligned}\iint_{\Pi_T} \Big(|u-c|\, \sigma_t^\varepsilon + \operatorname{sign}(u-c)(f(\gamma(x),u) - f(\gamma(x),c))\sigma_x^\varepsilon + |A(u)-A(c)|\, \sigma_{xx}^\varepsilon \Big)\, dt\, dx \\ - \iint_{\Pi_T} \operatorname{sign}(u-c) f(\gamma(x),c)_x \sigma^\varepsilon \, dt\, dx \geq 0.\end{aligned} \tag{4.20}$$

We claim that the entropy inequality (4.20) actually holds for all $c \in \mathbf{R}$. To prove this, we start by writing (4.20) in the form

$$\iint_{\Pi_T} \operatorname{sign}(u-c) f(\gamma(x),c)_x \sigma^\varepsilon \, dt\, dx \leq \mathcal{I}(c), \quad \forall c \in \mathbf{R} \setminus \Theta, \tag{4.21}$$

where

$$\mathcal{I}(c) := \iint_{\Pi_T} \Big(|u-c|\, \sigma_t^\varepsilon + \operatorname{sign}(u-c)(f(\gamma(x),u) - f(\gamma(x),c))\sigma_x^\varepsilon + |A(u)-A(c)|\, \sigma_{xx}^\varepsilon \Big)\, dt\, dx.$$

Observe that $\mathcal{I}(c)$ is a continuous function of $c \in \mathbf{R}$. Our goal now is to show that (4.21) also holds for $c \in \Theta$, so fix $c \in \Theta$, and let $\mathcal{E}_c = \{(x,t) \in \Pi_T | u(x,t) = c\}$. By the second part of Lemma 4.3, we can construct two sequences $\{\underline{c}_\nu\}_{\nu=1}^\infty$, $\{\overline{c}_\nu\}_{\nu=1}^\infty$ such that $\underline{c}_\nu, \overline{c}_\nu \in \mathbf{R} \setminus \Theta$ and

$$\begin{aligned} &\underline{c}_\nu < c, \quad \underline{c}_\nu \uparrow c \text{ as } \nu \to \infty, \quad \operatorname{sign}(u - \underline{c}_\nu) \to \operatorname{sign}(u-c) \text{ a.e. in } \Pi_T \setminus \mathcal{E}_c, \\ &\overline{c}_\nu > c, \quad \overline{c}_\nu \downarrow c \text{ as } \nu \to \infty, \quad \operatorname{sign}(u - \overline{c}_\nu) \to \operatorname{sign}(u-c) \text{ a.e. in } \Pi_T \setminus \mathcal{E}_c. \end{aligned} \tag{4.22}$$

Inequality (4.21) holds for each $\underline{c}_\nu$, and we can rewrite (4.21) as

$$\iint_{\Pi_T \setminus \mathcal{E}_c} \operatorname{sign}(u - \underline{c}_\nu) f(\gamma(x), \underline{c}_\nu)_x \sigma^\varepsilon \, dt\, dx + \iint_{\mathcal{E}_c} \operatorname{sign}(u - \underline{c}_\nu) f(\gamma(x), \underline{c}_\nu)_x \sigma^\varepsilon \, dt\, dx \leq \mathcal{I}(\underline{c}_\nu). \tag{4.23}$$

For $(x,t) \in \mathcal{E}_c$, $\underline{c}_\nu < c = u(x,t)$, and so we can rewrite this as

$$\iint_{\Pi_T \setminus \mathcal{E}_c} \operatorname{sign}(u - \underline{c}_\nu) f(\gamma(x), \underline{c}_\nu)_x \sigma^\varepsilon \, dt\, dx + \iint_{\mathcal{E}_c} f(\gamma(x), \underline{c}_\nu)_x \sigma^\varepsilon \, dt\, dx \leq \mathcal{I}(\underline{c}_\nu). \tag{4.24}$$

Since f_γ is continuous,

$$f(\gamma(x), \underline{c}_\nu)_x \sigma^\varepsilon \to f(\gamma(x), c)_x \sigma^\varepsilon \text{ a.e. in } \Pi_T \setminus \mathcal{E}_c.$$

With this observation, along with (4.22), when $\underline{c}_\nu \uparrow c$ in inequality (4.24), we get

$$\iint_{\Pi_T \setminus \mathcal{E}_c} \operatorname{sign}(u-c) f(\gamma(x),c)_x \sigma^\varepsilon \, dt\, dx + \iint_{\mathcal{E}_c} f(\gamma(x),c)_x \sigma^\varepsilon \, dt\, dx \leq \mathcal{I}(c). \tag{4.25}$$

A similar calculation using $\overline{c}_\nu$ yields the inequality

$$\iint_{\Pi_T \setminus \mathcal{E}_c} \operatorname{sign}(u-c) f(\gamma(x),c)_x \sigma^\varepsilon \, dt\, dx - \iint_{\mathcal{E}_c} f(\gamma(x),c)_x \sigma^\varepsilon \, dt\, dx \leq \mathcal{I}(c). \tag{4.26}$$

We next add the pair of inequalities (4.25) and (4.26), and then divide by two, giving us

$$\iint_{\Pi_T \setminus \mathcal{E}_c} \operatorname{sign}(u-c) f(\gamma(x),c)_x \sigma^\varepsilon \, dt\, dx \leq \mathcal{I}(c). \tag{4.27}$$

Recalling that $\operatorname{sign}(u-c) f(\gamma(x),c)_x \sigma^\varepsilon = 0$ on $\mathcal{E}_c$, we can extend the domain of integration to all of Π_T, and so

$$\iint_{\Pi_T} \operatorname{sign}(u-c) f(\gamma(x),c)_x \sigma^\varepsilon \, dt\, dx \leq \mathcal{I}(c). \tag{4.28}$$

We have shown that inequality (4.21) holds for $c \in \Theta$, and so the proof of the claim is complete.

We now add the two entropy inequalities (4.19) and (4.20), using $\rho^\varepsilon + \sigma^\varepsilon = \phi$, along with $\rho^\varepsilon(\xi_m, t) = \phi(\xi_m, t)$ to get

$$\begin{aligned}
&\iint_{\Pi_T} \Big(|u-c|\,\phi_t + \operatorname{sign}(u-c)(f(\gamma(x),u) - f(\gamma(x),c))\phi_x + |A(u)-A(c)|\,\phi_{xx}\Big)\,dt\,dx \\
&\quad + \iint_{\Pi_T\setminus\{\xi_m\}_{m=1}^M} |f(\gamma(x),c)_x|\,\rho^\varepsilon\,dt\,dx - \iint_{\Pi_T} \operatorname{sign}(u-c) f(\gamma(x),c)_x \sigma^\varepsilon\,dt\,dx \\
&\quad + \int_0^T \sum_{m=1}^M |f(\gamma(\xi_m+),c) - f(\gamma(\xi_m-),c)|\,\phi(\xi_m,t)\,dt \ge 0.
\end{aligned} \tag{4.29}$$

Thanks to (4.18), when $\varepsilon \downarrow 0$, the integrals in the second line of this inequality converge according to

$$\begin{aligned}
&\iint_{\Pi_T\setminus\{\xi_m\}_{m=1}^M} |f(\gamma(x),c)_x|\,\rho^\varepsilon\,dt\,dx \to 0, \\
&\iint_{\Pi_T} \operatorname{sign}(u-c) f(\gamma(x),c)_x \sigma^\varepsilon\,dt\,dx \to \iint_{\Pi_T\setminus\{\xi_m\}_{m=1}^M} \operatorname{sign}(u-c) f(\gamma(x),c)_x \phi\,dt\,dx,
\end{aligned}$$

completing the proof. □

Recall that in Section 3 we discussed two specific instances of the initial value problem (1.1), establishing in each case the existence of traces at jumps in γ. In [28] we established convergence of the scheme (4.4) to a weak solution of those problems. Thanks to Lemma 4.4, we now know that these weak solutions are also entropy solutions, yielding the following well-posedness theorem.

Theorem 4.1 (well-posedness)**.** *In addition to the hypotheses listed in* [28]*, suppose that*

(1) *the crossing condition (Assumption 1.1) is satisfied,*
(2) *the parameter $\gamma(x)$ is piecewise C^1 with finitely many jumps (in γ and γ'),*
(3) *u^* is independent of γ.*

Then the special case of the initial value problems (1.1) *studied in* [28] *is well-posed. For each initial function $u_0(x)$ satisfying the conditions in that paper, there exists a unique entropy solution $u(x,t)$ defined on Π_T which is L^1 stable.*

Remark 4.1. In the context of the problem studied in [28], one important case where the crossing condition is satisfied is where the convective flux takes the form $f(\gamma, u) = \gamma g(u)$, with $\gamma > 0$, $g(0) = g(1) = 0$, and g concave with a single maximum.

5. Conservation laws

With no diffusion term present in (1.1), the resulting problem is the conservation law

$$u_t + f(\gamma(x), u)_x = 0, \qquad u(x,0) = u_0(x), \tag{5.1}$$

where the vector-valued coefficient $\gamma(x)$ and the flux function $f(\gamma, u)$ satisfy the same assumptions as before, and $u_0 \in L^\infty(\mathbf{R})$. Let us state what we mean by an entropy solution to (5.1).

Definition 5.1 (entropy solution)**.** *An entropy solution of the initial value problem of* (5.1) *is a function $u \in L^\infty(\Pi_T)$ that satisfies* (1.1) *in $\mathcal{D}'(\mathbf{R}\times[0,T))$ and the following Kružkov-type entropy inequality for all $0 \le \phi \in \mathcal{D}(\mathbf{R}\times[0,T))$:*

$$\begin{aligned}
&\iint_{\Pi_T} \Big(|u-c|\,\phi_t + \operatorname{sign}(u-c)(f(\gamma(x),u) - f(\gamma(x),c))\phi_x\Big)\,dt\,dx \\
&\quad - \iint_{\Pi_T\setminus\{\xi_m\}_{m=1}^M} \operatorname{sign}(u-c) f(\gamma(x),c)_x \phi\,dt\,dx \\
&\quad + \int_0^T \sum_{m=1}^M |f(\gamma(\xi_m+),c) - f(\gamma(\xi_m-),c)|\,\phi(\xi_m,t)\,dt \\
&\quad + \int_{\mathbf{R}} |u_0(x)-c|\,\phi(x,0)\,dx \ge 0, \qquad \forall c \in \mathbf{R}.
\end{aligned} \tag{5.2}$$

In our original definition of an entropy solution we require that the initial condition at $t = 0$ is satisfied in the strong L^1 sense, see (**D**.3). When proving convergence of approximate solution sequences without having BV estimates at our disposal (e.g., via the compensated compactness method), it is difficult to verify condition (**D**.3) for a limit function. To have a more flexible framework in which to prove convergence of approximate solutions, we have here in the hyperbolic context chosen to include the initial condition into the entropy formulation. Such a "weak" formulation of the initial condition is much easier to verify for limits of certain approximate solution sequences. This point was made explicit in [17, 16].

We have the following L^1 stability and uniqueness theorem for the conservation law case:

Theorem 5.1 (L^1 stability and uniqueness). *Let v and u be two entropy solutions to the initial value problem* (5.1) *with initial data $v_0, u_0 \in L^\infty(\mathbf{R})$, respectively. If f satisfies the crossing condition (Assumption 1.1), and we assume the existence of the right and left traces of $u(\cdot,t)$ and $v(\cdot,t)$ at the jump points in γ, then, for a.e. $t \in (0,T)$,*

$$\int_{-r}^{r} |v(x,t) - u(x,t)|\, dx \le C \int_{-r-\|f_u\|t}^{r+\|f_u\|t} |v_0(x) - u_0(x)|\, dx, \qquad \forall r \in \mathbf{R}, \tag{5.3}$$

for some finite constant $C > 0$. If $\gamma'(x) = 0$ for a.e. $x \in \mathbf{R}$, then $C = 1$.

Proof. Starting from (2.34) without the "$|A(v) - A(u)|\, \phi_{xx}$" term, (5.3) follows from Kružkov's classical arguments [32], once we know that for any entropy solution u the initial condition is satisfied in the following sense:

$$\lim_{\tau \downarrow 0} \frac{1}{\tau} \int_0^\tau \int_{-r}^{r} |u(x,t) - u_0(x)|\, dx\, dt = 0, \qquad \forall r > 0. \tag{5.4}$$

Of course, condition (**D**.3) implies (5.4). On the other hand, with the initial condition merely satisfied in the "weak" sense of (5.2), one needs a non-trivial argument to prove (5.4), which is the content of the proof of Lemma B.1 in the Appendix. □

In Theorem 5.1 it is assumed that the right and left traces of an entropy solution $u(\cdot,t)$ exist. We now discuss a special case of the initial value problem (5.1), specifically the problem studied in reference [41], for which we can prove the existence of these traces directly from the definition of an entropy solution, at least under the assumption of $BV_t(\Pi_T)$ regularity. For this problem the convective flux has the form

$$f(\gamma, u) = kg(u) - a, \qquad \gamma = (k,a) \in \mathbf{R}^2, \tag{5.5}$$

so we are dealing with the initial value problem for the scalar conservation law

$$\begin{cases} u_t + (k(x)g(u) - a)_x = 0, & (x,t) \in \Pi_T = \mathbf{R} \times (0,T), \\ u(x,0) = u_0(x), & x \in \mathbf{R}. \end{cases} \tag{5.6}$$

We assumed in [41] that $u_0(x) \in L^1(\mathbf{R}) \cup L^\infty(\mathbf{R}) \cup BV(\mathbf{R})$. The spatially varying coefficients were required to satisfy $a, k \in L^1_{loc}(\mathbf{R}) \cup L^\infty(\mathbf{R}) \cup BV(\mathbf{R})$, and for all $x \in \mathbf{R}$,

$$\underline{a} \le a(x) \le \overline{a}, \quad 0 \le \underline{k} \le k(x) \le \overline{k}.$$

In order to control the amplitude of the solution, i.e., $u \in L^\infty(\Pi_T)$, we made the following technical assumption, which is reasonable in most cases of interest, see [41] for a number of examples.

