On the global stability of contact discontinuity for compressible Navier-Stokes equations

Feimin Huang[1,2] Huijiang Zhao[3,4]

1. Department of Pure and Applied Mathematics, Graduate School of Information Science and Technology, Osaka University, Osaka 560-0043, Japan
2. Institute of Applied Mathematics, AMSS, Academia Sinica, Beijing 100080, China
3. Laboratory of Mathematical Physics, Wuhan Institute of Physics and Mathematics, Academia Sinica, Wuhan 430071, China
4. School of Political Science and Economics and Graduate School of Science and Engineering, Waseda University, Tokyo 169-8050, Japan

Email address:feiminhuang@hotmail.com hhjjzhao@hotmail.com

Abstract: The asymptotic behavior of the solutions toward the contact discontinuity for the one-dimensional compressible Navier-Stokes equations with a free boundary is investigated. It is shown that the viscous contact discontinuity introduced in [3] is asymptotic stable with arbitrarily *large* initial perturbation if the adiabatic exponent γ is near 1. The case the asymptotic state is given by a combination of viscous contact discontinuity and the rarefaction wave is further investigated. Both the strength of rarefaction wave and the initial perturbation can be arbitrarily *large*.

1 Introduction

We study in present paper the large time behavior of the solutions for the one-dimensional compressible Navier-Stokes (NS) equations. It is known that there have been a lot of works on this subject. Most of these results are concerned with the rarefaction wave and viscous shock wave. We refer to [4,6-9,11-15,17-19] and references therein. It is noted that Nishihara, Yang and Zhao recently established in [17,18] the global stability of strong rarefaction wave for the 3×3 equations. However few result is known for the contact discontinuity except [3] due to various difficulties. As an elementary hyperbolic wave, the asymptotic stability of contact discontinuity should be investigated. Liu and Xin [10,20] first studied in 1995 the nonlinear stability of contact discontinuity for an artificial viscosity system. It was shown in [10,20] that the contact discontinuity can not be the asymptotic state and a linear diffusive wave called viscous contact wave (or viscous contact discontinuity), which approximates the contact discontinuity on any finite time interval, instead dominates the large time behavior of the solutions. Motivated by [10,20], Huang, Matsumura and Shi [3] investigated the contact discontinuity case for a physical system—compressible NS equations, with a free boundary. Unlike the artificial viscous system, the contact discontinuity for compressible NS equations is approximated by a nonlinear diffusive wave. The nonlinear stability of the viscous contact wave was established if the initial perturbation is small(see [3]). This means the nonlinear stability of [3] is *local*. It is worthy to point out that the stability of [10,20] is also *local*. Thus, a natural question is whether the viscous contact wave constructed in [3] is still stable or not under arbitrarily *large* perturbation. Our main purpose of this paper is to give a positive answer to this question when the adiabatic exponent γ is near 1. Furthermore, the case the asymptotic state is given by a superposition of the viscous contact discontinuity and the rarefaction wave is also treated. In this situation, both the strength of rarefaction wave and the initial perturbation can be arbitrarily *large*.

We now formulate our main results. The 1-d compressible NS equations reads in *Lagrangian* coordinates:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = \mu(\dfrac{u_x}{v})_x, \\ (e + \dfrac{u^2}{2})_t + (pu)_x = (\kappa\dfrac{\theta_x}{v} + \mu\dfrac{uu_x}{v})_x, \end{cases} \tag{1.1}$$

where $u(x,t)$ is the velocity, $v(x,t) > 0$ the specific volume, $\theta(x,t)$ the absolute temperature, $\mu > 0$ the viscosity constant and $\kappa > 0$ the coefficient of heat conduction. Here we consider the perfect gas so that the pressure

$$p = \frac{R\theta}{v} = Av^{-\gamma}e^{\frac{\gamma-1}{R}s}, \tag{1.2}$$

and the internal energy $e = \frac{R}{\gamma-1}\theta + const.$, where s is the entropy, $\gamma > 1$ is the adiabatic exponent and A, R are positive constants. By $(1.2)_2$, the entropy s can also be regarded as a function of v and θ. Our initial and boundary conditions are

$$\begin{cases} \theta|_{x=0} = \theta_-, \\ (p(v,\theta) - \mu\dfrac{u_x}{v})(0,t) = p_0, \ t > 0, \\ (v,u,\theta)(x,0) = (v_0,u_0,\theta_0)(x) \to (v_+,u_+,\theta_+) \text{ as } x \to +\infty, \end{cases} \tag{1.3}$$

where $(1.3)_2$ means the gas is attached at the boundary $x = 0$ to the atmosphere with pressure p_0 (see [19]), $v_+ > 0, u_+, \theta_+ > 0$ are constants and $\theta_0(0) = \theta_-$ holds as a compatibility condition.

It is well known that the contact discontinuity is a unique Riemann solution

$$(\bar{V},\bar{U},\bar{\Theta})(x,t) = \begin{cases} (v_-,u_-,\theta_-), \ x < 0, \\ (v_+,u_+,\theta_+), \ x > 0, \end{cases} \tag{1.4}$$

of the following Riemann problem

$$
\begin{cases}
v_t - u_x = 0, \\
u_t + p_x = 0, \\
(e + \dfrac{u^2}{2})_t + (pu)_x = 0, \\
(v, u, \theta)(x, 0) = (v_-, u_-, \theta_-), x < 0, \\
(v, u, \theta)(x, 0) = (v_+, u_+, \theta_+), x > 0,
\end{cases}
\tag{1.5}
$$

if $u_- = u_+$ and $p_- = \frac{R\theta_-}{v_-} = p_+ = \frac{R\theta_+}{v_+}$. In view of [3], the contact discontinuity $(\bar{V}, \bar{U}, \bar{\Theta})$ in the half space $x > 0$ can be approximated by a viscous contact wave (V, U, Θ) satisfying

$$
\|(V - \bar{V}, U - \bar{U}, \Theta - \bar{\Theta})\|_{L^p(0,+\infty)} = O(\kappa)(1+t)^{\frac{1}{2p}}, \ all\ p \geq 1.
$$

Here

$$
V(x,t) = \frac{R}{p_+}\Theta(x,t), \quad U(x,t) = u_+ + \frac{\kappa(\gamma-1)}{\gamma R}\frac{\Theta(x,t)_x}{\Theta(x,t)}, \tag{1.6}
$$

and $\Theta(x,t) = \Theta(\xi), \xi = \frac{x}{\sqrt{1+t}}$ is the unique self similarity solution of

$$
\Theta_t = a(\frac{\Theta_x}{\Theta})_x, \Theta(0,t) = \theta_-, \Theta(+\infty, t) = \theta_+, \quad a = \frac{\kappa p_+(\gamma-1)}{\gamma R^2} > 0. \tag{1.7}
$$

From (1.7), $\Theta(\xi)$ satisfies

$$
-\frac{1}{2}\xi\Theta' = a(\frac{\Theta'}{\Theta})', \quad \Theta(0) = \theta_-, \Theta(+\infty) = \theta_+, \quad ' = \frac{d}{d\xi}. \tag{1.8}
$$

By the same lines as in [2,3], we have

$$
\begin{aligned}
&C_1(\gamma-1)^{-\frac{1}{2}}|\theta_+ - \theta_-| \leq |\Theta'(0)| \leq C_2(\gamma-1)^{-\frac{1}{2}}|\theta_+ - \theta_-|, \\
&|(|\Theta_{xx}|, \frac{1}{\sqrt{\gamma-1}}|\Theta_x|, \frac{1}{\gamma-1}|\Theta - \theta_+|)| \leq C_3 e^{-\frac{C_4\xi^2}{\gamma-1}}, \ as\ \xi \to \infty,
\end{aligned}
\tag{1.9}
$$

where $C_i, i = 1, 2, 3, 4$ are positive constants depending on $\theta_\pm$. By (1.9), the lemma 1.1 of [3] reads in the following style.