Assumption 5.1. *There are real numbers $\overline{w} \le \widetilde{w} \le \widetilde{v} \le \overline{v}$ such that*

(1) $\widetilde{w} \le \inf_x u_0(x) \le u_0(x) \le \sup_x u_0(x) \le \widetilde{v}$ *for all $x \in \mathbf{R}$;*
(2) $g \in C^1([\overline{w}, \overline{v}])$, *and g is monotone on both intervals $I_w = [\overline{w}, \widetilde{w}]$ and $I_v = [\widetilde{v}, \overline{v}]$;*
(3) *there are constants η and ξ such that the equations*

$$kg(v) - a = \eta, \qquad kg(w) - a = \xi \tag{5.7}$$

have solutions $v(k,a,x) \in [\widetilde{v}, \overline{v}]$, $w(k,a,x) \in [\overline{w}, \widetilde{w}]$ for all $k \in [\underline{k}, \overline{k}]$ and all $a \in [\underline{a}, \overline{a}]$.

Finally, we assumed that g has finitely many critical points

$$u_1^\star < u_2^\star < \cdots < u_\kappa^\star,$$

with $|g'(u)| > 0$ away from the critical points. We add to the assumptions of [41] the hypothesis given in Section 1 that $\gamma = (k,a)$ is piecewise smooth with jumps located at the finite number of points $\xi_1 < \xi_2 < \cdots < \xi_M$.

We established in [41] that with the assumptions of that paper, weak solutions in the sense of Definition 1.1 exist. Although we did not state it there, it is easily proven using the results of that paper that such solutions also satisfy $u \in BV_t(\Pi_T)$.

Lemma 5.1. *For the particular case of the initial value problem* (5.6) *discussed in* [41], *entropy solutions u in the sense of Definition 5.1 satisfying $u \in BV_t(\Pi_T)$ admit right and left traces at the jump points $\{\xi_m\}_{m=1}^M$ of γ.*

Proof. For this problem, we use the following form of the singular mapping:

$$\Psi(k,u) = k\int_{u_0^\star}^{u} |g'(w)|\,dw,$$

which is easily seen to be strictly increasing and Lipschitz continuous in both variables.

As in the proof of Lemma 3.3, we can use Definition 5.1 of an entropy solution, the assumption that $u \in BV_t(\Pi_T)$, and Lemma 3.1 with $A' \equiv 0$ to show that for all $c \in \mathbf{R}$,

$$\Big(k(x)\,\mathrm{sign}(u-c)(g(u)-g(c))\Big)_x \in \mathcal{M}(\Pi_T), \quad \Big(k(x)\,(g(u)-g(c))\Big)_x \in \mathcal{M}(\Pi_T).$$

By adding these two quantities, then dividing by two, we get

$$\Big(k(x)\mathrm{sign}_+(u-c)(g(u)-g(c))\Big)_x \in \mathcal{M}(\Pi_T). \tag{5.8}$$

Similarly, subtracting, and then dividing by two yields

$$\Big(k(x)\mathrm{sign}_-(u-c)(g(u)-g(c))\Big)_x \in \mathcal{M}(\Pi_T). \tag{5.9}$$

Let $u_0^\star$ and $u_{\kappa+1}^\star$ denote respectively the minimum and maximum values of the solution u. For convenience, and without loss of generality, we assume that

$$u_0^\star < u_1^\star, \qquad u_\kappa^\star < u_{\kappa+1}^\star.$$

Let $\sigma_\nu = \mathrm{sign}(g'(u))$ for $u \in (u_\nu^\star, u_{\nu+1}^\star)$, which we know is constant and nonzero. For any $\nu = 0,\dots,\kappa$, define

$$\begin{aligned}\Phi_\nu(k,u) = k\Big(&-\mathrm{sign}_-(u-u_{\nu+1}^\star)(g(u)-g(u_{\nu+1}^\star))\\ &+\mathrm{sign}_-(u-u_\nu^\star)(g(u)-g(u_\nu^\star)) + g(u_{\nu+1}^\star)-g(u_\nu^\star)\Big)\sigma_\nu.\end{aligned}$$

and note that $\partial_x\phi_\nu(k(x),u) \in \mathcal{M}(\Pi_T)$ by (5.9). It is easily checked that

$$\Phi_\nu(k,u) = k\mathrm{sign}_+(u-u_\nu^\star)\int_{u_\nu^\star}^{u} \chi_{[u_\nu^\star,u_{\nu+1}^\star]}(w)\,|g'(w)|\,dw,$$

and that Ψ has the following decomposition:

$$\Psi(k,u) = \sum_{\nu=0}^{\kappa}\Phi_\nu(k,u). \tag{5.10}$$

Since each $\partial_x\phi_\nu(k(x),u) \in \mathcal{M}(\Pi_T)$ by (5.9), it is clear from (5.10) that $\partial_x\Psi(k(x),u) \in \mathcal{M}(\Pi_T)$.

The rest of the proof is similar to that of Lemma 3.3 and need not be repeated. □

Remark 5.1. In [38] the uniqueness of the entropy solution for (5.6) with $u_0 \in L^\infty(\mathbf{R})$ is proved when $a \equiv 0$, $k(x)$ is piecewise constant with a single jump discontinuity, and $g(u) = u(1-u)$. In this special case, the authors point out that the existence of traces is in fact a consequence of a general result found in [42]. Indeed, this remains true for any $k(x)$ that is piecewise constant with a finite number of jump discontinuities and any *genuinely nonlinear* $g(u)$. In the general

	$f(\gamma_-,c) \le f(\gamma_+,c)$	$f(\gamma_-,c) \ge f(\gamma_+,c)$
$u_- \le c \le u_+$	$f(\gamma_+,u_+) \le f(\gamma_+,c)$	$f(\gamma_-,u_-) \le f(\gamma_-,c)$
$u_+ \le c \le u_-$	$f(\gamma_-,u_-) \ge f(\gamma_-,c)$	$f(\gamma_+,u_+) \ge f(\gamma_+,c)$

TABLE 2. Entropy jump conditions for conservation laws.

case with $g(u)$ possibly containing some linear segments, $k(x)$ continuously varying between jump discontinuities, and a nonzero $a(x)$, the result in [42] does not apply.

Combining the existence result from [41], the L^1 stability result of Theorem 2.1, and finally Lemma 5.1 providing the existence of traces, we have the following well-posedness theorem.

Theorem 5.2 (well-posedness). *In addition to the hypotheses listed in* [41], *suppose that the crossing condition (Assumption 1.1) is satisfied, and that the spatially varying parameter $\gamma(x) = (a(x), k(x))$ is piecewise C^1 with finitely many jumps (in γ and γ'). Then the particular case of the initial value problem* (5.6) *studied in* [41] *is well-posed. For each admissible initial function $u_0(x)$, there exists a unique entropy solution $u(x,t)$ defined on Π_T which is L^1 stable.*

Remark 5.2. Setting $k(x) \equiv 1$ in (5.6), we get the conservation law (1.13) with a source term. The flux in (1.13) has the form $f(a(x),u) = g(u) - a(x)$, from which it is clear that there are no flux crossings. Thus the crossing condition is satisfied (trivially). For (1.13), the Rankine-Hugoniot condition is

$$g(u_+) - g(u_-) = a_+ - a_-,$$

and the entropy jump condition (2.12) becomes

$$\operatorname{sign}(u_+ - c)\,(g(u_+) - g(c)) - \operatorname{sign}(u_- - c)(g(u_-) - g(c)) \le |a_+ - a_-|\,. \tag{5.11}$$

Theorem 5.2, as it applies to the problem (1.13), can be seen as a generalization of the existence and uniqueness results of [20] to the situation where g is not necessarily convex or concave.

We shall conclude this section by discussing a few consequences of the entropy conditions in Table 2. This table results by dropping the diffusion terms from Table 1. Like those appearing in Table 1 for the degenerate parabolic problem (1.1), the inequalities in Table 2 do not require that the flux satisfy any crossing condition. Table 2 can be interpreted geometrically. First take the case where $u_- < u_+$. Since $f(\gamma_-,u_-) = f(\gamma_+,u_+)$ (the Rankine-Hugoniot condition), the horizontal line segment (in the c-y plane) $y = f(\gamma_-,u_-) = f(\gamma_+,u_+)$ must lie below the graph of the function $y(c) = \max(f(\gamma_-,c), f(\gamma_+,c))$. In the case where $u_- > u_+$, the entropy condition derived from Table 2 is that the horizontal line segment $y = f(\gamma_-,u_-) = f(\gamma_+,u_+)$ must lie above the graph of the function $y(c) = \min(f(\gamma_-,c), f(\gamma_+,c))$. Considering the case where $\gamma_- = \gamma_+$, it is evident that Table 2 generalizes the classical condition E of Oleĭnik.

For smooth solutions of the conservation law (1.3), the following equation also holds:

$$f(\gamma(x),u)_t + f_u(\gamma(x),u) f(\gamma(x),u)_x = 0,$$

from which it is clear that the characteristic curves satisfy

$$\frac{dx}{dt} = f_u(\gamma(x),u),$$

and that the flux $f(\gamma(x),u)$ is constant along the characteristics. When γ is smooth, it is well known that the characteristics flow into an entropy-satisfying discontinuity (shock). In other words, if we have an entropy satisfying discontinuity occurring along a smooth curve which passes through the point (x_0,t_0), then with s as the shock speed and $\gamma_0 = \gamma(x_0)$,

$$f_u(\gamma_0,u_-) > s > f_u(\gamma_0,u_+). \tag{5.12}$$

In [40], for the conservation law (5.1) with

$$f(\gamma(x),u) = \gamma(x)g(u), \qquad g \text{ strictly concave},$$

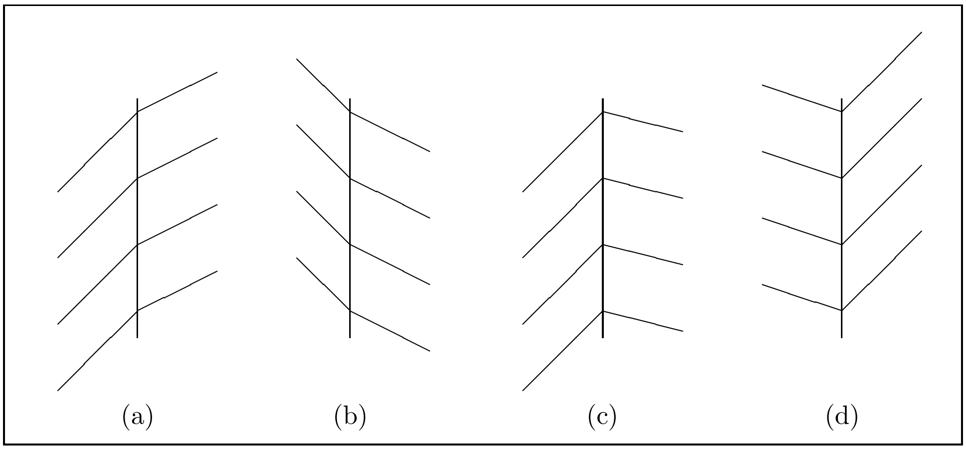

FIGURE 2. Geometric entropy condition for the characteristics ($\dot{x} = f_u(\gamma, u)$) at a jump in γ. (a), (b) Admissible discontinuities. (c) Admissible shock. (d) Entropy violation.

	$f(\gamma_-, c) \le f(\gamma_+, c)$	$f(\gamma_-, c) \ge f(\gamma_+, c)$
$u_- \le c \le u_+$	$f_u(\gamma_+, u_+) \le 0$	$f_u(\gamma_-, u_-) \ge 0$
$u_+ \le c \le u_-$	$f_u(\gamma_-, u_-) \ge 0$	$f_u(\gamma_+, u_+) \le 0$

TABLE 3. Necessary conditions for characteristics of conservation laws.

it was shown that for a piecewise smooth solution the appropriate entropy jump condition at a jump in γ is

$$\min(0, f_u(\gamma_-, u_-)) \max(0, f_u(\gamma_+, u_+)) = 0. \tag{5.13}$$

Condition (5.13) requires that the characteristics on at least one side of the jump in γ must flow into the discontinuity, a weaker requirement than the classical condition (5.12), see Figure 2. According to the following lemma, the entropy condition (5.13) is not limited to the rather special (but important) situation considered in [40].