Lemma 1.1. *If $|\theta_+ - \theta_-| \leq M(\gamma-1)$, then*

$$
\int_0^\infty \Theta_x^4 dx \leq C(\gamma-1)^{\frac{5}{2}}(1+t)^{-\frac{3}{2}}, \quad \int_0^\infty \Theta_{xx}^2 dx \leq C(\gamma-1)^{\frac{1}{2}}(1+t)^{-\frac{3}{2}}, \tag{1.10}
$$

$$
\int_0^\infty \Theta_{xxx}^2 dx \leq C(\gamma-1)^{-\frac{1}{2}}(1+t)^{-\frac{5}{2}}, \quad \int_0^\infty x(\Theta_x^2 + |\Theta_{xx}|)dx \leq C(\gamma-1), \tag{1.11}
$$

where the constant C only depends on M.

The main aim of present paper is to show the global stability of the viscous contact wave and the superposition of a viscous contact wave and a rarefaction wave. The precise statement of our first result is

Theorem 1.2. *Assume that* $p_0 = p_+$ *and* $|\theta_+ - \theta_-| \le M(\gamma - 1)$ *holds for some constant M. Assume that* (V, U, Θ) *is the viscous contact wave constructed in (1.6) and (1.7) and the initial data satisfies*

$$\begin{array}{l} 0 < M_0^{-1} \le v_0(x), \theta_0(x) \le M_0, \quad u_0(x) - U(x,0) \in H^1(0,\infty), \\ (v_0(x) - V(x,0), s_0(x) - S(x,0)) \in H_0^1(0,\infty) \end{array} \tag{1.12}$$

for some positive constant M_0*, where* $S = \frac{R}{\gamma-1}(\ln \frac{R\Theta}{A} + (\gamma-1)\ln V)$ *and* $s_0(x) = s(x,0)$*. Then there exists a positive constant* $\delta_0 > 0$ *such that if* $\gamma < 1 + \delta_0$*, the problem (1.1)-(1.3) has a unique global solution* $(v, u, \theta)(x,t)$ *satisfying*

$$(v - V, u - U, \theta - \Theta)(x,t) \in C(0, +\infty; H^1(0, +\infty)), \tag{1.13}$$

$$(v - V)_x(x,t) \in L^2(0, +\infty; L^2(0, +\infty)), \tag{1.14}$$

$$(u - U, \theta - \Theta)_x(x,t) \in L^2(0, +\infty; H^1(0, +\infty)), \tag{1.15}$$

and

$$\sup_{x \ge 0} |(v - V, u - U, \theta - \Theta)(x,t)| \to 0, \quad as \quad t \to +\infty. \tag{1.16}$$

Remark 1.3. The difference $|\theta_+ - \theta_-|$ is naturally bounded by $(\gamma - 1)$ multiplying some constant M from the physical point of view.

When $p_0 \ne p_+$, there are two subcases: the superposition of a viscous contact wave and a 3-rarefaction wave or that of a viscous contact wave and a 3-viscous shock wave. Here we only consider the previous one. In this situation, by the basic theory of hyperbolic system of conservation laws, there exists a unique point (v_m, u_m, θ_m) such that $p_0 = p_m = \frac{R\theta_m}{v_m}$ and (v_m, u_m, θ_m) belongs to the 3-rarefaction wave curve $R(v_+, u_+, \theta_+)$ in the phase plane, where

$$R(v_+, u_+, \theta_+) = \Big\{(v, u, \theta)\big| s = s_+ \,, u = u_+ - \int_{v_+}^{v} \lambda(\eta, s_+) d\eta, v > v_+\Big\}, \tag{1.17}$$

$$\lambda(v, s) = \sqrt{A\gamma v^{-\gamma-1} e^{\frac{\gamma-1}{R}s}}. \tag{1.18}$$

The precise statement of our second result is

Theorem 1.4. *Assume that there exists a unique point* (v_m, u_m, θ_m) *such that* $p_0 = p_m$ *and* $(v_m, u_m, \theta_m) \in R(v_+, u_+, \theta_+)$ *and* $|\theta_+ - \theta_m| + |\theta_m - \theta_-| \le M(\gamma - 1)$ *holds for some constant M. Assume that* $(V^{cd}, U^{cd}, \Theta^{cd})$ *is the viscous contact wave constructed in (1.6) and (1.7), where* (v_+, u_+, θ_+) *is replaced by*

(v_m,u_m,θ_m) and (V^r,U^r,Θ^r) is the smooth rarefaction wave constructed in (3.4) satisfying $S^r=s(V^r,\Theta^r)=s(v_+,\theta_+)=s_+$. Let

$$V=V^{cd}+V^r-v_m, U=U^{cd}+U^r-u_m, S=S^{cd}. \tag{1.19}$$

Assume that the initial data satisfies (1.12). Then there exists a positive constant $\delta_0>1$ such that if $\gamma<1+\delta_0$, the problem (1.1)-(1.3) has a unique global solution $(v,u,\theta)(x,t)$ satisfying (1.13)-(1.15). Furthermore,

$$\sup_{x\geq 0}|(v-V^{cd}-v^r+v_m, u-U^{cd}-u^r+u_m, s-S^{cd})(x,t)|\to 0, \tag{1.20}$$

where (v^r,u^r,θ^r),$(s^r=s(v^r,\theta^r)=s_+)$ is the 3-rarefaction wave uniquely determined by (3.1).

Our plan of this paper is as follows. In sec.2, the single contact discontinuity case is investigated. In sec.3, the case including the rarefaction wave is treated.

Notations. Throughout this paper, several positive generic constants are denoted by c,C without confusions. For function spaces, $H^l(\Omega)$ denotes the l-th order Sobolev space with its norm

$$\|f\|_l=(\sum_{j=0}^{l}\|\partial_x^j f\|^2)^{\frac{1}{2}}, \quad when\ \|\cdot\|:=\|\cdot\|_{L^2(\Omega)}. \tag{1.21}$$

The domain Ω will be often abbreviated without confusions.