Proposition 5.1. *Let u be an entropy solution. Under Assumptions 1.1 and 1.2, the characteristic condition* (5.13) *is satisfied at a jump in γ for a.e. $t \in [0, T]$.*

Proof. By forming difference quotients, and then letting $c \to u_-$, $c \to u_+$ in Table 2, we get the conditions on f_u shown in Table 3.

Take the case where there is no flux crossing between u_- and u_+. If $f(\gamma_+, \cdot) \ge f(\gamma_-, \cdot)$ Table 3 gives either $f_u(\gamma_-, u_-) \ge 0$ (if $u_+ \le u_-$) or $f_u(\gamma_+, u_+) \le 0$ (if $u_- \le u_+$). In either case (5.13) holds. Now suppose that there is a crossing, and first take $u_- \le u_+$, with a crossing point u_χ between them. Because of the crossing condition $f(\gamma_-, c) \ge f(\gamma_+, c)$ for $c \in [u_-, u_\chi]$. By Table 3, $f_u(\gamma_-, u_-) \ge 0$, implying (5.13). On the other hand, if $u_+ \le u_-$ with a crossing between them, then we have $f(\gamma_-, c) \ge f(\gamma_+, c)$ for $c \in [u_+, u_\chi]$. By Table 3, $f_u(\gamma_+, u_+) \le 0$, and (5.13) holds in this case also. □

Remark 5.3. We are not the first to conclude that the characteristic condition (5.13) should hold for physically relevant discontinuities at a jump in γ. For example, based on physical considerations, Kaaschieter [24] derives this condition for the conservation law governing two-phase flow in a heterogeneous porous medium.

6. A front tracking method

We briefly describe a front tracking algorithm to generate approximate solutions to the hyperbolic problem (5.1). This is done by replacing $\gamma(x)$ by a piecewise constant approximation γ^δ, and the (family of) flux function(s) by a piecewise linear (in u) and continuous approximation $f^\delta(\gamma, u)$. These approximations are carefully constructed so that for any $\bar{\gamma}$ in the set of values of γ^δ, and for any breakpoint[1] $\bar{u}$ of f^δ, then

$$f\left(\bar{\gamma}, \bar{u}\right) = f^\delta\left(\bar{\gamma}, \bar{u}\right).$$

Furthermore, the front tracking approximation $u^\delta(x,t)$ is a piecewise constant weak solution to the approximate equation

$$u_t^\delta + f^\delta\left(\gamma^\delta(x), u^\delta\right)_x = 0, \qquad u^\delta(x,0) = u_0^\delta(x). \tag{6.1}$$

The approximation u^δ is constructed by solving the initial Riemann problems defined by u_0^δ. By the construction of f^δ and γ^δ, the solution of such Riemann problems will be piecewise constant in x/t and take values among the breakpoints of f^δ. When discontinuities in u^δ collide, we solve the Riemann problem defined by this collision. For more information on the front tracking procedure for conservation laws of this type, see [30, 31, 19, 6].

In order to illuminate some properties of the front tracking approximation, we shall need some properties of the solution of the Riemann problem for (1.3) [18]. This is the initial value problem

$$\begin{cases} u_t + f_-(u)_x = 0, & x < 0, \\ u_t + f_+(u)_x = 0 & x > 0, \end{cases} \qquad u(x,0) = \begin{cases} u_-, & x < 0, \\ u_+, & x > 0, \end{cases} \tag{6.2}$$

where we for simplicity write $f_\mp = f(\gamma_\mp, u)$. Now let

$$G_-(u_-; u) = \begin{cases} \inf\left\{g \in C(\mathbf{R}),\ g(u_-) = f_-(u_-),\ g \text{ non-increasing}\right\}, & u \le u_-, \\ \sup\left\{g \in C(\mathbf{R}),\ g(u_-) = f_-(u_-),\ g \text{ non-increasing}\right\}, & u > u_-, \end{cases}$$

and

$$G_+(u_+; u) = \begin{cases} \sup\left\{g \in C(\mathbf{R}),\ g(u_+) = f_+(u_+),\ g \text{ nondecreasing}\right\}, & u \le u_+, \\ \inf\left\{g \in C(\mathbf{R}),\ g(u_+) = f_+(u_+),\ g \text{ non deceasing}\right\}, & u > u_+. \end{cases}$$

Now, by our assumptions[2] on f_- and f_+, there is a unique flux value $\tilde{f}$ and sets H_- and H_+ such that

$$G_-(u_-; H_-) = G_+(u_+; H_+) = \tilde{f}. \tag{6.3}$$

The value $\tilde{f}$ is the flux at $x = 0$. In order to solve the Riemann problem (6.2), we piece together the solutions of the two Riemann problems

$$\begin{cases} v_t + f_-(v)_x = 0, \quad v(x,0) = \begin{cases} u_-, & x < 0, \\ u_l, & x > 0, \end{cases} & x < 0, \\ w_t + f_+(w)_x = 0, \quad w(x,0) = \begin{cases} u_r, & x < 0, \\ u_+, & x > 0, \end{cases} & x > 0. \end{cases}$$

The limiting value u_l is chosen such that

$$\left|u_- - u_l\right| \quad \text{is minimized provided } u_l \in H_- \text{ and } G_-(u_-; u_l) = \tilde{f}. \tag{6.4}$$

and the other limiting value u_r such that

$$\left|u_+ - u_r\right| \quad \text{is minimized provided } u_r \in H_+ \text{ and } G_+(u_+; u_r) = \tilde{f}. \tag{6.5}$$

---

[1] A breakpoint is a point where f_u^δ is discontinuous in u.

[2] We can for instance assume that there exists $\alpha < \beta$ such that $f_\gamma(\gamma, \alpha) = f_\gamma(\gamma, \beta) = 0$, and that $u_0(x)$ takes values in $[\alpha, \beta]$.

The waves in v will have non-positive speed, and the waves in w nonnegative, hence the weak solution of (6.2) is defined by

$$u(x,t) = \begin{cases} v(x,t), & x < 0, \\ w(x,t), & x > 0. \end{cases} \tag{6.6}$$

Now note that if $u_- = u_l$ and $u_+ = u_r$, then $v(x,t) = u_l$ and $w(x,t) = u_r$, hence $u(x,t) = u(x,0)$. Therefore we find that

$$G_-(u_l; u_l) = G_+(u_r; u_r) = \tilde{f}.$$

Now we are in a position to prove the following entropy inequality:

Lemma 6.1. *Assume that u_l and u_r are chosen according to the above procedure, specifically that the minimal jump condition* (6.4) – (6.5) *holds, then*

$$F(\gamma_+, u_r) - F(\gamma_-, u_l) \le |f(\gamma_+, c) - f(\gamma_-, c)|, \qquad \forall c \in \mathbf{R}. \tag{6.7}$$

Proof. If $\operatorname{sign}(u_- - c) = \operatorname{sign}(u_+ - c)$ the the left hand side of (6.7) reads $\pm(f_+(c) - f_-(c))$, and the lemma holds trivially. We show the lemma if $u_- \le c \le u_+$, the opposite case is shown by the same arguments. Now the the left hand side of (6.7) reads

$$2\tilde{f} - (f_-(c) + f_+(c)),$$

and the lemma will hold if

$$f_-(c) \ge \tilde{f}, \quad \text{or} \quad f_+(c) \ge \tilde{f}. \tag{6.8}$$

Since $G_-(u_l; c)$ is non-increasing, and $G_+(u_r; c)$ is nondecreasing, and u_l and u_r are chosen according to the minimal jump condition, we have that

$$G_-(u_l; c) \le \tilde{f} \quad \text{for} \quad c \ge u_l, \quad \text{and} \quad G_+(u_r; c) \le \tilde{f} \quad \text{for} \quad c < u_+.$$

By the minimal jump condition, it then follows that either $G_-(u_-; \cdot)$ or $G_+(u_+; \cdot)$ (or both) is constant in $[u_-, u_+]$. If G_- is constant in this interval, then

$$f_-(c) \ge \tilde{f} \quad \text{for } c \in [u_-, u_+], \tag{6.9}$$

and if G_+ is constant, then

$$f_+(c) \ge \tilde{f} \quad \text{for } c \in [u_-, u_+]. \tag{6.10}$$

Hence (6.8) holds. □

Remark 6.1. Incidentally, this proof also demonstrates that the geometric entropy condition (5.13) holds for the solution of this Riemann problem, as long as $f_u(\gamma_\mp, u_\mp)$, is defined. However, for the front tracking approximation, the states u_l and u_r will be among the breakpoints of f^δ, hence f_u^δ is undefined at these points. Nevertheless, if we replace f_u in (5.13) by the one sided limit from "inside" the interval spanned by u_l and u_r, i.e., set

$$f_{-,u} = \lim_{\substack{u \to u_l \\ (u-u_l)(u_r-u_l) \ge 0}} f_u^\delta(\gamma_-, u), \quad \text{and} \quad f_{+,u} = \lim_{\substack{u \to u_r \\ (u_r-u)(u_r-u_l) \ge 0}} f_u^\delta(\gamma_+, u),$$

then the following equality holds

$$\min(0, f_{-,u}) \cdot \max(0, f_{+,u}) = 0, \qquad \text{if } u_l \ne u_r. \tag{6.11}$$

Next, let v^δ be another front tracking solution, i.e., another weak solution of (6.1) with initial values v_0^δ. Then since both v^δ and u^δ are piecewise constant, and thus certainly piecewise smooth, we can use the approach used when proving Theorem 2.1 to study

$$\|u^\delta(\cdot, t) - v^\delta(\cdot, t)\|_{L^1(\mathbf{R})}.$$

In order to show that this quantity is non-increasing in t, we need to establish (2.31) or (2.32), which in this case read

$$f^\delta\left(\gamma_-^\delta, v_-^\delta\right) \le f^\delta\left(\gamma_-^\delta, u_-^\delta\right), \tag{6.12}$$

$$f^\delta\left(\gamma_+^\delta, v_+^\delta\right) \le f^\delta\left(\gamma_+^\delta, u_+^\delta\right). \tag{6.13}$$

Recall that by assumption $u_-^\delta \geq v_-^\delta$ and $u_+^\delta \leq v_+^\delta$. We show either of (6.12)-(6.13) if $v_-^\delta \leq v_+^\delta$, the other case is similar. Now we have that one of (6.9) or (6.10) holds. We see that (6.9) implies (6.12) and (6.10) implies (6.13). Hence we have established

$$\|u^\delta(\cdot,t) - v^\delta(\cdot,t)\|_{L^1(\mathbf{R})} \leq \|u_0^\delta - v_0^\delta\|_{L^1(\mathbf{R})}. \tag{6.14}$$

If the sequences $\{v^\delta\}$ and $\{u^\delta\}$ converge in $L^1(\Pi_T)$ to u and v, then the limits are weak solutions of (1.3) with initial data $u_0(x)$ and $v_0(x)$ respectively. Thus also the limits are L^1 contractive, i.e., we have proved the following theorem:

Theorem 6.1. *Let u and v be weak solutions of* (1.3) *generated by front tracking. Then, for $t \geq 0$,*

$$\|u(\cdot,t) - v(\cdot,t)\|_{L^1(\mathbf{R})} \leq \|u_0 - v_0\|_{L^1(\mathbf{R})}. \tag{6.15}$$

Remark 6.2. Note that this theorem holds independently of whether the limits u and v are piecewise smooth, or even of bounded variation. Also, (6.15) holds independently of whether f satisfies any type of crossing condition. Furthermore, by (6.11), we see that if the limit u is piecewise smooth, then u will satisfy the geometric entropy condition (5.13), since $f_{-,u} \to f_u(\gamma_-, u_-)$ and $f_{+,u} \to f_u(\gamma_+, u_+)$ as $\delta \to 0$.

Now we shall show that the limit of any front tracking approximation to the general conservation law (5.1) satisfies the Kružkov type entropy condition (5.2). Thus we let u^δ be a weak solution to the approximate problem (6.1). We *assume* that u^δ can be constructed by front tracking. Introduce the singular mapping

$$z(\gamma, u) = \int_0^u |f_u(\gamma, v)|\, dv, \tag{6.16}$$

and set $z^\delta = z(\gamma^\delta, u^\delta)$. We shall assume that $\{z^\delta\}$ is a sequence of functions that satisfies the following three basic estimates [30, 31, 19, 6]:

$$\begin{aligned}
\|z^\delta\|_{L^\infty(\Pi_T)} &\leq C, \\
\|z^\delta(\cdot,t)\|_{BV(\mathbf{R})} &\leq C, \\
\|z(\cdot,t) - z(\cdot,s)\|_{L^1(\mathbf{R})} &\leq C(t-s),
\end{aligned}$$

for $t < T$, and where the constant C does not depend on t or on δ. Consequently, we have convergence in $L^1_{\mathrm{loc}}(\Pi_T)$ of z^δ along a subsequence. This also implies that

$$u^\delta \to u \quad \text{in } L^1_{\mathrm{loc}}(\Pi_T) \text{ as } \delta \to 0, \tag{6.17}$$

for some bounded function u.