2 Global stability of viscous contact wave

Proof of theorem 1.2. We put the perturbation $(\phi,\psi,\zeta,\varphi)(x,t)$ by

$$\begin{aligned} v(x,t)&=V(x,t)+\phi(x,t), \quad u(x,t)=U(x,t)+\psi(x,t),\\ \theta(x,t)&=\Theta(x,t)+\zeta(x,t), \quad s(x,t)=S(x,t)+\varphi(x,t), \end{aligned} \tag{2.1}$$

where $(V,U,\Theta)(x,t)$ is the viscous contact wave constructed in (1.6) and (1.7). By (1.6) and (1.7), we have

$$\begin{cases} V_t-U_x=0,\\ U_t+(\frac{R\Theta}{V})_x=\mu(\frac{U_x}{V})_x+F,\\ \frac{R}{\gamma-1}\Theta_t+p_+U_x=\kappa(\frac{\Theta_x}{V})_x+\mu\frac{U_x^2}{V}+G, \end{cases} \tag{2.2}$$

where

$$F=U_t-\mu(\frac{U_x}{V})_x=\frac{\kappa(\gamma-1)}{R\gamma}[(\ln\Theta)_{xt}-\mu(\frac{p_+}{R\Theta}(\ln\Theta)_{xx})_x], \tag{2.3}$$

$$G = -\mu \frac{U_x^2}{V} = -\frac{\mu p_+}{R\Theta}\Big(\frac{\kappa(\gamma-1)}{R\gamma}(\ln \Theta)_{xx}\Big)^2. \tag{2.4}$$

From lemma 1.1, we have

$$\|F\|_{L^1} \le C(\gamma-1)(1+t)^{-1}, \quad \|G\|_{L^1} \le C(\gamma-1)^{\frac{5}{2}}(1+t)^{-\frac{3}{2}}. \tag{2.5}$$

Substituting (2.2) into (1.1) and (1.3) yields

$$\begin{cases} \phi_t - \psi_x = 0, \\ \psi_t + \Big(\frac{R(\Theta+\zeta)}{V+\phi} - \frac{R\Theta}{V}\Big)_x = \mu\Big(\frac{U_x+\psi_x}{V+\phi} - \frac{U_x}{V}\Big)_x - F, \\ \frac{R}{\gamma-1}\zeta_t + \frac{R(\Theta+\zeta)}{V+\phi}(U_x+\psi_x) - p_+U_x \\ = \kappa\Big(\frac{\Theta_x+\zeta_x}{V+\phi}\Big)_x - \kappa\Big(\frac{\Theta_x}{V}\Big)_x + \mu\frac{(U_x+\psi_x)^2}{V+\phi} - \mu\frac{U_x^2}{V} - G, \\ \Big(\frac{R\theta_-}{V+\phi} - \mu\frac{U_x+\psi_x}{V+\phi}\Big)\Big|_{x=0} = p_+, \\ \zeta(0,t) = 0, \\ (\phi,\psi,\zeta)(x,0) = (v_0 - V, u_0 - U, \theta - \Theta)(x,0). \end{cases} \tag{2.6}$$

We shall prove theorem 1.2 by the local existence and the a priori estimate. We look for the solution (ϕ,ψ,ζ) in the solution space $X(0,+\infty)$, where

$$\begin{aligned} X(0,T) = \{&(\phi,\psi,\zeta) : (\phi,\zeta) \in C(0,T;H_0^1), \psi \in C(0,T;H^1), \tfrac{1}{2}M_1^{-1} \le v \\ &\le 2M_1, \tfrac{1}{4}M_0^{-1} \le \theta \le 4M_0,\ \phi_x \in L^2(0,T;L^2), (\psi,\zeta)_x \in L^2(0,T;H^1)\} \end{aligned} \tag{2.7}$$

for some $0 < T \le +\infty$, where the constant M_1 will be determined later. Since the local existence has already been established in [3], we only consider the a priori estimate here. We have

Proposition 2.1. *(A Priori Estimate). Assume that the conditions of theorem 1.2 hold, then there exists a positive constant δ_0 such that if $\gamma < 1 + \delta_0$ and $(\phi,\psi,\zeta) \in X(0,T)$ is a solution of (2.6) for some positive T, then the followings hold*

$$M_1^{-1} \le v(x,t) \le M_1, \quad \frac{1}{2}M_0^{-1} \le \theta(x,t) \le 2M_0, \tag{2.8}$$

$$\begin{aligned} &\|(\phi,\psi,\zeta)\|_1^2 + \int_0^t \{\|\phi_x\|^2 + \|(\psi_x,\zeta_x)\|_1^2\} d\tau \\ &\le C(M_0)(1 + \|(\phi_0,\psi_0,\varphi_0)\|_1^2), \end{aligned} \tag{2.9}$$

where the constant $C(M_0)$ depends on M, M_0 and the initial data, but does not depend on M_1.

Proposition 2.1 is proved by a series of lemmas.

Lemma 2.2. *It follows that*

$$
\begin{aligned}
&\|(\sqrt{\Phi(\frac{v}{V})},\psi,\frac{1}{\sqrt{\delta}}\zeta)(t)\|^2+\int_0^t\{\|(\frac{\psi_x}{v^{\frac12}},\frac{\zeta_x}{v^{\frac12}})(\tau)\|^2\}d\tau\\
\leq\ & C(M_0)\{\|(\phi_0,\psi_0,\sqrt{\delta}\varphi_0)\|_1^2+C(M_1)[\delta+\delta\int_0^t\|(\frac{\phi_x}{v^{\frac32}},\frac{\zeta_x}{v^{\frac12}})\|^2d\tau\\
&+\int_0^t\delta^2(1+\tau)^{-\frac85}\|(\psi,\frac{1}{\sqrt{\delta}}\zeta)\|^2(\tau)d\tau]\},
\end{aligned}
\tag{2.10}
$$

where $\delta=\gamma-1<1$.

Proof. Due to [3], we have

$$
\begin{aligned}
&(\frac12\psi^2+R\Theta\Phi(\frac{v}{V})+\frac{R}{\delta}\Theta\Phi(\frac{\theta}{\Theta}))_t+\frac{\kappa\Theta}{v\theta^2}\zeta_x^2+\frac{\mu\Theta\psi_x^2}{v\theta}\\
&+H_x+Q=-F\psi-\frac{\zeta G}{\theta},
\end{aligned}
\tag{2.11}
$$

where

$$\Phi(\sigma)=\sigma-1-\ln\sigma,\quad \Psi(\sigma)=\sigma^{-1}-1+\ln\sigma, \tag{2.12}$$

$$H=R[(\frac{\Theta+\zeta}{v}-\frac{\Theta}{V})\psi]-\mu[(\frac{U_x+\psi_x}{v}-\frac{U_x}{V})\psi]-\kappa\frac{\zeta}{\theta}(\frac{\theta_x}{v}-\frac{\Theta_x}{V}), \tag{2.13}$$

$$
\begin{aligned}
Q=\ &p_+\Psi(\frac{v}{V})U_x+\frac{p_+}{\delta}\Psi(\frac{\theta}{\Theta})U_x+\mu\psi_xU_x(\frac1v-\frac1V)-\frac{\zeta}{\theta}(p_+-p)U_x\\
&-\kappa\frac{\Theta_x}{\theta^2v}\zeta\zeta_x-\kappa\frac{\zeta_x\phi\Theta\Theta_x}{\theta^2vV}+\kappa\frac{\zeta\phi}{\theta^2vV}\Theta_x^2-\frac{\mu\zeta U_x^2}{\theta}(\frac1v-\frac1V)-\frac{2\mu\zeta}{v\theta}\psi_xU_x.
\end{aligned}
\tag{2.14}
$$