Using that u^δ is a weak solution to (6.1), it is not hard to show that the limit u is a weak solution to (5.1) if $u_0^\delta \to u$ as $\delta \to 0$. We aim to show that the limit u satisfies the generalized Kružkov entropy condition (5.2).

We shall require that the approximation $\gamma^\delta(x)$ also has discontinuity points for all $x \in \{\xi_m\}_{m=1}^M$ for all relevant δ. In addition to these discontinuities, for a fixed δ, γ^δ has discontinuities at $\{y_{i,j}\}$. These points are ordered so that

$$\xi_i = y_{i,0} < y_{i,1} < \cdots < y_{i,N_i} < y_{i,N_i+1} = \xi_{i+1},$$

for $i = 0, \ldots, N$. Let $\gamma_{i,j+1/2}$ denote the value of γ^δ in the interval $(y_{i,j}, y_{i,j+1})$, and set

$$\Delta x_{i,j} = |I_{i,j}|, \qquad j = 0, \ldots, N_i + 1,$$

where $I_{i,j}$ denotes the intervals

$$\begin{aligned}
I_{i,j} &= \left(\frac{y_{i,j-1} + y_{i,j}}{2}, \frac{y_{i,j} + y_{i,j+1}}{2}\right), \qquad \text{for } i = 1, \ldots, N_i, \\
I_{i,0} &= \left(\xi_i, \frac{\xi_i + y_{i,1}}{2}\right), \\
I_{i,N_i+1} &= \left(\frac{y_{i,N_i} + \xi_{i+1}}{2}, \xi_{N_i+1}\right).
\end{aligned}$$

Of course, these quantities all depend on δ, but for simplicity we omit this in our notation. We also assume that

$$\lim_{\delta\to 0} \frac{f^\delta\left(\gamma_{i,j+1/2},c\right)-f^\delta\left(\gamma_{i,j-1/2},c\right)}{\Delta x_{i,j}}\chi_{I_{i,j}}(x)=f(\gamma(x),c)_x, \tag{6.18}$$

where $\chi_{I_{i,j}}$ denotes the characteristic function of the interval $I_{i,j}$. To assume this convergence is not unreasonable since γ is continuously differentiable in (ξ_i,ξ_{i+1}). In what follows, we let $u_i^{\mp}$ and $u_{i,j}^{\mp}$ denote the left and right limits of u^δ at the points ξ_i and $y_{i,j}$ respectively. Since $u^\delta(\cdot,t)$ is piecewise constant, these limits exist.

In each interval $(y_{i,j},y_{i,j+1})$ u^δ is an entropy solution of the conservation law

$$u_t^\delta+f^\delta(\gamma_{i,j+1/2},u^\delta)_x=0,$$

and hence

$$\begin{aligned}
-\int_0^T\int_{y_{i,j}}^{y_{i,j+1}}&\left(\left|u^\delta-c\right|\varphi_t+F^\delta\left(\gamma_{i,j+1/2},u^\delta,c\right)\varphi_x\right)dx\,dt\\
&+\int_0^T\left(F^\delta\left(\gamma_{i,j+1/2},u_{i,j+1}^-,c\right)-F^\delta\left(\gamma_{i,j+1/2},u_{i,j}^+,c\right)\right)dt\\
&\qquad-\int_{y_{i,j}}^{y_{i,j+1}}\left|u^\delta(x,0)-c\right|\varphi(x,0)\,dx\le 0,
\end{aligned}$$

where

$$F^\delta(\gamma,u,c)=\operatorname{sign}(u-c)\left(f^\delta(\gamma,u)-f^\delta(\gamma,c)\right).$$

Summing this for $j=0,\dots,N_i$ we find that

$$\begin{aligned}
-\int_0^T\int_{\xi_i}^{\xi_{i+1}}&\left(\left|u^\delta-c\right|\varphi_t+F^\delta\left(\gamma^\delta(x),u^\delta,c\right)\varphi_x\right)dx\,dt-\int_{\xi_i}^{\xi_{i+1}}\left|u^\delta(x,0)-c\right|\varphi(x,0)\,dx\\
&+\int_0^T\left(F^\delta\left(\gamma_{i,N_i+1/2},u_{i+1}^-,c\right)\varphi(\xi_i,t)-F^\delta\left(\gamma_{i,1/2},u_i^+,c\right)\varphi(\xi_{i+1},t)\right)dt\\
&-\int_0^T\sum_{j=1}^{N_i}\left[F^\delta\left(\gamma_{i,j+1/2},u_{i,j}^+,c\right)-F^\delta\left(\gamma_{i,j-1/2},u_{i,j}^-,c\right)\right]\varphi(y_{i,j},t).\,dt\\
&\le 0.
\end{aligned}\tag{6.19}$$

Regarding the terms in the integrand in the last term in (6.19) we can write

$$\begin{aligned}
&F^\delta\left(\gamma_{i,j+1/2},u_{i,j}^+,c\right)-F^\delta\left(\gamma_{i,j-1/2},u_{i,j}^-,c\right)\\
&=\begin{cases}
-\operatorname{sign}\left(u_{i,j}^+-c\right)\left[f\left(\gamma_{i,j+1/2},c\right)-f\left(\gamma_{i,j-1/2},c\right)\right]\\
\qquad+\left\{\operatorname{sign}\left(u_{i,j}^+-c\right)-\operatorname{sign}\left(u_{i,j}^--c\right)\right\}\left(f_{i,j}^\times-f\left(\gamma_{i,j-1/2}\right)\right),\\
\text{or}\\
-\operatorname{sign}\left(u_{i,j}^--c\right)\left[f\left(\gamma_{i,j+1/2},c\right)-f\left(\gamma_{i,j-1/2},c\right)\right]\\
\qquad+\left\{\operatorname{sign}\left(u_{i,j}^+-c\right)-\operatorname{sign}\left(u_{i,j}^--c\right)\right\}\left(f_{i,j}^\times-f\left(\gamma_{i,j+1/2}\right)\right),
\end{cases}
\end{aligned}$$

where $f_{i,j}^\times=f(\gamma_{i,j+1/2},u_{i,j}^+)=f(\gamma_{i,j-1/2},u_{i,j}^-)$. If $\operatorname{sign}(u_{i,j}^+-c)=\operatorname{sign}(u_{i,j}^--c)$ the last terms in the above expressions are zero, while if $u_{i,j}^-\le c\le u_{i,j}^+$ then since these values are chosen according to the minimal jump entropy condition (6.4)-(6.5) we have that

$$\operatorname{sign}\left(u_{i,j}^+-c\right)-\operatorname{sign}\left(u_{i,j}^--c\right)=2\quad\text{and}\quad\begin{cases}f(\gamma_{i,j-1/2},c)\ge f_{i,j}^\times,\quad\text{or}\\ f(\gamma_{i,j+1/2},c)\ge f_{i,j}^\times,\end{cases}$$

and thus in this case one of the last terms must be non-positive. If $u_{i,j}^+ < c < u_{i,j}^-$ we use (6.4) – (6.5), to conclude that

$$\operatorname{sign}\left(u_{i,j}^+ - c\right) - \operatorname{sign}\left(u_{i,j}^- - c\right) = -2 \quad \text{and} \quad \begin{cases} f(\gamma_{i,j-1/2}, c) \le f_{i,j}^\times, & \text{or} \\ f(\gamma_{i,j+1/2}, c) \le f_{i,j}^\times, \end{cases}$$

and again we find that one of the last terms is nonpositive. If the first of these last terms is nonpositive for c between $u_{i,j}^-$ and $u_{i,j}^+$, we define $u_{i,j} = u^\delta(y_{i,j}, t) = u_{i,j}^+$, otherwise we define $u_{i,j} = u^\delta(y_{i,j}, t) = u_{i,j}^-$. Using these observations, we find that

$$\begin{aligned} -\int_0^T \int_{\xi_i}^{\xi_{i+1}} \left(\left|u^\delta - c\right| \varphi_t + F^\delta\left(\gamma^\delta(x), u^\delta, c\right)\varphi_x\right) dx\, dt - \int_{\xi_i}^{\xi_{i+1}} \left|u^\delta(x,0) - c\right| \varphi(x,0)\, dx \\ + \int_0^T \left(F^\delta\left(\gamma_{i,N_i+1/2}, u_{i+1}^-, c\right)\varphi(\xi_i, t) - F^\delta\left(\gamma_{i,1/2}, u_i^+, c\right)\varphi(\xi_{i+1}, t)\right) dt \\ + \int_0^T \sum_{j=1}^{N_i} \operatorname{sign}(u_{i,j} - c)\left[f\left(\gamma_{i,j+1/2}, c\right) - f\left(\gamma_{i,j-1/2}, c\right)\right]\varphi(y_{i,j}, t)\, dt \\ \le 0 \end{aligned} \tag{6.20}$$

Now $u_{i,j} = u^\delta(y_{i,j}^-, \cdot)$ or $u_{i,j} = u^\delta(y_{i,j}^+, \cdot)$, hence if we define $\bar{u}^\delta(x,t) = u_{i,j}(t)\chi_{I_{i,j}}(x)$, and set $\bar{z}^\delta = z_{i,j}^\delta(t)\chi_{I_{i,j}}$, we have that

$$\bar{z}^\delta\left(y_{i,j}, t\right) = z^\delta\left(y_{i,j}, t\right).$$

Now we claim that the sequence $\left\{\bar{z}^\delta\right\}$ is compact in $L^1_{\mathrm{loc}}(\Pi_T)$. Trivially we have that

$$\|\bar{z}^\delta\|_{L^\infty(\Pi_T)} \le \|z^\delta\|_{L^\infty(\Pi_T)} < C, \tag{6.21}$$

and

$$\|\bar{z}^\delta(\cdot,t)\|_{BV(\mathbf{R})} \le \|z^\delta(\cdot,t)\|_{BV(\mathbf{R})} \le C. \tag{6.22}$$

Furthermore,

$$\begin{aligned} \|\bar{z}^\delta(\cdot,t) - z^\delta(\cdot,t)\|_{L^1(\mathbf{R})} &= \int_{\mathbf{R}} \left|\bar{z}^\delta(x,t) - z^\delta(x,t)\right| dx \\ &= \sum_{i,j} \int_{y_{i,j-1/2}}^{y_{i,j+1/2}} \left|z^\delta\left(y_{i,j}, t\right) - z^\delta(y,t)\right| dy \\ &\le \sum_{i,j} \int_{y_{i,j-1/2}}^{y_{i,j+1/2}} \int_y^{y_{i,j}} \left|z_x^\delta(x,t)\right| dx\, dy \\ &\le \max_{i,j} |\Delta x_{i,j}|\, \|z^\delta(\cdot,t)\|_{BV(\mathbf{R})}. \end{aligned}$$

Setting $\Delta x = \max_{i,j} \Delta x_{i,j}$, we therefore find that

$$\begin{aligned} \|\bar{z}^\delta(\cdot,t) - \bar{z}^\delta(\cdot,s)\|_{L^1(\mathbf{R})} &\le \|z^\delta(\cdot,t) - z^\delta(\cdot,s)\|_{L^1(\mathbf{R})} + 2\Delta x \|z^\delta(\cdot,t)\|_{BV(\mathbf{R})} \\ &\le C((t-s) + \Delta x). \end{aligned} \tag{6.23}$$

By the bounds (6.21), (6.22) and (6.23), the sequence $\left\{\bar{z}^\delta\right\}$ converges along a subsequence (also labeled δ) and

$$\lim_{\delta\to 0} \bar{z}^\delta = \lim_{\delta \to 0} z^\delta = z.$$

Therefore, also $\lim_{\delta\to 0} \bar{u}^\delta = u$. Now define

$$\Delta_x f^\delta(x,c) = \frac{f\left(\gamma_{i,j+1/2}, c\right) - f\left(\gamma_{i,j-1/2}, c\right)}{\Delta x_{i,j}}, \qquad \text{for } x \in I_{i,j}.$$

Using this notation, the inequality (6.20) reads

$$
\begin{aligned}
-\int_0^T\int_{\xi_i}^{\xi_{i+1}} &\left|u^\delta - c\right| \varphi_t + F^\delta\left(\gamma^\delta(x), u^\delta, c\right)\varphi_x \,dx\,dt - \int_{\xi_i}^{\xi_{i+1}} \left|u^\delta(x,0) - c\right| \varphi(x,0)\,dx \\
&- \int_0^T \sum_{i=1}^M F^\delta\left(\gamma_i^+, u_i^+, c\right) - F^\delta\left(\gamma_i u_i^-, c\right)\varphi\left(\xi_i, t\right)\,dt \\
&+ \int_0^T\int_{\xi_i}^{\xi_{i+1}} \operatorname{sign}\left(\bar u^\delta - c\right)\Delta_x f^\delta(y,c) \sum_{j=1}^{N_i} \varphi\left(y_{i,j}, t\right)\chi_{I,j}(y)\,dy\,dt \\
&\le 0.
\end{aligned}
$$

Now we can add this for $i = 0, \ldots, M$ to obtain

$$
\begin{aligned}
-\iint_{\Pi_T} &\Big(\left|u^\delta - c\right| \varphi_t + F^\delta\left(\gamma^\delta(x), u^\delta, c\right)\varphi_x\Big)\,dx\,dt - \int_{\mathbf{R}} \left|u^\delta(x,0) - c\right|\varphi(x,0)\,dx \\
&- \int_0^T \sum_{i=1}^M \left[F^\delta\left(\gamma_i^+, u_i^+, c\right) - F^\delta\left(\gamma_i^-, u_i^-, c\right)\right]\varphi\left(\xi_i, t\right)\,dt \\
&+ \int_0^T \sum_{i=0}^M \int_{\xi_i}^{\xi_{i+1}} \operatorname{sign}\left(\bar u^\delta - c\right)\Delta_x f^\delta(x,c)\sum_{j=1}^{N_i}\varphi\left(y_{i,j}, t\right)\chi_{I_{i,j}}(x)\,dx\,dt \\
&\le 0. \qquad (6.24)
\end{aligned}
$$

Now clearly

$$
\Delta_x f^\delta(x,c)\sum_{j=1}^{N_i}\varphi\left(y_{i,j},t\right)\chi_{I_{i,j}}(x)\Delta x_{i,j} \to f(\gamma(x),c)_x \quad \text{as } \delta \to 0,
$$

in each interval (ξ_i, ξ_{i+1}). Furthermore, by Lemma 4.3

$$
\operatorname{sign}\left(\bar u^\delta - c\right) \to \operatorname{sign}(u - c),
$$

for almost all (x,t) and all but at most a countable set of c's.

Regarding the middle term of (6.24), each summand is bounded by

$$
\left|f^\delta\left(\gamma_i^+, c\right) - f^\delta\left(\gamma_i^-, c\right)\right|,
$$

since (u_i^-, u_i^+) satisfies the minimal jump entropy condition. Therefore by sending δ to 0 in (6.24) we find (F is defined in (1.8))

$$
\begin{aligned}
-\iint_{\Pi_T} &\Big(|u - c|\,\varphi_t + F(\gamma(x), u, c)\varphi_x\Big)\,dx\,dt + \iint_{\Pi_T \setminus \{\xi_m\}_{m=1}^M} \operatorname{sign}(u - c)\, f(\gamma(x), c)_x \varphi\,dx\,dt \\
&- \int_0^T \sum_{m=1}^M \left|f\left(\gamma\left(\xi_m +\right), c\right) - f\left(\gamma\left(\xi_m -\right), c\right)\right|\phi(\xi_m, t)\,dt \\
&\le 0,
\end{aligned}
$$

for all but a countable set of c's and all nonnegative test functions φ. This can be rewritten as

$$
\iint_{\Pi_T \setminus \{\xi_m\}_{m=1}^M} \operatorname{sign}(u - c)\, f(\gamma(x), c)_x \varphi\,dx\,dt \le \mathcal{I}(c),
$$

where $\mathcal{I}$ is a *continuous* function of c. Thus we can proceed as in the proof of Lemma 4.4 to conclude that (5.2) holds. We have proved the following theorem.

Theorem 6.2. *Let u be a weak solution of* (5.1) *constructed by the front tracking algorithm, that is, the front tracking approximation u^δ converges to u in $L^1_{\mathrm{loc}}(\Pi_T)$. Then u satisfies the Kružkov-type entropy condition* (5.2).

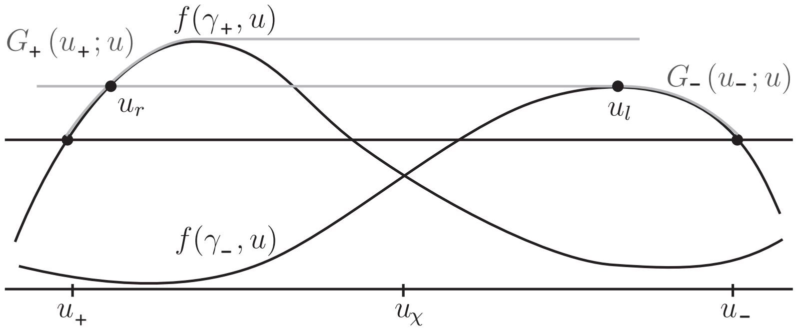

FIGURE 3. Violation of the crossing condition. The graph of $f(\gamma_+, u)$ lies above the graph of $f(\gamma_-, u)$ for u to the left of the crossing point u_χ.

7. A remark on the crossing condition

We will conclude this paper by trying to make clear why the entropy conditions of Table 2 are not strong enough to guarantee uniqueness of the entropy solution for the hyperbolic problem (5.1) without the crossing condition. Consider the situation shown in Figure 3, where the crossing condition is violated. Here we have $u_+ < u_\chi < u_-$ and the horizontal line connecting $f(\gamma_+, u_+)$ to $f(\gamma_-, u_-)$ lies above the graph of $y(c) = \min(f(\gamma_-, c), f(\gamma_+, c))$, so the entropy conditions of Table 2 are satisfied. Meanwhile, it is also clear that $f_u(\gamma_-, u_-) < 0 < f_u(\gamma_+, u_+)$, i.e., the characteristics flow away from the x axis on both sides of the shock. Intuitively, this situation implies an entropy violation (as in (d) of Figure 2), and in fact the proof of Theorem 2.1 breaks down in this situation. In Figure 3 we have also shown the functions G_- and G_+ (gray), and from these we see that the pair (u_-, u_+) is *not* the entropy solution of the Riemann problem. However the pair (u_l, u_r) (indicated in the figure) is, and between u_l and u_r and for this discontinuity the proof of Theorem 2.1 would not break down.

We have performed some numerical experiments to determine whether or not schemes of the type described in Section 4 actually admit such entropy violations. Our tests indicate that they do not. We conclude that the limits of such schemes satisfy additional entropy conditions that are not captured by the Kružkov-type entropy inequalities (4.9). Discovering these additional entropy conditions is one of our goals, and we plan to address this question in another paper.

Appendix A. A Kružkov-type integral inequality

The purpose of this section is to prove Theorem A.1 below, which was the starting point for proving L^1 stability of entropy solutions in Section 2. We prove Theorem A.1 by adapting Kružkov's "doubling of variables" proof as developed by Carrillo [8] for the homogeneous Dirichlet problem for degenerate parabolic equations with x-independent flux function, see [25] for the initial value problem with x dependent flux function. In particular, our presentation below follows [25] closely. For the convenience of the reader we repeat some of the arguments from [8, 25].

In what follows, we let

$$M := \|u\|_{L^\infty(\Pi_T)}, \qquad l = A(-M), \qquad L = A(M),$$

and define the function $A^{-1} : [l, L] \to \mathbf{R}$ by

$$A^{-1}(r) := \min\Big\{\xi \in [-M, M] \;\Big|\; A(\xi) = r\Big\}.$$

Note that $A^{-1}(\cdot)$ is a lower semicontinuous function. Next we introduce the sets

$$\mathcal{H} = \Big\{r \in [l, L] \;\Big|\; A^{-1}(\cdot) \text{ is discontinuous at } r\Big\}, \qquad \mathcal{P} = [l, L] \setminus \mathcal{H}.$$

We say that a function ϕ vanishes in a neighborhood of $x = \xi_m$ if

$$\phi(x, t) = 0, \qquad \forall (x, t) \in [\xi_m - h_m, \xi_m + h_m] \times [0, T], \tag{A.1}$$

for some $h_m > 0$, $m = 1, \dots, M$. If $\phi \in \mathcal{D}(\Pi_T)$ and (A.1) holds, we write $\phi \in \mathcal{D}\left(\Pi_T \setminus \{\xi_m\}_{m=1}^M\right)$.

Lemma A.1 (parabolic dissipation measure)**.** *Let u be a weak solution of* (1.1). *Then we have, for any $0 \le \phi \in \mathcal{D}\left(\Pi_T \setminus \{\xi_m\}_{m=1}^M\right)$,*

$$\begin{aligned}
&\iint_{\Pi_T} \Big(|u-c|\,\phi_t + \operatorname{sign}(u-c)\left(f(\gamma(x),u) - f(\gamma(x),c)\right)\phi_x + |A(u)-A(c)|\,\phi_{xx} \Big)\,dt\,dx \\
&\qquad - \iint_{\Pi_T \setminus \{\xi_m\}_{m=1}^M} \operatorname{sign}(u-c) f(\gamma(x),c)_x \phi\,dt\,dx \\
&\qquad\quad = \lim_{\varepsilon \downarrow 0} \iint_{\Pi_T} \operatorname{sign}_\varepsilon'\left(A(u)-A(c)\right)\left(\partial_x A(u)\right)^2 \phi\,dt\,dx,
\end{aligned} \tag{A.2}$$

for all $c \in \mathbf{R}$ such that $A(c) \in \mathcal{P}$. Here, for any $\varepsilon > 0$,

$$\operatorname{sign}_\varepsilon(w) = \begin{cases} -1, & w \le -\varepsilon, \\ w/\varepsilon, & -\varepsilon < w < \varepsilon, \\ 1 & w \ge \varepsilon. \end{cases}$$

Proof. Let ϕ and c be as in the lemma. For each $\varepsilon > 0$, $\operatorname{sign}_\varepsilon\left(A(u)-A(c)\right)\phi$ belongs to $L^2(0,T;H^1(\mathbf{R}))$ and it also holds that

$$\operatorname{sign}_\varepsilon\left(A(u)-A(c)\right)\phi = 0 \quad \text{on } [\xi_m - h_m, \xi_m + h_m] \times (0,T),\ m = 1,\dots,M. \tag{A.3}$$

Using $\operatorname{sign}_\varepsilon\left(A(u)-A(c)\right)\phi$ as a test function in (1.5) yields (see [8, 25] for details)

$$\begin{aligned}
&\underbrace{\iint_{\Pi_T} \left(\int_c^u \operatorname{sign}_\varepsilon\left(A(z)-A(c)\right)\,dz \right)\phi_t\,dt\,dx}_{I_1} + \underbrace{\iint_{\Pi_T} f(\gamma(x),u)\big[\operatorname{sign}_\varepsilon\left(A(u)-A(c)\right)\phi\big]_x\,dt\,dx}_{I_2} \\
&\qquad - \underbrace{\iint_{\Pi_T} A(u)_x \big[\operatorname{sign}_\varepsilon\left(A(u)-A(c)\right)\phi\big]_x\,dt\,dx}_{I_3} = 0.
\end{aligned}$$

We have for $c \in \mathbf{R}$ such that $A(c) \in \mathcal{P}$ (see [8, 25])

$$\lim_{\varepsilon \downarrow 0} I_1 = \iint_{\Pi_T} |u-c|\,\phi_t\,dt\,dx. \tag{A.4}$$

Using (A.3), it is not hard to check that

$$\begin{aligned}
I_2 = &\underbrace{\iint_{\Pi_T} \left(f(\gamma(x),u) - f(\gamma(x),c)\right)\big[\operatorname{sign}_\varepsilon\left(A(u)-A(c)\right)\phi\big]_x\,dt\,dx}_{I_{2,1}} \\
&\quad - \underbrace{\iint_{\Pi_T \setminus \{\xi_m\}_{m=1}^M} \operatorname{sign}_\varepsilon\left(A(u)-A(c)\right) f(\gamma(x),c)_x \phi\,dt\,dx}_{I_{2,2}}.
\end{aligned}$$

Since γ is regular off $\{\xi_m\}_{m=1}^M$, we have

$$\lim_{\varepsilon \downarrow 0} I_{2,2} = \iint_{\Pi_T \setminus \{\xi_m\}_{m=1}^M} \operatorname{sign}_\varepsilon(u-c)\, f(\gamma(x),c)_x \phi\,dt\,dx. \tag{A.5}$$

Here we have used that

$$\operatorname{sign}(A(u)-A(c)) = \operatorname{sign}(u-c) \quad \text{a.e. in } \Pi_T, \tag{A.6}$$

for each $c \in \mathbf{R}$ such that $A(c) \in \mathcal{P}$.