By the formula of $\Phi(\sigma)$, we know that $\Phi(1)=\Phi'(1)=0$, $\Psi(1)=\Psi'(1)=0$ and $\Phi(\sigma)$ is a strictly convex function. This yields that $\Phi(\sigma)>0$ and

$$
\begin{aligned}
&C_1(M_0)C_1(M_1)\phi^2\leq\Phi(\frac{v}{V})\leq C_2(M_0)C_2(M_1)\phi^2,\\
&|\Psi(\frac{\theta}{\Theta})|\leq C(M_0)C(M_1)\zeta^2,
\end{aligned}
\tag{2.15}
$$

for some constants $C_i(M_0),C_i(M_1),i=1,2$. Since

$$|Q|\leq\frac{\kappa\Theta}{4v\theta^2}\zeta_x^2+\frac{\mu\Theta}{4v\theta}\psi_x^2+C(M_0)C(M_1)[|U_x|(\phi^2+\frac1\delta\zeta^2)+(\phi^2+\zeta^2)\Theta_x^2], \tag{2.16}$$

and

$$
\begin{aligned}
&\int_0^\infty|F\psi|+|\frac{G\zeta}{\theta}|dx\leq\frac{\mu}{4^3M_0^2}\int_0^\infty\frac{\psi_x^2}{v}dx+\frac{\kappa}{4^4M_0^3}\int_0^\infty\frac{\zeta_x^2}{v}dx\\
&+C(M_0)C(M_1)[\delta(1+t)^{-\frac65}+\delta^2(1+t)^{-\frac85}\|(\psi,\frac{1}{\sqrt{\delta}}\zeta)\|^2]
\end{aligned}
\tag{2.17}
$$

due to (2.5), integrating (2.11) over $R_+\times(0,t)$ gives

$$
\begin{aligned}
&\|(\sqrt{\Phi(\frac{v}{V})},\psi,\frac{1}{\sqrt{\delta}}\zeta)\|^2+\int_0^t\int_0^\infty\frac{\psi_x^2}{v}+\frac{\zeta_x^2}{v}dxdt-\int_0^tH(0,t)dt\\
\leq\ & C(M_0)C(M_1)\{\int_0^t\int_0^\infty(\zeta^2+\phi^2)(|\Theta_{xx}|+\Theta_x^2)dxdt+\delta\\
&+\int_0^t\delta^2(1+\tau)^{-\frac85}\|(\psi,\frac{1}{\sqrt{\delta}}\zeta)\|^2d\tau\}+C(M_0)\|(\phi_0,\psi_0,\frac{1}{\sqrt{\delta}}\zeta_0)\|^2.
\end{aligned}
\tag{2.18}
$$

On the other hand, the boundary condition (1.2) exactly gives the value of $\phi(0,t)$. From [3] and the fact that $\phi_0(x)\in H_0^1(0,\infty)$, we have

$$\phi(0,t)=\phi_0(0)e^{-\frac{p_+}{\mu}t}=0, \tag{2.19}$$

which yields that $H(0,t)=0$. Note that

$$|\zeta(x,t)|\le x^{\frac{1}{2}}\|\zeta_x\|,\quad |\phi(x,t)|\le x^{\frac{1}{2}}\|\phi_x\|, \tag{2.20}$$

due to [16], applying Lemma 1.1 and the boundary condition (2.19), we have

$$\int_0^t\int_0^\infty(\zeta^2+\phi^2)(|\Theta_{xx}|+\Theta_x^2)dxdt\le C(M_0)C(M_1)\delta\int_0^t\|(\frac{\phi_x}{v^{\frac{3}{2}}},\frac{\zeta_x}{v^{\frac{1}{2}}})\|^2dt. \tag{2.21}$$

By (1.2) and (2.1), we have

$$\|\zeta_0\|_1\le C\delta\|(\phi_0,\varphi_0)\|_1. \tag{2.22}$$

Thus, combining (2.18)-(2.22) yields lemma 2.2.

Lemma 2.3. *There exists a small positive constant δ_0 such that if $\delta\le\delta_0$, then*

$$\begin{aligned}&\|(\phi,\psi,\frac{1}{\sqrt{\delta}}\zeta)(t)\|^2+\|\phi_x\|^2+\int_0^t\{\|(\phi_x,\psi_x,\zeta_x)(\tau)\|^2\}d\tau\\ \le\ & C(M_0)\{1+\|(\phi_0,\psi_0,\varphi_0)\|_1^2\}.\end{aligned} \tag{2.23}$$

Proof. Following [14], we introduce a new variable $\tilde{v}=\frac{v}{V}$. Then $(2.6)_2$ can be rewritten by the new variable as

$$(\mu(\frac{\tilde{v}_x}{\tilde{v}})-\psi)_t-p_x=F. \tag{2.24}$$

Multiplying (2.24) by $\frac{\tilde{v}_x}{\tilde{v}}$, we have

$$\begin{aligned}&(\frac{\mu}{2}(\frac{\tilde{v}_x}{\tilde{v}})^2-\psi\frac{\tilde{v}_x}{\tilde{v}})_t+(\psi\frac{\tilde{v}_t}{\tilde{v}})_x+\frac{R\theta}{v}(\frac{\tilde{v}_x}{\tilde{v}})^2-\frac{R}{v}\zeta_x\frac{\tilde{v}_x}{\tilde{v}}\\ &+\frac{R\theta}{v}(\frac{1}{\Theta}-\frac{1}{\theta})\Theta_x\frac{\tilde{v}_x}{\tilde{v}}=\frac{\psi_x^2}{v}+\psi_xU_x(\frac{1}{v}-\frac{1}{V})+F\frac{\tilde{v}_x}{\tilde{v}}.\end{aligned} \tag{2.25}$$

The Cauchy inequality yields that

$$|\frac{R}{v}\zeta_x\frac{\tilde{v}_x}{\tilde{v}}|+|\psi_xU_x(\frac{1}{v}-\frac{1}{V})|\le\frac{R\theta}{4v}(\frac{\tilde{v}_x}{\tilde{v}})^2+C(M_0)(\frac{\zeta_x^2}{v}+\frac{\psi_x^2}{v}+C(M_1)\phi^2U_x^2), \tag{2.26}$$

$$|\frac{R\theta}{v}(\frac{1}{\Theta}-\frac{1}{\theta})\Theta_x\frac{\tilde{v}_x}{\tilde{v}}|+|F\frac{\tilde{v}_x}{\tilde{v}}|\le\frac{R\theta}{4v}(\frac{\tilde{v}_x}{\tilde{v}})^2+C(M_0)C(M_1)(\zeta^2\Theta_x^2+|F|^2), \tag{2.27}$$

and

$$c_1\frac{\phi_x^2}{v^2}-C(M_0)C(M_1)\phi^2\Theta_x^2\le(\frac{\tilde{v}_x}{\tilde{v}})^2\le c_3\frac{\phi_x^2}{v^2}+C(M_0)C(M_1)\phi^2\Theta_x^2. \tag{2.28}$$

Similar to (2.19), we also have $\left(\frac{\tilde{v}_t}{\tilde{v}}\right)(0,t) = \left(\frac{\phi_t}{v}\right)(0,t) = 0$. Note that the right hand sides of (2.26)-(2.28) have already been investigated in Lemma 2.2. Integrating (2.25) over $R_+ \times (0,t)$ and using Lemma 2.2, (2.15), (2.26)-(2.28) and the fact that

$$|\psi\frac{\tilde{v}_x}{\tilde{v}}| \le \frac{\mu}{4}(\frac{\tilde{v}_x}{\tilde{v}})^2 + C\psi^2,$$

we get

$$\begin{aligned} &\|(\sqrt{\Phi(\frac{v}{V})}, \psi, \frac{1}{\sqrt{\delta}}\zeta)(t)\|^2 + \|\frac{\tilde{v}_x}{\tilde{v}}\|^2 + \int_0^t \{\|(\frac{\phi_x}{v^{\frac{3}{2}}}, \frac{\psi_x}{v^{\frac{1}{2}}}, \frac{\zeta_x}{v^{\frac{1}{2}}})(\tau)\|^2\}d\tau \\ \le\ & C(M_0)\big\{\|(\phi_0,\psi_0,\sqrt{\delta}\varphi_0)\|_1^2 + C(M_1)[\delta + \delta\int_0^t \|(\frac{\phi_x}{v^{\frac{3}{2}}}, \frac{\zeta_x}{v^{\frac{1}{2}}})(\tau)\|^2 d\tau \\ &+ \int_0^t \delta^2(1+\tau)^{-\frac{8}{5}}\|(\psi, \frac{1}{\sqrt{\delta}}\zeta)\|^2(\tau)d\tau]\big\}, \end{aligned} \tag{2.29}$$

We now choose a small positive constant $\delta_0 < 1$ so that $C(M_0)C(M_1)\delta_0 < \frac{1}{2}$. Then for any $\delta < \delta_0$, the Gronwall's inequality yields

$$\begin{aligned} &\|(\sqrt{\Phi(\frac{v}{V})}, \psi, \frac{1}{\sqrt{\delta}}\zeta)(t)\|^2 + \|\frac{\tilde{v}_x}{\tilde{v}}\|^2 + \int_0^t \{\|(\frac{\phi_x}{v^{\frac{3}{2}}}, \frac{\psi_x}{v^{\frac{1}{2}}}, \frac{\zeta_x}{v^{\frac{1}{2}}})(\tau)\|^2\}d\tau \\ \le\ & C(M_0)\big\{1 + \|(\phi_0,\psi_0,\sqrt{\delta}\varphi_0)\|_1^2\big\}. \end{aligned} \tag{2.30}$$

We note here that the constant $C(M_0)$ does not depend on M_1.

We now use the method of [14] and (2.30) to show $(2.8)_1$. To this end, let

$$\eta(\tilde{v}) = \int_1^{\tilde{v}} \frac{\sqrt{\Phi(\sigma)}}{\sigma}d\sigma, \quad \Phi(\sigma) = \sigma - 1 - \ln\sigma. \tag{2.31}$$

Since

$$\eta(\tilde{v}) \to \begin{cases} -\infty, & \tilde{v} \to 0+, \\ +\infty, & \tilde{v} \to +\infty, \end{cases} \tag{2.32}$$

and

$$|\eta(\tilde{v}(x,t))| = |\int_{+\infty}^x \frac{\partial}{\partial y}\eta(\tilde{v}(y,t))dy| \le \int_{R_+} (\Phi(\frac{v}{V}) + (\frac{\tilde{v}_x}{\tilde{v}})^2)(x,t)dx, \tag{2.33}$$

the inequality (2.30) yields that there exists a positive constant M_2 which only depends on M, M_0 and the initial data such that

$$M_2^{-1} \le v(x,t) \le M_2. \tag{2.34}$$

Let $M_1 = M_2$, then $(2.8)_1$ is proved. After we obtain the a priori estimate (2.34), it is easy to imply (2.23) from (2.15) and (2.30).

Lemma 2.4. *It follows that*

$$\|(\psi_x, \zeta_x)(t)\|^2 + \int_0^t \{\|(\psi_{xx}, \zeta_{xx})(\tau)\|^2\}d\tau \le C(M_0)(1 + \|(\phi_0,\psi_0,\zeta_0)\|_1^2). \tag{2.35}$$

Proof. We first estimate the term ψ_x. Multiplying $(2.6)_2$ by $-\psi_{xx}$, we have

$$\begin{aligned}&(\frac{1}{2}\psi_x^2)_t-(\psi_t\psi_x)_x+\mu\frac{\psi_{xx}^2}{v}-p_x\psi_{xx}\\&+\mu(\frac{U_x}{v}-\frac{U_x}{V})_x\psi_{xx}+\mu\psi_x(\frac{1}{v})_x\psi_{xx}=F\psi_{xx}.\end{aligned}\tag{2.36}$$

Since p_+ is constant, we have

$$|p_x\psi_{xx}|\le\frac{\mu}{4v}\psi_{xx}^2+C(M_0)(\phi^2+\zeta^2)\Theta_x^2+C(M_0)(\phi_x^2+\zeta_x^2).\tag{2.37}$$

On the other hand,

$$\begin{aligned}&|\mu(\tfrac{U_x}{v}-\tfrac{U_x}{V})_x\psi_{xx}|+|\mu\psi_x(\tfrac{1}{v})_x\psi_{xx}|\\\le\ &\frac{\mu}{4v}\psi_{xx}^2+C(M_0)\delta^2(\Theta_x^4+\Theta_{xx}^2+\Theta_{xxx}^2)+C(M_0)(\phi_x^2+\psi_x^2)\\&+C(M_0)|\phi_x||\psi_x||\psi_{xx}|.\end{aligned}\tag{2.38}$$

We compute by lemma 2.3

$$\begin{aligned}\int_0^\infty|\phi_x||\psi_x||\psi_{xx}|dx&\le\sup\{|\psi_x|\}\|\phi_x\|\|\psi_{xx}\|\\&\le\sqrt{2}\|\psi_{xx}\|^{\frac{3}{2}}\|\psi_x\|^{\frac{1}{2}}\|\phi_x\|\\&\le\mu\|\frac{\psi_{xx}}{2v^{\frac{1}{2}}}\|^2+C(M_0)\|\psi_x\|^2.\end{aligned}\tag{2.39}$$

Since $\psi_x(0,t)=\phi_t(0,t)=0$, integrating (2.36) over $R_+\times(0,t)$ and using (2.37)-(2.39) and Lemmas 1.1 and 2.3 imply

$$\|\psi_x(t)\|^2+\int_0^t\|\psi_{xx}(\tau)\|^2d\tau\le C(M_0)(1+\|(\phi_0,\psi_0,\zeta_0)\|_1^2).\tag{2.40}$$

We now estimate ζ_x. Multiplying $(2.5)_3$ by $-\zeta_{xx}$, we have

$$(\frac{R}{2\delta}\zeta_x^2)_t-(\frac{R}{\delta}\zeta_t\zeta_x)_x+\kappa\frac{\zeta_{xx}^2}{v}+\mu(\frac{u_x^2}{v}-\frac{U_x^2}{V})\zeta_{xx}+\tilde{Q}=G\zeta_{xx},\tag{2.41}$$

where

$$\tilde{Q}=\ -(p-p_+)U_x\zeta_{xx}+\kappa\zeta_x(\frac{1}{v})_x\zeta_{xx}-p\psi_x\zeta_{xx}+\kappa(\frac{\Theta_x}{v}-\frac{\Theta_x}{V})_x\zeta_{xx}.\tag{2.42}$$

By the same method for the estimate on ψ_x, we obtain

$$\|\frac{\zeta_x}{\delta^{\frac{1}{2}}}(t)\|^2+\int_0^t\|\zeta_{xx}(\tau)\|^2d\tau\le C(M_0)(1+\|(\phi_0,\psi_0,\zeta_0)\|_1^2).\tag{2.43}$$

We omit the details here. Combining (2.40) and (2.43) yields lemma 2.4.

By lemmas 2.3 and 2.4, we have

$$|\zeta(x,t)|_{L^\infty} \le C\|\zeta\|_1 \le C(M_0)\delta^{\frac{1}{2}}(1+\|(\phi_0,\psi_0,\zeta_0)\|_1). \tag{2.44}$$

Choosing δ suitably small yields

$$\frac{1}{2}M_0^{-1} \le \theta(x,t) = \Theta(x,t) + \zeta(x,t) \le 2M_0. \tag{2.45}$$

Thus proposition 2.1 is obtained from (2.34), (2.45) and lemmas 2.3 and 2.4. Theorem 1.2 is easy from the local existence and proposition 2.1.