Next, we write

$$
\begin{aligned}
I_{2,1} = &\underbrace{\iint_{\Pi_T} \operatorname{sign}_\varepsilon\left(A(u)-A(c)\right)\left(f(\gamma(x),u)-f(\gamma(x),c)\right)\phi_x\,dt\,dx}_{I_{2,1}^1} \\
&+ \underbrace{\iint_{\Pi_T} \left(f(\gamma(x),u)-f(\gamma(x),c)\right)\operatorname{sign}_\varepsilon'\left(A(u)-A(c)\right)A(u)_x\phi\,dt\,dx}_{I_{2,1}^2}\,.
\end{aligned}
\tag{A.7}
$$

By (A.6), we have

$$
\lim_{\varepsilon\downarrow 0} I_{2,1}^1 = \iint_{\Pi_T} \operatorname{sign}(u-c)\left(f(\gamma(x),u)-f(\gamma(x),c)\right)\phi_x\,dt\,dx. \tag{A.8}
$$

Continuing, first replacing integration over Π_T in $I_{2,1}^2$ by integration over $\Pi_T\setminus\{\xi_m\}_{m=1}^M$ and keeping in mind that c is such that $A(c)\in\mathcal{P}$, we note that

$$
I_{2,1}^2 = \underbrace{\iint_{\Pi_T\setminus\{\xi_m\}_{m=1}^M} \mathcal{Q}_\varepsilon(\gamma(x),A(u))_x\phi\,dt\,dx}_{I_{2,1}^{2,1}} - \underbrace{\iint_{\Pi_T\setminus\{\xi_m\}_{m=1}^M} \mathcal{Q}_{\varepsilon\gamma}(\gamma(x),A(u))\cdot\gamma'(x)\phi\,dt\,dx}_{I_{2,1}^{2,2}},
$$

where $\mathbf{R}\ni z\mapsto \mathcal{Q}_\varepsilon(\gamma(x),z)$ is defined as

$$
\begin{aligned}
\mathcal{Q}_\varepsilon(\gamma(x),z) &= \int_0^z \operatorname{sign}_\varepsilon'\left(\zeta-A(c)\right)\Big(f(\gamma(x),A^{-1}(\zeta))-f(\gamma(x),A^{-1}(A(c)))\Big)\,d\zeta \\
&= \frac{1}{\varepsilon}\int_{\min(z,A(c)-\varepsilon)}^{\min(z,A(c)+\varepsilon)} \Big(f(\gamma(x),A^{-1}(\zeta))-f(\gamma(x),A^{-1}(A(c)))\Big)\,d\zeta,
\end{aligned}
$$

and $\mathbf{R}\ni z\mapsto \mathcal{Q}_{\varepsilon\gamma}(\gamma(x),z)$ is similarly defined by replacing the scalar function f by the vector f_γ. Since $\mathbf{R}\ni u\mapsto f(\gamma(x),u)$ and $\mathbf{R}\ni u\mapsto f_\gamma(\gamma(x),u)$ are locally Lipschitz continuous,

$$
\lim_{\varepsilon\downarrow 0}\mathcal{Q}_\varepsilon(\gamma(x),z)=0, \qquad \lim_{\varepsilon\downarrow 0}\mathcal{Q}_{\varepsilon\gamma}(\gamma(x),z)=0, \qquad \forall z\in[l,L]
$$

for any $x\in\mathbf{R}$. Using (A.3) when doing an integration by parts, it follows

$$
I_{2,1}^{2,1} = -\iint_{\Pi_T\setminus\{\xi_m\}_{m=1}^M} \mathcal{Q}_\varepsilon(\gamma(x),A(u))\phi_x\,dt\,dx,
$$

so that by the dominated convergence theorem $\lim_{\varepsilon\downarrow 0} I_{2,1}^{2,1}=0$. Since γ,γ' are uniformly bounded functions away from the discontinuity points $\{\xi_m\}_{m=1}^M$, another application of the dominated convergence theorem gives $\lim_{\varepsilon\downarrow 0} I_{2,1}^{2,2}=0$. Summarizing, we have

$$
\lim_{\varepsilon\downarrow 0} I_{2,1}^2 = 0. \tag{A.9}
$$

Finally, thanks to (A.3), we have

$$
\begin{aligned}
\lim_{\varepsilon\downarrow 0} I_2 = &\iint_{\Pi_T} \operatorname{sign}_\varepsilon(u-c)\,A(u)_x\phi_x\,dt\,dx \\
&+ \lim_{\varepsilon\downarrow 0}\iint_{\Pi_T} \operatorname{sign}_\varepsilon'\left(A(u)-A(c)\right)\left(A(u)_x\right)^2\phi\,dt\,dx.
\end{aligned}
\tag{A.10}
$$

Collecting (A.4), (A.5), (A.8), (A.9), and (A.10), we obtain the desired equality (A.2). □

Theorem A.1 (A Kružkov-type integral inequality)**.** *For any two entropy solutions $v=v(x,t)$ and $u=u(x,t)$ the integral inequality* (2.22) *holds for any $0\le\phi\in\mathcal{D}\left(\Pi_T\setminus\{\xi_m\}_{m=1}^M\right)$. The right-hand side in* (2.22) *is zero if $\gamma'(x)=0$ for a.e. $x\in\mathbf{R}$ (γ is piecewise constant).*

Proof. Let $0 \le \phi \in \mathcal{D}\left(\left(\Pi_T \setminus \{\xi_m\}_{m=1}^M\right)^2\right)$, $\phi = \phi(x,t,y,s)$, $v = v(x,t)$, and $u = u(y,s)$. We shall need the "hyperbolic" sets

$$\mathcal{E}_v = \Big\{(x,t) \in \Pi_T : A(v(x,t)) \in \mathcal{H}\Big\}, \qquad \mathcal{E}_u = \Big\{(y,s) \in \Pi_T : A(u(y,s)) \in \mathcal{H}\Big\}.$$

Observe that

$$\operatorname{sign}(v-u) = \operatorname{sign}(A(v)-A(u)) \quad \text{a.e. in } \Big[\Pi_T \times (\Pi_T \setminus \mathcal{E}_v)\Big] \bigcup \Big[(\Pi_T \setminus \mathcal{E}_u) \times \Pi_T\Big] \tag{A.11}$$

and

$$A(v)_x = 0 \text{ a.e. in } \mathcal{E}_v, \qquad A(u)_y = 0 \text{ a.e. in } \mathcal{E}_u. \tag{A.12}$$

From the definition of entropy solution for $v = v(x,t)$ with $c = u(y,s)$ and Lemma A.1 for $v = v(x,t)$, we get (see also [8, 25])

$$\begin{aligned}
&-\iint_{\Pi_T} \Big(|v-u|\phi_t + F(\gamma(x),v,u)\phi_x + |A(v)-A(u)|\,\phi_{xx}\Big)\,dt\,dx \\
&\quad + \iint_{\Pi_T \setminus \{\xi_m\}_{m=1}^M} \operatorname{sign}(v-u) f(\gamma(x),u)_x \phi\,dt\,dx \\
&\qquad \le -\lim_{\varepsilon \downarrow 0} \iint_{\Pi_T \setminus \mathcal{E}_u} \operatorname{sign}'_\varepsilon\left(A(v)-A(u)\right)\big(A(v)_x\big)^2 \phi\,dt\,dx,
\end{aligned} \tag{A.13}$$

where F is defined in (1.8). Integrating (A.13) over $(y,s) \in \Pi_T$ and then using the first part of (A.12), we find that

$$\begin{aligned}
&-\iiiint_{(\Pi_T)^2} \Big(|v-u|\phi_t + F(\gamma(x),v,u)\phi_x + |A(v)-A(u)|\,\phi_{xx}\Big)\,dt\,dx\,ds\,dy \\
&\quad + \iiiint_{(\Pi_T \setminus \{\xi_m\}_{m=1}^M)^2} \operatorname{sign}(v-u) f(\gamma(x),u)_x \phi\,dt\,dx\,ds\,dy \\
&\qquad = -\lim_{\varepsilon \downarrow 0} \iiiint_{(\Pi_T \setminus \mathcal{E}_u)\times(\Pi_T \setminus \mathcal{E}_v)} \operatorname{sign}'_\varepsilon\left(A(v)-A(u)\right)\big(A(v)_x\big)^2 \phi\,dt\,dx\,ds\,dy,
\end{aligned} \tag{A.14}$$

where we have used the obvious fact

$$\begin{aligned}
&\iiiint_{\Pi_T \times (\Pi_T \setminus \{\xi_m\}_{m=1}^M)} \operatorname{sign}(v-u) f(\gamma(x),u)_x \phi\,dt\,dx\,ds\,dy \\
&\quad = \iiiint_{(\Pi_T \setminus \{\xi_m\}_{m=1}^M)^2} \operatorname{sign}(v-u) f(\gamma(x),u)_x \phi\,dt\,dx\,ds\,dy.
\end{aligned}$$

Similarly, for the entropy solution $u = u(y,s)$ with $c = v(x,t)$, using Lemma A.1 for $u = u(y,s)$ and the second part of (A.12),

$$\begin{aligned}
&-\iiiint_{(\Pi_T)^2} \Big(|u-v|\phi_t + F(\gamma(y),u,v)\phi_x + |A(u)-A(v)|\,\phi_{xx}\Big)\,dt\,dx\,ds\,dy \\
&\quad + \iiiint_{(\Pi_T \setminus \{\xi_m\}_{m=1}^M)^2} \operatorname{sign}(u-v) f(\gamma(y),v)_y \phi\,dt\,dx\,ds\,dy \\
&\qquad = -\lim_{\varepsilon \downarrow 0} \iiiint_{(\Pi_T \setminus \mathcal{E}_u)\times(\Pi_T \setminus \mathcal{E}_v)} \operatorname{sign}'_\varepsilon\left(A(u)-A(v)\right)\big(A(u)_x\big)^2 \phi\,dt\,dx\,ds\,dy,
\end{aligned} \tag{A.15}$$

Note that we can write, for each $(x,t) \in \Pi_T \setminus \{\xi_m\}_{m=1}^M$,

$$\begin{aligned}
&F(\gamma(x),v,u)\phi_x - \operatorname{sign}(v-u) f(\gamma(x),u)_x \phi \\
&\quad = \operatorname{sign}(v-u)\left(f(\gamma(x),v) - f(\gamma(y),u)\right)\phi_x - \operatorname{sign}(v-u)\Big[\left(f(\gamma(x),u) - f(\gamma(y),u)\right)\phi\Big]_x,
\end{aligned}$$

so that

(A.16)
$$
\begin{aligned}
-&\iiiint_{(\Pi_T)^2} F(\gamma(x),v,u)\phi_x \,dt\,dx\,ds\,dy + \iiiint_{(\Pi_T\setminus\{\xi_m\}_{m=1}^M)^2} \operatorname{sign}(v-u) f(\gamma(x),u)_x\phi \,dt\,dx\,ds\,dy \\
&= -\iiiint_{(\Pi_T\setminus\{\xi_m\}_{m=1}^M)^2} \Big(F(\gamma(x),v,u)\phi_x - \operatorname{sign}(v-u) f(\gamma(x),u)_x\phi\Big)\,dt\,dx\,ds\,dy \\
&= -\iiiint_{(\Pi_T\setminus\{\xi_m\}_{m=1}^M)^2} \operatorname{sign}(v-u)\left(f(\gamma(x),v)-f(\gamma(y),u)\right)\phi_x \,dt\,dx\,ds\,dy \\
&\qquad + \iiiint_{(\Pi_T\setminus\{\xi_m\}_{m=1}^M)^2} \operatorname{sign}(v-u)\Big[\left(f(\gamma(x),u)-f(\gamma(y),u)\right)\phi\Big]_x \,dt\,dx\,ds\,dy \\
&= -\iiiint_{(\Pi_T)^2} \operatorname{sign}(v-u)\left(f(\gamma(x),v)-f(\gamma(y),u)\right)\phi_x \,dt\,dx\,ds\,dy \\
&\qquad + \iiiint_{(\Pi_T\setminus\{\xi_m\}_{m=1}^M)^2} \operatorname{sign}(v-u)\Big[\left(f(\gamma(x),u)-f(\gamma(y),u)\right)\phi\Big]_x \,dt\,dx\,ds\,dy.
\end{aligned}
$$

Similarly, writing, for each $(y,s)\in\Pi_T\setminus\{\xi_m\}_{m=1}^M$,
$$
\begin{aligned}
&F(\gamma(y),u,v)\phi_y - \operatorname{sign}(u-v) f(\gamma(y),v)_y\phi \\
&\quad = \operatorname{sign}(v-u)\left(f(\gamma(x),v)-f(\gamma(y),u)\right)\phi_y - \operatorname{sign}(v-u)\Big[\left(f(\gamma(x),v)-f(\gamma(y),v)\right)\phi\Big]_y,
\end{aligned}
$$
we get

(A.17)
$$
\begin{aligned}
-&\iiiint_{(\Pi_T)^2} F(\gamma(y),u,v)\phi_y \,dt\,dx\,ds\,dy + \iiiint_{(\Pi_T\setminus\{\xi_m\}_{m=1}^M)^2} \operatorname{sign}(u-v) f(\gamma(y),v)_y\phi \,dt\,dx\,ds\,dy \\
&= -\iiiint_{(\Pi_T)^2} \operatorname{sign}(v-u)\left(f(\gamma(x),v)-f(\gamma(y),u)\right)\phi_y \,dt\,dx\,ds\,dy \\
&\qquad + \iiiint_{(\Pi_T\setminus\{\xi_m\}_{m=1}^M)^2} \operatorname{sign}(v-u)\Big[\left(f(\gamma(x),v)-f(\gamma(y),v)\right)\phi\Big]_y \,dt\,dx\,ds\,dy.
\end{aligned}
$$