3 Superposition of viscous contact wave and rarefaction wave

Proof of theorem 1.4. In this case, there exists a unique point $(v_m, u_m, \theta_m) \in R(V_+, u_+, \theta_+)$ such that $p_0 = p_m$ and the superposition of the viscous contact wave connecting (v_-, u_m, θ_-) with (v_m, u_m, θ_m) and the 3-rarefaction wave connecting (v_m, u_m, θ_m) with (v_+, u_+, θ_+), where $v_- = \frac{R\theta_-}{p_0}$. The 3-rarefaction wave $(v^r, u^r, \theta^r)(\frac{x}{t})$ connecting (v_m, u_m, θ_m) and (v_+, u_+, θ_+) is the weak solution of the Riemann problem

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v,\theta)_x = 0, \\ (e(v,\theta) + \frac{u^2}{2})_t + (p(v,\theta)\,u)_x = 0, \\ (v_0, u_0, \theta_0)(x) = \begin{cases} (v_m, u_m, \theta_m), x < 0, \\ (v_+, u_+, \theta_+), x > 0. \end{cases} \end{cases} \tag{3.1}$$

Since the rarefaction wave is a weak solution, it is necessary to construct a smooth approximate rarefaction wave $(V^r, U^r, \Theta^r)(x,t)$ of $(v^r, u^r, \theta^r)(\frac{x}{t})$ in $\mathbb{R}_+ \times (0, +\infty)$. To this end, we apply the idea of [4] to construct the smooth rarefaction wave. The advantage of this kind smooth wave is that the boundary effect can be exactly eliminated. We first construct the solution $w(x,t)$ of the following problem

$$\begin{cases} w_t + ww_x = 0, \quad (x,t) \in \mathbb{R} \times (0, +\infty), \\ w\big|_{t=0} = \begin{cases} w_-, & x < 0, \\ w_- + \tilde{w}K_q \displaystyle\int_0^{\varepsilon x} z^q e^{-z} dz, & x \ge 0, \end{cases} \end{cases} \tag{3.2}$$

where $w_\pm = \lambda(v_\pm, s_+)$, $s_+ = s(v_+, \theta_+)$, $\tilde{w} = w_+ - w_-$, K_q is a constant such that $K_q \int_0^{+\infty} z^q e^{-z} dz = 1$ for large constant $q \ge 8$ and ε is a small positive constant. We have the following properties of $w(x,t)$ due to [4].

Lemma 3.1.[4] *Let $0 < w_- < w_+$, then the problem (3.2) has a unique smooth solution $w(x,t)$ satisfying*

i. $w_- \le w(x,t) < w_+$, $w_x \ge 0$, *for* $x \ge 0, t \ge 0$.
ii. *For any* $p(1 \le p \le +\infty)$, *there exists a constant* $C_{p,q}$ *such that for* $t \ge 0$,

$$\|w_x(\cdot,t)\|_{L^p} \le C_{p,q} \min\left(\tilde{w}\varepsilon^{1-\frac{1}{p}}, \tilde{w}^{\frac{1}{p}} t^{-1+\frac{1}{p}}\right),$$
$$\|w_{xx}(\cdot,t)\|_{L^p} \le C_{p,q} \min\left(\tilde{w}\varepsilon^{2-\frac{1}{p}}, \tilde{w}^{\frac{1}{q}} \varepsilon^{1-\frac{1}{p}+\frac{1}{q}} t^{-1+\frac{1}{q}}\right).$$

iii. *when* $x \le w_- t$, $w(x,t) - w_- = w_x(x,t) = w_{xx}(x,t) = 0$.
iv. $\limsup_{t\to+\infty, x\in\Re} \left|w(x,t) - w^R(x,t)\right| = 0$.

Here $w^R(x,t)$ *is the Riemann solution of the scalar equation (3.2) with the initial data* $w_0(x) = w_-$, *if* $x < 0$ *and* $w_0(x) = w_+$, *if* $x > 0$.

The smooth approximation $(\tilde{V}^r, \tilde{U}^r, \tilde{\Theta}^r)$ to $(v^r, u^r, \theta^r)(x,t)$ is given by

$$\begin{cases} \tilde{S}^r(x,t) = s_+, \\ \lambda\big(\tilde{V}^r(x,t), s_+\big) = w(x,t), \\ \tilde{U}^r(x,t) = u_+ - \displaystyle\int_{v_+}^{\tilde{V}^r(x,t)} \lambda\big(\eta, s_+\big) d\eta. \end{cases} \tag{3.3}$$

Setting

$$\big(V^r, U^r, \Theta^r\big)(x,t) := \big(\tilde{V}^r, \tilde{U}^r, \tilde{\Theta}^r\big)(x, t+t_0)\Big|_{x\ge0}, \tag{3.4}$$

where the constant $t_0 > 1$ will be given later. Then we have

$$\begin{cases} V_t^r - U_x^r = 0, \\ U_t^r + p(V^r, \Theta^r)_x = 0, \\ \big(e(V^r, \Theta^r) + \dfrac{1}{2}(U^r)^2\big)_t + (p(V^r, \Theta^r)\, U^r)_x = 0, \\ (V^r, U^r, \Theta^r)\Big|_{x=0} = (v_m, u_m, \theta_m), \\ (V^r, U^r, \Theta^r)\Big|_{t=0} = (V_0^r, U_0^r, \Theta_0^r)(x) = (V^r, U^r, \Theta^r)(x, 0). \end{cases} \tag{3.5}$$

Due to Lemma 3.1, (V^r, U^r, Θ^r) has the following properties.

Lemma 3.2. *The smooth rarefaction wave* $(V^r, U^r, \Theta^r)(x,t)$ *satisfies, if* $q \ge p$,

i. $U_x^r(x,t) \ge 0, |U_x^r| \le C\varepsilon$, *for* $t \ge 0, x \ge 0$,
ii. *For any* $p(1 \le p \le +\infty)$, *there exists a constant* $C_{p,q}$ *such that*
$\|(V_x^r, U_x^r, \Theta_x^r)\|_{L^p(x\ge0)} \le C_{p,q} \min\{\varepsilon^{1-\frac{1}{p}}, (t_0+t)^{-1+\frac{1}{p}}\}$,
$\|(V_{xx}^r, U_{xx}^r, \Theta_{xx}^r)\|_{L^p(x\ge0)} \le C_{p,q} \min\{\varepsilon^{2-\frac{1}{p}}, (t_0+t)^{-1+\frac{1}{q}}\}, \quad t \ge 0$.
iii. $(V^r, U^r, \Theta^r)|_{x\le\lambda(v_m, s_+)t} = (v_m, u_m, \theta_m)$,
$(V_x^r, U_x^r, \Theta_x^r, V_{xx}^r, U_{xx}^r, \Theta_{xx}^r)|_{x\le\lambda(v_m,s_+)t} = 0$.
iv. $\limsup\limits_{t\to+\infty, x\in\{x\ge0\}} |(V^r, U^r, \Theta^r)(x,t) - (v^r, u^r, \theta^r)(\frac{x}{t})| = 0$.

Let $(V^{cd}, U^{cd}, \Theta^{cd})$ be the viscous contact wave constructed in (1.6) and (1.7), where (v_+, u_+, θ_+) is replaced by (v_m, u_m, θ_m). Then $(V^{cd}, U^{cd}, \Theta^{cd})$ satisfies

$$\begin{cases} V_t^{cd} - U_x^{cd} = 0, \\ U_t^{cd} + (\frac{R\Theta^{cd}}{V^{cd}})_x = \mu(\frac{U_x^{cd}}{V^{cd}})_x + F^{cd}, \\ \frac{R}{\gamma-1}\Theta_t^{cd} + p_0 U_x^{cd} = \kappa(\frac{\Theta_x^{cd}}{V^{cd}})_x + \mu\frac{(U_x^{cd})^2}{V^{cd}} + G^{cd}, \end{cases} \tag{3.6}$$

where

$$F^{cd} = U_t^{cd} - \mu(\frac{U_x^{cd}}{V^{cd}})_x, \quad G^{cd} = -\mu\frac{(U_x^{cd})^2}{V^{cd}}. \tag{3.7}$$

Let

$$\begin{pmatrix} V \\ U \\ S \end{pmatrix}(x,t) = \begin{pmatrix} V^{cd}(x,t) + V^r(x,t) - v_m \\ U^{cd}(x,t) + U^r(x,t) - u_m \\ S^{cd}(x,t) \end{pmatrix}, \tag{3.8}$$

and

$$(\phi, \psi, \theta, \varphi)(x,t) = (v - V, u - U, \theta - \Theta, s - S)(x,t), \tag{3.9}$$

where $\Theta(x,t) = \frac{A}{R}V^{-\gamma+1}e^{\frac{\gamma-1}{R}S}$ satisfies $\Theta(0,t) = \theta_-$. Then the system (1.1) is rewritten as

$$\begin{cases} \phi_t - \psi_x = 0, \\ \psi_t + [p(v,\theta) - p(V,\Theta)]_x = \mu(\frac{u_x}{v})_x - \mu(\frac{U_x}{V})_x + F, \\ \frac{R}{\gamma-1}\zeta_t + p\psi_x + (p - p(V,\Theta))U_x \\ = (\frac{\kappa\theta_x}{v})_x + \frac{\mu u_x^2}{v} - (\frac{\kappa\Theta_x}{V})_x - \frac{\mu U_x^2}{V} + G, \\ (\frac{R\theta_-}{V+\phi} - \mu\frac{U_x + \psi_x}{V+\phi})|_{x=0} = p_0, \\ \zeta(0,t) = 0, \\ (\phi, \psi, \zeta)|_{t=0} = (v - V, u - U, \theta - U)(x,0) =: (\phi_0, \psi_0, \zeta_0)(x). \end{cases} \tag{3.10}$$

where

$$F = -[p(V,\Theta) - p(V^r, \Theta^r))]_x + [\mu\left(\frac{U_x}{V}\right)_x - U_t^{cd}] =: -F_1 + F_2, \tag{3.11}$$

$$\begin{aligned} G = &- \left[p(V,\Theta)U_x - p_0 U_x^{cd} - p(V^r,\Theta^r)U_x^r\right] \\ &+ [(\frac{\kappa\Theta_x}{V})_x + \frac{\mu U_x^2}{V} - (\frac{\kappa\Theta_x^{cd}}{V^{cd}})_x] =: -G_1 + G_2. \end{aligned} \tag{3.12}$$

To prove theorem 1.4, it is sufficient to show the same a priori estimate as proposition 2.1. We shall follow the same idea of §2 to achieve our goal. Similar to (2.11), we have

$$\begin{aligned} &(\frac{1}{2}\psi^2 + R\Theta\Phi(\frac{v}{V}) + \frac{R}{\delta}\Theta\Phi(\frac{\theta}{\Theta}))_t + \frac{\kappa\Theta}{v\theta^2}\zeta_x^2 + \frac{\mu\Theta\psi_x^2}{v\theta} \\ &+ H_x + Q_1 U_x^r + Q_2 = F\psi + \frac{\zeta G}{\theta}, \end{aligned} \tag{3.13}$$

where Φ, Ψ and H are defined in §2 and

$$\begin{array}{l} Q_1 = p(V,S)(\dfrac{p}{p(V,S)} - 1 + \gamma(\tilde{v}-1) - \dfrac{\delta}{R}(s-S)) \\ Q_2 = Q_1 U_x^{cd} + V p(V,S)(-\dfrac{\delta}{R}\Phi(\dfrac{v}{V}) + \Psi(\dfrac{\theta}{\Theta}))S_t^{cd} + \mu\psi_x U_x(\dfrac{1}{v} - \dfrac{1}{V}) \\ -\kappa\dfrac{\Theta_x}{\theta^2 v}\zeta\zeta_x - \kappa\dfrac{\zeta_x\phi\Theta\Theta_x}{\theta^2 vV} + \kappa\dfrac{\zeta\phi}{\theta^2 vV}\Theta_x^2 - \dfrac{\mu\zeta U_x^2}{\theta}(\dfrac{1}{v} - \dfrac{1}{V}) - \dfrac{2\mu\zeta}{v\theta}\psi_x U_x. \end{array} \tag{3.14}$$

Note that $p(v,s) = Av^{-\gamma}e^{\frac{\gamma-1}{R}s}$ is convex to v and s. Thus we have $Q_1 > 0$ and $Q_1 U_x^r \geq 0$ due to lemma 3.2. By the definition, we have

$$S_t^{cd} = \frac{R\gamma}{\delta V^{cd}} V_t^{cd} = \frac{R\gamma}{\delta V^{cd}} U_x^{cd}, \tag{3.15}$$

and

$$R\Theta_x = p(V,S)(-\delta + \frac{\gamma V}{V^{cd}})V_x^{cd} - \delta p(V,S)V_x^r. \tag{3.16}$$

We obtain

$$\begin{array}{rl} |Q_2| \leq & \dfrac{\kappa\Theta}{4v\theta^2}\zeta_x^2 + \dfrac{\mu\Theta}{4v\theta}\psi_x^2 + C(M_0)C(M_1)\{(\phi^2+\zeta^2)(|\Theta_{xx}^{cd}| + |\Theta_x^{cd}|^2) \\ & +(|U_x^r|^2 + \delta^2|V_x^r|^2)(\zeta^2+\phi^2)\}. \end{array} \tag{3.17}$$

Here do not need to estimate the terms involving the contact wave in (3.17) because we have already done in the previous section. By lemma 3.2, we have

$$\begin{array}{rl} & \displaystyle\int_0^\infty (|U_x^r|^2 + |V_x^r|^2)(\zeta^2+\phi^2) \\ \leq & C(M_0)C(M_1)(t_0+t)^{-1}(\|\phi\|\|\phi_x\| + \|\zeta\|\|\zeta_x\|) \\ \leq & \dfrac{\mu}{4^4 M_0^3}\|\dfrac{\zeta_x}{v^{\frac{1}{2}}}\|^2 + C(M_0)C(M_1)t_0^{-\frac{1}{4}}\{\|\dfrac{\phi_x}{v^{\frac{3}{2}}}\| \\ & +(1+t)^{-\frac{3}{2}}\|\zeta\|^2 + (1+t)^{-\frac{3}{2}}\|\sqrt{\Phi(\dfrac{v}{V})}\|^2\}. \end{array} \tag{3.18}$$