Before we continue, let us introduce the notations
$$
\begin{aligned}
&\partial_{t+s} = \partial_t + \partial_s, \qquad \partial_{x+y} = \partial_x + \partial_y, \\
&\partial_{x+y}^2 = \left(\partial_x + \partial_y\right)^2 = \partial_x^2 + 2\partial_x\partial_y + \partial_y^2.
\end{aligned}
$$
Taking (A.16) and (A.17) into account when adding (A.15) and (A.14) yields

(A.18)
$$
\begin{aligned}
-&\iiiint_{(\Pi_T)^2} \Bigg(|v-u|\partial_{t+s}\phi + \operatorname{sign}(v-u)\big(f(\gamma(x),v)-f(\gamma(y),u)\big)\partial_{x+y}\phi \\
&\qquad\qquad + |A(v)-A(u)|\,\partial_{x+y}^2\phi\Bigg)\,dt\,dx\,ds\,dy \\
&+ \iiiint_{(\Pi_T\setminus\{\xi_m\}_{m=1}^M)^2} \operatorname{sign}(v-u)\Bigg(\Big[\left(f(\gamma(x),u)-f(\gamma(y),u)\right)\phi\Big]_x \\
&\qquad\qquad + \partial_y\Big[\left(f(\gamma(x),v)-f(\gamma(y),v)\right)\phi\Big]\Bigg)\,dt\,dx\,ds\,dy \\
&\le -\lim_{\varepsilon\downarrow 0}\iiiint_{(\Pi_T\setminus\mathcal{E}_u)\times(\Pi_T\setminus\mathcal{E}_v)} \operatorname{sign}_\varepsilon'\left(A(v)-A(u)\right)\Big(\left(A(v)_x\right)^2 + \left(A(u)_y\right)^2\Big)\phi\,dt\,dx\,ds\,dy \\
&\qquad - \iiiint_{(\Pi_T)^2} |A(v)-A(u)|\,2\partial_x\phi_y\,dt\,dx\,ds\,dy := \mathrm{RHS}.
\end{aligned}
$$

After performing a couple of integration by parts in the second term of RHS, then using (A.12), and finally adding the result to the first term of RHS, we find that (see [8, 25] for details) RHS ≤ 0.

We next introduce a non-negative function $\delta \in C_0^\infty$, satisfying $\delta(\sigma) = \delta(-\sigma)$, $\delta(\sigma) = 0$ for $|\sigma| \geq 1$, and $\int_{\mathbf{R}} \delta(\sigma)\, d\sigma = 1$. For $\rho > 0$ and $z \in \mathbf{R}$, let $\delta_\rho(z) = \frac{1}{\rho}\delta\left(\frac{z}{\rho}\right)$. We take our test function $\phi = \phi(x,t,y,s)$ to be of the form

$$\phi(x,t,y,s) = \varphi\left(\tfrac{x+y}{2}, \tfrac{t+s}{2}\right)\delta_\rho\left(\tfrac{x-y}{2}\right)\delta_\rho\left(\tfrac{t-s}{2}\right), \tag{A.19}$$

where $0 \leq \varphi \in \mathcal{D}\left(\Pi_T \setminus \{\xi_m\}_{m=1}^M\right)$ satisfies

$$\varphi(x,t) = 0, \quad \forall (x,t) \in [\xi_m - h_m, \xi_m + h_m] \times [0,T], \tag{A.20}$$

for some small $h_m > 0$, $m = 1, \dots, M$. By making sure that

$$\rho < \min_{m=1,\dots,M} h_m, \tag{A.21}$$

one can check that ϕ belongs to $\mathcal{D}\left(\left(\Pi_T \setminus \{\xi_m\}_{m=1}^M\right)^2\right)$.

We have

$$\begin{aligned}
\partial_{t+s}\phi(x,t,y,s) &= \partial_{t+s}\varphi\left(\tfrac{x+y}{2}, \tfrac{t+s}{2}\right)\delta_\rho\left(\tfrac{x-y}{2}\right)\delta_{\rho_0}\left(\tfrac{t-s}{2}\right),\\
\partial_{x+y}\phi(x,t,y,s) &= \partial_{x+y}\varphi\left(\tfrac{x+y}{2}, \tfrac{t+s}{2}\right)\delta_\rho\left(\tfrac{x-y}{2}\right)\delta_\rho\left(\tfrac{t-s}{2}\right),\\
\partial_{x+y}^2\phi(x,t,y,s) &= \partial_{x+y}^2\varphi\left(\tfrac{x+y}{2}, \tfrac{t+s}{2}\right)\delta_\rho\left(\tfrac{x-y}{2}\right)\delta_\rho\left(\tfrac{t-s}{2}\right).
\end{aligned} \tag{A.22}$$

Equipped with (A.22), (A.18) yields (after some work)

$$\begin{aligned}
&-\iiiint_{(\Pi_T)^2}\Big(\overline{I}_{\mathrm{time}}(x,t,y,s) + \overline{I}_{\mathrm{conv}}(x,t,y,s) + \overline{I}_{\mathrm{diff}}(x,t,y,s)\Big)\delta_\rho\left(\tfrac{x-y}{2}\right)\delta_\rho\left(\tfrac{t-s}{2}\right)\, dt\, dx\, ds\, dy\\
&\quad\leq \iiiint_{(\Pi_T \setminus \{\xi_m\}_{m=1}^M)^2}\Big(\overline{I}^1_{\mathrm{flux}}(x,t,y,s) + \overline{I}^2_{\mathrm{flux}}(x,t,y,s) + \overline{I}^3_{\mathrm{flux}}(x,t,y,s)\Big)\, dt\, dx\, ds\, dy,
\end{aligned} \tag{A.23}$$

where

$$\begin{aligned}
\overline{I}_{\mathrm{time}}(x,t,y,s) &= |v(x,t) - u(y,s)|\partial_{t+s}\varphi\left(\tfrac{x+y}{2}, \tfrac{t+s}{2}\right),\\
\overline{I}_{\mathrm{conv}}(x,t,y,s) &= \mathrm{sign}(v(x,t) - u(y,s))\begin{bmatrix} f(\gamma(x), v(x,t)) \\ -f(\gamma(y), u(y,s)) \end{bmatrix}\partial_{x+y}\varphi\left(\tfrac{x+y}{2}, \tfrac{t+s}{2}\right),\\
\overline{I}_{\mathrm{diff}}(x,t,y,s) &= |A(v(x,t)) - A(u(y,s))|\,\partial_{x+y}^2\varphi\left(\tfrac{x+y}{2}, \tfrac{t+s}{2}\right)\\
\overline{I}^1_{\mathrm{flux}}(x,t,y,s) &= -\mathrm{sign}(v(x,t) - u(y,s))\\
&\qquad\times\begin{bmatrix} \gamma'(x)\cdot f_\gamma(\gamma(x), u(y,s)) \\ -\gamma'(y)\cdot f_\gamma(\gamma(y), v(x,t)) \end{bmatrix}\varphi\left(\tfrac{x+y}{2}, \tfrac{t+s}{2}\right)\delta_\rho\left(\tfrac{x-y}{2}\right)\delta_\rho\left(\tfrac{t-s}{2}\right),\\
\overline{I}^2_{\mathrm{flux}}(x,t,y,s) &= -\mathrm{sign}(v(x,t) - u(y,s))\delta_\rho\left(\tfrac{x-y}{2}\right)\delta_\rho\left(\tfrac{t-s}{2}\right)\\
&\qquad\times\begin{bmatrix} \partial_x\varphi\left(\tfrac{x+y}{2}, \tfrac{t+s}{2}\right)\left(f(\gamma(x), u(y,s)) - f(\gamma(y), u(y,s))\right) \\ +\partial_y\varphi\left(\tfrac{x+y}{2}, \tfrac{t+s}{2}\right)\left(f(\gamma(x), v(x,t)) - f(\gamma(y), v(x,t))\right) \end{bmatrix},\\
\overline{I}^3_{\mathrm{flux}}(x,t,y,s) &= \begin{bmatrix} F(\gamma(x), v(x,t), u(y,s)) \\ -F(\gamma(y), v(x,t), u(y,s)) \end{bmatrix}\varphi\left(\tfrac{x+y}{2}, \tfrac{t+s}{2}\right)\delta_\rho\left(\tfrac{x-y}{2}\right)_x\delta_\rho\left(\tfrac{t-s}{2}\right).
\end{aligned}$$

The functions $\overline{I}^i_{\mathrm{flux}}$, $i = 1,2,3$, are defined only on $\left(\Pi_T \setminus \{\xi_m\}_{m=1}^M\right)^2$.

We now use the change of variables

$$\tilde{x} = \tfrac{x+y}{2}, \quad \tilde{t} = \tfrac{t+s}{2}, \quad z = \tfrac{x-y}{2}, \quad \tau = \tfrac{t-s}{2},$$

which maps $(\Pi_T)^2$ into $\Omega \subset \mathbf{R}^4$, where

$$\Omega = \left\{\left(\tilde{x}, \tilde{t}, z, \tau\right) \in \mathbf{R}^4 : 0 < \tilde{t} \pm \tau < T\right\}.$$

Moreover, this change of variables maps $(\Pi_T)^2 \setminus \{\xi_m\}_{m=1}^M$ into Ω_ξ, where

$$\Omega_\xi = \left\{\left(\tilde{x}, \tilde{t}, z, \tau\right) \in \Omega : \tilde{x} \pm z \neq \xi_m, m = 1, \dots, M\right\}.$$

With this change of variables,

$$\partial_{t+s}\varphi\left(\tfrac{x+y}{2},\tfrac{t+s}{2}\right)=\partial_{\tilde t}\varphi\left(\tilde x,\tilde t\right),\qquad \partial_{x+y}\varphi(x,t,y,s)=\partial_{\tilde x}\varphi\left(\tilde x,\tilde t\right).$$
$$\partial^2_{x+y}\varphi\left(\tfrac{x+y}{2},\tfrac{t+s}{2}\right)=\partial^2_{\tilde x}\varphi\left(\tilde x,\tilde t\right)=\varphi\left(\tilde x,\tilde t\right)_{\tilde x\tilde x}.$$

We may now write (A.23) as

$$\begin{aligned}&-\iiiint_{\Omega}\Big(I_{\text{time}}\left(\tilde x,\tilde t,z,\tau\right)+I_{\text{conv}}\left(\tilde x,\tilde t,z,\tau\right)+I_{\text{diff}}\left(\tilde x,\tilde t,z,\tau\right)\Big)\delta_\rho(z)\delta_\rho(\tau)\,d\tilde t\,d\tilde x\,d\tau\,dz\\&\le\iiiint_{\Omega_\xi}\Big(I^1_{\text{flux}}\left(\tilde x,\tilde t,z,\tau\right)+I^2_{\text{flux}}\left(\tilde x,\tilde t,z,\tau\right)+I^3_{\text{flux}}\left(\tilde x,\tilde t,z,\tau\right)\Big)\,d\tilde t\,d\tilde x\,d\tau\,dz,\end{aligned}\tag{A.24}$$

where

$$\begin{aligned}I_{\text{time}}\left(\tilde x,\tilde t,z,\tau\right)&=\left|v\left(\tilde x+z,\tilde t+\tau\right)-u\left(\tilde x-z,\tilde t-\tau\right)\right|\partial_{\tilde t}\varphi\left(\tilde x,\tilde t\right),\\
I_{\text{conv}}\left(\tilde x,\tilde t,z,\tau\right)&=\operatorname{sign}\left(v\left(\tilde x+z,\tilde t+\tau\right)-u\left(\tilde x-z,\tilde t-\tau\right)\right)\\
&\qquad\times\begin{bmatrix}f\left(\gamma\left(\tilde x+z\right),v\left(\tilde x+z,\tilde t+\tau\right)\right)\\-f\left(\gamma\left(\tilde x-z\right),u\left(\tilde x-z,\tilde t-\tau\right)\right)\end{bmatrix}\partial_{\tilde x}\varphi\left(\tilde x,\tilde t\right),\\
I_{\text{diff}}\left(\tilde x,\tilde t,z,\tau\right)&=\left|A\left(v\left(\tilde x+z,\tilde t+\tau\right)\right)-A\left(u\left(\tilde x-z,\tilde t-\tau\right)\right)\right|\partial^2_{\tilde x}\varphi\left(\tilde x,\tilde t\right)\\
I^1_{\text{flux}}\left(\tilde x,\tilde t,z,\tau\right)&=-\operatorname{sign}\left(v\left(\tilde x+z,\tilde t+\tau\right)-u\left(\tilde x-z,\tilde t-\tau\right)\right)\\
&\qquad\times\begin{bmatrix}\gamma'\left(\tilde x+z\right)\cdot f_\gamma\left(\gamma\left(\tilde x+z\right),u\left(\tilde x-z,\tilde t-\tau\right)\right)\\-\gamma'\left(\tilde x-z\right)\cdot f_\gamma\left(\gamma\left(\tilde x-z\right),v\left(\tilde x+z,\tilde t+\tau\right)\right)\end{bmatrix}\varphi\left(\tilde x,\tilde t\right)\delta_\rho(z)\delta_\rho(\tau),\\
I^2_{\text{flux}}\left(\tilde x,\tilde t,z,\tau\right)&=-\operatorname{sign}\left(v\left(\tilde x+z,\tilde t+\tau\right)-u\left(\tilde x-z,\tilde t-\tau\right)\right)\\
&\qquad\times\left(\begin{bmatrix}f\left(\gamma\left(\tilde x+z\right),u\left(\tilde x-z,\tilde t-\tau\right)\right)\\-f\left(\gamma\left(\tilde x-z\right),u\left(\tilde x-z,\tilde t-\tau\right)\right)\end{bmatrix}\right.\\
&\qquad\qquad\left.+\begin{bmatrix}f\left(\gamma\left(\tilde x+z\right),v\left(\tilde x+z,\tilde t+\tau\right)\right)\\-f\left(\gamma\left(\tilde x-z\right),v\left(\tilde x+z,\tilde t+\tau\right)\right)\end{bmatrix}\right)\partial_{\tilde x}\varphi\left(\tilde x,\tilde t\right)\delta_\rho(z)\delta_\rho(\tau),\\
I^3_{\text{flux}}\left(\tilde x,\tilde t,z,\tau\right)&=\begin{bmatrix}F\left(\gamma\left(\tilde x+z\right),v\left(\tilde x+z,\tilde t+\tau\right),u\left(\tilde x-z,\tilde t-\tau\right)\right)\\-F\left(\gamma\left(\tilde x-z\right),v\left(\tilde x+z,\tilde t+\tau\right),u\left(\tilde x-z,\tilde t-\tau\right)\right)\end{bmatrix}\varphi\left(\tilde x,\tilde t\right)\partial_z\delta_\rho(z)\delta_\rho(\tau).\end{aligned}$$

The functions I^i_{flux}, $i=1,2,3$, are defined only on Ω_ξ.

By repeatedly employing Lebesgue's differentiation theorem, to obtain the following limits

$$\begin{aligned}\lim_{\rho\downarrow0}\iiiint_{\Omega}I_{\text{time}}\left(\tilde x,\tilde t,z,\tau\right)\delta_\rho(z)\delta_\rho(\tau)\,d\tilde t\,d\tilde x\,d\tau\,dz&=\iint_{\Pi_T}\left|v(x,t)-u(x,t)\right|\partial_t\varphi(x,t)\,dt\,dx,\\
\lim_{\rho\downarrow0}\iiiint_{\Omega}I_{\text{conv}}\left(\tilde x,\tilde t,z,\tau\right)\delta_\rho(z)\delta_\rho(\tau)\,d\tilde t\,d\tilde x\,d\tau\,dz&=\iint_{\Pi_T}F(\gamma(x),u,v)\partial_x\varphi(x,t)\,dt\,dx,\\
\lim_{\rho\downarrow0}\iiiint_{\Omega}I_{\text{diff}}\left(\tilde x,\tilde t,z,\tau\right)\delta_\rho(z)\delta_\rho(\tau)\,d\tilde t\,d\tilde x\,d\tau\,dz&=\iint_{\Pi_T}\left|A(v(x,t))-A(u(x,t))\right|\partial_x^2\varphi(x,t)\,dt\,dx.\end{aligned}$$

Let consider the term I^1_{flux}. Note that

$$I^1_{\text{flux}}\left(\tilde x,\tilde t,z,\tau\right)=0,$$

if $\tilde x\in[\xi_m-h_m,\xi_m+h_m]$ for some $m=1,\dots,M$, since then $\varphi\left(\tilde x,\tilde t\right)=0$ for any $\tilde t$, or if $|z|\ge\rho$. On the other hand, if $\tilde x\notin[\xi_m-h_m,\xi_m+h_m]$ for all $m=1,\dots,M$, then $\tilde x\pm z<\xi_m$ or $\tilde x\pm z>\xi_m$ for some $m=1,\dots,M$, at least when $|z|<\rho$ and ρ satisfies (A.21). Since γ is C^1 off $\{\xi_m\}_{m=1}^M$, we can safely send $\rho\downarrow0$:

$$\begin{aligned}\lim_{\rho\downarrow0}\iiiint_{\Omega_\xi}I^1_{\text{flux}}\left(\tilde x,\tilde t,z,\tau\right)\,d\tilde t\,d\tilde x\,d\tau\,dz&=\iint_{\Pi_T\setminus\{\xi_m\}_{m=1}^M}\gamma'(x)\cdot F_\gamma(\gamma(x),v(x,t),u(x,t))\varphi(x,t)\,dt\,dx.\\
&\le\text{Const}\iint_{\Pi_T}\left|v(x,t)-u(x,t)\right|\varphi(x,t)\,dt\,dx.\end{aligned}$$

Next, we consider the term I^2_{flux}. As with the previous term, $I^2_{\text{flux}} = 0$ if $\tilde{x} \in [\xi_m - h_m, \xi_m + h_m]$ for some $m = 1, \dots, M$. If $\tilde{x} \notin [\xi_m - h_m, \xi_m + h_m]$ for all $m = 1, \dots, M$, then, if $|z| < \rho$ and (A.21) holds, $\tilde{x} \pm z < \xi_m$ or $\tilde{x} \pm z > \xi_m$ for some $m = 1, \dots, M$, so that

$$|\gamma(\tilde{x}+z) - \gamma(\tilde{x}-z)| \leq \text{Const}\,|z|,$$

since γ' is bounded away from $\{\xi_m\}_{m=1}^M$. Consequently, if ρ satisfies (A.21),

$$\begin{aligned}\left|I^2_{\text{flux}}\left(\tilde{x},\tilde{t},z,\tau\right)\right| &\leq \text{Const.}\,|\gamma(\tilde{x}+z) - \gamma(\tilde{x}-z)|\left|\partial_{\tilde{x}}\varphi\left(\tilde{x},\tilde{t}\right)\right|\delta_\rho(z)\delta_\rho(\tau)\\ &\leq \text{Const.}\,|z|\left|\partial_{\tilde{x}}\varphi\left(\tilde{x},\tilde{t}\right)\right|\delta_\rho(z)\delta_\rho(\tau), \qquad \forall\left(\tilde{x},\tilde{t},z,\tau\right) \in \Omega_\xi,\end{aligned}$$

and thus

$$\lim_{\rho\downarrow 0} \iiiint_{\Omega_\xi} I^2_{\text{flux}}\left(\tilde{x},\tilde{t},z,\tau\right)\, d\tilde{t}\, d\tilde{x}\, d\tau\, dz = 0.$$

Finally, we consider the term I^3_{flux}. Using again the fact that φ satisfies (A.20) and that ρ satisfies (A.21), we have

$$\begin{aligned}\left|I^3_{\text{flux}}\left(\tilde{x},\tilde{t},z,\tau\right)\right| &\leq \text{Const.}\,|z|\left|v\left(\tilde{x}+z,\tilde{t}+\tau\right) - u\left(\tilde{x}-z,\tilde{t}-\tau\right)\right|\varphi\left(\tilde{x},\tilde{t}\right)|\partial_z\delta_\rho(z)|\\ &\leq \text{Const.}\left|v\left(\tilde{x}+z,\tilde{t}+\tau\right) - u\left(\tilde{x}-z,\tilde{t}-\tau\right)\right|\varphi\left(\tilde{x},\tilde{t}\right)\tfrac{1}{\rho}\,\mathbf{1}_{|z|<\rho},\end{aligned}$$

for all $\left(\tilde{x},\tilde{t},z,\tau\right) \in \Omega_\xi$. Here we have used that

$$|z|\,|\partial_z\delta_\rho(z)| = |z|\,\tfrac{1}{\rho^2}\left|\delta'\left(\tfrac{z}{\rho}\right)\right| \leq \tfrac{|z|}{\rho}\tfrac{1}{\rho}\max|\delta'|\,\mathbf{1}_{|z|<\rho} \leq \tfrac{1}{\rho}\,\mathbf{1}_{|z|<\rho}.$$

From this it follows that

$$\lim_{\rho\downarrow 0} \iiiint_{\Omega_\xi} I^2_{\text{flux}}\left(\tilde{x},\tilde{t},z,\tau\right)\, d\tilde{t}\, d\tilde{x}\, d\tau\, dz \leq \text{Const} \iint_{\Pi_T} |v(x,t) - u(x,t)|\,\varphi(x,t)\, dt\, dx,$$

where the constant depends, among other things, on δ'. This concludes the proof of (2.22) and the theorem. □

Appendix B. A remark on the initial condition

The proof of the lemma below (it is used in the proof of Theorem 5.1) is a reproduction of the proof of a corresponding result in [16]. We include it for the convenience of the reader.

Lemma B.1. *Let u be an entropy solution of* (5.1) *in the sense of Definition 5.1 and with $u_0 \in L^\infty(\mathbf{R})$. Then statement* (5.4) *holds.*

Proof. For $\tau \in (0,T)$, let $\beta_\tau(t)$ be the Lipschitz function defined by

$$\beta_\tau(t) = \begin{cases}\frac{1}{\tau}(\tau - t), & \text{if } t \in [0,\tau],\\ 0, & \text{if } t \in (\tau, T].\end{cases}$$

Let $\alpha_r(x)$ be the function defined in (2.35), and let $\delta_\rho(\cdot)$ be a standard unit mass mollifier on $\mathbf{R}$ with support in $(-\rho,\rho)$ (see (A.19) in the appendix). With $\phi(x,t) = \alpha_r(x)\delta_\rho(x-y)\beta_\tau(t)$, $y \in \mathbf{R}$, as test function in (5.2) (after a standard regularization argument), $c = u_0(y)$, and after having integrated the result over $y \in \mathbf{R}$, we obtain

$$I_1 + I_2 + I_3 + I_4 + I_5 \geq 0,$$

where (F is the Kružkov entropy flux (1.8))

$$I_1 = -\frac{1}{\tau}\int_0^\tau \int_{\mathbf{R}}\int_{\mathbf{R}} |u(x,t)-u_0(y)|\,\alpha_r(x)\delta_\rho(x-y)\,dx\,dy\,dt,$$

$$I_2 = \int_0^\tau \int_{\mathbf{R}}\int_{\mathbf{R}} F(\gamma(x),u(x,t),u_0(y))\left(\alpha_r(x)\delta_\rho(x-y)\right)_x \beta_\tau(t)\,dx\,dy\,dt,$$

$$I_3 = \int_0^\tau \int_{\mathbf{R}}\int_{\mathbf{R}\setminus\{\xi_m\}_{m=1}^M} \operatorname{sign}(u(x,t)-u_0(y)) f(\gamma(x),u_0(y))_x \alpha_r(x)\delta_\rho(x-y)\beta_\tau(t)\,dx\,dy\,dt,$$

$$I_4 = \int_0^\tau \int_{\mathbf{R}} \sum_{m=1}^M \left| f\left(\gamma\left(\xi_m+\right),u_0(y)\right) - f\left(\gamma\left(\xi_m-\right),u_0(y)\right)\right| \alpha_r(\xi_m,t)\delta_\rho(\xi_m-y)\beta_\tau(t)\,dt\,dy,$$

$$I_5 = \int_{\mathbf{R}}\int_{\mathbf{R}} \left|u_(x)-u_0(y)\right| \alpha_r(x)\delta_\rho(x-y)\beta_\tau(t)\,dx\,dy.$$

From this it follows that

$$\frac{1}{\tau}\int_0^\tau \int_{\mathbf{R}} |u(x,t)-u_0(x)|\,\alpha_r(x)\,dx\,dt \le \left|\widetilde{I}_1\right| + \sum_{\ell=2}^4 |I_\ell| + |I_6|,$$

where

$$\begin{aligned}\left|\widetilde{I}_1\right| &:= \left| I_1 + \frac{1}{\tau}\int_0^\tau \int_{\mathbf{R}}\int_{\mathbf{R}} |u(x,t)-u_0(x)|\,\alpha_r(x)\,dx\,dt\right| \\ &\le \int_{\mathbf{R}}\int_{\mathbf{R}} |u_0(x)-u_0(y)|\,\alpha_r(x)\delta_\rho(x-y)\,dx\,dy \\ &\le \int_{\mathbf{R}} \|u_0(\cdot-y)-u_0(\cdot)\|_{L^1(-r,r)}\delta_\rho(y)\,dy.\end{aligned}$$

Since $u_0 \in L^1_{\mathrm{loc}}(\mathbf{R})$, the right-hand side of the last inequality tends to zero as $\rho \downarrow 0$. By the same argument, $|I_5|$ tends to zero as $\rho \downarrow 0$. Hence, given any $\varepsilon > 0$, we can always pick a $\rho_0 > 0$ (independent of τ) such that $\left|\widetilde{I}_1\right| + |I_5| < \varepsilon/2$ with $\rho = \rho_0$. Clearly, with $\rho = \rho_0$ fixed, we can always find a small enough τ_0 such that $\sum_{\ell=2}^5 |I_\ell| \le \varepsilon/3$ for any $\tau < \tau_0$. Consequently, $0 \le \frac{1}{\tau}\int_0^\tau \int_{-r}^r |u(x,t)-u_0(x)|\,dx\,dt \le \varepsilon$ for any $\tau < \tau_0$. This concludes the proof. □

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