We now estimate the terms of right hand side of (3.13). Since

$$\begin{array}{l} |F_1| \leq C(M_0)\left(|V^r - v_m| + |\Theta^r - \theta_m|\right)|\Theta_x^{cd}| \\ \qquad +C(M_0)\left(|V^{cd} - v_m| + |\Theta^{cd} - \theta_m|\right)(|\Theta_x^r|), \end{array} \tag{3.19}$$

and $\Theta_x^r = V^r - v_m = \Theta^r - \theta_m = 0$ for any $x < \lambda(v_m, s_+)t$ due to lemma 3.2, lemma 1.1 gives

$$\|F_1\|_{L^1} \leq C(M_0)\delta e^{-\frac{Ct^2}{\delta(1+t)}}, \tag{3.20}$$

when t is large. On the other hand,

$$|F_2| \leq C(M_0)(|U_t^{cd}| + |U_{xx}^{cd}| + |U_x^{cd}V_x^{cd}| + |U_{xx}^r| + |U_x^r||V_x^{cd}| + |U_x^{cd}V_x^r|). \tag{3.21}$$

Similar to (3.20), we have from lemmas 1.1 and 3.2,

$$\int_0^\infty |U_{xx}^r| + |U_x^r||V_x^{cd}| + |U_x^{cd}V_x^r|dx \leq C(M_0)((1+t)^{-1+\frac{1}{q}} + \delta e^{-\frac{Ct^2}{\delta(1+t)}}), \tag{3.22}$$

with some constant $q \geq 8$. Thus we have

$$
\begin{aligned}
&\int_0^\infty |F\psi| dx \leq \frac{\mu}{4^3 M_0^2}\int_0^\infty \frac{\psi_x^2}{v} dx + C(M_0)C(M_1)\{\delta(1+t)^{-\frac{6}{5}} \\
&+ t_0^{-\frac{1}{16}}(1+t)^{-\frac{17}{16}} + \delta e^{-\frac{Ct^2}{\delta(1+t)}} + [t_0^{-\frac{1}{8}}(1+t)^{-\frac{9}{8}} + \delta^2(1+t)^{-\frac{8}{5}} \\
&+ \delta^2 e^{-\frac{Ct^2}{\delta(1+t)}}]\|\psi\|^2\}.
\end{aligned} \tag{3.23}
$$

In the same way, we obtain

$$
\begin{aligned}
&\int_0^\infty |G\frac{\zeta}{\theta}| dx \leq \frac{\mu}{4^4 M_0^3}\int_0^\infty \frac{\zeta_x^2}{v} dx + C(M_0)C(M_1)\{\delta(1+t)^{-\frac{6}{5}} \\
&+ t_0^{-\frac{1}{16}}(1+t)^{-\frac{17}{16}} + \delta e^{-\frac{Ct^2}{\delta(1+t)}} + [t_0^{-\frac{1}{8}}(1+t)^{-\frac{9}{8}} + \delta^2(1+t)^{-\frac{8}{5}} \\
&+ \delta^2 e^{-\frac{Ct^2}{\delta(1+t)}}]\|\frac{\zeta}{\sqrt{\delta}}\|^2\}.
\end{aligned} \tag{3.24}
$$

The details are omitted. Thus integrating (3.13) over $R_+ \times (0,t)$, combining (2.18), (3.14)-(3.24) and using the fact that $H(0,t) = 0$ yields

$$
\begin{aligned}
&\|(\sqrt{\Phi(\frac{v}{V})}, \psi, \frac{1}{\sqrt{\delta}}\zeta)(t)\|^2 + \int_0^t \{\|(\frac{\psi_x}{v^{\frac{1}{2}}}, \frac{\zeta_x}{v^{\frac{1}{2}}})(\tau)\|^2\} d\tau \\
\leq\ & C(M_0)\big\{\|(\phi_0, \psi_0, \sqrt{\delta}\varphi_0)\|_1^2 + C(M_1)[\delta + t_0^{-\frac{1}{16}} \\
&+ (\delta + t_0^{-\frac{1}{4}})\int_0^t \|(\frac{\phi_x}{v^{\frac{3}{2}}}, \frac{\zeta_x}{v^{\frac{1}{2}}})\|^2 d\tau \\
&+ \int_0^t (t_0^{-\frac{1}{8}} + \delta^2)(1+t)^{-\frac{9}{8}}\|(\sqrt{\Phi(\frac{v}{V})}, \psi, \frac{1}{\sqrt{\delta}}\zeta)\|^2 dt]\big\}.
\end{aligned} \tag{3.25}
$$

Following the same procedure as in lemma 2.3, we have

$$
\begin{aligned}
&\|(\sqrt{\Phi(\frac{v}{V})}, \psi, \frac{1}{\sqrt{\delta}}\zeta)(t)\|^2 + \|\frac{\tilde{v}_x}{\tilde{v}}\|^2 + \int_0^t \{\|(\frac{\phi_x}{v^{\frac{3}{2}}}, \frac{\psi_x}{v^{\frac{1}{2}}}, \frac{\zeta_x}{v^{\frac{1}{2}}})\|^2\} dt \\
\leq\ & C(M_0)\big\{\|(\phi_0, \psi_0, \sqrt{\delta}\varphi_0)\|_1^2 + C(M_1)[\delta + t_0^{-\frac{1}{16}} \\
&+ (\delta + t_0^{-\frac{1}{4}})\int_0^t \|(\frac{\phi_x}{v^{\frac{3}{2}}}, \frac{\zeta_x}{v^{\frac{1}{2}}})\|^2 dt \\
&+ \int_0^t (t_0^{-\frac{1}{8}} + \delta^2)(1+t)^{-\frac{9}{8}}\|(\sqrt{\Phi(\frac{v}{V})}, \psi, \frac{1}{\sqrt{\delta}}\zeta)\|^2 dt]\big\}.
\end{aligned} \tag{3.26}
$$

We now choose some constants $\delta_0 < 1$ and $t_0 > 1$ such that

$$
C(M_0)C(M_1) \times \max\{\delta_0, t_0^{-\frac{1}{16}}\} < \frac{1}{2}. \tag{3.27}
$$

Then for any $\delta \leq \delta_0$, the Gronwall's inequality yields

$$
\begin{aligned}
&\|(\sqrt{\Phi(\frac{v}{V})}, \psi, \frac{1}{\sqrt{\delta}}\zeta)(t)\|^2 + \|\frac{\tilde{v}_x}{\tilde{v}}\|^2 + \int_0^t \{\|(\frac{\phi_x}{v^{\frac{3}{2}}}, \frac{\psi_x}{v^{\frac{1}{2}}}, \frac{\zeta_x}{v^{\frac{1}{2}}})(\tau)\|^2\} d\tau \\
\leq\ & C(M_0)\big\{1 + \|(\phi_0, \psi_0, \sqrt{\delta}\varphi_0)\|_1^2\big\}.
\end{aligned} \tag{3.28}
$$

Due to (2.31)-(2.33), there exists a positive constant M_2 which only depends on M, M_0 and the initial data such that

$$M_2^{-1} \le v(x,t) \le M_2. \tag{3.29}$$

Let $M_1 = M_2$, then we obtain $(2.8)_1$ and

$$\begin{aligned} & \|(\phi, \psi, \frac{1}{\sqrt{\delta}}\zeta)(t)\|^2 + \|\phi_x\|^2 + \int_0^t \{\|(\phi_x, \psi_x, \zeta_x)(\tau)\|^2\} d\tau \\ \le\ & C(M_0)\big\{1 + \|(\phi_0, \psi_0, \varphi_0)\|_1^2\big\}. \end{aligned} \tag{3.30}$$

The estimates on ζ_x are omitted here because it is similar to lemma 2.4. Thus we obtain the same a priori estimate as proposition 2.1. Theorem 1.4 is easy from the local existence and the a priori estimate.

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